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SMOOTH REGULAR NEIGHBORHOODS

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Introduction

Henry Whitehead [14, 15] investigated what he termed the *regular neighborhoods* of a subcomplex of a combinatorial manifold, and demonstrated the existence and uniqueness (up to combinatorial equivalence) of such a neighborhood. Our purpose here is to prove analogous theorems for subcomplexes of a smooth manifold. If K is a subcomplex of a C^∞ n-manifold M, a *smooth regular neighborhood* of K is a subset N of M that satisfies these two conditions:

(1) N is an n-dimensional, closed, C^∞ submanifold of M.

(2) N is a regular neighborhood of K in some smooth triangulation of M.

We prove that such neighborhoods always exist, and that any two are diffeomorphic (Theorem 1). As applications, we show that if two smooth manifolds M_1 and M_2 of the same simple homotopy type are differentiably imbedded in a high dimensional euclidean space, their closed tubular neighborhoods are diffeomorphic (Theorem 5). If there is a simple homotopy equivalence $M_1 \to M_2$ covered by a tangent bundle equivalence, then $M_1 \times D^k$ and $M_2 \times D^k$ are diffeomorphic for large k (Theorem 6). If a smooth manifold M is combinatorially equivalent to the n-sphere S^n, and if M bounds a π-manifold, then $M \times D^3$ and $S^n \times D^3$ are diffeomorphic (Theorem 7). An unknotting theorem, proved also by Smale and Kosinski, is obtained (Theorem 8).

Barry Mazur has also developed a theory of smooth neighborhoods. (Ann. of Math., to appear). The neighborhoods studied here are defined by a *geometric* condition, namely, that of collapsibility to the complex; Mazur's neighborhoods, in contrast, are defined by *algebraic* conditions on the inclusion map of the complex. Since these conditions are implied by collapsibility, Mazur's class of neighborhoods is larger, andh is uniqueness theorem more powerful.

Notation and definitions

Euclidean n-space is denoted by R^n, the closed unit interval [0, 1] by I. A manifold is *smooth* (or *differential*) if it is C^∞; such a manifold M has tangent bundle $T(M)$. Bundle equivalence is symbolized by $\sim$, homotopy by $\simeq$, diffeomorphism by $\approx$, differentiable isotopy by $\cong$, and combinatorial equivalence by $\equiv$. If A is a compact, unbounded, differential submanifold of a manifold B, then $U(A)$ or $U(A, B)$ is a *closed tubular neighborhood* of A in B, i.e., $\{x \in B \mid d(x, A) \leqq \varepsilon\}$ for a suitable $\varepsilon > 0$, where d is a global metric on B defined by a riemannian metric. The boundary of X is $\dot{X}$; the interior is int X. For *simple homotopy type* and related ideas, see [14, 15]. If K is a subcomplex of a combinatorial manifold M, then $N(K)$ denotes the *second regular neighborhood* of K, i.e., the union of those closed simplices of the second barycentric subdivision of M that meet K. *A smooth triangulation* of a differential manifold M is a C^∞ triangulation in the sense of [17]. A subcomplex of a smooth triangulation of M is called a subcomplex of M. Any identity map is denoted 1. A map $M \to M$ is *piecewise regular* if each closed simplex of some smooth triangulation is mapped diffeomorphically.

The main theorem

We recall that a *regular neighborhood* N of a subcomplex K of a combinatorial n-manifold M is a subcomplex of M which is also an n-manifold, and which *collapses* to K. This means that one can find subcomplexes $A_0, \cdots, A_p$ and closed simplices $\sigma_1, \cdots, \sigma_p$ such that $N = A_0$, $K = A_p$, $A_{i-1} = A_i \cup \sigma_i$, and $A_i \cap \sigma_i$ is the union of all but one of the faces of σ_i. (Here subcomplex, simplex, etc., refer to some rectilinear subdivision of M.) It is not necessary for N to be a neighborhood of K.

Now suppose that M is a differential n-manifold and that K is a subcomplex. A *smooth regular neighorhood* of K is defined to be a subcomplex N of M which is a smooth submanifold of M, and which is a regular neighborhood of K in some smooth triangulation of M.

For the next theorem, K is a finite subcomplex of a smooth n-manifold M. We assume that $\dot{M}$ is void.

THEOREM 1. (a) *There is a smooth regular neighborhood of* K.

(b) *If* N_1 *and* N_2 *are smooth regular neighborhoods of* K, *there is a diffeomorphism* $h: M \to M$ *such that* $hN_1 = N_2$ *and* $h \cong 1$.

(c) *If* $K \subset \operatorname{int} N_1 \cap \operatorname{int} N_2$ *and* W *is an open set of* M *containing* $N_1 \cup N_2$, *the diffeomorphism* h *can be chosen so that* $h(x) = x$ *if* $x \in K$ *or* $x \in M - W$.

PROOF. To prove (a) it suffices to establish the following stronger result:

(1a′) *Let* L *be a regular neighborhood of* K *and let* $U \supset L$ *be an open set. There is a piecewise regular homeomorphism* $\psi: M \to M$ *such that*

(i) ψL *is a smooth regular neighborhood of* K

(ii) $\psi | M - U = 1$

(iii) $\psi | K = 1$ *if* $K \subset \operatorname{int} L$.
To establish (1a′), we apply Theorem 2.5 of [4], which immediately gives the desired homeomorphism.

(For completeness, we outline the construction of ψ. There is a piecewise linear homeomorphism $M \to M$ taking L onto $N(L)$, which is contained in U if we first subdivide M sufficiently finely. This is because $N(L) - \operatorname{int} L \equiv \dot{N}(L) \times I$. Now $\dot{N}(L)$ is pushed by a piecewise regular map onto a smooth submanifold by means of the natural vector field transverse to $\dot{N}(L)$.)

To prove part (b) of Theorem 1, we assume that $K \subset \operatorname{int}(N_1 \cap N_2)$. There is no loss of generality, since from (1a′) we infer that N_i (for $i = 1, 2$) has a smooth regular neighborhood N'_i such that $N'_i - \operatorname{int} N_i \equiv \dot{N}_i \times I$, and by Thom [13] or Munkres [10] we conclude that $N'_i - \operatorname{int} N_i \approx \dot{N}_i \times I$. Hence there are diffeomorphisms $g_i \colon M \to M$ with $g_i N_i = N'_i$ and $g_i \cong 1$. Clearly $K \subset \operatorname{int}(N'_1 \cap N'_2)$.

This argument also proves that if $A \subset M$ is a smooth submanifold of dimension n and if A' is a smooth regular neighborhood of A, there is a diffeomorphism $g \colon M \to M$ such that $gA = A'$ and $g \cong 1$. This fact is used below.

Next we observe that since N_i collapses to K, $N(N_i)$ collapses to $N(K)$. This is implied by Lemma 11 of J. H. C. Whitehead [14]. Replacing $N(N_i)$ and $N(K)$ by smooth regular neighborhoods as in (1a′), we conclude that there are smooth regular neighborhoods A_i of N_i and C of K such that A_i is also a smooth regular neighborhood of C. Moreover, if $K \subset \operatorname{int}(N_1 \cap N_2)$, as we may assume, then $C \subset \operatorname{int}(A_1 \cap A_2)$. Therefore these are diffeomorphisms $f_i, h_i \colon M \to M$ such that $h_i N_i = A_i$ and $f_i A_i = C$, and all four diffeomorphisms are $\cong 1$. Thus $(f_2 h_2)^{-1} f_1 h_1$ takes N_1 onto N_2, proving (b).

Part (c) follows easily from the above constructions.

COROLLARY 2. *If K collapses to a subcomplex K', then K and K' have diffeomorphic smooth regular neighborhoods.*

PROOF. If N is a smooth regular neighborhood of K, N collapses to K', and (b) applies.

Actually, the proof of (b) proves the following.

COROLLARY 3. *Let K collapse to K'. Let N and N' be smooth regular neighborhoods of K and K', respectively, such that $K \subset \operatorname{int} N$, $K' \subset \operatorname{int} N'$. Then there is a diffeomorphism $h \colon M \to M$ such that $hN = N'$ and $h \cong 1$.*

REMARK. The following result can be proved. We do not use it in the present paper, and the proof is omitted.

THEOREM. *Let* K_1 *and* K_2 *be finite subcomplexes of an unbounded smooth manifold* M *with smooth regular neighborhoods* N_1 *and* N_2 *respectively. Suppose there is a piecewise linear isotopy of* $N(K)$ *in* M *carrying* K_1 *onto* K_2. *Then* $N_1 \approx N_2$.

The isotopy referred to is a piecewise linear imbedding $G: N(K)\times I \to M \times I$ such that $G(x, 0) = x$, $G(N(K)\times t) \subset M \times t$ for all $x \in N(K)$ and $t \in I$.

Applications

The following lemma is frequently assumed, but there seems to be no proof in the literature.

LEMMA 4. *Let* M *be a smooth n-dimensional submanifold of a smooth p-dimensional manifold* V; *we assume that* M *and* V *are unbounded and that* M *is compact. Then* M *is a subcomplex of* V *and* $U(M, V)$ *is a smooth regular neighborhood of* M.

PROOF. We first find a smooth triangulation of V making M a subcomplex. Take V as being smoothly imbedded in R^q. Whitehead showed in [17] that there is a rectilinear subcomplex A of R^q whose vertices are in V such that $A \subset U(V, R^q)$ and the orthogonal retraction π_V: $U(V, R^q)\to V$ restricts to A to yield a smooth triangulation $\pi_V: A \to V$. Likewise there is a rectilinear subcomplex B of R^q contained in $U(M, R^q)$ whose vertices are in M, and such that $\pi_M: B \to M$ is a smooth triangulation of M. Now instead of using the $(q-p)$-planes normal to V to define the retraction π_V, we may use instead an approximating family of $(q-p)$-planes which is piecewise linear, considering the family as a function defined on the complex A. We assume that π_V is defined in this way. We also assume that B is contained in a neighborhood of V so small that each of the $(q-p)$-planes meets B in just one point, and is transverse to B at that point. This defines a piecewise linear imbedding $\alpha: B \to A$. Then $\pi_V\alpha: B \to V$ is an approximation to $\pi_M: B \to M$, and $\pi_V\alpha(B)$ is a subcomplex of V. By using [17], matters can be so arranged that $\pi_V\alpha(B)$ lies in $U(M, V)$ and is cut transversely, and at exactly one point, by each radius of $U(M, V)$. (Here we assume V is given the riemannian metric induced from the standard metric of R^q.) There is now no difficulty in pushing $\pi_V\alpha(B)$ homeomorphically onto M by a homeomorphism h: $V \to V$ that maps each simplex of rectilinear subdivision of $\pi_V(A)$ diffeomorphically. Thus M is a subcomplex of V.

There is a family γ of curves in $N(M)$ transverse to $N(M)^{\cdot}$ and to M, and such that each $x \in N(M) - M$ lies on a unique such curve, γ_x. We may choose a tubular neighborhood $U = U(M, V)$ of radius so small that $U \subset \text{int } N(M)$, and each point of $\dot{U}$ lies on a unique curve in γ, and that each γ_x cuts $\dot{U}$ transversely. This implies that $N(M) - \text{int } U \equiv N(M)^{\cdot} \times I$. It follows easily that U collapses to M; hence U is a smooth regular neighborhood of M, and the lemma is proved.

Now let M_1 and M_2 be smooth n-manifolds, compact and with void boundaries.

THEOREM 5. *Let* $f\colon M_1 \to M_2$ *be a simple homotopy equivalence. Let* $g_i\colon M_i \to V (i = 1, 2)$ *be smooth imbeddings in a differential manifold* V *such that* $g_1 \simeq g_2 f$. *If* $\dim V \geqq 2n + 5$, *then* $U(g_1 M_1) \approx U(g_2 M_2)$.

PROOF. We may assume that $g_1 M_1 \cap g_2 M_2 = \varnothing$. By Lemma 4, we assume that $g_1 M_1 \cup g_2 M_2$ is a subcomplex of V. Since f is a simple homotopy equivalence, there is a complex K collapsing to M_1 and M_2; we may take M_1 and M_2 disjoint in K, and $\dim X \leqq n + 2$. The imbedding $M_1 \cup M_2 \to V$ can be extended to a map $K \to V$ for homotopical reasons, and to an imbedding $K \to V$ for dimensional reasons. Thus there is a subcomplex K of V collapsing to both $g_1 M_1$ and $g_2 M_2$. It follows from Corollary 2 that $g_1 M_1$ and $g_2 M_2$ have smooth regular neighborhoods that are diffeomorphic. Now apply Lemma 4 again.

A version of the following result was announced by Mazur [18].

THEOREM 6. *Let* $f\colon M_1 \to M_2$ *be a simple homotopy equivalence such that* $f^* T(M_2) \sim T(M_1)$. *Then* $M_1 \times D^k \approx M_2 \times D^k$ *for* $k \geqq n + 5$.

PROOF. We apply Theorem 5, taking $V = M_2 \times D^k$. The map $f \times 0\colon M_1 \to M_2 \times D^k$ can be approximated by a smooth imbedding $g_1\colon M_1 \to M_2 \times D^k$. Reasoning as in [8], we see that $g_1 M_1$ has a trivial normal bundle. Obviously $g_2 M_2$ has also, where $g_2\colon M_2 \to M_2 \times D^k$ is given by $g(x) = (x, 0)$. By Theorem 5, $U(M_1) \approx U(M_2)$. By the triviality of the normal bundles, $U(M_i) \approx M^i \times D^k$, which completes the proof.

REMARK. Whitehead [14, 15] proved that every homotopy equivalence $K_1 \to K_2$ is simple if $\pi_1(K_1)$ is cyclic of order 1, 2, 3, 4 or ∞.

The next two applications rely on rather deep combinatorial results of Zeeman [19].

THEOREM 7. *Let* M *be a smooth manifold combinatorially equivalent to* S^n.

(a) *If* $g\colon M \to R^{n+k}$ *is a smooth imbedding,* $U(gM) \approx S^n \times D^k$ *for* $k \geqq 3$.

(b) *If* M *bounds a compact* π*-manifold,* $M \times D^k \approx S^n \times D^k$ *for* $k \geqq 3$.

PROOF. (a) By Lemma 4, there is a smooth triangulation τ of R^{n+k} in which gM is a subcomplex. By Whitehead's uniqueness theorem for smooth triangulations [17], τ can be chosen so as to be isomorphic to a rectilinear triangulation; thus there is a homeomorphism $h: R^{n+k} \to R^{n+k}$ which is diffeomorphic on each simplex of τ, and which takes gM onto a combinatorial n-sphere Σ. By Zeeman [19], Σ bounds a combinatorial $(n+1)$-cell $E_0 \subset R^{n+k}$ provided $k \geqq 3$. Thus $h^{-1}E_0 = E$ is a subcomplex of R^{n+k} bounded by gM. We may flatten part of an $(n+1)$-simplex σ of E; let D be a small $(n+1)$-disk in the flat part of σ. Put $A = E - \text{int } D$. We can triangulate A so that it collapses to both $\dot{D}$ and gM. By Corollary 2 and Lemma 4, $U(gM) \approx U(\dot{D})$, and clearly $U(\dot{D}) \approx S^n \times D^k$. Part (b) follows from (a) once we show that there is a smooth imbedding $M \to R^{n+k}$ with trivial normal bundle. If M bounds a compact π-manifold, this follows from [3, 5].

The following unknotting theorem strengthens Theorem 7a, and has also been proved by A. Kosinski and (independently) by S. Smale.

We take S^n as a submanifold of S^{n+k}.

THEOREM 8. *Let M be a smooth manifold combinatorially equivalent to S^n, and let $g: M \to S^{n+k}$ be a smooth imbedding. If $k \geqq 3$, there is a diffeomorphism $h: S^{n+k} \to S^{n+k}$ such that*

(a) $h \cong 1$

(b) $hU(gM) = U(S^n)$.

PROOF. The proof is essentially the same as that of Theorem 7a, applying Corollary 3 instead of Corollary 2.

Remarks

1. Smale [11, 12] has shown that an n-manifold is combinatorially equivalent to S^n if it is homotopically equivalent and $n \geqq 5$.

2. Not all homotopy spheres bound π-manifolds [7].

3. M. Kervaire [6] has proved that a smoothly imbedded S^n in R^{n+k} always has a trivial normal bundle if $k > (n+1)/2$; see [2] for another proof. On the other hand, A. Haefliger has demonstrated the existence of an imbedding $S^{11} \to R^{17}$ with a non-trivial normal bundle.

4. Every homotopy n-sphere is imbeddable in R^{n+k} for $k > (n+1)/2$, according to Haefliger [1].

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