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On stretch factors of pseudo-Anosov maps

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On stretch factors of pseudo-Anosov maps

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by

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On stretch factors of pseudo-Anosov maps

Joshua Charles Pankau

Dedicated to my wife Jacqueline. Her boundless love and constant support gave me the strength to see this journey through.

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Abstract

On stretch factors of pseudo-Anosov maps

by

Joshua Charles Pankau

In 1974, Thurston proved that, up to isotopy, every automorphism of closed orientable surface is either periodic, reducible, or pseudo-Anosov. The latter case has lead to a rich theory with applications ranging from dynamical systems to low dimensional topology. Associated with every pseudo-Anosov map is a real number $\lambda > 1$ known as the *stretch factor*. Thurston showed that every stretch factor is an algebraic unit but it is unknown exactly which units can appear as stretch factors. Though this question remains open, we provide a partial answer by showing a large class of units are obtainable as stretch factors using a construction due to Thurston. We will show that every Salem number has a power that is the stretch factor of a pseudo-Anosov map arising from Thurston's construction, and then we will use the techniques to generalize to a much larger class of units. We also show that every totally real number field K is of the form $K = \mathbb{Q}(\lambda + \lambda^{-1})$, where λ is the stretch factor of a pseudo-Anosov map arising from Thurston's construction. Finally, we develop a new method of constructing closed orientable surfaces from positive integer matrices that will be crucial to proving the above results.

Contents

Chapter 1

Introduction

The group of automorphisms of a surface modulo isotopy, also known as the mapping class group of a surface, first appeared in the early parts of the 20th century. This was a natural object to study as the burgeoning ideas of Topology were beginning to mix with the rigid world of algebra. The mapping class group offered another algebraic tool that could be used to study properties of surfaces. Some early contributors to this study include German born mathematician Max Dehn, whose work will appear frequently in the chapters to come, and Danish born mathematician Jakob Nielsen, who studied the induced action of surface automorphisms on the boundary of hyperbolic space. Dehn introduced the concept of the Dehn twist automorphisms and proved that the mapping class group is finitely generated by these. Nielsen provided a classification of mapping class group elements but supposedly due to lack of organization this result went unnoticed for decades.

In the 1970's, American mathematician William P. Thurston studied the mapping class group from the theory of measured foliations, which he was developing. Thurston proved that elements of the mapping class group fall into three categories: periodic, reducible, and pseudo-Anosov. Upon announcing his findings it was discovered that Nielsen had made similar discoveries, albeit from a different perspective. Now, this result is known as the celebrated Nielsen-Thurston classification. The work herein will be primarily concerned with the pseudo-Anosov case.

A homeomorphism ϕ from a closed orientable surface S_g to itself is called *pseudo-Anosov* if there is a pair of transverse measured folations $\mathcal{F}_s$ and $\mathcal{F}_u$ of S_g in which ϕ stretches $\mathcal{F}_u$ by a real number $\lambda > 1$, and contracts $\mathcal{F}_s$ by a factor of λ^{-1}. The number λ is known as the *stretch factor* of ϕ. Thurston showed in [19] that the stretch factor of any pseudo-Anosov map is an algebraic unit whose degree over $\mathbb{Q}$ is bounded above by $6g - 6$. The question as to exactly which algebraic units appear as stretch factors of pseudo-Anosov maps has remained open for the last 40 years.

For the most part, the study of this question has been from the perspective of general constructions of pseudo-Anosov maps. Thurston provided such a construction, known unsurprisingly as *Thurston's construction*, which describes pseudo-Anosov maps as products of Dehn twists around simple closed curve that divide the surface into disks. We will give a detailed overview of this construction in Chapter 2 but similar discussions of this construction can be found in either [8], Exposé 13, or [7]. In [15], Penner describes a similar but different construction involving products of Dehn twists. These constructions can produce a large class of pseudo-Anosov maps, and share some overlap but there are pseudo-Anosov maps attainable by one construction and not the other. It is also known that these constructions cannot attain every pseudo-Anosov map though. In a recent paper [18], Shin and Strenner showed that if λ is a stretch factor of a pseudo-Anosov map coming from Penner's construction then λ does not have Galois conjugates on the unit circle. Whereas, we will show that if λ is a stretch factor coming from Thurston's construction then $\mathbb{Q}(\lambda + \lambda^{-1})$ is a totally real number field.

In this treatise we will focus on Thurston's construction, and a certain type of algebraic unit known as a *Salem number*. Salem numbers have complex conjugates on the unit circle so they cannot arise as stretch factors from Penner's construction. On the other hand, there are Salem numbers that appear as stretch factors from Thurston's construction, and many of the smallest known stretch factors are Salem numbers. A natural question to ask is if we can get every Salem number as a stretch factor, and we will show this is true up to some power of the Salem number. The methods and tools we will develop to prove this result about Salem numbers we will then generalize to prove the following.

Theorem A. *Let $\lambda > 1$ be a real algebraic unit with $[\mathbb{Q}(\lambda) : \mathbb{Q}] = n$, $\lambda + \lambda^{-1}$ totally real, and every Galois conjugate of λ lies in between λ^{-1} and λ in absolute value. Then there is a k such that λ^k is the stretch factor of a Thurston construction pseudo-Anosov homeomorphism of the surface $S_{(n+e)^2-(n+e)+1}$, where $e \in \{0, 1, 2, 3\}$.*

In other words, we will show that if $\lambda > 1$ is an algebraic unit satisfying the known restrictions of Thurston's construction then some power of λ is a stretch factor of a pseudo-Anosov map coming from Thurston's construction.

Since we know that $\mathbb{Q}(\lambda + \lambda^{-1})$ is a totally real number field when λ is a stretch factor from Thurston's construction, it is natural to ask which totally real number fields arise this way. We also prove:

Theorem B. *Every totally real number field is of the form $K = \mathbb{Q}(\lambda + \lambda^{-1})$, where λ is the stretch factor of a pseudo-Anosov map arising from Thurston's construction.*

Important to the proof of both of these theorems will be the idea of constructing a closed orientable surface from a nonsingular, positive, integer matrix. We describe such a process and prove the following in chapter 3.

Theorem C. *Given an $n \times n$ nonsingular, positive, integer matrix Q, there is a closed orientable surface S_g containing two tight filling multicurves, A and B, whose intersection matrix is Q. Furthermore, if all the entries of Q are greater than and equal to 2, then the genus of this surface is $g = n^2 - n + 1$.*

The structure of the dissertation is as follows: In chapter 2 we will discuss curves on surfaces, and define the mapping class group, then provide a description of Thurston's construction. In chapter 3 we describe a construction of a surface from a positive integer matrix and prove Theorem C. In chapter 4 we define Salem numbers, develop the tools necessary to show that every Salem number has a power that is a stretch factor, and then generalize those results to prove Theorem A. We also introduce some ideas from algebraic number theory and use them to prove Theorem B. In chapter 5 we talk about possible future work that can be done with these results. We also provide an appendix that contains various proofs and discussions that would otherwise bog down the flow of this dissertation.

Chapter 2

The Mapping Class Group

2.1 Curves on Surfaces

Simple closed curves on a surface play a critical role in studying the mapping class group, as automorphisms of the surface send simple closed curves to simple closed curves. You can cut a surface along a simple closed curve to obtain a simpler surface, and they can be used to give the surface a cell structure. These notions, as well as the notion of counting intersections between two simple closed curves, will be important to the presented work so we now take some time to develop some of the theory about simple closed curves. Throughout this dissertation, we will let S_g denote the closed orientable surface of genus g.

2.1.1 Definitions

Definition 2.1.1. *A* ***simple closed curve*** *on a surface is an embedding* $\alpha : \mathbb{S}^1 \to S_g$.

We will blur the lines between the map α and its image, and simply refer to the curve as α.

Definition 2.1.2. *A simple closed curve is said to be* ***essential*** *if it is not homotopic to a point.*

We will often be looking at collections of disjoint pairwise non-isotopic essential simple closed curves, and call such a collection a ***multicurve***. It is a well known fact that the maximum number of disjoint pairwise non-isotopic essential simple closed curves on a genus g surface is $3g-3$. This is often referred to as the pants decomposition.

An important feature of simple closed curves is that we can use them to cut the surface into disks. For example cutting the torus along a longitudinal simple closed curve, and then along the meridian gives us one disk.

Definition 2.1.3. *A pair of multicurves $A = \{\alpha_1, ..., \alpha_n\}$ and $B = \{\beta_1, ..., \beta_m\}$ are said to* ***fill*** *S_g if $S_g - A \cup B$ is a union of topological disks.*

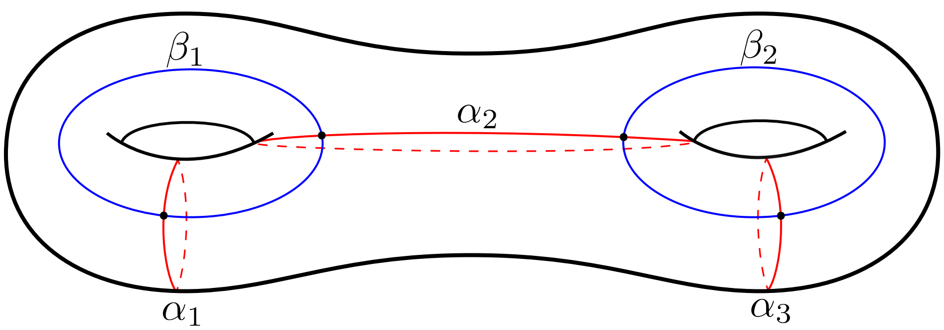


Figure 2.1: A pair of filling mulitcurves on S_2.

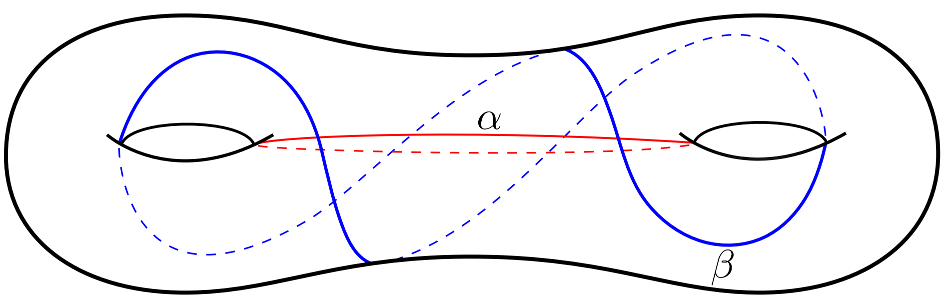


Figure 2.2: A pair of curves that fill S_2.

2.1.2 Intersection Number

We now discuss two natural ways of counting intersection of curves on a orientable surface. Since we can always homotope any two curves so that they are transverse, we will assume that every intersection of the two curves are transverse intersections. Let α and β be two oriented curves on S_g. If we choose an orientation for the surface, then the basis vectors for the tangent lines to α and β in the direction of their orientation give a basis for the tangent plane at the point of intersection. If this basis lies in the orientation class of the basis given by our choice of orientation for the surface, then we will assign a $+1$ to this intersection. Otherwise we assign a -1 to the intersection. This choice of $+1$ or -1 will be called the **index** of the intersection.

Definition 2.1.4. *The **algebraic intersection number** between two simple closed curves α and β on S_g, denoted by $\hat{i}(\alpha, \beta)$ is the sum of the indices of each intersection.*

The algebraic intersection number only depends on the homology class of the curves, so we can talk about the algebraic intersection number of the free homotopy classes a and b of α and β, respectively. Another way of counting intersection between two curves is to count the minimal number of intersections between any two curves in the free homotopy class of each curve.

Definition 2.1.5. *The **geometric intersection number** between two simple closed curves α and β, denoted by $i(\alpha, \beta)$ is the minimal number of intersections between any two curves in the free homotopy classes a and b of α and β, respectively. That is*

$$i(\alpha, \beta) = \min \{ |\alpha' \cap \beta'| \ : \ \alpha' \in a, \beta' \in b \}$$

In general, the algebraic intersection is easier to calculate, but we will often calculate the geometric intersection number by exhibiting two curves who already meet a minimal number of times. We give a name to such curves:

Definition 2.1.6. *Two curves α and β are said to be in* ***minimal position*** *if* $i(\alpha, \beta) = |\alpha \cap \beta|$.

There is a combinatorial condition for deciding whether two curves are in minimal position. This condition gives a procedure for taking curves α and β which are not in minimal position, to homotopic curves α' to β' which are in minimal position. First a definition:

Definition 2.1.7. *Two curves α and β are said to form a* ***bigon*** *if there is an embedded disk whose boundary is the union an arc of α and an arc of β intersecting in exactly two points.*

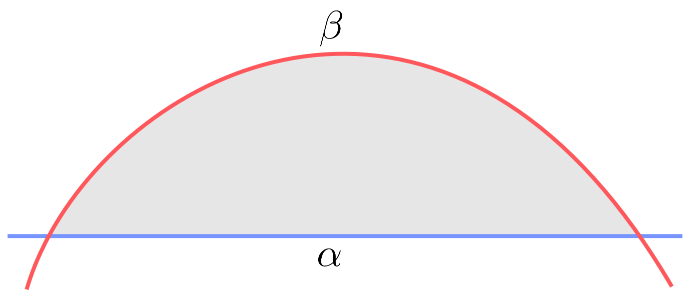


Figure 2.3: A bigon formed by two curves α and β.

Proposition 2.1 (Bigon Criterion)**.** *Two curves are in minimal position if and only if they do not form a bigon.*

With this proposition in mind we can take two simple closed curves α and β and isotope them to be in minimal position. We do this as follows: A bigon formed by an arc of α and an arc of β bounds a disk, and we can either isotope α or β across that disk until we have reduced the number of intersections by 2, provided there are no other arcs of α or β within the bigon. If we are in such a situation, then since we are talking about simple closed curves we know that any arc of α or β that enters the interior of the bigon must also leave the bigon. So there will be an inner bigon formed by another arc of α and β. These curves can only intersect a finite number of times so there must be an *inner most* bigon, see Figure 2.4 below. Now, we isotope across the inner most bigon to reduce the number of intersections by 2, then repeat this process until α and β no longer form any bigons. The resulting curves will now be in minimal position.

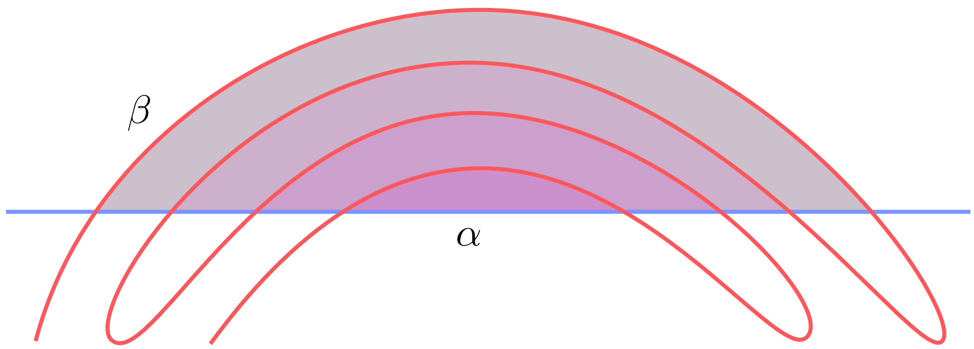


Figure 2.4: The curves α and β form several nested bigons. Isotoping β across the inner most bigon, and then repeating this process will result with two curves that are in minimal position.

We can extend this idea of minimal position to multicurves.

Definition 2.1.8. *Two multicurves $A = \{\alpha_1, ..., \alpha_m\}$ and $B = \{\beta_1, ..., \beta_n\}$ are said to be **tight** if each α_i and β_j are in minimal position.*

For example the curves in Figure 2.1 are tight multicurves, as well as fill the surface. Having a pair of multicurves on S_g that are tight and fill the surface is helpful because this defines a cell structure on S_g. The 1-skeleton is formed by vertices that are the intersection points of the multicurves, and the edges are the arcs of curves connecting two vertices. Since the multicurves fill the surface, the complement of the 1-skeleton is a collection of topological disks, hence we have given a cell structure to S_g. We will use this cell structure during our discussion of Thurston's construction in subsection 2.3.3.

Finally, we can use the cell structure of S_g defined by tight filling multicurves to compute an important invariant of the surface known as the ***Euler characteristic***, denoted $\chi(S_g)$. For a closed orientable surface, $\chi(S_g) = 2 - 2g$ and given a cell structure on S_g with V vertices, E edges, and F faces, then we have that $V - E + F = 2 - 2g$. This equation will allow us to compute the genus of surfaces that we construct from integer matrices in chapter 3.

2.2 The Mapping Class Group

2.2.1 Definitions

Let S_g denote the closed orientable surface of genus g and define the set $\text{Homeo}^+(S_g)$ to be the set of orientation preserving homeomorphisms of S_g. We can endow this set with the following topology: For any compact subset K and open subset U of S_g define

$$V(K,U) = \left\{\phi \in \text{Homeo}^+(S_g) \mid \phi(K) \subset U\right\}$$

In general, it is not true that the intersection of two of these sets can be written as a union of the $V(K,U)$, so these sets do not form a base for a topology, but they do form a subbase for a topology. We call this topology the ***Compact Open Topology***. Now let $\pi_0(Homeo^+(S_g))$ denote the set of path components of $\text{Homeo}^+(S_g)$. Now a path between two homeomorphisms f_0 and f_1 defines a homotopy f_t between the two maps, where $f_t \in \text{Homeo}^+(S_g)$ for all $t \in [0,1]$. Such a homotopy is called an ***isotopy***.

Isotopy defines an equivalence relation on $\text{Homeo}^+(S_g)$ and the equivalence classes, also called the ***isotopy classes***, are precisely the elements of $\pi_0(\text{Homeo}^+(S_g))$. If we let $[f]$ denote the isotopy class of f, then we can endow the set of isotopy classes with a group structure where the operation is given by $[f] \cdot [g] = [f \circ g]$. We are now ready to define the mapping class group of a surface S_g.

Definition 2.2.1. *The* ***Mapping Class Group*** *of S_g, denoted $Mod(S_g)$, is equal to $\pi_0(Homeo^+(S_g))$ endowed with the above group structure.*

The mapping class in general is a very complicated object, but we will see in the next section that it is actually finitely generated. Though we are only defining the mapping class group for closed orientable surfaces, this group can be defined for any surface, with slight modifications to the definition. We now give some typical examples of mapping class groups that also highlight the modifications to the definition.

Example 2.2.1. Let D^2 denote the closed disk homeomorphic to the closed unit disk in $\mathbb{R}^2$, then $\text{Mod}(D^2)$ is trivial. What we mean here is that every automorphism of D^2 is isotopic to the identity map. The modification to the definition of mapping class here is that we only consider orientation preserving homeomorphisms that fix the boundary pointwise. The idea is that any $f \in \text{Homeo}^+(D^2)$ can be isotoped so that action of f_t does the entirety of f's action on the disk of radius $1-t$, and fixes every point outside this disk, which gives an isotopy from f to the identity map. This is known as the Alexander trick.

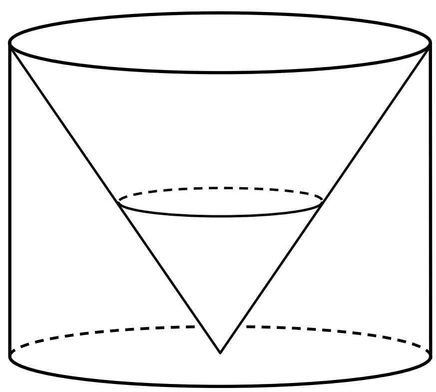


Figure 2.5: Visualization of the Alexander trick.

Example 2.2.2. Let D^2_n denote the closed disks with n punctures. Then $\text{Mod}(D^2_n)$ is isomorphic to the n-strand braid group, B_n. The modification to the definition here is that each $f \in \text{Homeo}^+(D^2_n)$ permutes the punctures. Now the idea is that if we forget about the punctures being deleted points, and just think about the homeomorphisms of D^2 that must permute those points, then applying the Alexander trick gives an isotopy where the permuted punctures move around the interior of the disk and eventually return to their starting position. This gives us a way of identifying an element of $\mathrm{Mod}(D_n^2)$ with an n-strand braid, and in fact, this identification gives an isomorphism between $\mathrm{Mod}(D_n^2)$ and B_n.

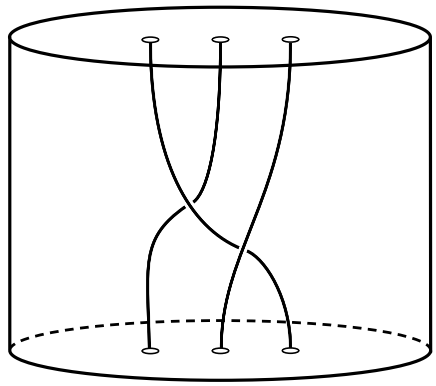


Figure 2.6: An element of B_3 traced out by the punctures through the isotopy.

Example 2.2.3. The mapping class group of the Torus is of particular interest not only because it is isomorphic to an easily understood group, but the structure of this group gives insight into the structure of the mapping class group of higher genus surfaces. By studying the action of an element of $\mathrm{Mod}(T^2)$ on $H_1(T^2;\mathbb{Z}) \cong \mathbb{Z}^2$ it can be shown

$$\mathrm{Mod}(T^2) \cong \mathrm{SL}(2,\mathbb{Z})$$

Since $\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}^2)$, the group of orientation preserving isometries of $\mathbb{H}^2$, then we can identify each element of $\mathrm{Mod}(T^2)$ with an isometry of $\mathbb{H}^2$, hence every element of the mapping class group of the Torus is associated with either an elliptic, parabolic, or hyperbolic element of $\mathrm{Isom}^+(\mathbb{H}^2)$. This association gives a wealth of information about element of the mapping class elements of the Torus, and without going too deep into the theory, the upshot of this correspondence is as follows. Let $f \in \mathrm{Mod}(T^2)$, and let τ be the corresponding element of $\mathrm{PSL}(2,\mathbb{R})$, then

1. If τ is an elliptic element, $|\mathrm{tr}(\tau)| < 2$, then f has finite order. We call such a mapping class *periodic*.

2. If τ is a parabolic element, $|\text{tr}(\tau)| = 2$, then f fixes the isotopy class of an essential simple closed curve on T^2. This type of mapping class is called *reducible*.

3. If τ is a hyperbolic element, $|\text{tr}(\tau)| > 2$, then f stretches T^2 along one foliation (to be discussed in the coming sections) and contracts T^2 along another foliation. We call this type of mapping class *Anosov*, first introduced by D.V. Anosov in [1].

As we will see in the next subsection, the classification of mapping class elements on S_g will be a similar list, with the notion of Anosov mapping classes being generalized to higher genus surfaces. These generalized Anosov mapping classes will be called *pseudo-Anosov*, a term coined by Thurston in [19].

2.2.2 Nielsen-Thurston Classification

In the previous subsection we saw that elements of $\text{Mod}(T^2)$ fall into three types. In this subsection we will define these mapping class elements for any genus surface S_g, and we also state the Nielsen-Thurston classification. We begin by defining *periodic* elements of $\text{Mod}(S_g)$.

Definition 2.2.2. *A mapping class $f \in Mod(S_g)$ is called **periodic** if there is a representative ϕ of f such that ϕ^n is isomorphic to the identity map for some n.*

Example 2.2.4. An example of a periodic element is what as known as a *hyperelliptic involution*, which is obtained by rotating the surface by π around an axis. See Figure 2.7 on the next page.

Example 2.2.5. Another example of a periodic element of S_g is obtained by arranging the tori components of the surface to be centered on the vertices of a regular g-gon, and then rotating the surface clockwise $2\pi/g$. See Figure 2.8 on the next page for $g = 3$.

We now define *reducible* mapping classes of S_g.

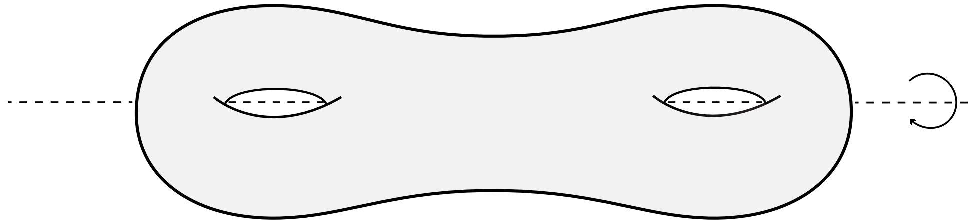


Figure 2.7: Rotating S_2 by π around the given axis is an order two element of $\mathrm{Mod}(S_2)$ known as a hyperelliptic involution.

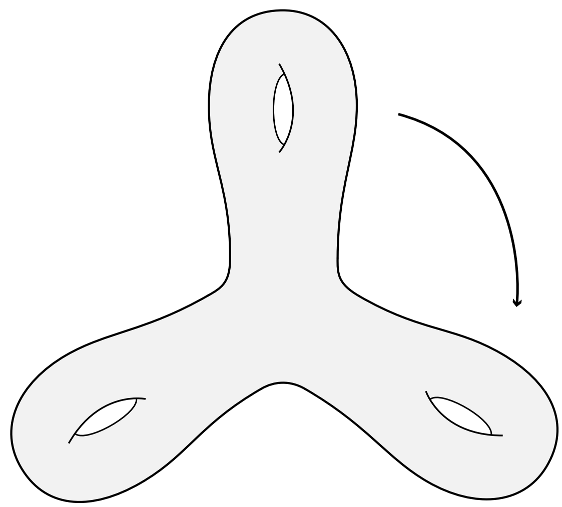


Figure 2.8: An order 3 element of $\mathrm{Mod}(S_3)$ obtained by rotating the surface as shown.

Definition 2.2.3. *A mapping class f is called* ***reducible*** *if there is a collection of disjoint non-isotopic essential simple closed curves $\{\gamma_1, ..., \gamma_n\}$, and a representative ϕ of f such that $\{\phi(\gamma_i)\} = \{\gamma_i\}$. That is, ϕ permutes the curves γ_i.*

These are called reducible because we can cut the surface along the curves γ_i and ϕ will induce homeomorphisms on the smaller pieces that do not fix any curves. It is worth noting that there is overlap between periodic and reducible mapping classes.

Example 2.2.6. The hyperelliptic involution shown in Figure 2.7 is both a periodic and reducible mapping class. Another example is an important mapping class is known as a Dehn twist, which we will discuss in more detail in the next subsection. The basic idea of a Dehn twist is that points in an annular neighborhood of a simple closed curve α are "twisted" around α, where every point outside of the annulus is left fixed. See Figures 2.9 and 2.10 below. The Dehn twist about α is denoted by T_α.

The final type of mapping class, known as a ***pseudo-Anosov*** mapping class, is also the one that will be the main focus of this treatise. The definition will require several more terms to be defined before we can state it, so we will hold off on the exact definition of pseudo-Anosov mapping classes until subsection 2.3.2. Like the notion of Anosov mapping classes of $\mathrm{Mod}(T^2)$, the idea is that a pseudo-Anosov mapping class "stretches" S_g along one foliation of the surface, while "contracting" the surface along another foliation.

Example 2.2.7. Let $\{\alpha_1, \alpha_2, \alpha_3\}$ and $\{\beta_1, \beta_2\}$ be the pair of tight, filling multicurves on S_2 as seen in figure 2.1. In Example 4.1.1 we will verify that the product of Dehn twists $T^2_{\alpha_1} T^2_{\alpha_2} T_{\alpha_3} T^2_{\beta_1} T^2_{\beta_2}$ is pseudo-Anosov.

With these three mapping classes in mind we are ready to state the classification theorem for mapping class group elements known as the Nielsen-Thurston classification.

Theorem 2.2 (Nielsen-Thurston Classification)**.** *For $g \geq 0$, every $f \in Mod(S_g)$ is either periodic, reducible, or pseudo-Anosov. Furthermore, a pseudo-Anosov mapping class is neither periodic nor reducible.*

The important content of this theorem is that every irreducible, infinite order mapping class has a representative that is pseudo-Anosov. A proof for this theorem can be found in [7]. The tools to prove this classification were discovered by Nielsen in the 1930's, but supposedly his work was lengthy and unorganized, so went unnoticed for decades. It wasn't until after Thurston announced a proof from the perspective of measured foliation theory that mathematicians rediscovered Nielsen's work.

2.2.3 Dehn Twists

Dehn twists are an important mapping class not just in terms of the work herein, but because they also generate the mapping class group as we will see below.

Before we state the definition of Dehn twist, we note that every simple closed curve on S_g has a neighborhood N that is homeomorphic to $\mathbb{S}^1 \times I$, where I is the unit interval. Let $\phi : \mathbb{S}^1 \times I \to N$ be a homeomorphism, so we can assign coordinates of the form $(e^{2\pi i\theta}, t)$ to points of N.

Definition 2.2.4. *Let α be a simple closed curve on S_g. The* ***Dehn twist*** *about α, denoted by T_α, is the homeomorphism of the form*

$$T_\alpha(x) = \begin{cases} (e^{2\pi i(\theta - t)}, t) & x = (e^{2\pi i\theta}, t) \in N \\ x & x \notin N \end{cases}$$

Our choice of twisting to the right is merely a preference. We could have chosen to twist to the left in which case the $\theta - t$ in the definition would be replaced by $\theta + t$. The idea here is that any curve transverse to α is twisted to the right all the way around the annular neighborhood of α, while leaving fixed every point outside of the annulus.

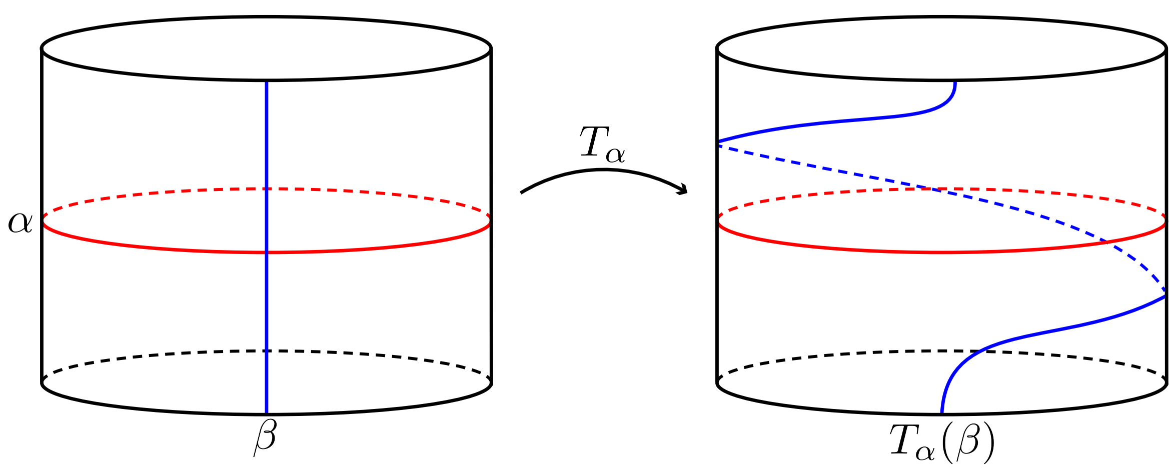


Figure 2.9: Dehn twist about α.

In the next figure we visualize a Dehn twist a genus 2 surface.

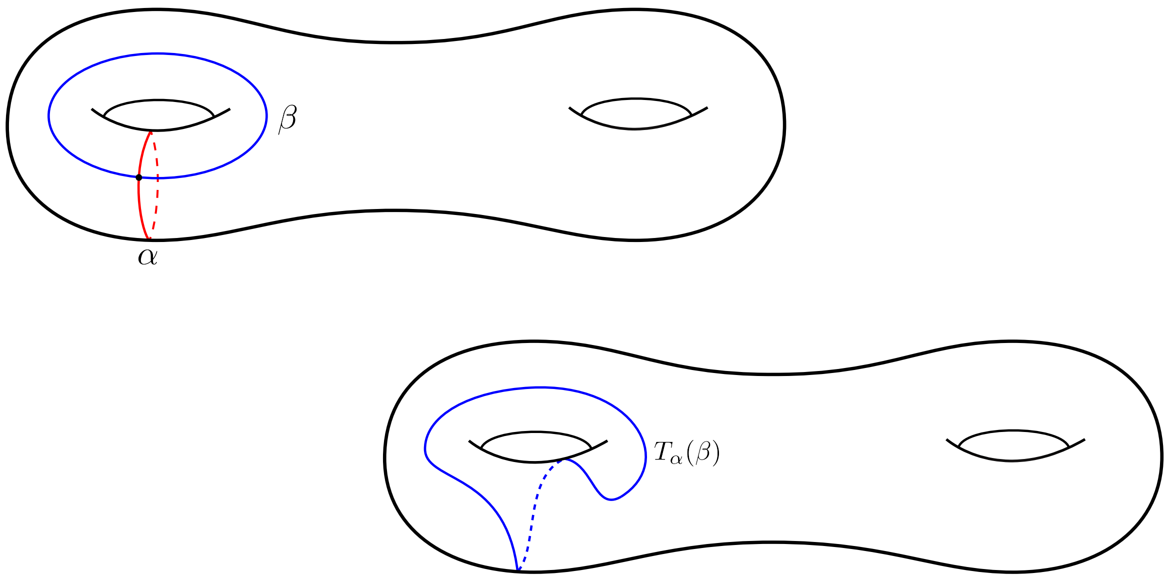


Figure 2.10: The effect on β after Dehn twisting about α.

Dehn twists are infinite order elements of $\mathrm{Mod}(S_g)$, a proof for which can be found in [7]. The idea is to compare the image of a simple closed curve β transverse to α with its image $T_\alpha(\beta)$ and you see that the geometric intersection between β and $T_\alpha(\beta)$ has increased. In fact,

$$i(\beta, T_\alpha^k(\beta)) = k \cdot i(\alpha, \beta)^2$$

Dehn twists are important in our work because certain products of Dehn twists turn out to be pseudo-Anosov as we will see when we discuss Thurston's construction. However, products of Dehn twists giving pseudo-Anosov mapping classes is not a coincidence as the next theorem shows.

Theorem 2.3 (Dehn-Lickorish theorem)**.** *For $g \geq 0$ $\mathrm{Mod}(S_g)$ is generated by finitely many Dehn twists about nonseparating simple closed curves.*

We see that up to isotopy, every surface automorphism is a product of Dehn twists. This result was first proved by Dehn in the 1920s, and was published in [5] in 1938. Dehn initially proved that the mapping class group of S_g is generated by $2g(g-1)$ Dehn twists.

In 1964 Lickorish independently discovered this result, but also improved upon it by showing that $\text{Mod}(S_g)$ is generated by Dehn twists about $3g-1$ nonseparating simple closed curves [12]. Fifteen years later in 1979, Humphries [9] pushed this result to the limit by showing the Dehn twists about $2g+1$ nonseparating curves sufficed to generate $\text{Mod}(S_g)$ and that any set of Dehn twists that generate the mapping class group must have at least $2g+1$ elements.

2.3 Pseudo-Anosov Maps

Pseudo-Anosov mapping classes are 'generic' in the sense that the probability of a random word in the Dehn-Lickorish generators not being pseudo-Anosov decays exponentially as the length of the word increases. This was proven by Rivin in [16]. In this section we will give the full definition of a pseudo-Anosov mapping class, discuss some known properties, and describe a general construction of pseudo-Anosov maps due to Thurston.

2.3.1 Measured Foliations

Before we give the full definition we need to discuss the concept of *measured singular foliations.*

Definition 2.3.1. *A* ***singular foliation*** $\mathcal{F}$ *of* S_g *is a decomposition of* S_g *into a union of disjoint subsets, called the* ***leaves*** *of* $\mathcal{F}$*, and a finite set of points of* S_g*, called the* ***singular points*** *of* $\mathcal{F}$*, such that the following two conditions hold:*

1. *For each nonsingular point* $p \in S_g$*, there is a smooth chart from a neighborhood of* p *to* $\mathbb{R}^2$ *that takes leaves to horizontal line segments. The transition maps between any two of these charts are smooth maps of the form* $(x, y) \mapsto (f(x, y), g(y))$*, in other words, the transition maps take horizontal lines to horizontal lines.*

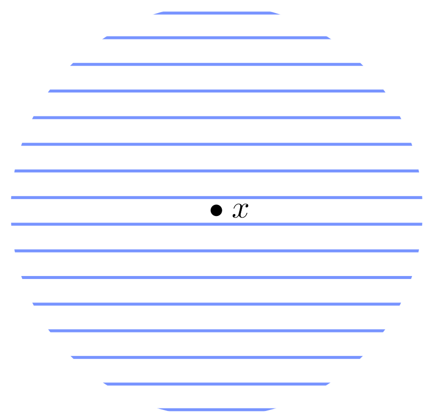


Figure 2.11: Leaves of a foliation around a nonsingular point.

2. *For each singular point $p \in S_g$, there is a smooth chart from a neighborhood of p to $\mathbb{R}^2$ that takes leaves to the level sets of a k-pronged saddle, $k \geq 3$.*

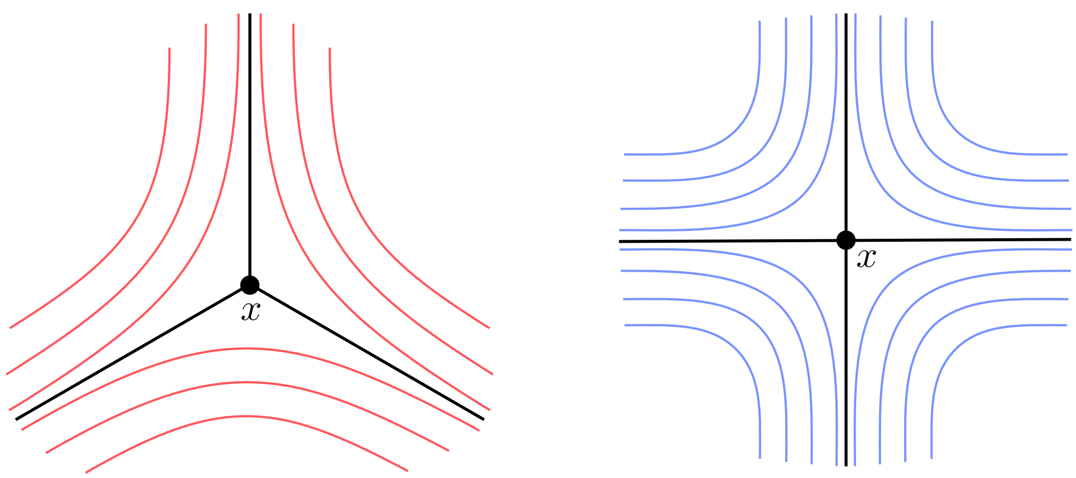


Figure 2.12: A neighborhood of a 3 prong and 4 prong singular point.

A singular foliation is said to be ***orientable*** if the leaves can be consistently oriented, that is, if each leaf can be oriented so that nearby leaves can be similarly oriented. It is not difficult to see that a foliation is ***locally orientable*** if and only if each of its singularities has an even numbed of prongs. This can be see from Figure 2.12 as choosing orientations for the three prongs in the left picture will lead to inconsistent choices for orientations of the nearby leaves, while this inconsistency does not arise in the right picture. We will use the word 'foliation' to mean singular foliation.

Definition 2.3.2. *Two foliations $\mathcal{F}_1, \mathcal{F}_2$ of S_g are said to be* ***transverse*** *if they have the same singular set, and every intersection between leaves of $\mathcal{F}_1$ and $\mathcal{F}_2$ are transverse intersections.*

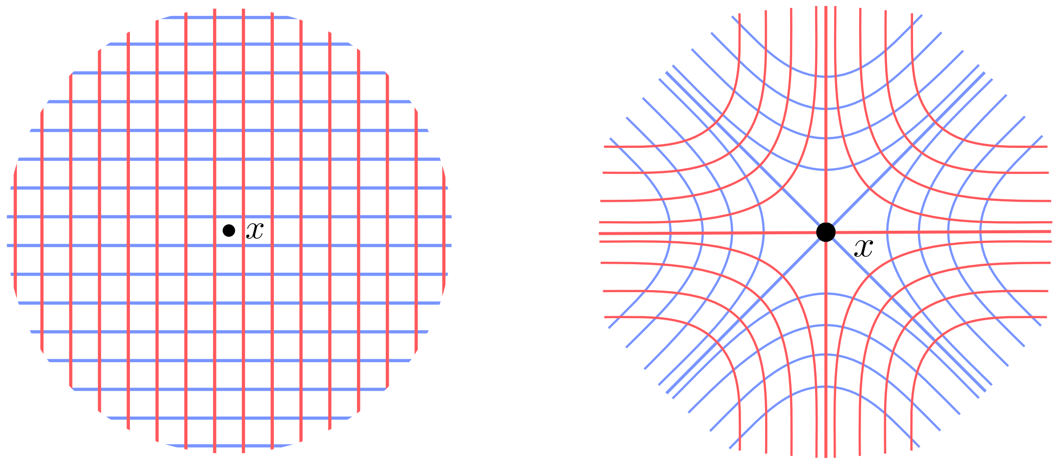


Figure 2.13: Leaves of the transverse foliations around a nonsingular point(left) and a singular point(right).

Definition 2.3.3. *A* ***transverse measure*** *μ on a foliation $\mathcal{F}$ is a function that assigns a positive real number to each smooth arc transverse to $\mathcal{F}$, so that μ is invariant under* ***leaf-preserving isotopy****. A foliation equipped with a transverse measure μ, denoted $(\mathcal{F}, \mu)$, is called a* ***measured foliation****.*

A *leaf preserving isotopy* between two curves α and β that are transverse to $\mathcal{F}$, is an isotopy $H(s,t)$ between α and β where $H(s_0, t)$ is contained in a single leaf of $\mathcal{F}$ for each $s_0 \in I$. There is a natural action of $\mathrm{Homeo}(S_g)$ on the set of measured foliations of S_g, that is, if $\phi \in \mathrm{Homeo}(S_g)$ and if $(\mathcal{F}, \mu)$ is a measured foliation of S_g, then the action of ϕ on $(\mathcal{F}, \mu)$ is given by

$$\phi \cdot (\mathcal{F}, \mu) = (\phi(\mathcal{F}), \phi_*(\mu)),$$

where $\phi_*(\mu)(\alpha) = \mu(\phi^{-1}(\alpha))$. Hence, $\mathrm{Mod}(S_g)$ acts on the set of isotopy classes of measured foliations (the quotient of the set of measured foliations by $\mathrm{Homeo}_0(S_g)$).

2.3.2 Pseudo-Anosov Mapping Classes

We are now ready to give the full definition of a pseudo-Anosov mapping class.

Definition 2.3.4. *A mapping class $f \in Mod(S_g)$ is called **pseudo-Anosov** if there is a representative $\phi \in f$, a real number $\lambda > 1$, along with a pair of transverse measured folations $(\mathcal{F}_s, \mu_s)$ and $(\mathcal{F}_u, \mu_u)$, called the **stable** and **unstable** foliations, such that*

$$\phi \cdot (\mathcal{F}_u, \mu_u) = (\mathcal{F}_u, \lambda\mu_u) \quad \phi \cdot (\mathcal{F}_s, \mu_s) = (\mathcal{F}_2, \lambda^{-1}\mu_s).$$

*The number λ is known as the **stretch factor** of f.*

In Thurston's announcement of his proof of the classification of mapping class elements [19], Thurston also proved the following.

Theorem 2.4. *If λ is the stretch factor a pseudo-Anosov mapping class on S_g, then λ is an algbraic unit such that $[\mathbb{Q}(\lambda) : \mathbb{Q}] \leq 6g - 6$.*

By ***algebraic unit*** we mean that both λ and λ^{-1} are algebraic integers, that is, that there are monic integer polynomials $p(x)$, $q(x)$ such that λ is a root of $p(x)$ and λ^{-1} is a root of $q(x)$. Thurston proved Theorem 2.4 by showing that λ is the largest eigenvalue, in absolute value, of a Perron-Frobenius matrix (see Appendix A for a short discussion of such matrices). If an eigenvalue of matrix A is the largest in absolute value, than that eigenvalue is often called the ***dominating eigenvalue***. A consequence of a stretch factor λ being the dominating eigenvalue of a Perron-Frobenius matrix is that λ is what is known as a ***Perron unit***, which is a real algebraic integer greater than 1, and that the absolute value of its Galois conjugates all lie in the interval (λ^{-1}, λ).

We see that stretch factors of pseudo-Anosov maps fit into a very restricted class of numbers, and according to [7] McMullen has conjectured that a real number $\lambda > 1$ is a stretch factor of a pseudo-Anosov map if and only if λ is a Perron unit. There has not been much progress in proving or disproving this claim, but our work ahead (particularly Theorem A) gives a partial answer towards the affirmative.

2.3.3 Thurston's Construction

We now give an overview of Thurston's construction of pseudo-Anosov maps. Our goal is to provide the basic framework of the construction and establish some of the key ideas that we will use in future chapters. For a more in depth discussion see [8]. We also provide a proof that if λ is a stretch factor of a pseudo-Anosov map arising from this construction then $\mathbb{Q}(\lambda + \lambda^{-1})$ is a totally real number field. We now give an overview of Thurston's construction following the discussion given in [13].

Theorem 2.5 (Thurston's Construction)**.** *Suppose that $A = \{\alpha_1, ..., \alpha_n\}$, $B = \{\beta_1, ..., \beta_m\}$ are tight, filling multicurves on S_g. To each $\alpha_i \in A$ assign an integer $n_i > 0$ and to each $\beta_j \in B$ assign an integer $m_j > 0$. Then the maps*

$$T_A = \prod_i T_{\alpha_i}^{n_i} \quad \textit{and} \qquad T_B = \prod_j T_{\beta_j}^{m_j}$$

can be represented by matrices in $PSL(2;\mathbb{R})$ and any word $\theta = w(T_A, T_B)$ which corresponds to a hyperbolic class $[\theta] \in SL(2;\mathbb{R})$ is pseudo-Anosov with stretch factor the larger of the two eigenvalues.

The first step in this construction is to give a branched flat structure to S_g, that is, a way to view the surface as a flat manifold except at a finite number of points. We do this by describing a way of decomposing the surface into glued up rectangles. Let $A = \{\alpha_1, ..., \alpha_n\}$ and $B = \{\beta_1, ..., \beta_m\}$ be tight, filling multicurves on S_g. $A \cup B$ defines a cell structure on S_g since the complement of $A \cup B$ is a union of disks. We define a dual cell structure of $A \cup B$ where we take a co-vertex for each cell, a co-edge for each edge, and a co-cell for each vertex. In a neighborhood of each vertex, we see four segments originating from the vertex so each co-cell will have four sides. We can think of the dual cell structure as placing rectangles on each vertex and then identify sides of the rectangles that have an arc of some α_i or β_j between them.

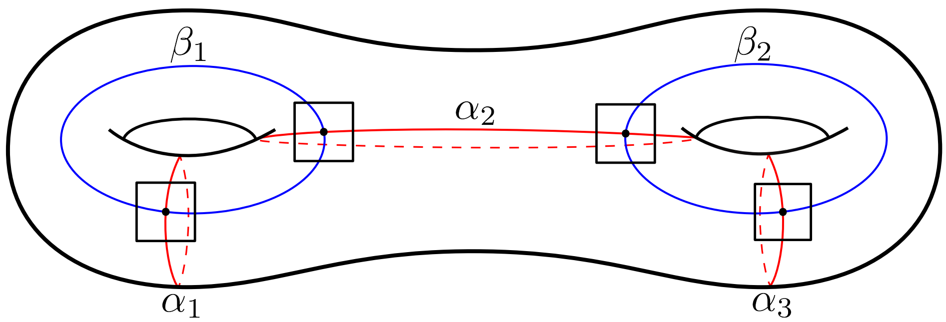


Figure 2.14: A pair of tight, filling multicurves on S_2 with co-cells at each vertex.

Assigning lengths to sides of the rectangles allows us to identify each co-cell with a rectangle in $\mathbb{R}^2$. So we can view S_g as a union of these metric rectangles, hence we have given S_g a branched flat structure.

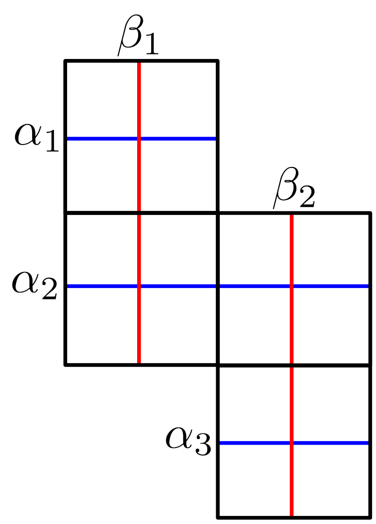


Figure 2.15: S_2 as a union of rectangles.

There is quite a bit of choice for the length of the sides of each rectangle and this freedom can be used to show the following:

Proposition 2.6. *Assign a positive integer n_i to each $\alpha_i \in A$ and a positive integer m_j*

to β_j. Then there is a branched flat structure on S_g for which both the maps

$$T_A = \prod_i T_{\alpha_i}^{n_i} \quad \textit{and} \quad T_B = \prod_j T_{\beta_j}^{m_j}$$

are affine.

By *affine* we mean that lines in the branched flat structure are mapped to lines. The Dehn twists about the α_i and β_j act on the branched flat structure by skewing the rectangles, so we wish to find a choice for side lengths so that the slope of the sides after twisting n_i times about α_i is constant in i. Similarly, twisting m_j times about β_j is constant in j. We require that the height of each rectangle lying along α_i to have constant height h_i and that each rectangle lying along β_j has constant width ℓ_j.

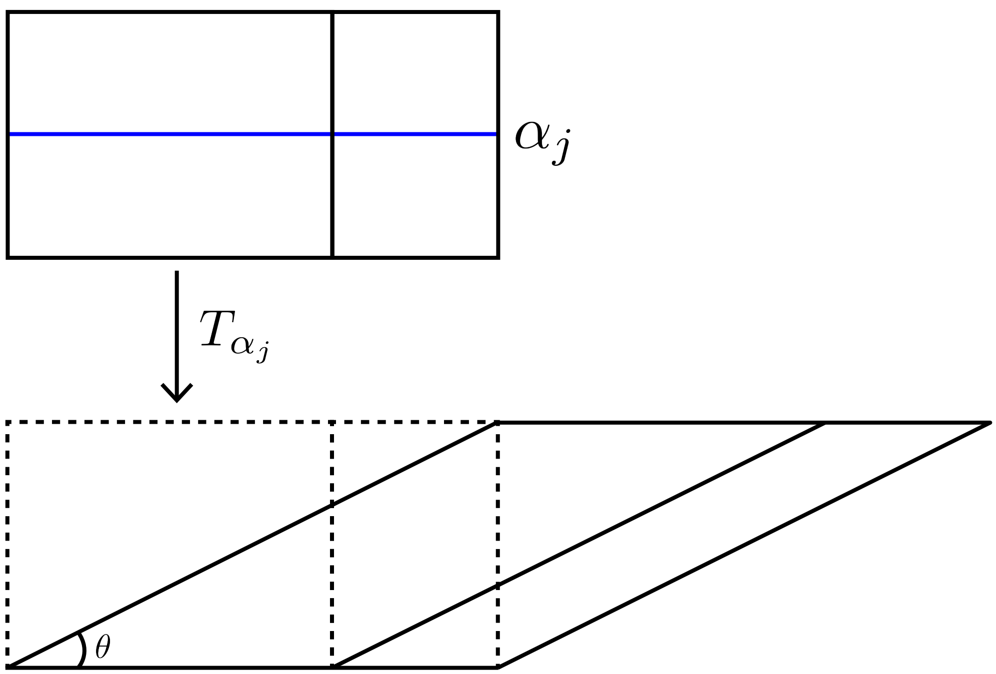


Figure 2.16: Dehn twist about α_j acting on the branched flat structure skew the rectangular strips.

If we let $i(\beta_r, \alpha_s) = Q_{rs}$ then it can be shown that T_A is affine if

$$\tan(\theta) = \frac{h_i}{n_i \sum_r \ell_r Q_{ri}} \quad \text{for all } i,$$

where θ is the angle between the image of the vertical edges along α_i with the horizontal edges along α_i after skewing by T_A. Similarly, the condition that T_B is affine is

$$\tan(\phi) = \frac{\ell_j}{m_j \sum_s h_s Q_{sj}} \quad \text{for all } j.$$

It can be shown that the choice of ℓ_j are from a Perron-Frobenius eigenvector with eigenvalue ν of the matrix $MQNQ^T$ where $Q_{ij} = i(\beta_i, \alpha_j)$ and M and N are diagonal matrices whose diagonal entries are m_j and n_i, respectively. The h_i come from a Perron-Frobenius eigenvector, also with eigenvalue ν, of the matrix QNQ^TM. These give dimensions of the rectangles in which T_A and T_B are affine. With our chosen convention of twisting, these lengths give the following representations (after some rescaling of the ℓ_i):

$$[T_A] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad [T_B] = \begin{bmatrix} 1 & 0 \\ -\nu & 1 \end{bmatrix}$$

which are matrices in $PSL(2;\mathbb{R})$. Now any word $\phi = w(T_A, T_B)$ such that $[\phi] \in SL(2;\mathbb{R})$ is hyperbolic ($|\text{tr}([\phi])| > 2$) will be pseudo-Anosov. Since $[\phi]$ is hyperbolic it has two real eigenvalues, λ and λ^{-1}, the stretch factor of ϕ is $|\lambda|$, and all lines parallel to the eigenspaces descend to transverse measured foliations on S_g. We end this section by proving the following:

Theorem 2.7. *If λ is the stretch factor of a pseudo-Anosov map arising from Thurston's construction then $\mathbb{Q}(\lambda + \lambda^{-1})$ is a totally real number field.*

Proof. Let A and B be tight, filling multicurves on a closed orientable surface, and let

M, N and Q be nonnegative integer matrices defined as above. Consider the matrix

$$\Lambda = \begin{bmatrix} \mathbf{0} & M^{1/2}QN^{1/2} \\ N^{1/2}Q^T M^{1/2} & \mathbf{0} \end{bmatrix}$$

Λ is symmetric so it has real eigenvalues which means

$$\Lambda^2 = \begin{bmatrix} M^{1/2}QNQ^T M^{1/2} & \mathbf{0} \\ \mathbf{0} & N^{1/2}Q^T MQN^{1/2} \end{bmatrix}$$

has nonnegative eigenvalues, hence $M^{1/2}QNQ^T M^{1/2}$ has nonnegative eigenvalues. Conjugating this matrix by $M^{1/2}$ gives $MQNQ^T$, thus $MQNQ^T$ has nonnegative eigenvalues. This tells us that ν is totally nonnegative, so $\mathbb{Q}(\nu)$ is totally real. Now if λ is a stretch factor arising from the above construction then λ is a root of degree 2 polynomial in $\mathbb{Q}(\nu)[x]$ of the form

$$x^2 - (\lambda + \lambda^{-1})x + 1$$

so $\lambda + \lambda^{-1} \in \mathbb{Q}(\nu)$, and therefore $\mathbb{Q}(\lambda + \lambda^{-1})$ is also totally real. ■

Chapter 3

Technical Results

3.1 Constructing Surfaces from Integer Matrices

In this chapter we describe a new construction in which we build a closed orientable surface S from a nonsingular positive integer matrix Q, where on S are two tight, filling multicurves A and B whose intersection matrix is Q. Our ability to transition from an integer matrix to a closed surface not only serves as an important link between the algebraic results of this dissertation and the topological ones, but the construction raises interesting combinatorial questions as well. In the first section we will describe the construction and prove its claimed properties, and in the next section we will discuss combinatorial data, and determine the genus of the resulting surface.

3.1.1 The Construction

Theorem 3.1. *Given an $n \times n$ nonsingular positive integer matrix Q, there is a closed orientable surface S with tight, filling multicurves $A = \{\alpha_1, ..., \alpha_n\}$ and $B = \{\beta_1, ..., \beta_n\}$ such that $Q_{ij} = i(\beta_i, \alpha_j)$.*

Proof. We start by taking n rectangular strips where the jth strip is divided into $\sum_{i=1}^{n} Q_{ij}$ rectangles each oriented clockwise. Since the matrix QQ^T is positive then it has a positive eigenvector with entries ℓ_i. All vertical edges along the jth strip will have length ℓ_j, where as ℓ_1 will be the width of the first Q_{1j} rectangles, and ℓ_2 will be the width of the next Q_{2j} rectangles, etc. We now take the central curve of each strip and call it α_j for $j = 1, ..., n$. Now we construct curves β_i as follows: For a fixed $i \in \{1, ..., n\}$ there are Q_{ij} rectangles of size $\ell_i \times \ell_j$ lying along the jth strip, and we imagine a curve that passes from the top of the left most $\ell_i \times \ell_j$ rectangle through the bottom, then wraps back around the jth strip and passes through the next $\ell_i \times \ell_j$ rectangle and continues this way until it reaches the right most $\ell_i \times \ell_j$ rectangle. The curve then continues to the $j + 1$st strip and wraps around each $\ell_i \times \ell_{j+1}$ rectangle in similar fashion. After the curve wraps around each strip it closes up, so we have a simple closed curve which we call β_i. This gives us gluing instructions where we glue two edges of distinct rectangles, matching orientation, if they are connected by an arc of some β_i. We also identify the two vertical ends of each strip.

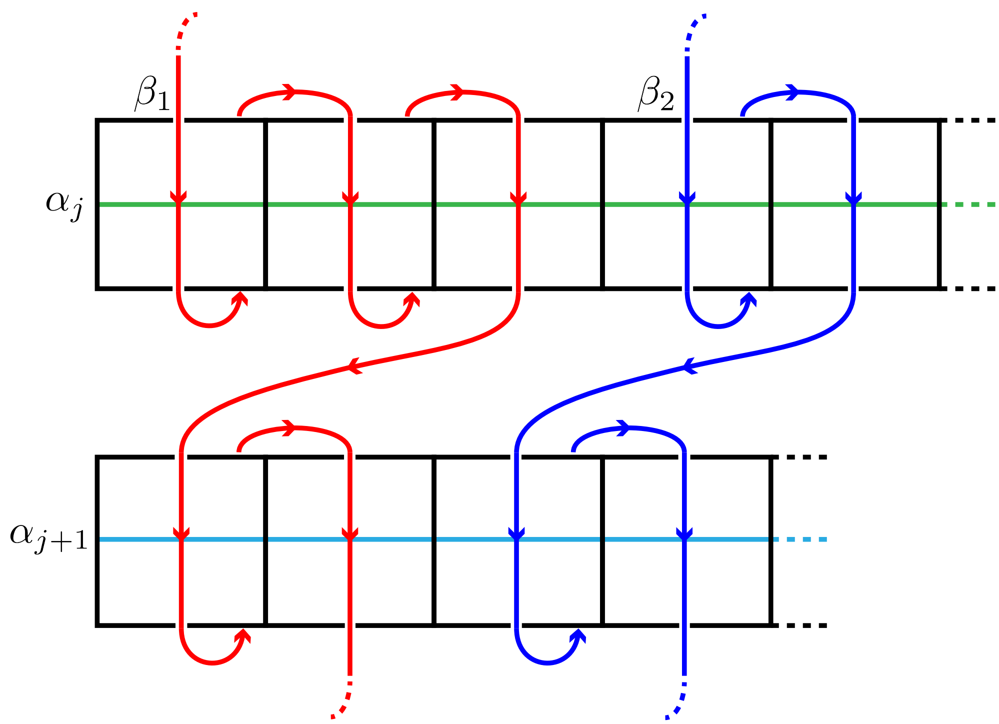


Figure 3.1: A piece of the cell structure. Edges of different cells are identified if there is an arc of some β_i between them.

Since we glue these strips together where each edge is identified with another, always matching orientation, then we get a closed orientable surface S. By construction, every intersection of β_i with α_j has the same sign so the geometric and algebraic intersection numbers of these curves are the same. This ensures that no β_i forms a bigon with any α_j, hence the multicurves $A = \{\alpha_1, ..., \alpha_n\}$ and $B = \{\beta_1, ..., \beta_n\}$ are tight. The condition that the matrix be nonsingular ensures that no two rows or columns are the same, which means that none of the β_i intersect an α_j in exactly the same way, so no two components of A or no two components of B are parallel. For if two components of A are parallel, say α_i and α_j, then $i(\beta_k, \alpha_i) = i(\beta_k, \alpha_j)$ for all k, which says that the ith column of Q is equal to the jth column, which is not possible if Q is invertible. Finally, taking the dual cell structure gives a cell structure of S whose vertices are the points of intersection between each α_i and β_j, and whose edges are arcs of some α_i or β_j. Hence, the complement of $A \cup B$ are disks, so A and B fill S. ■

Remark. We can do this construction more generally where you label each box along a α_j strip with some ordering of each of the β_i's, making sure that β_i appears Q_{ij}-times. Then you make some choice of how to connect the various β_i boxes across each strip, keeping in mind that the intersections must all have the same sign to ensure there are no bigons. For example, you can have β_1 wrap around α_1 twice then hit α_2 once, then α_3 twice, back up to α_1 once, etc. until all the β_1 boxes have been connected. This will still give an orientable surface with A and B as tight filling multicurves that intersect the correct number of times. For this more general construction, the genus of the surface is difficult to ascertain as different choices in arranging the curves can change the surface you obtain. For the construction given in the proof of Theorem 3.1 we can explicitly find the genus, as we will see in the next subsection.

Question 3.2. For a given nonsingular positive integer matrix Q, what is the minimum genus surface that can be constructed, using the general construction, having tight filling multicurves A and B where $Q_{ij} = i(\beta_i, \alpha_j)$?

3.1.2 Proof of Theorem C

In this subsection we give a proof of Theorem C by computing the genus of the surface resulting from the construction given in the previous subsection. In order for us to compute the genus of the surface, we will determine the Euler characteristic of the surface based on the cell structure given by the above construction. First we prove the following:

Proposition 3.3. *Given a surface S_g constructed as in Theorem 3.1 from a nonsingular positive integer matrix Q, let V be the number of vertices, and F the number of faces of the cell structure endowed on S_g via the construction. Then $\chi(S_g) = V - F$.*

Proof. This is a simple counting argument: By construction the number of faces, F, is the sum of the entries of Q. Now each edge of the F rectangles are identified to exactly one other edge, so the $4F$ edges across the rectangles are reduced to $2F$ edges in the cell structure, hence $E = 2F$. Now if V is the number of vertices then since the Euler characteristic is $\chi(S_g) = V - E + F$ we immediately see that $\chi(S_g) = V - F$ ■

Remarks. Since the Euler characteristic of a surface is always even, then we see that V and F must have the same parity. We will use this calculation of the Euler characteristic to compute the genus of the constructed surface and finish the proof of Theorem C, which we now restate.

Theorem C. *Given an $n \times n$ nonsingular, positive, integer matrix Q, there is a closed orientable surface S_g containing two tight filling multicurves, A and B, whose intersection matrix is Q. Furthermore, if all the entries of Q are greater than and equal to 2, then the genus of this surface is $g = n^2 - n + 1$.*

Before we prove this, we will consider the following example. It will not only verify the conclusion of Theorem C, but the discussion included will serve as a guide for proving the theorem in general.

Example 3.1.1. Consider the matrix

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

We claim that the genus of the surface constructed as above will be $g = 7$. We compute the genus using the Euler characteristic of the resulting cell-structure given to S_g by this construction. The number of faces is the sum of the entries of the matrix, so $F = 18$, and now we just need to determine the number of vertices. We can count the number of vertices by following the gluing instructions

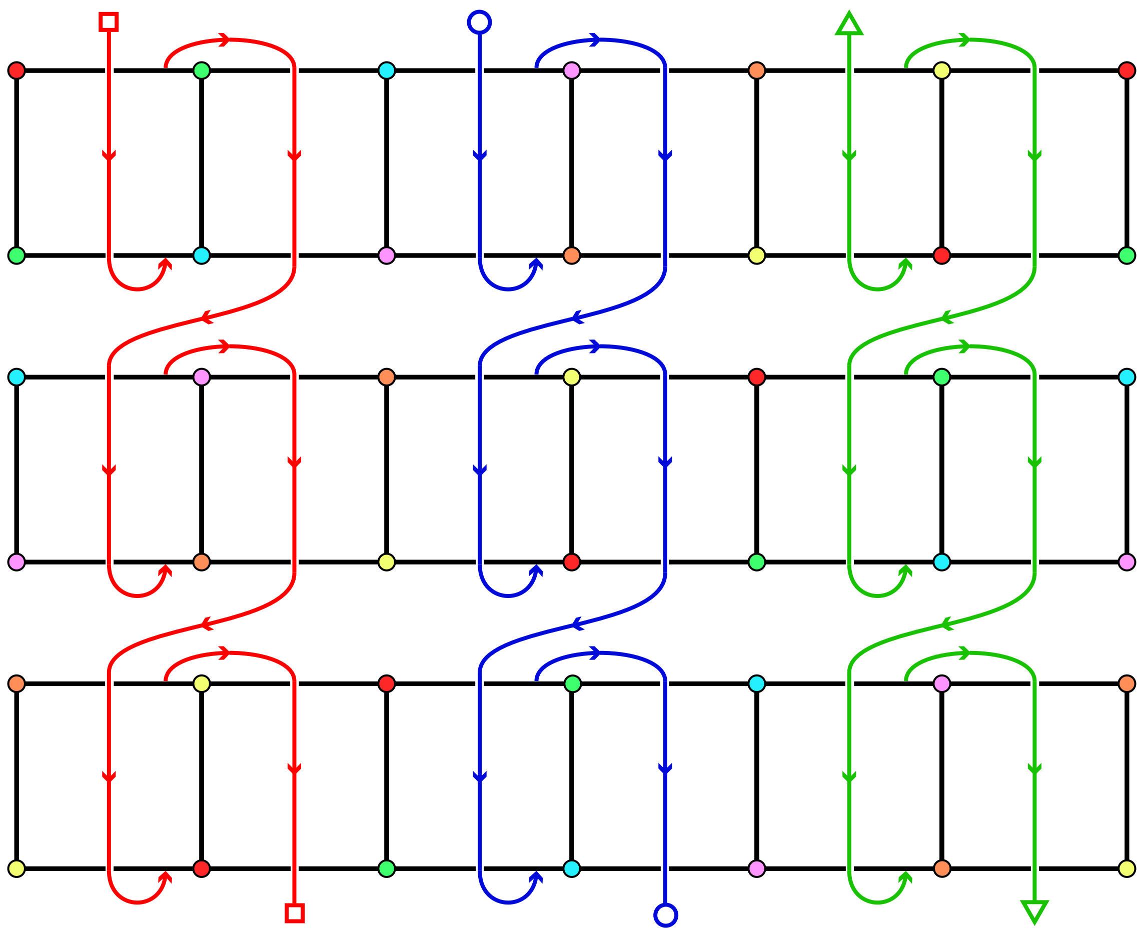


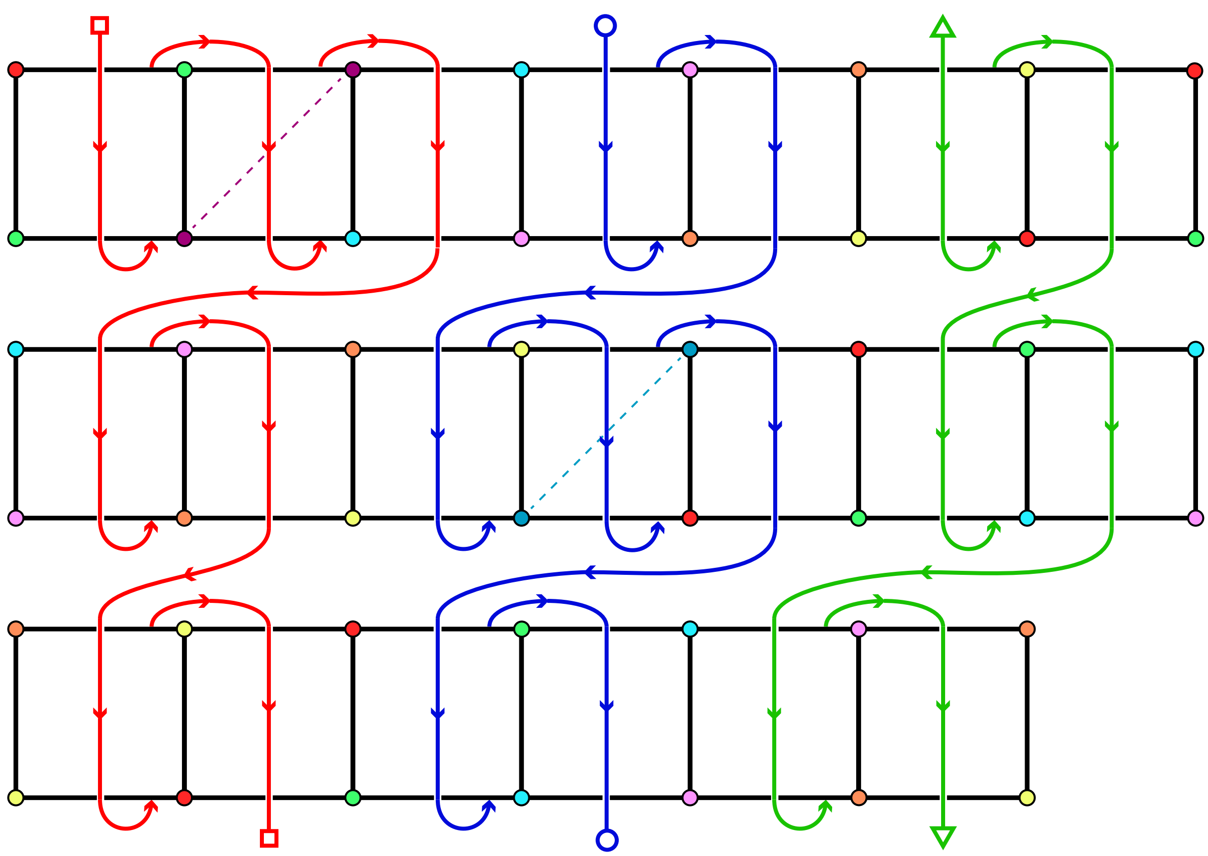
Figure 3.2: Each vertex on the left of each strip corresponds to a unique gluing orbit.

Here we see that $V = 6$, so $\chi(S_g) = 2 - 2g = -12$ and we see that $g = 7$ as expected. In the above image we can see that each of the 2 left most vertices of each strip correspond to distinct vertices of our cell structure when we follow them through the gluing. So we see that the 42 vertices amongst the three strips are partitioned into 6 sets containing 7 elements, which we call the ***gluing orbits***.

Now let us consider the matrix

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

Here we have increased two of the entries by 1, and we claim this has no effect on the genus of the surface. The idea is that each time we increase an entry by 1 we are not only adding one face, but also one vertex, a change that will cancel out when computing the Euler characteristic.



Here we can see that there are 46 vertices across all three strips, and there are still 6 gluing orbits of size 7, but now there are two additional gluing orbits of size 2. Therefore, the number of vertices in our cell structure is $V = 8$, and then number of faces is $F = 20$ which shows $\chi(S_g) = -12$. Hence, $g = 7$ again.

We are now ready to prove Theorem C:

Proof. We will start along the lines of the above example by looking at the matrix $[2]_{n\times n}$, the $n \times n$ matrix whose entries are all 2. Now, we have n strips with central curve α_i where each strip is decomposed into $2n$ square rectangles whose width and height are both 1 unit. Before identifying edges connected by an arc of some β_j, there are $4n + 2$ vertices along each strip, so in total there are $2n(2n + 1)$ vertices before gluing. Let $t_{i,j}$ denote the jth vertex along the top of the ith strip, and $b_{i,j}$ denotes the jth vertex along the bottom of the ith strip. With this notation in mind and by following the gluing process we can write out the gluing orbits of the left most vertices of each strip. The left most vertex across the top of the ith strip is $t_{i,1}$ and has gluing orbit

$$\left\{t_{i,1}, b_{i-1,2}, ..., t_{1,(2i-1)}, b_{n,2i}, ..., t_{i,2n+1}\right\}$$

and the gluing orbit for $b_{i,1}$ is

$$\{b_{i,1}, t_{i,2}, ..., t_{1,2i}, b_{n,2i+1}, ..., b_{i,2n+1}\}$$

Note that the right most index always increase by 1, therefore each of the $2n$ vetrices along the left sides of the strips have gluing orbits of length $2n + 1$. This implies that there are $2n$ vertices in the cell structure after gluing, hence $V = 2n$ and we see that the Euler characteristic of the constructed surface is $\chi(S_g) = V - F = 2n - 2n^2$. On the other hand $\chi(S_g) = 2 - 2g$ so solving for g shows that the genus of the surface constructed by the process of Theorem 3.1 is

$$\boxed{g = n^2 - n + 1}$$

We must now show that if each $Q_{ij} \geq 2$ then the genus of the constructed surface is still $n^2 - n + 1$. We need only see that if we increase one entry of $[2]_{n\times n}$ that the genus does not change. Suppose that for a fixed k and ℓ we have that $Q_{k\ell} = 3$, and the rest of the entries are 2. Now we have introduced a gluing orbit of length two $\{b_{k,2\ell}, t_{k,(2\ell+1)}\}$

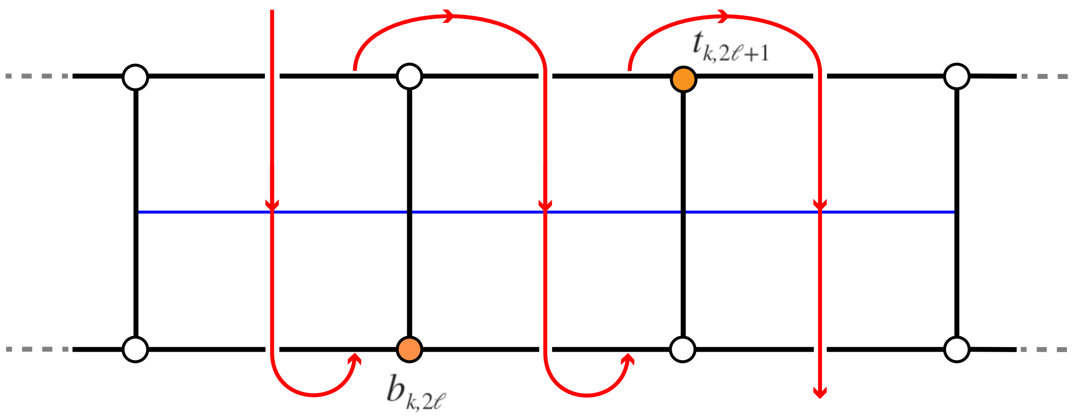


Figure 3.3: The filled vertices $b_{k,2\ell}$ and $t_{k,2\ell+1}$ form a gluing orbit of length 2.

But the gluing orbits of the left most vertices remain the same length, with the gluing orbits containing vertices of the form $b_{k,2\ell+s}$ or $t_{k,2\ell+1+s}$ now contain vertices of the form $b_{k,2\ell+1+s}$ or $t_{k,2\ell+2+s}$. This means that there are $2n+1$ gluing orbits, so the cell structure of S_g has $2n+1$ vertices, and there are $2n^2+1$ faces so again we have that $\chi(S_g) = 2n - 2n^2$. Therefore, $g = n^2 - n + 1$.

This argument shows that when the entries of Q are at least 2 then increasing an entry by 1 has the effect of increasing the number of vertices of the cell structure by 1 but since you are only adding 1 face then the Euler characteristic doesn't change, and hence the genus of the constructed surface does not change. Hence, if Q is an $n \times n$ integer matrix with each entry at least 2, then the genus of the constructed surface is $g = n^2 - n + 1$. ■

Chapter 4

Results

4.1 Stretch Factors Coming From Thurston's Construction

In this section we define Salem numbers, and look at two situations in which Salem numbers can arise as stretch factors coming from Thurston's construction. We will develop some algebraic results as needed and ultimately show that every Salem number has a power that is the stretch factor of a pseudo-Anosov map coming from Thurston's construction.

4.1.1 Salem Numbers and Initial Observations

We begin by first defining Salem numbers, and looking at two observations that relate Salem numbers to Thurston's construction.

Definition 4.1.1. *A real algebraic unit $\lambda > 1$ is called a Salem number if λ^{-1} is a Galois conjugate, and all other conjugates lie on the unit circle.*

From this definition we can deduce the following properties of Salem numbers:

Proposition 4.1. *Let λ be a Salem number. Then,*

1. *λ^k is a Salem number for any positive integer k.*

2. *$\lambda + \lambda^{-1}$ is a totally real algebraic integer.*

3. *The Galois conjugates of $\lambda + \lambda^{-1}$ lie in the interval $(-2, 2)$.*

Since $\lambda + \lambda^{-1}$ is totally real, it is at least plausible that there are Salem numbers that are stretch factors arising from Thurston's construction. In [11] Leininger finds Salem numbers that are stretch factors of pseudo-Anosov maps coming from Thurston's construction and asks if it is possible to obtain every Salem number as a stretch factor. The following observation gives a condition for when a Salem number arises as a stretch factor from Thurston's construction.

Observation 4.2. *There are Salem numbers that arise as stretch factors of pseudo-Anosov maps arising from Thurton's construction.*

Suppose you have two tight, filling multicurves A and B where $|B| = 2$ and $|A| = k$, then $MQNQ^T$ is a 2×2 matrix. If $m_1 = m_2$, then M commutes with QNQ^T, and hence $MQNQ^T$ is symmetric. This matrix has two positive eigenvalues ν and μ, where $\nu > \mu$ and which are roots of an integer polynomial

$$x^2 - ax + b$$

where $\nu + \mu = a$ and $\nu\mu = b$. As in the discussion of Thurston's contruction, the maps T_A and T_B are represented by the matrices

$$[T_A] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad [T_B] = \begin{bmatrix} 1 & 0 \\ -\nu & 1 \end{bmatrix}$$

and the product of these matrices gives

$$[T_A][T_B] = \begin{bmatrix} 1-\nu & 1 \\ -\nu & 1 \end{bmatrix}$$

The characteristic polynomial is $x^2 - (2-\nu)x + 1$, and we also have that the polynomial

$$f(x) = (x^2 - (2-\nu)x + 1)(x^2 - (2-\mu)x + 1)$$

is an integer polynomial. Now, if we assume $|2-\nu| > 2$ and $|2-\mu| < 2$ then $f(x)$ will have two real roots, λ and λ^{-1}, and two complex roots on the unit circle. This tells us that λ is a Salem number of degree 4 over the rationals.

Though this is a specific condition, we at least have a scenario where we know the resulting stretch factor will be a Salem number. We illustrate this observation with the following example.

Example 4.1.1. Let $A = \{\alpha_1, \alpha_2, \alpha_3\}$ and $B = \{\beta_1, \beta_2\}$ be the tight, filling multicurves on S_2 as in figure 2.1. We will choose $m_1 = m_2 = n_1 = n_2 = 2$ and $n_3 = 1$. Then

$$MQNQ^T = \begin{bmatrix} 8 & 4 \\ 4 & 6 \end{bmatrix},$$

where $\nu = 7 + \sqrt{17}$ and $\mu = 7 - \sqrt{17}$. We can see that $|2-\nu| > 2$ and $|2-\mu| < 2$, so $T_A T_B$ is a pseudo-Anosov map whose stretch factor is the Salem number

$$\lambda = \frac{5 + \sqrt{17} + \sqrt{38 + 10\sqrt{17}}}{2}.$$

As much as the above is meant to highlight that Salem numbers arise without much effort from Thurston's construction, we also see that Salem numbers can arise as stretch factors when $MQNQ^T$ is symmetric. An obvious simplification then is if Q is symmetric and $M = N = I$. We offer the following observation.

Observation 4.3. *Salem numbers can arise as stretch factors when Q is symmetric and $M = N = I$.*

Suppose S_g is a surface with tight, filling multicurves A and B with $|A| = |B|$ whose intersection matrix Q (which is a square symmetric matrix) has $\lambda + \lambda^{-1}$ as an eigenvalue with positive eigenvector $\mathbf{v}$. Let $M = N = I$, then we have

$$MQNQ^T = Q^2.$$

So we get that

$$[T_A] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad [T_B] = \begin{bmatrix} 1 & 0 \\ -(\lambda + \lambda^{-1})^2 & 1 \end{bmatrix}$$

which gives

$$[T_A][T_B] = \begin{bmatrix} 1 - (\lambda + \lambda^{-1})^2 & 1 \\ -(\lambda + \lambda^{-1})^2 & 1 \end{bmatrix}$$

which has characteristic polynomial $x^2 + (\lambda^2 + \lambda^{-2})x + 1$ whose roots are $-\lambda^2$ and $-\lambda^{-2}$. Hence, up to projectivization, $T_A T_B$ has λ^2 as its stretch factor.

With the above observations in mind, we wish to determine which Salem numbers can arise as stretch factors of a pseudo-Anosov map coming from Thurston's construction. Our goal for the next couple of subsections will be to develop enough theory to ultimately prove the following:

Theorem 4.4. *Let $\lambda > 1$ be a Salem number with $[\mathbb{Q}(\lambda) : \mathbb{Q}] = n$. Then there is a positive integer k such that λ^k is the stretch factor of a pseudo-Anosov homeomorphism coming from Thurston's construction on the surface $S_{(n+e)^2-(n+e)+1}$, where $e \in \{0, 1, 2\}$.*

4.1.2 Algebraic results

In this subsection we collect several algebraic results that will be important in the proofs of both Theorem A and Theorem B. We begin with a theorem due to Estes which shows that we can get any totally real algebraic integer as an eigenvalue of rational symmetric matrix. A proof for this may be found in [6]

Theorem 4.5 (Estes)**.** *Let η be a totally real algebraic integer of degree n over $\mathbb{Q}$ with minimal polyomial $f(x)$. Then η is an eigenvalue of a rational symmetric matrix of size $(n+e)\times(n+e)$ whose characteristic polynomial is $f(x)(x-1)^e$, and $e \in \{0,1,2\}$.*

Since any stretch factor λ coming from Thurston's construction must have $\lambda+\lambda^{-1}$ totally real, Theorem 4.5 will serve as an important starting point to linking algebraic units with that property to Thurston's construction. Note that if λ is a Salem number then by Theorem 4.5 there is a rational symmetric matrix Q whose eigenvalues are $\lambda+\lambda^{-1}$, the Galois conjugates of $\lambda+\lambda^{-1}$, and 1. Now, $\lambda+\lambda^{-1}$ and its conjugates each have multiplicity 1, whereas 1 may have multiplicity 0, 1 or 2. We are mostly concerned that $\lambda+\lambda^{-1}$ has multiplicity 1 because then its eigenspace is one-dimensional, which will be an important fact later when we describe an infinite family of matrices having the same eigenspaces.

Now, what we really want is for the eigenspace of $\lambda+\lambda^{-1}$ to be spanned by a positive vector, the reason for this will become apparent later. Theorem 4.5 makes no claims about the eigenvectors of Q but with the following proposition we can assume without loss of generality that $\lambda+\lambda^{-1}$ has a positive eigenvector.

Proposition 4.6. *The set $O(n;\mathbb{Q})$, orthogonal matrices with rational entries, is a dense subgroup of $O(n)$. Consequently, $SO(n;\mathbb{Q})$ is a dense subgroup of $SO(n)$.*

A proof for this can be found in [17] and we provide a proof in the Appendix. We use Proposition 4.6 as follows: Since every $U \in SO(n)$ has the property that $U^{-1} = U^T$, then conjugating a symmetric matrix by an $SO(n)$ matrix remains symmetric. Q has eigenvalue $\lambda + \lambda^{-1}$, so we conjugate Q by an $SO(n)$ matrix so the resulting matrix has a positive eigenvector corresponding to $\lambda + \lambda^{-1}$. We then perturb the entries of U so that they are all rational and now the resulting matrix is still rational and symmetric having a positive eigenvector $\mathbf{v}$ corresponding to $\lambda + \lambda^{-1}$.

We end this section with a proof the following theorem that gives us a condition for when we can raise a rational matrix to a high enough power so that the entries are all integers. This will allow us to use the construction described in Chapter 3 to build a surface from this matrix and link our algebraic results back to topological information.

Proposition 4.7. *Let $M \in M_n(\mathbb{Q})$ such that $\det(M) = \pm 1$ and the characteristic polynomial of M has integer coefficients. Then some power of M is integral.*

Proof. Since we can replace M with M^2 then without loss of generality we can assume $\det(M) = 1$. Let Ω be the rational canonical form of M, which by assumption has integer entries. Thus, there is a nonsingular rational matrix A such that $A^{-1}MA = \Omega$. Now $A = \frac{1}{c}P$ and $A^{-1} = \frac{1}{d}P'$ where

$$PP' = cdI$$

c, d are nonzero integers and P, P' are integer matrices. Let $\overline{\Omega}$ be the matrix obtained from Ω by reducing its entries modulo cd. Since $\det(\Omega) = 1$ then $\det(\overline{\Omega}) = 1$, so $\overline{\Omega} \in SL(n; \mathbb{Z}/cd\mathbb{Z})$. Since $SL(n; \mathbb{Z}/cd\mathbb{Z})$ is a finite group there is a positive integer k such that $\Omega^k \equiv I \mod cd$. That is, there an integer matrix B such that $\Omega^k = I + cdB$. Now

$$\begin{aligned} M^k &= A\Omega^k A^{-1} \\ &= I + cdABA^{-1} \\ &= I + PBP' \end{aligned}$$

where PBP' is an integer matrix, and thus M^k is an integer matrix. ■

4.1.3 Proof of Theorem 4.4

Throughout this subsection we will be assuming that $\lambda > 1$ is a Salem number. By Proposition 4.1 we know that λ^k is a Salem number for all positive integers k, $\lambda + \lambda^{-1}$ is a totally real, and that all Galois conjugates of $\lambda + \lambda^{-1}$ lie in the interval $(-2, 2)$. We now want a rational matrix having λ as an eigenvalue with the goal of finding a power of that matrix that has integer entries. Consider the matrix

$$\mathcal{M} = \begin{bmatrix} Q & -I \\ I & \mathbf{0} \end{bmatrix},$$

where I is the $(n+e) \times (n+e)$ identity matrix. We will use powers of this matrix to find a positive symmetric integer matrix having $\lambda^k + \lambda^{-k}$ as an eigenvalue for some k. We will now establish several important properties:

Proposition 4.8. *The characteristic polynomial of* $\mathcal{M}$ *is* $p(x)(x^2 - x + 1)^e$ *where* $p(x)$ *is the minimal polynomial of* λ *over* $\mathbb{Q}$*. Therefore,* $\mathcal{M}$ *has integral characteristic polynomial, and* $\det(\mathcal{M}) = 1$.

Proof. We will first show that μ is an eigenvalue for $\mathcal{M}$ if and only if $\mu + \mu^{-1}$ is an eigenvalue for Q. If μ is an eigenvalue for $\mathcal{M}$ then there is a vector $\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$ such that

$$\mathcal{M} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mu\mathbf{x} \\ \mu\mathbf{y} \end{bmatrix}$$
$$\begin{bmatrix} Q\mathbf{x} - \mathbf{y} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mu\mathbf{x} \\ \mu\mathbf{y} \end{bmatrix}$$

so $\mathbf{y} = \mu^{-1}\mathbf{x}$, and thus we have $Q\mathbf{x} = (\mu + \mu^{-1})\mathbf{x}$, therefore $\mu + \mu^{-1}$ must be an eigenvalue of Q. Now, if $\mu + \mu^{-1}$ is an eigenvalue of Q with corresponding eigenvector $\mathbf{x}$, then we have

$$\mathcal{M}\begin{bmatrix}\mathbf{x}\\ \mu^{-1}\mathbf{x}\end{bmatrix} = \begin{bmatrix}Q\mathbf{x}-\mu^{-1}\mathbf{x}\\ \mathbf{x}\end{bmatrix}$$
$$= \begin{bmatrix}(\mu+\mu^{-1})\mathbf{x}-\mu^{-1}\mathbf{x}\\ \mu\cdot\mu^{-1}\mathbf{x}\end{bmatrix}$$
$$= \mu\begin{bmatrix}\mathbf{x}\\ \mu^{-1}\mathbf{x}\end{bmatrix}$$

Hence, μ is an eigenvalue of $\mathcal{M}$. Therefore, since $\lambda+\lambda^{-1}$ and its conjugates are eigenvalues of Q then λ and its conjugates are eigenvalues of $\mathcal{M}$. Also, since 1 is an eigenvalue with multiplicity e then $\mathcal{M}$ must have an eigenvalue μ such that $\mu+\mu^{-1}=1$, in other words $\mu^2-\mu+1=0$, hence

$$\mu = \frac{1+i\sqrt{3}}{2} \quad \text{and} \quad \mu^{-1} = \frac{1-i\sqrt{3}}{2}$$

(where μ is a primitive 6th root of unity) and if $\mathbf{y}$ is an eigenvector of Q corresponding to 1 then $\begin{bmatrix}\mathbf{y}\\ \mu^{-1}\mathbf{y}\end{bmatrix}$ and $\begin{bmatrix}\mathbf{y}\\ \mu\mathbf{y}\end{bmatrix}$ are eigenvectors of $\mathcal{M}$ corresponding to μ and μ^{-1}, respectively. So if 1 has multiplicity e, then μ and μ^{-1} are eigenvalues of $\mathcal{M}$ both with multiplicity e. Therefore, we have shown that the characteristic polynomial of $\mathcal{M}$ has the desired form. Finally, since λ and μ are algebraic units then the characteristic polynomial of $\mathcal{M}$ is integral, and the product of the eigenvalues is 1 so $\det(\mathcal{M})=1$. ■

The fact that $\det(\mathcal{M})=1$ is true for any square matrix Q since I and $\mathbf{0}$ commute we have

$$\det(\mathcal{M}) = \det(Q\cdot\mathbf{0}+I^2) = 1$$

but most importantly this tells us that $\mathcal{M}^{-1}$ exists. It is easy to check that

$$\mathcal{M}^{-1} = \begin{bmatrix} \mathbf{0} & I \\ -I & Q \end{bmatrix}$$

and notice that

$$\mathcal{M} + \mathcal{M}^{-1} = \begin{bmatrix} Q & \mathbf{0} \\ \mathbf{0} & Q \end{bmatrix}$$

We will show that this behavior holds for all powers of $\mathcal{M}$.

Proposition 4.9 (Skew Property)**.** *$\mathcal{M}^k + \mathcal{M}^{-k}$ is a block diagonal matrix of the form $\begin{bmatrix} \mathcal{Q}_k & \mathbf{0} \\ \mathbf{0} & \mathcal{Q}_k \end{bmatrix}$ for any integer k. Here $\mathcal{Q}_k$ is a rational symmetric matrix whose characteristic polynomial is $g_k(x)(x-a)^e$, where $g_k(x)$ is the minimal polynomial of $\lambda^k + \lambda^{-k}$, and $a \in \{-2, -1, 1, 2\}$.*

Proof. First we will show that for any k we have $\mathcal{M}^k = \begin{bmatrix} Q_k & -Q_{k-1} \\ Q_{k-1} & Q_k - Q \cdot Q_{k-1} \end{bmatrix}$ where Q_k is an integral combination of powers of Q. We define $Q_0 = I$ and $Q_1 = A$. We will proceed by induction: For $k = 2$ we get

$$\mathcal{M}^2 = \begin{bmatrix} Q^2 - I & -Q \\ Q & -I \end{bmatrix}$$

So $Q_2 = Q^2 - I$, and $-I = Q_2 - Q \cdot Q_1$. Now assume that this form holds for k, then

$$\mathcal{M}^k = \begin{bmatrix} Q_k & -Q_{k-1} \\ Q_{k-1} & Q_k - Q \cdot Q_{k-1} \end{bmatrix}$$

in which case we have

$$\mathcal{M}^{k+1} = \begin{bmatrix} Q & -I \\ I & \mathbf{0} \end{bmatrix} \begin{bmatrix} Q_k & -Q_{k-1} \\ Q_{k-1} & Q_k - Q \cdot Q_{k-1} \end{bmatrix}$$
$$= \begin{bmatrix} Q \cdot Q_k - Q_{k-1} & -Q_k \\ Q_k & -Q_{k-1} \end{bmatrix}$$

so if $Q_{k+1} = Q \cdot Q_k - Q_{k-1}$ then we have that

$$\begin{bmatrix} Q_{k+1} & -Q_k \\ Q_k & Q_{k+1} - Q \cdot Q_k \end{bmatrix}$$

A similar inductive argument shows that Q^{-k} has the form $\begin{bmatrix} Q_k - Q \cdot Q_{k-1} & Q_{k-1} \\ -Q_{k-1} & Q_k \end{bmatrix}$ for all k. Therefore, we have that

$$\mathcal{M}^k + \mathcal{M}^{-k} = \begin{bmatrix} 2Q_k - Q \cdot Q_{k-1} & \mathbf{0} \\ \mathbf{0} & 2Q_k - Q \cdot Q_{k-1} \end{bmatrix}$$

so $\mathcal{Q}_k = 2Q_k - Q \cdot Q_{k-1}$, which is an integral combination of powers of Q so $\mathcal{Q}_k$ is a rational symmetric matrix. Now that we have established the skew-property where $\mathcal{M}^k + \mathcal{M}^{-k}$ is a block diagonal matrix where the $(1,1)$-block and the $(2,2)$-block are equal rational symmetric matrices, then we can see that not only are the eigenvalues of $\mathcal{M}^k + \mathcal{M}^{-k}$ of the form $\nu_k + \nu_k^{-1}$, where ν_k is an eigenvalue of $\mathcal{M}^k$ but also that $\nu_k + \nu_k^{-1}$ is an eigenvalue of $\mathcal{M}^k + \mathcal{M}^{-k}$ if and only if $\nu_k + \nu_k^{-1}$ is an eigenvalue of each diagonal block.

We immediately get from this that $\lambda^k + \lambda^{-k}$ and its conjugates are eigenvalues of $\mathcal{Q}_k$. Since $\mathcal{M}^k$ has μ^k and μ^{-k} as eigenvalues then we need to determine the possibilities for $\mu^k + \mu^{-k}$. Since μ is a primitive 6th root of unity we have the following chart where $g_k(x)$

denotes the minimal polynomial of $\lambda^k + \lambda^{-k}$:

$k \equiv b \mod 6$	$\mu^k + \mu^{-k}$	Characteristic Polynomial of $\mathcal{Q}_k$
$b = 0$	2	$g_k(x)(x-2)^e$
$b = 1, 5$	1	$g_k(x)(x-1)^e$
$b = 2, 4$	-1	$g_k(x)(x+1)^e$
$b = 3$	-2	$g_k(x)(x+2)^e$

Therefore, we have proved Proposition 4.9. ■

Since $\mathcal{Q}_k$ is an integral combination of powers of Q, then it is clear that the eigenspaces of $\mathcal{Q}_k$ and Q are exactly the same for all k. Therefore, since Q has a positive eigenvector $\mathbf{v}$ corresponding to $\lambda + \lambda^{-1}$, then $\mathcal{Q}_k$ also has eigenvector $\mathbf{v}$ corresponding to $\lambda^k + \lambda^{-k}$. The goal now is to use this, in conjunction with the next proposition, to show that there is a k where $\mathcal{Q}_k$ is a positive symmetric integral matrix.

Since $\mathcal{M}$ is a rational matrix with determinant 1 and integer characteristic polynomial, then by Proposition 4.7 there is a k such that $\mathcal{M}^k$ is an integer matrix. This means that $\mathcal{Q}_k$ is a symmetric integer matrix having $\lambda^k + \lambda^{-k}$ as an eigenvalue. We have now shown that every Salem number has a power k such that $\lambda^k + \lambda^{-k}$ is an eigenvalue of a symmetric integral matrix. Now we want to show that we can raise k high enough to get a positive matrix.

$\mathcal{Q}_k$ is symmetric so we know there is an orthonormal basis of eigenvectors of $\mathcal{Q}_k$. Since there is a positive eigenvector $\mathbf{v}$ corresponding to $\lambda^k + \lambda^{-k}$ then for any standard basis vector $\mathbf{e}_i$

$$\mathbf{e_i} = c_i \mathbf{v} + \mathbf{w}_i$$

where $\mathbf{w}_i$ lies in the orthogonal complement of Span$\{\mathbf{v}\}$, which is spanned by the other eigenvectors of $\mathcal{Q}_k$. Thus, for each i we have $c_i = \mathbf{v} \cdot \mathbf{e_i} > 0$. Applying $\mathcal{Q}_k$ to both sides gives

$$\mathcal{Q}_k\mathbf{e}_i = c_i(\lambda^k + \lambda^{-k})\mathbf{e}_i + \mathcal{Q}_k\mathbf{w}_i$$

since λ^k is a Salem number, the conjugates of $\lambda^k + \lambda^{-k}$ are real numbers in the interval $(-2, 2)$. As we see above the vectors making up $\mathbf{w}_i$ are each scaled by a number in the interval $[-2, 2]$, but $\lambda^k + \lambda^{-k}$ grows without bound, so eventually $\mathcal{Q}_k\mathbf{e}_i$ is a positive vector for any i.

So, if we start with an integer k such that $\mathcal{M}^k$ is integral, then $\mathcal{M}^{nk}$ is integral for any positive integer n, so choose n large enough so that $\mathcal{Q}_{nk}$ is a positive symmetric integer matrix. Therefore, we have shown that for any Salem number λ there is a positive integer k such that $\lambda^k + \lambda^{-k}$ is an eigenvalue of a positive symmetric integral matrix.

With this we can finish the proof of Theorem 4.4 as follows: Since $\mathcal{Q}_k$ is a nonsingular positive symmetric integer matrix having $\lambda^k + \lambda^{-k}$ as an eigenvalue with positive eigenvector $\mathbf{v}$, we can use Theorem 3.1 to find a surface with tight, filling multicurves A and B whose intersection matrix is $\mathcal{Q}_k$. By Proposition 3.2, the genus of this surface is $g = (n + e)^2 - (n + e) + 1$, where $n = [\mathbb{Q}(\lambda) : \mathbb{Q}]$. Choosing $M = N = I$ and following discussion proceeding observation 2, we see that $T_A T_B$ is a pseudo-Anosov map coming from Thurston's construction with stretch factor λ^{2k}.

Brief Summary. Let λ be a Salem number. Using Estes we find a rational symmetric matrix Q having $\lambda + \lambda^{-1}$, its conjugates, and 1 as eigenvalues. Without loss of generality we can assume that Q has a positive eigenvector $\mathbf{v}$ corresponding to $\lambda + \lambda^{-1}$ since we can always conjugate Q by an $SO(n; \mathbb{Q})$ matrix to rotate an eigenvector of $\lambda + \lambda^{-1}$ into the first orthant. We define the matrix $\mathcal{M} = \begin{bmatrix} Q & -I \\ I & \mathbf{0} \end{bmatrix}$ which has λ, its conjugates, and the 6th roots of unity μ and μ^{-1} as eigenvalues. Thus, $\mathcal{M}$ has determinant 1, integer characteristic polynomial, $\mathcal{M}^{-1}$ exists and has the skew-property where $\mathcal{M}^k + \mathcal{M}^{-k}$ is a block diagonal matrix where the $(1,1)$ and $(2,2)$ blocks are equal; call these blocks $\mathcal{Q}_k$.

$\mathcal{Q}_k$ has $\lambda^k + \lambda^{-k}$ as an eigenvalue, and all other eigenvalues lie in the interval $[-2, 2]$ and by above arguments we can power up $\mathcal{M}$ so that $\mathcal{Q}_k$ is positive and integral. Using Theorem 3.1, we know there is a surface with two tight filling multicurves A and B having $\mathcal{Q}_k$ as their intersection matrix. Applying Thurston's construction with $M = N = I$ we get a pseudo-Anosov map $T_A T_B$ having λ^{2k} as its stretch factor.

4.1.4 Proof of Theorem A

We have already done much of the heavy lifting required to prove Theorem A, as many of the results and ideas used in the proof of theorem 4.4 can be repurposed as we will explain below. First we will restate Theorem A.

Theorem A. *Let $\lambda > 1$ be a real algebraic unit with $[\mathbb{Q}(\lambda) : \mathbb{Q}] = n$, $\lambda + \lambda^{-1}$ totally real, and every Galois conjugate of λ lies in between λ^{-1} and λ in absolute value. Then there is a k such that λ^k is the stretch factor of a Thurston construction pseudo-Anosov homeomorphism of the surface $S_{(n+e)^2-(n+e)+1}$, where $e \in \{0, 1, 2\}$.*

For an algebraic integer such that $\lambda + \lambda^{-1}$ is totally real, we can apply Theorem 4.5 to obtain a rational, symmetric matrix Q whose characteristic polynomial is the minimal polynomial of $\lambda + \lambda^{-1}$ times $(x-1)^e$, where $e \in \{0, 1, 2\}$. Then, like before, we define the matrix

$$\mathcal{M} = \begin{bmatrix} Q & -I \\ I & \mathbf{0} \end{bmatrix}$$

This matrix has λ as an eigenvalue, and determinant 1. It also has an integer characteristic polynomial so by Proposition 4.7 we know that there is a positive integer k such that $\mathcal{M}^k$ is an integer matrix. Now,

$$\mathcal{M}^k + \mathcal{M}^{-k} = \begin{bmatrix} \mathcal{Q}_k & 0 \\ 0 & \mathcal{Q}_k \end{bmatrix}$$

has $\lambda^k + \lambda^{-k}$ as an eigenvalue. Note that this argument shows that $\lambda^k + \lambda^{-k}$ is totally real for all k. Now, in order for us to be in a situation where we can apply Thurston's construction we need to know that $\lambda^k + \lambda^{-k}$ is the dominating eigenvalue for all sufficiently large k.

Proposition 4.10. *If λ is an algebraic unit as described above, then there is some ℓ such that $\lambda^k + \lambda^{-k}$ is the dominating eigenvalue for $\mathcal{Q}_k$ for all $k > \ell$.*

Proof. Let $\sigma_i(\lambda)$ denote the Galois conjugates of λ. By assumption we have that both of the following inequalities hold

$$\lambda^{-1} < |\sigma_i(\lambda)| < \lambda \quad \text{and} \quad \lambda^{-1} < |\sigma_i(\lambda^{-1})| < \lambda$$

then certainly for all k we have

$$\lambda^{-k} < |\sigma_i(\lambda^k)| < \lambda^k \quad \text{and} \quad \lambda^{-k} < |\sigma_i(\lambda^{-k})| < \lambda^k.$$

Since λ^k is larger than both $|\sigma_i(\lambda^k)|$, and $|\sigma_i(\lambda^{-k})|$, then we have that

$$\frac{|\sigma_i(\lambda^k)| + |\sigma_i(\lambda^{-k})|}{\lambda^k} \to 0 \text{ as } k \to \infty$$

So, there is an ℓ such that for all $k > \ell$, we have

$$|\sigma_i(\lambda^k)| + |\sigma_i(\lambda^{-k})| < \lambda^k$$

and therefore,

$$|\sigma_i(\lambda^k) + \sigma_i(\lambda^{-k})| < \lambda^k + \lambda^{-k}$$

holds for all $k > \ell$. ■

We have shown that $\lambda^k + \lambda^{-k}$ is the dominating eigenvalue of $\mathcal{Q}_k$ for all $k > \ell$. Since $\mathcal{Q}_k$ is symmetric for all k and all share the same eigenspaces. This means that there is an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{n+e}\}$ for $\mathbb{R}^{n+e}$ where each $\mathbf{v}_i$ is an eigenvector of $\mathcal{Q}_k$, for all k. Let $\mathbf{v}_1$ be a positive eigenvector that spans the eigenspace of $\lambda^k + \lambda^{-k}$, for all k. We can write each standard basis vector in the form

$$\mathbf{e}_j = c_j \mathbf{v}_1 + \mathbf{w}_j$$

where $c_j > 0$ and $\mathbf{w}_j$ is a linear combination of $\{\mathbf{v}_2, ..., \mathbf{v}_{n+e}\}$. Since we have that $\lambda^k + \lambda^{-k}$ is the dominating eigenvalue for $\mathcal{Q}_k$, for large enough k, then for all k sufficiently large, we have

$$\mathcal{Q}_k \mathbf{e}_j \approx c_j(\lambda^k + \lambda^{-k})\mathbf{v}_1$$

So for all sufficiently large k we have that the entries of $\mathcal{Q}_k \mathbf{e}_j$ are positive, for all j. Therefore, the entries of $\mathcal{Q}_k$ are positive. So take k large enough so that $\mathcal{Q}_k$ is a nonsingular, positive, integer, symmetric matrix, and just like in the proof of Theorem 4.4 we can now apply Theorem C to build a closed orientable surface of genus $g = (n+e)^2 - (n+e) + 1$ with a pair of tight, filling multicurves A and B whose intersection matrix is $\mathcal{Q}_k$. We can now apply Thurston's construction with $M = N = I$ and obtain the following representations for T_A and T_B:

$$[T_A] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad [T_B] = \begin{bmatrix} 1 & 0 \\ -(\lambda^k + \lambda^{-k}) & 1 \end{bmatrix}$$

Hence, $T_A T_B$ is a pseudo-Anosov map having λ^{2k} as its stretch factor. Therefore, we have shown that given a real algebraic unit λ satisfying the hypotheses of Theorem A, then there is a positive integer k and a Thurston construction pseudo-Anosov homeomorphism of the surface of genus $g = (n+e)^2 - (n+e) + 1$ having λ^{2k} as its stretch factor. Hence, we have proven Theorem A.

4.2 Totally real number fields arising from Thurston's Construction

In this section we establish some final results, develop some background algebraic number theory and prove Theorem B. We start this section by proving a condition for when a real symmetric matrix will have a positive power and then conclude this section by showing that any rational symmetric matrix with a dominating eigenvalue larger than 1 is conjugate to a rational symmetric matrix that has a positive power.

4.2.1 Algebraic Number Theory

Proposition 4.11. *If Q is a real symmetric matrix with a unique dominating eigenvalue $\lambda > 1$ and a positive eigenvector $\mathbf{v}$, then there is some power of Q that is positive.*

Proof. The proof of this is very similar to the proof that $\mathcal{Q}_k$ is eventually positive. Every standard basis vector can be written as

$$\mathbf{e}_i = c_i\mathbf{v} + \mathbf{w}_i$$

with $c_i > 0$ for each i, and applying Q^k to both sides gives

$$Q^k\mathbf{e}_i = c_i\lambda^k\mathbf{v} + Q^k\mathbf{w}_i$$

Since $\lambda > \mu$ for all other eigenvalues μ then the sum $c_i\lambda^k\mathbf{v} + Q^k\mathbf{w}_i$ is eventually a positive vector. Hence, $Q^k\mathbf{e}_i$ is eventually positive for all i. So there is a k such that Q^k is a positive matrix. ∎

Proposition 4.12. *If $Q \in M_n(\mathbb{Q})$ is symmetric, and has a unique dominating eigenvalue $\lambda > 1$ with corresponding eigenvector $\mathbf{v}$. Then there is a matrix $U \in SO(n;\mathbb{Q})$ such that UA^kU^T is a positive symmetric rational matrix for some positive integer k.*

Proof. By Proposition 4.6 $SO(n;\mathbb{Q})$ is dense in $SO(n)$, so we can conjugate Q by an $SO(n,\mathbb{Q})$ matrix U so that λ now has a positive eigenvector. By Proposition 4.11 we know there is some k such that UA^kU^T is a positive symmetric rational matrix. ■

Before getting to the proof of Theorem B we will develop some algebraic number theory that we will use to prove the following:

Theorem 4.13. *Given a totally real number field K, there is an algebraic unit $\eta \in K$ such that $\eta > 1$, $K = \mathbb{Q}(\eta)$, $K = \mathbb{Q}(\eta^m)$ for all positive integers m, and all conjugates of η are positive, less than 1.*

First a couple definitions:

Definition 4.2.1. *Given a basis $\mathbf{e}_1, ..., \mathbf{e}_n$ for $\mathbb{R}^n$, then the $\mathbf{e}_i$ form a basis for a free $\mathbb{Z}$-module L of rank n, namely,*

$$L = \mathbb{Z}\mathbf{e}_1 \oplus ... \oplus \mathbb{Z}\mathbf{e}_n$$

*A set L constructed this way is called a **lattice** in $\mathbb{R}^n$.*

It is a well known fact that every number field K of degree n over $\mathbb{Q}$ has exactly n embeddings into $\mathbb{C}$. Since we are talking about totally real number fields, these embeddings will be into $\mathbb{R}$. Let $\sigma_1, ..., \sigma_n$ be these embeddings, where σ_1 denotes the inclusion embedding.

Definition 4.2.2. *Let K be a totally real number field of degree n. We define the **logarithmic embedding** of K into $\mathbb{R}^n$ by*

$$\lambda(x) = (\log|\sigma_1(x)|, ..., \log|\sigma_n(x)|)$$

for all nonzero $x \in K$. Note: that $\lambda(xy) = \lambda(x) + \lambda(y)$, so λ is a homomorphism from the multiplicative group K^* to the additive group $\mathbb{R}^n$.

The logarithmic embedding is used to prove the Dirichlet Unit Theorem, which gives a complete description of the unit group of a number field. A proof for this theorem can be found in [2], but we will just state it:

Theorem 4.14 (Dirichlet Unit Theorem)**.** *Let K be a number field, r_1 is the number of real embbedings, and r_2 is the number of complex embeddings (up to conjugacy). Then the unit group U of K is isomorphic to $G \times \mathbb{Z}^{r_1+r_2-1}$, where G is a finite cyclic group consisting of all the roots of unity in K.*

Since we are considering number fields K that are totally real then if $[K : \mathbb{Q}] = n$ we have $r_1 = n$, $r_2 = 0$ and $G = \{-1, 1\}$. Hence, the unit group of K is isomorphic to $\{-1, 1\} \times \mathbb{Z}^{n-1}$. That is, there are units $u_1, ..., u_{n-1}$ such that every unit of K is of the form

$$\pm u_1^{m_1} \cdots u_{n-1}^{m_{n-1}}$$

where m_i are integers.

The logarithmic embedding maps the unit group U of K to the hyperplane

$$H = \left\{ (x_1, ..., x_n) \;\middle|\; \sum_{i=1}^{n} x_i = 0 \right\}.$$

In fact, if $u_1, ..., u_{n-1}$ are the generators for U then $\{\lambda(u_1), ..., \lambda(u_{n-1})\}$ is a basis for H, hence $\lambda(U)$ is a lattice in H.

4.2.2 Proof of Theorem B

In this subsection we give the proof of Theorem B, but first we will prove Theorem 4.13.

Proof of Theorem 4.13. **Step 1**: We start by proving the following claim:

Claim: *Suppose η is a unit that generates the field, which is bigger than 1, and all its Galois conjugates are less than* 1 *in absolute value. Then $K = \mathbb{Q}(\eta^m)$ for any positive integer m.*

Proof. Let $\{\sigma_2(\eta), ..., \sigma_n(\eta)\}$ be the Galois conjugates of η. If there is a positive integer m such that $\mathbb{Q}(\eta^m)$ is a proper subfield of $\mathbb{Q}(\eta)$, then η^m has the following Galois conjugates (reordering if necessary) $\{\sigma_2(\eta)^m, ..., \sigma_k(\eta)^m\}$, where $k < n$. Since η and η^m are algebraic units we have that

$$|\eta\sigma_2(\eta)\cdots\sigma_n(\eta)| = 1$$

and

$$|\eta^m\sigma_2(\eta)^m\cdots\sigma_k(\eta)^m| = 1.$$

This tells us that

$$|\eta\sigma_2(\eta)\cdots\sigma_k(\eta)| = 1$$

therefore

$$|\sigma_{k+1}(\eta)\cdots\sigma_n(\eta)| = 1$$

which is impossible since $|\sigma_i(\eta)| < 1$ for $i = 2, ..., n$. Therefore, $K = \mathbb{Q}(\eta^m)$ for any positive integer m. ■

Step 2: Now we will find such a unit. Let $u_1, ..., u_{n-1}$ be positive generators for the unit group of K. Since $\{\lambda(u_1), ..., \lambda(u_{n-1})\}$ is a basis for H then we can take rational numbers $a_1, ..., a_{n-1}$ such that

(1) $\sum_{i=1}^{n-1} a_i \log|u_i| > 1$

(2) $\sum_{i=1}^{n-1} a_i \log|\sigma_j(u_i)| < 0, \ 2 \leq j \leq n-1$

(3) The entries of $a_1\lambda(u_1) + ... + a_{n-1}\lambda(u_{n-1})$ sum to 0

(4) All entries of $a_1\lambda(u_1) + ... + a_{n-1}\lambda(u_{n-1})$ are distinct.

Clearing denominators gives us an integer combination

$$b_1\lambda(u_1) + ... + b_{n-1}\lambda(u_{n-1})$$

that also has the above properties. Now take $u \in U$ to be the element

$$u = u_1^{b_1} \cdots u_{n-1}^{b_{n-1}}$$

By construction no two entries of $\lambda(u)$ are identical, hence $\sigma_i(u) \neq \sigma_j(u)$ for $i \neq j$, so u has n distinct Galois conjugates, therefore the minimal polynomial of u over $\mathbb{Q}$ has degree n. Hence, $K = \mathbb{Q}(u)$. Also, by construction, $\log|u| > 1$ so $|u| > 1$ but $|\sigma_i(u)| < 1$ for all $2 \leq i \leq n$, so by step 1 any power of u generates K. Let $\eta = u^2$, then $\eta > 1$, all its conjugates are positive less than 1, $K = \mathbb{Q}(\eta)$ and $K = \mathbb{Q}(\eta^m)$ for all positive integers m. ■

We now have all the pieces to prove the following:

Lemma 4.15. *Let K be a totally real number field. Then there is an algebraic unit η such that $K = \mathbb{Q}(\eta)$ and η is the dominant eigenvalue of a positive symmetric integral matrix that is the intersection matrix of a pair of tight, filling multicurves on some closed orientable surface.*

Proof. Let K be a totally real number field, then by Theorem 4.13 we can find an algebraic unit ζ so that $\zeta > 1$, $K = \mathbb{Q}(\zeta^m)$ for all positive integers m, and all conjugates of ζ are positive less than 1. By Theorem 4.5 there is a rational symmetric matrix B having ζ as its unique dominating eigenvalue greater than 1, and whose characteristic polynomial is $f(x)(x-1)^e$, where $f(x)$ is the minimal polynomial of ζ over $\mathbb{Q}$. Note that $f(x)$ has integer coefficients and since ζ is a unit then the constant term of $f(x)$ is 1, so $\det(B) = \pm 1$.

Now, by Proposition 4.12 we can conjugate B by an $SO(n+e;\mathbb{Q})$ matrix U where there is a positive integer k such that $Q = UB^kU^T$ is a positive rational matrix, without loss of generality assume k is even. Now, Q is a positive rational matrix whose eigenvalues are ζ^k, its conjugates, and 1, hence the coefficients of the characteristic polynomial of Q are integers, and $\det(Q) = (\pm 1)^k = 1$.

Applying Proposition 4.7, we know there is some positive integer ℓ so that Q^ℓ is integral and if we let $\eta = \zeta^{k\ell}$ then $K = \mathbb{Q}(\eta)$ where η is the dominating eigenvalue of Q^ℓ, which is a positive symmetric integral matrix. Now apply Theorem C to find a closed orientable surface with multicurves $A = \left\{\alpha_1, ..., \alpha_{(n+e)}\right\}$ and $B = \left\{\beta_1, ..., \beta_{(n+e)}\right\}$ such that $i(\beta_i, \alpha_j) = Q^\ell_{ij}$. ■

We end this chapter by proving Theorem B, which we restate here.

Theorem B. *Every totally real number field is of the form $K = \mathbb{Q}(\lambda + \lambda^{-1})$, where λ is the stretch factor of a pseudo-Anosov map arising from Thurston's construction.*

Proof. Let K be a totally real number field. By Lemma 4.15 we can find a unit α such that $K = \mathbb{Q}(\eta)$ and η is the dominating eigenvalue of a positive symmetric integral matrix Q that is the intersection matrix of a pair of tight, filling multicurves, A and B, on a closed orientable surface S_g. Without loss of generality we can assume $\eta > 2$. Applying Thurston's construction with $M = N = I$ we have η^2 as the dominating eigenvalue of $MQNQ^T = Q^2$, and the following representations of T_A and T_B:

$$[T_A] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad [T_B] = \begin{bmatrix} 1 & 0 \\ -\eta^2 & 1 \end{bmatrix}$$

Multiplying these matrices gives

$$[T_A][T_B] = \begin{bmatrix} 1 - \eta^2 & 1 \\ -\eta^2 & 1 \end{bmatrix}$$

Since $\eta^2 > 4$ then we have that $|tr([T_A][T_B])| = |2-\eta^2| > 2$. So $T_A T_B$ is a pseudo-Anosov map with stretch factor

$$\lambda = \frac{(\eta^2 - 2) + \eta\sqrt{\eta^2 - 4}}{2}$$

Hence, $\lambda + \lambda^{-1} = \eta^2 - 2$, and so $\mathbb{Q}(\lambda + \lambda^{-1}) = \mathbb{Q}(\eta^2 - 2) = \mathbb{Q}(\eta^2) = K$. ■

Chapter 5

Future Direction

In this chapter we discuss potential future research directions

5.1 Refining Genus

As we saw in chapter 3, we can construct closed, orientable surfaces from positive integer matrices. We described gluing instructions, and were able to determine the genus of the resulting surface. We made no claim that the gluing instructions we used produced the smallest genus surface possible. The gluing operations we chose were for convenience, as it enabled us to keep track of the number of vertices, and compute the Euler characteristic of the surface. As we saw in chapter 3, the construction can be done much more generally with the only restriction we impose is that each the intersection of the curves β_i and α_j all have the same sign. With this more general construction in mind, we can ask the following question.

Question 5.1. Giving a positive symmetric matrix Q, what is the smallest genus surface we can construct using the general construction described in chapter 3?

Here we offer a few examples that highlights the fact that the specific gluing instructions we used to prove Theorem 3.1 does not necessarily give rise to the smallest possible genus surface.

Example 5.1.1. Consider $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$, which from the proof of Theorem C we know that the gluing instructions from Theorem 3.1 will result in a genus 3 surface. Now we if follow different gluing instructions we may be able to obtain a genus 2 surface. Instead of the gluing instructions being that β_i wraps around a strip two times, before moving to the next strip or closing up, we will require that each β_i meets each strip exactly once before restarting at the top of the first strip and passing through the next rectangle in each strip.

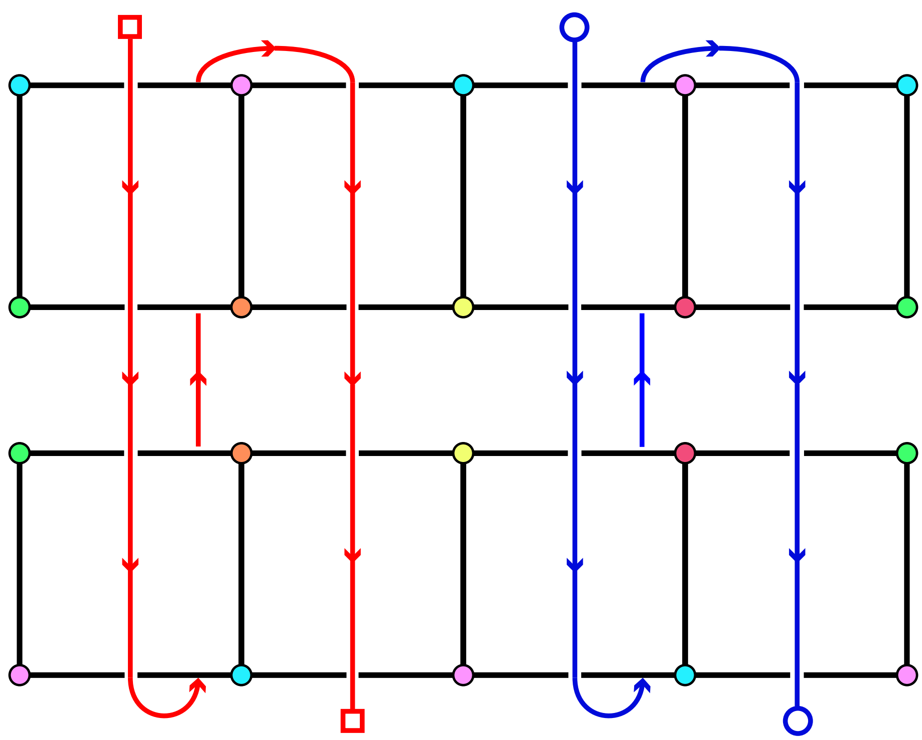


Figure 5.1: Illustration of the new gluing instructions. Each β_i must pass through each strip once before wrapping back around.

As we can see by the above illustration, these gluing instructions give rise to six gluing orbits, which in turn gives 6 vertices in the resulting cell structure. Hence, we have 6 vertices and 8 faces so the Euler characteristic is -2, therefore, the genus of the resulting surface is 2.

These gluing instructions give a smaller genus surface, but this idea of passing through each strip before wrapping back to the top does not always give the smallest possible surface as the next example shows.

Example 5.1.2. This time consider the matrix $\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$, which by our original gluing operations would give rise to a surface of genus 3. Now if we follow the gluing instructions given in example 5.1.1, each β_i must pass through each strip once before returning to the top, or closing up. We obtain the following diagram.

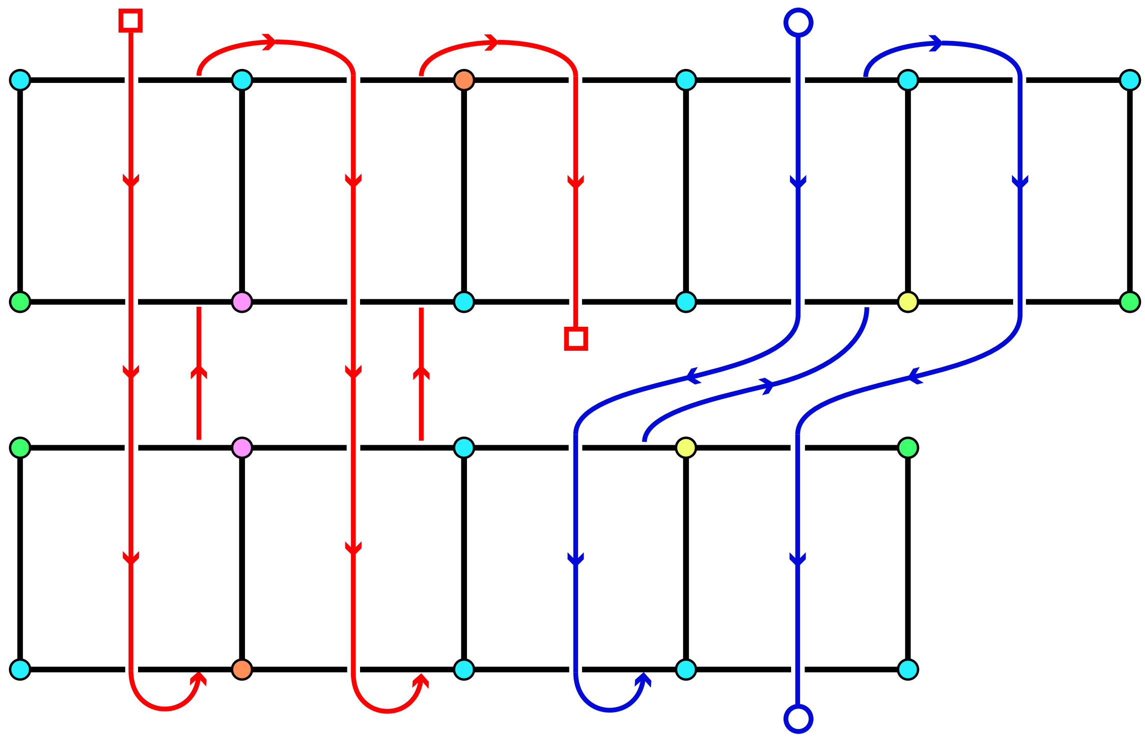


Figure 5.2

From here we can see that there are 5 vertices in the resulting cell structure as well as 9 faces. Thus, the Euler characteristic is -4, so the genus of the resulting surface is 3. We see that increasing one of the entries of the matrix had the effect of decreasing the number of vertices, which increased the genus. So then we ask is there gluing instructions that would result in a genus 2 surface? Consider the gluing instructions given by the following diagram.

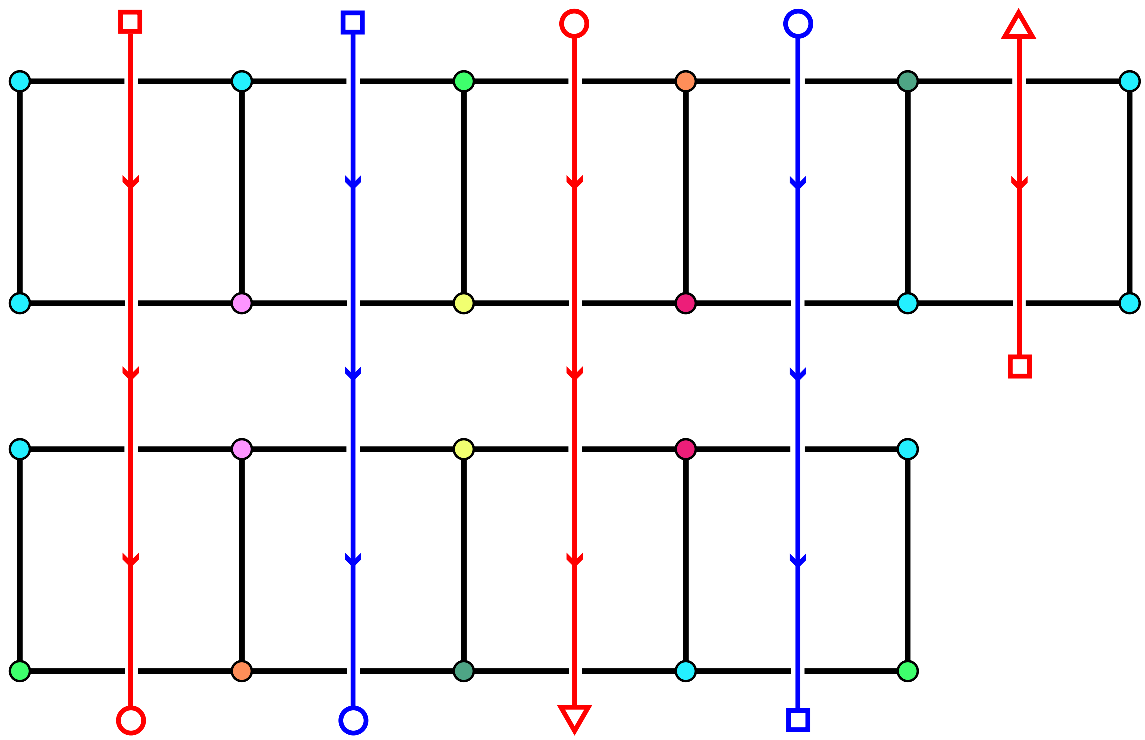


Figure 5.3: An alternate set of gluing instructions.

Here, β_1 passes through the left most rectangle in each strip, but skips a rectangle during each successive pass through each strip. Similarly for β_2. With these gluing instructions we see that there are 7 vertices and 9 faces, hence, the Euler characteristic is -2 and again we have that the genus of the resulting surface is 2.

As we see from these low level examples that we can certainly find different gluing instructions that can lower the genus of the resulting surface. It would be interesting to know if there some systematic way of choosing the gluing instructions so that the resulting genus is as small as possible.

5.2 Removing Powers

The work done in Chapter 4 makes significant progress to toward classifying stretch factors of pseudo-Anosov maps, but the methods used therein have the drawback that they produce powers of algebraic units that are stretch factors and not the units themselves. More specifically, we showed that every real algebraic unit $\lambda > 1$ satisfying the assumptions of Theorem A admits a power that is a stretch factor of a pseudo-Anosov map. So the question becomes

Question 5.2. If λ is algebraic unit satisfying the assumptions of theorem A, is λ the stretch factor of some pseudo-Anosov map?

It is unclear how we can adapt the methods of Chapter 4 to avoiding the need to power up λ. Even if we know that $\lambda + \lambda^{-1}$ is the dominating eigenvalue of a positive integer matrix, then our methods guarantee that λ^2 is a stretch factor. At the moment it seems unlikely that the methods of Chapter 4 are enough to obtain λ as a stretch factor.

There has been other recent work towards determining when a given algebraic unit is the stretch factor of a pseudo-Anosov map. In [3], the authors show that if you have an algebraic unit λ that is the dominating eigenvalue of an integer matrix satisfying certain technical conditions then there is a pseudo-Anosov map having the λ as its stretch factor. It may be worthwhile to explore any interplay between our results that can be used to answer Question 5.2.

5.3 Minimal Stretch Factor Problem

Every closed orientable surface S_g has a minimal stretch factor δ_g among pseudo-Anosov mapping classes of S_g. In [14] Penner showed that $2^{(1/12g-2)} \leq \delta_g \leq (2+\sqrt{3})^{1/g}$. Though upper and lower bounds on δ_g are known, we only know the minimal stretch factor for genus 1 and genus 2, with the genus 2 case proven in 2008 by Cho and Ham in [4]. If you restrict to pseudo-Anosov mapping classes with orientable foliations then the minimal stretch factor amongst those mapping classes is known up to genus 5 and they are all Salem numbers, see [10].

Finding minimal stretch factors is an interesting problem in its own right, and seeing Salem numbers appearing as minimal stretch factors is intriguing, given the results of this dissertation. It may be worth exploring ways we can expand our results to seek out minimal stretch factors, and see if the trend of minimal stretch factor Salem numbers continues.

Appendix A

Miscellaneous Theory

In this appendix we discuss some topics that we did not spend time developing in the earlier chapters of this dissertation, either because knowledge was assumed, or because the discussion would have taken us too far afield of where we were heading.

A.1 Surfaces

We begin with a very brief review of terminology related to surfaces, as well as state the classification of closed orientable surfaces.

A.1.1 Properties of Surfaces

A ***Surface*** is a two dimensional manifold, such as a sphere or a torus. Typically surfaces are assumed to be connected, and often are assume to be compact, which rules out planes or surfaces with deleted points, also known as ***punctured surfaces***. Surfaces have several important characteristic the first being what is known as the ***genus*** of the surface which is defined as the number of cuts along non-intersecting simple closed curves that leave the surface connected. For example the Torus has genus 1 since we can cut along a longitudinal simple closed curve and the surface remains connected, but any further cut will result in a disconnected surface.

A surface S may have a ***boundary***, denoted $\partial(S)$, which is the set of points such that every neighborhood of these points is homeomorphic to an open subset of the upper half plane $\mathbb{H} = \{(x, y) \mid y \geq 0\}$ (thought of as a subspace of $\mathbb{R}^2$) which intersection with the set $\partial\mathbb{H} = \{(x, 0) \mid x \in \mathbb{R}\}$ is nonempty. The boundary of a compact surface is a compact one dimensional manifold, so it is homeomorphic to either a closed interval, or a circle $\mathbb{S}^1$. A compact surface without boundary is said to be ***closed***.

A final property of surfaces that we discuss is whether a surface is ***orientable*** or not. A surface is said to be orientable if there is a choice of normal vectors at every point of the surface that varies continuously. In essence, this means we can walk around the surface pointing upwards and when we return to where we started we will still be pointing in the same direction. For example, the torus is orientable but the Möbius strip is not. In fact, a surface is nonorientable if and only if it contains an embedded Möbius strip.

A.1.2 Classification of Surfaces

The notion of the ***connect sum*** of surfaces gives us a way to construct new surfaces out of old ones. The connect sum of two closed surfaces S and S' is the surface obtained by deleting an open disk from each surface and then gluing the boundary of a cylinder to the boundary circles of the deleted disks. The idea of connect sum is used in the classification of surfaces, which is a famous result often attributed to Möbius. We state the closed, orientable version.

Theorem A.1. *Every closed, orientable surface is homeomorphic to the connect sum of a 2-dimensional sphere with $g \geq 0$ tori.*

Now associated with each surface is a number which is an invariant of the homeomorphism class of the surface known as the **Euler characteristic** of the surface. The *Euler characteristic* of a genus g surface S_g, denoted $\chi(S_g)$, is

$$\chi(S_g) = 2 - 2g$$

Two surfaces are homeomorphic if and only if they have the same Euler characteristic. Another way to calculate the Euler characteristic is to give a cell structure to S_g with V vertices, E edges, and F faces and we have that

$$\chi(S) = V - E + F$$

A.2 Perron-Frobenius Theorem

A key part of Thurston's construction and crucial to the proof that stretch factors are algebraic units is a particular type of matrix known as a Perron-Frobenius matrix, which we now define.

Definition A.2.1. *A square matrix Q is called **Perron-Frobenius** if it has nonnegative real entries and there is some positive integer k such that Q^k has positive entires. Such a matrix is sometimes called **primitive**.*

These matrices are so named Perron-Frobenius because they are the focus of the Perron-Frobenius Theorem, which asserts the following.

Theorem A.2 (Perron-Frobenius Theorem)**.** *Let Q be a primitive matrix, then Q has a unique positive real eigenvalue that is larger in absolute value than the other eigenvalues, as well as a corresponding positive eigenvector. The unique positive real eigenvalue is known as the **Perron-Frobenius eigenvalue**, and a corresponding positive eigenvector is known as a **Perron-Frobenius eigenvector**.*

This theorem was first proved for matrices with positive entries by German mathematician Oskar Perron in the early 1900's, and was extended to primitive matrices a few years later by another German mathematician Georg Frobenius. This theorem has seen applications in many areas of mathematics, but for us it appears during our discussion of Thurston's construction when we define the primitive matrix $MQNQ^T$. This theorem guarantees that $MQNQ^T$ has a unique positive eigenvalue ν with a positive eigenvector, which gives us the choice of side lengths for the rectangles in the branched flat structure so that the products of Dehn twists act affinely.

A.3 Orthogonal Matrices

An important step in several theorems was conjugating a rational matrix by a $SO(n;\mathbb{Q})$ matrix to obtain a positive eigenvector. We made claims about the density of $SO(n;\mathbb{Q})$ in $SO(n)$, but we did not say much more. Since that step is pivotal to the proofs of the main theorems, we will now take the time to fully justify it. We start by defining orthogonal matrices.

Definition A.3.1. *Let A be a real $n \times n$ matrix, then we say A is* ***orthogonal*** *if $A^T A = I$.*

Immediately from this definition we have the following:

1. $A^T = A^{-1}$, and thus the inverse of an orthogonal matrix is orthogonal.
2. The columns of A form an orthonormal basis for $\mathbb{R}^n$
3. $\det(A) = \pm 1$
4. If B is also another orthogonal matrix then AB is an orthogonal matrix.
5. If $||\cdot||$ denotes the Euclidean norm, then $||A\mathbf{v}|| = ||\mathbf{v}||$ for every $\mathbf{v} \in \mathbb{R}^n$

Properties (1) and (4) tell us that the set of orthogonal matrices form a group under matrix multiplication. We denote this group by $O(n)$. An important subgroup is the orthogonal matrices with determinant 1, denoted by $SO(n)$. The goal of this section is to prove the following two theorems:

Theorem A.3. *$O(n;\mathbb{Q})$ is dense in $O(n)$, and consequently, $SO(n;\mathbb{Q})$ is dense in $SO(n)$.*

Once we have proven Theorem A.3 we want to show that we can conjugate a rational symmetric matrix so that a given eigenvalue has a positive eigenvector. More specifically we prove:

Theorem A.4. *If $A \in M_n(\mathbb{Q})$ is symmetric, and has an eigenvalue $\lambda > 1$ with corresponding eigenvector $\mathbf{v}$. Then there is a matrix $B \in SO(n;\mathbb{Q})$ such that $B^{-1}A^kB$ is a positive matrix for some positive integer k.*

Before we can talk about dense subgroups we need to give a topology to $O(n)$. The topology we want is the one induced by the ***Frobenius Norm***.

Definition A.3.2. *If A is an $n \times n$ matrix, then the **Frobenius Norm** of A, denoted by $||A||_F$, is*

$$||A||_F = \left(\sum_{i,j} a_{ij}^2\right)^{1/2}$$

It is not hard to see that for any $\mathbf{v} \in \mathbb{R}^n$ and any $n \times n$ matrix A

$$||A\mathbf{v}|| \leq ||A||_F||\mathbf{v}||.$$

Now, the Frobenius norm induces a topology on $O(n)$ which has the following sets as a basis:

$$B_\epsilon(A) = \{C \in O(n) \mid \epsilon > 0,\ ||A - C||_F < \epsilon\}.$$

The next step to proving Theorem A.3 is for us to show that every orthogonal matrix is the product of some number of *reflection matrices*, which we now define:

Let $\mathbf{u}$ be a unit vector in $\mathbb{R}^n$ then the **reflection matrix** of $\mathbf{u}$ is

$$H_{\mathbf{u}} = I - 2\mathbf{u}\mathbf{u}^T$$

A reflection matrix is its own inverse, and is symmetric, hence reflection matrices are Orthogonal matrices. We now prove the following theorem:

Theorem A.5. *Let $A \in O(n)$ for $n \geq 2$, then there are reflection matrices $H_{\mathbf{v}_1}, ..., H_{\mathbf{v}_k}$ such that $A = H_{\mathbf{v}_1} H_{\mathbf{v}_2} \cdots H_{\mathbf{v}_k}$, for $1 \leq k \leq n$. In other words, every orthogonal matrix is the product of at most n reflections.*

(Note: For $n = 1$ the only two Orthogonal matrices are $[-1], [1]$ and the only reflection matrix is $[-1]$ but $[1] = [-1][-1]$, hence $[1]$ is a product of 2 reflections.)

Proof. We proceed by induction:

For $n = 2$, let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be an orthogonal matrix, now pick a reflection matrix $H_{\mathbf{v}}$ such that

$$H_{\mathbf{v}} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

which is possible since $\begin{bmatrix} a \\ c \end{bmatrix}$ is a unit vector. Since the product of orthogonal matrices is orthogonal then we know the columns of $H_{\mathbf{v}} A$ are orthonormal. Since the first column of $H_{\mathbf{v}} A$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ then the second column must be $\begin{bmatrix} 0 \\ \pm 1 \end{bmatrix}$.

If the second column of $H_{\mathbf{v}} A$ is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then $H_{\mathbf{v}} A = I$ and thus $A = H_{\mathbf{v}}$ and we are done.

If the second column of $H_{\mathbf{v}}A$ is $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ then we have

$$H_{\mathbf{v}}A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = H_{\mathbf{e}_2}$$

Thus, $A = H_{\mathbf{v}}H_{\mathbf{e}_2}$, and again we are done.

Now assume for some n that any $(n-1) \times (n-1)$ orthogonal matrix is the product of at most $n-1$ reflection matrices. Let A be in $O(n)$, then $A\mathbf{e}_1$ is the first column of A, now pick a reflection matrix $H_{\mathbf{v}_1}$ such that

$$H_{\mathbf{v}_1}A\mathbf{e}_1 = \mathbf{e}_1$$

Thus,

$$H_{\mathbf{v}_1}A = \begin{bmatrix} 1 & * \\ \mathbf{0} & B \end{bmatrix}$$

Where B is an $(n-1) \times (n-1)$ orthogonal matrix. Now since the columns of $H_{\mathbf{v}_1}A$ are orthogonal then we know that the row vector $*$ must have all zero entries. Thus,

$$H_{\mathbf{v}_1}A = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & B \end{bmatrix}$$

If $B = I_{n-1}$ then $A = H_{\mathbf{v}_1}$ and we are done. Otherwise, by the induction hypothesis, we know that there are unit vectors $\widetilde{\mathbf{v}}_2, ..., \widetilde{\mathbf{v}}_k \in \mathbb{R}^{n-1}$, $2 \leq k \leq n$, such that

$$B = H_{\widetilde{\mathbf{v}}_2} \cdots H_{\widetilde{\mathbf{v}}_k}$$

Let $\mathbf{v}_i = \begin{bmatrix} 0 \\ \tilde{\mathbf{v}}_i \end{bmatrix}$, then define

$$H_{\mathbf{v}_i} = I - 2\mathbf{v}_i\mathbf{v}_i^T = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & H_{\tilde{\mathbf{v}}_i} \end{bmatrix}$$

Then we have that

$$\begin{aligned} H_{\mathbf{v}_1}A &= \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & B \end{bmatrix} \\ &= \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & H_{\tilde{\mathbf{v}}_2} \cdots H_{\tilde{\mathbf{v}}_k} \end{bmatrix} \\ &= \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & H_{\tilde{\mathbf{v}}_2} \end{bmatrix} \cdots \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & H_{\tilde{\mathbf{v}}_k} \end{bmatrix} \\ &= H_{\mathbf{v}_2} \cdots H_{\mathbf{v}_k} \end{aligned}$$

Therefore, $A = H_{\mathbf{v}_1}H_{\mathbf{v}_2} \cdots H_{\mathbf{v}_k}$, so A is the product of at most n reflection matrices. ■

Once we have proven the next proposition then we will be ready to prove Theorem A.3:

Proposition A.6. *Rational points on the unit n sphere, S^n, centered at the origin, are dense in S^n.*

Proof. Stereographic projection is the homeomorphism from $S^n \setminus \{\mathbf{e}_{n+1}\} \to \mathbb{R}^n$ such that

$$(x_1, ..., x_{n+1}) \mapsto \left(\frac{x_1}{1 - x_{n+1}}, ..., \frac{x_n}{1 - x_{n+1}}\right)$$

whose inverse is the map

$$\mathbf{a} = (a_1, ..., a_n) \mapsto \left(\frac{2a_1}{||\mathbf{a}||^2 + 1}, ..., \frac{2a_n}{||\mathbf{a}||^2 + 1}, \frac{||\mathbf{a}||^2 - 1}{||\mathbf{a}||^2 + 1}\right)$$

Now it should be clear that stereographic projection gives a bijection between rational points of $S^n \setminus \{\mathbf{e}_{n+1}\}$ and $\mathbb{Q}^n$. Thus, since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$ then the rational points of $S^n \setminus \{\mathbf{e}_{n+1}\}$ are dense in $S^n \setminus \{\mathbf{e}_{n+1}\}$, hence the rational points of S^n are dense in S^n. ■

We are now ready to prove Theorem A.3:

Proof. Let $A \in O(n)$, then by Theorem A.5

$$A = \prod_{i=1}^{k} H_{\mathbf{v}_i}$$

for some $k \leq n$, and some unit vectors $\mathbf{v}_1, ..., \mathbf{v}_k$.

Let $\delta > 0$ and for each i choose a unit vector $\mathbf{u}_i$ such that $\mathbf{u}_i$ has rational coordinates and

$$\|\mathbf{v}_i - \mathbf{u}_i\|_\infty \leq \frac{\delta}{8n^2}$$

which is possible by Proposition A.6 since we can pick a $\mathbf{u}_i$ close enough to $\mathbf{v}_i$ such that the absolute value of the difference of all the entries is as small as we want.

Now, Define

$$U = \prod_{i=1}^{k} H_{\mathbf{u}_i}$$

Obviously U has rational entries. Now,

$$\begin{aligned}
||U - A||_F &= \left\| \prod_{i=1}^{k} H_{\mathbf{u}_i} - \prod_{i=1}^{k} H_{\mathbf{v}_i} \right\|_F \\
&= \left\| \prod_{i=1}^{k} H_{\mathbf{u}_i} - H_{\mathbf{v}_1} \prod_{i=2}^{k} H_{\mathbf{u}_i} + H_{\mathbf{v}_1} \prod_{i=2}^{k} H_{\mathbf{u}_i} - \prod_{i=1}^{k} H_{\mathbf{v}_i} \right\|_F \\
&= \left\| (H_{\mathbf{u}_1} - H_{\mathbf{v}_1}) \prod_{i=2}^{k} H_{\mathbf{u}_i} + H_{\mathbf{v}_1} \left(\prod_{i=2}^{k} H_{\mathbf{u}_i} - \prod_{i=2}^{k} H_{\mathbf{v}_i} \right) \right\|_F \\
&\leq \left\| (H_{\mathbf{u}_1} - H_{\mathbf{v}_1}) \prod_{i=2}^{k} H_{\mathbf{u}_i} \right\|_F + \left\| H_{\mathbf{v}_1} \left(\prod_{i=2}^{k} H_{\mathbf{u}_i} - \prod_{i=2}^{k} H_{\mathbf{v}_i} \right) \right\|_F \\
&= \|H_{\mathbf{u}_1} - H_{\mathbf{v}_1}\|_F + \left\| \prod_{i=2}^{k} H_{\mathbf{u}_i} - \prod_{i=2}^{k} H_{\mathbf{v}_i} \right\|_F .
\end{aligned}$$

The last equality due to repeated uses of the fact that $\|AB\|_F = \|BA\|_F = \|A\|_F$ for any orthogonal matrix A and any $n \times n$ matrix B. Now, repeatedly applying the above argument gives

$$\|U - A\|_F \leq \sum_{i=1}^{k} \|H_{\mathbf{u}_i} - H_{\mathbf{v}_i}\|_F. \tag{1}$$

Now we wish to bound each $\|H_{\mathbf{u}_i} - H_{\mathbf{v}_i}\|_F$. We can do this by the following: Let a_m denote the mth coordinate of a vector $\mathbf{a}$, then

$$\begin{aligned}
\|H_{\mathbf{u}_i} - H_{\mathbf{v}_i}\|_F &= \|(I - 2\mathbf{u}_i\mathbf{u}_i^T) - (I - 2\mathbf{v}_i\mathbf{v}_i^T)\|_F \\
&= \left(\sum_{m,\ell} (2u_{im}u_{i\ell} - 2v_{im}v_{i\ell})^2\right)^{1/2} \\
&\leq \left(n^2 \cdot \max_{m,\ell} \left\{(2u_{im}u_{i\ell} - 2v_{im}v_{i\ell})^2\right\}\right)^{1/2} \\
&= n \cdot \max_{m,\ell} \{|2u_{im}u_{i\ell} - 2v_{im}v_{i\ell}|\}
\end{aligned}$$

where the term

$$\max_{m,\ell} \{|2u_{im}u_{i\ell} - 2v_{im}v_{i\ell}|\}$$

is what is known as the **infinity norm** of a matrix or vector, that is, the infinity norm of a matrix or vector is just the maximum among the absolute value of its entries, and is denoted by $\|\cdot\|_\infty$. Hence, the above inequality becomes

$$\|H_{\mathbf{u}_i} - H_{\mathbf{v}_i}\| \leq n\|H_{\mathbf{u}_i} - H_{\mathbf{v}_i}\|_\infty \tag{2}$$

By above, the absolute value of the $m\ell$th entry of $H_{\mathbf{u}_i} - H_{\mathbf{v}_i}$ is

$$\begin{aligned}
|2u_{im}u_{i\ell} - 2v_{im}v_{i\ell}| &= |2u_{im}u_{i\ell} - 2v_{im}u_{i\ell} + 2v_{im}u_{i\ell} - 2v_{im}v_{i\ell}| \\
&= |2u_{i\ell}(u_{im} - v_{im}) + 2v_{im}(u_{i\ell} - v_{i\ell})| \\
&\leq 2|u_{i\ell}| \cdot |u_{im} - v_{im}| + 2|v_{im}| \cdot |u_{i\ell} - v_{i\ell}|
\end{aligned}$$

$$\leq 4 \max_m \{|u_{im} - v_{im}|\}$$
$$= 4\|\mathbf{u}_i - \mathbf{v}_i\|_\infty$$

The second inequality comes from the fact that $H_{\mathbf{u}_i}$ and $H_{\mathbf{v}_i}$ are orthogonal matrices and hence each entry is less than or equal to 1 in absolute value. The above calculation shows that

$$\|H_{\mathbf{u}_i} - H_{\mathbf{v}_i}\|_\infty \leq 4\|\mathbf{u}_i - \mathbf{v}_i\|_\infty \tag{3}$$

Putting $(1), (2), (3)$ and (4) together gives

$$\begin{aligned}\|U - A\|_F &\leq \sum_{i=1}^{k} 4\|\mathbf{u}_i - \mathbf{v}_i\|_\infty \\ &\leq \sum_{i=1}^{k} \frac{\delta}{2n} \\ &= \frac{k}{2n}\delta \\ &\leq \frac{\delta}{2} \\ &< \delta\end{aligned}$$

Thus, $O(n;\mathbb{Q})$ is dense in $O(n)$. Note that $\det(U) = \det(A) = \pm 1$, and so if A is in $SO(n)$ then U is also in $SO(n)$, hence $SO(n;\mathbb{Q})$ is dense in $SO(n)$. ■

Now we are ready to prove Theorem A.4

Proof. It is not hard to see that $SO(n)$ acts transitively on S^{n-1}, the unit sphere of $\mathbb{R}^n$ centered at the origin. If $\mathbf{x}, \mathbf{y} \in S^{n-1}$ then $\mathbf{x}^T\mathbf{x} = \mathbf{y}^T\mathbf{y} = 1$. Now, we wish to find a reflection matrix that sends x to y(it will also send y to x). We define the following unit vector:

$$\mathbf{u} = \frac{1}{\sqrt{2(1 - \mathbf{x}^T\mathbf{y})}}(\mathbf{x} - \mathbf{y})$$

and now show that $H_{\mathbf{u}}(\mathbf{x}) = \mathbf{y}$: (Note: $\mathbf{x}^T\mathbf{y} = \mathbf{y}^T\mathbf{x}$)

$$\begin{aligned}
H_{\mathbf{u}}(\mathbf{x}) &= \mathbf{x} - 2\mathbf{u}\mathbf{u}^T\mathbf{x} \\
&= \mathbf{x} - \frac{2}{2(1-\mathbf{x}^T\mathbf{y})}(\mathbf{x}-\mathbf{y})(\mathbf{x}^T - \mathbf{y}^T)\mathbf{x} \\
&= \mathbf{x} - \frac{1}{(1-\mathbf{x}^T\mathbf{y})}(\mathbf{x}-\mathbf{y})(1-\mathbf{y}^T\mathbf{x}) \\
&= \frac{\mathbf{x}(1-\mathbf{x}^T\mathbf{y}) - (\mathbf{x}-\mathbf{y})(1-\mathbf{x}^T\mathbf{y})}{1-\mathbf{x}^T\mathbf{y}} \\
&= \frac{\mathbf{x} - \mathbf{x}\mathbf{x}^T\mathbf{y} - \mathbf{x} + \mathbf{x}\mathbf{x}^T\mathbf{y} + \mathbf{y}(1-\mathbf{x}^T\mathbf{y})}{1-\mathbf{x}^T\mathbf{y}} \\
&= \mathbf{y}
\end{aligned}$$

Thus, any point of S^{n-1} can be sent to any other point of S^{n-1} via a reflection matrix. Hence, there is a reflection matrix $H_{\mathbf{v}}$ that sends $\mathbf{x}$ to $-\mathbf{y}$. Therefore, $H_{\mathbf{y}}H_{\mathbf{v}}$ is an $SO(n)$ matrix that sends $\mathbf{x}$ to $\mathbf{y}$. Thus, $SO(n)$ acts transitively on S^{n-1}.

Now, given a symmetric matrix $A \in M_n(\mathbb{Q})$ with a unique dominating eigenvalue $\lambda > 1$ with eigenvector $\mathbf{v}$, if $\mathbf{v}$ has negative entries then by above we know we can find a $U' \in SO(n)$ matrix such that $U'\mathbf{v}$ has positive entries. Also, $U'\mathbf{v}$ is an eigenvector of $U'AU'^{-1}$ corresponding to λ. Chose $\epsilon > 0$ so that $B_\epsilon(U\mathbf{v})$ is contained in the first orthant and pick a $U \in SO(n;\mathbb{Q})$ such that

$$\|U' - U\|_F < \frac{\epsilon}{\|\mathbf{v}\|}$$

Then,

$$\begin{aligned}
\|U'\mathbf{v} - U\mathbf{v}\| &\le \|U' - U\|_F\|\mathbf{v}\| \\
&< \frac{\epsilon}{\|\mathbf{v}\|}\|\mathbf{v}\| \\
&= \epsilon
\end{aligned}$$

Hence, $U\mathbf{v} \in B(U'\mathbf{v})$ so $U\mathbf{v}$ is positive. Note that UAU^{-1} is a rational matrix. Since UAU^{-1} has a unique dominating eigenvalue $\lambda > 1$ and a positive eigenvector $U\mathbf{v}$ then by Proposition 4.11 we know that there is a positive integer k such that UA^kU^{-1} is positive. $\blacksquare$

Bibliography

[1] D. V. Anosov. *Geodesic flows on closed Riemann manifolds with negative curvature.* Proceedings of the Steklov Institute of Mathematics, No. 90 (1967). Translated from the Russian by S. Feder. American Mathematical Society, Providence, R.I., 1969, pp. iv+235.

[2] Robert B. Ash. *A course in algebraic number theory.* Dover Publications Inc., Mineola, NY, 2010, pp. 49–56. ISBN: 978-0-486-47754-1; 0-486-47754-1.

[3] Hyungryul Baik, Ahmad Rafiqi, and Chenxi Wu. "Constructing pseudo-Anosov maps with given dilatations". In: *Geom. Dedicata* 180 (2016), pp. 39–48. ISSN: 0046-5755. DOI: 10.1007/s10711-015-0089-1. URL: https://doi.org/10.1007/s10711-015-0089-1.

[4] Jin-Hwan Cho and Ji-Young Ham. "The minimal dilatation of a genus-two surface". In: *Experiment. Math.* 17.3 (2008), pp. 257–267. ISSN: 1058-6458. URL: http://projecteuclid.org/euclid.em/1227121381.

[5] Max Dehn. *Papers on group theory and topology.* Translated from the German and with introductions and an appendix by John Stillwell, With an appendix by Otto Schreier. Springer-Verlag, New York, 1987, pp. viii+396. ISBN: 0-387-96416-9. DOI: 10.1007/978-1-4612-4668-8. URL: https://doi.org/10.1007/978-1-4612-4668-8.

[6] Dennis R. Estes. “Eigenvalues of symmetric integer matrices”. In: *J. Number Theory* 42.3 (1992), pp. 292–296. ISSN: 0022-314X. DOI: 10.1016/0022-314X(92)90094-6. URL: http://dx.doi.org/10.1016/0022-314X(92)90094-6.

[7] Benson Farb and Dan Margalit. *A primer on mapping class groups*. Vol. 49. Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 2012, pp. xiv+472. ISBN: 978-0-691-14794-9.

[8] Albert Fathi, François Laudenbach, and Valentin Poénaru. *Thurston's work on surfaces*. Vol. 48. Mathematical Notes. Translated from the 1979 French original by Djun M. Kim and Dan Margalit. Princeton University Press, Princeton, NJ, 2012, pp. xvi+254. ISBN: 978-0-691-14735-2.

[9] Stephen P. Humphries. “Generators for the mapping class group”. In: *Topology of low-dimensional manifolds (Proc. Second Sussex Conf., Chelwood Gate, 1977)*. Vol. 722. Lecture Notes in Math. Springer, Berlin, 1979, pp. 44–47.

[10] Erwan Lanneau and Jean-Luc Thiffeault. “On the minimum dilatation of pseudo-Anosov homeromorphisms on surfaces of small genus”. In: *Ann. Inst. Fourier (Grenoble)* 61.1 (2011), pp. 105–144. ISSN: 0373-0956. DOI: 10.5802/aif.2599. URL: https://doi.org/10.5802/aif.2599.

[11] Christopher J. Leininger. “On groups generated by two positive multi-twists: Teichmüller curves and Lehmer's number”. In: *Geom. Topol.* 8 (2004), pp. 1301–1359. ISSN: 1465-3060. DOI: 10.2140/gt.2004.8.1301. URL: https://doi.org/10.2140/gt.2004.8.1301.

[12] W. B. R. Lickorish. “A finite set of generators for the homeotopy group of a 2-manifold”. In: *Proc. Cambridge Philos. Soc.* 60 (1964), pp. 769–778.

[13] D. D. Long. “Constructing pseudo-Anosov maps”. In: *Knot theory and manifolds (Vancouver, B.C., 1983)*. Vol. 1144. Lecture Notes in Math. Springer, Berlin, 1985,

pp. 108–114. DOI: 10.1007/BFb0075014. URL: http://dx.doi.org/10.1007/BFb0075014.

[14] R. C. Penner. “Bounds on least dilatations”. In: *Proc. Amer. Math. Soc.* 113.2 (1991), pp. 443–450. ISSN: 0002-9939. DOI: 10.2307/2048530. URL: https://doi.org/10.2307/2048530.

[15] Robert C. Penner. “A construction of pseudo-Anosov homeomorphisms”. In: *Trans. Amer. Math. Soc.* 310.1 (1988), pp. 179–197. ISSN: 0002-9947. DOI: 10.2307/2001116. URL: http://dx.doi.org/10.2307/2001116.

[16] Igor Rivin. “Walks on groups, counting reducible matrices, polynomials, and surface and free group automorphisms”. In: *Duke Math. J.* 142.2 (2008), pp. 353–379. ISSN: 0012-7094. DOI: 10.1215/00127094-2008-009. URL: https://doi.org/10.1215/00127094-2008-009.

[17] Eric Schmutz. “Rational points on the unit sphere”. In: *Cent. Eur. J. Math.* 6.3 (2008), pp. 482–487. ISSN: 1895-1074. DOI: 10.2478/s11533-008-0038-4. URL: http://dx.doi.org/10.2478/s11533-008-0038-4.

[18] Hyunshik Shin and Balázs Strenner. “Pseudo-Anosov mapping classes not arising from Penner’s construction”. In: *Geom. Topol.* 19.6 (2015), pp. 3645–3656. ISSN: 1465-3060. DOI: 10.2140/gt.2015.19.3645. URL: http://dx.doi.org/10.2140/gt.2015.19.3645.

[19] William P. Thurston. “On the geometry and dynamics of diffeomorphisms of surfaces”. In: *Bull. Amer. Math. Soc. (N.S.)* 19.2 (1988), pp. 417–431. ISSN: 0273-0979. URL: https://doi.org/10.1090/S0273-0979-1988-15685-6.