Title

Some results on gradient Ricci solitons and complete Kähler manifolds with nonnegative curvature

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Publication Date

2013

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA, SAN DIEGO

Some results on gradient Ricci solitons and complete Kähler manifolds with nonnegative curvature

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

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2013

The dissertation of Bo Yang is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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2013

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ACKNOWLEDGEMENTS

First of all I would like to thank Ben Chow and Lei Ni for being my advisors and various helpful suggestions during my PhD study. It is a great pleasure to express my gratitude to them for their generous support during my first year and two summers at UCSD.

I am also grateful to both Ben Chow and Peng Lu for their teaching during our collaboration on gradient Ricci solitons.

I am indebted to Fangyang Zheng for his generosity to share many insights on uniformization theory in higher dimensions. He kindly agreed to collaborate on rotationally symmetric Kähler-Ricci metrics with nonnegative curvature. It is hard to imagine that I would work anything out on this topic without his warm encouragement.

Also I want to thank Ben Weinkove for his excellent teaching on complex geometry at UCSD and constant help, Thanks also go to Pengfei Guan, Xiaojun Huang, Steven Lu, and Li Sheng Tseng for their interests and support, to Albert Chau for helpful discussions on uniformization theory on various occasions, to Xiaohua Zhu whose teaching encouraged me to pursue a PhD degree.

Thanks also go to many teachers at UCSD, including Mark Gross, Ken Intriligator, and Congjun Wu for their kind help to sit in my doctoral committee, and Li-Tien Cheng for his kind help on my teaching at UCSD. I am grateful to many old and new friends for interesting discussions on mathematics on various occasions in the past years, in particular Guoyi Xu for dicussions on Ricci flow, Pun Wai Tong and Zezhou Zhang for some informal seminars we enjoyed at UCSD. It is also a great pleasure to thank Ron Evans, Kimberly Eaton, and Lois Stewart for they kind help on my application for travel support for several math conferences!

Most importantly I thank my family for their understanding, love, and constant support; and to Qiong for all the happiness we have together so far and all that is to come.

Main results of Chapter 1 are joint work with Ben Chow and Peng Lu. They were published in Comptes Rendus Mathématique in 2011, Proceedings of the American Mathematical Society in 2012, and Annals of Global Analysis and

Geometry in 2012.

Parts of Chapter 2 are joint work with Fangyang Zheng which was published in Communications in Analysis and Geometry 2013. I also include my own work which appeared in Mathematische Annalen 2013. The remaining results of Chapter 2 are from my ongoing project and it will appear as a separate paper and be submitted to a journal for consideration for publication in the future.

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Holomorphic functions and geometry of complete noncompact Kähler manifolds with nonnegative curvature. To appear.

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ABSTRACT OF THE DISSERTATION

Some results on gradient Ricci solitons and complete Kähler manifolds with nonnegative curvature

by

Bo Yang

Doctor of Philosophy in Mathematics

University of California, San Diego, 2013

Professor Lei Ni, Chair
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In this dissertation we study two problems related to Ricci flow on complete noncompact manifolds. In Chapter 1 we report joint work with Ben Chow and Peng Lu on volume growth, lower bounds on scalar curvature, and upper bounds on Ricci curvature for complete gradient Ricci solitons. Most of our results are obtained without any assumption on the soliton metrics. We wish they could be useful for further study on complete gradient Ricci solitons under general assumptions. In Chapter 2 we obtain several results on complete Kähler manifolds with nonnegative curvature. The uniformization problem of such manifolds has been an important topic in complex geometry. We construct new examples of such metrics

and discuss their connection with Kähler-Ricci flow. This is partly joint work with Fangyang Zheng. We also explore the interaction between function theory and metric geometry on general Kähler manifolds with nonnegative bisectional curvature. This part is selected from my ongoing project and complete results will appear elsewhere.

Chapter 1

Complete noncompact gradient Ricci solitons

An important aspect of Riemannian geometry is to study the interaction between curvature and topology of Riemannian manifolds. Hamilton [56] introduced Ricci flow in 1982 to prove that every closed 3-manifold with positive Ricci curvature is diffeomorphic to a 3-spherical form. Ever since then Ricci flow has been proved to be a powerful tool to study geometry of manifolds. The most important application is Perelman's proof of the Poincaré Conjecture and the Geometrization Conjecture for 3-manifolds ([95], [96], and [97]). In higher dimensions, there is also some important progress on Ricci flow. Notably results include the theorem of Böhm and Wilking [6] that closed manifolds with positive curvature operators are space forms and the differentiable sphere theorem due to Brendle and Schoen [10].

In Chapter 1 we focus on complete gradient Ricci solitons which are self-similar solutions to Ricci flows on complete noncompact manifolds. There has been much progress on understanding geometric structures on gradient Ricci solitons with in the past years. In the following sections we will study volume growth and curvature behaviors of complete gradient Ricci solitons. These are joint work with B. Chow and P. Lu ([39], [41], and [41]).

The organization of Chapter 1 is as follows: In Section 1.1 we review the definition of gradient Ricci solitons and emphasize their importance in the study of Ricci flow. In Section 1.2 we discover a neat monotonicity formulation in the spirit of Bishop-Gromov volume comparison to prove that asymptotic volume ratio (AVR) of any shrinking soliton is well-defined, a byproduct is a necessary and sufficient condition on AVR $= 0$. Lower bounds on scalar curvature on shrinking solutions and steady solitons are studied in Section 1.3. In Section 1.4 we prove integral upper bounds on the Ricci curvature along modified geodesics on non-expanding solitons. At the end of Chapter 1 we discuss some more recent progress on classification of Ricci solitons and related questions.

1.1 Introduction

Let $(\mathcal{M}^n, g)$ be a Riemannian manifold and f a real-valued function on it, we call $(\mathcal{M}^n, g, f)$ a gradient Ricci soliton if

$$Ric(g) + \nabla\nabla f + \frac{\epsilon}{2} g = 0. \tag{1.1}$$

Here $\epsilon \in \mathbb{R}$ is constant. We say g is shrinking, steady, and expanding if $\epsilon < 0$, $\epsilon = 0$, and $\epsilon > 0$ respectively. And we also call the function f a *potential function* of gradient Ricci soliton. For simplicity we can normalize ϵ to be 1, 0, or -1. Throughout this dissertation we also call shrinking (steady, expanding) gradient Ricci solitons shrinkers (steadies, expanders) for short. It is well known (See [60] for example.) that $R+|\nabla f|^2+\epsilon f$ is constant on any gradient Ricci solitons. Unless specified otherwise we shall always assume that f is normalized in the sense that $R + |\nabla f|^2 + \epsilon f = 0$ holds on $\mathcal{M}$.

Complete gradient Ricci solitons are self-similar solutions to the Ricci flow. To be more precise, suppose $(\mathcal{M}^n, g_0, f)$ is a complete gradient Ricci soliton, then there exists s solution $g(t)$ to the Ricci flow with $g(t) = g_0$ and a family of diffeomorphism $\varphi(t)$ such that:

$$\begin{cases} \frac{\partial}{\partial t}\varphi(t)(x) = \frac{1}{1+\epsilon\, t}\nabla_{g_0} f(\varphi(t)(x)) \\ g(t) = (1+\epsilon\, t)\varphi(t)^* g_0 \\ \frac{\partial}{\partial t} g(t) = -2Ric(g(t)) \end{cases} \tag{1.2}$$

We remark that the above statement follows from a standard calculation (See [38] for example.) and a nice observation due to Zhang [121] that the completeness of g_0 implies the completeness of $\nabla_{g_0} f$.

If $(\mathcal{M}^n, g)$ is a Kähler manifold and f a real-valued function on it, we can define a Kähler-Ricci soliton structure by requiring the $(1,0)$ part of the complexified gradient vector field of the Ricci soliton metric to be a holomorphic vector field. Therefore $(\mathcal{M}^n, g, f)$ is called a Kähler-Ricci soliton if in the holomorphic coordinates

$$R_{i\bar{j}} + f_{i\bar{j}} + \epsilon g_{i\bar{j}} = 0 \text{ and } f_{ij} = f_{\bar{i}\bar{j}} = 0. \tag{1.3}$$

Historically, the concept of Ricci solitons was first introduced by Hamilton [57] when he studied Ricci flow on compact surfaces. Ever since then there has been a lot of progress on understanding geometric structure of Ricci solitons. By the work of Hamilton [59] and [60], Perelman [95], and others, gradient Ricci solitons are often expected to model finite time singularity formation of the Ricci flow. See [3] for a neckpinch forming a shrinker and [54] and [2] for a degenerate neckpinch forming a steady soliton, respectively.

In another viewpoint, Ricci solitons, as a natural generalization of Einstein metrics, is an important object for study in geometric analysis. One expect the methods to study Einstein metrics also works to study geometry and topology of Ricci solitons. It is also of interests to physicists as well and are called quasi-Einstein in physics literature (See [94] for example.)

It is also interesting to mention that Ricci solitons are related to Bakry-Émery Ricci tensor of metric measure space ([4]). Therefore the study of Ricci solitons should have applications in function theoretic, spectral, and other aspects of general metric measure spaces. We refer readers to [77], [78] and reference therein for some recent developments on this direction.

Due to their importance in the study of Ricci flow and related subjects, it is interesting to understand the geometry and topology of Ricci solitons and even look for a possible complete classification. The first step of this program is to construct more examples of gradient Ricci solitons. One typical method of constructing Ricci solitons is to impose suitable symmetry conditions on both the underlying manifolds and the soliton metrics and then reduce the soliton equation

(1.1) to ordinary differential equations. See [66], [13] for the compact case and [64], [13], [14], [94], [48], and [43] for the complete noncompact case. In the Kähler case, Wang and Zhu [106] constructed Kähler-Ricci solitons on toric Fano Kähler manifolds by solving equations of Monge-Ampère type.

There has been much recent work on gradient Ricci solitons including [47], [19], [20], [76], [77] and [78] to name only a few. Here we only list some notable classification results. Perelman classified 3-dimensional κ-noncollapsed shrinking solitons with bounded and nonnegative sectional curvature. Ni and Wallach [91] and Naber [80] improved by dropping κ-noncollapsing assumption and replaced nonnegative sectional curvature by nonnegative Ricci curvature. It follows from [91] and the localized Hamilton-Ivey estimates of [27] that 3-dimensional nonflat complete noncompact shrinking soliton must be a quotient of the round cylinder. In dimension 4, Naber [80] classified shrinking solitons with bounded and nonnegative curvature operator. There are also some classification results in high dimensions under the assumption of locally conformal flatness. See [18] and [34] for example.

We refer the readers to see the review paper [16] or the forthcoming book [37] for a more detailed account on recent progress on gradient Ricci solitons.

1.2 Asymptotic volume ratio of shrinkers

Recall that the asymptotic volume ratio (AVR) of a complete noncompact Riemannian manifold $(\mathcal{N}^n, h)$ is defined by

$$\mathrm{AVR}(h) \doteqdot \lim_{r\to\infty} \frac{\mathrm{Vol}\, B(p,r)}{\omega_n r^n} \tag{1.4}$$

if the limit exists, where $B(p,r)$ denotes the geodesic ball in $\mathcal{N}$ with center p and radius r and where ω_n is the volume of the unit Euclidean n-ball. It is easy to check that the AVR(h) is independent of the choice of p. Moreover, if h has nonnegative Ricci curvature, then this limit exists by the Bishop–Gromov volume comparison theorem.

In this section we study asymptotic volume ratio for complete noncompact shrinking gradient Ricci solitons. In particular we establish a monotonicity formula for volume of sublevel sets for potential functions on any Ricci shrinkers. The direct consequence is that AVR is well-defined on any shrinkers. We also give a necessary and sufficient condition for complete noncompact gradient Ricci shrinkers to have positive AVR.

It was proved by Chen [27] (see also [121]) that complete ancient solutions to the Ricci flow, and in particular shrinkers, must have nonnegative scalar curvature. As a consequence, the potential function f satisfies the estimate:

$$0 \leq f(x) \leq \left(\frac{1}{2}r(x) + f(O)^{\frac{1}{2}}\right)^2, \tag{1.5}$$

where $r(x)$ denotes the distance function to a fixed point O in $\mathcal{M}$. Cao and Zhou [19] proved that there exists a positive constant C which depends on the dimension n, $\sup_{y \in B(O,1)} |\nabla f|(y)$, and the minimum of the Ricci curvature Rc_g in the ball $B(O,1)$ such that f satisfies the lower estimate:

$$f(x) \geq \frac{1}{4}\left(r(x) - C\right)^2 \tag{1.6}$$

for $x \in \mathcal{M} - B\left(O, C\right)$ (see Fang, Man, and Zhang [47] for related estimates). In fact, carefully following the proof of [19] and integrating by parts yield:

$$f(x) \geq \frac{1}{4}\left[\left(r(x) - 4n - 2f\left(O\right)^{\frac{1}{2}} + \frac{4}{3}\right)_+\right]^2, \tag{1.7}$$

where $c_+ \doteqdot \max(c, 0)$. Recently Haslhofer and Müller [61] further observed that if the reference point O is chosen to be a global minimum point of f (its existence is ensured by (1.5) and (1.6)), then one obtains improved estimates with constants depending only on n:

$$\frac{1}{4}\left[(r(x) - 5n)_+\right]^2 \leqslant f(x) \leqslant \frac{1}{4}(r(x) + \sqrt{2n})^2. \tag{1.8}$$

Define the functions

$$\mathrm{V} : \mathbb{R} \to [0, \infty), \qquad \mathrm{R} : \mathbb{R} \to [0, \infty)$$

by

$$\mathrm{V}(c) \doteqdot \int_{\{f<c\}} d\mu, \qquad \mathrm{R}(c) \doteqdot \int_{\{f<c\}} R\, d\mu.$$

In [19], the following ODE relating $\mathrm{V}(c)$ and $\mathrm{R}(c)$ was established

$$0 \leq \frac{n}{2}\,\mathrm{V}(c) - \mathrm{R}(c) = c\,\mathrm{V}'(c) - \mathrm{R}'(c). \tag{1.9}$$

Cao and Zhou [19] proved the following using (1.9) and aided by an observation of Munteanu [75] (see [61] for an improvement).

Theorem 1.2.1. *Any complete noncompact shrinking gradient Ricci soliton must have at most Euclidean volume growth, i.e.,* $\limsup_{r\to\infty} \frac{\mathrm{Vol}\, B(O,r)}{\omega_n r^n}$ *is finite.*

Note that an earlier result by Carrillo and Ni [20] states that any nonflat shrinker with nonnegative Ricci curvature must have zero AVR. Based on Cao and Zhou's work, Zhang [120] proved a sharp upper bound on the volume growth of shrinkers under the assumption that $R \geq \delta$ for some constant $\delta > 0$. More recently, C.-W. Chen [33] proved that the AVR of a shrinker is bounded from below by some $c > 0$ if the average scalar curvature satisfies $\frac{1}{\mathrm{Vol}\, B(O,r)} \int_{B(O,r)} R\, d\mu \leq r^{\alpha}$, where α is a negative constant (see also [19] for a similar result in the case where $\alpha = 0$).

Theorem 1.2.2. *Let* $(\mathcal{M}^n, g, f)$ *be a complete noncompact shrinking gradient Ricci soliton. Then* $\mathrm{AVR}(g)$ *exists and is finite (by [61], it is bounded by a constant depending only on* n*). Moreover,* $\mathrm{AVR}(g) > 0$ *if and only if* $\int_{n+2}^{\infty} \frac{\mathrm{R}(c)}{c\,\mathrm{V}(c)} dc < \infty$.

Proof. Let $\mathrm{P}(c) \doteqdot \frac{\mathrm{V}(c)}{c^{\frac{n}{2}}} - \frac{\mathrm{R}(c)}{c^{\frac{n}{2}+1}}$ and $\mathrm{N}(c) \doteqdot \frac{\mathrm{R}(c)}{c\,\mathrm{V}(c)}$. Note that $\frac{\mathrm{R}(c)}{\mathrm{V}(c)}$ is the average scalar curvature over the set $\{f < c\}$. The ODE (1.9) implies

$$\mathrm{P}'(c) = -\left(1 - \frac{n+2}{2c}\right) \frac{\mathrm{R}(c)}{c^{\frac{n}{2}+1}} = -\frac{\left(1 - \frac{n+2}{2c}\right)\mathrm{N}(c)}{1 - \mathrm{N}(c)}\,\mathrm{P}(c). \tag{1.10}$$

Since $0 \leq \mathrm{R}(c) \leq \frac{n}{2}\,\mathrm{V}(c)$ by (1.9), we have

$$\left(1 - \frac{n}{2c}\right) \frac{\mathrm{V}(c)}{c^{\frac{n}{2}}} \leq \mathrm{P}(c) \leq \frac{\mathrm{V}(c)}{c^{\frac{n}{2}}}. \tag{1.11}$$

Hence, by the bounds (1.5) and (1.6) for f,

$$2^n \omega_n\,\mathrm{AVR}(g) = \lim_{c\to\infty} \frac{\mathrm{V}(c)}{c^{n/2}} = \lim_{c\to\infty} \mathrm{P}(c),$$

which exists by (1.10).

Integrating (1.10) yields

$$\mathrm{P}(c) = \mathrm{P}(n+2)\ e^{-\int_{n+2}^{c} \frac{\left(1-\frac{n+2}{2c}\right)\mathrm{N}(c)}{1-\mathrm{N}(c)}dc} \tag{1.12}$$

for $c \geq n+2$. From $\frac{\mathrm{R}(c)}{\mathrm{V}(c)} \leq \frac{n}{2}$ it is easy to see that for any $c \in [n+2, \infty)$ we have

$$\frac{1}{2}\int_{n+2}^{c} \mathrm{N}\,(c)\,dc \leq \int_{n+2}^{c} \left(1 - \frac{n+2}{2c}\right) \frac{\mathrm{N}\,(c)}{1-\mathrm{N}\,(c)} dc \leq 2\int_{n+2}^{c} \mathrm{N}\,(c)\,dc. \tag{1.13}$$

If $\int_{n+2}^{\infty} \mathrm{N}\,(c)\,dc = \infty$, then by (1.12) we have $\mathrm{AVR}(g) = \frac{1}{2^n \omega_n} \lim_{c\to\infty} \mathrm{P}(c) = 0$.

If $\int_{n+2}^{\infty} \mathrm{N}\,(c)\,dc < \infty$, then by (1.12) and (1.13), we have

$$\mathrm{P}(c) \geq \mathrm{P}(n+2)\ e^{-2\int_{n+2}^{\infty} \mathrm{N}(c)dc} > 0.$$

Hence $\mathrm{AVR}(g) > 0$. □

Remark 1.2.3. *It follows from the proof of Theorem 1.2.2 that we have a precise formula for* AVR *in terms of scalar curvature. Note that* AVR *is bounded from above by a constant which only depends on n, The natural question is whether it is actually bounded by* AVR *of the standard Euclidean space. It is also interesting to know more about the behavior of* AVR *among all shrinkers. One may compare a result of Yokota [119] which says there is a gap for* AVR *if we consider complete ancient solutions to the Ricci flow with bounded nonnegative Ricci curvature.*

1.3 Lower bounds on the scalar curvature

In this section we focus on lower bounds on the scalar curvature of shrinking solution and steady solitons. To more precise, we prove that recent work of Ni and Wilking [92] yields the sharp result that a noncompact nonflat Ricci shrinker has at most quadratic scalar curvature decay. We also prove a similar result for certain noncompact steady gradient Ricci solitons.

It was proved by Chen [27] that $R \geq 0$ for Ricci shrinkers. If a Ricci shrinker is not isometric to Euclidean space, then $R > 0$ (see Pigola, Rimoldi, and Setti [99] and Zhang [120]). Recently, Ni and Wilking [92] proved that on any noncompact nonflat Ricci shrinker and for any $\delta > 0$, there exists a constant $C_\delta > 0$ such that $R(x) \geqslant C_\delta d(x,O)^{-2-\delta}$ wherever $d(x,O)$ is sufficiently large. The purpose of this note is to observe the following version of their result and a similar result for certain noncompact steady gradient Ricci solitons.

Theorem 1.3.1. *Let $(\mathcal{M}^n, g, f)$ be a complete noncompact nonflat gradient shrinking soliton with the potential function f normalized in the sense that $R+|\nabla f|^2 - f = 0$. Then for any given point $O \in \mathcal{M}$ there exists a constant $C_0 > 0$ such that $R(x)d(x,O)^2 \geqslant C_0^{-1}$ wherever $d(x,O) \geqslant C_0$. Consequently, the asymptotic scalar curvature ratio of g is positive.*

Proof. Recall the estimate on the potential function on any complete shrinker due to Cao and Zhou mentioned in Section 1.2. Namely there exists a positive constant C_1 such that f satisfies the estimate:

$$\frac{1}{4}\left[(d(x,O)-C_1)_+\right]^2 \leq f(x) \leq \frac{1}{4}\left(d(x,O)+2f(O)^{\frac{1}{2}}\right)^2, \tag{1.14}$$

where $c_+ \doteqdot \max(c,0)$. Define the f-Laplacian $\Delta_f \doteqdot \Delta - \nabla f \cdot \nabla$. We have $0 < R + |\nabla f|^2 = f = \frac{n}{2} - \Delta_f f$. Recall that (see [45] for example)

$$\Delta_f R = -2\left|\mathrm{Rc}\right|^2 + R. \tag{1.15}$$

Note that

$$\Delta_f\left(f^{-1}\right) = f^{-1} - f^{-2}\left(\frac{n}{2} - 2\frac{|\nabla f|^2}{f}\right), \tag{1.16}$$

$$\Delta_f\left(f^{-2}\right) = 2f^{-2} - f^{-3}\left(n - 6\frac{|\nabla f|^2}{f}\right). \tag{1.17}$$

Using (1.15) and (1.16), we compute for any $c > 0$

$$\Delta_f\left(R - cf^{-1}\right) \leqslant R - cf^{-1} + cf^{-2}\left(\frac{n}{2} - 2\frac{|\nabla f|^2}{f}\right). \tag{1.18}$$

Define $\phi \doteqdot R - cf^{-1} - cnf^{-2}$. By (1.17) we obtain

$$\Delta_f \phi \leqslant \phi - cnf^{-3}\left(\frac{f}{2} - n\right) - cf^{-4}\left(2f + 6n\right)|\nabla f|^2. \tag{1.19}$$

Choosing $c > 0$ sufficiently small, we have $\phi > 0$ inside $B(O, C_1 + 3n)$, where C_1 is as in (1.6). If $\inf_{\mathcal{M}-B(O,C_1+3n)} \phi \doteqdot -\delta < 0$, then by (1.6) there exists

$\rho > C_1 + 3n$ such that $\phi > -\frac{\delta}{2}$ in $\mathcal{M} - B(O,\rho)$. Thus a negative minimum of ϕ is attained at some point x_0 outside of $B(O, C_1 + 3n)$. By the maximum principle, evaluating (1.19) at x_0 yields $\frac{f(x_0)}{2} - n \leq 0$. However, (1.6) implies that $f(x_0) \geqslant \frac{9n^2}{4}$, a contradiction. We conclude that $R \geq cf^{-1} + cnf^{-2}$ on $\mathcal{M}$. The theorem follows from (1.14). □

Remark 1.3.2. *Feldman, Ilmanen, and Knopf [48] constructed complete noncompact Kähler–Ricci shrinkers on the total spaces of k-th powers of tautological line bundles over the complex projective space $\mathbb{CP}^{n-1}$ for $0 < k < n$. These examples, which have Euclidean volume growth and quadratic scalar curvature decay, show that Theorem 1 is sharp.*

By a similar argument we prove the following result regarding steady gradient Ricci solitons. See [7], [18], [55], [60], and [76] for some earlier works on the qualitative aspects of steady Ricci solitons.

Theorem 1.3.3. *Let $(\mathcal{M}^n, g, f, 0)$ be a complete steady gradient Ricci soliton with $R + |\nabla f|^2 = 1$. If $\lim_{x\to\infty} f(x) = -\infty$ and $f \leq 0$, then $R \geq \frac{1}{\sqrt{\frac{n}{2}+2}} e^f$.*

Proof. Note that on steady gradient Ricci solitons we have $\Delta_f f = -1$, $\Delta_f R = -2|\mathrm{Rc}|^2 \leq -\frac{2}{n}R^2$, and $\Delta_f(e^f) = -R\,e^f$. For $c \in \mathbb{R}$,

$$\Delta_f\left(R - ce^f\right) \leq -\frac{2}{n}R^2 + cR\,e^f \leq \frac{nc^2}{8}e^{2f}.$$

Using $\Delta_f(e^{2f}) = 2e^{2f}(1 - 2R)$, we compute for $b \in \mathbb{R}$ that

$$\Delta_f\left(R - ce^f - be^{2f}\right) \leq \left(\frac{nc^2}{8} - 2b + 4bR\right)e^{2f}. \tag{1.20}$$

Suppose $R - ce^f - be^{2f}$ is negative somewhere. Then, since $R \geq 0$ by [27] and $\lim_{x\to\infty} e^{f(x)} = 0$ by hypothesis, a negative minimum of $R - ce^f - be^{2f}$ is attained at some point. By (1.20) and the maximum principle, at such a point we have

$$0 \leq \frac{nc^2}{8} - 2b + 4bR < \frac{nc^2}{8} - 2b + 4b(c+b)$$

since $f \leq 0$. Given $c \in (0, \frac{1}{2}]$, the minimizing choice $b = \frac{1-2c}{4}$ yields $\frac{(1-2c)^2}{4} < \frac{nc^2}{8}$. We obtain a contradiction by choosing $c = \frac{1}{\sqrt{\frac{n}{2}+2}}$. □

Remark 1.3.4. *Given a steady Ricci soliton* $(\mathcal{M}^n, g, f, 0)$ *with* $R+|\nabla f|^2 = 1$ *and* $O \in \mathcal{M}$*, since* $|\nabla f| \leq 1$*, we have* $f(x) \geq f(O) - d(x, O)$ *on* $\mathcal{M}$*. For the cigar soliton* $(\mathbb{R}^2, \frac{4(dx^2+dy^2)}{1+x^2+y^2})$ *we have* $R = e^f$ *assuming* $\max_{x\in\mathbb{R}^2} f(x) = 0$*. See [110] and [76] for an estimate for the potential functions of steady gradient Ricci solitons.*

Remark 1.3.5. *We notice that Fernández-Lopez and García-Río [51] studied lowers bounds on scalar curvatures on certain steady solitons. Their results, without assuming that the potential function decays uniformly at infinity, works on steadies with nonnegative curvature.*

1.4 Integral upper bounds on the Ricci curvature along modified geodesics on non-expanding solitons

Following previous discussions on lower bounds on Ricci solitons, we now turn to the upper bounds of complete shrinking and steady solitons.

Establishing upper bounds for the curvatures of gradient Ricci solitons is basic to controlling their geometry and topology. One example of this is that a better than quadratic in distance upper bound for the scalar curvature of a shrinker suffices to imply that the shrinker is of finite topological type (see [47] and [19]). Estimates exist for the scalar curvature (uniform in the case of steady solitons and quadratic in the case of shrinking solitons) but are not as strong for the more informative Ricci or Riemann curvatures.

It is very likely that gradient Ricci shrinking solitons necessarily have bounded Riemmanian curvature tensor. For example, It was proved in [79] that any complete Ricci shrinker with bounded Ricci curvature must have Riemann curvature with polynomial growth. In general, an optimistic conjecture in [37] states that any singularity model (i.e. the complete ancient solution which arises as a limit of blow up of finite singular solutions) must have bounded curvature. A solution to such a problem, either positive or negative, will greatly improve our understanding on geometry of gradient Ricci solitons.

To be more precise, in this section we define an energy functional $\mathcal{J}$ associated to a smooth function ϕ on a complete Riemannian manifold. This energy function is motivated by Li and Yau [71] and similar to Perelman [95]. As an application, we deduce integral Ricci curvature upper bounds along modified geodesics for complete steady and shrinking gradient Ricci solitons. Because of the ubiquity of such geodesics, this yields significant evidence for desired pointwise estimates for the Ricci curvatures.

In more detail, the novelty of this paper is to use a standard second variation argument to obtain a geodesic integral bound for the f-Laplacian of the scalar curvature. In essence, the Hessian of R is 'created', traced, and then converted to $|\mathrm{Rc}|^2$ by the Ricci soliton equation (see (1.21) and (1.22) below). Since locally, modified geodesics are near Riemannian geodesics, there is hope of strengthening the estimates in this paper.

1.4.1 The functional $\mathcal{J}$ and its first and second variations

Throughout this subsection $(\mathcal{M}^n, g)$ is a complete Riemannian manifold and $\phi\colon \mathcal{M} \to \mathbb{R}$ be a smooth function with a uniform lower bound. Below, all objects on $\mathcal{M}$ are assumed to be C^∞ unless otherwise specified. For a path $\gamma\colon [0, \bar{s}] \to \mathcal{M}$, where $\bar{s} > 0$, define the functional (see Li and Yau [71])

$$\mathcal{J}(\gamma) = \int_0^{\bar{s}} \left(|\gamma'(s)|^2 + 2\phi(\gamma(s)) \right) ds.$$

Since ϕ has a uniform lower bound, for any $x, y \in \mathcal{M}$ and $\bar{s} > 0$, among all paths joining x to y, there exists a minimizer of $\mathcal{J}$.

When $\phi = \frac{R}{2}$, where R denotes the scalar curvature, $\mathcal{J}$ is almost the *steady* version $\mathcal{L}_0$ of the $\mathcal{L}$-length, defined by $\mathcal{L}_0(\gamma) = \int_0^{\bar{\tau}} \left(|\gamma'(\tau)|^2_{g(\tau)} + R_{g(\tau)}(\gamma(\tau)) \right) d\tau$, where $g(\tau)$ is a solution to the backward Ricci flow. Note that Perelman's original $\mathcal{L}$-length [95] is commonly viewed as the *shrinker* version and is most relevant to finite time singularity formation (see also [49] for the *expander* version). For a steady soliton there is a subtle difference between $\mathcal{J}$ and $\mathcal{L}_0$, due to the fact that $\mathcal{J}$ is for the static form and $\mathcal{L}_0$ is for the dynamic form. This difference, whose effect is not just a diffeomorphism change, is essential for our purposes and manifests itself in that for the second variation of $\mathcal{J}$ along a minimizer, the main term is the Hessian of R, whereas for $\mathcal{L}_0$ the main term is Hamilton's matrix Harnack expression without the $\frac{\mathrm{Rc}}{\tau}$ term.

Given a family of paths $\gamma_u : [0,\bar{s}] \to \mathcal{M}$, $u \in (-\varepsilon,\varepsilon)$, define $\Gamma : [0,\bar{s}] \times (-\varepsilon,\varepsilon) \to \mathcal{M}$ by $\Gamma(s,u) = \gamma_u(s)$ and define $S = \Gamma_*(\partial/\partial s)$ and $U = \Gamma_*(\partial/\partial u)$. The well-known first variation formula is

$$\frac{1}{2}\frac{d}{du}\Big|_{u=0} \mathcal{J}(\gamma_u) = \int_0^{\bar{s}} \left(\langle \nabla_S U, S\rangle + U(\phi)\right) ds = \int_0^{\bar{s}} \langle U, -\nabla_S S + \nabla\phi\rangle\, ds + \langle U, S\rangle|_0^{\bar{s}}.$$

Hence the critical points γ of $\mathcal{J}$ on paths with fixed endpoints, called ϕ-geodesics, are the paths which satisfy $\nabla_S S = \nabla\phi$. Then $S|S|^2 = 2\langle \nabla_S S, S\rangle = 2S(\phi)$ implies $|S|^2 - 2\phi = C = \text{const}$ on γ.

Let Rm denote the Riemann curvature tensor. The standard second variation formula is

$$\begin{aligned}\frac{1}{2}\frac{d^2}{du^2}\Big|_{u=0} \mathcal{J}(\gamma_u) = &\int_0^{\bar{s}} \left(|\nabla_S U|^2 - \langle \mathrm{Rm}(S,U)U,S\rangle + \nabla\nabla\phi(U,U)\right) ds \\ &- \int_0^{\bar{s}} \langle \nabla_U U, \nabla_S S - \nabla\phi\rangle\, ds + \langle \nabla_U U, S\rangle|_0^{\bar{s}},\end{aligned}$$

where we used $\nabla\nabla\phi(U,U) = U(U(\phi)) - (\nabla_U U)(\phi)$. So when γ_0 is a ϕ-geodesic,

$$\frac{1}{2}\frac{d^2}{du^2}\Big|_{u=0} \mathcal{J}(\gamma_u) - \langle \nabla_U U, S\rangle|_0^{\bar{s}} = \int_0^{\bar{s}} \left(|\nabla_S U|^2 - \langle \mathrm{Rm}(S,U)U,S\rangle + \nabla\nabla\phi(U,U)\right) ds.$$

Now assume $\zeta : [0,\bar{s}] \to \mathbb{R}$ is a piecewise smooth function which vanishes at the endpoints, let $\{e_i\}_{i=1}^n$ be a parallel orthonormal frame along γ_0, take $U = \zeta e_i$, and sum over i and each smooth partition of U. Note that $\nabla_U U$ is continuous along γ_0. If $\gamma = \gamma_0$ is ϕ-minimal, then

$$0 \le \frac{1}{2}\sum_{i=1}^n \frac{d^2}{du_i^2}\Big|_{u_i=0} \mathcal{J}(\gamma_{u_i}) = \int_0^{\bar{s}} \left(n(\zeta')^2 + \zeta^2\Delta\phi - \zeta^2\,\mathrm{Rc}(S,S)\right) ds,$$

where Rc is the Ricci curvature tensor.

Let $f : \mathcal{M} \to \mathbb{R}$. By $\nabla\nabla f(S,S) = S(S(f)) - \langle \nabla_S S, \nabla f\rangle$, and $\nabla_S S = \nabla\phi$, we have

$$\int_0^{\bar{s}} \zeta^2\,\mathrm{Rc}(S,S)\, ds = \int_0^{\bar{s}} \zeta^2\,\mathrm{Rc}_f(S,S)\, ds + 2\int_0^{\bar{s}} \zeta\zeta'\langle \nabla f, S\rangle\, ds + \int_0^{\bar{s}} \zeta^2\langle \nabla\phi, \nabla f\rangle\, ds,$$

where $\mathrm{Rc}_f = \mathrm{Rc} + \nabla\nabla f$ and we integrated by parts. Let $\Delta_f = \Delta - \nabla f \cdot \nabla$ be the f-Laplacian. Then

$$-\int_0^{\bar{s}} \zeta^2 \Delta_f \phi ds + \int_0^{\bar{s}} \zeta^2 \mathrm{Rc}_f (S, S)\, ds \leq \int_0^{\bar{s}} \left(n \left(\zeta'\right)^2 - 2\zeta\zeta' \left\langle \nabla f, S \right\rangle \right) ds. \quad (1.21)$$

In the following subsection we shall apply (1.21) to study gradient Ricci solitons.

1.4.2 Application to steady gradient Ricci solitons

We prove the geodesic integral estimate for the Ricci curvature of a steady soliton In this subsection. (See (1.24) below.)

Theorem 1.4.1. *Let $(\mathcal{M}^n, g, f)$ be a nonflat complete steady Ricci soliton:* $\mathrm{Rc}_f = 0$*, with f normalized in the sense that $R + |\nabla f|^2 = 1$. If $x, y \in \mathcal{M}$ are two points with $d(x,y) > 4$, then there exists a point z with $d(y,z) \leq \frac{\sqrt{2}}{2} d(x,y)$ such that $|\mathrm{Rc}|(z) \leq \frac{C_1}{\sqrt{d(x,y)}}$ for some constant C_1 only depending on n. In particular, there exists a sequence of points z_i tending to infinity such that $|\mathrm{Rc}(z_i)| \leq C_1 (d(x, z_i))^{-1/2}$.*

Similar estimates hold for ∇R since $|\nabla R| = 2 \left|\mathrm{Rc}\left(\nabla f\right)\right| \leq 2 \left|\mathrm{Rc}\right|$.

Proof. It follows from [27] that $R > 0$. Let $c > 0$ and $2\phi = cR$.[1] Since $-\Delta_f R = 2\left|\mathrm{Rc}\right|^2$, $|\nabla f| \leq 1$, and $|S| = \sqrt{C + cR} \leq \sqrt{C + c}$, on a minimal $\frac{1}{2}cR$-geodesic we have

$$\int_0^{\bar{s}} \zeta^2 \left|\mathrm{Rc}\right|^2 ds \leq \frac{n}{c} \int_0^{\bar{s}} \left(\zeta'\right)^2 ds + \frac{2\sqrt{C + c}}{c} \int_0^{\bar{s}} \left|\zeta\zeta'\right| ds. \quad (1.22)$$

Given $x, y \in \mathcal{M}$ with $\bar{s} \doteqdot d\left(x, y\right) > 4$, let $\gamma : [0, \bar{s}] \to \mathcal{M}$ be a minimal $\frac{1}{2}cR$-geodesic from x to y, and let $\bar{\gamma} : [0, \bar{s}] \to \mathcal{M}$ be a minimal geodesic from x and y. Then

$$C\bar{s} \leq \int_0^{\bar{s}} \left(\left|\gamma'\left(s\right)\right|^2 + cR(\gamma(s)) \right) ds \leq \int_0^{\bar{s}} \left(\left|\bar{\gamma}'\left(s\right)\right|^2 + cR(\bar{\gamma}(s)) \right) ds \leq (1 + c)\bar{s}, \quad (1.23)$$

Hence $C \leq 1 + c$.

---

[1] Note that the integral curves to ∇f are $-\frac{R}{2}$-geodesics.

If $\zeta(s) = s$ for $s \in [0,1]$, $\zeta(s) = 1$ for $s \in [1, \bar{s} - 1]$, and $\zeta(s) = \bar{s} - s$ for $s \in [\bar{s} - 1, \bar{s}]$, then by (1.22),

$$\int_0^{\bar{s}} \zeta(s)^2 |\mathrm{Rc}|^2 (\gamma(s))\, ds \leq \frac{2n + 2\sqrt{1+2c}}{c}. \tag{1.24}$$

Note that for any $s_0 \in [\bar{s}/2, \bar{s}]$

$$d(\gamma(s_0), y) \leq \int_{s_0}^{\bar{s}} \sqrt{C + cR(\gamma(s))} ds \leq \frac{\bar{s}}{2}\sqrt{1+2c}. \tag{1.25}$$

Pick a $c \in (0, \frac{1}{2}]$; then $\frac{\bar{s}}{2}\sqrt{1+2c} \leq \frac{\sqrt{2}}{2}\bar{s}$. By (1.24) we have

$$\inf_{s \in [\bar{s}/2, \bar{s}-1]} |\mathrm{Rc}|^2 (\gamma(s)) \leq \frac{1}{\frac{\bar{s}}{2} - 1} \int_{\bar{s}/2}^{\bar{s}-1} |\mathrm{Rc}|^2 (\gamma(s))\, ds < \frac{4(2n + 2\sqrt{1+2c})}{\bar{s}c}.$$

This implies that there is a point z with $d(z,y) \leq \frac{\sqrt{2}}{2} d(x,y)$ such that $|\mathrm{Rc}|(z) < \frac{C_1}{\sqrt{d(x,y)}} \leq \frac{C_1(1+\frac{\sqrt{2}}{2})^{1/2}}{\sqrt{d(x,z)}}$, where $C_1 \doteqdot \left(\frac{8n+8\sqrt{1+2c}}{c}\right)^{1/2}$. □

As a corollary, $\liminf_{z\to\infty} |\mathrm{Rc}|(z) = 0$. This is a result of Fernández-Lopez and García-Río [50]. By different methods, Munteanu and Sesum [76] and Wu [110] obtained $\liminf_{z\to\infty} R(z) = 0$.

1.4.3 Application to noncompact shrinking gradient Ricci solitons

In this subsection we prove for shrinkers a Ricci curvature estimate analogous to that in the previous discussion. (See (1.29) below.)

Theorem 1.4.2. *Let $(\mathcal{M}^n, g, f)$ be a complete noncompact shrinking gradient Ricci soliton:* $\mathrm{Rc}_f = \frac{1}{2}g$, *with f normalized so that $R + |\nabla f|^2 - f = 0$. Let O be a point such that $f(O) = \min_{\mathcal{M}} f \leq \frac{n}{2}$. Then there exists a constant C_2 which only depends on n and $f(O)$ such that for any $y \in \mathcal{M}$ with $d(O,y) > 4$, there exists a point $z \in \mathcal{M}$ with $d(z,y) \leq \frac{\sqrt{2}}{2} d(O,y)$ such that $|\mathrm{Rc}|(z) \leq C_2(d(O,y)+1)$. In particular there exists a sequence of points z_i tending to infinity such that $|\mathrm{Rc}|(z_i) \leq C_2(d(O,z_i)+1)$.*

Consequently $|\nabla R|(z_i) \leq C_2(d(O,z_i)+1)^2$ holds since $|\nabla R| = 2|\mathrm{Rc}(\nabla f)|$ and $|\nabla f|(z) \leq \frac{d(O,z)}{2} + f(O)^{1/2}$.

Proof. We only need to consider the case $R > 0$ (see [99] and [120]). Let $c > 0$ and $2\phi = c\frac{R}{f}$. From $\Delta_f R = -2\left|\mathrm{Rc}\right|^2 + R$, $\Delta_f f = \frac{n}{2} - f$, and $\nabla R = 2\,\mathrm{Rc}\left(\nabla f\right)$, we compute

$$\begin{aligned}\Delta_f \frac{R}{f} &= \frac{R}{f^2}(2f - \frac{n}{2}) - 2\frac{\left|\mathrm{Rc}\right|^2}{f} - 4\frac{\mathrm{Rc}(\nabla f, \nabla f)}{f^2} + 2\frac{R|\nabla f|^2}{f^3} \\ &\leq -\frac{|\,\mathrm{Rc}\,|^2}{f} + 4\frac{(1+\sqrt{n})^2}{f},\end{aligned} \tag{1.26}$$

where we have used $\frac{2R}{f} - \frac{|\mathrm{Rc}|^2}{2f} \leq \frac{2n}{f}$ and $-\frac{|\mathrm{Rc}|^2}{2f} - 4\frac{\mathrm{Rc}(\nabla f, \nabla f)}{f^2} \leq 8\frac{|\nabla f|^4}{f^3}$. Hence it follows from (1.21) that

$$\begin{aligned}&\frac{c}{2}\int_0^{\bar{s}} \zeta^2 \left(\frac{|\,\mathrm{Rc}\,|^2}{f} - 4\frac{(1+\sqrt{n})^2}{f}\right) ds + \frac{1}{2}\int_0^{\bar{s}} \zeta^2 \left|S\right|^2 ds \\ &\leq 2n - \int_0^{\bar{s}} 2\zeta\zeta' \left\langle \nabla f, S\right\rangle ds.\end{aligned} \tag{1.27}$$

Given $y \in \mathcal{M}$ with $\bar{s} \doteqdot d\left(O, y\right) > 4$, let $\gamma : [0, \bar{s}] \to \mathcal{M}$ be a minimal $\frac{1}{2}c\frac{R}{f}$-geodesic from O to y. Note that $\frac{R}{f} \leq 1$. By a similar argument as in (1.23) we get $C \leq 1 + c$.

Let $r = d\left(\cdot, O\right)$. Then $\left|\nabla f\right|(z) \leq \sqrt{f\left(z\right)} \leq \sqrt{\frac{n}{2}} + r\left(z\right)$. We have $|S| \leq \sqrt{C+c}$ and $r\left(\gamma\left(s\right)\right) \leq \min\{s\sqrt{C+c}, r\left(y\right) + \left(\bar{s} - s\right)\sqrt{C+c}\}$. Define $\zeta(s) = s$ for $s \in [0,1]$, $\zeta(s) = 1$ for $s \in [1, \bar{s}-1]$, and $\zeta(s) = \bar{s} - s$ for $s \in [\bar{s}-1, \bar{s}]$. Then

$$\begin{aligned}-\int_0^{\bar{s}} \zeta\zeta' \left\langle \nabla f, S\right\rangle ds &\leq \int_0^1 s\sqrt{f\left(\gamma\left(s\right)\right)}\left|S\left(s\right)\right| ds + \int_{\bar{s}-1}^{\bar{s}} \left(\bar{s}-s\right)\sqrt{f\left(\gamma\left(s\right)\right)}\left|S\left(s\right)\right| ds \\ &\leq \tfrac{1}{2}\sqrt{C+c}\left(\sqrt{2n} + r\left(y\right) + 2\sqrt{C+c}\right).\end{aligned} \tag{1.28}$$

Let $A = \sqrt{C+c}$; then $A \leq \sqrt{1+2c}$. Since $f(\gamma(s)) \geq f(O)$ and $\bar{s} = d\left(O, y\right)$, from (1.27) and (1.28) we have

$$\int_0^{\bar{s}} \frac{\zeta^2 |\,\mathrm{Rc}\,|^2}{f} ds \leq \frac{4(1+\sqrt{n})^2 r\left(y\right)}{f\left(O\right)} + \frac{4(\sqrt{n}+A)^2}{c} + \frac{2Ar\left(y\right)}{c}. \tag{1.29}$$

Let $c \in (0, \frac{1}{2})$. By an argument similar to (1.25) we have $d\left(\gamma\left(s_0\right), y\right) \leq \frac{\sqrt{2}}{2} r\left(y\right)$ for $s_0 \in [\frac{\bar{s}}{2}, \bar{s}]$. Thus

$$\frac{(\frac{r(y)}{2} - 1)\min_{s\in[\frac{\bar{s}}{2}, \bar{s}-1]} |\,\mathrm{Rc}\,|^2\left(\gamma\left(s\right)\right)}{\left(\sqrt{\frac{n}{2}} + Ar\left(y\right)\right)^2} \leq \int_{\frac{1}{2}\bar{s}}^{\bar{s}-1} \frac{|\,\mathrm{Rc}\,|^2\left(\gamma\left(s\right)\right)}{f\left(\gamma\left(s\right)\right)} ds \leq \mathrm{Const}\left(r\left(y\right) + 1\right),$$

where we have used $f(\gamma(s)) \leq \sqrt{\frac{n}{2}} + r(\gamma(s)) \leq \sqrt{\frac{n}{2}} + A\bar{s}$. Therefore there exists $C_2 < \infty$, only depending on n and $f(O)$, such that for any $y \in \mathcal{M}$ with $r(y) > 4$, there exists a point $z \in \mathcal{M}$ with $d(z, y) \leq \frac{\sqrt{2} r(y)}{2}$ such that $|\operatorname{Rc}|(z) \leq C_2 (r(y) + 1) \leq C_2 \left((1 - \frac{\sqrt{2}}{2})^{-1} r(z) + 1 \right)$. □

Finally, we make a remark about $\inf_\gamma \mathcal{J}(\gamma)$. Define the function $\rho(x, y, \bar{s}) \doteqdot \inf_\gamma \mathcal{J}(\gamma)$, where the infimum is over $\gamma : [0, \bar{s}] \to \mathcal{M}$ from x to y. Let $\gamma \doteqdot \gamma_0$ be a minimal ϕ-geodesic. Since $\nabla_S S = \nabla \phi$, we have $\frac{1}{2} \frac{d}{du}\big|_{u=0} \mathcal{J}(\gamma_u) = \langle U, S \rangle |_0^{\bar{s}}$. Thus $\nabla \rho(x, y, \bar{s}) = 2S(\bar{s}) = 2\gamma'(\bar{s})$. By Lemma 3.1 of [71], we have $\frac{\partial \rho}{\partial \bar{s}} + \frac{1}{4} |\nabla \rho|^2 = 2\phi$ in the weak sense. Since ρ is similar to Perelman's reduced distance, we now derive a heat-type inequality satisfied by a natural quantity expressed in terms of ρ.

Let $\{e_i\}_{i=1}^n$ be a parallel orthonormal frame along $\gamma \doteqdot \gamma_0$. Summing $U = \frac{s}{\bar{s}} e_i$ over i in the second variation formula, we obtain

$$\frac{1}{2} \Delta_y \rho \leq \frac{n}{\bar{s}} + \int_0^{\bar{s}} \left(-\frac{s^2}{\bar{s}^2} \operatorname{Rc}(S, S) + \frac{s^2}{\bar{s}^2} \langle \nabla f, \nabla \phi \rangle + \frac{s^2}{\bar{s}^2} \Delta_f \phi \right) ds.$$

Since

$$\begin{aligned} -\int_0^{\bar{s}} \frac{s^2}{\bar{s}^2} \operatorname{Rc}(S, S) ds &= \int_0^{\bar{s}} \frac{s^2}{\bar{s}^2} S(S(f)) \, ds - \int_0^{\bar{s}} \frac{s^2}{\bar{s}^2} \langle \nabla_S S, \nabla f \rangle \, ds \\ &= \langle S, \nabla f \rangle (\bar{s}) - \frac{2}{\bar{s}} f(y) + \frac{2}{\bar{s}^2} \int_0^{\bar{s}} f(\gamma(s)) ds - \int_0^{\bar{s}} \frac{s^2}{\bar{s}^2} \langle \nabla \phi, \nabla f \rangle ds, \end{aligned}$$

we have

$$\frac{1}{2} \Delta_y \rho(x, y, \bar{s}) \leq \frac{n}{\bar{s}} + \int_0^{\bar{s}} \frac{s^2}{\bar{s}^2} \Delta_f \phi ds + \langle S, \nabla f \rangle (\bar{s}) + \frac{2}{\bar{s}^2} \int_0^{\bar{s}} (f(\gamma(s)) - f(y)) \, ds.$$

Thus $\Phi = \bar{s}^{-\frac{n}{2}} e^{-\frac{\rho}{4}}$ satisfies in the weak sense (note $\langle S, \nabla f \rangle (\bar{s}) = \frac{1}{2} \langle \nabla \rho, \nabla f \rangle$)

$$\left(\frac{\partial}{\partial \bar{s}} - (\Delta_y)_f + \frac{\phi}{2} \right) \Phi \leq \frac{\Phi}{2} \left(\int_0^{\bar{s}} \frac{s^2}{\bar{s}^2} \Delta_f \phi ds + \frac{2}{\bar{s}^2} \int_0^{\bar{s}} (f(\gamma(s)) - f(y)) \, ds \right).$$

1.5 Further discussions

After our work discussed in previous sections, we notice there has been some important further progress on the study of gradient Ricci solitons. In particular, Brendle [8] proved that any three-dimensional steady gradient Ricci soliton which is non-flat and κ-noncollapsed is isometric to the Bryant soliton up to scaling. This confirms a claim made by Perelman [95]. In another paper [9] he also proved a similar result in higher dimensions with additional assumption on the asymptotic behaviors of soliton metrics along infinity.

It was proved by Munteanu and Wang [78] that any shrinking solitons must be of at least linear volume growth. The result, sharp on cylinders, could be viewed a natural generalization of the result of Calabi and Yau on volume growth of complete manifolds with nonnegative Ricci curvature.

We also note that a complete classification result in dimension 4 and higher is still open. The result of Naber [80] mentioned in the introduction assumes bounded and nonnegative curvature operator. For Kähler-Ricci solitons, the first question is to understand their holomorphic structures. In this direction, Chau-Tam [21] proved that any complete gradient Kähler-Ricci soliton with nonnegative Ricci curvature is biholormophic to $\mathbb{C}^n$ if it is steady and the scalar curvature attains the its maximum at some point or if it is expanding, The steady case was also proved by Bryant [11] independently. In [112] we generalize their result to the case of nonnegative Ricci curvature.

Theorem 1.5.1 ([112])**.** *Let $(\mathcal{M}^n, g, f)$ be a complete noncompact steady gradient Kähler-Ricci soliton with non-negative Ricci curvature. Assume that its scalar curvature attains a positive maximum along a compact complex submanifold $\mathcal{K}$ with codimension 1 and the Ricci curvature is positive away from $\mathcal{K}$. Then $\mathcal{M}^n$ is biholomorphic to a holomorphic line bundle over $\mathcal{K}$.*

Examples of steady solitons satisfying the assumption in Theorem 1.5.1 include the ones on canonical line bundles over Fano Kähler-Einstein manifolds. Note that these new examples are not locally conformally flat in general, and steady ones can have nonnegative Ricci curvature.

In view of Theorem 1.5.1, it is possible to study complex structure of Kähler-Ricci solitons in complex dimension 2. In particular, let $(\mathcal{M}^2, g, f)$ denote a complete noncompact Kähler-Ricci soliton and ∇f is the associated holomorphic vector field, and let $\mathcal{Z}_f$ denote the zero locus of ∇f. It is known ([11]) that $\mathcal{Z}_f$ is a disjoint union of nonsingular complex totally geodesic submanifolds. Can we get the information on complex structures on $\mathcal{M}^2$ by studying $\mathcal{Z}_f$? We plan to study Kähler-Ricci solitons in complex dimension 2 along this direction in the future.

Main results in Chapter 1, including Theorem 1.2.1, Theorem 1.3.1, Theorem 1.4.1, and Theorem 1.4.2, are joint work with Ben Chow and Peng Lu. They were published in Comptes Rendus Mathématique in 2011, Proceedings of the American Mathematical Society in 2012, and Annals of Global Analysis and Geometry in 2012.

Chapter 2

Complete noncompact Kähler manifolds with nonnegative holomorphic bisectional curvature

One of the central problems in complex geometry is to generalize the classical uniformization theorem for Riemann surfaces to complex manifolds in higher dimensions. A natural program is to study Kähler manifolds under suitable curvature assumptions. For example, the Frankel conjecture and the generalized Frankel conjecture address the structure of compact Kähler manifolds with positive or nonnegative holomorphic bisectional curvature. In the noncompact case a similar question asks if any complete noncompact Kähler manifolds with positive bisectional curvature is biholomorphic to complex Euclidean space. In the negative curvature case it is believed that the universal cover of any compact Kähler manifold with negative sectional curvature is biholomorphic to a bounded domain in Euclidean space. Those problems provided much of the driving force for the development of complex differential geometry and several complex variables and have been subjects of intensive research in the past years.

Uniformization problems are closely related to the function theory on complete Kähler manifolds with curvature assumptions. In the nonnegative bisectional curvature case, although a sharp dimension estimate on spaces of holomorphic functions with polynomial growth was proved, the finite generation of the ring of such holomorphic functions is unknown. Hopefully a better understanding of generators of this ring could give the manifold a nice embedding which has further geometric consequences. In the negative curvature case, there is an long-standing open problem which asks if there exist nontrivial bounded holomorphic functions on complete Kähler manifolds with sectional curvature bounded by two negative constants.

In Chapter 2, we focus on analytic and geometric properties of complete noncompact Kähler manifolds with nonnegative curvature. In particular, we construct new examples of $U(n)$-invariant Kähler metrics with nonnegative and unbounded curvature and give their applications to a problem of Yau. We also study the interplay of function theory and metric geometry on general Kähler manifolds with nonnegative bisectional curvature.

In more detail, we review previous results on examples and general theory on complete noncompact Kähler manifolds with nonnegative curvature in Section 2.1. In Section 2.3 we construct new examples of complete Kähler metrics with nonnegative and unbounded curvature and study various levels of positivity on the curvature for those metrics. Applications include a counterexample to a problem of Yau in Section 2.2 and a rigidity result on rotationally symmetric Kähler metrics with positive complex sectional curvature in Subsection 2.3.2. Section 2.4 and Subsection 2.4.5 are devoted to proving several results on $U(n)$-invariant Kähler-Ricci flow with nonnegative and unbounded curvature. At the end of Chapter 2 we study the interaction of function theory and geometry on complete Kähler manifolds with nonnegative bisectional curvature. We are able to derive some information on the volume growth and curvature decay from the growth of holomorphic functions and canonical sections on such manifolds.

2.1 Background and the work of Wu and Zheng

The central question from uniformization on complete noncompact Kähler manifolds with nonnegative curvature is the uniformization conjecture whose most general form is due to Yau [116]. The conjecture states that any complete noncompact Kähler manifold with positive holomorphic bisectional curvature is biholomorphic to complex Euclidean space. Some related questions were also asked by Greene and Wu [53] and Siu [105]. For example, on noncompact Kähler manifolds with positive holomorphic bisectional curvature Siu and Yau (See p508 [105].) proved that there are holomorphic functions and holomorphic vector fields with any prescribed values to any finite order at any finite number of points. However, it is not known if such manifolds are necessarily Stein. Note that the uniformization conjecture asks the uniqueness of complex structure of $\mathbb{C}^n$ under the assumption of positive curvature, it is naturally desirable to understand the metric geometry of Kähler manifold with nonnegative curvature.

The first result on the conjecture was by Mok, Siu and Yau [74]. On a complete noncompact Kahler manifold has Euclidean volume growth and nonnegative sectional curvature with pointwise faster than quadratic decay, they solved the Poincaré-Lelong equation $\sqrt{-1}\partial\bar{\partial}u = Ric$ and used it to prove that such a manifold is holomorphically isometric to $\mathbb{C}^n$. Later Mok [72] considered noncompact Kahler manifolds with positive bisectional curvature, Euclidean volume growth, and pointwise quadratic scalar curvature decay. Using the L^2-method of Andreotti and Vesentini [1], and Hörmander [62] and techniques in algebraic geometry, Mok proved that, that such a manifold is biholomorphic to an affine algebraic variety in $\mathbb{C}^N$ for some large N. If if the complex dimension is two, he proved that the manifold is biholomorphic to $\mathbb{C}^2$ we in addition assume that it is of positive sectional curvature.

Later the Poincaré-Lelong equation was solved in much weaker assumptions and applied to study the structure of Kähler manifold with nonnegative curvature, see for example Ni and Tam [88] and [90]. More recently, Ni [86] proved any complete noncompact Kahler manifold with nonnegative bisectional curvature whose scalar has faster than quadratic decay in the average sense (No global curvature upper bounds assumed!) must be flat.

Another powerful tool to study the uniformization problem is Kähler-Ricci flow on complete noncompact manifolds. Shi [101] studied Ricci flow on complete noncompact Riemannian manifolds with bounded curvature and established the short time existence. Shi ([102], [103]) first used Kähler-Ricci flow to study the uniformization problem. A quick remark is that there are extra difficulties to study Ricci flow on complete noncompact manifolds, and his methods is different from these for Kähler-Ricci flow on compact Kähler manifolds. The best results on uniformization conjecture obtained so far is due to Chau-Tam [23]. Their result states that any complete noncompact Kähler manifold with bounded nonnegative bisectional curvature and Euclidean volume growth is biholomorphic to $\mathbb{C}^n$.

Besides the above mentioned, some other notable results include the result of Chen and Zhu [32] on volume growth and curvature decay of Kähler manifold with nonnegative bisectional curvature. Ni and Tam [88] introduced the heat deformation of plurisubharmonic functions to prove new structure theorems for Kähler manifold with nonnegative bisectional curvature. A sharp dimension estimate on spaces of holomorphic functions with polynomial growth was established in Ni [81]. Notably, their results imply that complete Kähler metrics with nonnegative bisectional curvature can not exist on the total space of any holomorphic vector bundle E over $\mathbb{CP}^n$ or any other compact Hermitian symmetric space, except when E is the trivial bundle.

We refer to [53], [74], [72], [102], [103], [29], [31], [23], [24], [25] and references therein for detailed progress on the uniformization conjecture.

Perhaps one of the major reasons that made the problem so resilient is the lack of examples. Before the work of Wu and Zheng [108], there are only three types of examples of complete Kähler metrics with positive bisectional curvature (See [65], [13], [14].) and they are all constructed on $\mathbb{C}^n$ with $U(n)$-symmetry. Recently Wu and Zheng [108] gave a complete characterization of complete $U(n)$ invariant Kähler metrics on $\mathbb{C}^n$ with positive bisectional curvature and was first able to demonstrate the abundance of such metrics.

Now let us discuss main results in [108]: Follow the notations in [108]. Let $z = (z_1, \cdots, z_n)$ be the standard coordinate on $\mathbb{C}^n$ and $r = |z|^2$. A $U(n)$-invariant Kähler metric on $\mathbb{C}^n$ has the Kähler form

$$\omega = \frac{\sqrt{-1}}{2}\partial\overline{\partial}\mathcal{P}(r) \tag{2.1}$$

where $\mathcal{P}(r) \in C^\infty[0, +\infty)$. Under the local coordinates, the metric has components:

$$g_{i\bar{j}} = f(r)\delta_{ij} + f'(r)\bar{z}_i z_j. \tag{2.2}$$

We further denote:

$$f(r) = \mathcal{P}'(r), \qquad h(r) = (rf)'. \tag{2.3}$$

It can be checked that the from ω will give a complete Kähler metric on $\mathbb{C}^n$ if and only if

$$f > 0, \quad h > 0, \quad \int_0^{+\infty} \frac{\sqrt{h}}{\sqrt{r}} dr = +\infty. \tag{2.4}$$

Now if we compute the components of the curvature tensor at $(z_1, 0, \cdots, 0)$ under the orthonormal frame $\{e_1 = \frac{1}{\sqrt{h}}\partial_{z_1}, e_2 = \frac{1}{\sqrt{f}}\partial_{z_2}, \cdots, e_n = \frac{1}{\sqrt{f}}\partial_{z_n}\}$, then denote A, B, C respectively:

$$A = R_{1\bar{1}1\bar{1}} = -\frac{1}{h}(\frac{rh'}{h})', \; B = R_{1\bar{1}i\bar{i}} = \frac{f'}{f^2} - \frac{h'}{hf}, \; C = R_{i\bar{i}i\bar{i}} = 2R_{i\bar{i}j\bar{j}} = -\frac{2f'}{f^2}, \tag{2.5}$$

where we assume $2 \leq i \neq j \leq n$, It is easy to check all other components of curvature tensor are zero.

Let $\mathcal{M}_n$ denote the space of all $U(n)$ invariant complete Kähler metrics on $\mathbb{C}^n$ with positive bisectional curvature.

Theorem 2.1.1 (**Characterization of $\mathcal{M}_n$ by the ABC function**). *Suppose $n \geq 2$ and h is a smooth positive function on $[0, +\infty)$ satisfying (2.4), then(2.1) gives a complete Kähler metric with positive (nonnegative) bisectional curvature if and only if A, B, C are positive (nonnegative). Moreover, it is of positive sectional curvature iff $D \doteq AC - B^2 > 0$, and is of positive complex curvature operator iff $D_n \doteq \frac{n}{2(n-1)}AC - B^2 > 0$.*

If we define a smooth function ξ on $[0, +\infty)$ by

$$\xi(r) = -\frac{rh'(r)}{h}, \tag{2.6}$$

then h determines ξ uniquely. On the other hand, note that ξ determines h by $h(r) = h(0)e^{\int_0^r \frac{\xi(t)}{t}dt}$, hence ω up to scaling. The following interesting theorem in [108] reveals that the space $\mathcal{M}_n$ is in fact quite large.

Theorem 2.1.2 (Characterization of $\mathcal{M}_n$ by the ξ function). *Suppose $n \geq 2$ and h is a smooth positive function on $[0, +\infty)$, then the form defined by (2.1) gives a complete Kähler metric with positive bisectional curvature on $\mathbb{C}^n$ if and only if ξ defined by (2.6) satisfying*

$$\xi(0) = 0, \quad \xi' > 0, \quad \xi < 1. \tag{2.7}$$

Fix a metric ω in $\mathcal{M}_n$, the geodesic distance between the origin and a point $z \in \mathbb{C}^n$ is:

$$s = \int_0^r \frac{\sqrt{h}}{2\sqrt{r}} dr. \tag{2.8}$$

where $r = |z|^2$. We denote $B(s)$ the ball in $\mathbb{C}^n$ centered at the origin and with the radius s with respect to ω. It is further shown in [108] that:

$$\mathrm{Vol}(B(s)) = c_n (rf)^n. \tag{2.9}$$

where c_n is the Euclidean volume of the Euclidean unit ball in $\mathbb{C}^n$.

Using Theorem 2.2 Wu and Zheng further proved the following estimates on volume growth of geodesics ball $B(s)$ and the first Chern number for Kähler metrics in $\mathcal{M}_n$. Note that an estimate on volume growth of geodesics ball in the general case has been proved by Chen and Zhu [32].

Proposition 2.1.3 (Volume growth estimates for metrics in $\mathcal{M}_n$). *$rf = f(1) + 2\sqrt{h(1)}(s - s(1))$ for $r > 1$ and $rf \leq s^2$ for any $r \geq 0$. So there exists a constant C such that:*

$$Cs^n \leq \mathrm{Vol}(B(s)) \leq c_n s^{2n}. \tag{2.10}$$

for s large enough.

Proposition 2.1.4 (Bounding the first Chern number for metrics in $\mathcal{M}_n$). *Given any ω in $\mathcal{M}_n$ with $n \geq 1$, we have*

$$\int_{C^n} (Ric)^n = c_n \left(\frac{n\xi(+\infty)}{\pi}\right)^n \leq c_n \left(\frac{n}{\pi}\right)^n. \tag{2.11}$$

while $Vol(B(s)) = c_n v^n$.

Wu and Zheng [108] further introduced another function F in the following way: First we define $x = \sqrt{rh}$ and a nonnegative function y of r by

$$y(0) = 0, \qquad x'^2 + y'^2 = \frac{h}{4r}, \qquad y' > 0. \tag{2.12}$$

One can check that $x(r)$ is strictly increasing and then we may define $F(x)$ a function on $[0, x_0)$ by $y = F(x)$, where

$$x_0^2 = \lim_{r \to +\infty} rh = h(1) e^{\int_1^{+\infty} \frac{1-\xi}{r} dr}. \tag{2.13}$$

Extending F to $(-x_0, x_0)$ by letting $F(x) = F(-x)$, one can check that F is a smooth, even function on $|x| < x_0$. Starting with such a F satisfying certain conditions, one can recover the metric ω in a geometric way. See Section 5 in [108] for details. This result is summarized as the following theorem.

Theorem 2.1.5 (**Characterization of $\mathcal{M}_n$ by the F function**). *Suppose $n \geq 1$, there is a one to one correspondence of between the set $\mathcal{M}_n$ and the set of $\mathcal{F}$ of smooth, even function $F(x)$ defined on $(-x_0, x_0)$ satisfying*

$$F(0) = 0, \qquad F'' > 0, \qquad \lim_{x \to x_0} F(x) = +\infty. \tag{2.14}$$

Denote $v = rf$, one can rewrite s and $\mathrm{Vol}(B(s))$ in terms of F:

$$s = \int_0^x \sqrt{1 + (F'(\tau))^2} d\tau, \qquad v \doteqdot \int_0^x 2\tau \sqrt{1 + (F'(\tau))^2} d\tau.$$

$$\mathrm{Vol}(B(s)) = c_n v^n = c_n \left(\int_0^x 2\tau \sqrt{1 + (F'(\tau))^2} d\tau \right)^n. \tag{2.15}$$

Rewrite A, B, and C defined in (2.5) in terms of F:

$$A = \frac{F'F''}{2x(1 + F'^2)^2}, \qquad B = \frac{x^2}{v^2} - \frac{1}{v\sqrt{1 + F'^2}}, \qquad C = \frac{2}{v} - \frac{2x^2}{v^2}. \tag{2.16}$$

Recall the scalar curvature at the point $z = (z_1, 0, \cdots, 0)$ is given by

$$R = A + 2(n-1)B + \frac{1}{2} n(n-1) C. \tag{2.17}$$

Using (2.17), (2.16), (2.15), and a careful integration by parts, Wu and Zheng [108] proved the following relation between average scalar curvature decay and volume growth of geodesic balls. See also Chen and Zhu [32] for a related result on any complete Kähler manifold with positive bisectional curvature.

Proposition 2.1.6 (**Estimates on average scalar curvature for metrics in** $\mathcal{M}_n$). *Given any complete Kähler metric* ω *in* $\mathcal{M}_n$ *with* $n \geq 2$*, there exists a constant* $C_1 > 0$ *such that*

$$\frac{1}{C_1(1+v)} \leq \frac{1}{\mathrm{Vol}(B(s))} \int_{B(s)} R(s) w^n \leq \frac{C_1}{1+v}. \tag{2.18}$$

while $\mathrm{Vol}(B(s)) = c_n v^n$.

2.2 Applications to a problem of Yau

Shing-Tung Yau asked the following question in [115] (See p278, Problem 9 from Section I "Metric geometry"):

Question 2.2.1. *Given an* n*-dimensional complete manifold with nonnegative Ricci curvature. Let* $B(r)$ *be the geodesic ball around some point* p *and* σ_k *be the* k*-th elementary symmetric function of the Ricci tensor, then is it true that* $r^{-n+2k} \int_{B(r)} \sigma_k$ *has an upper bound when* r *tends to infinity? This should be considered as a generalization of the Cohn-Vossen inequality.*

On Kähler manifolds one can have a similar question.

Question 2.2.2. *On a complete Kähler manifold with complex dimension* n *and nonnegative Ricci curvature, if we denote* ω *and Ric the Kähler form and the Ricci form respectively, one would like to ask if* $r^{-2n+2k} \int_{B(r)} Ric^k \wedge \omega^{n-k}$ *is bounded for any* $1 \leq k \leq n$ *when* r *goes to infinity.*

As an application of the work of Wu and Zheng [108] discussed in Section 2.1, we exhibit counterexamples to Question 2.2.1 when $2 \leq k \leq n$ (Now the real dimension is $2n$, Question 1.1 is meaningful when $1 \leq k \leq n$). More precisely, We will show that for any complex dimension $n \geq 2$, $\int_{B(r)} \sigma_n$ always blows up when r goes to infinity for all nonflat metrics in $\overline{\mathcal{M}}_n$; for any $2 \leq k < n$, we can find a metric in $\overline{\mathcal{M}}_n$ such that $r^{-2n+2k} \int_{B(r)} \sigma_k$ is unbounded when r large (See Theorem 2.2.6). The essential reason is that for a carefully constructed metric from $\overline{\mathcal{M}}_n$, terms containing the square of radial curvature from the expression of σ_k ($k > 1$)

make $r^{-n+2k}\int_{B(r)}\sigma_k$ unbounded when r large. Note that each eigenvalue of real Ricci curvature of a Kähler manifold is of multiplicity 2. Since our methods rely on this special property of σ_k in the case of $k>1$, Question 2.2.1 is still open in $k=1$. It is interesting to note that the recent work of Petrunin [98] implies that Question 2.2.1 is affirmative for $k=1$ under the stronger assumption of nonnegative sectional curvature.

We also show that Question 2.2.2 is affirmative for any complete $U(n)$ invariant Kähler metrics on $\mathbb{C}^n$ with nonnegative bisectional curvature (See Theorem 2.2.8). In fact, Question 2.2.2 is closely related to the uniformization of general Kähler manifolds with nonnegative curvature, see [32] for example.

Section 2.2 is divided into two parts. First we follow [108] to prove parallel results to characterize $U(n)$-invariant Kähler metrics with nonnegative bisectional curvature. In the latter part we use these results to construct counterexamples to Question 2.2.1 when $2\le k\le n$.

2.2.1 $U(n)$-invariant Kähler metrics with nonnegative bisectional curvature

Let $\overline{\mathcal{M}}_n$ denote the space of all $U(n)$ invariant complete Kähler metrics on $\mathbb{C}^n$ with nonnegative bisectional curvature. First we state a generalization of Theorem 2.1.2 to the space $\overline{\mathcal{M}}_n$.

Proposition 2.2.3 (**Characterization of $\overline{\mathcal{M}}_n$ by the ξ function**). *Suppose $n\ge 2$ and h is a smooth positive function on $[0,+\infty)$, then the form defined by (2.1) gives a complete Kähler metric with nonnegative bisectional curvature if and only if ξ defined by (2.6) satisfying*

$$\xi(0)=0,\quad \xi'\ge 0,\quad \xi\le 1. \tag{2.19}$$

Proof of Proposition 2.2.3. In view of Theorem 2.1.1, the point is to show that the ξ satisfying the assumption in Proposition 2.2.3 is equivalent to the nonnegativity of A, B, and C. the original proof of Theorem 2.2 due to Wu and Zheng [108] works here. In fact, the proof shows that the completeness of the metric and the nonnegativity of A must imply the nonnegativity of B and C. This could be considered as a strong rigidity property of nonnegatively curved metrics with $U(n)$-symmetry. Now we only sketch the necessary part. First from (2.6) we know $\xi(0) = 0$. Note that (2.6) and Theorem 2.1 imply

$$A = \frac{\xi'}{h} \geq 0 \tag{2.20}$$

which leads to $\xi' \geq 0$.

To prove $\xi \leq 1$, argument by contradiction as in [108]. Assume $\lim_{r\to+\infty} = b > 1$, then take $\delta_0 > 0$ such that $1 + \delta_0 < b$. It follows that there exists $r_0 > 0$ with $\xi(r_0) \geq 1 + \delta_0$. Thus integrating (2.6) leads to $h(r) = h(0) \exp \int_0^r \frac{\xi}{r} dr \leq \frac{c}{r^{1+\delta_0}}$ which contradicts to the completeness of the metric (2.4). □

It also follows from the original proof of Proposition 2.3 and 2.4 due to Wu and Zheng that the same conclusion holds for the space $\overline{\mathcal{M}}_n$. Namely, for any metric ω in $\overline{\mathcal{M}}_n$, $Cs^n \leq \mathrm{Vol}(B(s)) \leq c_n s^{2n}$ holds for s sufficiently large. and $\int_{C^n} (Ric)^n \leq c_n (\frac{n}{\pi})^n$ is true. We remark here that the estimate on lower bounds of the volume growth of $B(s)$ here can not be true for an arbitrary complete non-compact Kähler manifolds with nonnegative bisectional curvature. For example, take $\Sigma_1 \times \mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1$ where Σ_1 is a capped cylinder on one end and $\mathbb{CP}^1$ is the complex projective plane with the standard metric.

Next we state another generalization of Theorem 2.5 to $\overline{\mathcal{M}}_n$.

Theorem 2.2.4 (**Characterization of $\overline{\mathcal{M}}_n$ by the F function**). *Suppose $n \geq 1$, there is a partition of the set $\overline{\mathcal{M}}_n \setminus \{g_e\} = S_1 \cup S_2 \cup S_3$ where g_e is the standard Euclidean metric on $\mathbb{C}^n$ such that:*

(1) S_1 corresponds to the set of $\mathcal{F}$ of smooth, even function $F(x)$ defined on $(-\infty, +\infty)$ defined above satisfying

$$F(0) = F'(0) = 0, \qquad F'' \geq 0, \qquad F'(\infty) < +\infty, \qquad F(\infty) = +\infty. \tag{2.21}$$

S_1 consists of nonflat Kähler metrics in $\overline{\mathcal{M}}_n$ whose geodesic balls have Euclidean volume growth.

(2) S_2 corresponds to the set of $\mathcal{F}$ of smooth, even function $F(x)$ on $(-x_0, x_0)$ (where x_0 is either finite or $+\infty$) satisfying

$$F(0) = F'(0) = 0, \qquad F'' \geq 0, \qquad F'(x_0) = F(x_0) = +\infty. \tag{2.22}$$

S_2 includes Kähler metrics in $\overline{\mathcal{M}}_n$ whose geodesic balls have strictly less than Euclidean volume growth and bisectional curvatures in the radial direction strictly positive along a sequence of points in $\mathbb{C}^n$ tending to infinity.

(3) For any metric $\omega \in S_3$, there exists a positive real number r_0 such that $r_0 = \inf\{r : \xi(r) = 1\}$ and a corresponding positive real number x_0 such that there exists a smooth even function $F(x)$ defined on $(-x_0, x_0)$ such that

$$F(0) = F'(0) = 0, \qquad F'' \geq 0, \qquad F'(x_0) = +\infty, \qquad F(x_0) < \infty, \tag{2.23}$$

S_3 is the set of metrics with geodesic balls having half Euclidean volume growth and whose bisectional curvatures in the radial direction vanish outside a compact set. A standard example in complex dimension 1 is a capped cylinder on one end.

Proof of Theorem 2.2.4. The proof of Theorem 2.2.4 is based on a modification of the proof of Theorem 2.5 in [108]. From Proposition 2.2.3, we know for any Kähler metric in $\overline{\mathcal{M}}_n$, there exists a corresponding $\xi(r)$ on $[0, +\infty)$ with $\xi(0) = 0, \xi' \geq 0$, and $\xi \leq 1$. Denote $r_0 = \inf\{r : \xi(r) = 1\}$.

Recall the definition of x and y in (2.12), $x = \sqrt{rh}$ and $x'(r)^2 + y'(r)^2 = \frac{h}{4r}$ with $y(0) = 0$ and $y' \geq 0$. It is easy to check:

$$\frac{dx}{dr} = (1 - \xi)\sqrt{\frac{h}{4r}}, \tag{2.24}$$

then we know $x(r)$ and $y(r)$ are both nondecreasing with respect to r.

(Case I) $r_0 = +\infty$. From (2.24) we know $x(r)$ is strictly increasing on $[0, +\infty)$, then we can define $F(x)$ by $y = F(x)$ on $x \in (-x_0, x_0)$ after an even extension by letting $F(-x) = F(x)$. It follows that

$$F(0) = 0, \quad F'(x) \geq 0, \quad 1 + [F'(x)]^2 = \frac{1}{(1 - \xi)^2}. \tag{2.25}$$

Note that $0 \leq \xi(r) < 1$ is nondecreasing on $(-\infty, +\infty)$, we conclude that $F'' \geq 0$.

Moreover, (2.24) and (2.25) implies:

$$\begin{aligned}\lim_{x\to x_0} F(x) &= \int_0^{x_0}\sqrt{\frac{1}{(1-\xi)^2}-1}dx \\ &= \int_0^{+\infty}\sqrt{1-(1-\xi)^2}\sqrt{\frac{h}{4r}}dr \\ &\geq \sqrt{1-(1-\xi(a))^2}\int_a^{+\infty}\sqrt{\frac{h}{4r}}dr.\end{aligned} \tag{2.26}$$

for any $a > 0$. Note that the integral in the last step of (2.26) is distance function (2.8). we conclude $F(x_0) = \infty$ if and only if $\xi(+\infty) > 0$. Note that the latter condition is satisfied when ω is nonflat.

We further divide our discussion into two subcases:

(Subcase Ia) $0 < \xi(+\infty) < 1$. In this case we have F' is bounded on $(-x_0, x_0)$ and $x_0 = +\infty$. Moreover, we will show that the geodesic balls of $(\mathbb{C}^n, \omega)$ has Euclidean volume growth. We follow the method of Wu and Zheng (See P528 of [108]). Note that (2.8),(2.9), $(rf)'(r) = h$ and $(rh)'(r) = h(1-\xi)$, it follows from the L'Hospital's rule that:

$$\begin{aligned}\lim_{s\to+\infty}\frac{\mathrm{Vol}(B(s))}{s^{2n}} &= \lim_{r\to+\infty}\frac{c_n(rf)^n}{s^{2n}} \\ &= \lim_{r\to+\infty} c_n\left(\frac{\sqrt{rf}}{s}\right)^{2n} \\ &= c_n(1-\xi(+\infty))^n\end{aligned} \tag{2.27}$$

(Subcase Ib) $\xi(+\infty) = 1$, It follows from (2.27) that in this case the geodesic balls of $(\mathbb{C}^n, \omega)$ has strictly less than Euclidean volume growth. Since $A = \frac{\xi'}{h}$, $\xi(0) = 0$, and $\xi(+\infty) = 1$ we also know that bisectional curvatures in the radial direction strictly positive along at least a sequence of points in $\mathbb{C}^n$ tending to infinity.

(Case II) $r_0 > 0$ is finite. Note that $(rh)' = h(1-\xi)$, we conclude that $x_0 = \lim_{r\to+\infty}\sqrt{rh}$ is finite and $x_0^2 = r_0 h(r_0)$. This implies that $F(x)$ is well defined on $(-x_0, x_0)$ with $F(x_0) < +\infty$. Since $A = \frac{\xi'}{h}$ we conclude that bisectional curvatures in the radial direction vanishes outside a compact set in $\mathbb{C}^n$. Next we proceed to show that the geodesic balls of $(\mathbb{C}^n, \omega)$ has half Euclidean volume growth. Again the methods follows from [108] (See p528 of [108]).

$$\lim_{s\to+\infty}\frac{\mathrm{Vol}(B(s))}{s^n} = \lim_{r\to+\infty} c_n(\frac{rf}{s})^n = \lim_{r\to+\infty} c_n(2\sqrt{rh})^n = 2^n c_n x_0^n. \tag{2.28}$$

Denote S_1, S_2, and S_3 the sets of metrics in the above three cases (Subcase Ia), (Subcase Ib), and (Case II) respectively, we have proved Theorem 2.2.4.

□

Next we gives some more explicit description of S_3. Given any metric ω in S_3, $\frac{d(rh)}{dr} = (1-\xi)h$, $h=(rf)'$ and $\xi(r)=1$ when $r>r_0$, then:

$$rf|_{r_0}^{r} = \int_{r_0}^{r}\frac{r_0 h(r_0)}{r}dr, \tag{2.29}$$

which further implies:

$$rf = x_0^2\ln\frac{r}{r_0} + r_0 f(r_0). \tag{2.30}$$

Now we compute A, B, C with (2.5) in Section 2 when $r\geq r_0$:

$$A = -\frac{1}{h}(\frac{rh'}{h})' = \frac{\xi'}{h} = 0, \tag{2.31}$$

$$B = \frac{f'}{f^2} - \frac{h'}{hf} = \frac{1}{r}(\frac{(rf)'-f}{f^2} - \frac{rh'}{hf}) = \frac{1}{r}(\frac{h}{f^2} - \frac{1-\xi}{f}) = \frac{h}{rf^2} = \frac{x_0^2}{r^2f^2}. \tag{2.32}$$

$$C = -\frac{2f'}{f^2} = 2\frac{rf-rh}{(rf)^2} = 2\frac{x_0^2(\ln\frac{r}{r_0}-1)+r_0f(r_0)}{[x_0^2\ln\frac{r}{r_0}+r_0f(r_0)]^2} \tag{2.33}$$

We also see the distance function for metrics in S_3:

$$s(r) = \int_0^{r_0}\sqrt{\frac{h}{4r}}dr + \frac{x_0}{2}\ln\frac{r}{r_0}. \tag{2.34}$$

Now one can estimate the average of A, B, and C in $B(s)$ from (2.17), (2.31), (2.32), (2.33), (2.9), and (2.34). Namely, if $n\geq 2$, for any metric in S_3 there exists a constant C_1 such that

$$\frac{1}{C_1 rf} \leq \frac{1}{\mathrm{Vol}(B(s))}\int_{B(s)} R\,\omega^n \leq \frac{C_1}{rf}, \tag{2.35}$$

where $\mathrm{Vol}(B(s)) = c_n(rf)^n$ and f is defined in (2.3).

If ω is a nonflat Kähler metric in $S_1 \cup S_2$, we see from Theorem 2.2.4 that F must have $F'(x_0) > 0$. Then the formula of A, B, and C in terms of F is exactly the same as (2.16) derived in [108] (See p536 in [108]). Follow the proof of Proposition 2.6 in [108], we get the same conclusion. We summarize the above discussion as the following result.

Proposition 2.2.5. *When $n \geq 2$, given any non flat metric in $\overline{\mathcal{M}}_n$, there exists a constant $C_1 > 0$ such that*

$$\frac{1}{C_1(1+rf)} \leq \frac{1}{\mathrm{Vol}(B(s))} \int_{B(s)} R\, w^n \leq \frac{C_1}{1+rf}. \tag{2.36}$$

where $\mathrm{Vol}(B(s)) = c_n(rf)^n$.

2.2.2 Counterexamples to Question 2.2.1

Now we state the result regarding Question 2.2.1.

Theorem 2.2.6. *Given any $n \geq 2$, any nonflat Kähler metric in $\overline{\mathcal{M}}_n$ has $\int_{B(s)} \sigma_n \omega^n$ unbounded when s goes to infinity. Moreover, if $2 \leq k < n$ one can construct a complete Kähler metric ω from $S_1 \subset \overline{\mathcal{M}}_n$ with bounded curvature on $\mathbb{C}^n$ such that $\frac{1}{s^{2n-2k}} \int_{B(s)} \sigma_k\, \omega^n$ is unbounded when s tends to infinity.*

Proof of Theorem 2.2.6. It follows from (2.5) that for any metric in $\overline{\mathcal{M}}_n$ we have Ricci curvature at z given by:

$$\lambda = R_{1\bar{1}} = A + (n-1)B,\ \mu = R_{i\bar{i}} = B + \frac{n}{2}C \quad 2 \leq i \leq n. \tag{2.37}$$

Note that we are now working on the Kähler manifolds $\mathbb{C}^n$ and the Ricci tensor is J-invariant where J is the standard complex structure on $\mathbb{C}^n$. Therefore the Ricci tensor in the real case has eigenvalue λ of multiplicity 2 and μ of multiplicity $2n-2$. Let σ_k denote the k-th elementary symmetric function of the Ricci curvature tensor.

First note that Question 1.1 are true for any metric $\omega \in \overline{\mathcal{M}}_n$ when $k = 1$. Since $\sigma_1 = 2R$ where R is the scalar curvature in the Kähler case, it follows from

Proposition 2.2.5 and the upper bound of the volume growth of $B(s)$ after Proposition 2.2.3 that $\frac{1}{s^{2n-2}}\int_{B(s)} R\omega^n$ is bounded when r tending to infinity. Therefore, we focus on Question 1.1 in the case of $2 \leq k \leq n$, Note that in this case σ_k of the Ricci tensor is a linear combination of $\lambda^2\mu^{k-2}$, $\lambda\mu^{k-1}$, and μ^k. To sum up, σ_k is a linear combination of three types of quantities:

(Type I) $A^2B^iC^j$, $AB^{1+i}C^j$, and $B^{2+i}C^j$ when $i \geq 0$, $j \geq 0$, and $i+j = k-2$.

(Type II) AB^iC^j and $B^{1+i}C^j$ when $i \geq 0$, $j \geq 0$, and $i + j = k - 1$.

(Type III) B^iC^j when $i \geq 0$, $j \geq 0$, and $i + j = k$.

We divide the proof of Theorem 2.2.6 into two cases.

(**Case I**) If $k = n$, we only need to look at the term C^n contained in σ_n. Recall that if for any fixed complete Kähler metric ω in $S_1 \cup S_2$, we may assume that there exists $0 < M_1 < x_0$ such that $F'(x) \geq C_0$ where $C_0 = F'(M_1) > 0$ when $x \geq M_1$. we have the expression of C from (2.16):

$$\begin{aligned} C &= \frac{2v - 2x^2}{v^2} \qquad (2.38) \\ &= \frac{\int_0^x 2\tau(\sqrt{1 + F'(\tau)^2} - 1)d\tau}{v^2} \\ &\geq \frac{\int_{M_1}^x 2\tau(\frac{F'(\tau)^2}{\sqrt{1+F'(\tau)^2}+1})d\tau}{v(\int_{M_1}^x 2\tau\sqrt{1 + F'(\tau)^2}d\tau + \int_0^{M_1} 2\tau\sqrt{1 + F'(\tau)^2}d\tau)}. \end{aligned}$$

Note that we have $1 \leq \frac{F'(x)}{C_0}$ when $x \geq M_1$.

$$C \geq \frac{\frac{C_0}{1+\sqrt{C_0^2+1}}I(x)}{v(\sqrt{\frac{1}{C_0^2} + 1}I(x) + M_1^2\sqrt{1 + C_0^2})} \geq \frac{C_1}{v}, \qquad (2.39)$$

where

$$C_1 = \frac{\frac{C_0}{1+\sqrt{C_0^2+1}}I(x)}{\sqrt{\frac{1}{C_0^2} + 1}I(x) + M_1^2\sqrt{1 + C_0^2}}, \qquad I(x) = \int_{M_1}^x 2\tau F'(\tau)d\tau. \qquad (2.40)$$

Since $I(x)$ goes to ∞ and C_1 is bounded when x tends to x_0, we conclude that there exists a C_2 and M_2 such that when $x > M_2$,

$$C \geq \frac{C_2}{v}. \qquad (2.41)$$

We remark that (2.41) is used in the proof of Proposition 2.3 in [108].

There exists a constant C_3 only depending on n such that:

$$\begin{aligned}\int_{B(s)} \sigma_n \omega^n &\geq C_3 \int_{B(s)} C^n \omega^n \\ &= C_3 c_n \int_0^x C^n dv^n \\ &\geq C_3 c_n \int_{v(M_2)}^{v(x)} (\frac{C_2}{v})^n dv^n \\ &= n C_3 c_n C_2^n \ln \frac{v(x)}{v(M_2)}\end{aligned} \tag{2.42}$$

It follows that (2.42) is unbounded when x tends to x_0 since $v(x_0) = +\infty$.

To sum up, we show that for any non flat Kähler metric ω in $S_1 \cup S_2$, $\int_{\mathbb{C}^n} \sigma_n \omega^n$ is ∞. If $\omega \in S_3$, it follows from (2.30) and (2.33) that $\lim_{s \to +\infty} \int_{B(s)} \sigma_n \omega^n$ is unbounded when s goes to infinity. Therefore, Question 1.1 is false when $k = n$ for any non flat Kähler metric ω in $\overline{\mathcal{M}}_n$.

(**Case II**) If $2 \leq k < n$, for any fixed nonflat Kähler metric ω in S_1, we may assume that there exist C_4 and M_3 such that $F'(x) \leq C_4$ for all $x \in (-x_0, x_0)$ and $F'(x) \geq \frac{1}{C_4}$ when $x \geq M_3$. Then it follows from a similar argument as in (2.41), we may further assume that there exist C_5 and M_3 such that for any $x \geq M_3$

$$C \geq \frac{C_5}{v}. \tag{2.43}$$

Since $A = \frac{F'F''}{2x(1+F'^2)^2}$, we conclude that A and $\frac{F''(x)}{x}$ are equivalent. If we can construct a Kähler metric ω in S_1 such that

$$\frac{1}{s^{2n-2k}} \int_{B(s)} (\frac{F''(x)}{x})^2 (\frac{C_5}{v})^{k-2} \omega^n \tag{2.44}$$

is unbounded when s tends to ∞, then so is $\frac{1}{s^{2n-2k}} \int_{B(s)} A^2 C^{k-2} \omega^n$. Therefore $\frac{1}{s^{2n-2k}} \int_{B(s)} \sigma_k \omega^n$ will be unbounded when s tends to ∞.

Let us rewrite (2.44):

$$\begin{aligned}&\int_{B(s)} (\frac{F''(x)}{x})^2 (\frac{C_5}{v})^{k-2} \omega^n \\ =\ & c_n C_5^{k-2} \int_0^x (\frac{F''(\tau)}{\tau})^2 (\frac{1}{v})^{k-2} dv^n \\ =\ & 2n c_n C_5^{k-2} \int_0^x \frac{1}{\tau} (F''(\tau))^2 v^{n-k+1} \sqrt{1 + (F'(\tau))^2} d\tau.\end{aligned} \tag{2.45}$$

Since $s = \int_0^x \sqrt{1+(F'(\tau))^2}d\tau$, $v = \int_0^x 2\tau\sqrt{1+(F'(\tau))^2}d\tau$ and $F'(x) \leq C_4$ we know s and x are equivalent, v and x^2 are equivalent. In order to estimate (2.45), it suffices to estimate the following.

$$\int_0^x (F'')^2 \tau^{2(n-k)+1}\, d\tau. \tag{2.46}$$

To sum up, if there exists a function $\delta(x) \in C^\infty[0,+\infty)$, such that

$$\lim_{x\to x_0} \frac{1}{x^{2n-2k}} \int_0^x \delta^2(\tau)\, \tau^{2(n-k)+1}\, d\tau = +\infty, \qquad \int_0^{+\infty} \delta(x)\, dx < +\infty. \tag{2.47}$$

Then we can solve $F(x)$ with $F''(x) = \delta(x)$ with the initial value $F(0) = F'(0) = 0$, it will follow from Theorem 2.2.4 that we can construct a complete Kähler metric ω in S_1 such that

$$\frac{1}{s^{2n-2k}} \int_{B(s)} \sigma_k\, \omega^n \tag{2.48}$$

is unbounded when s tends to infinity. Hence both Question 1.1 and 1.2 can not be true when $2 \leq k < n$.

In fact such a $\delta(x)$ is not hard to construct. Consider $\bar{\delta}(x)$ defined by the following with q an integer to be determined.

$$\bar{\delta}(x) = \begin{cases} 2 & x \in [2, 2+(\frac{1}{2})^q] \\ \vdots & \vdots \\ l & x \in [l, l+(\frac{1}{l})^q] \\ \vdots & \vdots \\ 0 & x \in [0,+\infty) \setminus (\cup_{l\geq 2}[l, l+(\frac{1}{l})^q]) \end{cases} \tag{2.49}$$

Now set $q = \frac{5}{2}$, it is easy to verify that $\bar{\delta}(x)$ satisfies (2.47). Choose $\delta(x)$ as a suitable smoothing of $\bar{\delta}(x)$ on $[0,+\infty)$ which also satisfies (2.47), we will get the counterexample. It is straightforward to see that the result metric $\omega \in S_1$ has bounded curvature on $\mathbb{C}^n$.

□

It follows from Theorem 2.2.4 and Proposition 2.2.5 that for any complete Kähler metric $(\mathbb{C}^n, \omega)$ with $\omega \in \overline{\mathcal{M}_n}$ with Euclidean volume growth has quadratic average scalar curvature decay. Note that the same result for any complete Kähler manifolds with bounded nonnegative bisectional curvature and Euclidean volume growth has been proved by Ni [83] and [85]. We refer readers to the introduction of Ni [85] and Chau and Tam [23] for the important role of this result in the study of the function theory and the uniformization conjecture on complete Kähler manifolds with positive bisectional curvature. Now we construct another example which implies that in general only assuming Euclidean volume growth one can not expect the same rate of decay for L^p norm of curvature for any $p > 1$.

Proposition 2.2.7. *For any $n \geq 2$ and any $p > 1$, there exists a metric $\omega \in \overline{\mathcal{M}}_n$ such that the geodesic balls in $(\mathbb{C}^n, \omega)$ has Euclidean volume growth. Moreover,*

$$\frac{s^2}{\mathrm{Vol}(B(s))} \int_{B(s)} \Big[Rm(\frac{\partial}{\partial s}, J\frac{\partial}{\partial s}, J\frac{\partial}{\partial s}, \frac{\partial}{\partial s})\Big]^p \omega^n \tag{2.50}$$

is unbounded as s goes to infinity. Here we denote $\frac{\partial}{\partial s}$ to the unit radial direction on $\mathbb{C}^n$.

Proof of Proposition 2.2.7. For a given metric ω in S_1, as a similar argument in (Case II) of the proof of Theorem 2.2.6, it suffices to show that we can find a smooth function $\eta(x)$ on $[0, +\infty)$ such that

$$\lim_{x \to +\infty} \frac{1}{x^{2n-2}} \int_0^x \eta^p(\tau)\, \tau^{2n-1-p}\, d\tau = +\infty, \qquad \int_0^{+\infty} \eta(x)\, dx < +\infty. \tag{2.51}$$

Consider $\bar{\eta}(x)$ defined by the following where α and β are two integers to be determined.

$$\bar{\eta}(x) = \begin{cases} 2^\alpha & x \in [2, 2 + (\frac{1}{2})^\beta] \\ \vdots & \vdots \\ l^\alpha & x \in [l, l + (\frac{1}{l})^\beta] \\ \vdots & \vdots \\ 0 & x \in [0, +\infty) \setminus (\cup_{l \geq 2} [l, l + (\frac{1}{l})^q]) \end{cases} \tag{2.52}$$

Pick any $\alpha > 1$ and $1 + \alpha < \beta < p(\alpha - 1) + 2$, then $\bar{\eta}$ defined above satisfies (2.51). It is not hard to find $\eta(x)$ from a suitable smoothing of $\bar{\eta}(x)$ which will result in the desired metric ω. Note that $(\mathbb{C}^n, \omega)$ we constructed has unbounded curvature on $\mathbb{C}^n$. □

We proceed to show that Question 2.2.2 is true for $\overline{\mathcal{M}}_n$.

Theorem 2.2.8. *For any metric $\omega \in \overline{\mathcal{M}}_n$, then $s^{-2n+2k}\int_{B(s)} Ric^k \wedge \omega^{n-k}$ is bounded when s goes to infinity.*

Proof of Proposition 2.2.8. First we remark that it directly follows from analogues of Proposition 2.3, 2.4 and 2.6 in [108] for the space $\overline{\mathcal{M}}_n$ (See Proposition 2.2.5 and the paragraph after Proposition 2.2.3) that Question 1.3 is true for $k = 1$ and $k = n$. It suffices to show that $s^{-2n+2k}\int_{B(s)} Ric^k \wedge \omega^{n-k}$ is bounded for any $2 \le k < n$.

Note that for $2 \le k < n$, $Ric^k \wedge \omega^{n-k}$ is a linear combination of $\lambda\mu^{k-1}$ and μ^k. It turns out that we only needs to show that $\frac{1}{s^{2n-2k}}\int_{B(s)} P(A, B, C)\omega^n$ is bounded when s goes to infinity. Here P is a monomial of the following two types:

(Type I) AB^iC^j, and $B^{1+i}C^j$ when $i \ge 0$, $j \ge 0$, and $i + j = k - 1$.

(Type II) B^pC^q when $p \ge 0$, $q \ge 0$, and $p + q = k$.

First we consider any Kähler metric ω in $S_1 \cup S_2$. Note that (2.16) implies that

$$B \le \frac{x^2}{v^2} \le \frac{1}{v}, \quad C \le \frac{2}{v}. \tag{2.53}$$

Then we have the following when $p + q = k$,

$$\begin{aligned} & \frac{1}{s^{2n-2k}}\int_{B(s)} B^pC^q\omega^n & (2.54)\\ \le\ & 2^q c_n \frac{1}{s^{2n-2k}}\int_0^{v(x)} \frac{1}{v^k} n v^{n-1} dv \\ \le\ & \frac{2^q n c_n (\int_0^x 2\tau\sqrt{1+(F'(\tau))^2}d\tau)^{n-k}}{(n-k)(\int_0^x \sqrt{1+(F'(\tau))^2}d\tau)^{2n-2k}}. \end{aligned}$$

According to the L'Hospital's rule, (2.54) has the limit when x tends to x_0:

$$\lim_{x\to x_0} \frac{(\int_0^x 2\tau\sqrt{1+(F'(\tau))^2}d\tau)^{n-k}}{(\int_0^x \sqrt{1+(F'(\tau))^2}d\tau)^{2n-2k}} = \left(\frac{2}{\sqrt{1+\lim_{x\to x_0} F'(x)}}\right)^{n-k}. \tag{2.55}$$

We conclude that $\frac{1}{s^{2n-2k}}\int_{B(s)} B^pC^q\omega^n$ is bounded when s goes to infinity.

Next we turn to the term AB^iC^j, integrate by parts as in the original proof of Proposition 2.6 in [108].

$$
\begin{aligned}
& \int_{B(s)} AB^iC^j\omega^n && (2.56)\\
= \quad & c_n\int_0^x \frac{F'F''}{2\tau(1+(F')^2)^2}\frac{1}{v^{k-1}}nv^{n-1}2\tau\sqrt{1+(F')^2}d\tau \\
= \quad & c_n\int_0^x \frac{F'F''}{(1+(F')^2)^{\frac{3}{2}}}nv^{n-k}d\tau \\
= \quad & c_n\int_0^v nv^{n-k}d(-\frac{1}{\sqrt{1+(F')^2}}) \\
= \quad & (-c_nnv^{n-k}\frac{1}{\sqrt{1+(F')^2}})|_0^v + c_nn(n-k)\int_0^v \frac{1}{\sqrt{1+(F')^2}}v^{n-k-1}dv \\
\leq \quad & c_nnv^{n-k}.
\end{aligned}
$$

It follows from (2.55) that

$$
\frac{1}{s^{2n-2k}}\int_{B(s)} AB^iC^j\omega^n \tag{2.57}
$$

is bounded when s tends to infinity.

It remains to verify that Question 2.2.2 is true when $2 \leq k < n$ for any metric $\omega \in S_3$. Note that in this case we have (2.32), (2.33), (2.34), and $A = 0$ outside a compact set for metrics in S_3, it follows from a straightforward calculation that $\frac{1}{s^{2n-2k}}\int_{B(s)} Ric^k \wedge \omega^{n-k}$ is bounded when s goes to infinity. Hence we finish the proof of Proposition 2.2.8. □

After the first draft of this paper. Professor Shing-Tung Yau [117] suggested that the original conjecture be considered in the stronger assumption of nonnegative sectional curvature. In the following section we will give further answers to these questions after developing a more complicated result to construct $U(n)$-invariant Kähler metrics with positive curvature.

2.3 New examples of $U(n)$-invariant Kähler metrics with positive curvature

In this section we study new examples of $U(n)$-invariant complete Kähler metrics on $\mathbb{C}^n$ with positive and unbounded sectional curvature.

Let $\mathcal{M}_n$ ($\mathcal{N}_n$, $\mathcal{K}_n$) denote the space of all $U(n)$-invariant complete Kähler metrics on $\mathbb{C}^n$ with positive bisectional curvature (positive sectional curvature, positive complex curvature operator). See Subsection 2.3.2 for the definition of complex curvature operator on Kähler manifolds. We will show the following theorems.

Proposition 2.3.1. *There are examples in $\mathcal{N}_n$ and $\mathcal{K}_n$ with unbounded curvature.*

It is shown in [108] that examples of metrics with unbounded curvature in $\mathcal{M}_n$ can be easily constructed. In fact, their results implies that one can perturb any metric in $\mathcal{M}_n$ such that the new one have scalar curvature blows up with any given rate along a prescribed sequence along the infinity. However, if we require stronger assumptions on the positivity of the curvature, more careful analysis on the perturbation are needed. In fact, we construct examples in Proposition 2.3.1 by perturbing a particularly chosen metric in $\mathcal{K}_n$, although we believe that such a choice should not be essential.

As an application of the perturbation argument developed in the proof of Proposition 2.3.1, we prove

Proposition 2.3.2. *Fix any $n > 2$. For any integer $2 \leq k \leq n$, there exists a complete Kähler metric in $\mathcal{N}_n$ such that $r^{-2n+2k} \int_{B(O,r)} \sigma_k$ is unbounded when r tends to infinity. Here σ_k denotes the k-th elementary symmetric function of the eigenvalues of the Ricci tensor and $B(O,r)$ is the geodesic ball of radius r centered at a fixed point O.*

This problem is related to Yau's Question 2.2.1 [115] and his suggestion mentioned at the end of Section 2.2. In Subsection 2.3.1, we are able to construct examples in $\mathcal{N}_n$ satisfying the conclusion of Proposition 2.3.2 with a perturbation argument. A careful choice of the perturbation function will allow us to keep the perturbed metric having positive curvature while making σ_k oscillate large enough such that $r^{-n+2k} \int_{B(r)} \sigma_k$ unbounded when r is large.

It is interesting to compare Proposition 2.3.2 with the result of Petrunin [98] which shows a sharp difference in the case of $k = 1$.

Theorem 2.3.3 (Petrunin [98]). *Let (M^n, g) be a complete Riemannian manifold with nonnegative sectional curvature, then for any $O \in M$ and $r > 0$, there exists a constant $c(n)$ which depends only on n such that*

$$r^{2-n} \int_{B(O,r)} R \, dVol(g) \leq c(n).$$

2.3.1 Proof of Proposition 2.3.1 and 2.3.2

Step 1: Set up and the generating function $p(x)$

Now assume $F : [0, x_0) \to [0, \infty)$ is a smooth strictly convex function where $0 < x_0 \leq +\infty$, $F(0) = F'(0) = 0$, and $F(x_0) = +\infty$. Write

$$p(x) = \sqrt{1 + (F'(x))^2}, \quad v(x) = \int_0^x 2\tau p(\tau) d\tau.$$

We will call $p(x)$ *the generating function* for the metric in $\mathcal{M}_n$. One can rewrite ABC in terms of $p(x)$:

$$A = \frac{F'F''}{2x(1 + (F')^2)^2} = \frac{p'}{2x\, p^3}. \tag{2.58}$$

$$B = \frac{1}{v^2}(x^2 - \frac{v}{\sqrt{1 + (F'(x))^2}}) = \frac{1}{v^2}(x^2 - \frac{v}{p}). \tag{2.59}$$

$$C = \frac{2}{v^2}(v - x^2). \tag{2.60}$$

The problem of characterizing the space $\mathcal{N}_n$ or $\mathcal{K}_n$ can be reduced to the following problem:

Question 2.3.4. *Let $0 < x_0 \leq \infty$, and let $p(x)$ be a smooth strictly increasing function on $[0, x_0)$ with $p(0) = 1$, $p'(0) = 0$, $p''(0) > 0$ and $\int_0^{x_0} p(\tau) d\tau = +\infty$. Can we find $p(x)$ such that*

$$\frac{p'v^2}{xp^3}(v - x^2) > (x^2 - \frac{v}{p})^2 \quad or \quad \frac{n}{2n-2}\frac{p'v^2}{xp^3}(v - x^2) > (x^2 - \frac{v}{p})^2 \tag{2.61}$$

holds for all $x \in (0, x_0)$? Here $v(x) = \int_0^x 2\tau p(\tau) d\tau$. Note that $\mathcal{N}_n$ and $\mathcal{K}_n$ are the same when $n = 2$. In general the second named author [122] proved that any complete Kähler surface with positive sectional curvature must have positive complex curvature operator, See Subsection 2.3.2 for more discussion.

Step 2: Perturbation on $p(x)$ and Estimates on D

First we recall the following examples in [108].

Example 2.3.5 ([108])**.** *$\xi = \frac{cr}{1+r}$ where $r \in [0, +\infty)$. Then for any $0 < c < 1$ the corresponding metric lies in $\mathcal{M}_n$ and has maximal volume growth; it lies in $\mathcal{K}_n$ if $0 < c \leq \frac{1}{2}$ while it dose not have positive sectional curvature for any $\frac{1}{2} < c < 1$.*

Let us fix $c = \frac{1}{2}$. A routine calculation shows:

$$A = \frac{1}{2(1+r)^{\frac{3}{2}}}, \;\; B = \frac{1}{4(1+r)}, \quad C = \frac{1}{2\sqrt{1+r}}, \;\; D = \frac{3}{16(1+r)^2}. \tag{2.62}$$

$$s = \int_0^r \frac{1}{2\sqrt{r(1+r)^{\frac{1}{2}}}} dr \sim r^{\frac{1}{4}}. \tag{2.63}$$

$$p = \frac{1+r}{1+\frac{r}{2}}, \;\; p'(x) = \frac{r^{\frac{1}{2}}(1+r)^{\frac{5}{4}}}{(1+\frac{r}{2})^3}, \quad v = 2(\sqrt{1+r} - 1). \tag{2.64}$$

$$x^2 = \frac{r}{\sqrt{1+r}}, \quad r = \frac{x^4 + \sqrt{x^8 + 4x^4}}{2} \tag{2.65}$$

Our goal is to perturb the function $p(x)$ to produce a Kähler metric in $\mathcal{N}_n$ with unbounded curvature along a sequence of points tending to infinity.

Define

$$\widetilde{p} = p + \Phi, \quad E = \int_0^x 2\tau\Phi(\tau) d\tau. \tag{2.66}$$

Here we assume that Φ is nonnegative nondecreasing function on $[0, +\infty)$ and vanishes in a small neighborhood of 0. These conditions ensure that the new function $\widetilde{p}(x)$ will be a generating function for some metric in $\mathcal{M}_n$. Note that we have $\widetilde{v} = \int_0^x 2\tau \widetilde{p}(\tau) d\tau = v + E$, so it is straightforward to get the curvature components of the new metric generated by $\widetilde{p}$:

$$\widetilde{A} = \frac{\widetilde{p}'}{2x\,\widetilde{p}^3} = \frac{p'(x) + \Phi'(x)}{2x\,(p+\Phi)^3}. \tag{2.67}$$

$$\widetilde{B} = \frac{1}{\widetilde{v}^2}(x^2 - \frac{\widetilde{v}}{\widetilde{p}}) = \frac{x^2}{(v+E)^2} - \frac{1}{(v+E)(p+\Phi)}. \tag{2.68}$$

$$\widetilde{C} = \frac{2}{\widetilde{v}^2}(\widetilde{v} - x^2) = \frac{2}{v+E} - \frac{2x^2}{(v+E)^2}. \tag{2.69}$$

The point of **Step 2** is to estimate $\widetilde{D} \doteq \widetilde{A}\widetilde{C} - \widetilde{B}^2$ via the following lemma.

Lemma 2.3.6. *There exists $\delta_0 > 0$ and two constants $C(\delta_0)$ and x_0 both only depending on δ_0 such that for any $0 < \delta \leq \delta_0$ and any smooth increasing function $\Phi(x)$ defined on $[0, +\infty)$ with $\Phi = 0$ in a small neighborhood of 0 and*

$$\lim_{x \to +\infty} \Phi(x) = \delta, \quad \int_0^{+\infty} \Phi'(\tau)\tau^2 d\tau \leq C(\delta_0)\delta, \tag{2.70}$$

we always have $\widetilde{D} > 0$ for any $x > x_0$.

Assume we have proved Lemma 2.3.6, we pick any $0 < \delta < \delta_0$ and any function Φ satisfying the assumption in Lemma 2.3.6. Consider $\widetilde{D}$ inside the ball with $0 \leq x \leq 2x_0$. Since $\widetilde{D}$ converges to $D > 0$ uniformly on compact sets $B(O, 2x_0)$ as $\delta \to 0$. Shrinking δ if necessary, we conclude that there exists a function $\Phi(x)$ defined on $[0, +\infty)$ such that

$$\lim_{x \to +\infty} \Phi(x) = \delta, \quad \int_0^{+\infty} \Phi'(\tau)\tau^2 d\tau \leq C(\delta_0)\delta, \tag{2.71}$$

such $\widetilde{D} > 0$ for this particular $\Phi(x)$ inside the ball with $0 \leq x \leq 2x_0$. Therefore, $\widetilde{D} > 0$ everywhere and the generating function $\widetilde{p}(x)$ give a metric with positive sectional curvature.

Proof of Lemma 2.3.6. Note that:

$$\begin{aligned} & \widetilde{D} - D \qquad (2.72) \\ = \; & (A + \widetilde{A} - A)(C + \widetilde{C} - C) - (B + \widetilde{B} - B)^2 - (AC - B^2) \\ = \; & (\widetilde{A} - A)C + (\widetilde{C} - C)A + (\widetilde{A} - A)(\widetilde{C} - C) - 2B(\widetilde{B} - B) - (\widetilde{B} - B)^2 \end{aligned}$$

Step 1 of Proof of Lemma 2.3.6: estimating $\widetilde{A} - A$.

It follows from (2.58) and (2.67) that

$$\widetilde{A} - A = \frac{\Phi'}{2x(p + \Phi)^3} - \frac{p'(3p^2\Phi + 3p\Phi^2 + \phi^3)}{2xp^3(p + \Phi)^3}. \tag{2.73}$$

Then we know $\widetilde{A} - A \geq 0$ at points where

$$\Phi'(x) \geq \Phi \frac{p'(x)}{p^3}(\Phi^2 + 3\Phi p + 3p^2). \tag{2.74}$$

Otherwise, we have

$$-\frac{p'(x)}{2xp^6}(\delta^3+6\delta^2+12\delta)\le \widetilde{A}-A\le 0 \tag{2.75}$$

where we use $1\le p\le 2$.

Note that $\frac{P'(x)}{2xp^3}\sim O(\frac{1}{r^{\frac{3}{2}}})$. we conclude that there exists x_0 and $c(\delta)$ only depends on δ such that

$$\text{If } \widetilde{A}-A\le 0, \text{ then } -\frac{c(\delta)}{r^{\frac{3}{2}}}\le \widetilde{A}-A\le 0 \text{ whenever } x\ge x_0. \tag{2.76}$$

Step 2 of Proof of Lemma 2.3.6: estimating $\widetilde{C}-C$.

It follows from (2.60) and (2.69) that

$$\widetilde{C}-C=\frac{-2E}{v(v+E)}+2x^2\frac{E^2+2vE}{v^2(v+E)^2} \tag{2.77}$$

Note that $v\le 2\sqrt{r}$ and $x^2\le\sqrt{r}$.

$$\frac{2E}{v(v+E)}+x^2\frac{E^2+2vE}{v^2(v+E)^2}\le\frac{\delta}{2\sqrt{r}}+\frac{2\delta+\delta^2}{4\sqrt{r}}. \tag{2.78}$$

Similarly we conclude that there exists $c(\delta)$ only depends on δ such that

$$|\widetilde{C}-C|\le\frac{c(\delta)}{\sqrt{r}}. \tag{2.79}$$

Step 3 of Proof of Lemma 2.3.6: estimating $\widetilde{B}-B$.

It follows from (2.59) and (2.68) that

$$\widetilde{B}-B=-x^2\frac{E^2+2Ev}{(v+E)^2v^2}+\frac{Ep+E\Phi+v\Phi}{vp(v+E)(p+\Phi)}. \tag{2.80}$$

Now plug in $v=2(\sqrt{1+r}-1)$, $x^2=\frac{r}{\sqrt{r+1}}$ and $p=\frac{1+r}{1+\frac{r}{2}}$. Denote $\widetilde{B}-B=I_1-I_2$ where

$$I_1 = \frac{r}{\sqrt{r+1}}\frac{E^2+2E\cdot 2(\sqrt{r+1}-1)}{[2(\sqrt{1+r}-1)+E]^2[2(\sqrt{r+1}-1)]^2}, \tag{2.81}$$

$$I_2 = \frac{E(\frac{1+r}{1+\frac{r}{2}})+E\Phi+2(\sqrt{r+1}-1)\Phi}{2(\sqrt{r+1}-1)[2(\sqrt{r+1}-1)+E](\frac{1+r}{1+\frac{r}{2}})[\frac{1+r}{1+\frac{r}{2}}+\Phi]}. \tag{2.82}$$

The following claim is crucial in Step 3 of Proof of Lemma 2.3.6.

Claim 2.3.7.

$$\lim_{r\to+\infty}\sqrt{r}I_1 = \lim_{r\to+\infty}\sqrt{r}I_2 = \frac{\delta^2+4\delta}{4(2+\delta)^2}, \tag{2.83}$$

$$r|I_1 - I_2| \leq c(\delta) \tag{2.84}$$

for constant $c(\delta)$ and $x \geq x_0$ where x_0 only depends on δ .

Proof of Claim 2.3.7. (2.83) is straightforward if we note

$$\lim_{r\to+\infty}\frac{E}{\sqrt{r}} = \lim_{x\to+\infty}\Phi(x) = \delta. \tag{2.85}$$

Introduce:

$$\begin{aligned} I_3 &\doteq r[E^2 + 2E\cdot 2(\sqrt{r+1}-1)]2(\sqrt{r+1}-1)(\frac{1+r}{1+\frac{r}{2}}) \\ &\cdot[2(\sqrt{r+1}-1)+E][\frac{1+r}{1+\frac{r}{2}}+\Phi] \end{aligned} \tag{2.86}$$

$$\begin{aligned} I_4 &\doteq \sqrt{1+r}[E+2(\sqrt{r+1}-1)]^2[2(\sqrt{r+1}-1)]^2 \\ &\cdot[E(\frac{1+r}{1+\frac{r}{2}})+E\Phi+2(\sqrt{r+1}-1)\Phi] \end{aligned} \tag{2.87}$$

To prove (2.84) we will show there exists constants $c(\delta)$ and x_0 which only depends on δ

$$\frac{|I_3 - I_4|}{r^{\frac{5}{2}}} \leq c(\delta) \tag{2.88}$$

for any $x \geq x_0$.

Note the fact that $\sqrt{r+1}-1 = \sqrt{r}-C_1$, $\frac{1+r}{1+\frac{r}{2}} = 2-\frac{C_2}{r}$, and $\sqrt{1+r}-\sqrt{r} = \frac{C_3}{\sqrt{r}}$ where C_1, C_2 and C_3 are bounded functions of r.

$$\begin{aligned} I_3 &= r[E^2+4E\sqrt{r}-2C_1E][2\sqrt{r}-2C_1] \\ &\quad [2-\frac{C_2}{r}][2\sqrt{r}+E-2C_1][2+\Phi-\frac{C_2}{r}] \\ I_4 &= [\sqrt{r}+\frac{C_3}{\sqrt{r}}][E+2\sqrt{r}-2C_1]^2[2\sqrt{r}-2C_1]^2 \\ &\quad [2E+E\Phi+2\sqrt{r}\Phi-\frac{C_2}{r}E-2C_1\Phi] \end{aligned} \tag{2.89}$$

It is straightforward to check that:

$$\frac{|I_3 - I_4|}{r^{\frac{5}{2}}} = 16(\delta+2)(E-\sqrt{r}\Phi)+c(\delta), \tag{2.90}$$

where $c(\delta)$ only depends on δ.

A further integration of parts shows:

$$\begin{aligned} E - \sqrt{r}\Phi &= \int_0^x 2\tau\Phi(\tau)d\tau - \sqrt{r}\Phi \\ &= x^2\Phi(x) - \int_0^x \Phi'(x)\tau^2 d\tau - \sqrt{r}\Phi. \\ &= O(\frac{1}{\sqrt{r}})\Phi(x) - \int_0^x \Phi'(\tau)\tau^2 d\tau. \end{aligned} \tag{2.91}$$

To sum up, we need

$$\int_0^{+\infty} \Phi'(\tau)\tau^2 d\tau \leq c(\delta) \tag{2.92}$$

to ensure that $|\widetilde{B} - B| \leq \frac{C(\delta)}{r}$ for r large. Claim 2.3.7 is proved. □

To sum up, we proved Lemma 2.3.6. □

Now we are ready to prove both Proposition 2.3.1 and 2.3.2 by the perturbation method established above.

Proof of Proposition 2.3.1. Now that Lemma 2.3.6 is proved, it is easy to prescribe a suitable perturbation function $\Phi(x)$ to prove Proposition 2.3.1.

Define the following function η on $[0, +\infty)$ with α and β to be determined.

$$\eta(x) = \begin{cases} 2^\alpha & x \in [2, 2 + (\frac{1}{2})^\beta] \\ 3^\alpha & x \in [3, 3 + (\frac{1}{3})^\beta] \\ \vdots & \vdots \\ 0 & x \in [0, +\infty) \setminus (\cup_{l \geq 2}[l, l + (\frac{1}{l})^\beta]) \end{cases} \tag{2.93}$$

.

Set $\alpha > 1$ and $\beta > \alpha + 3$, then it is easy to check.

$$\int_0^{+\infty} \eta(\tau)d\tau < \infty, \quad \int_0^x \eta(\tau)\tau^2 d\tau < \infty, \tag{2.94}$$

After a suitable smoothing of η, still denoted by η for simplicity, choose

$$\Phi'(x) = \delta \frac{\eta(x)}{\int_0^{+\infty} \eta(\tau)d\tau}. \tag{2.95}$$

Here we choose a sufficiently small $\delta \leq \delta_0$ where δ_0 is defined in Lemma 2.3.6.

Note that

$$\widetilde{A} = \frac{p'(x) + \Phi'(x)}{2x\,(p+\Phi)^3}.$$

will be unbounded along a sequence going to infinity. This will produce a metric in $\mathcal{N}_n$ with unbounded curvature. The proof of Proposition 2.3.1 is done since a similar perturbation estimates on D_n works.

□

Proof of Proposition 2.3.2. Fix any fixed $2 \leq k \leq n$, it suffices to show that we can find $p(x)$ defined in Question 2.3.4 with $x_0 = +\infty$ such that both (2.61) and the following hold.

$$\limsup_{x\to+\infty} \frac{1}{x^{2n-2k}} \int_0^x \frac{p^2}{p^2-1}[p'(\tau)]^2 \tau^{2(n-k)+1} d\tau = +\infty. \tag{2.96}$$

Compare with the proof to Proposition 2.3.1, we only need to show: Fix any fixed $n > 2$ and $2 \leq k \leq n$, we can find a smooth increasing function $\Phi(x)$ defined on $[0, +\infty)$ with $\Phi = 0$ at a small neighborhood of 0. such that there exists a constant δ small enough and C independent of δ:

$$\lim_{x\to+\infty} \Phi(x) = \delta, \quad \int_0^{+\infty} \Phi'(\tau)\tau^2 d\tau \leq C\delta, \tag{2.97}$$

$$\limsup_{x\to+\infty} \frac{1}{x^{2n-2k}} \int_0^x [\Phi'(\tau)]^2 \tau^{2(n-k)+1} d\tau = +\infty. \tag{2.98}$$

That could be done by assuming $\alpha + 3 < \beta < 2\alpha + 2$ on the function η defined in the proof of Proposition 2.3.1.

□

2.3.2 Various levels of positivity on the curvature

In this subsection, we discuss the different levels of positivity for metrics in $\mathcal{M}_n$.

Start with an arbitrary Kähler manifold (M^n, g) of complex dimension n. At any $p \in M$, the complexified tangent space $T = T_pM \otimes_{\mathbb{R}} \mathbb{C}$ splits as $V \oplus \overline{V}$, with $V \cong \mathbb{C}^n$ the space of all type $(1,0)$ tangent vectors. Extend the Riemannian curvature tensor R of g linearly over $\mathbb{C}$. The Kählerness of g implies that the curvature operator of g, as a symmetric bilinear form on $\Lambda^2 T = \Lambda^2 V \oplus (V \otimes \overline{V}) \oplus \Lambda^2 \overline{V}$, vanishes on the first and the third components of the right hand side. It can be identified with the following Hermitian bilinear form Q on $V \otimes \overline{V}$ defined by:

$$Q(\xi, \overline{\eta}) = \sum_{i,j,k,l=1}^{n} R_{i\bar{j}k\bar{l}} \xi_{i\bar{j}} \overline{\eta_{l\bar{k}}} \ ,$$

where $R_{i\bar{j}k\bar{l}}$ are the components of R under a unitary basis $\{e_1, \ldots, e_n\}$ of V, and with $\xi = \sum \xi_{i\bar{j}} e_i \otimes \overline{e_j}$ and $\eta = \sum \eta_{i\bar{j}} e_i \otimes \overline{e_j}$. We will call Q the *complex curvature operator* of g.

For any $1 \leq k \leq n$, we will say that (M^n, g) have *k-positive curvature*, or Q is k-positive, if $Q(\xi, \bar{\xi}) > 0$ for any $0 \neq \xi \in V \otimes \overline{V}$ with $\mathrm{rank}(\xi) \leq k$. Here the rank of ξ is defined to be the rank of the matrix $(\xi_{i\bar{j}})$ under any basis e of V. Similarly, one can define the notion of k-nonnegative, k-negative, or k-nonpositive.

So 1-positive means that g has positive bisectional curvature, and n-positive means that g has positive complex curvature operator. We point out that Kähler manifold (M^n, g) has 2-positive curvature means exactly that g has *positive complex sectional curvature*, namely, for any non-zero element $\sigma \in V \otimes \overline{V}$, it holds that $-R(\sigma, \overline{\sigma}) > 0$.

When g has 2-positive (2-nonnegative) curvature, its sectional curvature must be positive (or nonnegative), but the converse may not be true in general, when the complex dimension $n > 2$. (For $n = 2$ it is proved in [122] that the positivity of sectional curvature is equivalent to the positivity of the complex curvature operator).

For our metric in $\mathcal{M}_n$, however, we will show that the 2-positivity is always equivalent to the positivity of the sectional curvature of g. In other words, any metric in $\mathcal{N}_n$ will have positive complex sectional curvature. This result gives an intuitive explanation why we are able to prove that nonnegative sectional curvature is preserved under $U(n)$-invariant Kähler-Ricci flow $\mathbb{C}^n$ in Section 2.4.

Theorem 2.3.8. *Let g be a complete $U(n)$-invariant Kähler metric on $\mathbb{C}^n$. If g has positive (nonnegative) sectional curvature everywhere, then it will have 2-positive (2-nonnegative) curvature everywhere.*

Proof of Theorem 2.3.8. Let us use the notations discussed in the early part of this section. Let $g \in \mathcal{N}_n$, we have $D = AC - B^2 > 0$ everywhere. Fix a point $z = (z_1, 0, \ldots, 0)$. We want to show that $Q(\xi, \bar{\xi}) > 0$ for any $\xi \neq 0$ with rank at most 2. Under the unitary tangent frame $\{e_1, \ldots, e_n\}$ at z, the only non-zero components of the curvature tensor are $R_{i\bar{i}j\bar{j}}$. Denote it by P_{ij}, then $P_{11} = A$, $P_{1i} = B$, $P_{ii} = C$, and $P_{ij} = \frac{1}{2}C$ for any $2 \leq i \neq j \leq n$.

For convenience, let us choose a new frame $\tilde{e}_1 = \rho e_1$, $\tilde{e}_i = e_i$ for $2 \leq i \leq n$, where $\rho = \frac{\sqrt{C}}{\sqrt{2B}}$. Then under the new frame $\tilde{e}$, the only non-zero components of the curvature tensor are $\tilde{R}_{i\bar{i}j\bar{j}} = \tilde{P}_{ij}$, where

$$\tilde{P}_{ij} = \frac{C}{2}(1 + \delta_{ij}) + \delta_{i1}\delta_{j1}\frac{C}{2}(\frac{AC}{2B^2} - 2).$$

Now write $\xi = \sum_{i,j=1}^{n} \xi_{i\bar{j}} \tilde{e}_i \overline{\tilde{e}_j}$. We have

$$\begin{aligned} \frac{2}{C}Q(\xi, \bar{\xi}) &= \frac{2}{C}\sum_{i,j=1}^{n} \tilde{P}_{ij}\xi_{i\bar{i}}\overline{\xi_{j\bar{j}}} + \frac{2}{C}\sum_{i\neq j} \tilde{P}_{ij}|\xi_{i\bar{i}}|^2 \\ &= (\frac{AC}{2B^2} - 2)|\xi_{1\bar{1}}|^2 + |\sum_{i=1}^{n} \xi_{i\bar{i}}|^2 + \sum_{i,j=1}^{n} |\xi_{i\bar{j}}|^2 \end{aligned} \tag{2.99}$$

In particular, when $\xi_{1\bar{1}} = 0$ and $\xi \neq 0$, we have $Q(\xi, \bar{\xi}) > 0$. So scale ξ if necessary, we may assume from now on that $\xi_{1\bar{1}} = 1$. Notice that any unitary change on the subframe $\{\tilde{e}_2, \ldots, \tilde{e}_n\}$ will not affect the components of the curvature tensor, so we may assume that the lower right $(n-1) \times (n-1)$ block of the matrix $(\xi_{i\bar{j}})$ only have non-zero entries in its first two rows. In particular, only the first three elements on the diagonal line of $(\xi_{i\bar{j}})$ might be non-zero. Let us denote the upper left 3×3 corner of $(\xi_{i\bar{j}})$ by

$$E = \begin{pmatrix} 1 & v_1 & v_2 \\ u_1 & x & z \\ u_2 & 0 & y \end{pmatrix} \tag{2.100}$$

Notice that by performing a unitary change of $\{e_2, e_3\}$ if necessary, we could make the $(3,2)$-entry zero. Since the trace of the matrix $(\xi_{i\bar{j}})$ is $1+x+y$, we have

$$\begin{aligned} & \frac{2}{C}Q(\xi,\bar{\xi}) \\ \geq \;& (\frac{AC}{2B^2}-2)+1+|u|^2+|v|^2+|x|^2+|y|^2+|z|^2+|1+x+y|^2 \\ = \;& (\frac{AC}{2B^2}-2)+1+f \\ = \;& \frac{1}{2}(\frac{AC}{B^2}-1)-\frac{1}{2}+f \;>\; -\frac{1}{2}+f \end{aligned} \tag{2.101}$$

since $D = AC - B^2 > 0$. Here we wrote $|u|^2 = |u_1|^2 + |u_2|^2$ and $|v|^2 = |v_1|^2 + |v_2|^2$. Since the rank of ξ is at most 2, we have

$$\det E = xy + zu_2v_1 - xu_2v_2 - yu_1v_1 = 0. \tag{2.102}$$

Our goal is to show that $f \geq \frac{1}{2}$ for any $w = (u,v,x,y,z) \in \mathbb{C}^7$ with $\det E = 0$. Assume the contrary, namely, $\inf f = c < \frac{1}{2}$. Take a sequence of points $\{w_k\}$ in $V = \{\det E = 0\} \subset \mathbb{C}^7$ such that $f(w_k) \to a$. Since f dominates the square of the Euclidean distance of $\mathbb{C}^7$, $\{w_k\}$ is bounded, thus having a subsequence converging a point $w \in V$. We have $f(w) = c < \frac{1}{2}$. Let us fix such a point w, and we will derive a contradiction from this.

First notice that $x \neq 0$, since otherwise $f \geq |y|^2 + |1+y|^2 \geq \frac{1}{2}$. Similarly, $y \neq 0$. Next, notice that when $u_2 = 0$, we have $x = u_1v_1$, thus $|u|^2 + |v|^2 \geq 2|u_1v_1| = 2|x|$, which leads to

$$\begin{aligned} f \;&\geq\; 2|x| + |x|^2 + |y|^2 + |1+x+y|^2 \\ &\geq\; 2|x| + |x|^2 + \frac{1}{2}|1+x|^2 \;\geq\; \frac{1}{2}, \end{aligned} \tag{2.103}$$

contradicting with the assumption that $f(w) < \frac{1}{2}$. So we must have $u_2 \neq 0$, and similarly, $v_1 \neq 0$. Near the point w, V is the smooth hypersurface in $\mathbb{C}^7$ given by the graph of

$$z = x\frac{v_2}{v_1} + y\frac{u_1}{u_2} - \frac{xy}{u_2v_1}$$

and w is a local minimum point of the function f, now viewed as a function of (u,v,x,y). The first order derivatives of f are all zero at w, from these equations we get

$$v_1\bar{v}_2 = -x\bar{z}, \qquad u_2\bar{u}_1 = -y\bar{z} \tag{2.104}$$
$$|u_2|^2 - |v_2|^2 = |v_1|^2 - |u_1|^2 = |z|^2 \tag{2.105}$$
$$2|x|^2 + x(1+\bar{y}) + |z|^2 + |u_1|^2 = 0 \tag{2.106}$$
$$2|y|^2 + y(1+\bar{x}) + |z|^2 + |v_2|^2 = 0 \tag{2.107}$$

If $z = 0$, then by (2.104), we have $v_2 = u_1 = 0$, so $u_2 = v_1 = 0$ by (2.105), a contradiction. So $z \neq 0$ at w. So again by (2.104) we know that $u_1v_2 \neq 0$ at w as well. Let us write $\alpha = v_1\bar{v}_2$ and $\beta = u_2\bar{u}_1$. Then we have $x = -\frac{\alpha}{\bar{z}}$ and $y = -\frac{\beta}{\bar{z}}$. Plug them into (2.106) and (2.107) in the above, and write $a = |\frac{u_2}{v_2}|^2 > 1$ and $b = |\frac{v_1}{u_1}|^2 > 1$, note that (2.105) we get

$$z = (a+1)\bar{\alpha} + \bar{\beta} = \bar{\alpha} + (b+1)\bar{\beta}. \tag{2.108}$$

Thus $a\bar{\alpha} = b\bar{\beta}$, which is just $\alpha = \beta$ since (2.104). Hence $a = b$.

Let us write $|u_1|^2 = |v_2|^2 = \rho > 0$. Then we have $|u_2|^2 = |u_1|^2 = a\rho$, and $|z|^2 = (a-1)\rho > 0$. Since $z = (a+2)\bar{\alpha}$, we get $(a-1)\rho = (a+2)^2 a\rho^2$, so

$$\rho = \frac{a-1}{a(a+2)^2} > 0. \tag{2.109}$$

We also have $x = y = -\frac{1}{a+2}$. So at w, we have

$$f = 2\rho + 2a\rho + (a-1)\rho + \frac{2}{(a+2)^2} + (1 - \frac{2}{a+2})^2 = \frac{a^3 + 3a^2 - 1}{a\,(a+2)^2} \tag{2.110}$$

On the other hand, since w is on $V = \{\det E = 0\}$, we have

$$(a+2)\bar{\alpha} = -\frac{2}{a+2}(\frac{\bar{\alpha}}{a\rho}) - \frac{1}{(a+2)^2}\frac{1}{u_2v_1}$$

Therefore

$$[(a+2) + \frac{2}{a\rho(a+2)}] = -\frac{1}{(a+2)^2 a\rho v_2 u_2}$$

So $u_2v_2 = -|u_2v_2| = -\sqrt{a}\rho$, and we get

$$\rho = \frac{1}{\sqrt{a}(a+1)(a+2)}. \tag{2.111}$$

Compare (2.111) with (2.109), we get

$$a^2 - 1 = \sqrt{a}(a+2).$$

Since $a > 1$, we have

$$\begin{aligned} & 2(a^3 + 3a^2 - 1) - a(a+2)^2 \\ = \; & a(a^2-1) + 2(a^2-1) - 3a \\ = \; & \sqrt{a}(a+2)^2 - 3a \; > \; (a+2)^2 - 3a > 0, \end{aligned}$$

Thus $f(w) > \frac{1}{2}$ by (2.110), a contradiction. Thus we proved Theorem 2.3.8. $\square$

Remark 2.3.9. *In view of the proof of Theorem 2.1.1 and 2.3.8, it is possible that for any $U(n)$-invariant Kähler metric on $\mathbb{C}^n$ the necessary condition for k-positive complex curvature operator ($2 < k < n$) $\frac{k}{2(k-1)}AC - B^2 > 0$ is also sufficient.*

Also it is still open if one can construct an example of $U(n)$-invariant Kähler metric with positive sectional curvature but not positive curvature operator. The curvature characterization in Theorem 2.1.1 provides good evidence that such an example should exist.

2.4 U(n)-invariant Kähler-Ricci flow with non-negative curvature

The main result of this section is that various levels of nonnegative curvature and asymptotic volume ratio are both preserved along $U(n)$-invariant Kähler-Ricci flow on $\mathbb{C}^n$ with nonnegative curvature. Note that we do not assume any upper bounds on curvature along Kähler-Ricci flow. We also discuss the short existence problem of Kähler-Ricci flow with $U(n)$-symmetry.

2.4.1 Introduction

As we mentioned in Section 2.1, the Ricci flow has been a powerful tool to study the uniformization problem since Shi's work [102] and [103]. The best results obtained to date is the theorem of Chau and Tam [23] which solves the conjecture for manifolds with bounded curvature and Euclidean volume growth. All the important progress along the uniformization problem via the Ricci flow approach assume upper bounds of curvatures since they all rely on the long time existence theorem of Kähler-Ricci flow proved by Shi [102] and [103], which in turn is based on Shi's short time existence result of Ricci flow on complete manifolds with bounded curvatures [101].

So far, all the available examples of complete Kähler metrics with positive bisectional curvature are constructed on $\mathbb{C}^n$ with $U(n)$-symmetry. In particular it is known from [108] and our discussion in Section 2.3 that there are many examples of Kähler metrics with positive but unbounded curvature. In general, it is unknown whether one can prove an existence theorem of Kähler-Ricci flow on any complete Kähler manifold with positive holomorphic bisectional curvature (without assuming boundedness of curvatures). Such a result, if exists, could be very helpful to study uniformization problem for Kähler manifolds with positive but unbounded curvature. For the Riemannian Ricci flow, the recent exciting work of Cabezas-Rivas and Wilking [12] proved a short time existence theorem for any complete noncompact manifold with nonnegative complex sectional curvature.

Recall that in previous sections, we discussed $U(n)$-invariant examples of Kähler metrics with unbounded and positive bisectional curvature. These examples motivate us to consider the following question:

Question 2.4.1. *Starting with any complete $U(n)$-invariant Kähler metric on $\mathbb{C}^n$ with nonnegative holomorphic bisectional curvature, does there always exist a complete solution to the Kähler-Ricci flow with $U(n)$-symmetry?*

Recall that Chen [27] shows that the nonnegativity of the scalar curvature is preserved for any complete solution to the Ricci flow. Motivated by the new cut-off techniques developed in [27], we show that any complete Kähler-Ricci flow on $\mathbb{C}^n$ with $U(n)$-symmetry preserves various levels of nonnegative curvatures. In general it is known that Kähler-Ricci flow preserves nonnegativity of holomorphic bisectional curvature on compact manifolds ([5] and [73]) or complete manifolds with bounded curvature ([102]). The point here is that by assuming the $U(n)$-symmetry we do not require any upper bounds on curvatures.

Theorem 2.4.2. *Let $g(t)$ be a complete solution of Kähler-Ricci flow on $\mathbb{C}^n$ with $U(n)$-symmetry for $t \in [0,T]$, if the holomorphic bisectional curvature (sectional curvature, or complex curvature operator) of the initial metric $g(0)$ is nonnegative, so is that of $g(t)$ for any $t \in [0,T]$. Furthermore, if $g(0)$ has holomorphic bisectional curvature (sectional curvature, or complex curvature operator) positive somewhere, then $g(t)$ has positive bisectional curvature (sectional curvature, or complex curvature operator) on $\mathbb{C}^n \times (0,T]$.*

In fact our method shows that any nonnegative curvature conditions which lies between nonnegative sectional curvature and nonnegative complex curvature operator is always preserved by complete Kähler-Ricci flow on $\mathbb{C}^n$ with $U(n)$-symmetry. It is worth noting that in general, the nonnegative sectional curvature may not be preserved by Ricci flow on noncompact manifolds [82]. This is well explained by our Theorem2.3.8 in the previous section. Recall that it was shown in [93] that nonnegative complex sectional curvature is preserved along Ricci flow on closed manifolds. Such an invariance curvature condition under Ricci flow is useful in the proof of differentiable sphere theorem due to Brendle and Schoen [10]. Our Theorem 2.3.8 demonstrates a somewhat surprising rigidity phenomenon for $U(n)$-invariant Kähler metrics on $\mathbb{C}^n$ with positive curvature.

The asymptotic volume ratio (AVR) is another interesting quantity whose invariance under the Ricci flow is first studied by Hamilton [60]. There has been various generalizations of Hamilton's result in the context of Ricci flow and Kähler-Ricci flow, see [102], [32], [29], [88], [118], and [100].

Theorem 2.4.3. *Let $g(t)$ be a complete $U(n)$-invariant Kähler-Ricci flow on $\mathbb{C}^n \times [0,T]$ where $g(0)$ is of nonnegative holomorphic bisectional curvature, then* AVR *of $(\mathbb{C}^n, g(t))$ is constant on $[0,T]$. In fact, for any $t \in [0,T]$*

$$\lim_{s \to +\infty} \frac{V_t(B_t(O,s))}{V_0(B_0(O,s))} = 1,$$

where $B_t(O,s)$ denotes the geodesic ball of radius s for the metric $g(t)$ centered at the origin.

Of course it is desirable if one can answer Question 2.4.1 affirmatively. Kähler-Ricci flow equation on $\mathbb{C}^n$ with $U(n)$-symmetry can be reduced to a non-linear equation of fast diffusion type. In the case of $n = 1$ such equations has been studied extensively from the PDE perspective, we refer the readers to check the review paper [63] and reference therein. However, in higher dimensions it seems very hard to solve such equations directly due to the high nonliearities involved. In the last part of the paper we study the Riemannian Ricci flow constructed by Cabezas-Rivas and Wilking [12] with the initial metric being a $U(n)$-invariant Kähler metric with nonnegative complex sectional curvature on $\mathbb{C}^n$. Note that by Theorem **??** we only need to assume nonnegative sectional curvature. Recall that Cabezas-Rivas and Wilking constructed such a Ricci flow after obtaining some delicate estimates on curvature evolution of the Ricci flows emanating from a sequence of double covers which converges to the original manifold in the sense of Cheeger-Gromov. Such double covers, obtained by gluing two copies of geodesic balls in the original manifold with increasing radii after identifying the boundary and perturbing the inner region nearby, are topologically spheres. Moreover, they are endowed with metrics with nonnegative complex sectional curvature. Ricci flows on those closed manifolds instantaneously evolve their curvatures into positive complex sectional curvatures (p.6 in [12]), thus destroy the Kähler structures even when the initial metrics are Kähler on some open sets. Apriori it is not clear if one can get a complete Kähler-Ricci flow after taking limits on those closed Ricci flows. In Subsection 2.4.5, we are able to show Cabezas-Rivas and Wilking's Ricci flow is indeed a Kähler-Ricci flow under some extra technical assumptions. We believe that those assumptions could be improved by more refined analysis on the curvature evolution for the Ricci flow on those double covers. We plan to study the existence of complete Kähler-Ricci flow with positive curvature in a general context in future works.

Section 2.4 is organized as follows: We introduce $U(n)$-invariant Kähler-Ricci flow equation in Section 2.4.2, then we prove that any complete Kähler-Ricci flow on $\mathbb{C}^n$ with $U(n)$-symmetry preserves nonnegativity of various levels of curvature and the asymptotic volume ratio in Subsection 2.4.3 and Subsection 2.4.4. In Subsection 2.4.5, we discuss the Ricci flow constructed by Cabezas-Rivas and Wilking [12] with the initial metric being $U(n)$-invariant Kähler metrics with nonnegative sectional curvature on $\mathbb{C}^n$. We end up with an existence theorem for Kähler-Ricci flow with $U(n)$-symmetry under extra assumptions.

2.4.2 The $U(n)$-invariant Kähler-Ricci flow equation

Throughout this section we will follow the notation from the Section 1. Pick any metric in $\mathcal{M}_n$, at the point $(z_1, 0, \cdots, 0)$ on $\mathbb{C}^n$, under the orthonormal frame $\{e_1 = \frac{1}{\sqrt{h}}\partial_{z_1}, e_2 = \frac{1}{\sqrt{f}}\partial_{z_2}, \cdots, e_n = \frac{1}{\sqrt{f}}\partial_{z_n}\}$, the non-zero components of the Ricci curvature are:

$$Ric(e_1, \bar{e}_1) = A + (n-1)B;\ Ric(e_i, \bar{e}_i) = B + \frac{n}{2}C; \tag{2.112}$$

where $2 \leq i \leq n$.

Assume that there exists a complete solution to the Ricci flow with $U(n)$-symmetry with initial metric in $\mathcal{M}_n$, we have:

$$\begin{cases} \frac{\partial h(z,t)}{\partial t} = -(A + (n-1)B)h(z,t) \\ \frac{\partial f(z,t)}{\partial t} = -(B + \frac{n}{2}C)f(z,t) \end{cases} \tag{2.113}$$

Recall (2.5) and (2.6), we can simplify (2.113).

$$\begin{cases} \frac{\partial h(r,t)}{\partial t} = \frac{\partial}{\partial r}(\frac{r\frac{\partial h}{\partial r}}{h}) - (n-1)(\frac{h^2}{rf^2} - \frac{h(1-\xi)}{rf}); \\ \frac{\partial f(r,t)}{\partial t} = -\frac{\xi}{r} - \frac{n-1}{r} + (n-1)\frac{h}{rf}. \end{cases} \tag{2.114}$$

It is easy to see that the first equation in (2.114) can be derived from the second. To sum up, to get a complete Ricci flow with $U(n)$-symmetry it suffices to solve $f(r,t)$ on $[0, +\infty) \times [0, T)$ such that.

$$\begin{cases} \frac{\partial f(r,t)}{\partial t} = \frac{2f_r + rf_{rr}}{f + rf_r} + (n-1)\frac{f_r}{f}. \\ f(r,t) > 0, \quad f + rf_r > 0, \quad \int_0^{+\infty} \frac{\sqrt{f + rf_r}}{\sqrt{r}} dr = +\infty. \end{cases} \tag{2.115}$$

When $n = 1$, it reduces to the rotationally symmetric Ricci flow equation on $\mathbb{R}^2$. In general, the Ricci flow equation on $\mathbb{R}^2$ is related to a fast diffusion equation which was first studied by Wu [109] (see [63] and reference therein for further developments). In particular, the recent work [52] and [12] implies that there always exists a complete solution to the Ricci flow with long time existence starting from a nonnegatively curved $\mathbb{R}^2$ and its curvature becomes bounded instantaneously. In particular, Wu's result [109] implies that, assuming rotational symmetry, such a flow converges after modified by diffeomorphisms on $\mathbb{R}^2$. It was further proved in [109] that the limiting metric is Hamilton's cigar soliton if circumference at infinity of the initial metric is finite and it is the standard flat metric on $\mathbb{R}^2$ if the asymptotic volume ratio is positive.

In higher dimensions, Fan [46] studied the uniqueness and convergence of $U(n)$-invariant Kähler-Ricci flow on $\mathbb{C}^n$ with positive bisectional curvature. However, his result assumed upper bounds on curvatures and used the short existence theorem of Shi [103] and some earlier convergence results of Chau-Tam [22].

2.4.3 Nonnegativity of curvatures are preserved

Theorem 2.4.4. *Let $g(t)$ $t \in [0,T]$ be a complete solution of the Kähler-Ricci flow on $\mathbb{C}^n$ with $U(n)$-symmetry, if the holomorphic bisectional curvature of the initial metric $g(0)$ is nonnegative, so is that of $g(t)$ for any $t \in [0,T]$. Moreover, if $g(0)$ has positive holomorphic bisectional curvature somewhere, then $g(t)$ has positive holomorphic bisectional curvature on $\mathbb{C}^n \times (0,T]$.*

Proof of Theorem 2.4.4. We only need to prove that the nonnegativity of the bisectional curvature is preserved, since the strong maximum principle is a local result and its proof is standard (See [58] and p.193-195 of [36]). Given any complete $U(n)$-invariant Kähler-Ricci flow solution $g(t)$ on $\mathbb{C}^n$, we have a time-dependent orthonormal moving frame $\{e_1(t) = \frac{1}{\sqrt{h(t)}}\partial_{z_1}, e_2(t) = \frac{1}{\sqrt{f(t)}}\partial_{z_2}, \cdots, e_n(t) = \frac{1}{\sqrt{f(t)}}\partial_{z_n}\}$ at the point $z = (z_1, 0, \cdots, 0)$. Denote

$$\begin{aligned} A(z,t) &= Rm_{g(t)}(e_1(t), \bar{e}_1(t), e_1(t), \bar{e}_1(t)), \quad (2.116)\\ B(z,t) &= Rm_{g(t)}(e_1(t), \bar{e}_i(t), e_1(t), \bar{e}_i(t)),\\ C(z,t) &= Rm_{g(t)}(e_i(t), \bar{e}_i(t), e_i(t), \bar{e}_i(t)), \end{aligned}$$

where $2 \leq i \leq n$. We have the evolution equation for bisectional curvature tensor

$$\begin{cases} (\frac{\partial}{\partial t} - \Delta)A = A^2 + 2(n-1)B^2, \\ (\frac{\partial}{\partial t} - \Delta)B = -B^2 + AB + \frac{n}{2}BC, \\ (\frac{\partial}{\partial t} - \Delta)C = \frac{n}{2}C^2 + 2B^2. \end{cases} \tag{2.117}$$

Based on the following lemma, it suffices to show that $A(t) \geq 0$ is preserved along $g(t)$.

Lemma 2.4.5 (Theorem 2, p.525 in [108], see also Proposition 3.1 in [111])**.** *For any complete Kähler metric ω on $\mathbb{C}^n$ with $U(n)$-symmetry, the bisectional curvature of ω is nonnegative if and only if $A \geq 0$ everywhere.*

Chen [27] proved that the nonnegativity of the scalar curvature is always preserved along any complete solution to the Ricci flow (without assuming upper bounds on curvatures). The method is to apply the maximum principle for $u = \varphi(\frac{d_t(x,x_0)}{ar_0})R(x,t)$ where φ is a suitable cut-off function and $a > 0$ is a sufficiently large constant. Recall that the scalar curvatures evolves by $(\frac{\partial}{\partial t} - \Delta)R = 2|Ric|^2 \geq \frac{2}{n}R^2$, which is very similar to the evolution equation of A in (2.117). Therefore, we can prove that $A \geq 0$ is preserved by the same method in [27]. □

Recall that Theorem 2.1.1 gives a characterization of nonnegativity of various curvatures via A, B, and C. In particular, any complete $U(n)$-invariant Kähler metric with nonnegative bisectional curvature has nonnegative sectional curvature (nonnegative complex curvature operator) if and only if $D = AC - B^2 \geq 0$ ($D_n = \frac{n}{2(n-1)}AC - B^2 \geq 0$).

Theorem 2.4.6. *Let $g(t)$ be a complete solution of the Kähler-Ricci flow on $\mathbb{C}^n$ with $U(n)$-symmetry for $t \in [0,T]$. If the sectional curvature (complex curvature operator) of the initial metric $g(0)$ is nonnegative, so is that of $g(t)$ for any $t \in (0,T]$. Moreover, if $g(0)$ has sectional curvature (complex curvature operator) positive somewhere, then $g(t)$ has positive sectional curvature (complex curvature operator) on $\mathbb{C}^n \times (0,T]$.*

Proof of Theorem 2.4.6. First we will prove that the nonnegativity of sectional curvature is preserved. Suppose there is a point (z_0, t_0) where $0 < t_0 \leq T$ where the sectional curvature is negative along some real 2-plane, then $D(z_0,t_0)=AC-B^2<0$. By picking $r_0>0$ small enough we may assume that $Ric(z,t)\le\frac{n-1}{r_0^2}$ for any $z\in B_{t_0}(z_0,r_0)$ where $B_{t_0}(z_0,r_0)$ is with respect to $g(t_0)$.

$$\begin{aligned}&(\frac{\partial}{\partial t}-\Delta)(AC-B^2) \qquad (2.118)\\ =\;& [(\frac{\partial}{\partial t}-\Delta)A]C+[(\frac{\partial}{\partial t}-\Delta)C]A-2B[(\frac{\partial}{\partial t}-\Delta)B]\\ &-2\nabla A\cdot\nabla C+2|\nabla B|^2\\ =\;& A^2C+(n-2)B^2C+\frac{n}{2}C^2A+2B^3-2\nabla A\cdot\nabla C+2|\nabla B|^2.\end{aligned}$$

Let φ is a fixed smooth cut-off non-increasing function such that $\varphi=1$ on $(-\infty,1]$ and $\varphi=0$ on $[2,+\infty)$. Moreover,

$$-2<\varphi'\le 0,\quad |\varphi''|+\frac{(\varphi')^2}{\varphi}\le 32. \qquad (2.119)$$

Define $u(z,t)\doteqdot\varphi(\frac{d_t(z,z_0)}{ar_0})D(z,t)$, where $a>0$ will be a sufficiently large number.

$$\begin{aligned}&(\frac{\partial}{\partial t}-\Delta)u \qquad (2.120)\\ =\;& \varphi'\frac{1}{ar_0}\Big[(\frac{\partial}{\partial t}-\Delta)d_t\Big]D+\varphi\Big[(\frac{\partial}{\partial t}-\Delta)D\Big]-2\nabla\varphi\cdot\nabla D-\varphi''\frac{D}{(ar_0)^2}\end{aligned}$$

Denote $u_{min}(t)=min_{z\in\mathbb{C}^n}u(z,t)$, so $u_{min}(t_0)\le u(z_0,t_0)<0$. Assume that there exists (z_1,t_1) such that $u(z_1,t_1)=min_{t\in[0,T]}u_{min}(t)<0$. Now we compute the right hand side of (2.120) at the space-time point (z_1,t_1). For simplicity, let us call it $Q(z_1,t_1)$.

First of all, Lemma 8.3 from Perelman [95] implies:

$$(\frac{\partial}{\partial t}-\Delta)\ d_{t_1}(z,z_0)\ge-\frac{5(n-1)}{3r_0}, \qquad (2.121)$$

whenever $d_{t_1}(z,z_0)>r_0$.

The definition of (z_1,t_1) implies $\nabla u(z_1,t_1)=0$. Therefore $\nabla D=-\frac{\nabla\varphi}{\varphi}D$ and $\nabla A=\frac{1}{C}(\nabla D+2B\nabla B-A\nabla C)$.

It follows from (2.58), (2.59), and (2.60) and a straightforward calculation that

$$\nabla_s B=\frac{2x}{v}(A-2B),\quad \nabla_s C=\frac{2x}{v}(2B-C). \qquad (2.122)$$

See also Section in Appendix for a detailed proof of (2.122).

$$
\begin{aligned}
& Q(x_1,t_1) && (2.123)\\
\geq\ & \varphi\Big\{A^2C+(n-2)B^2C+\frac{n}{2}C^2A+2B^3-2\nabla A\cdot\nabla C+2|\nabla B|^2\Big\}\\
& -\frac{10(n-1)\varphi'}{3ar_0^2}D+\frac{2(\varphi')^2}{(ar_0)^2}D-\frac{\varphi''D}{(ar_0)^2}\\
=\ & \varphi\Big[A^2C+(n-2)B^2C+\frac{n}{2}C^2A+2B^3\Big]\\
& +\varphi\Big[-\frac{2}{C}\nabla D\cdot\nabla C-\frac{4B}{C}\nabla B\cdot\nabla C+\frac{2A}{C}|\nabla C|^2+2|\nabla B|^2\Big]\\
& -\frac{10(n-1)\varphi'}{3ar_0^2}D+\frac{2(\varphi')^2}{(ar_0)^2}D-\frac{\varphi''D}{(ar_0)^2}\\
\geq\ & \varphi\Big[A^2C+(n-2)B^2C+\frac{n}{2}C^2A+2B^3\Big]\\
& +\varphi\frac{4x^2}{v^2}\frac{2}{C}\Big[A^2C+AC^2+8B^3-6ABC\Big]\\
& -\frac{\varphi\prime}{\varphi}\frac{1}{ar_0}\frac{2x}{Cv}|2B-C|D-\frac{10(n-1)\varphi'}{3ar_0^2}D+\frac{2(\varphi')^2}{(ar_0)^2}D-\frac{\varphi''D}{(ar_0)^2}
\end{aligned}
$$

Claim 2.4.7. *At the point* (z_1,t_1)

$$A^2C+(n-2)B^2C+\frac{n}{2}C^2A+2B^3\geq[B^2-AC]^{\frac{3}{2}}=|D|^{\frac{3}{2}}, \tag{2.124}$$

$$A^2C+AC^2+8B^3-6ABC\geq 0 \tag{2.125}$$

Proof of Claim 2.4.7. Note that $B^2>AC$ at the point (z_1,t_1), (2.124) can be verifed by a straightforward calculation. (2.125) simply follows from the the arithmetic and geometric mean inequality. □

It follows from (2.123) that

$$\frac{d^-u_{min}(t)}{dt}|_{t=t_1}\geq|u|^{\frac{3}{2}}+[-\frac{\varphi'}{ar_0}C_1-\frac{\varphi'}{ar_0^2}C_2+\frac{(\varphi')^2C_3}{\varphi(ar_0)^2}+\frac{|\varphi''|}{(ar_0)^2}]u \tag{2.126}$$

where C_1, C_2 and C_3 are all constants depending only on the $g(t)$ restricted to a compact subset $\mathbb{C}^n\times[0,T]$.

On the other hand, the choice of the point (z_1,x_1) implies $\frac{d^-u_{min}(t)}{dt}\leq 0$. We conclude that $\sqrt{|u(x_1,t_1)|}\leq\frac{C_5}{ar_0}+\frac{C_6}{(ar_0)^2}$. Therefore, we have

$$D(x_0,t_0)\geq u(x_1,t_1)\geq-[\frac{C_5}{ar_0}+\frac{C_6}{(ar_0)^2}]^2. \tag{2.127}$$

Now let a goes to infinity, we get $D(z_0, t_0) \geq 0$, which contradicts to the choice of (z_0, t_0). Therefore, the first part of Theorem 2.4.6 is proved.

It remains to show that the condition $D_n = \frac{n}{2(n-1)}AC - B^2 \geq 0$ is preserved. Let us write $D_\lambda = \lambda AC - B^2$. We will next prove that, for any $\lambda \in [\frac{1}{2}, 1]$, the condition $D_\lambda \geq 0$ is preserved under the $U(n)$-invariant Kähler-Ricci flow. We show it by using a successive approximation on λ.

Follow the proof above for the preservation of the condition $D_1 \geq 0$, a similar computation shows:

$$\begin{aligned}
& (\frac{\partial}{\partial t} - \Delta)[\varphi(\frac{d_t(z, z_0)}{ar_0})D_\lambda(z, t)] \qquad (2.128) \\
\geq\ & \varphi\Big[\lambda A^2C + (2\lambda(n-1) - n)B^2C + \frac{\lambda n}{2}C^2A + 2B^3 - 2(1-\lambda)AB^2\Big] \\
& + \varphi\frac{4x^2}{v^2}\frac{2}{C}\Big[\lambda A^2C + AC^2 + 8B^3 - (2+4\lambda)ABC - (4-4\lambda)AB^2\Big] \\
& - \frac{\varphi\prime}{\varphi}\frac{1}{ar_0}\frac{2x}{Cv}|2B - C|D_\lambda - \frac{10(n-1)\varphi'}{3ar_0^2}D_\lambda + \frac{2(\varphi')^2}{(ar_0)^2}D_\lambda - \frac{\varphi'' D_\lambda}{(ar_0)^2}.
\end{aligned}$$

Denote

$$\begin{aligned}
I_1(\lambda) &= \lambda A^2C + (2\lambda(n-1) - n)B^2C + \frac{\lambda n}{2}C^2A + 2B^3 - 2(1-\lambda)AB^2 \\
I_2(\lambda) &= A^2C + AC^2 + 8B^3 - (2+4\lambda)ABC - (4-4\lambda)AB^2
\end{aligned}$$

The key observation is that:

Lemma 2.4.8. *There exists a decreasing sequence $\{\lambda_k\}$ with the following property:*

(1) $\lambda_0 = 1$ and $\frac{1}{2} < \lambda_k < 1$ for any $k > 1$.

(2) Assume $\lambda_{k+1}AC < B^2 \leq \lambda_k AC$ at the point (z_1, t_1), then:

$$I_1(\lambda_{k+1}) \geq [B^2 - \lambda_{k+1}AC]^{\frac{3}{2}} = |D_{\lambda_{k+1}}|^{\frac{3}{2}}, \qquad (2.129)$$

$$I_2(\lambda_{k+1}) \geq 0 \qquad (2.130)$$

Proof of Lemma 2.4.8. Denote $K \doteqdot \frac{AC}{B^2}$ where $\frac{1}{\lambda_k} \leq K < \frac{1}{\lambda_{k+1}}$.

$$\begin{aligned}
& I_2(\lambda_{k+1}) \qquad (2.131) \\
=\ & A^2C + AC^2 + 8B^3 - (2+4\lambda_{k+1})ABC - (4-4\lambda_{k+1})AB^2 \\
=\ & AB^2\Big\{\lambda_{k+1}K^2(\frac{B}{A})^2 + [8 - (2+4\lambda_{k+1})K]\frac{B}{A} + K - 4(1-\lambda_{k+1})\Big\}
\end{aligned}$$

To ensure that $I_2(\lambda_{k+1})$ is nonnegative, it suffices to have $\frac{1}{\lambda_k} \geq 4(1-\lambda_{k+1})$ and $\lambda_{k+1} \geq \frac{1}{2}$. This motivates us to define the sequence $\{\lambda_k\}$ recursively by $\lambda_{k+1} = 1 - \frac{1}{4\lambda_k}$ with $\lambda_1 = 1$. In fact it is easy to check that $\lambda_k = \frac{k+1}{2k}$.

$$\begin{aligned} & I_1(\lambda_{k+1}) \\ = \ & \lambda_{k+1}A^2C + (2\lambda_{k+1}(n-1)-n)B^2C + \frac{\lambda_{k+1}n}{2}C^2A + 2B^3 - 2(1-\lambda_{k+1})AB^2 \\ = \ & AB^2\Big\{\big[2\lambda_{k+1}(n-1) - n + \frac{\lambda n}{2}K\big]K(\frac{B}{A})^2 + 2\frac{B}{A} + \lambda_{k+1}K - 2(1-\lambda_{k+1})\Big\} \end{aligned} \tag{2.132}$$

and

$$[B^2 - \lambda_{k+1}AC]^{\frac{3}{2}} = (1-\lambda_{k+1}K)B^3. \tag{2.133}$$

It follows from (2.132) and (2.133) that $\frac{\lambda_{k+1}}{\lambda_k} \geq 2 - 2\lambda_{k+1} \geq 0$ and $2\lambda_{k+1}(n-1) - n + \frac{n}{2}\frac{\lambda_{k+1}}{\lambda_k} \geq 0$ will suffice to prove (2.129). It is straightforward to check that those two inequalities are satisfies if we let $\lambda_k = \frac{k+1}{2k}$. □

From Lemma 2.4.8 and the proof of the preservation of the condition $D_1 \geq 0$, we can argue inductively to show that $D_\lambda \geq 0$ is preserved for any $\frac{1}{2} < \lambda \leq 1$. Thus we have proved that $D_n = \frac{n}{2(n-1)}AC - B^2$ is preserved along the Ricci flow with $U(n)$-symmetry. To sum up, we have shown that any nonnegative curvature condition which lies between nonnegative sectional curvature and nonnegative complex curvature operator is preserved along any complete solution to Kähler-Ricci flow on $\mathbb{C}^n$ with $U(n)$-symmetry.

□

2.4.4 The asymptotic volume ratio is preserved

Asymptotic volume ratio (AVR), which measures the cone angle at infinity, is an important quantity in the study of Ricci flow. Hamilton [60] proved that AVR is constant on any complete Ricci flow with bounded nonnegative Ricci curvature if the Riemannian curvature decays pointwisely to zero along infinity on each time slice. In the case of Kähler-Ricci flow, Shi [102] proved that the maximal volume growth is preserved for a complete Kähler-Ricci flow with bounded bisectional curvature in the space time and with the scalar curvature of the initial metric having average quadratic decay. Shi's result was improved by weaker assumptions on the scalar curvature decay in more recent works, see Chen-Zhu [31], Chen-Tang-Zhu [29] and Ni-Tam [88] for exmaple. There are also similar results for the Ricci flow on Riemannian manifolds with nonnegative bounded curvature operator (Yokota [118] and Schulze-Simon [100]). In particular, Ni-Tam [88] proved that the order of volume growth of geodesics balls in each time slice keeps constant on complete Kähler-Ricci flow with bounded nonnegative bisectional curvature in space-time. Our main result in this subsection is to remove the upper bound of curvature with the help of $U(n)$-symmetry.

Theorem 2.4.9. *Let $g(t)$, $t \in [0,T]$, be a complete solution to the Kähler-Ricci flow on $\mathbb{C}^n$ with $U(n)$-symmetry. Assume that the bisectional curvature of $g(t)$ is nonnegative for any $t \in [0,T]$. Then for any $t \in [0,T]$*

$$\lim_{s\to+\infty} \frac{V_t(B_t(O,s))}{V_0(B_0(O,s))} = 1 \tag{2.134}$$

where $B_t(O,s)$ denotes the geodesic ball of radius s for the metric $g(t)$ centered at the origin. In particular, the AVR *of $(\mathbb{C}^n, g(t))$ is constant for any $t \in [0,T]$.*

Recall that for any fixed metric g_0 in $\mathcal{M}_n$, at the point $z = (z_1, 0, \cdots, 0)$ on $\mathbb{C}^n$, under the orthonormal frame $\{e_1 = \frac{1}{\sqrt{h}}\partial_{z_1}, e_2 = \frac{1}{\sqrt{f}}\partial_{z_2}, \cdots, e_n = \frac{1}{\sqrt{f}}\partial_{z_n}\}$:

$$Ric(e_1, \bar{e}_1) = A + (n-1)B, \tag{2.135}$$

and the distance between the origin and a point z is given by

$$s = \int_0^r \frac{\sqrt{h}}{2\sqrt{r}} dr \tag{2.136}$$

where $r = |z|^2$. The following lemma gives an integral bound for the Ricci curvature along the radial geodesic. It further implies a lower bound on the radial distance function along $U(n)$-invariant Kähler-Ricci flow.

Lemma 2.4.10. *There exists a constant C_0 which only depends on g_0 restricted on the fixed coordinate ball such that*

$$\int_0^s Ric(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}) ds \leq C_0. \tag{2.137}$$

Proof of Lemma 2.4.10.

$$\int_0^s Ric(e_1, \bar{e}_1)ds = \int_0^r \frac{[A+(n-1)B]\sqrt{h}}{2\sqrt{r}}dr \tag{2.138}$$

Recall (2.58),(2.59), and (2.60) and let $a = \sqrt{h(1)} > 0$.

$$\begin{aligned} \int_1^r \frac{A\sqrt{h}}{2\sqrt{r}}dr &= \int_a^x \frac{F'F''}{2\tau[1+F'(\tau)^2]^{\frac{3}{2}}}d\tau \\ &= \int_a^x \frac{1}{2\tau}(\frac{-1}{\sqrt{1+F'(\tau)^2}})'d\tau \\ &\leq \frac{1}{2a} - \frac{1}{2x\sqrt{1+F'(x)^2}} - \int_a^x \frac{1}{2\tau^2\sqrt{1+F'(\tau)^2}}d\tau \\ &\leq \frac{1}{2a} \end{aligned} \tag{2.139}$$

$$\begin{aligned} &\int_1^r \frac{B\sqrt{h}}{2\sqrt{r}}dr \\ = &\int_a^x \frac{1}{v^2}(\tau^2 - \frac{v}{\sqrt{1+F'(\tau)^2}})\sqrt{1+F'(\tau)^2}d\tau \\ \leq &\frac{1}{2}\frac{a}{v(a)} + \frac{1}{2}\int_a^x \frac{1}{v}[-1 + \frac{v}{\tau^2\sqrt{1+F'^2}} - \frac{v}{\tau}(\frac{1}{\sqrt{1+F'^2}})']d\tau \\ \leq &\frac{1}{2}\frac{a}{v(a)} + \frac{1}{2}(\int_a^x \frac{2}{\tau^2}d\tau + \frac{1}{a}) \\ \leq &\frac{1}{2}\frac{a}{v(a)} + \frac{3}{2a}. \end{aligned} \tag{2.140}$$

□

As in the work of Shi [103], we define

$$F(x,t) = \log(\frac{\det g(x,t)}{\det g(x,0)}). \tag{2.141}$$

and Let Δ_t and dV_t denote the Laplacian operator and the volume element with respect to $g(t)$.

Lemma 2.4.11. *Let $g(t)$, $t \in [0,T]$, be a complete solution to the Kähler-Ricci flow on $\mathbb{C}^n$ with $U(n)$-symmetry and nonnegative bisectional curvature. Then for* *any fixed* $t \in (0,T]$, *there exists a constant* C_0 *which only depends on* n *and* $g(t)$ *restricted to a compact set of* $\mathbb{C}^n \times [0,t]$, *such that:*

$$\fint_{B_0(O,s)} (1-e^{F(x,t)})dV_0 \le \frac{C_0 t}{s^2}\Big[\int_0^{10s} [s\fint_{B_0(O,s)} R(x,0)dV_0]ds + s\Big]. \tag{2.142}$$

Proof of Lemma 2.4.11. It will follows from a slight modification of the method in Theorem 2.2 on p.132 of Ni-Tam [88]. In particular we need their estimate (2.148) whose proof is also included for the convenience of the reader. In [88] the result was stated under the assumption that $g(t)$ has bounded curvature along $[0,T]$, here we are able to remove the assumption on curvature bounds with aid of the $U(n)$-symmetry.

First we have the following inequality from Shi [103]:

$$\Delta_0 F(x,t) \le R(x,0) + e^F \frac{\partial F(x,t)}{\partial t}. \tag{2.143}$$

Let $G_s(x,y)$ be the positive Green's function on $B_0(O,s)$ with zero boundary value. Integrate (2.143) over $B_0(O,s)\times[0,t]$, we get for $B_0 = B_0(O,s)$ that:

$$\begin{aligned} &\int_{B_0} G_s(O,x)(1-e^{F(x,t)})dV_0 \qquad (2.144)\\ \le\ & t\int_{B_0} G_s(O,x)R(x,0)dV_0 - \int_0^t \int_{B_0} G_s(O,x)\Delta_0 F(x,\tau)dV_0\, d\tau \\ =\ & t\int_{B_0} G_s(O,x)R(x,0)dV_0 + \int_0^t \Big[F(O,\tau) + \int_{\partial B_0} F(x,\tau)\frac{\partial G_s(O,x)}{\partial s}dA_0\Big]d\tau \\ \le\ & t\Big[\int_{B_0} G_s(O,x)R(x,0)dV_0 - F(x,t)|_{\partial B_0}\Big], \end{aligned}$$

where dA_0 is the area element for ∂B_0.

Note that in (2.144) we used the Green's formula and the facts that $F(x,t) \le 0$ is non-increasing and $\int_{\partial B_0(O,s)} \frac{\partial G_s(O,x)}{\partial s} dA_0 = -1$.

As in [103], [31] and [88], considering $\mathbb{C}^n \times \mathbb{C}^2$, we may assume that $(\mathbb{C}^n, g(0))$ admits a minimal positive Green's function such that

$$\alpha \frac{d^2(x,y)}{V(x,d(x,y))} \le G(x,y) \le \frac{1}{\alpha}\frac{d^2(x,y)}{V(x,d(x,y))}. \tag{2.145}$$

for some $\alpha > 0$ which depends only on n.

Then we can apply the mean value inequality in [87] (see the proof of Theorem 1.1 and Theorem 2.1 on p.345-348 of [87]) to conclude that

$$\begin{aligned}\int_{B_0(O,s)} G_s(O,x)R(x,0)dV_0 &\leq \int_{B_0(O,s)} G(O,x)R(x,0)dV_0 \\ &\leq c(n,\alpha)\int_0^{2s}\Big[s\fint_{B_0(O,s)} R(x,0)dV_0\Big]d\tau\end{aligned} \tag{2.146}$$

$$s^2\fint_{B_0(O,s)}(1-e^{F(x,t)})dV_0 \leq c(n,\alpha)\int_{B_0(O,5s)} G_s(O,x)(1-e^{F(x,t)})dV_0 \tag{2.147}$$

Now (2.144) becomes

$$s^2\fint_{B_0(O,s)}(1-e^{F(x,t)})dV_0 \leq c(n)\,t\Big[\int_0^{10s}[s\fint_{B_0(O,s)} R(x,t)dV_0]ds - F(x,t)\Big]. \tag{2.148}$$

We remark that similar estimates as (2.148) have been used in [102], [31], [29].

The proof of Lemma 2.4.11 will be done if we can prove

$$-F(x,t)|_{\partial B_0(0,s)} \leq C_0 s \tag{2.149}$$

for a constant C_0 which depends only on $g(t)$ restricted to a compact set in $\mathbb{C}^n$.

Since $(\mathbb{C}^n, g(t))$ has $U(n)$-symmetry and positive curvature, we have for any $r \geq 1$

$$s = \int_0^r \frac{1}{2}\sqrt{\frac{h(r)}{r}}dt \geq \frac{\sqrt{h(1)}}{2}\log r \tag{2.150}$$

Recall the definition of F in (2.141),

$$\begin{aligned} F &= \log h(x,t) + (n-1)\log f(x,t) \\ &\quad -\log h(x,0) - (n-1)\log f(x,0) \end{aligned} \tag{2.151}$$

$$\begin{aligned} -rF_r &\leq \frac{-rh_r(x,t)}{h(x,t)} + (n-1)\frac{-rf_r(x,t)}{f(x,t)} \\ &\leq \xi(+\infty,t) + \frac{f(r,t)-h(r,t)}{f(r,t)} \leq n \end{aligned} \tag{2.152}$$

Now from (2.152) and (2.150) we have

$$-F(r,t) \leq n\log r - F(1,t) \leq C_0 s, \tag{2.153}$$

which completes the proof of Lemma 2.4.11. □

Proof of Theorem 2.4.9. Theorem 2.4.9 can be proved by arguing similarly as the proof of Theorem 2.2 in [88]. First note that Lemma 2.4.10 and the volume comparison theorem will imply that the AVR of $g(t)$ is non-increasing on $[0,T]$.

As in [31] and [88], we have

$$V_t(B_t(O,s)) \geq \int_{B_0(O,s)} dV_t \geq V_0(B_t(O,s)) - \int_{B_0(O,s)} (1-e^{F(x,t)})dV_0. \tag{2.154}$$

Recall that we have decay estimates of the average scalar curvature for $U(n)$-invariant Kähler metrics on $\mathbb{C}^n$ with nonnegative curvature, see Theorem 7 in [108] or Proposition 3.3 in [111]). We conclude from Lemma 2.4.11 that the AVR of $g(t)$ is non-decreasing on $[0,T]$. Hence it must be constant for any $t \in [0,T]$. In fact, the proof shows that the precise order of volume growth of $(\mathbb{C}^n, g(t))$ is constant on $[0,T]$, i.e. (2.134) must hold. □

2.4.5 Discussions on the existence of $U(n)$-invariant Kähler-Ricci flow

The theorem of Cabezas-Rivas and Wilking.

Let us recall the following result by Cabezas-Rivas and Wilking [12]:

Theorem 2.4.12 (Cabezas-Rivas and Wilking [12])**.** *Let (M^n, g) be an open manifold with nonnegative complex sectional curvature. Then there exists a constant $T>0$ which depends on n and g such that there exists a complete solution of Ricci flow $g(t)$ with nonnegative complex sectional curvature starting from g on $[0,T]$. In addition, if we assume*

$$\inf\{Vol_g B_g(p,1) : p \in M\} = v_0 > 0, \tag{2.155}$$

then T can be chosen as $T(n,v_0)$ and the scalar curvature of $(M, g(t))$ is bounded by $\frac{c(n,v_0)}{t}$ on $(0, T(n,v_0)]$.

Ricci flow on double covers with rotational symmetry

Let us consider $(\mathbb{C}^n, g_0)$, where g_0 is a complete $U(n)$-invariant Kähler metric with nonnegative complex sectional curvature. In fact by Theorem 2.3.8 we only need to assume that g_0 has nonnegative sectional curvature. Let $B(O,i)$ denote the geodesic ball with radius i centered at the origin. The double cover D_i is a closed manifold obtained by gluing two copies of $B(O,i)$ after identifying the boundary and perturbing the inner region nearby. To be more precise, define a smooth function ϕ_i: $(-\infty,i)\to\mathbb{R}$ with $\phi_i=0$ on $(-\infty,i-\epsilon]$ and $\phi_i(i)=1$, ϕ_i', $\phi_i''>0$ on $(i-\epsilon,i)$, and ϕ_i is left continuous at i and left derivatives of its inverse vanishes at 1. Then define $F_i : D_i\to\mathbb{C}^n\times\mathbb{R}$ by $F_i(z)=(z,\phi_i(s(z)))$ on one copy of $B(O,i)$ and $F_i(z)=(z,2-\phi_i(s(z)))$ on the other. It can be checked this will realize D_i as a closed smooth hypersurface in $\mathbb{C}^n\times\mathbb{R}$. The induced metric of D_i from the product metric of $\mathbb{C}^n\times\mathbb{R}$, denoted by g_i, has nonnegative complex sectional curvature. It follows from Proposition 4.1 in [12] that (D_i,g_i,O) converges to $(\mathbb{C}^n,g_0,O)$ smoothly in the sense of Cheeger-Gromove convergence (see Cabezas-Rivas and Wilking [12] p.7-8 for the details of the results mentioned above.)

The general Theorem 2.4.12 is proved by Cabezas-Rivas and Wilking by establishing some delicate curvature estimates on the closed Ricci flow evolved from those double covers. In this section, under the extra assumption of the rotational symmetry of $(\mathbb{C}^n,g_0)$, we will compute the curvatures of (D_i,g_i) and its evolution under the Ricci flow more explicitly.

Fix a point $p=(x_1,\cdots,x_{2n})$ on (D_i,g_i) where $x_k=0$ for $2\le k\le 2n$. Define $r=\sum_{k=1}^{2n}|x_k|^2$ and the function $s(r)=\int_0^r\frac{\sqrt{h(r)}}{2\sqrt{r}}dr$. Note that $s(r)$ is the distance function only inside the ball $B(O,i-\epsilon)$. Under the local coordinates $\{x_1,\cdots,x_{2n}\}$, the metric (D_i,g_i) has components:

$$
\begin{aligned}
&g_i(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_1})=h(2+(\phi_i'(s))^2),\ \ g_i(\frac{\partial}{\partial x_{n+1}},\frac{\partial}{\partial x_{n+1}})=2h. \qquad (2.156)\\
&g_i(\frac{\partial}{\partial x_k},\frac{\partial}{\partial x_k})=g_i(\frac{\partial}{\partial x_{n+k}},\frac{\partial}{\partial x_{n+k}})=2f.\quad (2\le k\le n)\\
&g_i(\frac{\partial}{\partial x_p},\frac{\partial}{\partial x_q})=0\ \ (p\ne q).
\end{aligned}
$$

At the same point we calculate the nonzero components of the second fundamental form of (D_i,g_i) with respect to $\mathbb{C}^n\times\mathbb{R}$. Note that we abuse the notations again by denoting $\frac{\partial}{\partial x_k}$ for $dF_i(\frac{\partial}{\partial x_k})$.

$$\Pi(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_1})=\frac{\sqrt{2}\phi_i''(s)}{\sqrt{2+(\phi_i')^2}}h \tag{2.157}$$
$$\Pi(\frac{\partial}{\partial x_{n+1}},\frac{\partial}{\partial x_{n+1}})=\frac{\sqrt{2}\phi_i'(s)}{\sqrt{2+(\phi_i')^2}}(\sqrt{\frac{h}{r}}-\sqrt{\frac{r}{h}}h_r),$$
$$\Pi(\frac{\partial}{\partial x_k},\frac{\partial}{\partial x_k})=\Pi(\frac{\partial}{\partial x_{n+k}},\frac{\partial}{\partial x_{n+k}})=\frac{\sqrt{2}\phi_i'(s)}{\sqrt{2+(\phi_i')^2}}\sqrt{\frac{h}{r}}.$$

The Gauss equation gives the curvature of (D_i, g_i) under the complexified local coordinates $z_k = x_k + \sqrt{-1}x_{n+k}$. We should caution here that these z_k are only holomorphic inside the ball $B(O, i-\epsilon) \subset D_i$. We list all the nonzero curvature components below.

$$R_{1\bar{1}1\bar{1}} = h^2A + \Pi_{1\bar{1}}^2 - |\Pi_{11}|^2, \quad R_{1\bar{1}k\bar{k}} = hfB, \tag{2.158}$$
$$R_{k\bar{k}k\bar{k}} = f^2C + \Pi_{k\bar{k}}^2, \quad R_{k\bar{k}l\bar{l}} = \frac{1}{2}f^2C,$$
$$R_{\bar{k}\bar{1}1k} = \Pi_{k\bar{k}}\Pi_{1\bar{1}}, \quad R_{\bar{k}11k} = \Pi_{k\bar{k}}\Pi_{11}.$$

Under the orthonormal frame $e_1 = \frac{1}{\sqrt{h(2+(\phi_i'(s))^2)}}\partial_{x_1},\ e_2 = \frac{1}{\sqrt{2f}}\partial_{x_2}, \cdots,$ $e_{n+1} = \frac{1}{\sqrt{2h}}\partial_{z_{n+1}}, \cdots,\ e_{n+2} = \frac{1}{\sqrt{2f}}\partial_{x_{n+2}},\ \cdots,\ e_{2n} = \frac{1}{\sqrt{2f}}\partial_{x_{2n}}$ and its corresponding complexification $(\omega_1, \cdots, \omega_n)$ with $\omega_k = \frac{1}{\sqrt{2}}(e_k - \sqrt{-1}e_{n+k})$, it is easy to see that

$$g_i(\omega_k, \bar{\omega}_l) = \delta_{kl}, \quad g_i(\omega_k, \omega_l) = 0 \tag{2.159}$$

It is easy to derive the formula for the curvature components of (D_i, g_i) under the orthonormal frame ω_k. It has similar type of nonzero components as that under the coordinates z_k as in (2.158).

To simplify the computation of the curvature evolution of the Ricci flow on the double cover, we use the Uhlenbeck trick to evolve the above complexified orthonormal frame ω_k to get a time-dependent orthonormal frame $(\omega_1(t), \cdots, \omega_n(t))$ with the property that (2.159) holds for any t.

Lemma 2.4.13. *The complete Ricci flow $(\mathbb{C}^n, g(t))$ constructed in Theorem 2.4.12 is a Kähler-Ricci flow on $\mathbb{C}^n$ with $U(n)$-symmetry if $R_{AB\gamma\delta}(g(t)) = 0$ everywhere under the time-dependent orthonormal frame $(\omega_1(t), \cdots, \omega_n(t))$. Here A, B are any indices either barred or unbarred while γ and δ are unbarred from 1 to n.*

Proof of Lemma 2.4.13. The fact that $g(t)$ being Kähler is proved by Shi (See p.138-p.142 [103]). Here the $U(n)$-symmetry follows from the rotational symmetry of $(\mathbb{C}^n, g_0)$ and $(D_i, g_i(t))$. □

Conditions that ensure the existence of Kähler-Ricci flow

From now on we focus our discussion in the complex dimension $n = 2$, we emphasize that it is purely for the sake of convenience and all results mentioned below can be generalized to higher dimensions.

The following two results study the curvature evolution of the closed Ricci flow $(D_i, g_i(t))$ under the time-dependent orthnormal frame $(\omega_1(t), \omega_2(t))$. Note that the only non-vanishing curvature components of the initial metric $(D_i, g_i(0))$ are $R_{1\bar{1}1\bar{1}}, R_{1\bar{1}2\bar{2}}, R_{2\bar{2}2\bar{2}}, R_{\bar{2}112}$, and $R_{\bar{2}\bar{1}12}$.

Lemma 2.4.14. *The curvatures of $(D_i, g_i(t))$ satisfy:*

$$R_{1\bar{1}12} = R_{1\bar{1}2\bar{1}} = R_{2\bar{2}1\bar{2}} = R_{2\bar{2}21} = 0 \tag{2.160}$$

Proof of Lemma 2.4.14. We have the following curvature evolution equations via the Uhlenbeck trick.

$$\begin{aligned}(\frac{\partial}{\partial t} - \Delta)R_{1\bar{1}21} &= -R_{1\bar{1}EG}R_{\bar{E}\bar{G}21} + 2R_{E1G1}R_{\bar{E}\bar{1}\bar{G}2} - R_{E1G2}R_{\bar{E}\bar{1}\bar{G}1} \qquad (2.161)\\ &= Q_1(R_{1\bar{1}21}, R_{1\bar{1}1\bar{2}}, R_{1\bar{2}21}).\end{aligned}$$

Similarly:

$$(\frac{\partial}{\partial t} - \Delta)R_{1\bar{1}1\bar{2}} = Q_2(R_{1\bar{1}21}, R_{1\bar{1}1\bar{2}}, R_{2\bar{2}1\bar{2}}, R_{2\bar{2}12}) \tag{2.162}$$

$$(\frac{\partial}{\partial t} - \Delta)R_{2\bar{2}12} = Q_3(R_{1\bar{1}21}, R_{1\bar{1}1\bar{2}}, R_{2\bar{2}1\bar{2}}, R_{2\bar{2}12}) \tag{2.163}$$

$$(\frac{\partial}{\partial t} - \Delta)R_{2\bar{2}2\bar{1}} = Q_4(R_{1\bar{1}21}, R_{1\bar{1}1\bar{2}}, R_{2\bar{2}1\bar{2}}, R_{2\bar{2}12}) \tag{2.164}$$

Notice that $R_{1\bar{1}12} = R_{1\bar{1}2\bar{1}} = R_{2\bar{2}1\bar{2}} = R_{2\bar{2}21} = 0$ holds at $t = 0$, so the lemma follows from the maximum principle for solutions to parabolic equations on closed manifolds. □

Similarly we can prove the following lemma.

Lemma 2.4.15. *The curvatures of* $(D_i, g_i(t))$ *satisfy:*

$$R_{1221} = R_{2\bar{1}21} = R_{2\bar{1}2\bar{1}} = 0 \tag{2.165}$$

After passing the time-dependent orthnormal frame $(\omega_1(t), \omega_2(t))$ defined on $(D_i, g_i(t))$ to the limit Ricci flow $(\mathbb{C}^2, g(t))$, we have a time-dependent orthnormal frame on $(\mathbb{C}^2, g(t))$. Let $(\omega_1(t), \omega_2(t))$ still denote the frame $(\mathbb{C}^2, g(t))$ for simplicity, the above lemmas show that all curvatures components of the type $R_{AB\gamma\delta}(g(t))$ of $(\mathbb{C}^2, g(t))$ vanish except $R_{\bar{2}112}$ and $R_{\bar{2}\bar{1}12}$.

Theorem 2.4.16. *Assume* $(\mathbb{C}^2, g_0)$ *is a complete* $U(2)$*-invariant Kähler manifold with nonnegative sectional curvature. Moreover, assume that the condition (2.155) holds.(e.g. (2.155) is satisfied when* $(\mathbb{C}^2, g_0)$ *has Euclidean volume growth.) Then Cabezas-Rivas and Wilking's Ricci flow in Theorem 2.4.12 satisfies either*

$$R_{\bar{2}112} = R_{\bar{2}\bar{1}12} = 0 \tag{2.166}$$

everywhere on $\mathbb{C}^2 \times [0, T(2, v_0)]$ *or* $P(x,t) > 0$ *everywhere and*

$$\liminf_{r_0 \to +\infty, t_0 \to 0} \frac{\inf_{B_{t_0}(O, r_0)} P(x, t_0)}{\sup_{B_t(O,r_0)\times(0,t_0]} P(x,t)} = 0 \tag{2.167}$$

where $P(x,t) \doteq \sqrt{|R_{\bar{2}112}|^2 + |R_{\bar{2}\bar{1}12}|^2}$.

Proof of Theorem 2.4.16. The proof is motivated by a result of Chen (Theorem 3.1 on p.371 [27]), where an important interior estimate for the Ricci flow was proved. See also Simon [104] for related works.

First note that $R_{\bar{2}\bar{1}12} \geq 0$ since $(\mathbb{C}^2, g(t))$ has nonnegative complex sectional curvature. If $R_{\bar{2}\bar{1}12} = 0$ at one point (x,t), then by the strong maximum principle it vanishes everywhere, which further implies $R_{\bar{2}112} = 0$ from the evolution equation of $R_{\bar{2}\bar{1}12}$. Now assume that $P(x,t) > 0$ everywhere and there exists a constant $\epsilon_0 > 0$ such that

$$\liminf_{r_0 \to +\infty, t_0 \to 0} \frac{\inf_{B_{t_0}(O, r_0)} P(x, t_0)}{\sup_{B_t(O,r_0)\times(0,t_0]} P(x,t)} \geq \epsilon_0 \tag{2.168}$$

We will show that there exists a constant C independent of r_0 such that

$$P(x,t) \leq \frac{C}{r_0^2} \tag{2.169}$$

whenever $0 < t \leq T(n, v_0)$ and $d_t(x, O) \leq \frac{r_0}{2}$.

Assume that (2.169) is not true. Then there will exist a sequence $\{r_n\}$ tending to infinity and a sequence of points (x_n, t_n) with $P(x_n, t_n) \geq \frac{4^n}{r_n^2}$ where $t_n \to 0$ and $d_{t_n}(x_n, O) \leq \frac{r_n}{2}$.

It follows from a point-picking technique of Perelman [95] (see also p.372 of [27]) that for any fixed $B > 0$, picking n large enough if necessary, we can choose $(\bar{x}_n, \bar{t}_n)$ such that $0 < \bar{t}_n \leq t_n$, $d_{\bar{t}_n}(\bar{x}_n, O) \leq \frac{3}{4} r_n$, and $\bar{Q}_n \doteq R(\bar{x}_n, \bar{t}_n) \geq \frac{4^n}{r_n^2}$, Moreover,

$$0 \leq R(x,t) \leq 4\bar{Q}_n \tag{2.170}$$

wherever $0 < \bar{t}_n \leq t_n$ and $d_t(x, O) \leq d_{\bar{t}_n}(\bar{x}_n, O) + B\bar{Q}_n^{-\frac{1}{2}}$.

As in p.372 of [27], construct a smooth function $\psi : \mathbb{R} \to [0,1]$ such that $\psi = 1$ on $(-\infty, d_{\bar{t}_n}(\bar{x}_n, O) + \frac{B}{2}\bar{Q}_n^{-\frac{1}{2}}]$ and $\psi = 0$ on $[d_{\bar{t}_n}(\bar{x}_n, O) + B\bar{Q}_n^{-\frac{1}{2}}, +\infty)$, Moreover, one can assume that $|\psi'| \leq 4\frac{\bar{Q}_n^{\frac{1}{2}}}{B}$ and $|\psi''| + \frac{|\psi'|^2}{\psi} \leq 32\frac{\bar{Q}_n}{B^2}$.

Denote by $u(x,t) = \psi(d_t(x,O))(|R_{\bar{2}112}|^2 + |R_{\bar{2}\bar{1}12}|^2)$. We have

$$\begin{aligned}
& (\frac{\partial}{\partial t} - \Delta)u \qquad (2.171)\\
= \ & \Big\{\psi'[(\frac{\partial}{\partial t} - \Delta)d_t] + (\frac{|\psi\prime|^2}{\psi} - \psi'')\Big\}(|R_{\bar{2}112}|^2 + |R_{\bar{2}\bar{1}12}|^2)\\
& +2\psi[|R_{\bar{2}112}|^2(2R_{\bar{2}\bar{1}12} + R_{1\bar{1}2\bar{2}} - R_{1\bar{1}2\bar{2}} + R_{2\bar{2}2\bar{2}})]\\
& +2\psi[{R_{\bar{2}\bar{1}12}}^3 + R_{\bar{2}\bar{1}12}|R_{\bar{2}112}|^2 + {R_{\bar{2}\bar{1}12}}^2(2R_{1\bar{1}2\bar{2}} + R_{1\bar{1}2\bar{2}} + R_{2\bar{2}2\bar{2}})]\\
& -2\psi(|\nabla R_{\bar{2}\bar{1}12}|^2 + |\nabla R_{\bar{2}\bar{1}12}|^2)
\end{aligned}$$

$$\frac{d^+ u_{max}(t)}{dt} \leq (16\frac{\bar{Q}_n}{B} + 32\frac{\bar{Q}_n}{B^2})P(x,t) + 32u_{max}(t)\bar{Q}_n. \tag{2.172}$$

In the above we used, by (2.170), that $(\frac{\partial}{\partial t} - \Delta)d_t \geq -4\bar{Q}_n^{\frac{1}{2}}$ when $d_t(x,O) \geq \bar{Q}_n^{\frac{1}{2}}$, from Lemma 8.3 on p.20 of [95].

It follows from (2.172) that:

$$\frac{d^+}{dt}(e^{-32\bar{Q}_n t}u_{max}(t)) \leq 64\frac{\bar{Q}_n}{B}e^{-32\bar{Q}_n} \sup_{B_t(O,r_0)\times(0,\bar{t}_n]} P(x,t) \tag{2.173}$$

where we use $d(x,t) \leq d_{\bar{t}_n}(\bar{x}_n, O) + B\bar{Q}_n^{-\frac{1}{2}} \leq r_0$ for n large.

Note that Theorem 2.4.12 guarantees that $\bar{t}\bar{Q} \leq c(2, v_0)$, so (2.173) implies that

$$u_{max}(\bar{t}) \leq \frac{e^{32c(2,v_0)} - 1}{2B} \sup_{B_t(O,r_0)\times(0,\bar{t}_n]} P(x,t) \tag{2.174}$$

However, since B can be chosen to be arbitrarily large, we get a contradiction with the assumption (2.168). Such a contradiction implies that (2.169) must be true. However (2.169) again contradicts to the assumption $P(x,t) > 0$ after taking r_0 approaching to the infinity. To sum up, we have completed the proof of Theorem 2.4.16. □

Note that that the constant $c(n, v_0)$ in Theorem 2.4.12 will be 0 if v_0 is the volume of a standard unit ball in Euclidean space. Indeed, starting from a standard flat metric on $\mathbb{R}^n$, it is easy to see that (from Theorem 3.1 of [27] for example) that Cabezas-Rivas and Wilking's Ricci flow is a flat one. Intuitively it is reasonable to expect that $c(n, v_0)$ is close to zero if v_0 is close to the volume of a unit Euclidean ball.

Assume that $(\mathbb{C}^2, g(t))$ is a Kähler-Ricci flow with $U(n)$-symmetry. Then we certainly have $R_{\bar{2}112} = R_{\bar{2}\bar{1}12} = 0$. In addition, by Lemma 2.4.10, we know that the term $\int_0^r Ric(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}, t)ds$ is bounded in the space-time. The following proposition shows that conversely the boundedness of $R_{\bar{2}112}$, $R_{\bar{2}\bar{1}12}$, and $\int_0^r Ric(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}, t)ds$ in spacetime will be sufficient to ensure the existence of a Kähler-Ricci flow, provided that $c(2, v_0)$ is suitably small.

Proposition 2.4.17. *Under the same assumption as in Theorem 2.4.16, assume that the constant (2.155) satisfies* $c(2, v_0) \leq \frac{1}{32}$*, and there exists a constant* t_1 *such that* $R_{\bar{2}\bar{1}12}$ *and* $R_{\bar{2}112}$ *are bounded on* $\mathbb{C}^2 \times (0, t_1]$*. In addition, assume that there exists a constant* C_1 *which depends on* $g(t)$ *for* $t \in (0, t_1]$ *such that*

$$\int_0^r Ric(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}, t)ds \leq C_1 \tag{2.175}$$

holds for all $r > 0$ *and* $t \in (0, t_1]$*. Then the Ricci flow* $(\mathbb{C}^2, g(t))$ *constructed by Cabezas-Rivas and Wilking is a* $U(n)$*-invariant Kähler-Ricci flow as long as it exists.*

Proof of Proposition 2.4.17. Under the assumption (2.175), tracing the proof of Lemma 8.3 on p.20 of [95], we will get that, for any $t \in (0, t_1]$, the distance function $d_t(x, O)$ satisfies the following estimates:

$$(\frac{\partial}{\partial t} - \Delta_t)d_t(x, O) \geq -4(C_1 + \frac{1}{r_0}) \tag{2.176}$$

whenever $d_t(x,O) > r_0$.

Define $v(x,t) = \eta P(x,t) = \eta(\frac{d_t(x,O)}{r_0})(|R_{\bar{2}112}|^2 + |R_{\bar{2}\bar{1}12}|^2)$, where η is a cut-off function with $\eta = 1$ on $(-\infty, 1]$, $\eta = 0$ on $[2, +\infty)$, and $|\eta'| + |\eta''| + \frac{|\eta'|^2}{\eta} \leq 16$.

A similar computation shows that there exists a constant $C_2 > 0$ such that

$$\frac{d^+ v_{max}(t)}{dt} \leq 32\frac{c(2,v_0)}{t} v_{max}(t) + (\frac{32}{r_0^2} + \frac{4C_1}{r_0})C_2 \tag{2.177}$$

where we use the boundedness of $P(x,t)$.

Note that we have assumed that $32c(2,v_0) < 1$. Integrating (2.177) on $[0,t_1]$, we will arrive at:

$$v_{max}(t_1) \leq \frac{t_1}{1 - 32c(2,v_0)}(\frac{32}{r_0^2} + \frac{4C_1}{r_0})C_2. \tag{2.178}$$

By taking $r_0 \to +\infty$, we get $P(x,t) = 0$ everywhere. Now the Proposition 2.4.17 follows from Lemma 2.4.13. □

Remark 2.4.18. *We believe that a more refined analysis on curvature evolution on the double covers will imply that either (2.175) is automatically true or (2.167) contradicts to $P(x,t) > 0$ on the limit flow $(\mathbb{C}^2, g(t))$. We will study the existence of complete Kähler-Ricci flow in a more general context in future works.*

2.5 Holomorphic functions on noncompact Kähler manifolds with nonnegative curvature

Throughout this section we work on a general complete Kähler manifold with nonnegative holomorphic bisectional curvature. We discuss some questions related to the function theory and geometric properties of complete Kähler manifold with nonnegative holomorphic bisectional curvature. The results here are mainly from our forthcoming paper [113].

2.5.1 Definition of minimal growth orders

Let $(\mathcal{M}^n, g)$ be a complete Kähler manifold with complex dimension n and nonnegative holomorphic bisectional curvature. Recall that a holomorphic function

f is *of polynomial growth* if there exists $\alpha \geq 0$ and $C(\alpha, f)$ such that

$$|f(x)| \leq C(d^{\alpha}(x, O) + 1)$$

where $d(x, O)$ is the distance with respect to a fixed point $O \in M$.

Define the *minimal growth order* of holomorphic function of polynomial growth on $(\mathcal{M}^n, g)$.

$$d_{min} = \inf\{\alpha > 0 | \exists \text{ a nonconstant } f, C, \text{and}, \alpha \, \text{s.t. } |f(x)| \leq C(d^{\alpha}(x, O) + 1)\}.$$

Note that it is equivalent to define d_{min} as:

$$d_{min} = \inf_f \{\limsup_{r \to +\infty} \frac{\log |f(x)|}{\log d(x, O)} | \text{where } f \text{ is a nonconstant holomorphic function}\}.$$

More generally, one may define Hadamard's order of any holomorphic function by

$$\mathrm{Ord}_H(f) = \limsup_{r \to +\infty} \frac{\log \log(\sup_{B(r)} |f(x)|)}{\log r}. \tag{2.179}$$

It is well-known from Cheng-Yau [35]) that there are no non-constant sublinear growth holomorphic functions on $(\mathcal{M}^n, g)$. In our notation it means $d_{min} \geq 1$.

If $(\mathcal{M}^n, g)$ admits a non-constant holomorphic function with polynomial growth, we can define $K(\mathcal{M}^n)$ which is the *transcendence degree* of the quotient field of holomorphic function with polynomial growth. Such a quantity, similar to algebraic dimension for compact complex manifolds, is important in function theory on complete Kähler manifolds with nonnegative curvature. A theorem of Ni [81] says that $1 \leq K(\mathcal{M}^n) \leq n$ for any complete noncompact Kähler manifold $(\mathcal{M}^n, g)$ with nonnegative bisectional curvature.

Similarly we can define the *minimal growth order* D_{min} for holomorphic sections of the canonical line bundle over complete Kähler manifold with nonnegative bisectional curvature:

$$D_{min} = \inf_s \{\limsup_{r \to +\infty} \frac{\log ||s(x)||}{\log d(x, O)} | \, s \text{ is any nonconstant holomorphic section of } K_M\}.$$

For the sake of convenience, we set $d_{min} = +\infty$ or $D_{min} = +\infty$ if $(\mathcal{M}^n, g)$ does not admit any holomorphic functions of polynomial growth or there are none sections of polynomial growth on K_M.

In Section 2.5 we are interested in how to relate d_{min}, D_{min}, $K(\mathcal{M}^n)$, and the differential geometry of $(\mathcal{M}^n, g)$.

2.5.2 Kähler manifolds with quadratic average scalar curvature decay

In this section we review some results on Kähler manifolds with quadratic average scalar curvature decay. These are the class of Kähler manifolds on which the uniformization theory is most successful so far.

As we mentioned before, Chau and Tam [23] proved that any complete noncompact Kähler manifold with bounded and nonnegative bisectional curvature and Euclidean volume growth is biholomorphic to $\mathbb{C}^n$. It is interesting to note that Ni ([83] and [85]) proved that any complete noncompact Kähler manifold with bounded and nonnegative bisectional curvature and Euclidean volume growth must have uniformly quadratic average scalar curvature decay. Later Chau-Tam ([24] and [26]) further studied uniformization problem on Kähler manifolds with bounded and nonnegative bisectional curvature and uniformly quadratic average scalar curvature decay, instead of assuming Euclidean volume growth. In particular, a uniform lower bound on the the injectivity radius along Kähler-Ricci flow with initial data being such manifolds was proved in [26].

Now we observe that the lower bound on the the injectivity radius along the Kähler-Ricci flow established in [26] implies that the equivalence of Euclidean volume growth and uniformly quadratic average scalar curvature decay for Kähler manifolds with bounded and nonnegative bisectional curvature.[1]

Proposition 2.5.1. *let $(\mathcal{M}^n, g)$ be a complete noncompact n-dimensional Kähler manifold with nonnegative bisectional curvature and Ricci curvature is positive at one point O. Assume that there exists C which is independent on O such that*

$$\frac{1}{V(B(O,r))}\int_{B(O,r)} R(x)\, dVol(x) \le \frac{C}{(1+r)^2}. \tag{2.180}$$

Then $(\mathcal{M}^n, g)$ has Euclidean volume growth.

Proof of Proposition 2.5.1. It follows from Shi [103] that the Kähler-Ricci flow with the initial metric $(\mathcal{M}^n, g)$ in our proposition has long time existence and

[1] I would like to thank Lei Ni for sharing me with this observation.

is the scalar curvature satisfies $R(x,t) \leq \frac{C_1}{t}$ for some C_1. Recently Chau-Tam (p6298 [26]) proved that at a fixed point p the injectivity radius of $g(t)$ satisfies $\mathrm{inj}(g(t),p) \geq C_2\sqrt{t}$ for some constant C_2. Denote $B_t(p,r)$ the geodesic ball with respect to $g(t)$, a scaling argument and volume comparison lead to:

$$Vol_t(B_t(p, \frac{C_1\sqrt{t}}{2})) \geq C_3(\frac{C_1\sqrt{t}}{2})^{2n} \tag{2.181}$$

for some C_3 depending on n, C_1, and C_2.

It follows from Lemma 8.3 p20 [95] we have an upper bound for the changing distance:

$$\frac{d}{dt}d_t(p,q)|_{t=t_0} \geq -C(n)(C_1C_4 + \frac{1}{C_4})\frac{1}{\sqrt{t_0}} \tag{2.182}$$

whenever $d_{t_0}(p,q) > 2C_4\sqrt{t_0}$. Here C_4 will be determined later.

Note that $d_{t_0}(p,q)$ is nonincreasing on t, we conclude that for any $t_0 > 0$

$$B_0\Big(p, \Big[C_1 + C(n)(C_1C_4 + \frac{1}{C_4})\Big]\sqrt{t_0}\Big) \supseteq B_{t_0}(p, \frac{C_1\sqrt{t_0}}{2}) \setminus B_{t_0}(p, 2C_4\sqrt{t_0}). \tag{2.183}$$

It follows that

$$\begin{aligned} & Vol_0\Big(B_0\big(p, \Big[C_1 + C(n)(C_1C_4 + \frac{1}{C_4})\Big]\sqrt{t_0}\big)\Big) \qquad (2.184) \\ \geq\ & C_3(\frac{C_1\sqrt{t_0}}{2})^{2n} - \omega_{2n}(2\sqrt{C_4}\sqrt{t_0})^{2n} \end{aligned}$$

Note that t_0 is arbitrary, clearly (2.184) implies that $(\mathcal{M}^n, g)$ has Euclidean volume growth after we choose C_4 small enough. □

Remark 2.5.2. *In view of Cao's splitting theorem [15], we may assume that the universal cover of* $(\mathcal{M}^n, g)$ *does no split other than Ricci curvature is quasi-positive in Proposition 2.5.1. In general, without assuming the upper bound of curvature, it is related to a conjecture of Wu-Zheng [107].*

We would like to mention that there are recent progresses on solutions to Poincaré-Lelong equation without global upper bounds on curvatures due to Ni and Tam [90]. Their result, combined with Corollary 5.2 in [81], implies the following result.

Proposition 2.5.3 (Ni [81], Ni and Tam [90]). *Let $(\mathcal{M}^n, g)$ be a complete non-compact n-dimensional Kähler manifold with nonnegative (but possibly unbounded) bisectional curvature and Ricci curvature is positive at one point O. Assume that there exists C which depends on O such that*

$$\frac{1}{V(B(O,r))}\int_{B(O,r)} R(x)\, dVol(x) \leq \frac{C}{(1+r)^2}. \tag{2.185}$$

Then the transcendence degree $K(\mathcal{M}^n) = n$ (i.e. there are global holomorphic function with polynomial growth giving local coordinates at O) and $\pi_1(\mathcal{M}^n)$ is finite.

2.5.3 A conjecture on the precise measurement assuming the existence of non-constant holomorphic functions with polynomial growth

Let $(\mathcal{M}^n, g)$ be a complete n-dimensional Kähler manifold with nonnegative bisectional curvature. In this subsection we are interested in the metric geometry of such manifolds on which one can find a non-constant holomorphic function with polynomial growth.

Function theory on Kähler manifolds with nonnegative curvature turns out to useful in understanding geometry of such manifolds. It is well-known ([105],[72], [32], and the introduction in [85]) that one can construct nontrivial holormophic functions on Kähler manifold with positive bisectional curvature.

Now we will restate the above result in a slightly general context and also include a proof by L^2 methods in Appendix B for the sake of convenience. It also relates the existence of holomorphic functions of polynomial growth with the value of the transcendence degree $K(\mathcal{M}^n)$.

Proposition 2.5.4. *Let $(\mathcal{M}^n, g)$ be a complete noncompact Kähler manifold with nonnegative bisectional curvature, assume that the universal cover $(\widetilde{\mathcal{M}}^n, \widetilde{g})$ does not split, then*

(1) $(\mathcal{M}^n, g)$ admits a nonconstant holomorphic function with Hadamard's order $d_H \leq 1$. The canonical line bundle K_M on $\mathcal{M}^n$ admits nontrival sections with at most exponential growth.

(2) Either the transcendence degree $K(\mathcal{M}^n) = 0$ or $K(\mathcal{M}^n) = n$. Moreover, in the case of $K(\mathcal{M}^n) = n$, at any point $O \in \mathcal{M}^n$ there exists n holomorphic functions with polynomial growth giving local coordinates at O and $\pi_1(\mathcal{M}^n)$ is finite.

Remark 2.5.5. *We remark that Proposition 2.5.4 is sharp in two aspects.*

In (1), there are examples of Kähler manifolds with nonnegative bisectional curvature without any nonconstant holomorphic functions. Recall that a toroidal group X is $\mathbb{C}^n/\Lambda_k$ where $\Lambda_k \subset \mathbb{C}^n$ is some lattice of rank $n+1 \leq k \leq 2n$ with the property that any holomorphic functions on X are constant. Those noncompact toroidal groups are $(\mathbb{C}^{2n-k})^*$-bundle over a complex torus $\mathbb{T}_C^{k-n}$ and their tangent bundles are trivial. A standard example, due to Cousin [42], is $\mathbb{C}^2/\Lambda_3$ where Λ_3 is generated by $(1,0)$, $(0,1)$, and $(\sqrt{-1}, a+\sqrt{-1})$ where a is irrational.

In (2) we can not assume "$(\mathcal{M}^n, g)$ itself does not split" instead. Consider $\mathbb{C}^1 \times \mathbb{C}^{n-1}$ with the product metric $g_e \times h$ where g_e is the Euclidean metric on $\mathbb{C}^1$ and h is a $U(n-1)$-invariant Kähler metric with positive bisectional curvature. Define a cyclic group $\Gamma = <\gamma>$ where $\gamma(\xi, z_1, ..., z_{n-1}) = (\xi + k, z_1 e^{2\pi k\sqrt{-1}}, ..., z_{n-1}e^{2\pi k\sqrt{-1}})$. Note that γ have no fixed points and $\mathcal{M} = (\mathbb{C}^1 \times \mathbb{C}^{n-1})/\Gamma$ whose induced metric has nonnegative bisectional curvature. Chau-Tam (p6302 [26]) use such examples to motivate their discussion on fibre bundle structures for Kähler manifolds with nonnegative curvature. Now we observe that $K(\mathcal{M}) = n-1$ if we pick $k = \frac{1}{p}$ for a prime number p and h with Euclidean volume growth.

Now assume that $(\mathcal{M}^n, g)$ satisfy the assumption in Proposition 2.5.4 and it admits a nonconstant holomorphic function with polynomial growth. Since $K(\mathcal{M}^n) = n$, we can also define *the minimal order* $(d_{min}^{(1)}, ..., d_{min}^{(n)})$ for the set of every n holomorphic functions with polynomial growth which gives local coordinates at a fixed point. Namely, for all such $\{f_1, ..., f_n\}$ with $\limsup_{r\to+\infty} \frac{\log|f_i(x)|}{\log r(x)}$ nondecreasing for $1 \leq i \leq n$, we define:

$$(d_{min}^{(1)}, ..., d_{min}^{(n)}) = \inf_{\{f_1,...,f_n\}} \{\limsup_{r\to+\infty} \frac{\log|f_1(x)|}{\log r(x)}, ..., \limsup_{r\to+\infty} \frac{\log|f_n(x)|}{\log r(x)}\}$$

We caution that it is apriori unclear if $(d_{min}^{(1)}, ..., d_{min}^{(n)})$ can be obtained by a set of holomorphic functions on $\mathcal{M}^n$, but it is true for any complete Kähler metric on $\mathbb{C}^n$ with nonnegative curvature and $U(n)$-symmetry. Note that $d_{min} = d^{(1)}_{min}$, in this sense those $\{(d^{(1)}_{min}, ..., d^{(n)}_{min}\}$ could be viewed as a refinement on d_{min}. Note that we can show that the volume growth of $(\mathcal{M}^n, g)$ satisfies $\mathrm{Vol}(B(r)) \geq C[\frac{r^2}{\log r}]^n$ for r sufficiently large and some constant $C > 0$ using a technique in [28]. Next we propose some conjectures on determining the exact values for the AVR and ASCD of $(\mathcal{M}^n, g)$:

Recall that on any Riemannian manifold $(\mathcal{N}^n, g)$ with nonnegative curvature one can define the asymptotic volume ratio (AVR) and the average scalar curvature decay (ASCD):

$$\mathrm{AVR}(\mathcal{N}^n, g) = \lim_{r\to+\infty} \frac{\mathrm{Vol}(B(O,r))}{\omega_n r^n},$$
$$\mathrm{ASCD}(\mathcal{N}^n, g) = \limsup_{r\to+\infty} \frac{r^2}{\mathrm{Vol}(B(O,r))} \int_{B(O,r)} R(x) dVol(x),$$

where ω_n the volume of unit ball in $\mathbb{R}^n$. Note that both definitions are independent of the choice of O.

Conjecture 2.5.6. *Let $(\mathcal{M}^n, g)$ be a complete simply-connected n-dimensional Kähler manifold with nonnegative bisectional curvature and its universal cover $(\widetilde{\mathcal{M}}^n, \widetilde{g})$ does not split, then is it true that there exists a constant $c(n, d^{(1)}_{min}, ..., d^{(n)}_{min})$ such that*

$$\frac{1}{c(n, d^{(1)}_{min}, ..., d^{(n)}_{min})} \leq \mathrm{AVR}(\mathcal{M}^n, g) = c(n, d^{(1)}_{min}, ..., d^{(n)}_{min}). \tag{2.186}$$

In fact, a more ambitious conjecture is that if

$$\mathrm{AVR}(\mathcal{M}^n, g) = \prod_{i=1}^{n} \frac{1}{d^{(i)}_{min}}? \tag{2.187}$$

Here $(d^{(1)}_{min}, ..., d^{(n)}_{min})$ are minimal growth orders for the set of every n holomorphic functions of polynomial growth $\{f_1, ..., f_n\}$ which are linearly independent at a fixed point $O \in \mathcal{M}^n$.

Remark 2.5.7. *A result of Li [67] states that any Riemannian manifold $(\mathcal{M}^n, g)$ with nonnegative Ricci curvature has finite fundamental group and satisfies*

$$\mathrm{AVR}(\mathcal{M}^n, g) = \frac{1}{|\pi_1(\mathcal{M}^n)|} \mathrm{AVR}(\widetilde{\mathcal{M}}^n, \widetilde{g}).$$

Therefore, if Conjecture 2.5.6 is true, note the fact that AVR *of metric product is equal to the product of* AVR *of each factor, then we would get a complete picture of* AVR *of complete Kähler manifolds with nonnegative bisectional curvature in terms of the growth orders of holomorphic functions and the order of its fundamental group.*

We further speculate:

Conjecture 2.5.8. *Let $(\mathcal{M}^n, g)$ be a complete n-dimensional Kähler manifold with nonnegative bisectional curvature whose universal cover $(\widetilde{\mathcal{M}}^n, \widetilde{g})$ does not split, then $D_{min} = \sum_{i=1}^n d_{min}^{(i)} - n$ and there exists a constant $c(n)$ such that*

$$\frac{1}{c(n)} D_{min} \leq \mathrm{ACSD}(\mathcal{M}^n, g) \leq c(n) D_{min}. \tag{2.188}$$

Again an optimistic conjecture is that:

$$\mathrm{ASCD}(\mathcal{M}^n, g) = n D_{min}? \tag{2.189}$$

Can we change $\limsup$ *in (2.186) to* $\lim$*?*

Remark 2.5.9. *Both Conjectures 2.5.6 and 2.5.8 are affirmative for any $U(n)$-invariant Kähler metric with nonnegative bisectional curvature on $\mathbb{C}^n$. In this case $d_{min}^{(i)} = d_{min}$ for any $1 \leq i \leq n$ and $D_{min} = n(d_{min} - 1)$. Both conjectures can be viewed as a quantified version of the conjectures in p931 [81]. Note that both conjectures are almost true for simply connected Riemann surface with nonnegative curvature, see more discussion in Appendix A.*

We have the following result to characterize manifolds with $d_{min} = 1$.

Proposition 2.5.10. *Let $(\mathcal{M}^n, g)$ be a complete n-dimensional Kähler manifold with nonnegative bisectional curvature and $d_{min} = 1$, then $(\mathcal{M}^n, g)$ must split as $\mathcal{N}^{n-1} \times \mathbb{C}$.*

Proof of Propostion 2.5.10. Suppose that for any $k \in \mathbb{N}$ there exists a holomorphic functions f_k and C_k such that

$$|f_k(x)| \leq C_k (d(x, O))^{1+\frac{1}{k}}.$$

We will show that we can choose some f_{N_0} from $\{f_k\}_{k=1}^{\infty}$ such that the following is true for some new constants $\tilde{C}_k$:

$$|f_{N_0}(x)| \leq \tilde{C}_k (d(x,O))^{1+\frac{1}{k}}. \tag{2.190}$$

Denote $S_N = \text{Span}\{f_N, f_{N+1}, f_{N+2}, ...\}$, the dimension estimates on holomorphic functions with polynomial growth ([81], [28]) imply that $\lim_{N\to+\infty} \dim S_N$ exists and it is finite. Therefore, we can find two natural numbers N_0 and l so that $\dim S_N = l$ for any $N \geq N_0$. Note that S_N is non-increasing when N increases, we conclude that $f_{N_0} \in S_N$ for any $N \geq N_0$, which proves (2.190). Now Propostion 2.5.10 follows Ni-Tam's Liouville type theorem for plurisubharmonic functions on Kähler manifolds with nonnegative curvature, see Theorem 0.3 p460 [88]. □

Proposition 2.5.11. *Let $(\mathcal{M}^n, g)$ be a complete n-dimensional Kähler manifold with nonnegative bisectional curvature, then there exists a constant $C(n)$ such that*

$$\text{ASCD}(\mathcal{M}^n, g) \leq C(n) D_{min}. \tag{2.191}$$

In particular, by Ni's gap theorem [86], any nonflat complete Kähler manifold with nonnegative bisectional curvature must have $D_{min} > 0$.

Proof of Proposition 2.5.11. In fact Propostion 2.5.11 is no more than a reformulation of Corollary 2.3 on p.930 in [81] in terms of D_{min}. We follow the same proof, which will be sketched for the sake of convenience.

For any $\epsilon > 0$ there exists a holomorphic section $s_\epsilon \in H^0(\mathcal{M}^n, K_M)$ and a constant C_ϵ so that

$$|s_\epsilon(x)| \leq C_\epsilon (d(x,O))^{D_{min}+\epsilon}.$$

Recall that Poincaré-Lelong implies that $\Delta \log ||s||^2(x) \geq R(x)$. Let $v(x,t)$ be the solution to the heat equation $(\frac{\partial}{\partial t} - \Delta)v(x,t) = 0$ with initial value $v(x,0) = \log ||s_\epsilon||^2(x)$. Denote $w(x,t) = \frac{\partial}{\partial t} v(x,t)$, it follows from Lemma 3.1 p926 [81] imply that $\frac{\partial}{\partial t}(tw(x,t)) \geq 0$.

By the heat kernel estimate of Li-Yau, there exists constants $c_1(n)$ and $c_2(n)$ which only depend on n such that

$$\frac{c_1(n)}{\text{Vol}(B(x,\sqrt{t}))} e^{-\frac{d^2(x,y)}{3t}} \leq H(x,y,t) \leq \frac{c_2(n)}{\text{Vol}(B(x,\sqrt{t}))} e^{-\frac{d^2(x,y)}{5t}}. \tag{2.192}$$

From now on let us use $C(n)$ denote any constant which only depends on n, it follows from (2.192) that:

$$w(x,t) \geq \frac{C(n)}{\mathrm{Vol}(B(x,\sqrt{t}))} \int_{B(O,\sqrt{t})} R(x) dVol(x) \tag{2.193}$$

On the other hand, the sharp large time asymptotics for $v(x,t)$:

$$\limsup_{t\to+\infty} \frac{v(x,t)}{\log t} \leq D_{min} + \epsilon \tag{2.194}$$

is proved in Lemma 4.2 on p.934 in [81] (for the case of Euclidean volume growth) and Lemma 2.1 on p.1438 in [28] (for the general case).

Therefore, $\frac{\partial}{\partial t}(tw(x,t)) \geq 0$, together with (2.193) and (2.194) imply that

$$\limsup_{t\to+\infty} \frac{t}{\mathrm{Vol}(B(x,\sqrt{t}))} \int_{B(O,\sqrt{t})} R(x) dVol(x) \leq C(n)(D_{min} + \epsilon). \tag{2.195}$$

We prove (2.191) by taking ϵ goes to zero in the above. The recent gap theorem in Ni [86] shows that any Kähler manifold with nonnegative bisectional curvature satisfies $\mathrm{ASCD}(\mathcal{M}^n, g) = 0$ if and only if $(\mathcal{M}^n, g)$ is flat. In other words any nonflat complete Kähler manifold with nonnegative bisectional curvature must have $D_{min} > 0$. □

Proposition 2.5.12. *Let $n > 1$ and $(\mathcal{M}^n, g)$ be a complete n-dimensional Kähler manifold with nonnegative bisectional curvature with maximum volume growth, then*

$$\mathrm{ASCD}(\mathcal{M}^n, g) \leq 4^n n! e^{\frac{1}{4}} \frac{2n-1}{n-1} D_{min}. \tag{2.196}$$

Moreover, assume that $\forall k \in \mathbb{N}$, there exists $r(k) > 0$ and $s_k \in H^0(\mathcal{M}^n, K_M)$ such that $||s_k|| \geq d(x,O)^{D_{min}-\frac{1}{k}}$ whenever $d(x,O) \geq r(k)$, and $[s_k] \to 0$ in the sense that $\int_{\{s_k=0\}} H(x,y,t) dVol_y \to 0$ when $k \to +\infty$. then we have

$$\mathrm{ASCD}(\mathcal{M}^n, g) \geq \frac{1}{4e^{\frac{1}{4}} + (n-1)! 4^n e^{\frac{1}{4}} - 4} D_{min}. \tag{2.197}$$

Proof of Propostion 2.5.12. A sharp estimates on heat kernel on complete manifolds with nonnegative Ricci curvature and Euclidean volume growth was proved by Li-Tam-Wang [70]. Their result implies that for $(\mathcal{M}^n, g)$ in our proposition with $\alpha = \mathrm{AVR}(\mathcal{M}^n, g)$ and any fixed $\delta > 0$ there exists a constant $C = C(n, \alpha)$ such that

$$\frac{\omega_{2n}\delta^{2n}d^{2n}(x,y)}{\mathrm{Vol}(B(x,\delta d))}e^{-\frac{1+9\delta}{4t}d^2(x,y)} \le H(x,y,t) \le (1 + C(\delta+\beta))\frac{\omega_{2n}}{\alpha(4\pi t)^n}e^{-\frac{1-\delta}{4t}d^2(x,y)}.$$

Here $\beta : [0, +\infty) \to \mathbb{R}$ is defined by

$$\beta(d(x,y)) = \delta^{-2n} \max_{r \ge (1-\delta)d(x,y)} \Big\{1 - \delta^{n(2n+1)}\frac{\mathrm{Vol}(B(p,r))}{\mathrm{Vol}(B(p,\delta^{2n+1}r))}\Big\}. \tag{2.198}$$

Note that $\lim_{d(x,y)\to+\infty} \beta(d(x,y)) = 0$.

Now follow the notation in the proof of Propostion 2.5.11 we will give an explicit form of the constant $C(n)$ in (2.193) with the aid of the above sharp lower bound on the heat kernel. Precisely we will show that

$$\limsup tw(x,t) \ge \frac{\omega_{2n}}{(4\pi)^n}e^{-\frac{1}{4}}(\frac{n-1}{2n-1}) \limsup \frac{t}{\mathrm{Vol}(B(x,\sqrt{t}))}\int_{B(O,\sqrt{t})} R(x)dVol(x)$$

Note that (2.199) will imply (2.196) if we follow the proof Proposition 2.5.11. Indeed, the above lower bound on the heat kernel implies that

$$\begin{aligned} & t\int_{\mathcal{M}^n} H(x,y,t)R(y)dVol(y) \qquad (2.199)\\ \ge\ & \frac{\omega_{2n}\delta_{2n}}{(4\pi)^n t^{n-1}}\int_0^{+\infty}\frac{e^{-\frac{1+9\delta}{4t}r^2}}{\mathrm{Vol}(x,\delta r)}\int_{\partial B(x,r)} R(y)dA(y)\,dr \\ \ge\ & \frac{\omega_{2n}\delta_{2n}e^{-\frac{1+9\delta}{4}}}{(4\pi)^n t^{n-1}\,\mathrm{Vol}(x,\delta\sqrt{t})}\int_0^{\sqrt{t}} r^{2n}d(\int_{B(x,r)} R(y)dVol(y)) \\ =\ & \frac{\omega_{2n}e^{-\frac{1+9\delta}{4}}}{(4\pi)^n}I(x,t) \end{aligned}$$

where

$$I(x,t) = \delta^{2n}\frac{t^n\int_{B(x,\sqrt{t})} R(y)dVol(y) - \int_0^{\sqrt{t}} 2nr^{2n-1}\int_{B(x,r)} R(y)dVol(y)dr}{t^{n-1}\,\mathrm{Vol}(B(x,\delta\sqrt{t}))} \tag{2.200}$$

It is straightforward to check that

$$\limsup_{t\to+\infty} I(x,t) \ge \frac{n-1}{2n-1}\limsup_{t\to+\infty}\frac{t\int_{B(x,\sqrt{t})} R(y)dVol(y)}{\mathrm{Vol}(B(x,\sqrt{t}))}$$

Now we turn to the second part, the Poincaré-Lelong equation implies that

$$\Delta \log ||s_k||^2(x) = g^{i\bar{j}}[s_k]_{i\bar{j}} + R(x) \ge R(x) \tag{2.201}$$

Let $v_k(x,t)$ be the solution to the heat equation with initial value $v_k(x,0) = \log ||s_k||^2(x)$. Denote $w_k(x,t) = \frac{\partial}{\partial t} v_k(x,t)$, then we have $\frac{\partial}{\partial t}(tw(x,t)) \geq 0$.

Applying the above upper bound on heat kernel and the assumption that $[s_k] \to 0$ as $k \to +\infty$, we get:

$$\begin{aligned} &\lim_{t\to+\infty} tw_k(x,t) \\ \leq\ & \frac{\omega_{2n}}{(4\pi)^n}\Big[(1-e^{-\frac{1}{4}})\limsup_{t\to+\infty} tk(x,\sqrt{t}) + \int_{\sqrt{t}}^{+\infty} e^{-\frac{r^2}{4t}}\frac{r^{2n+1}}{2t^{n+1}}k(x,r)dr\Big] \end{aligned} \tag{2.202}$$

where $k(x,r) = \frac{1}{\mathrm{Vol}(B(x,r))}\int_{B(O,r)} R(x)dVol(x)$.

Similarly, we have

$$\liminf_{t\to+\infty} \frac{v_k(x,t)}{\log t} \geq \frac{\omega_{2n}}{(4\pi)^n} e^{-\frac{1}{4}}\frac{1}{4}(D_{min} - \frac{1}{k}). \tag{2.203}$$

which implies that

$$\lim_{t\to+\infty} tw_k(x,t) \geq \frac{\omega_{2n}}{(4\pi)^n} e^{-\frac{1}{4}}\frac{1}{4}(D_{min} - \frac{1}{k}). \tag{2.204}$$

(2.203) and (2.204) together imply that

$$\begin{aligned} &\frac{\omega_{2n}}{(4\pi)^n} e^{-\frac{1}{4}}\frac{1}{4}(D_{min} - \frac{1}{k}) \\ \leq\ & \frac{\omega_{2n}}{(4\pi)^n}\Big[(1-e^{-\frac{1}{4}})\limsup_{t\to+\infty} tk(x,\sqrt{t}) + \int_{\sqrt{t}}^{+\infty} e^{-\frac{r^2}{4t}}\frac{r^{2n+1}}{2t^{n+1}}k(x,r)dr\Big] \end{aligned} \tag{2.205}$$

Now a simple argument by contradiction will lead to

$$\limsup_{r\to+\infty} r^2 k(x,r) \geq \frac{1}{4e^{\frac{1}{4}} + (n-1)!4^n e^{\frac{1}{4}} - 4} D_{min} \tag{2.206}$$

In fact, by a monotonicity property on $r^2k(x,r)$ proved by Ni (p756 [86]) we further conclude that

$$\liminf_{r\to+\infty} r^2 k(x,r) \geq C(n) D_{min}$$

for some constant $C(n)$. □

2.5.4 Volume growth and curvature decay in view of holomorphic functions

Chen-Zhu [32] proved that any complete noncompact Kähler manifold with quasi-positive bisectional curvature has at least half-Euclidean volume growth and at least linear decay if assuming bisectional curvature positive everywhere. See also a generalization of this result to the case of nonnegative curvautre by Ni-Tam [88]. It was further asked by Chen-Zhu (see [108] p.519) if there is a connection between volume growth rate and the decay rate of average curvature for complete positively curved Kähler metrics. Wu-Zheng [108] shown that for any $U(n)$-invariant Kähler metric with positive bisectional curvature on $\mathbb{C}^n$ there exists a constant C such that $\frac{1}{C(1+v(r))} \leq \frac{1}{Vol(B(O,r))} \int_{B(O,r)} R(x) dVol_x \leq \frac{C}{1+v(r)}$ and $Vol(B(O,r)) = \omega_{2n}(v(r))^n$. Here ω_{2n} is the volume of the ball unit in Euclidean space $\mathbb{R}^{2n}$ and $v(r)$ a smooth function of r. In particular, there is another constant c such that $cr \leq v(r) \leq r^2$ for r large enough.

Using Wu-Zheng's theorem [108] on rotationally symmetric Kähler metric with positive curvature on $\mathbb{C}^n$, one can construct metrics with the volume growth like $(r \log r)^n$ and $\frac{r^{2n}}{(\log r)^n}$. Motivated by these examples, we define the order of volume growth of any complete Kähler manifold $(\mathcal{M}^n, g)$ with nonnegative bisectional curvature.

$$K_+ = \limsup_{r \to +\infty} \frac{\log \mathrm{Vol}(B(r))}{n \log r}.$$

$$K_- = \liminf_{r \to +\infty} \frac{\log \mathrm{Vol}(B(r))}{n \log r}.$$

Next we introduce the notion of *the minimal Hadamard's order* d_0 on $(\mathcal{M}^n, g)$.

$$d_0 = \inf\{d > 0 | \exists \text{ a nonconstant holomorphic function } f \text{ s.t.} \, \mathrm{Ord}_H(f) = d\}.$$

Note that it is apriori unclear if d_0 can be exactly attained by a holomorphic function on $(\mathcal{M}^n, g)$. But it is true for complete $U(n)$-invariant Kähler metric on $\mathbb{C}^n$ with quasi-positive bisectional curvature. Also it is direct to see $1 \leq K_- \leq K_+ \leq 2$ using Chen-Zhu's result [32], however it is unclear if it is always the case that

$K_- = K_+$ even assuming the metric being rotationally invariant, see Appendix A for more discussion.

Observation 2.5.13. *On any $(\mathcal{M}^n, g)$ with complete noncompact Kähler metric with nonnegative curvature, assume that the universal cover $(\widetilde{\mathcal{M}}^n, \widetilde{g})$ does not split, then $K_- \geq 2 - d_0$. Moreover, if $Vol(B(O,r)) \geq C_1 r^{n\alpha}$ for some $C_1 > 0$ and $1 < \alpha \leq 2$ implies that $\int_{B(O,r)} R(x) d\mu(x) \leq \frac{C_2}{r^\alpha}$, then $K_- = 2 - d_0$.*

Proof of Observation 2.5.13. If we assume that $(\mathcal{M}^n, g)$ has quasi-positive bisectional curvature, then it is not hard to show that $K_- \geq 2 - d_0$ by a technique developed in [28]. In fact, their method can be used to show that $\mathrm{Vol}(B(r)) \geq C r^{(2-a)n}$ if there exists a nonconstant holomorphic function $f(x)$ with $\log(|f(x)| + 1) \leq C(r^a + 1)$ for some $0 < a \leq 1$. Obviously, $K_- = 2 - d_0$ is true for any complete $U(n)$-invariant Kähler metric on $\mathbb{C}^n$ with nonnegative bisectional curvature in view of Wu-Zheng's result mentioned above. □

2.5.5 Minimal orders and Type-III solutions to the Kähler-Ricci flow

The main purpose of this subsection is to discuss minimal growth orders of holomorphic functions in the context of Kähler-Ricci flow.

Type-III solutions to the Kähler-Ricci flow

According to Hamilton [60], a complete solution $(\mathcal{M}, g(t))$ where $t \in [0, +\infty)$ to the Ricci flow on a complete manifold $\mathcal{M}$ is called Type-III if

$$\sup_{\mathcal{M} \times [0,+\infty)} t|Rm|(x,t) < \infty.$$

Type-III solutions are important in the singularity analysis of the Ricci flow and it is closely related to gradient expanding solitons, see [60], [14], and [30].

It is known from the work of Shi [103] that Kähler-Ricci flow on manifolds with bounded nonnegative bisectional curvature curvature and uniformly quadratic average scalar curvature decay is type-III. Under the extra assumption of Euclidean volume growth Chau-Tam [23] further proved that the blowing down of such a flow converges to an expanding Kähler-Ricci soliton with nonnegative curvature. Such a result is important in their proof of the uniformization theorem on complete Kähler manifolds with Euclidean volume growth and bounded nonnegative bisectional curvature curvature. There is a slight generalization of this result of Ni [84]. Note that Ni's result ([83] and [85]) and Proposition 2.5.1 have established the equivalence of Euclidean volume growth and quadratic decay of average scalar curvature for Kähler manifolds with bounded and nonnegative bisectional curvature. Recently Schulze-Simon [100] proved the Riemannian analogue of this result. Namely they show that there exists a long-time Ricci flow solution starting from Riemannian manifolds with Euclidean volume growth and bounded nonnegative curvature operator with its blowing down being an expanding soliton. In the following theorem we collect these results on Kähler-Ricci flow on manifold with bounded nonnegative curvature and Euclidean volume growth.

Theorem 2.5.14 ([83], [23], and [100]). *Suppose that $(\mathcal{M}^n, g_0)$ is a complete noncompact Kähler manifold with bounded nonnegative curvature and Euclidean volume growth.*

(1) The Kähler-Ricci flow with initial metric g_0 has a long time existence $g(t)$ and $\alpha = \mathrm{AVR}(\mathcal{M}^n, g(t))$ is invariant along $g(t)$, and there exists a constant $c(n,\alpha)$ such that $R(x,t) \le \frac{c(n,\alpha)}{t}$ for any x and t.

(2) For any point $x \in \mathcal{M}^n$, let $\{\lambda_1(x,t), ..., \lambda_n(x,t)\}$ denote the eigenvalues of Ricci curvature $Ric(x,t)$ with respect to $g(t)$ in the nondecreasing order, then $t\lambda_i(x,t)$ is nondecreasing, therefore $\lim_{t\to+\infty} t\lambda_i(x,t)$ exists.

(3) Fix any point p be on $\mathcal{M}^n$, given any $t_k \to +\infty$, define $g_k(t) = \frac{1}{t_k} g(t_k t)$, the pointed sequence $(\mathcal{M}^n, g_k(t), p)$ sub-sequentially converges to a gradient expanding Kähler-Ricci soliton $(\mathcal{N}^n, h(t), O)$ with nonnegative bisectional curvature.

(4) The expanding soliton in (3) satisfies $\mathrm{AVR}(\mathcal{N}, h(1)) = \mathrm{AVR}(\mathcal{M}^n, g_0)$.

Proof of Theorem 2.5.14. Briefly speaking (1) is a consequence of Shi's short time existence result to Kähler-Ricci flow [101] and Perelman's Corollary 11.6 [95]. The proof has been given in [83]. (2) and (3) has been proved in [23]. (4) could be proved following the method in [100]. Precisely, (4) was proved in [100] for Riemannian manifolds with bounded and nonnegative curvature operator and Euclidean volume growth. The point is show $\mathrm{AVR}(\mathcal{N}, h(t)) \leq \mathrm{AVR}(\mathcal{M}^n, g_0)$ since the reverse direction easily follows from the blowup down procedure and the fact that AVR is invariant along Kähler-Ricci flow with bounded curvature. To prove this, let $(\mathcal{M}^n, g_k(t), p)$ be the sequence converging to $(\mathcal{N}^n, h(t), O)$ in (3) and denote the asymptotic cone (X, d_X, O) the Gromov-Hausdorff limit of $(\mathcal{M}^n, g_k(0), p)$ for that particular choice of t_k. Note that we don't know the uniqueness of asymptotic cones for Kähler manifolds with nonnegative bisectional curvature and Euclidean volume growth. Following [100] one can prove that $(\mathcal{M}^n, g_k(t), p)$ converges to (X, d_X, O) in the sense of Gromov-Hausdorff as t goes to zero. Now $\mathrm{AVR}(\mathcal{N}, h(t)) \leq \mathrm{AVR}(\mathcal{M}^n, g_0)$ will follow from Cheeger-Colding's volume convergence theorem on noncollasping manifolds with lower Ricci bounds under Gromov-Hausdorff convergence. □

We have the following proposition to relate ASCD and limiting behaviors of Type-III Kähler-Ricci flow.

Proposition 2.5.15. *Suppose that $(\mathcal{M}^n, g_0)$ is a complete noncompact Kähler manifold with bounded nonnegative curvature and Euclidean volume growth, by Theorem 2.5.14 there exists a complete solution $g(t)$ ($t \in [0, +\infty)$) to the Kähler-Ricci flow with initial metric g_0. Then for any point $x \in \mathcal{M}^n$, $\lim_{t\to+\infty} t\lambda_i(x,t)$ and $\lim_{t\to+\infty} tR(x,t)$ is independent of the choice of x. Denote $\beta = \lim_{t\to+\infty} tR(x,t)$, then there exists a constant $C(n)$ such that*

$$\frac{1}{C(n)} \mathrm{ASCD}(\mathcal{M}^n, g(t)) \leq \beta \leq C(n)\, \mathrm{ASCD}(\mathcal{M}^n, g(t))$$

holds for any $t \in [0, +\infty)$.

Proof of Proposition 2.5.15. Define

$$F(x,t,T) = \log\Big[\frac{\det(g_{i\bar{j}}(x,t+T))}{\det(g_{i\bar{j}}(x,T))}\Big]$$

and we have $\frac{d}{dt}F(x,t) = -R(x,t)$, Such a quantity measures the change of volume element along $g(t)$ and its estimates proves to be important to study the long time behavior of the flow since Shi's work [103]. Indeed our proof relies on a version of such estimates due to Ni-Tam [89].

More precisely, denote $\mathbb{m}(t) = \inf_{x\in\mathcal{M}^n} F(x,t,T)$ and

$$k_T(x,r) = \frac{1}{\mathrm{Vol}_T(B_T(x,r))}\int_{B_T(x,r)} R(y,T)dVol_T(y),$$

then it is proved (p124 [89]) that there exists a constant $C(n)$ such that for any $A>0$

$$\begin{aligned} &\frac{1}{C(n)}\int_0^{\sqrt{t}} sk_T(x,s)ds \le -F(x,t,T) \quad (2.207)\\ &C(n)\Big[(1+\frac{t(1-\mathbb{m}(t))}{A^2})\Big]\int_0^A sk_T(x,s)ds - \frac{t\mathbb{m}(t)(1-\mathbb{m}(t))}{A^2}\Big]. \end{aligned}$$

On one hand, since $\frac{d}{dt}F(x,t) = -R(x,t)$ and $tR(x,t)$ is nondecreasing, we have

$$\lim_{t\to+\infty}\frac{-F(x,t)}{\log t} = \lim_{t\to+\infty} tR(x,t) = \beta(x) \quad (2.208)$$

Pick $A^2 = 2C(n)t(1-\mathbb{m}(t))$ as in p127 [89], the second half of (2.207) becomes

$$-F(x,t) + \frac{1}{2}\mathbb{m}(t) \le (2C_1+1)\int_0^A sk_T(x,s)ds. \quad (2.209)$$

Pick a sequence $t_j \to +\infty$ such that $\lim_{j\to+\infty}\frac{-\mathbb{m}(t_j)}{2\log t_j} = \limsup_{t\to+\infty}\frac{-\mathbb{m}(t)}{2\log t}$, then pick $x_j \in \mathcal{M}^n$ with $-F(x_j,t_j) \ge -\mathbb{m}(t_j) - \frac{1}{j}$. It is direct to check that

$$\limsup_{t\to+\infty}\frac{-\mathbb{m}(t)}{2\log t} = \lim_{j\to+\infty}\frac{-F(x_j,t_j)+\frac{1}{2}\mathbb{m}(t)}{\log t_j}. \quad (2.210)$$

Note that $\mathrm{ASCD}(g(t)) = \limsup_{r\to+\infty} r^2 k_T(x,r)$ are independent of choice of base point O. Therefore, for any $\epsilon > 0$, $t_j^2 k_T(x_j,t_j) \le \mathrm{ASCD}(g(t)) + \epsilon$. It is proved ([86]) that $r^2 k_T(x,r)$ is nondecreasing mudulo a constant which only depends on the n. That is to say that, thre exists constants $C_2(n)$ and $C_3(n)$ on n so that $t^2 k_T(x,t) \le C_3(n) t_1^2 k_T(x,t_1)$ for any $t \le C_2(n)t_1$.

$$\begin{aligned}
&\limsup_{j\to+\infty} \frac{(2C_1+1)\int_0^A sk_T(x_j,s)ds}{\log t} \\
\leq\ & \limsup_{j\to+\infty}(2C_1+1)\limsup_{j\to} \frac{C_3(n)\big[\,\mathrm{ASCD}(g(t))+\epsilon\big]\log(\sqrt{2C_1 t_j(1-\mathbb{t}_{\mathbb{j}})})}{\log t_j} \\
\leq\ & \frac{1}{2}(2C_1+1)C_3(n)\big[\,\mathrm{ASCD}(g(t))+\epsilon\big]
\end{aligned} \tag{2.211}$$

where in the last step we use $-\mathbb{m}(t) = O(\log t)$.

To sum up, we have proved that $\sup_{x\in\mathcal{M}^n}\beta(x) \leq C(n)\,\mathrm{ASCD}(g(T))$ for some $C(n)$. A similar argument on the first half of (2.207) will give $\inf_{x\in\mathcal{M}^n}\beta(x) \geq \frac{1}{C(n)}\,\mathrm{ASCD}(g(T))$. Since T can be arbitrary, (2.207) is proved once we can show $\beta(x)$ is constant.

Indeed, we can show $\lim_{t\to+\infty} t\lambda_i(x,t)$ is independent of the choice of x. Fix and t and let x and y are two points and $\gamma(s)$ ($s \in [0, d_t(x,y)]$) the unit speed minimizing geodesic with respect to $g(t)$ connecting x and y, and $v(s,t)$ being any parallel transport along $\gamma(s)$ such that $v(0,t)$ is the unitary eigenvector corresponding to $\lambda_i(x,t)$. Then we have

$$\Big| tRic(v(d_t(x,y),t),\bar{v}(d_t(x,y),t))|_y - tRic(v(0,t),\bar{v}(0,t))|_x \Big| \leq t|\nabla Ric| d_t(x,y) \tag{2.212}$$

Recall Shi's estimate $|\nabla Ric|^2 \leq \frac{C}{t^3}$ ([103]), the above implies that

$$\lim_{t\to+\infty} tRic(v(d_t(x,y),t),\bar{v}(d_t(x,y),t))|_y = \beta(x)$$

Since we can set $v(0,t)$ to be any unitary vector at x in (2.212), it follows from the minimax principle for eigenvalues that the limits of $\{t\lambda_1(x,t),...,t\lambda_n(x,t)\}$ and $\{t\lambda_1(y,t),...,t\lambda_n(y,t)\}$ must match. □

It is direct to see that the minimal orders $\{d^{(1)}_{min},...,d^{(n)}_{min}\}$ and D_{min} are all invariant under the Type-III Kähler-Ricci flow. Motivated by Proposition 2.5.15, we propose the following conjecture.

Conjecture 2.5.16. *Suppose that $(\mathcal{M}^n, g_0)$ is a complete noncompact Kähler manifold with bounded nonnegative curvature and Euclidean volume growth. Denote*

$g(t)$ ($t \in [0, +\infty)$) the complete solution Kähler-Ricci flow with initial metric g_0 from Theorem 2.5.14, then

(1) Denote $\beta = \lim_{t\to+\infty} tR(x,t)$ and $\gamma_i = \lim_{t\to+\infty} t\lambda_i(x,t)$, can we show that $D_{min} = \beta$ and

$$\{d_{min}^{(1)}, ..., d_{min}^{(n)}\} = \{\gamma_1 + 1, ..., \gamma_n + 1\}?$$

(2) $\mathrm{ASCD}(\mathcal{M}^n, g(t))$ is invariant along $g(t)$.

Expanding Ricci solitons with nonnegative Ricci curvature

In view of Theorem 2.5.14, it is interesting to understand the geometry of gradient expanding Ricci solitons with nonnegative curvature. Gradient Ricci solitons of expanding type has been studied intensively in recent years, see [13], [14], and [79] for example. In this subsection we focus on expanding Ricci solitons with nonnegative Ricci curvature. First of all, it is known that any gradient Ricci soliton of expanding type with nonnegative Ricci curvature is diffeomorphic to the Euclidean space. Moreover, Chau-Tam [21] proved it is biholomrphic to the complex Euclidean space if it is a Kähler-Ricci soliton.

Next we give the estimates of the potential function and the asymptotic volume ratio.

Lemma 2.5.17. *Suppose that $(\mathcal{N}^n, g, f)$ is a complete noncompact gradient expanding Ricci soliton with $Ric + Hess\, f = g$ and nonnegative Ricci curvature, and we set the normalization $R + |\nabla f|^2 = 2f$ and denote O to be the unique critical point of f whose existence is ensured by the assumption of nonnegative Ricci curvature, then $\mathcal{N}^n$ is diffeomorphic to $\mathbb{R}^n$ and the potential function f satisfies:*

$$\frac{1}{2}d(x,O)^2 + f(O) \leq f(x) \leq (\frac{1}{\sqrt{2}}d(x,O) + \sqrt{f(O)})^2 \tag{2.213}$$

and the scalar curvature satisfy the exponential decay:

$$R(x) \geq R(O)e^{-(f(x)-f(O))}.$$

Proof of Lemma 2.5.17. This seems well-known and the proof is quite direct. □

We procced to prove an upper bound for $\mathrm{AVR}(\mathcal{N}^n, g)$ assuming nonnegative sectional curvature.

Proposition 2.5.18. *Let $(\mathcal{N}^n, g, f)$ be a complete noncompact nonflat gradient expanding Ricci soliton as in Lemma 2.5.17. In addition, assume $(\mathcal{N}^n, g)$ has nonnegative sectional curvature. Then there exists a constant $c(n)$ which only depends on the dimension such that $0 < \mathrm{AVR}(\mathcal{N}^n, g) \le \frac{c(n)}{c_0}$ where $c_0 = f(O) = \frac{R(O)}{2}$.*

Proof of Proposition 2.5.18. In fact one only need to assume nonnegative scalar curvature to show that any expanding Ricci soliton has at least Euclidean volume growth, see p6307 [25]. Here we are interested in a more explicit bound on $\mathrm{AVR}(\mathcal{N}^n, g)$ in terms of the maximum of scalar curvature.

Define the functions

$$\mathrm{V} : \mathbb{R} \to [0, \infty), \qquad \mathrm{R} : \mathbb{R} \to [0, \infty)$$

by

$$\mathrm{V}(c) \doteqdot \int_{\{f<c\}} d\mu, \qquad \mathrm{R}(c) \doteqdot \int_{\{f<c\}} R \, d\mu.$$

Similarly as shrinking solitons treated in [19], we can prove the following ODE relating $\mathrm{V}(c)$ and $\mathrm{R}(c)$.

$$n \, \mathrm{V}(c) + \mathrm{R}(c) = 2c \, \mathrm{V}'(c) - \mathrm{R}'(c). \tag{2.214}$$

It is direct to check

$$\frac{d}{dc}\Big[\frac{\mathrm{V}(c)}{c^{\frac{n}{2}}}(2 - \frac{\mathrm{R}(c)}{c\,\mathrm{V}(c)})\Big] = (\frac{n+2}{2c} + 1)\frac{\mathrm{R}(c)}{c^{\frac{n}{2}+1}} \tag{2.215}$$

Note that (2.213) shows that $f(x)$ is comparable to $\frac{1}{2}d(x, O)^2$ and that $R(x)$ attains the maximum at O, denoting $c_0 = f(O)$, an integration shows

$$\mathrm{AVR}(\mathcal{N}^n, g) = \frac{1}{\omega_n 2^{\frac{n}{2}+1}} \int_{c_0}^{+\infty} (\frac{n+2}{2c} + 1)\frac{\mathrm{R}(c)}{c^{\frac{n}{2}+1}} dc \tag{2.216}$$

Now recall the following result due to Petrunin [98]: Let (M^n, g) be any complete Riemannian manifold with nonnegative sectional curvature, then for any $O \in M$ and $r > 0$, there exists a constant $c(n)$ such that

$$\int_{B(O,r)} R \; dVol(g) \le c(n) r^{n-2},$$

In view of (2.213), we have

$$\frac{\mathrm{R}(c)}{c^{\frac{n}{2}+1}} \leq \frac{\int_{B(x,\sqrt{2(c-c_0)})} R\, dVol}{c^{\frac{n}{2}+1}} \leq \frac{c(n)}{c} \tag{2.217}$$

Now Proposition 2.5.18 follows from (2.216) and (2.217). □

Corollary 2.5.19. *Suppose that $(\mathcal{M}^n, g_0)$ is a complete noncompact Riemmanian manifold with bounded and nonnegative curvature operator and Euclidean volume growth. By Schulze-Simon [100] the Ricci flow solution $g(t)$ with initial metric g_0 has the long time existence and it is Type-III. Denote $\beta = \sup_{x\in\mathcal{M}^n}\beta(x) = \lim_{t\to+\infty} tR(x,t)$, then $0 < \mathrm{AVR}(\mathcal{M}^n, g) \leq \frac{c(n)}{\beta}$. The same conclusion is true for Kähler manifolds with bounded and nonnegative complex curvature operator and Euclidean volume growth, and $\beta(x)$ is independent of x.*

Geometry of non-negatively curved Kähler-Ricci expanding soliton

In this part, we focus on gradient expanding Kähler-Ricci soliton with nonnegative curvature. As mentioned in previous subsection, the complex structure of such solitons is well-understood, Chau and Tam [21] proved any gradient expanding Kähler-Ricci soliton with nonnegative Ricci curvature is biholomorphic to $\mathbb{C}^n$. Recall that the main theorem of Chau and Tam [23] states any complete noncompact Kähler manifold with bounded and nonnegative bisectional curvature and Euclidean volume growth is biholomorphic to $\mathbb{C}^n$. In view of Theorem 2.5.14, those results could be viewed as a stability result on complex structures when blowing down Kähler-Ricci flow on such manifolds. Again Theorem 2.5.14 tells us that AVR is also preserved when we blow down such a Kähler-Ricci flow. These results naturally lead us to further investigate the metric geometry of such manifolds in the context of this blowing down procedure.

We will solve the minimal order for holomorphic functions and sections of canonical line bundles on complete gradient expanding Kähler-Ricci solitons with nonnegative bisectional curvature. Therefore we confirm Conjecture 2.5.16 for expanding solitons with nonnegative bisectional curvature.

Theorem 2.5.20. *Assume that $(\mathcal{N}^n, g, f)$ is a complete noncompact gradient expanding Kähler-Ricci soliton with $R_{i\bar{j}} + g_{i\bar{j}} = f_{i\bar{j}}$ and nonnegative Ricci curvature,* *Let us set the normalization $R + |\nabla f|^2 = 2f$ and $\{\mu_1, ..., \mu_n\}$ eigenvalues of Ricci curvature at the critical point O of f in the nondecreasing order, then*

$$\{d_{min}^{(1)}, ..., d_{min}^{(n)}\} = \{\mu_1 + 1, ..., \mu_n + 1\}.$$

Furthermore, if we assume $(\mathcal{N}^n, g)$ has nonnegative bisectional curvature, then $D_{min} = R(O) = \sum_{i=1}^n d_{min}^{(i)} - n$ where $R(O)$ is the maximum of scalar curvature.

Proof of Theorem 2.5.20. It is known from [21] that the expanding Kähler-Ricci soliton in our Theorem 2.5.20 must be biholomorphic to $\mathbb{C}^n$. In fact, we can use Bryant [11] to construct the so-called Poincaré coordinate system on $(\mathcal{N}^n, g)$. Indeed, it is proved in [11] that there exist local coordinates $(\mathcal{U}_O, z_i)$ at O such that the complexified vector field $\nabla f = \sum_i (1 + \mu_i) z_i \frac{\partial}{\partial z_i}$, then one can extend $(\mathcal{U}_O, z_i)$ to the whole $(\mathcal{N}^n, g)$ by defining

$$z_i(\varphi(q, t)) = e^{(1+\mu_i)t} z_i(q) \tag{2.218}$$

for any $q \in \mathcal{U}_O$ where $\varphi(q, t)$ is the holomorphic flow generated by ∇f. It is easy to see this definition is independent of choice of q and $\{z_i\}$ gives a global holomorphic coordinate on $(\mathcal{N}^n, g)$.

Next we will solve the growth orders for holomorphic functions $\{z_1, ..., z_n\}$. First we will solve the growth order for holomorphic functions z_k along any integral curves of ∇f through a point q on $(\mathcal{N}^n, g)$ with $z_k(q) \neq 0$. It follows from $R + |\nabla f|^2 = 2f$ that

$$f(\varphi(q, t)) - f(q) = \int_0^t |\nabla f(\varphi(q, s))|^2 ds \leq \int_0^t 2f(\varphi(q, s)) ds. \tag{2.219}$$

Now a simple integration on (2.219) leads to $f(\varphi(q, t)) \leq f(q) e^{2t}$.

Therefore,

$$\limsup_{t \to +\infty} \frac{\log z_k(\varphi(q, t))}{\log d(\varphi(q, t), O)} = \limsup_{t \to +\infty} \frac{\log z_k(\varphi(q, t))}{\log \sqrt{2f(\varphi(q, t), O)}} \geq 1 + \mu_i \tag{2.220}$$

Next we will prove $\limsup_{x \to +\infty} \frac{\log z_k(x)}{\log d(x, O)} \leq \mu_k + 1$. Let $\{x_i\}$ be any sequence of points on $(\mathcal{N}^n, g)$ tending to infinity, it suffices to show $\limsup_{i \to +\infty} \frac{\log z_k(x_i)}{\log d(x_i, O)} \leq \mu_k + 1$. Unlike the straightforward estimate (2.219) of f restricted an integral curve, we need to be more careful to prove a uniform estimate which holds outside certain level set of f.

Note that $\sqrt{2f}$ is comparable to distance function and O is the unique critical point of f, we see that $f(\varphi(q,t)) \to +\infty$ as $t \to +\infty$. Pick any $a > R(O)$ and denote $b = \sup_{f(y)=a} R(y)$ and $c = \sup_{f(y)=a} |z_k(y)|$. For every point x_i, the integral curve $\varphi(x_i, t)$ converges to O as $t \to -\infty$. Let $\varphi(q, -t_i)$ denote the first intersection point of the integral curve $\varphi(x_i, t)$ and the level set $\{f = a\}$. we have the following lower estimate of f:

$$f(x_i) - f(\varphi(x_i, -t_i)) \geq \int_{-t_i}^{t} (2f(\varphi(x_i, s)) - b) ds \tag{2.221}$$

for any $-t_i \leq t \leq 0$.

It follows from (2.221) that $f(x_i) \geq \frac{1}{2}e^{2t_i}(2a - b) + \frac{b}{2}$, which immediately leads to

$$\limsup_{i\to+\infty} \frac{\log z_k(x_i)}{\log d(x_i, O)} \leq \limsup_{i\to+\infty} \frac{2\log(ce^{(1+\mu_k)t_i})}{\log\left[(e^{2t_i})(2a - b) + b\right]} \leq \mu_k + 1. \tag{2.222}$$

To sum up, we have proved that $\limsup_{x\to+\infty} \frac{\log z_k(x)}{\log d(x,O)} = \mu_k + 1$ for any $1 \leq k \leq n$.

Now we observe that Poincaré coordinates give the minimal growth orders for holomorphic functions on $\mathcal{N}^n$. Otherwise, there exist

$$\{h_1(z_1, ...z_n), ..., h_n(z_1, ..., z_n)\}$$

which are n holomorphic functions with polynomial growth which are linearly independent at O, with its growth orders $\{\nu_1, ..., \nu_n\}$. Assume that both $\{\mu_1 + 1, ..., \mu_n + 1\}$ and $\{\nu_1, ..., \nu_n\}$ are in a non-decreasing order, let i_0 denote the first $1 \leq i \leq n$ such that $\nu_i < \mu_i + 1$, If $i_0 = 1$ then we will show that h_1 must be constant.

Indeed, assume that h_1 has growth order ν_1 strictly less than $\mu_1 + 1$, note that $\mathcal{N}^n$ is biholomorphic to $\mathbb{C}^n$, then the Cauchy estimate implies at any point $p = (w_1, ..., w_n)$ in terms of Poincaré coordinates:

$$\frac{\partial h_1}{\partial z_1}(w_1, ..., w_n) \leq \frac{1}{r} \sup_{q\in D} |h_1(q)| \tag{2.223}$$

where $D = \{(z_1, w_2, ..., w_n) \in \mathcal{N}^n \mid |z_1 - w_1| = r\}$ for any $r > 0$.

Now by a similar argument as when we derive (2.221) one conclude that for any $q \in D$ there exists some $t(q) > 0$ such that

$$r \geq |z_1(q)| - |w_1(p)| \geq C_1 e^{(1+\mu_1)t(q)} - |w_1(p)| \tag{2.224}$$

for some constant C_1. In the meantime,

$$|f(q)| \leq C_2(d(q,O)^{\nu_1}) \leq C_2[\sqrt{2f(q)}]^{\nu_1} \leq C_3 e^{\nu_1 t(q)}. \tag{2.225}$$

Take r goes to infinity, it follows from (2.223), (2.224), and (2.225) that

$$\frac{\partial h_1}{\partial z_1}(w_1, ..., w_n)$$

vanishes everywhere and h_1 is constant, which it is impossible since

$$\{h_1(z_1, ...z_n), ..., h_n(z_1, ..., z_n)\}$$

are linearly independent at O.

If $i_0 > 1$ then a same argument implies that $\frac{\partial f_{i_0}}{\partial z_i}$ must be zero for any $i_0 \leq i \leq n$. It follows that $f_j = f_j(z_1, ..., z_{i_0 - 1})$ for any $1 \leq j \leq i_0$. it shows that $\{f_1, ..., f_n\}$ are linearly dependent at O.

Next we solve D_{min}. It is well-known that any holomorphic vector bundle over $\mathbb{C}^n$ are trivial. Therefore any holomorphic sections of $K_{\mathcal{N}^n}$ will be the form of

$$f(z_1, ..., z_n) dz_1 \wedge dz_2, ..., \wedge dz_n$$

where f is a global holormophic function on $\mathcal{N}^n$. We only need to prove that $s_0 = dz_1 \wedge dz_2, ..., \wedge dz_n$ has the growth order exactly $R(O) = \sum_{i=1}^n \mu_i$. Our calculation relies on this following idenity:

$$\frac{d^2}{dt^2}\Big[-\log\det g_{i\bar{j}}(\varphi(q,t))\Big] = -2\frac{\partial^2}{\partial z_i \partial \bar{z}_j}(\log\det g_{i\bar{j}}) f^i f^{\bar{j}} = 2Ric(\nabla f, \nabla f). \tag{2.226}$$

Now pick any sequence of points x_i which goes to infinity, a further integration leads to

$$\frac{d}{dt}\Big[-\log\det(\varphi(x_i,t))\Big] = 2R(O) - 2R(\varphi(x_i,t)) \tag{2.227}$$

where we use $\frac{d}{dt}[-\log\det(\varphi(q,t))]|_{t=-\infty}=0$. Recall (2.221), we are able to solve the growth order of s_0 along the integral curve of ∇f:

$$\limsup_{i\to+\infty}\frac{-\frac{1}{2}\log\det g_{i\bar{j}}(x_i)}{\log d(x_i,O)}=\limsup_{t\to+\infty}\frac{-\frac{1}{2}\log\det g_{i\bar{j}}(\varphi(q,t))}{\frac{1}{2}\log(2f)}\leq R(O). \tag{2.228}$$

In fact, the above argument also implies that

$$\limsup_{i\to+\infty}\frac{-\frac{1}{2}\log\det g_{i\bar{j}}(x_i)}{\log d(x_i,O)}\geq R(O)-\lim_{t\to+\infty}R(x_i) \tag{2.229}$$

Now if we assume in addition that $(\mathcal{N}^n,g)$ has nonnegative bisectional curvature, it follows from [32] that the scalar curvature of $(\mathcal{N}^n,g)$ has at least linear average decay. Therefore, there exists a sequence of points z_i such that $R(z_k)\to 0$ which z_k tends to infinity. We may replace x_i in (2.229) by z_k to conclude that

$$D_{min}=\limsup_{z\to+\infty}\frac{\log||s_0||(z)}{\log d(z,O)}=\limsup_{z\to+\infty}\frac{-\frac{1}{2}\log\det g_{i\bar{j}}(z)}{\log d(z,O)}=R(O). \tag{2.230}$$

□

Remark 2.5.21. *In the proof of Theorem 2.5.20 we show that any holomorphic function f on $(\mathcal{N}^n,g,f)$ with growth order $\mu_k\leq d<\mu_{k+1}+1$ where $1\leq k\leq n-1$ must be $f=f(z_1,...,z_k)$. This is of course not necessarily true without working in Poincaré coordinates.*

The only known examples of Kähler-Ricci expanders with nonnegative Ricci curvature are those rotationally symmetric ones on $\mathbb{C}^n$. In particular in the complex dimension 2 case for any $\lambda>1$ one can construct such examples $(\mathbb{C}^2,g,f)$ whose holomorphic vector field $\nabla f=\lambda z_1\frac{\partial}{\partial z_1}+\lambda z_1\frac{\partial}{\partial z_1}$, here (z_1,z_2) are the global Poincaré coordinates on $\mathbb{C}^2$. On the other hand, let $(w_1,w_2)=\varphi(z_1,z_2)=(z_1,z_2+z_1^2)\in Aut(\mathbb{C}^2)$, on $(\mathbb{C}^2,g)$ with global coordinates (w_1,w_2) we have $z_2(w_1.w_2)=w_2-w_1^2$ has growth order λ while w_1 and w_2 have growth orders λ and 2λ respectively.

Section 2.2 is an exposition of my own work which appeared in Mathematische Annalen 2013. Section 2.3 and Section 2.4 are from a joint work with Fangyang Zheng which was published in Communications in Analysis and Geometry 2013. The remaining results of Chapter 2 are from my ongoing project and it will appear as a separate paper [113] and be submitted elsewhere for consideration for publication in the near future.

Appendix A

Additional results on $U(n)$-invariant Kähler metrics with nonnegative curvature

In this appendix we discuss some additional results on $U(n)$-invariant Kähler metrics with nonnegative curvature. Section A.1 is devoted to proving an important lemma used in the proof of Theorem 2.4.6. In Section A.2, as an illustration of the minimal orders for holomorphic functions on complete Kähler manifolds with nonnegative bisectional curvature, we give a detailed discussion on the case of $U(n)$ invariant Kähler metrics on $\mathbb{C}^n$ with positive curvature. (Similar results follows if we only assuming nonnegative bisectional curvature.) In Section A.4 we give some observations on holomorphic functions on complete noncompact Riemann surfaces which mainly follow from previous works of Li and Tam.

A.1 Proof of (2.122)

Recall that $\overline{\mathcal{M}}_n$ denote the space of all $U(n)$ invariant complete Kähler metrics on $\mathbb{C}^n$ with nonnegative bisectional curvature. In this section we will prove (2.122) which is rephrased as follows:

Lemma A.1.1. *For any metric in $\overline{\mathcal{M}}_n$, we have*

$$\nabla_s B = \frac{2x}{v}(A-2B) \text{ and } \nabla_s C = \frac{2x}{v}(2B-C), \tag{A.1}$$

where s denotes the radial geodesic distance.

Proof of Lemma A.1.1. If we assume that our metric is in $\mathcal{M}_n$, then the proof of (A.1) is just a matter of a standard calculation following (2.16) and (2.15).

However, since we are working on $\overline{\mathcal{M}}_n$ which can not always be be characterized by F, we have to discuss other cases arising from Theorem 2.2.4. More precisely, We have to prove that (A.1) is true for any metric in S_3 from Theorem 2.2.4. For the convenience of the reader, we also include the verification of (A.1) for metrics in S_1 and S_2.

Indeed, given any metrics in S_1 and S_2, it follows from (2.16) and (2.15) that

$$\nabla_s B = \frac{\frac{\partial B}{\partial x}}{\sqrt{1+(F'(x))^2}} \tag{A.2}$$

and

$$\begin{aligned}\frac{\partial B}{\partial x} &= -\frac{2x\sqrt{1+(F'(x))^2}}{v^2}\left(\frac{x^2}{v} - \frac{1}{\sqrt{1+(F'(x))^2}}\right) \\ &\quad + \frac{1}{v}\left(\frac{2x}{v} - \frac{2x^3\sqrt{1+(F'(x))^2}}{v^2} + \frac{F'F''}{(\sqrt{1+(F'(x))^2})^3}\right). \end{aligned} \tag{A.3}$$

Therefore

$$\begin{aligned}\nabla_s B &= -\frac{4x^3}{v^3} + \frac{4x}{v^2\sqrt{1+(F'(x))^2}} + \frac{F'F''}{v[1+(F'(x))^2]^2} \\ &= \frac{2x}{v}(A-2B). \end{aligned} \tag{A.4}$$

And a similar calculation works for $\nabla_s C$.

Next we will verify (2.16) for metric in S_3. Since now the metric can not be globally parameterized by F, we turn to use the parameter r outside a compact set. Indeed, it follows from (2.31), (2.32), (2.33), and (2.34) that

$$\nabla_s B = \frac{\frac{\partial B}{\partial r}}{\frac{\partial s}{\partial r}} = \frac{2\sqrt{r}}{\sqrt{h}}\frac{\partial B}{\partial r}, \tag{A.5}$$

and

$$\frac{\partial B}{\partial r} = \frac{-2hx_0^2}{(rf)^3}. \tag{A.6}$$

Therefore,

$$\nabla_s B = \frac{-4\sqrt{rh}}{rf} B = -\frac{4x}{v} B. \tag{A.7}$$

Similarly we have

$$\frac{\partial C}{\partial r} = 2\frac{\partial}{\partial r}[\frac{rf - rh}{(rf)^2}], \tag{A.8}$$

and

$$\nabla_s C = 4\sqrt{rh}\frac{2h - f}{r^2 f^3} = \frac{2x}{v}(2B - C). \tag{A.9}$$

In view of Theorem 2.2.4, we have proved Lemma A.1.1.

□

A.2 Holormophic functions and volume growth

We also have the following result relating the growth of the coordinate function z_i to the volume growth of the geodesic balls with respect to the metric ω in $\overline{\mathcal{M}}_n$.

Proposition A.2.1. *Given any metric $\omega \in \overline{\mathcal{M}}_n$, if the coordinate function z_i for some $1 \leq i \leq n$ has polynomial growth with respect to ω, then the geodesic balls of $(\mathbb{C}^n, \omega)$ have Euclidean volume growth.*

Proof of Proposition A.2.1. Assume some coordinate function z_i for some $1 \leq i \leq n$ has polynomial growth with respect to ω in $\overline{\mathcal{M}}_n$. it follows from ω being rotationally symmetric that there exists some integer α and constant $C_6 > 0$ such that:

$$r = |z|^2 \leq C_6 s^\alpha. \tag{A.10}$$

From Theorem 2.2.4 it suffices to show that $\omega \in S_1$, namely $F'(x)$ bounded when x goes to x_0. First we note that ω can not be from S_3 from the explicit formula (2.34) on the distance with respect to metrics in S_3 in Theorem 3.1.

Plugging (2.15) into (A.10) leads to:

$$r \leq C_6(\int_0^x \sqrt{1 + (F'(\tau))^2} d\tau)^\alpha. \tag{A.11}$$

Note that:

$$\frac{dr}{dx} = \frac{2r}{(1-\xi)x} = \frac{2r\sqrt{1+(F'(x))^2}}{x}. \tag{A.12}$$

Solve r in terms of x from (A.12) and plug into (A.11):

$$e^{2\int_{C_7}^{x} \frac{\sqrt{1+(F'(\tau))^2}}{\tau} d\tau} \leq C_6\left(\int_0^x \sqrt{1+(F'(\tau))^2} d\tau\right)^{\alpha} \tag{A.13}$$

for any $C_7 \leq x < x_0$. Here C_7 is the value of x which corresponds to $r = 1$.

It is not hard to show that $F'(x)$ is bounded for all $x \in (0, x_0)$ from (A.13). First we see that x_0 must be infinity. Otherwise, the left hand side

$$e^{2\int_{C_7}^{x} \frac{\sqrt{1+(F'(\tau))^2}}{\tau} d\tau} \geq e^{\frac{2}{x_0}\int_{C_7}^{x} \sqrt{1+(F'(\tau))^2} d\tau},$$

then (A.13) could not be true since exponential functions can not be bounded from above by any polynomial as $\int_{C_7}^{x} \sqrt{1+(F'(\tau))^2} d\tau$ goes to infinity. Next we show that $F'(x)$ is bounded for all $x \in (0, +\infty)$. It follows from (A.13) that

$$\frac{2\int_{C_7}^{x} \frac{\sqrt{1+(F'(\tau))^2}}{\tau} d\tau}{\alpha \ln \int_0^x \sqrt{1+(F'(\tau))^2} d\tau + \ln C_6} \tag{A.14}$$

should be bounded by 1 when x tends to infinity.

It is easy to see that (A.14) has a limit when x goes to infinity.

$$\lim_{x\to+\infty} \frac{2\int_{C_7}^{x} \frac{\sqrt{1+(F'(\tau))^2}}{\tau} d\tau}{\alpha \ln \int_0^x \sqrt{1+(F'(\tau))^2} d\tau + \ln C_6} = \frac{2}{\alpha}\sqrt{1+[\lim_{x\to+\infty} F'(\tau)]^2}, \tag{A.15}$$

which implies that $F'(x)$ is bounded for all x in $[0, +\infty)$. Therefore $\omega \in S_1$. $\square$

Remark A.2.2. *It was conjectured by Ni (See Conjecture 3.1 on p.931 in [81].) that Proposition A.2.1 is true in general for any complete Kähler manifold with quasi-positive bisectional curvature.*

As an illustration of minimal orders for holomorphic functions on complete Kähler manifolds with nonnegative bisectional curvature, we give a detailed discussion on the case of $U(n)$ invariant Kähler metrics on $\mathbb{C}^n$ with positive curvature. Similar results follows if we only assuming nonnegative bisectional curvature.

First recall the following formulas will be directly from [108].

$$v = \int_0^x 2\tau\sqrt{1+(F'(\tau))^2}d\tau. \tag{A.16}$$

$$s = \int_0^x \sqrt{1+(F'(\tau))^2}d\tau$$

$$\log|z| = \int_C^x \frac{\sqrt{1+(F'(\tau))^2}}{\tau}d\tau$$

Note that the coordinate function gives the minimal order growth for any $U(n)$-invariant Kähler metric with positive bisectional curvature on $\mathbb{C}^n$. Follow the definition in the next section, we have $K_- = \liminf_{x\to+\infty}\frac{\log v}{\log s}$, $K_+ = \limsup_{x\to+\infty}\frac{\log v}{\log s}$ and $d_0 = \limsup\frac{\log(\log|z|)}{\log s} = \limsup\frac{\log(\int_C^x \frac{\sqrt{1+(F'(\tau))^2}}{\tau})d\tau}{\log s}$.

It is easy to see that $1 \le K_- \le K_- \le 2$ and $0 \le d_0 \le 1$,

Example A.2.3. *(1) Two extreme cases: $K_+ = K_- = d_0 = 1$ when the metric is a cigar (i.e. x_0 is finite); $K_+ = K_- = 2$ and $d_0 = 0$ when the metric has Euclidean volume growth. (i.e. $x_0 = +\infty$ and $F'(x_0)$ is finite.)*

(2) Two almost extreme examples: when $F(x) = \int_0^x \sqrt{\log^2(\tau+e)-1}d\tau$ then $d_0 = 0$ and $K_+ = K_- = 2$, the volume growth is like $\mathrm{Vol}(B(O,s)) \sim \frac{s^{2n}}{(\log s)^n}$. When $F(x) = e^x - x - 1$ then $d_0 = K_+ = K_- = 1$, and the volume growth is like $\mathrm{Vol}(B(O,s)) \sim (s\log s)^n$.

Now we give a summary of our calculation in the $U(n)$-symmetry case.

Result 1. *Assume that $x_0 = +\infty$ and $F'(x_0) = +\infty$. we have the following results.*

(1)

$$1 \le 1 + \liminf_{x\to+\infty}\frac{F(x)}{xF'(x)} \le K_- \le K_+ \le 1 + \limsup_{x\to+\infty}\frac{F(x)}{xF'(x)} \le 2.$$

(2) If $\lim_{x\to+\infty}\frac{\int_C^x \frac{F'(\tau)}{\tau}d\tau}{F'(x)}$ exists, then

$$d_0 = \frac{1}{1+\lim_{x\to+\infty}\frac{\int_C^x \frac{F'(\tau)}{\tau}d\tau}{F'(x)}}$$

In general, we only have:

$$d_0 \le \frac{1}{1+\liminf_{x\to+\infty}\frac{\int_C^x \frac{F'(\tau)}{\tau}d\tau}{F'(x)}}$$

(3) If $\lim_{x\to+\infty}\frac{F(x)}{xF'(x)}=a_0$*, then* $\lim_{x\to+\infty}\frac{\int_C^x\frac{F'(\tau)}{\tau}d\tau}{F'(x)}$ *exists and the limit is* $\frac{a_0}{1-a_0}$*. Therefore,* $K_-=K_+=2-d_0=1+a_0$*.*

(4) In general, then we have

$$1-\limsup_{x\to+\infty}\frac{F(x)}{xF'(x)}\le 2-K_+\le 2-K_-=d_0\le 1-\liminf_{x\to+\infty}\frac{F(x)}{xF'(x)}$$

Remark A.2.4. *Does there exist an example of* $U(n)$ *invariant Kähler metric on* $\mathbb{C}^n$ *with positive bisectional curvature such that* $K_-<K_+$*? It is probably true in view that* $\lim_{x\to+\infty}\frac{F(x)}{xF'(x)}$ *does not necessarily exist.*

Result 2. *Recall* F *is the generating function for a* $U(n)$*-invariant Kähler metric positive bisectional curvature on* $\mathbb{C}^n$*.*

(1) $2-K_-\le d_0$ *can be proved simply by the Cauchy-Schwarz inequality. Of course, as mentioned earlier it is true in general without* $U(n)$*-symmetry on* $\mathbb{C}^n$*.*

(2) In view of [108], the generating function $F(x)$ *relates any* $U(n)$*-invariant Kähler metric positive bisectional curvature on* $\mathbb{C}^n$ *to a rotationally symmetric positive curved metric on* $\mathbb{R}^2$*. We see that* $2-K_-\ge d_0$ *holds from Corollary A.4.2. However it is unclear how to prove it by the direct calculation.*

A.3 More on $U(n)$-invariant Kähler-Ricci flows

First we recall the result of Wu-Zheng [108]. Let F is any smooth function, even function on $(-\infty,+\infty)$ such that

$$F(0)=F'(0)=0,\ F''\ge 0,\ F'(+\infty)<+\infty,\ \ F(+\infty)=+\infty,$$

then we can construct a complete $U(n)$-invariant Kähler metric g_0 on $\mathbb{C}^n$ with nonnegative bisectional curvature and Euclidean volume growth. Since such a correspondence is one to one, we may call this F *the generating function* of g_0. Now we state a necessary condition for $U(n)$-invariant Kähler metrics on $\mathbb{C}^n$ with positive bisectional curvature to be a positive-time slice of a Type-III solution to Kähler-Ricci flow with bounded and positive curvature.

Proposition A.3.1. *Suppose that* $(\mathbb{C}^n,g_0)$ *is a complete* $U(n)$*-invariant Kähler metric with bounded and nonnegative bisectional curvature and Euclidean volume* *growth, then there exists a complete $U(n)$-invariant solution $g(t)$ to Kähler-Ricci flow with the initial metric g_0 on $\mathbb{C}^n \times [0,+\infty)$. Assume that $(\mathbb{C}^n, g_1)$ is the t_1-slice of $g(t)$ for some number $t_1 > 0$, then the generating function $F(x)$ of g_1 must satisfy:*

$$\frac{n+1}{4}(F''(0))^2 t_1 \leq \sqrt{1+F'(+\infty)^2} - 1. \tag{A.17}$$

A.4 Holomorphic functions on Riemann surfaces

Let $(\mathcal{M}^2, g)$ denote a complete noncompact Riemann surface with nonnegative curvature. It is well known that $(\mathcal{M}^2, g)$ is biholomorphic to $\mathbb{C}^1$ unless it is flat.

Lemma A.4.1 (Li-Tam [68] and [69]). *Let $(\mathcal{M}^2, g)$ be a simply connected Riemann surface with nonnegative curvature. Note that $\mathcal{M}^2$ is diffeomorphic to $\mathbb{R}^2$, let $r_0(x)$ and $r(x)$ be the Euclidean and geodesic distance with respect to a fixed point O. then there exists C_1 and R_1 such that*

$$\frac{1}{C_1}\left(\int_1^{r(t)} \frac{t}{V(t)}dt - 1\right) \leq \log r_0(x) \leq C_1\left(\int_1^{r(t)} \frac{t}{V(t)}dt + 1\right) \tag{A.18}$$

for any $x \in \mathcal{M}^2 - B(O, R_0)$.

In particular, if $\mathrm{AVR}(\mathcal{M}^2, g) = \alpha > 0$, then

$$\lim_{r(x)\to+\infty} \frac{\log r(x)}{\log r_0(x)} = \alpha. \tag{A.19}$$

holds, and there exists C_2 and R_1 such that $\log r_0(x) \leq C_1 r(x)$ for any $x \in \mathcal{M}^2 - B(O, R_2)$.

Corollary A.4.2. *Let $(\mathcal{M}^2, g)$ be a simply-connected open Riemann surface with nonnegative curvature. then $d_0 = 2 - K_-$. Moreover, both Conjecture 2.5.6 and 2.5.8 are true except we only have $d_{min} - 1 \leq D_{min} \leq d_{min}$.*

Proof of Corollary A.4.2. It is a consequence of [68] and [69]. □

Proposition A.4.3. *Conjecture 2.5.16 is true for any complete noncompact Riemann surface with nonnegative curvature and Euclidean volume growth.*

Proof of Corollary A.4.2. It is a consequence of [12], [52], and [100]. □

Appendix B

Construction of holomorphic functions by the L^2–method

We will prove the following proposition mentioned in Section 2.5.

Proposition B.0.4. *Let $(\mathcal{M}^n, g)$ be a complete noncompact n-dimensional Kähler manifold with nonnegative bisectional curvature, assume that the universal cover $(\widetilde{\mathcal{M}}^n, \widetilde{g})$ does not split, then*

(1) $(\mathcal{M}^n, g)$ admits a nonconstant holomorphic function with its Hadamard's order $d_H \leq 1$. The canonical line bundle K_M on $\mathcal{M}^n$ admits nontrival sections with at most exponential growth.

(2) Either the transcendence degree $K(\mathcal{M}^n) = 0$ or $K(\mathcal{M}^n) = n$. Moreover, in the case of $K(\mathcal{M}^n) = n$, at any point $O \in \mathcal{M}^n$ there exists n holomorphic functions with polynomial growth which give local coordinates at O and $\pi_1(\mathcal{M}^n)$ is finite.

Proof of Proposition B.0.4 relies on the following general formulation of the well-known L^2-estimate of the $\bar{\partial}$ operator.

Theorem B.0.5 (See [1] and [44] for example)**.** *Let (M, g) be a complete Kähler manifold with dimension n and (L, h) be a Hermitian line bundle with nonnegative curvature on M. Suppose φ is a smooth function outside a discrete set S on M and for any point $p \in S$ there exists a constant C_p and a open neighborhood (U_p, z)*

such that $\varphi(z) = C_p \log |z|^2$ *on* U_p*. Assume that*

$$Ric(\omega_g) + \Theta(L, he^{-\varphi}) = Ric(\omega_g) + \Theta(L, h) + \sqrt{-1}\partial\bar{\partial}\varphi > \epsilon\omega_g \tag{B.1}$$

on $M \setminus S$ *for some continuous function* $\epsilon : M \to [0, 1]$*. Then for every smooth* L*-valued* $(0, 1)$ *form* θ *which satisfies*

$$\bar{\partial}\theta = 0 \ \textit{and} \int_M \frac{1}{\epsilon}\|\theta\|^2 e^{-\varphi}\omega_g^n < \infty, \tag{B.2}$$

there exists a smooth L*-valued function* u *on* M *such that*

$$\bar{\partial}u = \theta \ \textit{and} \int_M \|u\|^2 e^{-\varphi}\omega_g^n \leq \int_M \frac{1}{\epsilon}\|\theta\|^2 e^{-\varphi}\omega_g^n < \infty. \tag{B.3}$$

Remark B.0.6. *Part (1) of Proposition B.0.4 has been known if we assume holomorphic bisectional curvature is quasi-positive. (See Siu [105] for example.) Note that we use a slightly weaker assumption that the universal cover* $(\widetilde{\mathcal{M}}^n, \widetilde{g})$ *does not split. However, as we show below, applying a result by Ni and Tam our assumption will lead to the existence of strictly plurisubharmonic functions, which is exactly the necessary ingredient to apply the* L^2 *method.*

Proof of Proposition B.0.4. **Step 1**: We show the existence of a strictly plurisubharmonic function on $(\mathcal{M}^n, g)$.

Recall that the Busemann function $\beta(x)$ on Lipschitz with Lipschitz constant 1 on any complete Riemannian manifold. It is proved by Wu and it is a continuous plurisubharmonic function if $(\mathcal{M}^n, g)$ has nonnegative bisectional curvature. Let $v(x, t)$ be the solution of the heat equation with initial value $v(x, t) = \beta(x)$. It is proved by Ni and Tam [88] that for any $t > 0$, $|\nabla v(x, t)| \leq 1$ and the null space

$$\mathcal{K}(x, t) = \{w \in T_x^{1,0}(\mathcal{M}) \mid v_{\alpha\bar{\beta}} w^\alpha = 0 \text{ for all } \beta\}$$

of $v_{\alpha\bar{\beta}}$ is a parallel distribution on $\mathcal{M}$. Therefore the null space must be trivial since we assume $(\widetilde{\mathcal{M}}^n, \widetilde{g})$ does not split, which leads to a family of strictly plurisubharmonic function on $(\mathcal{M}^n, g)$. Consider the heat deformation of $max(0, \beta(x))$ we may assume there exists a strictly plurisubharmonic function ϕ such that

$$0 \leq \phi(x) \leq C(d(x, O) + 1) \tag{B.4}$$

for some constant C.

Step 2: The detailed construction of holomorphic functions and canonical sections by the L^2-method.

Fix any point p on $\mathcal{M}$ and denote (V_p, z) a holomorphic coordinates, we may assume that $p \in \overline{U} \subset W \subset \{z \in V_p \,|\, |z| < 1\} \subset V_p$ where both U and W are open neighborhoods at p. Define $\chi_p(x) = \rho(x) \log |z(x)|^2$ where $\rho(x) : \mathcal{M} \to \mathbb{R}$ a smooth function with $\rho(x) = 1$ on U and $\rho(x) = 0$ outside W. Note that $\chi_p(x)$ is now a well-defined smooth function on the whole $\mathcal{M}$.

Fix $d_0 \in \mathbb{Z}$ and pick $d \in \mathbb{Z}$ large enough, we define $\xi(x) = z^\alpha + O(|z|^{d+1})$ where α is a multi-index and $|\alpha| = d_0$ inside $\{z \in V_p \,|\, |z| < 1\}$, Note that $\xi(x)$ is holomorphic inside $\{z \in V_p \,|\, |z| \leq 1\}$, now extend it smoothly such that $\xi = 0$ outside $\{z \in V_p \,|\, |z| < 2\}$. It is clear that $\xi(x)$ is well-defined and $\bar{\partial}\,\xi$ is a $(0,1)$ form on $\mathcal{M}$.

Let $\mu = d + m + 2$, we claim that there exists a continuous function $\epsilon : \mathcal{M} \to [0,1]$ such that

$$\sqrt{-1}\partial\bar{\partial}\left[\mu(\phi(x) + \chi_p(x))\right] \geq \epsilon(x)\,\omega_g, \tag{B.5}$$

and

$$\int_{\mathcal{M}} \frac{1}{\epsilon(x)} |\bar{\partial}\,\xi|^2 e^{-\mu(\phi(x)+\chi_p(x))}\omega_g^n < \infty. \tag{B.6}$$

Indeed since $\phi(x)$ is strictly plurisubharmonic it is easy to find a continuous function $\epsilon(x)$ satisfying (B.5) and (B.6). Note that $\epsilon(x)$ will be positive inside U and outside W. Thought it may have zeros inside $W \setminus U$, (B.6) holds since $\bar{\partial}\,\xi$ vanishes inside $\{z \in V_p \,|\, |z| \leq 1\}$.

Now applying Theorem B.0.5 to the trivial line bundle $L = \mathcal{M} \times \mathbb{C}^1$ and the equation $\bar{\partial}\eta = \bar{\partial}\xi$, we conclude that there exists a smooth function on $\mathcal{M}$ such that:

$$\int_{\mathcal{M}} |\eta|^2 e^{-\mu(\phi(x)+\chi_p(x))}\omega_g^n < \infty. \tag{B.7}$$

First we note that (B.7) and our choice of μ implies that η has to vanish up to the order $d+1$; Second it follows from (B.7) and there exists a constant C_1 (B.4) which depends on C and μ such that

$$\int_{\mathcal{M}} |\eta|^2 \omega_g^n < e^{C_1(d(x,O)+1)}. \tag{B.8}$$

Denote $f = \xi - \eta$ then we get a nontrivial holomorphic function which has vanishing order d_0. Moreover, there exists some constant C_2

$$\int_{\mathcal{M}} |f|^2 \omega_g^n < e^{C_2(d(x,O)+1)}, \tag{B.9}$$

where we make use the assumption that ξ is compact supported.

Now it follow from the mean value inequality of holomorphic function that for some constant C_3

$$|f| < e^{C_3(d(x,O)+1)}. \tag{B.10}$$

That is to say f is of at most Hardamard's order 1.

It is direct to see that the same argument works if we choose $E = K_{\mathcal{M}}$ to be the canonical line bundle over $\mathcal{M}$. In this case, we will get nontrivial canonical section with Hardamard's order no more than 1.

Step 3: The alternative of the transcendence degree: $K(\mathcal{M}^n) = 0$ or n.

Suppose there exists a nonconstant holomorphic function of polynomial growth on $\mathcal{M}$, say f_0. It is well known that $\log(|f_0|^2 + 1)$ is a smooth plurisubharmonic function. Considering the heat deformation of $\log(|f_0|^2 + 1)$ as Step 1, the result of Ni and Tam will imply that we can get a strictly plurisubharmonic function on $\mathcal{M}$, then we can apply L^2 method as in Step 2 to construct holomorphic functions with precised local expansion at any given point p. It follows from Proposition 5.1 (p.940 in [81]) that $K(\mathcal{M}^n) = n$. Moreover, Corollary 0.1 in [81] implies that $\pi_1(\mathcal{M})$ is finite.

□

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