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Author
Krandel, Jared

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Lipschitz decompositions of domains with bilaterally flat boundaries

Jared Krandel

Abstract

We study classes of domains in $\mathbb{R}^{d+1}$, $d \geq 2$ with sufficiently flat boundaries which admit a decomposition or covering of bounded overlap by Lipschitz graph domains with controlled total surface area. This study is motivated by the following result proved by Peter Jones as a piece of his proof of the Analyst's Traveling Salesman Theorem in the complex plane: Any simply connected domain $\Omega \subseteq \mathbb{C}$ with finite boundary length $\mathcal{H}^1(\partial\Omega) < \infty$ can be decomposed into Lipschitz graph domains with total boundary length at most $M\mathcal{H}^1(\partial\Omega)$ for some $M > 0$ independent of Ω. In this paper, we prove an analogous Lipschitz decomposition result in higher dimensions for domains with Reifenberg flat boundaries satisfying a uniform beta-squared sum bound. We use similar techniques to show that domains with general Reifenberg flat or uniformly rectifiable boundaries admit similar Lipschitz decompositions while allowing the constituent domains to have bounded overlaps rather than be disjoint.

Contents

1 Introduction

1.1 Overview

In many areas of analysis, general domains which are somehow "close" or well-approximated by a *Lipschitz domain* tend to have many desirable properties.

Definition 1.1 (Lipschitz domains). We say that $\Omega \subseteq \mathbb{R}^{d+1}$ is a *Lipschitz domain* if for each $p \in \partial\Omega$, there exists $r > 0$ such that $B(p, r) \cap \partial\Omega$ is a Lipschitz graph.

For instance, the idea of finding good Lipschitz domains inside of more general domains has an important place in the study of harmonic measure in the plane and beyond [DJ90], [Dah77], [Bad10], [Azz18]. Lipschitz domains have similarly been used to give characterizations of rectifiability and uniform rectifiability [ABHM19], [ABHM17], [BH17], [GMT18]. The slightly stronger notion of a *Lipschitz graph domain* has also played an important role in quantitative geometric measure theory.

Definition 1.2 (Lipschitz graph domains). We say that an open, connected set $\Omega \subseteq \mathbb{R}^{d+1}$ is an M-*Lipschitz graph domain* if the following holds: There exists a composition of a translation, dilation, and rotation A with image domain $\widetilde{\Omega} = A(\Omega)$ such that there exists a function $r_{\widetilde{\Omega}} : \mathbb{S}^d \to \mathbb{R}^+$ with

$$\partial\widetilde{\Omega} = \left\{ r_{\widetilde{\Omega}}(\theta)\theta : \theta \in \mathbb{S}^d \right\}$$

and, for any $\theta, \psi \in \mathbb{S}^d$

$$|r_{\widetilde{\Omega}}(\theta) - r_{\widetilde{\Omega}}(\psi)| \leq M|\theta - \psi|,$$

$$\frac{1}{1+M} \leq r_{\widetilde{\Omega}}(\theta) \leq 1.$$

Intuitively, a Lipschitz graph domain is a "Lipschitz graph over a sphere". These domains appear in the following striking result due to Peter Jones which is the primary inspiration for this paper:

Theorem 1.3 ([Jon90] Theorem 2). *There exists a constant $M > 0$ such that the following holds: For any simply connected domain $\Omega \subseteq \mathbb{C}$ with $\mathcal{H}^1(\partial\Omega) < \infty$, there is a rectifiable curve Γ such that*

$$\Omega \setminus \Gamma = \bigcup_j \Omega_j$$

where Ω_j is an M-Lipschitz graph domain for each j, and

$$\sum_j \mathcal{H}^1(\partial\Omega_j) \leq M\mathcal{H}^1(\partial\Omega).$$

We informally say that Theorem 1.3 gives a *Lipschitz decomposition* of a domain Ω in the sense that Ω is written as a union of closures of disjoint Lipschitz graph domains with boundary lengths controlled by the boundary length of Ω. Also see [GJM92] for a similar result for minimal surfaces in $\mathbb{R}^n$. Despite being geometrically interesting in and of itself, Theorem 1.3 has an important place in the history of quantitative geometric measure theory because it is central to Jones's original proof of the Analyst's Traveling Salesman Theorem in $\mathbb{R}^2$. This central result gives a characterization of subsets of rectifiable curves and an estimate on their lengths in terms of a quantity called the *Jones beta number* which measures how close a subset $E \subseteq \mathbb{R}^2$ is to being linear locally.

Definition 1.4 (Jones beta number)**.** Let $E, Q \subseteq \mathbb{R}^2$ where Q has finite diameter. We define the β-number for E in the "window" Q by

$$\beta_E(Q) = \inf_L \sup_{x \in Q \cap E} \frac{\operatorname{dist}(x, L)}{\operatorname{diam}(Q)},$$

where L ranges over all affine lines in $\mathbb{R}^2$.

Theorem 1.5 (cf. [Jon90] Theorem 1, [Oki92] in $\mathbb{R}^n$, $n > 2$)**.** *Let $E \subseteq \mathbb{R}^2$. E is contained in a rectifiable curve if and only if*

$$\beta_E^2(\mathbb{R}^2) = \operatorname{diam}(E) + \sum_{Q \in \Delta(\mathbb{R}^2)} \beta_E(3Q)^2 \operatorname{diam}(Q) < \infty$$

where $\Delta(\mathbb{R}^2)$ is the set of all dyadic cubes in $\mathbb{R}^2$ and $3Q$ is the cube with the same center as Q but three times the side length. If Σ is a connected set of shortest length containing E, then

$$\beta_\Sigma^2(\mathbb{R}^2) \lesssim \mathcal{H}^1(\Sigma) \tag{1.1}$$

and

$$\beta_E^2(\mathbb{R}^2) \gtrsim \mathcal{H}^1(\Sigma). \tag{1.2}$$

There are now many results referred to as "Traveling Salesman Theorems" which share the general structure and philosophy of Theorem 1.5 but take place in different spaces such as Hilbert space [Sch07], Banach spaces [BM23a], [BM23b], Carnot groups [Li22], graph inverse limit spaces [DS16], and general metric spaces [DS21], [Hah05]. Many also apply to different geometric objects such as Jordan arcs [Bis22], Hölder curves [BNV19], higher-dimensional sets [AS18], [Hyd22a], [Hyd22b], [Ghi20], or measures [BS17], [BLZ23].

Jones proves Theorem 1.5 essentially as a corollary of Theorem 1.3. Roughly speaking, given a rectifiable curve $\Gamma \subseteq \mathbb{D}$, one can apply the Lipschitz decomposition result to each component of $\mathbb{D} \setminus \Gamma$ and use the boundaries of the produced Lipschitz graph domains to control the beta numbers of Γ. In fact, this shows that rectifiable curves in $\mathbb{R}^2$ admit extensions of controlled length which are quasiconvex: if one considers the union of the boundaries as a new curve $\widetilde{\Gamma} = \cup_j \partial\Omega_j \cup \Gamma$, then $\mathcal{H}^1(\widetilde{\Gamma}) \lesssim \mathcal{H}^1(\Gamma)$ and $\widetilde{\Gamma}$ is quasiconvex (see [AS12] for a generalization of this corollary to higher dimensions).

Jones's result is powerful, but it is confined to two dimensions. In this paper, we consider the following question:

Question 1.6. For $\Omega \subseteq \mathbb{R}^{d+1}$, $d > 1$, what geometric conditions on $\partial\Omega$ are sufficient for Ω to admit a Lipschitz decomposition?

One of the attractive features of Theorem 1.3 is the minimality of its assumptions on Ω; Jones only assumes simple connectivity and finite boundary length. These assumptions suffice essentially because they give access to a nicely behaved parameterization in the form of a conformal map $\varphi : \mathbb{D} \to \Omega$. The lack of similar conformal maps in higher dimensions precludes one from directly translating Jones's original argument from $\mathbb{R}^2$ to higher dimensions, but, by assuming stronger control of the geometry of $\partial\Omega$, one does get access to nicely behaved parameterizations which are sufficient replacements. The vital geometric condition on $\partial\Omega$ is called *Reifenberg flatness*, which states that $\partial\Omega$ is bilaterally close to a d-plane at all scales and all locations. This bilateral closeness is measured by the *bilateral beta number*.

Definition 1.7 (bilateral beta number)**.** For $E \subseteq \mathbb{R}^n$, P a $d-$plane, and B a ball, the *d-bilateral beta number relative to* P for E inside B is

$$b\beta_E^d(B,P) = \frac{2}{\operatorname{diam}(B)} d_H(B \cap E, B \cap P).$$

The full *bilateral beta number* for E inside B is then

$$b\beta_E^d(B) = \inf_{P\ d\text{-plane}} b\beta_E^d(B,P).$$

Definition 1.8 ((ϵ,d)-Reifenberg flatness)**.** For fixed $\epsilon > 0$ and $d, n \in \mathbb{N}$ with $0 < d < n$, we say a set $E \subseteq \mathbb{R}^n$ is *(ϵ,d)-Reifenberg flat* if, for all $x \in E$ and $r > 0$,

$$b\beta_E^d(B(x,r)) \leq \epsilon.$$

Sets that are (ϵ,d)-Reifenberg flat for small enough $\epsilon \leq \epsilon_0(d,n)$ admit bi-Hölder parameterizations which we informally call *Reifenberg parameterizations.* This was first shown by Reifenberg in [Rei60], but was later generalized by David and Toro [DT12] to produce parameterizations of Reifenberg flat sets "with holes" along with giving a condition under which the parameterization can be upgraded from bi-Hölder to bi-Lipschitz.

Theorem 1.9 (cf. [DT12] Theorem 1.10)**.** *For any $d, n \in \mathbb{N}$ with $0 < d < n$ and $0 < \tau < \frac{1}{10}$, there exists a constant $\epsilon_0(d,n)$ such that if $\epsilon \leq \epsilon_0$ and $0 \in E \subseteq \mathbb{R}^n$ is (ϵ,d)-Reifenberg flat, then there exists a bijection $g : \mathbb{R}^n \to \mathbb{R}^n$ satisfying the following conditions: For any $z, x, y \in \mathbb{R}^n$ with z arbitrary, $|x-y| \leq 1$,*

$$|g(z) - z| \leq \tau,$$

$$\frac{1}{4}|x-y|^{1+\tau} \leq |g(x) - g(y)| \leq 3|x-y|^{1-\tau},$$

and, for some d-plane P such that $b\beta_E(B(0,10),P) \leq \epsilon$,

$$E \cap B(0,1) = g(P) \cap B(0,1).$$

Given a domain $\Omega \subseteq \mathbb{R}^{d+1}$ such that $\partial\Omega$ is (ϵ,d)-Reifenberg flat, we use the Reifenberg parameterization g produced by Theorem 1.9 as a replacement for the conformal map in Jones's original argument to first prove the following new result

Theorem A. *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a domain with $0 \in \partial\Omega$. There exists $\epsilon_0(d) > 0$ such that for any $L > 0$, if $\epsilon \leq \epsilon_0$ and*

(i) $\partial\Omega$ is (ϵ,d)-Reifenberg flat,

(ii) $\sum_{k=1}^{\infty} \beta_{\partial\Omega}^{d,1}(B(x,2^{-k}))^2 \leq L$ for all $x \in \partial\Omega$,

then there exists a d-Ahlfors regular, d-rectifiable set Σ such that

$$\Omega \cap B(0,1) \setminus \Sigma = \bigcup_{j=1}^{\infty} \Omega_j$$

and there exists $L_1(\epsilon,L,d) > 0$ such that $\mathscr{L} = \{\Omega_j\}_{j \in J_{\mathscr{L}}}$ is a collection of disjoint L_1-Lipschitz graph domains. In addition, for any $y \in \partial\Omega \cap B(0,1)$ and $0 < r < 1$, we have

$$\sum_{j=1}^{\infty} \mathcal{H}^d(\partial\Omega_j \cap B(y,r)) \lesssim_{\epsilon,L,d} r^d.$$

See Definition 2.16 for the definition of $\beta^{d,1}_{\partial\Omega}(B(x,2^{-k}))$. Hypothesis (ii) is used to ensure that David and Toro's bi-Lipschitz condition for the Reifenberg parameterization is satisfied. If this hypothesis is not satisfied, then one can still run the proof of Theorem A to produce a collection of Lipschitz graph domains whose total boundary measure and Lipschitz constants blow up near where the sum in (ii) diverges. However, we conjecture that a result similar to Theorem A holds without assumption (ii).

If one is willing to weaken the conclusion of $\{\Omega_j\}$ being disjoint to having bounded overlap, then one can show that similar Lipschitz decompositions exist for domains with weaker assumptions on the boundary. We prove the following result of this type:

Theorem B. *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a domain with $0 \in \partial\Omega$. There exist constants $A(d), L(d), \epsilon(d) > 0$ such that if $0 \in \partial\Omega$ is (ϵ, d)-Reifenberg flat, then there exists a collection of L-Lipschitz graph domains $\{\Omega_j\}_{j\in\mathscr{L}}$ such that*

(i) $\Omega_j \subseteq \Omega$,

(ii) $\Omega \cap B(0,1) \subseteq \bigcup_{j=1}^{\infty} \Omega_j$,

(iii) $\exists C(d) > 0$ *such that* $\forall x \in \Omega$, $x \in \Omega_j$ *for at most C values of j,*

(iv) For any $y \in \partial\Omega \cap B(0,1)$ and $0 < r \leq 1$, we have

$$\sum_{j=1}^{\infty} \mathcal{H}^d(\partial\Omega_j \cap B(y,r)) \lesssim_{\epsilon,d,L} \mathcal{H}^d(\partial\Omega \cap B(y,Ar)).$$

To prove this result, we use a collection of $(1 + C\delta)$-bi-Lipschitz Reifenberg parameterizations to produce a large collection of disjoint Lipschitz graph domains with controlled boundaries and expand these domains with Whitney-type "buffer zones" to form a true covering of $\Omega \cap B(0,1)$. This method carries over to the well-known *d-uniformly rectifiable* sets of David and Semmes who give many different equivalent definitions of d-uniform rectifiability [DS93]. One such definition involves the *bilateral weak geometric lemma* (BWGL), which roughly says that E looks Reifenberg flat on most scales and locations.

Definition 1.10 (bilateral weak geometric lemma)**.** Given a family of Christ-David cubes $\mathscr{D}$ for E (see Theorem 2.10) and constants $M, \epsilon > 0$, define

$$\mathrm{BWGL}(M,\epsilon) = \{Q \in \mathscr{D} : b\beta^d_E(MB_Q) > \epsilon\}.$$

For $Q \in \mathscr{D}$, define

$$\mathrm{BWGL}(Q,M,\epsilon) = \sum_{\substack{R \subseteq Q \\ R \in \mathrm{BWGL}(M,\epsilon)}} \ell(R)^d.$$

We say that E satisfies the *bilateral weak geometric lemma* if for any $M, \epsilon > 0$, there exists a constant $C_0(M,\epsilon)$ such that for all $Q \in \mathscr{D}$,

$$\mathrm{BWGL}(Q,M,\epsilon) \leq C_0 \ell(Q)^d. \tag{1.3}$$

If E is (ϵ, d)-Reifenberg flat, then $\mathrm{BWGL}(Q,M,\epsilon) = 0$ for all Q and M. Equation (1.3) is often referred to as a *Carleson packing condition.* One can define a d-uniformly rectifiable set as a d-Ahlfors regular set which satisfies the BWGL.

Definition 1.11 (d-Ahlfors regularity)**.** We say that a set $E \subseteq \mathbb{R}^n$ is *d-Ahlfors regular* if E is closed and there exists a constant $C_0 > 0$ such that for any $x \in E$ and $0 < r < \mathrm{diam}(E)$, we have

$$C_0^{-1} r^d \leq \mathscr{H}^d(E \cap B(x,r)) \leq C_0 r^d.$$

Definition 1.12 (d-uniform rectifiability)**.** We say a d-Ahlfors regular set $E \subseteq \mathbb{R}^n$ is *d-uniformly rectifiable* if E satisfies the BWGL.

Using similar methods to those of the proof of Theorem B, we prove an analogue of Theorem B for d-uniformly rectifiable sets.

Theorem C. *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a domain with $0 \in \partial\Omega$. If $\partial\Omega$ is d-uniformly rectifiable, then there exists $L(d), A(d) > 0$ such that there exists a collection of L-Lipschitz graph domains $\{\Omega_j\}_{j \in J_{\mathscr{L}}}$ such that conclusions (i), (ii), (iii), and (iv) (with additional dependence on uniform rectifiability constants) of Theorem B hold.*

Uniform rectifiability was studied in detail by David and Semmes in [DS93] where the authors explore connections between the BWGL and numerous other equivalent definitions involving boundedness of singular integral operators, approximation by Lipschitz graphs (the existence of corona decompositions), "big piece" parameterizations by Lipschitz maps, and more. Uniform rectifiability has recently become of interest in the study of harmonic measure and the solvability of the homogeneous Dirichlet problem in rough domains. In [AHM$^+$20], the authors give a geometric characterization of open sets $\Omega \subseteq \mathbb{R}^{d+1}$ such that there exists $p < \infty$ such that the $L^p(\partial\Omega)$-Dirichlet problem is solvable given the background hypotheses that $\partial\Omega$ is d-Ahlfors-David regular and Ω satisfies the interior corkscrew condition. They prove that solvability is equivalent to $\partial\Omega$ being d-uniformly rectifiable and Ω satisfying a quantitative connectivity condition called the weak local John condition. A related line of research studies L^p solvability of inhomogeneous problems on rough domains. In the course of preparing this work, the author was notified of [MPT22] in which the authors study equivalences of solutions to boundary value problems in rough domains and show that the regularity problem for so-called DKP operators is L^p-solvable on certain geometrically nice domains. In the course of their study, the authors derive a very similar result to Theorem C with the added assumption that Ω satisfies the interior corkscrew condition and the added conclusion that the nice approximating domains are adapted to a DKP operator (see Section 4.3 of [MPT22] and see also [MT24] for an earlier version of their construction).

1.2 Outlines of the paper and proofs of the theorems

In Section 2, we introduce the necessary notation and basic facts about Reifenberg parameterizations, Whitney decompositions, Christ-David cubes, coronizations, Reifenberg flat sets, and uniformly rectifiable sets.

In Sections 3 and 4 we respectively prove Theorems A and Theorems B and C while taking for granted the results on Lipschitz graph domains proved in Section 5 and results on controlling the derivative of Reifenberg parameterizations proved in Section 6.

Roughly speaking, the proof of Theorem A in Section 3 proceeds as follows. The fact that $\partial\Omega$ is Reifenberg flat means that we can produce a Reifenberg parameterization $g : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ such that $g(\mathbb{R}^d \times \{0\}) \cap B(0,1) = \partial\Omega \cap B(0,1)$. The uniform bound on the beta-squared sum in condition (ii) of Theorem A ensures that g is $L'(d,L)$-bi-Lipschitz so that $\partial\Omega$ is in fact a bi-Lipschitz image, hence uniformly rectifiable. This means that there exists a Christ-David lattice $\mathscr{D}$ for $\partial\Omega$ with a graph coronization whose stopping time regions $\mathscr{F} = \{S\}$ consist of cubes well-approximated by Lipschitz graphs with Lipschitz constant small in terms of d and L' (this is a coronization that produces a corona decomposition). Proposition 6.7 implies that Dg is nearly constant on parts of its domain which are mapped into regions of Ω sitting "above" a stopping time region S on the scale of the cubes inside S. By the results of Section 5, g maps forward Lipschitz graph domains to Lipschitz graph domains when the change in Dg is small compared to the Lipschitz constants of the mapped domains. Therefore, to produce a Lipschitz decomposition of $\Omega \cap B(0,1)$, it suffices to produce a Lipschitz decomposition $\mathscr{L}_0$ (see Definition 3.8) of the domain of g into domains over which Dg is nearly constant so that the collection of images $\mathscr{L} = \{g(\mathcal{D}) : \mathcal{D} \in \mathscr{L}_0\}$ is a Lipschitz decomposition.

In order to form this decomposition, we produce a "coronization" of a lattice of Whitney boxes which parallels the coronization for $\mathscr{D}$ on $\partial\Omega$ (see 3.5). That is, we separate Whitney boxes into bad boxes which g maps near bad cubes in $\mathscr{D} \cap \mathscr{B}$ or cubes on the "edges" in scale and location of stopping time regions in $\mathscr{D}$. This decomposition then maps forward to a collection of domains whose total surface measure is bounded by the surface measure of $\partial\Omega$ plus the Carleson packing sums for the bad and "edge" cubes of $\mathscr{D}$.

The proofs of Theorems B and C both follow a single similar argument to that of Theorem A. In the Reifenberg flat case, the difference is that any single global Reifenberg parameterization g produced for the set is not in general bi-Lipschitz, so we have no uniform estimates on how g distorts any given cube. In the uniformly rectifiable case, we have no global Reifenberg parameterization because there can be many scales and locations at which Reifenberg flatness fails. In either case, we sidestep these by producing a collection of local $(1+\delta)$-bi-Lipschitz parameterizations by parameterizing pieces of the domain above stopping time regions in a graph coronization (see Definition 2.15) using single stopping time domains composed of Whitney cubes. By similar arguments, the surface measure of these domains is controlled by the surface masure of $\partial\Omega \cap B(0,1)$ plus the Carleson packing sum of the same bad set of cubes in $\mathscr{D} \cap \mathscr{B}$ and near "edges" of stopping time domains. We then fill parts of $\Omega \cap B(0,1)$ that are missed by these domains with "buffer zones" of cubes on the exterior of these domains as well as families of cubes which sit above surface cubes in the bad set. By similar reasoning, the surface measure of these domains is bounded by the same Carleson packing sums as above.

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2 Preliminaries

2.1 Conventions and basic definitions

Whenever we write $A \lesssim B$, we mean that there exists some constant C independent of A and B such that $A \leq CB$. If we write $A \lesssim_{a,b,c} B$ for some constants a, b, c, then we mean that the implicit constant C mentioned above is allowed to depend on a, b, c. We will sometimes write $A \simeq B$ to mean that both $A \lesssim B$ and $B \lesssim A$ hold.

In many computations, we use a constant C to denote a catch-all, general constant which is allowed to vary significantly from one line to the next.

Definition 2.1 (Hausdorff measure, Hausdorff distance, Nets)**.** For $F, E \subseteq \mathbb{R}^n,\ a \in \mathbb{R}^n$, we let

$$\begin{aligned} \operatorname{dist}(E,F) &= \inf\{|x-y| : x \in F,\ y \in E\}, \\ \operatorname{dist}(a,E) &= \operatorname{dist}(\{a\},E) \end{aligned}$$

and define

$$\operatorname{diam}(F) = \sup\{|x-y| : x, y \in F\}.$$

For any $r > 0$, we let

$$B(E,r) = \{x \in \mathbb{R}^{d+1} : \operatorname{dist}(x,E) < r\}.$$

For any subset $F \subseteq \mathbb{R}^{d+1}$, an integer $m \geq 0$, and constant $0 < \delta \leq \infty$, we define

$$\mathscr{H}^m(F) = \inf\left\{\sum \operatorname{diam}(E_i)^d : F \subseteq \bigcup E_i, \operatorname{diam}(E_i) < \delta\right\}.$$

The Hausdorff m-measure of F is defined as

$$\mathscr{H}^m(F) = \lim_{\delta \to 0} \mathscr{H}^m_\delta(F),$$

We will only use this in the case $m = d$, and we often use the notation $|F| = \mathscr{H}^d(F)$. We refer to the function $\mathscr{H}^m_\infty$ as the m-dimensional Hausdorff content. Given two closed sets $E, F \subseteq \mathbb{R}^{d+1}$, and a third set $B \subseteq \mathbb{R}^{d+1}$ we define the Hausdorff distance between E and F inside B as

$$d_B(E,F) = \frac{2}{\operatorname{diam} B} \max\left\{ \sup_{y \in E \cap B} \operatorname{dist}(y,F), \sup_{y \in F \cap B} \operatorname{dist}(y,E) \right\}.$$

Given a subset $E \subseteq \mathbb{R}^{d+1}$ and $r > 0$, we let $\operatorname{Net}(E,r)$ denote the set of r-nets of E. That is, $X \in \operatorname{Net}(E,r)$ if $X \subseteq E$ such that both

(i) For any $x \neq y \in X$, $|x-y| \geq r$,

(ii) $E \subseteq \bigcup_{x \in X} B(x,r)$.

2.2 Reifenberg parameterizations

In this section, we record the basic facts about Reifenberg parameterizations needed from [DT12].

2.2.1 Coherent Collections of Balls and Planes (CCBP)

Set $r_k = 10^{-k}$ and let $x_{j,k} \in \mathbb{R}^{d+1},\ j \in J_k$ satisfy

$$|x_{j,k} - x_{i,k}| \geq r_k. \tag{2.1}$$

Put $B_{j,k} = B(x_{j,k}, r_k)$ and for $\lambda > 0$ define $V_k^\lambda = \bigcup_{j \in J_k} \lambda B_{j,k} = \bigcup_{j \in J_k} B(x_{j,k}, \lambda r_k)$ where λB is always the ball with the same center as B and radius dilated by a factor of λ. We also assume

$$x_{j,k} \in V_{k-1}^2. \tag{2.2}$$

We will always use a d-plane as the initial surface Σ_0. We require

$$\operatorname{dist}(x_{j,0}, \Sigma_0) \leq \epsilon \text{ for } j \in J_0. \tag{2.3}$$

Finally, the coherent collection of planes is a collection of planes (in general of any dimension $m < d+1$, although here we only take d-planes) $P_{j,k}$ associated to $x_{j,k}$ such that the compatibility conditions

$$d_{x_{j,k},100r_k}(P_{i,k}, P_{j,k}) \leq \epsilon \text{ for } k \geq 0 \text{ and } i,j \in J_k \text{ such that } |x_{i,k} - x_{j,k}| \leq 100 r_k \tag{2.4}$$

$$d_{x_{i,0},100}(P_{i,0}, P_x) \leq \varepsilon \text{ for } i \in J_0 \text{ and } x \in \Sigma_0 \text{ such that } |x_{i,0} - x| \leq 2, \tag{2.5}$$

$$d_{x_{i,k},20r_k}(P_{i,k}, P_{j,k+1}) \leq \varepsilon \text{ for } i \in J_k \text{ and } j \in J_{k+1} \text{ such that } |x_{i,k} - x_{j,k+1}| \leq 2r_k. \tag{2.6}$$

With these conditions, we can define a CCBP

Definition 2.2. A CCBP is a triple $(\Sigma_0, \{B_{j,k}\}, \{P_{j,k}\})$ such that conditions (2.1), (2.2), (2.3), (2.4), (2.5), (2.6) are satisfied with ϵ sufficiently small in terms of d.

We first state a small modification of a lemma in [AS18] which gives criteria for a triple $(\Sigma_0, \{B_{j,k}\}, \{P_{j,k}\})$ to be a CCBP.

Lemma 2.3 (cf. [AS18] Theorem 2.5)**.** *For any* $k \in \mathbb{N} \cup \{0\}$*, let* $r_k = 10^{-k}$*. Let* $\{x_{j,k}\}_{j \in J_k}$ *be a collection of points such that for some d-plane* P_0 *we have*

$$\operatorname{dist}(x_{j,0}, P_0) < \epsilon,$$

$$|x_{j,k} - x_{i,k}| \geq r_k,$$

and, with $B_{j,k} = B(x_{j,k}, r_k)$*,*

$$x_{i,k} \in V_{k-1}^2$$

where

$$V_k^\lambda = \bigcup_{j \in J_k} \lambda B_{j,k}.$$

Let $P_{j,k}$ *be a d-plane such that* $x_{j,k} \in P_{j,k}$*. There is* $\epsilon_0 > 0$ *such that for any* $0 < \epsilon < \epsilon_0$*, if*

$$\epsilon'_k(x_{j,k}) \lesssim \epsilon \text{ for all } k \geq 0 \text{ and } j \in J_k$$

then $(P_0, \{B_{j,k}\}, \{P_{j,k}\})$ *is a CCBP. See (6.1) for the definition of the* ϵ'_k *numbers.*

CCBPs allow the construction of Reifenberg parameterizations which we will denote by the letter g. David and Toro give the following Theorem

Theorem 2.4 ([DT12] Theorems 2.15, 2.23)**.** *Let* $(\Sigma_0, \{B_{j,k}\}, \{P_{j,k}\})$ *be a CCBP with* ϵ *sufficiently small. Then there exists a bijection* $g : \mathbb{R}^n \to \mathbb{R}^n$ *such that*

$$g(z) = z \quad \text{when } \operatorname{dist}(z, \Sigma_0) \geq 2, \tag{2.7}$$

$$|g(z) - z| \leq C\varepsilon \quad \text{for } z \in \mathbb{R}^n, \tag{2.8}$$

$$\frac{1}{4}|z' - z|^{1+C\varepsilon} \leq |g(z') - g(z)| \leq 3|z' - z|^{1-C\varepsilon} \tag{2.9}$$

for $z, z' \in \mathbb{R}^n$ *such that* $|z' - z| \leq 1$*, and* $\Sigma = g(\Sigma_0)$ *is a* $C\varepsilon$*-Reifenberg flat set that contains the accumulation set*

$$E_\infty = \big\{x \in \mathbb{R}^n\,;\, x \text{ can be written as } x = \lim_{m \to +\infty} x_{j(m),k(m)}, \text{ with } k(m) \in \mathbb{N} \\ \text{and } j(m) \in J_{k(m)} \text{ for } m \geq 0, \text{ and } \lim_{m \to +\infty} k(m) = +\infty\big\}.$$

If in addition there exists $M > 0$ *such that*

$$\sum_{k \geq 0} \epsilon'_k(f_k(z))^2 \leq L \quad \text{for all } z \in \Sigma_0,$$

then g *is bi-Lipschitz: there is a constant* $C(n, d, L) \geq 1$ *such that*

$$C(n, d, L)^{-1}|z - z'| \leq |g(z) - g(z')| \leq C(n, d, L)|z - z'|.$$

2.2.2 The definition of g

Following Chapter 3 of [DT12], we take ψ_k to be a smooth function vanishing outside V_k^8 and $\theta_{j,k}$ to be a collection of smooth compactly supported functions in $10B_{j,k}$ such that $|\nabla^m \theta_{j,k}(y)| \leq C_m r_k^{-m}$ and $\psi_k(y) + \sum_{j \in J_k} \theta_{j,k}(y) = 1$. We then define a sequence of maps f_k by

$$f_0(y) = y, \; f_{k+1} = \sigma_k \circ f_k$$

where

$$\sigma_k(y) = y + \sum_{j \in J_k} \theta_{j,k}(y)\,[\pi_{j,k}(y) - y] = \psi_k(y)y + \sum_{j \in J_k} \theta_{j,k}(y)\,\pi_{j,k}(y),$$

where $\pi_{j,k}$ is orthogonal projection onto $P_{j,k}$. In our application, we only care about points inside V_k^8, so $\psi_k(y) = 0$ and the formula simplifies to

$$\sigma_k(y) = \sum_{j \in J_k} \theta_{j,k}(y)\,\pi_{j,k}(y).$$

The map σ_k also satisfies

$$|\sigma_k(y) - y| \leq C\epsilon r_k \tag{2.10}$$

for $k \geq 0$ and $y \in \Sigma_k$.

The map g is constructed by, roughly speaking, interpolating between adjacent maps in the sequence f_k at distance r_k from the surface $\Sigma_k = f_k(\Sigma_0)$. In order to construct this, David and Toro define a collection of linear isometries R_k on $\mathbb{R}^n$. The following proposition summarizes the properties of R_k that we need

Proposition 2.5 ([DT12] Proposition 9.29). *Let $\mathcal{R}$ denote the set of linear isometries of $\mathbb{R}^n$. Also set*

$$T_k(x) = T\Sigma_k(f_k(x)) \quad \text{for } x \in \Sigma_0 \text{ and } k \geq 0.$$

There exist C^1 mappings $R_k : \Sigma_0 \to \mathcal{R}$, with the following properties:

$$R_0(x) = I \text{ for } x \in \Sigma_0,$$

$$R_k(x)(T_0(x)) = T_k(x) \quad \text{for } x \in \Sigma_0 \text{ and } k \geq 0,$$

$$|R_{k+1}(x) - R_k(x)| \leq C\varepsilon \quad \text{for } x \in \Sigma_0 \text{ and } k \geq 0, \tag{2.11}$$

In addition, we record the bounds the distance between generations of tangent planes and between planes at different locations

Lemma 2.6 ([DT12] Lemma 9.2). *We have that for $k \geq 0$ and $x, x' \in \Sigma_0$ such that $|x' - x| \leq 10$,*

$$D(T\Sigma_{k+1}(f_{k+1}(x)), T\Sigma_k(f_k(x))) \leq C_1\varepsilon$$

$$D(T\Sigma_k(f_k(x')), T\Sigma_k(f_k(x))) \leq C_2\varepsilon\, r_k^{-1}|f_k(x') - f_k(x)|.$$

Now, following Chapter 10 in [DT12], we define a collection ρ_k of positive, smooth, radial functions such that $\sum_{k\geq 0} \rho_k(y) = 1$ for $y \in \mathbb{R}^n \setminus \{0\}$ and $\rho_k(y) = 0$ unless $r_k < |y| < 20r_k$. Because $[r_k, 20r_k] \cap [r_{k-2}, 20r_{k-2}] = [r_k, 20r_k] \cap [100r_k, 2000r_k] = \varnothing$, we always have at most two values of k such that $\rho_k(y) \neq 0$ for any fixed y. In order to single out specific values of k, we define functions $l, n : \mathbb{R}^+ \to \mathbb{N}$ by

$$l(y) = \min\{k \in \mathbb{N} : \rho_k(y) > 0\}, \tag{2.12}$$

$$n(y) = \max\{k \in \mathbb{N} : \rho_k(y) > 0\} = l(y) + 1. \tag{2.13}$$

More concretely, we have

$$n(y) = n \iff 20r_{n+1} = 2r_n < y \leq 20r_n \tag{2.14}$$

because then $\rho_{n+1}(y) = 0$ while $\rho_n(y) > 0$. Roughly speaking, $n(y)$ gives the index of the maps $f_{n(y)}$ which is most relevant for the behavior of g on points roughly distance $|y|$ from Σ_0. We will now assume $\Sigma_0 = \mathbb{R}^d$ and write

$$g(z) = \sum_{k \geq 0} \rho_k(y)\,\{f_k(x) + R_k(x) \cdot y\} \quad \text{for } z = (x, y)$$

We will commonly use the notation $z = (x, y)$ as understood above when discussing points in the domain of g.

2.3 Whitney cubes, Whitney boxes, and Christ-David cubes

We will make significant use of the standard Whitney decomposition of the upper half-space with respect to $\mathbb{R}^d \times \{0\} \subseteq \mathbb{R}^{d+1}$.

Definition 2.7 (Whitney cubes)**.** Define

$$\mathscr{W} = \left\{[k_1 2^{-n}, (k_1+1)2^{-n}] \times \cdots \times [k_d 2^{-n}, (k_d+1)2^{-n}] \times [2^{-n}, 2^{-n+1}] : k_1, \ldots, k_d, n \in \mathbb{Z}\right\}.$$

$\mathscr{W}$ consists of exactly the dyadic cubes in $\mathbb{R}^d \times [0,\infty)$ which satisfy $\ell(W) = h(W) = \operatorname{dist}(W, \mathbb{R}^d)$ where $\ell(W) = h(W)$ denote the *side length* of W and the *height* of W. Cubes $W, R \in \mathscr{W}$ have a natural partial order induced by distance to $\mathbb{R}^d \times \{0\}$. We define the projection $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d \times \{0\}$ by $\pi(x,y) = x$ where $(x,y) \in \mathbb{R}^d \times \mathbb{R}$ and write

$$W \leq R$$

if and only if $\pi(W) \subset \pi(R)$. If $h(W) = \frac{1}{2}h(R)$, we call W a *child* of R and R a *parent* of W. This gives a partial order on $\mathscr{W}$ which we use to define the *descendants* of W

$$D(W) = \{R \in \mathscr{W} : R \leq W\}.$$

This partial order imposes a natural tree structure on $\mathscr{W}$ which we will use in stopping time constructions. It will additionally be useful to refine the family of Whitney cubes into rectangular Whitney boxes in which the side length of the boxes in the first d-coordinate directions is allowed to vary.

Definition 2.8 (Whitney boxes)**.** We define the set of p-th order *Whitney boxes* by

$$\mathscr{R}_p = \left\{[k_1 2^{-p-n}, (k_1+1)2^{-p-n}] \times \cdots \times [k_d 2^{-p-n}, (k_d+1)2^{-p-n}] \times [2^{-n}, 2^{-n+1}] : k_1, \ldots, k_d, n \in \mathbb{Z}\right\}.$$

These are like Whitney cubes, but they have lengths along the first d coordinate directions contracted by a factor of 2^p. Given $R \in \mathscr{R}_p$, we call $\ell(R) = 2^{-p-n}$ the *side length* and $h(R) = 2^{-n} = \operatorname{dist}(R, \mathbb{R}^d)$ the *height* so that

$$\ell(R) = 2^{-p} h(R).$$

Any collection of Whitney boxes has a tree structure induced by the same partial order as in Definition 2.7. We set $\mathscr{R} = \cup_p \mathscr{R}_p$.

We will later construct stopping time regions composed of Whitney boxes in the upper half space. We will also need the following notion of "closeness".

Definition 2.9 (A-close subsets)**.** We call two subsets $W, R \subseteq \mathbb{R}^{d+1}$ *A-close* (as in [DS93] pg. 59) if the following hold:

$$\begin{aligned}
&\frac{1}{A}\operatorname{diam} W \leq \operatorname{diam} R \leq A \operatorname{diam} W, \\
&\operatorname{dist}(W, R) \leq A(\operatorname{diam} W + \operatorname{diam} R).
\end{aligned}$$

We will also use the notation

$$W \simeq_A R$$

when W is A-close to R.

We will also need families of partitions of $\partial\Omega \subseteq \mathbb{R}^{d+1}$ which function as dyadic cubes do for $\mathbb{R}^{d+1}$. These were originally devised by Christ in [Chr90], but the formulation given here is due to Hytonen and Martikainen from [HM12].

Theorem 2.10 (Christ-David cubes)**.** *Let X be a doubling metric space. Let X_k be a nested sequence of maximal ρ^k-nets for X where $\rho < 1/1000$ and let $c_0 = 1/500$. For each $k \in \mathbb{Z}$ there is a collection $\mathscr{D}_k$ of "cubes," which are Borel subsets of X such that the following hold.*

(i) $X = \bigcup_{Q\in\mathscr{D}_k} Q$.

(ii) If $Q, Q' \in \mathscr{D} = \bigcup \mathscr{D}_k$ *and* $Q \cap Q' \neq \emptyset$, *then* $Q \subseteq Q'$ *or* $Q' \subseteq Q$.

(iii) For $Q \in \mathscr{D}$, *let* $k(Q)$ *be the unique integer so that* $Q \in \mathscr{D}_k$ *and set* $\ell(Q) = 5\rho^{k(Q)}$. *Then there is* $\zeta_Q \in X_k$ *so that*

$$B(x_Q, c_0\ell(Q)) \subseteq Q \subseteq B(x_Q, \ell(Q))$$

and

$$X_k = \{x_Q : Q \in \mathscr{D}_k\}.$$

If in addition we assume $X \subseteq \mathbb{R}^{d+1}$ and X is $d-$Ahlfors-David regular, then these cubes also satisfy

(iv) $|Q| \simeq_d (\operatorname{diam} Q)^d \simeq_d \ell(Q)^d$.

For any $Q \in \mathscr{D}$, we will use the notation $Q^{(1)}$ to refer to the parent of Q

We will refer to any family of Christ-David Cubes for $\partial\Omega$ by $\mathscr{D}$ and define

$$B_Q = B(x_Q, \ell(Q)).$$

2.4 Coronizations for Reifenberg flat and uniformly rectifiable sets

The boundary measure bounds for our Lipschitz decompositions come from Carleson packing conditions for well-chosen *coronizations* of a Christ-David lattice for $\partial\Omega$. Coronizations essentially consist of a partition of $\mathscr{D}$ into "good" cubes $\mathscr{G}$ and "bad" cubes $\mathscr{B}$ and a further partition of $\mathscr{G}$ into disjoint *stopping time regions* $\mathscr{F} = \{S_i\}_i$.

Definition 2.11 (stopping time regions)**.** We call $S \subseteq \mathscr{G} \subseteq \mathscr{D}$ a *stopping time region* if it is *coherent*, i.e.

(i) There exists a "top cube" $Q(S) \in S$ such that $R \subseteq Q(S)$ for all $R \in S$,

(ii) If $Q \in S$ and $R \in \mathscr{D}$ satisfies $Q \subseteq R \subseteq Q(S)$, then $R \in S$,

(iii) If $Q \in S$, then either every child of Q is in S, or none of them are.

Remark 2.12. We note that Definition 2.11 makes sense with any well-ordered collection of subsets of $\mathbb{R}^{d+1}$ in place of $\mathscr{D}$. For instance, we will use the term stopping time region to refer to such collections of Whitney boxes with the partial order defined in Definition 2.7.

Definition 2.13 (Coronizations (cf. [DS93] Definition 3.13))**.** We say that a triple $(\mathscr{G}, \mathscr{B}, \mathscr{F})$ is a *coronization* of $\mathscr{D}$ if

(i) $\mathscr{F}$ is a collection of disjoint stopping time regions as in Definition 2.11 with $\mathscr{G} = \bigcup_{S\in\mathscr{F}} S$,

(ii) $\mathscr{G} \cup \mathscr{B} = \mathscr{D}$ and $\mathscr{G} \cap \mathscr{B} = \varnothing$,

(iii) $\mathscr{B}$ and $\{Q(S)\}_{S\in\mathscr{F}}$ satisfy Carleson packing conditions. That is, there exist constants $C_1, C_2 > 0$ such that for any $Q \in \mathscr{D}$

$$\sum_{\substack{R\in\mathscr{B}\\ R\subseteq Q}} \ell(R)^d \leq C_1\mathcal{H}^d(Q), \quad \text{and} \quad \sum_{\substack{S\in\mathscr{F}\\ Q(S)\subseteq Q}} \ell(Q(S))^d \leq C_2\mathcal{H}^d(Q).$$

The stopping time regions in coronizations collect scales and locations into good, "connected" packages on which $\partial\Omega$ behaves well. David and Semmes used the concept of a coronization to produce a definition of uniform rectifiability involving *corona decompositions*

Definition 2.14 (Corona decomposition (cf. [DS93] Definition 3.19))**.** We say that a set $E \subseteq \mathbb{R}^n$ admits a d-dimensional *corona decomposition* if for any constants $\eta, \theta > 0$, there exists a coronization $(\mathscr{G}, \mathscr{B}, \mathscr{F})$ of a d-dimensional lattice $\mathscr{D}$ for E such that for each $S \in \mathscr{F}$, there exists a d-dimensional Lipschitz graph $\Gamma(S)$ with Lipschitz constant less than η such that for each $x \in 2Q$ and $Q \in S$

$$\operatorname{dist}(x, \Gamma(S)) \leq \theta \operatorname{diam}(Q). \tag{2.15}$$

If one has an appropriate coronization, then one can use Reifenberg parameterizations to produce the approximating Lipschitz graphs in the definition of the corona decomposition directly. We call these specific good coronizations *graph coronizations*

Definition 2.15 (graph coronizations)**.** For constants $M, \epsilon, \delta > 0$, we say that $(\mathscr{G}, \mathscr{B}, \mathscr{F})$ is a d-dimensional $(M, \epsilon, \delta)-$*graph coronization* if it is a coronization such that $\mathscr{B} \supseteq \mathrm{BWGL}(M, \epsilon)$ and for each $S \in \mathscr{F}$ and $Q \in S$, there exists a d-plane $P_Q \ni x_Q$ such that

(i) $b\beta_E(MB_Q, P_Q) \leq 2b\beta_E(MB_Q) \leq 2\epsilon$,

(ii) $\sum_{x\in Q\in S} \beta_E^{d,1}(MB_Q)^2 \leq \epsilon^2$ for any $x \in Q(S)$.

(iii) $\mathrm{Angle}(P_Q, P_{Q(S)}) \leq \delta$,

Condition (ii) above uses the *content beta number* introduced by Azzam and Schul in [AS18]. This is closely related to the more standard L^p *beta numbers* used by David and Semmes in characterizing uniform rectifiability via the *strong geometric lemma*.

Definition 2.16 (L^p beta numbers and content beta numbers)**.** Let $B = B(x, r) \subseteq \mathbb{R}^{d+1}$ and let P be a d-plane. We define

$$\beta_{E,p}^d(B, P) = \left(\frac{1}{r^d}\int_{B\cap E}\left(\frac{\operatorname{dist}(y, P)}{r}\right)^p \, d\mathscr{H}^d(y)\right)^{1/p},$$

and we define the L^p *beta number* as

$$\beta_{E,p}^d(B) = \inf\{\beta_{E,p}^d(B, P) : P \text{ is a } d\text{-dimensional plane in } \mathbb{R}^{d+1}\}.$$

Similarly, we define

$$\beta_E^{d,p}(B, P) = \left(\frac{1}{r_B^d}\int_0^\infty \mathscr{H}_\infty^d\{x \in B \cap E : \operatorname{dist}(x, P) > tr_B\} t^{p-1} dt\right)^{1/p},$$

and we define the L^p *content beta number* as

$$\beta_E^{d,p}(B) = \inf\{\beta_E^{d,p}(B, P) : P \text{ is a } d\text{-dimensional plane in } \mathbb{R}^{d+1}\}.$$

If E is d-Ahlfors regular, then these two beta numbers are comparable with constants depending on d and the regularity constant.

Proposition 2.17 (cf. [DS93] Part I, Theorem 1.57 and Theorem 2.4; Part IV Proposition 2.1)**.** *Let $E \subseteq \mathbb{R}^{d+1}$ be d-Ahlfors regular for $d \geq 1$. The following are equivalent:*

(i) E is d-uniformly rectifiable,

(ii) E satisfies the strong geometric lemma: For any $Q \in \mathscr{D}$, $M > 1$, and $1 \leq p < \frac{2d}{d-2}$,

$$\sum_{R\subseteq Q} \beta_{E,p}^d(MB_R)^2 \ell(R)^d \lesssim_{M,d} \ell(Q)^d,$$

(iii) E admits a corona decomposition.

(iv) E admits an (M, ϵ, δ)-graph coronization for any $M, \epsilon, \delta > 0$.

The main tool we will use to create Lipschitz decompositions is the graph coronization. In Appendix A, we review the d-dimensional traveling salesman results of [AS18] and [Hyd22a] which give a similar analysis for general Reifenberg flat sets. By collecting these results, we prove the following proposition:

Proposition 2.18. *For any $d, n \in \mathbb{N}$ with $0 < d < n$, there exists $\epsilon_0(d, n), \delta(d, n) > 0$ such that if $\epsilon \leq \epsilon_0 \ll \delta^4$ and $E \subseteq \mathbb{R}^n$ is (ϵ, d)-Reifenberg flat, then E admits an (M, ϵ, δ)-graph coronization for any $M > 0$.*

We will use the existence of graph coronizations as in the previous two propositions to prove Theorems B and C.

We also record some important facts about using beta numbers to control the Hausdorff distance of planes. Given a set $E \subseteq \mathbb{R}^{d+1}$ and a Christ-David lattice $\mathscr{D}$ for E, we define epsilon numbers adapted to the lattice $\mathscr{D}$ and a collection of planes $\{P_Q\}_{Q\in\mathscr{D}}$. Fix $K = \frac{10^4}{\rho}$. We define

$$\epsilon(Q) = \sup\left\{ d_{KB_R}(P_U, P_R) : k(R) \in \{k(Q), k(Q) - 1\},\ k(U) = k(Q),\ x_Q \in \frac{K}{10}B_Q \cap \frac{K}{10}B_R \right\}.$$

This is essentially a version of David and Toro's ϵ_k' numbers which is adapted to a cube structure rather than a general collection of nets. Now, let $M \geq \frac{10K}{\rho^2}$. We will use these to control ϵ_k' in terms of $\beta^{d,1}(MB_Q)$ in the second lemma below. First, we give a recall a general result that allows one to bound the Hausdorff distance between planes by beta numbers:

Lemma 2.19 ([AS18] Lemma 2.16)**.** *Suppose $E \subseteq \mathbb{R}^n$ and B is a ball centered on E such that for all balls $B' \subseteq B$, $\mathscr{H}^d_\infty(B') \geq cr^d_{B'}$. Let P and P' be two d-planes. Then*

$$d_{B'}(P, P') \lesssim_{d,c} \left(\frac{r_B}{r_{B'}}\right)^{d+1} \beta^{d,1}_E(B, P) + \beta^{d,1}_E(B', P').$$

The next lemma applies this to bound $\epsilon(Q)$ by $\beta^{d,1}_E(MB_Q)$:

Lemma 2.20. *Let $\mathscr{D}$ be a Christ-David lattice for a lower content d-regular set E and $K, M > 0$ be constants such that $\frac{10^4}{\rho} \leq K \leq 10^{-1}\rho^2 M$. If $\{P_Q\}_{Q\in\mathscr{D}}$ is a family of d planes satisfying $\beta^{d,1}_E(2\rho^{-1}KB_Q, P_Q) \lesssim \beta^{d,1}_E(2\rho^{-1}KB_Q)$, then*

$$\epsilon(Q) \lesssim_{\rho,M,d} \beta^{d,1}_E(MB_Q).$$

Proof. Let $U, R \in \mathscr{D}$ be cubes which achieve the supremum in the definition of $\epsilon(Q)$. Then

$$\epsilon(Q) = d_{KB_R}(P_U, P_R).$$

We want to apply Lemma 2.19 with $B = B' = KB_R$. First, we prove some ball inclusions. We claim

$$KB_R \subseteq 2\rho^{-1}KB_U. \tag{2.16}$$

Indeed, we let $y \in KB_R$ and we compute

$$\begin{aligned}|y - x_U| &\leq |y - x_R| + |x_R - x_Q| + |x_Q - x_U| \\ &\leq K\ell(R) + \frac{K}{10}\ell(R) + \frac{K}{10}\ell(U) \leq 2K\ell(R) \leq 2\rho^{-1}K\ell(U).\end{aligned}$$

Second, we claim

$$2\rho^{-1}KB_U \subseteq MB_Q \text{ and } 2\rho^{-1}KB_R \subseteq MB_Q. \tag{2.17}$$

Because $\ell(R) \geq \ell(U)$, it suffices to prove $2\rho^{-1}KB_R \subseteq MB_Q$. We let $y \in 2\rho^{-1}KB_R$ and compute

$$|y - x_Q| \leq |y - x_R| + |x_R - x_Q| \leq 4\rho^{-1}K\ell(R) + \frac{K}{10}\ell(R) \leq 10K\rho^{-2}\ell(Q) < M\ell(Q).$$

Now, we apply Lemma 2.19 with $B = B' = KB_R$, then

$$\begin{aligned}
d_{KB_R}(P_U, P_R) &\lesssim \beta_E^{d,1}(KB_R, P_R) + \beta_E^{d,1}(KB_R, P_U) \\
&\lesssim_\rho \beta_E^{d,1}(2\rho^{-1}KB_R, P_R) + \beta_E^{d,1}(2\rho^{-1}KB_U, P_U) \\
&\lesssim \beta_E^{d,1}(2\rho^{-1}KB_R) + \beta_E^{d,1}(2\rho^{-1}KB_U) \\
&\lesssim_M \beta_E^{d,1}(MB_Q).
\end{aligned}$$

where the second line follows from (2.16), the third line follows from the hypothesis on P_Q, and the final line from (2.17). ■

3 The proof of Theorem A

Fix constants $\rho = \frac{1}{1000}$, $K = \frac{10^4}{\rho}$, $M = \frac{10K}{\rho^2}$, $A_0 = \frac{1000\sqrt{d}}{c_0\rho}$. Throughout this section, assume that $\Omega \subseteq \mathbb{R}^{d+1}$ satisfies the hypotheses of Theorem A. We begin by constructing a Reifenberg parameterization for $\partial\Omega \cap B(0,1)$

3.1 The CCBP adapted to $\mathscr{D}$

We want to form a CCBP adapted to the Christ-David lattice $\mathscr{D}$ for $\partial\Omega$ with the aim of applying David and Toro's bi-Lipschitz Reifenberg parameterization result Theorem 2.4. For any $k \in \mathbb{Z}$, let $s(k)$ be an integer such that

$$50\rho^{s(k)} \leq r_k < 50\rho^{s(k)-1} \tag{3.1}$$

We note that if $Q \in \mathscr{D}_{s(k)}$, then this means

$$10\ell(Q) \leq r_k < 10\rho^{-1}\ell(Q) \tag{3.2}$$

and

$$\frac{\rho}{5000}r_k \leq \frac{c_0}{10\rho}r_k \leq c_0\ell(Q) \leq \operatorname{diam} Q \leq \ell(Q) \leq \frac{r_k}{10}.$$

For any $k \geq 0$, define

$$Y_k = \{x_Q : Q \in \mathscr{D}_{s(k)},\ Q \cap B(0, A_0) \neq \varnothing\}, \tag{3.3}$$

$$X_k \in \operatorname{Net}(Y_k, r_k). \tag{3.4}$$

We enumerate $X_k = \{x_{j,k}\}_{j \in J_k}$ and often use the notation $x_{j,k} = x_Q = x_{Q_{j,k}}$. Let P_0 achieve the infimum in the definition of $b\beta_{\partial\Omega}(B(0, 10A_0))$ and define

$$\begin{aligned}
B_{j,k} &= B(x_{j,k}, r_k), \\
P_{j,k} &= P_{Q_{j,k}},
\end{aligned}$$

where $P_{Q_{j,k}} \ni x_{Q_{j,k}}$ are such that $b\beta_{\partial\Omega}^{d,1}(2\rho^{-1}KB_{Q_{j,k}}) \lesssim b\beta_{\partial\Omega}^{d,1}(2\rho^{-1}KB_{Q_{j,k}})$ as in the hypotheses of Lemma 2.20. We first show that $\epsilon_k'(x_{j,k})$ is controlled by $\epsilon(Q_{j,k})$.

Lemma 3.1. *Fix $k \geq 0$ and $Q \in \mathscr{D}_{s(k)}$. For any $z \in \mathbb{R}^{d+1}$ such that $|z - x_Q| < 200\rho^{-1}\ell(Q)$,*

$$\epsilon_k'(z) \leq K\epsilon(Q).$$

Proof. We first show that the supremum in the definition of $\epsilon(Q)$ is over a larger collection of pairs of planes than that in the definition of $\epsilon_k'(z)$. Let $i \in J_k$ be such that $z \in 10B_{i,k}$. Then by (3.2),

$$|x_Q - x_{i,k}| < |x_Q - z| + |z - x_{i,k}| < 200\rho^{-1}\ell(Q) + 10r_k < 300\rho^{-1}\ell(Q_{i,k}) < \frac{K}{10}\ell(Q)$$

because $K \geq 10^4\rho^{-1}$ and $\ell(Q) = \ell(Q_{i,k})$. Therefore, $x_Q \in \frac{K}{10}B_{Q_{i,k}}$. If instead $z \in 11B_{i,k-1}$ for some $i \in J_{k-1}$, then

$$|x_Q - x_{i,k-1}| < |x_Q - z| + |z - x_{i,k-1}| < 200\rho^{-1}\ell(Q) + 11r_{k-1} < 310\rho^{-1}\ell(Q_{i,k-1}) < \frac{K}{10}\ell\left(Q_{i,k-1}\right).$$

Therefore, $x_Q \in \frac{K}{10}B_{Q_{i,k-1}}$. In addition, for any admissible $x_{i,l}$ in the definition of $\epsilon_k'(z)$ we can write $100r_l \leq 1000\rho^{-1}\ell(Q_{i,l}) < K\ell(Q_{i,l})$ so that $100B_{i,l} \subseteq KB_{Q_{i,l}}$. Let $P_{i,k}$ and $P_{m,l}$ be planes which achieve the supremum in the definition of $\epsilon_k'(z)$. Then

$$d_{x_{m,l},100B_{m,l}}(P_{i,k}, P_{m,l}) \leq \frac{K\ell(B_{Q_{m,l}})}{100r_l} d_{KB_{Q_{m,l}}}(P_{Q_{i,k}}, P_{Q_{m,l}}) \leq K d_{KB_{Q_{m,l}}}(P_{Q_{i,k}}, P_{Q_{m,l}})$$

using the fact that $\ell(Q_{m,l}) < r_l$. ■

Applying this result for $z = x_{j,k}$ shows that $\epsilon_k'(x_{j,k}) \lesssim \epsilon(Q_{j,k})$ which we can use to prove that the triple $\mathscr{X} = (P_0, \{B_{j,k}\}, \{P_{j,k}\})$ is a CCBP.

Lemma 3.2. *$\mathscr{X}$ is a CCBP.*

Proof. We will use Lemma 2.3. First, we will show that for any $j \in J_0$, $\text{dist}(x_{j,0}, P_0) \lesssim \epsilon$. Indeed, $x_{j,0} = x_Q$ for some $Q \in \mathscr{D}_{s(0)}$ with $Q \cap B(0, A_0) \neq \varnothing$. Hence, $x_Q \in B(0, 2A_0) \cap \partial\Omega$ so that $b\beta_{\partial\Omega}(B(0, 10A_0), P_0) \leq \epsilon$ implies

$$\text{dist}(x_Q, P_0) \lesssim b\beta(B(0, 10A_0)) \cdot 10A_0 \lesssim_d \epsilon.$$

Now, we fix $k > 0$ and $j \in J_k$ and prove the following claim:

Claim: There exists $i \in J_{k-1}$ such that $x_{j,k} \in B_{i,k-1}$

Proof: Indeed, let $x_{j,k} = x_{Q_{j,k}}$. If $s(k) = s(k-1)$, then $Y_{k-1} = Y_k$ so that $x_{Q_{j,k}} \in Y_{k-1}$. The claim follows since X_{k-1} is an r_{k-1}-net for Y_{k-1}. If instead $s(k) > s(k-1)$, then $x_{Q_{j,k}^{(1)}} \in Y_{k-1}$ so that there exists $i \in J_{k-1}$ such that $x_{Q_{j,k}^{(1)}} \in B_{i,k-1}$. We have

$$\ell\left(Q_{j,k}^{(1)}\right) = 5\rho^{s(k-1)} \leq 5\rho^{s(k)-1} \leq r_{k-1}$$

so that

$$\text{dist}(x_{Q_{j,k}}, x_{i,k-1}) \leq \text{dist}(x_{Q_{j,k}}, x_{Q_{j,k}^{(1)}}) + \text{dist}(x_{Q_{j,k}^{(1)}}, x_{i,k-1}) \leq \ell\left(Q_{j,k}^{(1)}\right) + r_{k-1} \leq 2r_{k-1}$$

which proves $x_{Q_{j,k}} \in 2B_{i,k-1}$.

By Lemma 3.1, it suffices to show that $\epsilon(Q_{j,k}) \lesssim \epsilon$. But by the definition of $P_{Q_{j,k}}$ and Lemma 2.20, we have $\epsilon(Q_{j,k}) \lesssim_{M,d} \beta_{\partial\Omega}^{d,1}(MB_{Q_{j,k}}) \lesssim \epsilon$. ■

Since we've shown that $\mathscr{X}$ is a CCBP, Theorem 2.4 gives a Reifenberg parameterization $g : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ such that

$$g(P_0) \cap B(0,1) = \partial\Omega \cap B(0,1)$$

Without loss of generality, we can assume $P_0 = \mathbb{R}^d \times \{0\}$ and translate the Whitney decomposition $\mathscr{W}$ as in Definition 2.7 to a new decomposition $\mathscr{W}'$ such that $W_0 = [-2,2]^d \times [4,8] \in \mathscr{W}'$. We have that

$$\Omega \cap B(0,1) \subseteq g([-2,2]^d \times [0,8])$$

because $\partial\Omega$ is contained in the closure of $\cup_k X_k$, so in practice we only need to consider the set of descendants of W_0 to cover $\Omega \cap B(0,1)$:

$$\mathscr{W}_0 = \{W \in \mathscr{W}' : W \in D(W_0)\}. \tag{3.5}$$

We can now derive some useful properties of g.

Lemma 3.3 (Properties of g)**.**

(i) For any $x \in [-2,2]^d \times \{0\}$ and $n \in \mathbb{N}$,

$$f_n(x) \in V_n^8,$$

(ii) For any $z = (x,y) \in [-2,2]^d \times [0,8]$

$$(1 - C(d)\epsilon)|y| \leq \operatorname{dist}(g(z), \partial\Omega) \leq (1 + C(d)\epsilon)|y|. \tag{3.6}$$

(iii) For any $x \in [-2,2]^d \times \{0\}$ and $p, n \in \mathbb{N}$ with $p < n$, there exists a collection of cubes $Q_n \subseteq Q_{n-1} \subseteq \cdots \subseteq Q_p$ such that for any k with $p \leq k \leq n$, $\operatorname{dist}(g(x, r_k), Q_{s(k)}) \lesssim r_k$ and

$$\sum_{k=p}^{n} \epsilon'(f_k(x))^2 \lesssim_{M,\rho,d} \sum_{k=p}^{n} \beta_{\partial\Omega}^{d,1}(MB_{Q_k})^2.$$

In particular, g is $L'(L, \rho, M, d)$-bi-Lipschitz.

Proof. Let $z = (x,y)$ be as in (ii) with $n = n(y)$. We first prove (ii) with the added hypothesis that $f_n(x) \in V_n^8$. We will then prove (i) which will complete the proof of (ii).

Observe that

$$g(z) - f_n(x) = \sum_k \rho_k(y) \{f_k(x) - f_n(x) + R_k(x) \cdot y\}$$

where $|f_k(x) - f_n(x)| \lesssim \epsilon r_n$ and $R_k(x) \cdot y$ is a vector of norm $|y|$ which is orthogonal to the tangent plane $T_k(x)$ to Σ_k at $f_k(x)$. The fact that $f_n(x) \in V_n^8$ implies the existence of $Q \in \mathscr{D}_{s(n)}$ such that $|f_n(x) - x_Q| \leq 8r_n$. The fact that $b\beta_{\partial\Omega}(MB_Q, P_Q) \lesssim \epsilon$ combined with Lemma 6.1 and (6.4) implies

$$d_{f_n(x),19r_n}(\Sigma_n, \partial\Omega) \leq d_{f_n(x),19r_n}(\Sigma_n, P_Q) + d_{f_n(x),19r_n}(P_Q, \partial\Omega) \lesssim \epsilon. \tag{3.7}$$

We conclude $d_{f_n(x),19r_n}(T_n(x) + f_n(x), \partial\Omega) < C\epsilon$, which implies $(1 - C\epsilon)|y| \leq \operatorname{dist}(g(z), \partial\Omega) \leq (1 + C\epsilon)|y|$ as desired.

We now prove (i) by induction on n. For the base case $n = 0$, notice that $f_0(x) = x \in B(0, 5\sqrt{d}) \cap P_0$ so that $b\beta_{\partial\Omega}(B(0, 10A_0), P_0) \leq \epsilon$ implies the existence of $y \in \partial\Omega$ with $\operatorname{dist}(y, x) \lesssim_{A_0} \epsilon$. There exists $Q_0 \in \mathscr{D}_{s(0)}$ such that $y \in Q_0$ and $\operatorname{dist}(Q, 0) \leq 10\sqrt{d}$ so that x_{Q_0} is a member of the set Y_0 (see (3.3)) from which the maximal net X_0 forming the CCBP is taken. Notice that

$$|f_0(x) - x_{Q_0}| \leq |x - y| + |y - x_{Q_0}| \leq C\epsilon + \ell(Q_0) \leq 2r_0.$$

Hence, we are done if $x_{Q_0} \in X_0$. Otherwise, there exists $x_{Q_0'} \in X_0$ such that $|x_{Q_0} - x_{Q_0'}| \leq r_0$ so that $|f_0(x) - x_{Q_0'}| \leq 3r_0$ implying $f_0(x) \in V_0^3$. This proves the base case for (i).

For the inductive step, assume that $f_n(x) \in V_n^8$ for some $n \in \mathbb{N}$. Using (3.7), we find $y \in \partial\Omega$ such that

$$|f_{n+1}(x) - y| \leq |f_{n+1}(x) - f_n(x)| + |f_n(x) - y| \lesssim \epsilon r_{n+1}$$

and hence there exists $Q_{n+1} \in \mathscr{D}_{s(n+1)}$ with $\operatorname{dist}(Q_{n+1}, 0) \leq 10\sqrt{d}$ such that $|f_{n+1}(x) - x_{Q_{n+1}}| \leq 2r_{n+1}$. By a similar argument to the base case, this finishes the proof of (i).

To prove (iii), notice that $f(x) \in \partial\Omega$ so that there exists an infinite chain of (possibly repeating) cubes $Q_0 \supseteq Q_1 \supseteq \cdots \ni f(x)$ where $Q_k \in \mathscr{D}_{s(k)}$. We claim that

$$\epsilon'(f_k(x)) \lesssim \epsilon(Q_k) \lesssim \beta_{\partial\Omega}^{d,1}(MB_{Q_k}).$$

Indeed, by Lemmas 3.1 and 2.20, we only need to show that $|f_k(x) - x_{Q_k}| < 200\rho^{-1}\ell(Q_k)$ to verify the first inequality. But we have

$$\begin{aligned} |f_k(x) - x_{Q_k}| &\leq |f_k(x) - f(x)| + |f(x) - x_{Q_k}| \\ &\leq C\epsilon r_k + 10r_k + \ell(Q_k) \leq (C\epsilon + 100)\rho^{-1}\ell(Q_k) + \ell(Q_k) \leq 102\rho^{-1}\ell(Q_k) \end{aligned}$$

as desired. Because the set $\{n : s(k) = s(n)\}$ has a uniformly bounded number of elements in terms of ρ, it follows that

$$\sum_{k=p}^{n} \epsilon_k'(f_k(x))^2 \lesssim_{M,\rho} \sum_{k=p}^{n} \epsilon(Q_k)^2 \lesssim_{M,\rho,d} \sum_{k=p}^{n} \beta_{\partial\Omega}^{d,1}(MB_{Q_k})^2.$$

The claim that $\text{dist}(g(x, r_k), Q_k) \lesssim r_k$ follows from (ii). By the hypotheses on Ω, we have $\sum_{k=1}^{\infty} \epsilon'(f_k(x))^2 \lesssim \sum_{f(x)\in Q} \beta_{\partial\Omega}^{d,1}(MB_Q)^2 \leq L$ so that g is $L'(L, \rho, M, d)$-bi-Lipschitz by Theorem 2.4. ■

Now that we know that g is L'-bi-Lipschitz, we define $p(L') \in \mathbb{Z}$ such that

$$2^{p-1} \leq L' < 2^p \tag{3.8}$$

and we replace $\mathscr{W}_0$ with

$$\mathscr{R}_w = \{R \in \mathscr{R}_p : \exists W \in \mathscr{W}_0,\ R \subseteq W\}.$$

That is, $\mathscr{R}_w$ is the set of Whitney boxes R with $\ell(R) = 2^{-p}h(R)$ which are contained in some member of $\mathscr{W}_0$. This ensures that

$$L'\ell(R) = L'2^{-p}h(R) \leq h(R) \tag{3.9}$$

so that g does not stretch R across too far of a region on the scale of $h(R)$.

We say more about the shape of image boxes in the following lemma:

Lemma 3.4 (Image boxes). *For any $W \in \mathscr{R}_w$, we have*

$$(1 - C\epsilon)h(W) \leq \text{dist}(g(W), \partial\Omega) \leq (1 + C\epsilon)h(W), \tag{3.10}$$

$$(1 - C\epsilon)h(W) \leq \text{diam}\, g(W) \leq 5\sqrt{d}h(W). \tag{3.11}$$

There exists constants $C_0(L'), C_1(d)$ such that

$$B(g(c_W), C_0^{-1}h(W)) \subseteq g(W) \subseteq B(g(c_W), C_1 h(W)) \tag{3.12}$$

where c_W is the center of W.

Proof. We first note that (3.10) follows from (3.6) and the fact that $\text{dist}(W, \mathbb{R}^d) = h(W)$ by definition. To prove (3.11), let $z, z' \in R$ with $z = (x, y), z' = (x', y')$. We have

$$\begin{aligned} |g(x, y) - g(x', y')| &\leq |g(x, y) - g(x', y)| + |g(x', y) - g(x', y')| \\ &\leq L'|x - x'| + 2|y - y'| \\ &\leq L'\sqrt{d}\ell(R) + 2h(R) \\ &\leq 5\sqrt{d}h(R) \end{aligned}$$

The lower bound follows from (3.6) by considering the distance between images of points in the lower and upper faces of W. To prove (3.12), we first observe that each box $W \in \mathscr{R}$ contains a small ball $B(c_W, c(L')h(W))$ around its center. Since g is L'-bi-Lipschitz, we get a larger constant $C_0(L')$ such that the lower containment in (3.12) holds. The existence of $C_1(d)$ as in the upper containment follows from the upper inequality in (3.11). We also note that because g is injective and distinct boxes $R, W \in \mathscr{R}_w$ have disjoint interiors, we have

$$B(g(c_W), C_0^{-1}h(W)) \cap B(g(c_R), C_0^{-1}h(R)) = \varnothing. \tag{3.13}$$

■

3.2 Whitney coronizations and the Lipschitz decomposition

In Item (iii) of Lemma 3.3, we showed that the mapping g was bi-Lipschitz so that $\partial\Omega$ is locally a bi-Lipschitz image. Hence, $\partial\Omega$ is d-uniformly rectifiable and therefore has an (M, ϵ, δ)-graph coronization for arbitrarily small values of ϵ and δ by Proposition 2.17. Take $\epsilon'(d, L), \delta'(d, L) > 0$ fixed later sufficiently small and let $\mathscr{C} = (\mathscr{G}, \mathscr{B}, \mathscr{F})$ be an (M, ϵ', δ')-graph coronization for $\partial\Omega$.

The plan for the proof of Theorem A is to construct a "coronization" of $\mathscr{R}_w$ which "follows" the coronization $\mathscr{C}$ of $\partial\Omega$. That is, we will construct a triple

$$\mathscr{C}_w = (\mathscr{G}_w, \mathscr{B}_w, \mathscr{T})$$

of good boxes, bad boxes, and stopping time regions $\mathscr{T} = \{T\}_{T \in \mathscr{T}}$ (see Remark 2.12) partitioning $\mathscr{G}_w$ such that for each $T \in \mathscr{T}$, there exists some $S \in \mathscr{F}$ such that the images of all boxes in T under g are "surrounded" in scale and location by cubes in S.

Definition 3.5 (g-Whitney coronizations)**.** Let $g, \mathscr{R}_w$ be as above. We now give a partition of $\mathscr{R}_w$ into a bad set $\mathscr{B}_w$ and good set $\mathscr{G}_w$ which picks out all Whitney boxes whose images under g are "A_0-surrounded" by surface cubes within a single stopping time region $S \in \mathscr{F}$:

$$\mathscr{G}_w = \{W \in \mathscr{W} : \exists S \in \mathscr{F},\ \forall Q \in \mathscr{D} \text{ such that } Q \simeq_{A_0} g(W) \text{ we have } Q \in S\}, \tag{3.14}$$
$$\mathscr{B}_w = \mathscr{W} \setminus \mathscr{G}_w. \tag{3.15}$$

(See Definition 2.9.) Given a root box $W \in \mathscr{G}_w$, we define the stopping time region T_W with top cube W to be the maximal sub tree of $D(W) \cap \mathscr{G}_w$ such that for any $R \in T_W$, either all of its children are in T_W, or none are. Any such stopping time region has associated minimal cubes and stopped cubes

$$m(T_W) = \{R \in T_W : R \text{ has a child not in } T_W\},$$
$$\mathrm{Stop}(T_W) = \{R \in \mathscr{W} : R \text{ has a parent in } m(T_W)\}.$$

We initialize our construction with the lattice $\mathscr{R}_w$ and triple $(\mathscr{G}_w, \mathscr{B}_w, \mathscr{T}_0 = \varnothing)$. Given the k-th stage stopping time collection $\mathscr{T}_k$, we choose a root box $W \in \mathscr{G}_w \setminus \cup_{T \in \mathscr{T}_k} T$ and form the stopping time region T_W. We set $\mathscr{T}_{k+1} = \mathscr{T}_k \cup \{T_W\}$. Repeating this process inductively, we obtain a partition $\mathscr{T} = \bigcup_{k=1}^{\infty} \mathscr{T}_k$ of $\mathscr{G}_w$ into coherent stopping time regions. This gives the triple $\mathscr{C}_w = (\mathscr{G}_w, \mathscr{B}_w, \mathscr{T})$. We call $\mathscr{C}_w$ the *g-Whitney coronization* of $\mathscr{R}_w$ with respect to $\mathscr{C} = (\mathscr{G}, \mathscr{B}, \mathscr{F})$.

Remark 3.6 (improving the stopping time)**.** In this construction, we used Whitney boxes with side length $\ell(R) = 2^{-p}h(R)$ to ensure that for any $R \in \mathscr{R}_w$, $\operatorname{diam} g(R) \lesssim_d h(R)$. Without this condition or some other method of controlling the size of image boxes, we could have $z = (x, y),\ z = (x', y')$ such that $h(R) \ll |g(z) - g(z')|$ which would cause us to lose control of the change in Dg across R we desire in Lemma 3.11 below.

What we really want are image pieces of some kind which satisfy the conclusions of Lemma 3.4 along with small parameterization derivative change across the pieces as in Lemma 3.11 below. If one could form reasonable stopping time domains out of similar pieces whose images satisfy the conclusions of 3.4 with constant C_0 dependent only on d, this would essentially prove a version of Theorem A without hypothesis (ii). If g were $K(d)$-quasiconformal, then this could likely be accomplished by adding modifications to the stopping time by dynamically either combining or cutting apart children boxes for a given top box $W(T)$ along coordinate directions according to the size and shape of Dg inside. In general though, Dg can distort boxes so badly that coordinate boxes cannot be mapped forward appropriately in general, so one would need to devise a better way of partitioning the domain into pieces which are mapped forward well under a more wild parameterization.

We will use $\mathscr{C}_w$ to break up $\mathscr{R}_w$ into regions which will map forward under g to the Lipschitz graph domains we desire as in the conclusion of Theorem A

Definition 3.7 (Stopping time domains)**.** Let $\mathscr{C}_w = (\mathscr{G}_w, \mathscr{B}_w, \mathscr{T})$ be a g-Whitney coronization as above. For each $T \in \mathscr{T}$, we define a *stopping time domain*

$$\mathcal{D}_T = \bigcup_{W \in T} W.$$

For each $W \in \mathscr{B}_w$, we note that $\ell(W) = 2^{-p}h(W)$ where p is as in (3.8) and define a collection of associated trivial domains by chopping W into 2^p cubes of common side length $\ell(W)$. That is, assuming $W = [0, \ell(W)]^d \times [h(W), 2h(W)]$, we set

$$\mathscr{L}_W = \{[0, \ell(W)]^d \times [h(W) + k\ell(W), h(W) + (k+1)\ell(W)] : 0 \le k \le 2^p - 1\}.$$

The collection $\mathscr{L}' = \{\mathcal{D}_T\}_{T \in \mathscr{T}} \cup \bigcup_{W \in \mathscr{B}_w} \mathscr{L}_W$ is a partition of $\bigcup_{W \in \mathscr{R}} W = [-2,2]^d \times [0,8]$ up to finite overlaps on boundaries. Each cube domain $R_W \in \mathscr{L}_W$ is $C(d)$-Lipschitz graphical, but the domain $\mathcal{D}_T$ is not Lipschitz graphical in general. However, T consists of a coherent collection of boxes of a given ratio of side length to height $\ell(R) = 2^{-p}h(R)$. Therefore, applying a dilation A_p by a factor of 2^p in the first d coordinates gives a domain $A_p(\mathcal{D}_T)$ consisting of cubes. Proposition 5.1 then gives the existence of a d-rectifiable, d-Ahlfors upper regular set Σ_T such that

$$A_p(\mathcal{D}_T) \setminus \Sigma_T = \bigcup_{j \in J_T} A_p(\mathcal{D}_T^j)$$

where $\{A_p(\mathcal{D}_T^j)\}_{j \in J_T}$ is a collection of $C(d)$-Lipschitz graph domains with disjoint interiors. By Lemma 5.4, we then get the existence of a constant $C'(L,d)$ such that $\mathcal{D}_T^j$ is a $C'(L,d)$-Lipschitz graph domain. We then finally define

$$\mathscr{L}_0 = \left\{\mathcal{D}_T^j\right\}_{T \in \mathscr{T}, j \in J_T} \cup \bigcup_{W \in \mathscr{B}_w} \mathscr{L}_W.$$

We can now define the collection of Lipschitz graph domains $\mathscr{L}$ as desired in Theorem A:

Definition 3.8 (Lipschitz decomposition)**.** Let $\mathscr{L}_0$ be as in Definition 3.7. We define the *Lipschitz decomposition* of $\Omega \cap B(0,1)$ as

$$\mathscr{L} = \{g(\mathcal{D}) : \mathcal{D} \in \mathscr{L}_0\}. \tag{3.16}$$

In order to prove Theorem A, it suffices to prove Propositions 3.9 and 3.10 below.

Proposition 3.9. *Let Ω be as in Theorem A and $\mathscr{L} = \{\Omega_j\}_{j \in J_{\mathscr{L}}}$ be as in* (3.16)*. There exists $L_1(L,d,\epsilon) > 0$ such that for any $j \in J_{\mathscr{L}}$, Ω_j is an L_1-Lipschitz graph domain.*

To prove Proposition 3.9, we use the fact that the graph coronization $\mathscr{C}$ of $\partial\Omega$ and the Whitney coronization $\mathscr{C}_w$ of Definition 3.5 adapted to $\mathscr{C}$ were chosen so that Dg is very close to being constant on any given domain $\mathcal{D} \in \mathscr{L}_0$. This uses the explicit calculations for Dg given in Proposition 6.7. This means g distorts $\mathcal{D}$ only slightly such that $\mathcal{D}$ remains a Lipschitz graph domain (see Proposition

5.6). The refinement of Whitney cubes to smaller Whitney boxes ensures that $\operatorname{diam}(g(W)) \simeq_d h(W)$ holds for any box W so that $g(W)$ does not stretch across too long of a region of $\partial\Omega$ compared to its distance from $\partial\Omega$. If $W \in \mathscr{B}_W$, then this ensures that $Dg|_W$ varies at a rate determined at worst by the Reifenberg flatness constant ϵ. Because in this case, W is divided into the set $\mathscr{L}_W$ of cubes, which are $C(d)$-Lipschitz graph domains (note C is independent of L), g maps them forward to Lipschitz graph domains given that ϵ is fixed small enough with respect to d.

The construction of stopping time regions $\mathscr{T}$ proceeds in such a way that any $T \in \mathscr{T}$ is a coherent collection of (potentially long and thin) Whitney boxes such that the change in Dg on $\mathcal{D}_T$ is controlled by the geometry of $\partial\Omega$ inside some surface stopping time region $S \in \mathscr{F}$. These regions are defined such that $\partial\Omega$ looks like a Lipschitz graph with small constant $\epsilon'(L,d)$ and angle variation $\delta'(L,d)$ on the scale of cubes in S from which we derive that $Dg|_{\mathcal{D}_T}$ varies at a rate determined by $\delta'(L,d)$ (See Lemma 3.11), giving Lipschitz graphicality for domains in $\{g(\mathcal{D}_T^j)\}_{j\in J_T, T\in\mathscr{T}}$ by Proposition 5.6 again as long as ϵ', δ' are fixed small enough with respect to L and d.

Proposition 3.10. *Let Ω be as in Theorem A and $\mathscr{L} = \{\Omega_j\}_{j\in J_{\mathscr{L}}}$ be as in* (3.16). *For any $y \in \partial\Omega \cap B(0,1)$ and $0 < r < 1$, we have*

$$\sum_{j\in j_{\mathscr{L}}} \mathcal{H}^d(\partial\Omega_j \cap B(y,r)) \lesssim_{\epsilon,L,d} r^d. \tag{3.17}$$

To prove 3.10 we use the fact that the Whitney coronization is chosen in such a way that the images of boxes in the bad set $\mathscr{B}_w$ have surface measure controlled by the measure of the A_0-close bad cubes $\mathscr{B}$ or cubes in $\mathscr{D}$ on the "edges" of stopping time regions which we collect in the set $\mathscr{B}_e$ in (3.25) below. These cubes form a Carleson set (see Lemma 3.16) which gives Carleson packing type estimates for the surface measure of the image cubes $\{g(R_W)\}_{W\in\mathscr{B}_w,\ R_W\in\mathscr{L}_W}$. Because the only time we stop in the construction of $T \in \mathscr{T}$ is when we hit some $W \in \mathscr{B}_w$ the surface measure of domains in $\{g(\mathcal{D}_T)\}_{T\in\mathscr{T}}$ is controlled by the measure of nearby cubes in $\mathscr{B}_e$. The fact that g is bi-Lipschitz and preserves distances to the boundary means that the family $\{g(W)\}_{W\in\mathscr{R}}$ behaves in many ways like a Whitney decomposition itself (see Lemma 3.14) so that we can bound the number of image boxes which are A_0-close to any fixed bad cube $Q \in \mathscr{B}_e$, giving the desired Carleson type estimates.

3.3 Lipschitz bounds for Theorem A

The goal of this section is to prove Proposition 3.9. The following lemma allows us to control the change in Dg on any stopping time domain T.

Lemma 3.11 (Variation of Dg). *For any $T \in \mathscr{T}$ and $z, w \in \mathcal{D}_T$, we have*

$$|Dg(z) \cdot Dg(w)^{-1} - I| \le C\delta' \tag{3.18}$$

Proof. First, fix some $T \in \mathscr{T}$. We want to apply Proposition 6.7 with $M_0 \lesssim_d 1$ and $z = c(W(T)) = (x,y)$ by showing that $\mathcal{D}_T \subseteq G_z^{M_0}$. So, let $z' = (x',y') \in R \in T$ and let $n = n(y'),\ p = l(y)$. We need to prove the following three statements:

(i) $|f_p(x) - f_p(x')| \lesssim_d r_p$,

(ii) $\sum_{k=p}^{n} \epsilon'_k(f_k(x'))^2 \lesssim \epsilon'$,

(iii) $\operatorname{Angle}(T_k(x'), T_p(x')) \lesssim \delta'$.

We begin by observing that (i) follows from the fact that f_p is L'-bi-Lipschitz so that

$$|f_p(x) - f_p(x')| \le L'|x - x'| \le 2L'\sqrt{d}\ell(W(T)) \lesssim_d h(W(T)) \lesssim r_p$$

using (3.9). To prove (ii), let $Q_p \supseteq Q_{p+1} \supseteq \cdots \supseteq Q_n$ be the cubes given by Lemma 3.3. For any k with $p \le k \le n$ the fact that $\operatorname{dist}(g(x', r_k), Q_k) \lesssim r_k$ means that $(x', r_k) \in R \le W(T)$ with

$$\operatorname{diam} Q_k \ge c_0 \ell(Q_{k+1}) \ge \frac{c_0\rho}{10} r_k \ge \frac{c_0\rho}{200} h(R) \ge \frac{c_0\rho}{1000\sqrt{d}} \operatorname{diam}(R) = A_0 \operatorname{diam} R.$$

so that $Q_k \simeq_{A_0} R$. This means there exists $S \in \mathscr{F}$ such that $Q_k \in S$ for any k by the definition of the stopping time region T. We conclude that

$$\sum_{k=p}^{n} \epsilon_k'(f_k(x))^2 \lesssim \sum_{k=p}^{n} \beta_{\partial\Omega}^{d,1}(MB_{Q_k})^2 \lesssim \epsilon'.$$

To prove (iii), observe that

$$\operatorname{Angle}(T_p(x'), T_n(x')) \le \operatorname{Angle}(T_p(x'), P_{Q_p}) + \operatorname{Angle}(P_{Q_p}, P_{Q_n}) + \operatorname{Angle}(P_{Q_n}, T_n(x')) \lesssim \epsilon' + \delta' + \epsilon' \lesssim \delta'$$

where we used Lemma 6.1 and the fact that $Q_p, Q_n \in S$. ■

Using the results of Section 5, we can now prove Proposition 3.9.

Proof of Proposition 3.9. Every domain in $\mathscr{L}$ is either of the form $g(\mathcal{D}_T^j)$ for some $T \in \mathscr{T}, j \in J_T$ or $g(R_W)$ for some $W \in \mathscr{B}_w$, $R \in \mathscr{L}_W$. We first consider domains of the first form.

Let $T \in \mathscr{T}$ and let $A_p : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^{d+1}$ be given by $A_p(x, y) = (2^p x, y)$. By definition, the image stopping time region $\mathcal{D}_T' = A_p(\mathcal{D}_T)$ is composed of cubes and Proposition 5.1 implies there exists a constant $L_0(d)$ such that $\mathcal{D}_T'$ has a decomposition into L_0-Lipschitz graph domains which passes to a decomposition of $\mathcal{D}_T$ into $L_0'(d, L')$-Lipschitz graph domains $\{\mathcal{D}_T^j\}_{j \in J_T}$ by applying A_p^{-1}. Now, using Lemma 3.11, we see (3.18) holds on $\mathcal{D}_T$ so that by taking $\epsilon'(L', d), \delta'(L', d)$ sufficiently small, Proposition 5.6 implies $g(\mathcal{D}_T^j)$ is an $L_1(L', d)$-Lipschitz graph domain.

Now, let $W \in \mathscr{B}_w$ and $R_W \in \mathscr{L}_W$. The proof of Lemma 3.11 shows that $|Dg(z) \cdot Dg(w)^{-1} - I| \le C\epsilon$ using only the fact that $\partial\Omega$ is ϵ Reifenberg flat. Since R_W is a cube, it is a $C(d)$-Lipschitz graph domain so that Proposition 5.6 implies $g(R_W)$ is a $C'(d)$-Lipschitz graph domain as long as ϵ is sufficiently small with respect to d. ■

3.4 Surface area bounds for Theorem A

We now focus on proving Proposition 3.10. We will justify the name coronization by proving Carleson estimates for the g-Whitney coronization which will imply the desired estimates for our domains.

Definition 3.12 (C_0-Whitney family)**.** Let $\Omega_0 \subseteq \mathbb{R}^{d+1}$ be a domain and let $C_0 \ge 1$. We say that a collection $\mathscr{V}$ of subsets of Ω_0 is a C_0-*Whitney family* if for every $V \in \mathscr{V}$, we have

$$C_0^{-1} \operatorname{diam} V \le \operatorname{dist}(V, \Omega_0^c) \le C_0 \operatorname{diam} V, \tag{3.19}$$

there exists $c_V \in V$ such that

$$B(c_V, C_0^{-1} \operatorname{diam} V) \subseteq V, \tag{3.20}$$

and, if $V \ne V'$, then

$$B(c_V, C_0^{-1} \operatorname{diam} V) \cap B(c_{V'}, C_0^{-1} \operatorname{diam} V') = \varnothing. \tag{3.21}$$

Lemma 3.13. *Let $\Omega_0, \mathscr{V}$ be as in Definition 3.12. Let $A \ge 1$, $U \subseteq \mathbb{R}^{d+1}$ and set*

$$\mathscr{V}_{A,U} = \{V \in \mathscr{V} : V \simeq_A U\}.$$

Then,

$$\#(\mathscr{V}_{A,U}) \lesssim_{A, C_0, d} 1. \tag{3.22}$$

If $\mathscr{U}$ is a collection of subsets such that for any $V \in \mathscr{V}$, there exists $U \in \mathscr{U}$ such that $V \simeq_A U$, then

$$\sum_{V\in\mathscr{V}} (\operatorname{diam} V)^d \lesssim_{A,C_0,d} \sum_{U\in\mathscr{U}} (\operatorname{diam} U)^d. \tag{3.23}$$

Proof. For any $V \in \mathscr{V}$, we have

$$\operatorname{dist}(U,V) \leq A \operatorname{diam} U,$$

$$A^{-1} \operatorname{diam} U \leq \operatorname{diam} V \leq A \operatorname{diam} U.$$

Let $B_V = B(c_V, C_0^{-1} \operatorname{diam} V)$ and fix $u \in U$. It follows that $B_V \subseteq V \subseteq B(u, 3A \operatorname{diam} U)$ and $C_0^{-1} \operatorname{diam} V \geq (C_0 A)^{-1} \operatorname{diam} U$ so that $\{B_V\}_{V\in\mathscr{V}_{A,U}}$ is a collection of disjoint balls with radius $r(B_V) \geq (C_0 A)^{-1} \operatorname{diam} U$ contained in the ball $B(u, 3A \operatorname{diam} U)$ and hence has cardinality bounded in terms of C_0, A, and d. This proves (3.22).

To prove (3.23), notice that

$$\sum_{V\in\mathscr{V}} (\operatorname{diam} V)^d \leq \sum_{U\in\mathscr{U}} \sum_{V\in\mathscr{V}_{A,U}} (\operatorname{diam} V)^d \lesssim_A \sum_{U\in\mathscr{U}} \#(\mathscr{V}_{A,U})(\operatorname{diam} U)^d \lesssim_{A,C_0,d} \sum_{U\in\mathscr{U}} (\operatorname{diam} U)^d \qquad \blacksquare$$

We define

$$\mathcal{G}_0 = \{g(W) : W \in \mathscr{R}_w\} \tag{3.24}$$

and observe that $\mathcal{G}_0$ is a $\Lambda_0(L', d)$-Whitney family by equations (3.10) - (3.13):

Lemma 3.14. *There exists a constant $\Lambda_0(L', d) > 0$ such that $\mathcal{G}_0$ is a $\Lambda_0(L', d)$-Whitney family.*

Combining this fact with Lemma 3.13 will allow us to bound the surface measure of images of stopped boxes in terms of the side-length of A_0-close bad and stopped cubes in $\mathscr{D}$. The following two lemmas will give a Carleson packing condition on this bad subset $\mathscr{B}_e \subseteq \mathscr{D}$ defined in (3.25) below from which we will be able to conclude the desired surface measure bound (3.17). We begin with the following lemma due to David and Semmes.

Lemma 3.15 (cf. [DS93] Part I Lemma 3.27, (3.28))**.** *Let $A \geq 1$, let $\mathscr{D}$ be a Christ-David lattice with coronization $(\mathscr{G}, \mathscr{B}, \mathscr{F})$. Then,*

(a) The set

$$\mathscr{A} = \{Q \in \mathscr{G} : \exists Q' \in S' \neq S \ni Q \text{ such that } Q \simeq_A Q'\}$$

satisfies a Carleson packing condition.

(b) Suppose $\mathscr{H} \subseteq \mathscr{D}$ satisfies a Carleson packing condition. The set

$$\mathscr{H}_A = \{Q \in \mathscr{D} : \exists Q' \in \mathscr{H} \text{ such that } Q \simeq_{A_0} Q'\}$$

satisfies a Carleson packing condition.

This lemma will directly give us a Carleson packing condition on the set

$$\mathscr{B}_e = \mathscr{B} \cup \{Q \in \mathscr{G} : \exists Q' \in S' \neq S \ni Q \text{ such that } Q \simeq_{2A_0^2} Q'\}. \tag{3.25}$$

Lemma 3.16 ($\mathscr{B}_e$ Carleson packing condition)**.** *The family $\mathscr{B}_e$ satisfies a Carleson packing condition. For any $W \in \mathscr{B}_w$, there exists $Q_W \in \mathscr{B}_e$ such that $g(W) \simeq_{A_0} Q_W$.*

Proof. The fact that $\mathscr{B}_e$ satisfies a Carleson packing condition follows from Lemma 3.15. For the second statement, let $W \in \mathscr{B}_w$. By definition, $W \notin \mathscr{G}_w$ so that either

(i) $\exists Q \in \mathscr{B}$ such that $g(W) \simeq_{A_0} Q$,

(ii) $\exists S_1, S_2 \in \mathscr{F}$ such that $Q_1 \in S_1 \neq S_2 \ni Q_2$ with $g(W) \simeq_{A_0} Q_1$ and $g(W) \simeq_{A_0} Q_2$.

The first case gives the desired cube Q_W immediately. In the second case, a calculation using the definition of A_0-closeness shows that $Q_1 \simeq_{2A^2} Q_2$ so that $Q_1, Q_2 \in \mathscr{B}_e$ and we can set $Q_W = Q_1$. ■

We now fix $y \in \partial\Omega \cap B(0,1)$ and $0 < r \leq 1$. In order to pick out the pieces of the domains which actually intersect $B(y,r)$, for any $T \in \mathscr{T}$ we define

$$\mathscr{T}'_{y,r} = \{T \in \mathscr{T} : \Omega_T \cap B(y,r) \neq \varnothing\}.$$

We break up $\mathscr{T}'_{y,r}$ into regions with large and small top cubes:

$$\begin{aligned}\mathscr{T}_{L,r} &= \{T \in \mathscr{T}_{y,r'} : h(W(T)) > 10r\},\\ \mathscr{T}_{y,r} &= \mathscr{T}'_{y,r} \setminus \mathscr{T}_L.\end{aligned}$$

It is also convenient to collect all of the boundaries associated with a given stopping time domain $T \in \mathscr{T}$ into one set:

$$\mathcal{B}_T = \bigcup_{j \in J_T} \partial\Omega_T^j.$$

We note that $\mathcal{B}_T$ is d-upper Ahlfors regular by Proposition 5.1. Proposition 3.10 will follow from the following three lemmas below. The first gives a bound for the domains in $\mathscr{T}_{L,r}$ while the second gives a bound for those in $\mathscr{T}_{y,r}$.

Lemma 3.17.

$$\sum_{T \in \mathscr{T}_{L,r}} \mathcal{H}^d(\mathcal{B}_T \cap B(y,r)) \lesssim_{L',d} r^d \leq \mathcal{H}^d(\partial\Omega \cap B(y,r)).$$

Proof. We will show that $\#(\mathscr{T}_{L,r})$ is bounded independent of y and r. For any $T \in \mathscr{T}_{L,r}$ we claim that there exists some $W_T \in T$ such that $h(W_T) \simeq r$ and $\operatorname{dist}(g(W_T), y) \simeq r$. Indeed, by definition there exists $R_T \in T$ such that $g(R_T) \cap B(y,r) \neq \varnothing$. There then exists a box $W_T \in T$ with $W_T \geq R_T$ with the desired properties because of (3.6) and the inequality $h(W(T)) > 10r$. But, since the collection $\{g(W_T)\}_{T \in \mathscr{T}_{L,r}}$ is a Whitney family, it follows that $N = \#(\mathscr{T}_{L,r}) = \#(\{g(W_T)\}_{T \in \mathscr{T}_{L,r}}) \lesssim_{L',d} 1$. Therefore, since $\mathcal{B}_T$ is d-upper Ahlfors regular,

$$\sum_{T \in \mathscr{T}_{L,r}} \mathcal{H}^d(\mathcal{B}_T \cap B(y,r)) \lesssim_d \#(\mathscr{T}_{L,r}) r^d \lesssim_{L',d} r^d.$$ ■

We now handle the regions with small top boxes:

Lemma 3.18.

$$\sum_{T \in \mathscr{T}_{y,r}} \mathcal{H}^d(\mathcal{B}_T \cap B(y,r)) \lesssim_{L',d,\epsilon'} \mathcal{H}^d(\partial\Omega \cap B(y, A_0^2 r)) \lesssim_{L',d} r^d. \tag{3.26}$$

Proof. We first note that since $\mathcal{H}^d(\mathcal{B}_T) \lesssim_d \mathcal{H}^d(\partial\Omega_T)$, we have

$$\sum_{T \in \mathscr{T}_{y,r}} \mathcal{H}^d(\mathcal{B}_T \cap B(y,r)) \leq \sum_{T \in \mathscr{T}_{y,r}} \mathcal{H}^d(\mathcal{B}_T) \lesssim_d \sum_{T \in \mathscr{T}_{y,r}} \mathcal{H}^d(\partial\Omega_T).$$

Therefore, it suffices to prove $\sum_{T \in \mathscr{T}_{y,r}} \mathcal{H}^d(\partial\Omega_T) \lesssim_{L',d,\epsilon} \mathcal{H}^d(\partial\Omega \cap B(y, A_0^2 r))$.

For any $T \in \mathscr{T}_{y,r}$, (3.9) gives $\operatorname{diam} g(\operatorname{Bot}(W))) \lesssim_d h(W)$ so that Lemma 3.11 and the fact that g is L'-bi-Lipschitz give an analogue of (5.1):

$$\begin{aligned}\mathcal{H}^d(\partial\Omega_T) &\lesssim_{d,L'} \mathcal{H}^d(\partial\Omega_T \cap \partial\Omega) + \sum_{W \in m(T)} \mathcal{H}^d(g(\operatorname{Bot}(W)))\\ &\lesssim_d \mathcal{H}^d(\partial\Omega_T \cap \partial\Omega) + \sum_{W \in m(T)} h(W)^d.\end{aligned} \tag{3.27}$$

Now, $W \in m(T)$ implies that there exists a child $W' \in \operatorname{Stop}(T) \cap \mathscr{B}_w$ for which we have $Q \in \mathscr{B}_e$ with $g(W') \simeq_{A_0} Q$ by Lemma 3.16. For any $x \in Q$, we compute

$$\begin{aligned} |x-y| &\leq \operatorname{diam} Q + \operatorname{dist}(Q, g(W')) + \operatorname{diam} g(W') + \operatorname{dist}(y, g(W')) \\ &\leq 2A_0 \operatorname{diam} g(W') + 2A_0 \operatorname{diam} g(W') + \operatorname{diam} g(W') + 10r \\ &\leq 10\sqrt{d} A_0 h(W') + 10r \leq 100\sqrt{d} A_0 r \leq A_0^2 r \end{aligned} \tag{3.28}$$

This shows that $Q \subseteq B(y, A_0^2 r)$. Hence, applying Lemma 3.13 with $\mathscr{V} = \{g(W) : W \in m(T)\}$ and $\mathscr{U} = \{Q \in \mathscr{B}_e : Q \subseteq B(y, A_0^2 r)\}$, we get

$$\sum_{W \in m(T)} h(W)^d \lesssim_{A_0,d,L'} \sum_{\substack{Q \in \mathscr{B}_e \\ Q \subseteq B(y, A_0^2 r)}} \ell(Q)^d \lesssim_{d,\epsilon'} \mathcal{H}^d(\partial\Omega \cap B(y, A_0^2 r)) \tag{3.29}$$

where the last inequality follows from the Carleson packing condition for $\mathscr{B}_e$. By observing that $\partial\Omega_T \cap \partial\Omega \subseteq B(y, 50\sqrt{d}r)$ for any $T \in \mathscr{T}_{y,r}$ and $\mathcal{H}^d(\partial\Omega_T \cap \partial\Omega_{T'} \cap \partial\Omega) = 0$ for $T \neq T'$, (3.27) implies

$$\sum_{T \in \mathscr{T}_{y,r}} \mathcal{H}^d(\partial\Omega_T) \lesssim_{A_0,L',d,\epsilon'} \mathcal{H}^d(\partial\Omega \cap B(y, A_0^2 r)) \lesssim_{d,L'} r^d$$

using the fact that g is bi-Lipschitz and parameterizes $\partial\Omega$ in the last inequality. ■

Finally, we handle the boundaries of "trivial" cube domains associated to the bad boxes in $\mathscr{B}_w$. To do so, we collect the boundaries associated to fixed $W \in \mathscr{B}_w$ into the set

$$\mathcal{B}_W = \bigcup_{R \in \mathscr{L}_W} \partial R.$$

Lemma 3.19.

$$\sum_{W \in \mathscr{B}_w} \mathcal{H}^d(\mathcal{B}_W \cap B(y,r)) \lesssim_{L',d,\epsilon} \mathcal{H}^d(\partial\Omega \cap B(y, A_0^2 r)) \lesssim_{A_0,d,L'} r^d.$$

Proof. We first note that

$$\sum_{W \in \mathscr{B}_w} \mathcal{H}^d(\mathcal{B}_W \cap B(y,r)) \leq \sum_{\substack{W \in \mathscr{B}_w \\ g(W) \cap B(y,r) \neq \varnothing}} \mathcal{H}^d(\mathcal{B}_W) \lesssim_{L'} \mathcal{H}^d(\partial g(W)) \lesssim h(W)^d$$

using $\mathcal{H}^d(g(\operatorname{Bot}(W))) \lesssim_d h(W)^d$ as in (3.27) above. In addition, there exists some cube $Q \in \mathscr{B}_e$ such that $g(W) \simeq_{A_0} Q$ and, as in (3.28), $Q \subseteq B(y, A_0^2 r)$. Hence, we have

$$\begin{aligned} \sum_{W \in \mathscr{B}_w} \mathcal{H}^d(\mathcal{B}_W \cap B(y,r)) &\lesssim_d \sum_{\substack{W \in \mathscr{B}_w \\ g(W) \cap B(y,r) \neq \varnothing}} h(W)^d \lesssim_{A_0,d,L'} \sum_{\substack{Q \in \mathscr{B}_e \\ Q \subseteq B(y, A_0^2 r)}} \ell(Q)^d \\ &\lesssim_{d,\epsilon'} \mathcal{H}^d(\partial\Omega \cap B(y, A_0^2 r)) \lesssim_{d,L'} r^d. \end{aligned}$$ ■

Proof of Proposition 3.10. First, consider $\Omega_j \in \mathscr{L}$ such that either there exists j_0, T_0 such that $\Omega_j = \Omega_{T_0}^{j_0}$ or there exists $W \in \mathscr{B}_w$ and $R \in \mathscr{L}_W$ such that $\Omega_j = g(R)$. Therefore, we have

$$\begin{aligned} &\sum_{j \in J_{\mathscr{L}}} \mathcal{H}^d(\partial\Omega_j \cap B(y,r)) \\ &\quad\leq \sum_{T \in \mathscr{T}_{L,r}} \sum_{j \in J_T} \mathcal{H}^d(\partial\Omega_T^j \cap B(y,r)) + \sum_{T \in \mathscr{T}_{y,r}} \sum_{j \in J_T} \mathcal{H}^d(\partial\Omega_T^j \cap B(y,r)) + \sum_{W \in \mathscr{B}_w} \sum_{R \in \mathscr{L}_W} \mathcal{H}^d(\partial R \cap B(y,r)) \\ &\quad\lesssim \sum_{T \in \mathscr{T}_{L,r}} \mathcal{H}^d(\mathcal{B}_T \cap B(y,r)) + \sum_{T \in \mathscr{T}_{y,r}} \mathcal{H}^d(\mathcal{B}_T \cap B(y,r)) + \sum_{W \in \mathscr{B}_w} \mathcal{H}^d(\mathcal{B}_W \cap B(y,r)) \\ &\quad\lesssim_{L',d,\epsilon'} r^d \end{aligned}$$

by Lemmas 3.17, 3.18, and 3.19. ■

This completes the proof of Theorem A.

4 The proofs of Theorems B and C

We now turn to proving Theorems B and C. Both of these theorems will follow from the following result

Theorem 4.1. *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a domain. There exists constants $A(d), L(d), \epsilon_0(d) > 0$ such that if $\partial\Omega$ admits a d-dimensional graph coronization with $\epsilon \le \epsilon_0$, then there exists a collection $\mathscr{L} = \{\Omega_j\}_{j \in J_{\mathscr{L}}}$ of L-Lipschitz graph domains such that*

(i) $\Omega_j \subseteq \Omega$,

(ii) $\Omega \cap B(0,1) \subseteq \bigcup_{j=1}^{\infty} \Omega_j$,

(iii) $\exists C(d) > 0$ *such that* $\forall x \in \mathbb{R}^{d+1}$, $x \in \Omega_j$ *for at most* C *values of* j,

(iv) *For any* $y \in \partial\Omega \cap B(0,1)$ *and* $0 < r \le 1$, *we have*

$$\sum_{j=1}^{\infty} \mathcal{H}^d(\partial\Omega_j \cap B(y,r)) \lesssim_{\epsilon,d} \mathcal{H}^d(\partial\Omega \cap B(y, Ar)).$$

The proof will be via relatively minor modifications of the argument for Theorem A. The idea is to construct a collection of CCBPs with associated maps $\{g_i\}_{i \in I}$ where $g_i : \mathcal{D}_i \to \overline{\Omega}$ which individually parameterize only a little piece of $\overline{\Omega}$ at a time. These maps will be $(1+C\delta)$-bi-Lipschitz at the cost of introducing an outer "buffer zone" of domains in the image of these mappings having bounded overlap.

We now fix constants ρ, A_0, K as in Section 3 and set $A_1 = \max\left\{20A_0^2,\ \frac{2000\sqrt{d}A_0}{c_0\rho}\right\}$, $M = \max\{\frac{10K}{\rho^2},\ A_1^2\}$.

4.1 Local CCBPs adapted to $\mathscr{D}$

We will construct Reifenberg parameterizations as in subsection 3.1 centered around the points of a Whitney-like net $\mathcal{C}_0$ of $\Omega \cap B(0,1)$ rather than having a single global map.

For every $n \ge 0$, define

$$\begin{aligned} s_n &= 3 \cdot 2^{-n+1}, \\ D_n &= \{z \in B(0,1) : \operatorname{dist}(z, \partial\Omega) = s_n\}, \\ C_n &= \operatorname{Net}(D_n, s_n) = \{p_{i,n}\}_{i \in I_n}. \end{aligned}$$

Set $\mathcal{C}_0 = \bigcup_n C_n$.

Definition 4.2 (flat and non-flat points)**.** Let $p \in \Omega \cap B(0,1)$. Define

$$\mathscr{Q}_p = \left\{Q \in \mathscr{D} : Q \simeq_{10\sqrt{d}A_1} B\left(p, \frac{1}{2}\operatorname{dist}(p, \partial\Omega)\right)\right\}$$

We say that p is *flat* if there exists $S \in \mathscr{F}$ such that $\mathscr{Q}_p \subseteq S$. Otherwise, we say that p is *non-flat*. Given the set $\mathcal{C}_0$ above, we define the flat and non-flat points of $\mathcal{C}_0$ by

$$\begin{aligned} \mathcal{F}_0 &= \{p \in \mathcal{C}_0 : \exists S \in \mathscr{F},\ \mathscr{Q}_p \subseteq S\}, \\ \mathcal{N}_0 &= \mathcal{C}_0 \setminus \mathcal{F}_0. \end{aligned}$$

Fix $p \in \mathcal{F}_0$ and let $S_p \in \mathscr{F}$ be such that $\mathscr{Q}_p \subseteq S_p$. Without loss of generality, assume that $\operatorname{dist}(p, 0) = \operatorname{dist}(p, \partial\Omega) = 6 = s_0$. The fact that $p \in \mathcal{F}_0$ implies there exists $Q_p \in \mathscr{D}_{s(0)}$ with $\operatorname{dist}(p, Q_p) \leq 6$ and $c_0\rho \leq \operatorname{diam}(Q_p) \leq 1 = r_0$ so that $Q_p \in \mathscr{Q}_p$ because $A_1 \geq 10(c_0\rho)^{-1}$. Hence, $b\beta_{\partial\Omega}(MB_{Q_p}) \leq \epsilon$. Without loss of generality, suppose $P_{Q_p} = \mathbb{R}^d$ achieves the infimum in the definition of $b\beta_{\partial\Omega}(MB_{Q_p})$.

For any $k \geq 0$, let

$$Y_k^p = \{x_Q : Q \in S_p \cap \mathscr{D}_{s(k)}\}, \tag{4.1}$$
$$X_k^p \in \operatorname{Net}(Y_k^p, r_k). \tag{4.2}$$

We enumerate $X_k^p = \{x_{j,k}\}_{j \in J_k}$ and define

$$\begin{aligned} B_{j,k} &= B(x_{j,k}, r_k), \\ P_{j,k} &= P_{Q_{j,k}}, \\ \mathscr{L}_p &= (P_{Q_p}, \{B_{j,k}\}, \{P_{j,k}\}). \end{aligned}$$

where $P_{Q_{j,k}} \ni x_{Q_{j,k}}$ satisfy $\beta_{\partial\Omega}^{d,1}(2\rho^{-1}KB_{Q_{j,k}}, P_{Q_{j,k}}) \lesssim \beta_{\partial\Omega}^{d,1}(2\rho^{-1}KB_{Q_{j,k}}B_{Q_{j,k}})$ as in the hypotheses of Lemma 2.20. Using the fact that $Q \in S_p \subseteq \mathscr{G}$ so that $b\beta(MB_Q) \leq \epsilon$, a nearly identical argument to that of Lemma 3.2 gives that $\mathscr{L}_p$ is a CCBP:

Lemma 4.3. *For any $p \in \mathcal{F}_0$, $\mathscr{L}_p$ is a CCBP.*

We will now prove the following analogue of Lemma 3.3

Lemma 4.4 (properties of g_p)**.** *There exists a choice of constant $A_1 \lesssim_d A_0$ such that for any $z = (x, y) \in \widehat{\mathcal{D}}_p$, the following hold:*

(i) $f_{n(y)}(x) \in V_{n(y)}^8$,

(ii) $(1 - C\epsilon)|y| \leq \operatorname{dist}(g_p(z), \partial\Omega) \leq (1 + C\epsilon)|y|$.

(iii) For any $m \in \mathbb{N}$ with $m < n(y)$, there exists a collection of cubes $Q_{n(y)} \subseteq Q_{n(y)-1} \subseteq \cdots \subseteq Q_m$ such that for any k with $m \leq k \leq n$, $Q_k \in S_p$ and $\operatorname{dist}(g(x, r_k), Q_k) \lesssim r_k$ and

$$\sum_{k=m}^{n} \epsilon'(f_k(x))^2 \lesssim_{M,\rho,d} \sum_{k=m}^{n} \beta_{\partial\Omega}^{d,1}(MB_{Q_k})^2 \lesssim \epsilon.$$

Proof. The proof is similar to that of 3.3 with the only complication being that we need the map g_p to also behave nicely on the buffer region of A_0 close cubes to those in $\mathscr{W}_0$. We will prove this for fixed $z = (x, y)$ by first assuming that (i) holds and showing that items (ii) and (iii) hold. We will then prove item (i) by induction, considering the points $(x, r_k) \in \widehat{\mathcal{D}}_p$ for $0 \leq k < n(y)$ (assume without loss of generality that $h(W(T)) = 4$).

So, first assume that item (i) holds. Given this, item (ii) follows exactly as in Lemma 3.3 item (ii). Similarly, item (iii) follows as in Lemma 3.3 item (iii) by replacing the infinite chain of cubes with a chain terminating in $Q_{n(y)} \in \mathscr{D}_{s(n(y))} \cap S_p$.

We now prove item (i) by the induction discussed above. For the base case, recall that $f_0(x) = x$ so that $(x, y) \in \widehat{\mathcal{D}}_p$ means $\operatorname{dist}(x, W(T_p)) \leq 2A_0 \operatorname{diam} W(T)$. Since we've chosen M large enough, $x \in MB_{Q_p} \cap P_{Q_p}$ so that $\operatorname{dist}(x, \partial\Omega) \lesssim_M \epsilon$. This means $p \in \mathcal{F}$ implies that there exists $Q_0 \in \mathscr{D}_{s(0)} \cap S_p$ such that $|x - x_{Q_0}| \leq 2r_0$ from which the claim follows. We will finish the proof by proving the following claim:

Claim: For any $k < n(y)$, $f_k(x) \in V_k^8$ implies that $f_{k+1}(x) \in V_{k+1}^8$.

Proof: The fact that $(x, r_k) \in \widehat{\mathcal{D}}_p$ means that $(x, r_k) \in R_k \in \mathscr{W}_p'$ and there exists $W \in T_p$ such that $R_k \simeq_{A_0} W$. This gives $\operatorname{dist}(R_k, W) \lesssim_{A_0} h(W)$ and $r_k \simeq h(R_k) \simeq_d \operatorname{diam} R_k \simeq_{A_0} h(W)$. If now

$f_k(x) \in V_k^8$, then there exists $Q \in S_p \cap \mathscr{D}_{s(k)}$ such that $|f_k(x) - x_Q| \leq 8r_k$, so that $b\beta_{\partial\Omega}(MB_Q) \leq \epsilon$ implies there is $Q_{k+1} \in \mathscr{D}_{s(k+1)}$ with $|f_{k+1}(x) - x_{Q_{k+1}}| \leq 2r_{k+1}$. Applying item (ii) and (3.11) gives

$$\begin{aligned}
\operatorname{dist}(Q_{k+1}, g(W)) &\leq \operatorname{dist}(Q_{k+1}, g(R_k)) + \operatorname{diam} g(R_k) + \operatorname{dist}(g(R_k), g(W)) \\
&\leq 2\sqrt{d}h(R_k) + 2\sqrt{d}h(R_k) + A_0(\operatorname{diam} g(R_k) + \operatorname{diam} g(W)) \\
&\leq 5\sqrt{d}A_0 h(R_k) + 5\sqrt{d}A_0(h(R_k) + \operatorname{diam} g(W) \leq A_0^2 \operatorname{diam} g(W)
\end{aligned}$$

and

$$\operatorname{diam} Q_{k+1} \leq r_{k+1} \leq h(R) \leq A_0 h(W) \leq 2A_0 \operatorname{diam} g(W).$$

$$\operatorname{diam} Q_{k+1} \geq c_0 \ell(Q) \geq \frac{c_0\rho}{10} r_{k+1} \geq \frac{c_0\rho}{200} h(R_k) \geq \frac{c_0\rho}{200A_0} h(W) \geq \frac{c_0\rho}{1000\sqrt{d}A_0} \operatorname{diam} g(W).$$

Therefore, $Q_{k+1} \simeq_{A_1} g(W)$. By the definition of T_p, we then have $Q_{k+1} \in S_p$ so that $x_{Q_{k+1}} \in Y_{k+1}^p$ and $f_{k+1}(x) \in V_{k+1}^8$ as in Lemma 3.3 (i). ■

4.2 The Lipschitz decomposition with bounded overlaps

Hence, by Theorem 2.4, we get a Reifenberg parameterization $g_p : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ and we let $\mathscr{W}_p'$ be the Whitney decomposition of $\mathbb{H}^{d+1}$ such that $W_0 = [-2,2]^d \times [4,8] \in \mathscr{W}_p'$ so that $p = c(W_0) = (0,6)$ and let $\mathscr{W}_0 = \{W \in \mathscr{W}_p' : W \in D(W_0)$ as in (3.5). We now give a one-step version of the stopping time construction in Definition 2.11 to produce a single domain $\mathcal{D}_p$ and an extended version $\widehat{\mathcal{D}}_p$ which contains additional "buffer" cubes which g_p maps forward to approximating Lipschitz graph domains

Definition 4.5 (Stopping time regions around flat p)**.** Fix a constant $A_1 > 1$ and $p \in \mathcal{F}$ and form the map g_p and Whitney lattices $\mathscr{W}_p'$ and $\mathscr{W}_0$ as above. As in (3.14), we define

$$\mathscr{G}_p = \{W \in \mathscr{W}_0 : \forall Q \in \mathscr{D} \text{ such that } Q \simeq_{A_1} g_p(W) \text{ we have } Q \in S_p\}$$

By definition, $p \in \mathcal{F}_0$ implies $W_0 \in \mathscr{G}_p$. Define a single stopping time region $T_p \subseteq \mathscr{W}_0$ by setting T_p to be the maximal subtree of $D(W_0) \cap \mathscr{G}_p$ such that for any $R \in T_p$, either all of its children are in T_p or none are.

Definition 4.6 (Stopping time domains around flat p)**.** For any $p \in \mathcal{F}_0$, we define a *stopping time domain*

$$\mathcal{D}_p = \bigcup_{W \in T_p} W.$$

Additionally, we extend $\mathcal{D}_p$ by a "buffer" region of A_0-close cubes on the boundary of $\mathcal{D}_p$ by defining the *extended stopping time region* and *extended stopping time domain* by

$$\begin{aligned}
\widehat{T}_p &= \{W \in \mathscr{W}_p' : \exists R \in T_p,\ W \simeq_{A_0} R\}, \\
\widehat{\mathcal{D}}_p &= \bigcup_{W \in \widehat{T}_p} W.
\end{aligned}$$

We will carve up the image domains

$$\begin{aligned}
\Omega_p &= g_p(\mathcal{D}_p), \\
\widehat{\Omega}_p &= g_p(\widehat{\mathcal{D}}_p)
\end{aligned}$$

to construct one family of our desired Lipschitz graph domains in the conclusion of Theorem 4.1.

We will also need to construct Lipschitz graph domains around non-flat $q \in \mathcal{N}_0$. Because $\partial\Omega$ admits a graph coronization, there are a controlled number of such q so that we can cover the regions around them by "trivial" domains without adding too much total boundary.

Definition 4.7 (Trivial domains around non-flat q)**.** Fix once and for all an auxiliary Whitney decomposition $\widetilde{\mathscr{W}}$ of Ω. For any $q \in \mathcal{N}_0$, there exists a Whitney cube $W_q \in \widetilde{\mathscr{W}}$ such that $q \in W_q$ and

$$\operatorname{diam} W_q \leq \operatorname{dist}(q, \partial\Omega) \leq 8 \operatorname{diam} W_q.$$

We directly define

$$\begin{aligned}
\mathcal{D}_q &= \Omega_q = W_q, \\
\widehat{\mathcal{D}}_q &= \{W \in \widetilde{\mathscr{W}} : W \simeq_{A_0} W_q\}, \\
\widehat{\Omega}_q &= \bigcup_{W \in \widehat{\mathcal{D}}_q} W.
\end{aligned}$$

We will get our final collection of domains by choosing a well-spaced subsets $\mathcal{F} \subseteq \mathcal{F}_0$ and $\mathcal{N} \subseteq \mathcal{N}_0$ and carving up the domains in $\{\widehat{\Omega}_p\}_{p \in \mathcal{F}} \cup \{\widehat{\Omega}_q\}_{q \in \mathcal{N}}$.

To choose our collections $\mathcal{F}$ and $\mathcal{N}$, we put an ordering on the points of $\mathcal{C}_0$ by choosing some ordering on each finite set C_n and then imposing $p_n < p_m$ for any $p_n \in C_n,\ p_m \in C_m$ with $n < m$. $\mathcal{C}_0$ has a least element which we call c_0 and we define an auxiliary collection $\mathcal{P}_0 = \{p_0\}$. Given the definitions of $\mathcal{C}_0$ and $\mathcal{P}_0$, we define $\mathcal{C}_{n+1}$ and $\mathcal{P}_{n+1}$ inductively for any $n \geq 0$ by

$$\mathcal{C}_{n+1} = \mathcal{C}_n \setminus \left\{ p \in \mathcal{C}_n \ : \ \operatorname{dist}\left(p, \bigcup_{p' \in \mathcal{P}_n} \Omega_{p'}\right) < \frac{A_0}{30} \operatorname{dist}(p, \partial\Omega). \right\}, \tag{4.3}$$

$$\mathcal{P}_{n+1} = \mathcal{P}_n \cup \{p_{n+1}\}, \tag{4.4}$$

where p_{n+1} is the least element of $\mathcal{C}_{n+1}$ with respect to the ordering inherited from $\mathcal{C}_0$. Finally, put

$$\begin{aligned}
\mathcal{C} &= \bigcup_{n=0}^{\infty} \mathcal{P}_n, \\
\mathcal{F} &= \mathcal{F}_0 \cap \mathcal{C}, \\
\mathcal{N} &= \mathcal{N}_0 \cap \mathcal{C}.
\end{aligned}$$

We can now give the definition of our desired Lipschitz decomposition with bounded overlaps

Definition 4.8 (Lipschitz decomposition with bounded overlap)**.** For any $p \in \mathcal{F}$, Proposition 5.1 implies there exists an Ahlfors regular d-rectifiable set Σ_{T_p} such that

$$\mathcal{D}_p \setminus \Sigma_{T_p} = \bigcup_{j \in J_p} \mathcal{D}_p^j$$

where $\mathcal{D}_p^j$ is an $L_0(d)$-Lipschitz graph domain. We set $\Omega_p^j = g_p(\mathcal{D}_p^j)$ and define our *Lipschitz decomposition with bounded overlap*

$$\mathscr{L} = \{g_p(W)\}_{p \in \mathcal{F},\ W \in \widehat{\mathcal{D}}_p \setminus \mathcal{D}_p} \cup \{\Omega_p^j\}_{p \in \mathcal{F},\ j \in J_p} \cup \{R\}_{q \in \mathcal{N},\ R \in \widehat{\mathcal{D}}_q}. \tag{4.5}$$

In analogy to Propositions 3.9 and 3.10, we will finish the proof of Theorem 4.1 if we can prove the following propositions:

Proposition 4.9. *Let Ω be as in Theorem 4.1 and $\mathscr{L} = \{\Omega_j\}_{j \in J_{\mathscr{L}}}$ be as in* (4.5)*. There exists $L_1(d) > 0$ such that for any $j \in J_{\mathscr{L}}$, Ω_j is an L_1-Lipschitz graph domain. In addition, we have*

- *(i)* $\Omega_j \subseteq \Omega$,
- *(ii)* $\Omega \subseteq \bigcup_{j \in J_{\mathscr{L}}} \overline{\Omega_j}$,
- *(iii)* $\exists C(d) > 0$ *such that* $\forall x \in \Omega$, $x \in \Omega_j$ *for at most* C *values of* j.

Proposition 4.10. *Let Ω be as in Theorem 4.1 and $\mathscr{L} = \{\Omega_j\}_{j \in J_{\mathscr{L}}}$ be as in* (4.5)*. For any $y \in \partial\Omega \cap B(0,1)$ and $0 < r < 1$, we have*

$$\sum_{j \in j_{\mathscr{L}}} \mathcal{H}^d(\partial\Omega_j \cap B(y,r)) \lesssim_{\epsilon,d} \mathcal{H}^d(\partial\Omega \cap B(y, A_1 r)). \tag{4.6}$$

4.3 Lipschitz bounds and covering / overlap properties for Theorems B and C

In order to prove Propositions 4.9 and 4.10, we must show that the mapping g_p behaves on $\widehat{\mathcal{D}_p}$ as our single Reifenberg parameterization g did on each $\mathcal{D}_T$ in the setting of Theorem A.

The following analogue of Lemma 3.11 allows us to control the change in Dg_p on any extended stopping time domain $\widehat{\mathcal{D}}_p$.

Lemma 4.11 (Variation of Dg_p)**.** *For any $p \in \mathcal{F}$ and $z \in \widehat{\mathcal{D}}_p$, we have*

$$|Dg_p(z) - I| \leq C\delta \tag{4.7}$$

In particular, $g_p|_{\widehat{\mathcal{D}}_p}$ is $(1 + C\delta)$-bi-Lipschitz.

This result follows directly from the proof of Lemma 3.11. Equation (4.7) follows from the added observation that $p \in \mathcal{D}_p$ and $\operatorname{dist}(p, \partial\Omega) \geq 2$ (after normalizing) implies $Dg_p(p) = I$ so that the claim follows from (3.18) by taking $w = p$.

We now have enough to show that each domain in $\mathscr{L}$ as in (4.5) is Lipschitz graphical

Lemma 4.12. *There exists a constant $L_1(d) > 0$ such that Ω_j is an L_1-Lipschitz graph domain for all $j \in J_{\mathscr{L}}$.*

Proof. Each domain in the set $\{R\}_{q\in\mathcal{N},\ R\in\widehat{\mathcal{D}}_q}$ is a cube, which is an $L_0(d)$-Lipschitz graph domain trivially. Each domain Ω_j in the set $\{g_p(W)\}_{p\in\mathcal{F},\ W\in\widehat{\mathcal{D}}_p\setminus\mathcal{D}_p} \cup \{\Omega_p^j\}_{p\in\mathcal{F},\ j\in J_p}$ is the image under g_p of an L_0-Lipschitz graph domain. Therefore, by Lemma 4.11 and 5.6 there exists $L_1(d) > L_0$ such that that each such Ω_j is an L_1-Lipschitz graph domain. ■

In order to prove the remaining statements of Proposition 4.9, we first show that the buffer region $\widehat{\Omega}_p \setminus \Omega_p$ contains a cone around Ω_T with respect to the distance to $\partial\Omega$ for any $p \in \mathcal{C}$:

Lemma 4.13. *For any $p \in \mathcal{C}$, $\widehat{\Omega}_p$ contains a $\frac{A_0}{10}$-cone around Ω_p with respect to distance from $\partial\Omega$. That is,*

$$F = \left\{ w \in \Omega : \operatorname{dist}(w, \Omega_p) < \frac{A_0}{10} \min\left\{\operatorname{dist}(w, \partial\Omega), \operatorname{dist}(g_p(W(T_p)), \partial\Omega)\right\} \right\} \subseteq \widehat{\Omega}_p \tag{4.8}$$

Proof. First, suppose that $p \in \mathcal{F}$ and let $z \in F$. Since $\widehat{\Omega}_p = g_p(\widehat{\mathcal{D}}_p)$ where g_p is $(1 + C\delta)$-bi-Lipschitz by Lemma 4.11 and translates distance in the domain to $\mathbb{R}^d$ to distance to $\partial\Omega$ in the image by Lemma 4.4 (ii), it suffices to show

$$\left\{ z \in \Omega : \operatorname{dist}(z, \mathcal{D}_p) < \frac{A_0}{4} \min\left\{\operatorname{dist}(z, \mathbb{R}^d), \operatorname{dist}(W(T_p), \mathbb{R}^d)\right\} \right\} \subseteq \widehat{\mathcal{D}}_p \tag{4.9}$$

because the desired containment then follows by mapping (4.9) forward. Now, there exists $W \in T_p$ such that $\operatorname{dist}(z, W) = \operatorname{dist}(z, \mathcal{D}_p)$ and there exists a cube $W_z \in \mathscr{W}_p'$ such that $z \in W_z$. By the definition of $\widehat{\mathcal{D}}_p$, it suffices to show that $W \simeq_{A_0} W_z$. We estimate

$$\begin{aligned}\operatorname{dist}(W, W_z) &\leq \operatorname{dist}(z, \mathcal{D}_p) < \frac{A_0}{4} \min\{\operatorname{dist}(z, \mathbb{R}^d), \operatorname{dist}(W(T_p), \mathbb{R}^d)\} \\ &\leq \frac{A_0}{2} \min\{h(W_z), h(W(T_p))\} = \frac{A_0}{2} \min\{\ell(W_z), \ell(W(T_p))\}.\end{aligned} \tag{4.10}$$

Using this we get

$$\ell(W) = h(W) \leq \operatorname{dist}(W, W_z) + \operatorname{diam} W_z + h(W_z) \leq \left(\frac{A_0}{2} + \sqrt{d+1} + 1\right) \ell(W_z) \leq A_0 \ell(W_z)$$

given that $A_0 \geq 4\sqrt{d}$. A similar calculation shows that $\ell(W_z) \leq A_0 \ell(W)$ which completes the proof in the case when $p \in \mathcal{F}$. If $q \in \mathcal{N}$, then $\Omega_q = W_q$. Let $w \in F$ and let $W_w \in \widetilde{\mathscr{W}}$ with $w \in W_w$. By a similar computation to the above, one can show that $W_q \simeq_{A_0} W_w$ from which the result follows. ■

With the help of Lemma 4.13, we can prove the bounded overlap and covering properties of $\mathscr{L}$.

Lemma 4.14. *Let $p, p' \in \mathcal{C},\ p \neq p'$. The following hold:*

(i) $\Omega_p \cap \Omega_{p'} = \varnothing$,

(ii) $\Omega \cap B(0,1) \subseteq \bigcup_{p \in \mathcal{C}} \overline{\widehat{\Omega}}_p$,

(iii) $\exists C(d) > 0$ *such that* $\forall x \in \Omega,\ x \in \widehat{\Omega}_p$ *for at most* C *values of* j,

(iv) $\widehat{\Omega}_p \subseteq \Omega$.

Proof. We begin with proving (i). Using the partial order on $\mathcal{C}$, assume without loss of generality that $p' < p$. By the definition of $\mathscr{C}$, we have $\operatorname{dist}(p, \Omega_{p'}) \geq \frac{A_0}{30} \operatorname{dist}(p, \partial\Omega)$. We claim that

$$\Omega_p \subseteq B(p, 3\sqrt{d} \operatorname{dist}(p, \partial\Omega)) \subseteq B\left(p, \frac{A_0}{30} \operatorname{dist}(p, \partial\Omega)\right)$$

where the final inclusion follows because $A_0 \geq 120\sqrt{d}$. Indeed, if $p \in \mathcal{N}$, then $\Omega_p = W_p \ni p$ with $\operatorname{diam} W_p \leq \operatorname{dist}(p, \partial\Omega)$. If instead $p \in \mathcal{F}$, then $\Omega_p = g_p(\mathcal{D}_p)$ where $\mathcal{D}_p$ is composed of a union of cubes in the descendants $D(W(T_p))$ where $\operatorname{dist}(p, \partial\Omega) \geq \ell(W(T_p))$ so that the fact that g_p is $(1 + C\delta)$-bi-Lipschitz means $\sqrt{d+1} \operatorname{dist}(p, \Omega) \geq \operatorname{diam}(g_p(W))$ and $\operatorname{dist}(p, \Omega) \geq \operatorname{dist}(g_p(W), p)$ for any $W \in T_p$. The claim follows.

We now prove (ii). Let $z \in \Omega \cap B(0,1)$ and let $k \geq 0$ be such that $s_{k+1} \leq \operatorname{dist}(z, \partial\Omega) \leq s_k$. By the definition of C_k, there exists $p_k \in C_k$ such that

$$|z - p_k| \leq 3s_k = 6s_{k+1} \leq 6 \operatorname{dist}(z, \partial\Omega).$$

Now, if $p_k \in \mathcal{C}$, then by Lemma 4.13, $z \in \widehat{\Omega}_{p_k}$. Otherwise, $p_k \notin \mathcal{C}$ so that by (4.3) there exists $p \in \mathcal{C}$ such that $p < p_k$ and $\operatorname{dist}(p_k, \Omega_p) < \frac{A_0}{30} \operatorname{dist}(p_k, \partial\Omega) = \frac{A_0}{30} s_k = \frac{A_0}{15} s_{k+1}$. But then

$$\operatorname{dist}(z, \Omega_p) \leq |z - p_k| + \operatorname{dist}(p_k, \Omega_p) \leq 6s_{k+1} + \frac{A_0}{15} s_{k+1} \leq \frac{A_0}{10} \operatorname{dist}(z, \partial\Omega)$$

so that $z \in \widehat{\Omega}_p$ by Lemma 4.13 as long as $\operatorname{dist}(z, \Omega_p) \leq \frac{A_0}{10} \operatorname{dist}(g(W_p), \partial\Omega)$ which follows from the fact that $p < p_k$ (p is not a net point of smaller scale).

We now prove (iii). Let $z \in \Omega \cap B(0,1)$ and define $\mathcal{C}_z = \{p \in \mathcal{C} : z \in \widehat{\Omega}_p\}$. It suffices to prove

$$\#(\mathcal{C}_z) \lesssim_d 1$$

First, suppose $p \in \mathcal{C}_z \cap \mathcal{F}$. Then there exists $W \in \widehat{T}_p$ such that $z \in g_p(W)$ and the definition of $\widehat{\Omega}_p$ then implies that there exists $R_p \in T_p$ such that $R_p \simeq_{A_0} W$. Let $r_z = \operatorname{dist}(z, \partial\Omega)$. Lemmas 4.4 and 4.11 imply that $\operatorname{diam} g_p(R_p) \simeq_d \operatorname{dist}(g_p(R_p), \partial\Omega) \simeq_{A_0} r_z$ and there exists $C_0(d, A_0) > 0$ such that

$$B(g_p(c_{R_p}), C_0^{-1} r_z) \subseteq g_p(R_p) \subseteq B(z, C_0 r_z) \tag{4.11}$$

Since $\Omega_p \cap \Omega_{p'} = \varnothing$ for $p \neq p'$, we have

$$g_p(R_p) \cap g_{p'}(R_{p'}) = \varnothing. \tag{4.12}$$

it follows from (4.11) and (4.12) that $\#(\mathcal{C}_z \cap \mathcal{F}) \lesssim_{A_0, d} 1$. A similar argument shows that $\#(\mathcal{C}_z \cap \mathcal{N}) \lesssim_{d, A_0} 1$ from which the claim follows.

Item (iv) follows from Lemma 4.4 (ii). ∎

Remark 4.15 (Whitney family)**.** In fact, (4.11) and (4.12) in combination with Lemma 4.4 show that there exists a constant $\Lambda_1(d)$ such that the family

$$\mathcal{G}_1 = \bigcup_{\substack{p\in\mathcal{F}\\ W\in T_p}} g_p(W) \cup \bigcup_{q\in\mathcal{N}} W_q. \tag{4.13}$$

is a Λ_1-Whitney family in the sense of Definition 3.12 (compare with Lemma 3.14).

We can now finish the proof of Proposition 4.9.

Proof of Proposition 4.9. We showed the existence of L_1 such that Ω_j is L_1-Lipschitz graphical forany $j \in J_{\mathscr{L}}$ in Lemma 4.12. The fact that $\Omega_j \subseteq \Omega$ follows from Lemma 4.14 (iv) while $\Omega \subseteq \bigcup_{j\in J_{\mathscr{L}}} \Omega_j$ follows from Lemma 4.14 (ii). Finally, item (iii) of Proposition 4.9 follows from Lemma 4.14 (iii) because for each $p \in \mathcal{C}$, there is by definition at most one index j_p such that $x \in \Omega_{j_p} \subseteq \widehat{\Omega}_p$. ■

4.4 Surface area bounds for Theorems B and C

In this section, we prove Proposition 4.10. The proof is similar to that of Proposition 3.10 given Remark 4.15. Fix $y \in \partial\Omega \cap B(0,1)$ and $0 < r \leq 1$ and let $A_2 = 100\sqrt{d}A_0^2$, $A_3 = 50\sqrt{d}A_1A_2$. If $p \in \mathcal{F}$ is such that $\widehat{\Omega}_p \cap B(y,r) \neq \varnothing$, then there exists a cube R with $\ell(R) \leq 2r$ such that $g_p(R) \cap B(y,r) \neq \varnothing$ and $g_p(R) \simeq_{A_0} W$ with $W \in T_p$. Then

$$\operatorname{dist}(g_p(W),y) \leq \operatorname{dist}(g_p(W),g_p(R)) + \operatorname{diam} g_p(R) \leq A_0(1+A_0)\operatorname{diam} g_p(R) \leq 3\sqrt{d}A_0^2\ell(R) < 10\sqrt{d}A_0^2 r.$$

Therefore, since $A_2 > 50\sqrt{d}A_0^2$, we get that $\Omega_p \cap B(y, A_2 r) \neq \varnothing$. We set

$$\mathscr{T}'_{y,A_2r} = \{T_p : p \in \mathcal{F},\ \Omega_p \cap B(y,A_2r) \neq \varnothing\}.$$

The above discussion gives that $\widehat{\Omega}_p \cap B(y,r) \neq \varnothing \implies \Omega_p \cap B(y,A_2r) \neq \varnothing$, so it suffices to consider stopping time domains in the family $\mathscr{T}'_{y,A_2r}$. Break up $\mathscr{T}'_{y,A_2r}$ into regions with large and small top cubes:

$$\begin{aligned}\mathscr{T}_{L,A_2r} &= \{T_p \in \mathscr{T}'_{y,A_2r} : h(W(T_p)) > 10A_2r\},\\ \mathscr{T}_{y,A_2r} &= \mathscr{T}'_{y,A_2r} \setminus \mathscr{T}_{L,A_2r}.\end{aligned}$$

We also collect all of the boundaries of domains in our decomposition $\mathscr{L}$ associated with a given flat point $p \in \mathcal{F}$ into the set

$$\mathcal{B}_p = \bigcup_{j\in J_{T_p}} \partial\Omega_p^j \cup \bigcup_{W\in\widehat{\mathcal{D}}_p\setminus\mathcal{D}_p} g(\partial W). \tag{4.14}$$

We note that $\mathcal{B}_p$ is d-Ahlfors regular with constant depending on d and A_0 by Proposition 5.1 and the fact that each cube $W \subseteq \widehat{\mathcal{D}}_p \setminus \mathcal{D}_p$ is A_0-close to a cube $W' \in T_p$ with at least one face inside $\partial\mathcal{D}_p$.

We can then use the arguments of the previous section to get the following analogues of Lemmas 3.17 and 3.18.

Lemma 4.16.
$$\sum_{\substack{p\in\mathcal{F}\\ T_p\in\mathscr{T}_{L,A_2r}}} \mathcal{H}^d(\mathcal{B}_p \cap B(y,r)) \lesssim_d r^d \leq \mathcal{H}^d(\partial\Omega \cap B(y,r)).$$

Proof. It follows from the proof of Lemma 3.17 and the fact that $\mathcal{G}_1$ is a Whitney family (see Remark 4.15) that $\#(\mathscr{T}_{L,A_2r}) \lesssim_{A_2,d} 1$. Since $\mathcal{B}_p$ is d-Ahlfors regular, we have

$$\sum_{\substack{p\in\mathcal{F}\\ T_p\in\mathscr{T}_{L,A_2r}}} \mathcal{H}^d(\mathcal{B}_p\cap B(y,r)) \lesssim_{A_0,d} \#(\mathscr{T}_{L,A_2r})r^d \lesssim_{A_2,d} r^d. \qquad \blacksquare$$

We now handle the regions with small top boxes:

Lemma 4.17.

$$\sum_{\substack{p\in\mathcal{F}\\ T_p\in\mathscr{T}_{y,A_2r}}} \mathcal{H}^d(\mathcal{B}_p\cap B(y,r)) \lesssim_{d,\epsilon} \mathcal{H}^d(\partial\Omega\cap B(y,A_3r)). \tag{4.15}$$

Proof. We modify the proof of Lemma 3.18. We first observe that since $\mathcal{H}^d(\mathcal{B}_p) \lesssim_{A_0,d} \mathcal{H}^d(\partial\Omega_p)$, we have

$$\sum_{\substack{p\in\mathcal{F}\\ T_p\in\mathscr{T}_{y,A_2r}}} \mathcal{H}^d(\mathcal{B}_p\cap B(y,r)) \le \sum_{\substack{p\in\mathcal{F}\\ T_p\in\mathscr{T}_{y,A_2r}}} \mathcal{H}^d(\mathcal{B}_p) \lesssim_{A_0,d} \sum_{\substack{p\in\mathcal{F}\\ T_p\in\mathscr{T}_{y,A_2r}}} \mathcal{H}^d(\partial\Omega_p).$$

Therefore, it suffices to prove (4.15) with $\mathcal{B}_p\cap B(y,r)$ replaced by $\partial\Omega_p$. For any $T_p\in\mathscr{T}_{L,A_2r}$, we get

$$\mathcal{H}^d(\partial\Omega_p) \lesssim_d \mathcal{H}^d(\partial\Omega_p\cap\partial\Omega) + \sum_{W\in m(T_p)} h(W)^d \tag{4.16}$$

Now, $W\in m(T_p)$ implies that there exists a child $W'\in \mathrm{Stop}(T_p)$ for which we have $Q\in\mathscr{B}_e$ of (3.25) with $g(W')\simeq_{A_1} Q$ by Lemma 3.16. By replacing A_0 with A_1 and r with A_2r in (3.28), we get $Q\subseteq B(y,50\sqrt{d}A_1A_2r)\subseteq B(y,A_3r)$. Hence, applying Lemma 3.13 with $\mathscr{V}=\{g(W): W\in m(T_p),\ T_p\in\mathscr{T}_{y,A_2r}\}$ and $\mathscr{U}=\{Q\in\mathscr{B}_e : Q\subseteq B(y,A_3r)\}$, we get

$$\sum_{\substack{p\in\mathcal{F}\\ T\in\mathscr{T}_{y,A_2r}}} \sum_{W\in m(T_p)} h(W)^d \lesssim_d \sum_{\substack{Q\in\mathscr{B}_e\\ Q\subseteq B(y,A_3r)}} \ell(Q)^d \lesssim_{d,\epsilon} \mathcal{H}^d(\partial\Omega\cap B(y,A_3r)) \tag{4.17}$$

where the last inequality follows from the Carleson packing condition for $\mathscr{B}_e$. By observing that $\partial\Omega_p\cap\partial\Omega\subseteq B(y,50\sqrt{d}A_2r)$ and $\mathcal{H}^d(\partial\Omega_p\cap\partial\Omega_{p'}\cap\partial\Omega)=0$ for any $p\neq p'$, (4.16) implies

$$\sum_{T_p\in\mathscr{T}_{y,A_2r}} \mathcal{H}^d(\partial\Omega_p) \lesssim_{d,\epsilon} \mathcal{H}^d(\partial\Omega\cap B(y,A_3r)). \qquad \blacksquare$$

We also need to bound the surface measure associated to trivial domains around non-flat $q\in\mathcal{N}$. For any $q\in\mathcal{N}$, we define the set of boundaries

$$\mathcal{B}_q = \partial W_q \cup \bigcup_{\substack{W\in\widetilde{\mathscr{W}}\\ W\subseteq\widetilde{\mathcal{D}}_q\setminus\mathcal{D}_q}} \partial W.$$

We note that $\mathcal{H}^d(\mathcal{B}_q)\lesssim_{d,A_0} \ell(W_q)^d$.

Lemma 4.18.

$$\sum_{q\in\mathcal{N}} \mathcal{H}^d(\mathcal{B}_q\cap B(y,r)) \lesssim_{d,\epsilon} \mathcal{H}^d(\partial\Omega\cap B(y,A_3r))$$

Proof. Observe that $\mathcal{B}_q\cap B(y,r)\neq\varnothing$ implies there exists $Q\in\mathscr{B}_e$ such that $W_q\simeq_{10A_1} Q$ and $Q\subseteq B(y,A_3r)$ so that we have

$$\sum_{q\in\mathcal{N}} \mathcal{H}^d(\mathcal{B}_q\cap B(y,r)) \le \sum_{\substack{q\in\mathcal{N}\\ \mathcal{B}_q\cap B(y,r)\neq\varnothing}} \ell(W_q)^d \lesssim_{A_1,d} \sum_{\substack{Q\in\mathscr{B}_e\\ Q\subseteq B(y,A_3r)}} \ell(Q)^d \lesssim_{d,\epsilon} \mathcal{H}^d(\partial\Omega\cap B(y,A_3r)). \qquad \blacksquare$$

Proof of Proposition 4.10. $\Omega_j \in \mathscr{L}$ implies that there either there exists j_0, T_0 such that $\Omega_j = \Omega_{T_0}^{j_0}$ or $q \in \mathcal{N}$ such that $\Omega_j = R \in \widetilde{W}$ where $R \simeq_{A_0} W_q$. This means that

$$\begin{aligned}
&\sum_{j \in j_{\mathscr{L}}} \mathcal{H}^d(\partial\Omega_j \cap B(y,r)) \\
&\quad \leq \sum_{T \in \mathscr{T}_{L,A_2 r}} \sum_{j \in J_T} \mathcal{H}^d(\partial\Omega_T^j \cap B(y,r)) + \sum_{T \in \mathscr{T}_{y,A_2 r}} \sum_{j \in J_T} \mathcal{H}^d(\partial\Omega_T^j \cap B(y,r)) + \sum_{q \in \mathcal{N}} \mathcal{H}^d(\mathcal{B}_q \cap B(y,r)) \\
&\quad \lesssim \sum_{T \in \mathscr{T}_{L,A_2 r}} \mathcal{H}^d(\mathcal{B}_T \cap B(y,r)) + \sum_{T \in \mathscr{T}_{y,A_2 r}} \mathcal{H}^d(\mathcal{B}_T \cap B(y,r)) + \sum_{q \in \mathcal{N}} \mathcal{H}^d(\mathcal{B}_q \cap B(y,r)) \\
&\quad \lesssim_{L',d,\epsilon} \mathcal{H}^d(\partial\Omega \cap B(y, A_3 r))
\end{aligned}$$

by Lemmas 4.16, 4.17, and 4.18. ∎

5 Lipschitz graph domains

Because each stopping time domain is not necessarily a Lipcshitz graph domain, we will construct a d-Ahlfors regular, d-rectifiable set Σ_T which carves $\mathcal{D}_T$ into a collection of $c(d)$-Lipschitz graph domains. The images of these nicer domains under a Reifenberg parameterization whose derivative is nearly constant on the domain will then map them forward to Lipcshitz graph domains as desired in the conclusions of Theorems A, B, and C.

5.1 Carving up stopping time domains

We want to prove the following proposition:

Proposition 5.1. *There exists a constant $L_0(d) > 0$ such that for any stopping time region $T \subseteq \mathscr{W}$, there exists a d-Ahlfors upper regular set Σ_T which is a union of subsets of d-planes such that*

$$\mathcal{D}_T \setminus \Sigma_T = \bigcup_{j \in J_T} \mathcal{D}_T^j$$

where

$$\sum_{j \in J_T} \mathcal{H}^d(\partial\mathcal{D}_T^j) \lesssim_d \mathcal{H}^d(\partial\mathcal{D}_T) \simeq_d \mathcal{H}^d(\mathcal{D}_T \cap \mathbb{R}^d) + \sum_{W \in m(T)} \ell(W)^d \tag{5.1}$$

and $\mathcal{D}_T^j$ is an L_0-Lipschitz graph domain.

Remark 5.2. In Proposition 5.1, we only care that T is a coherent collection of cubes in the sense of Definition 2.11, not that they are produced by the specific g-Whitney coronization construction in Definition 3.5.

Σ_T will be defined as a union of more local sets Σ_W for $W \in m(T)$. The basic idea is to use a "cover" emanating from the bottom face of every minimal cube W downwards at a $\frac{\pi}{4}$ angle with the vertical in order to turn the jagged right angles made by stopped cubes into smoother $\frac{\pi}{4}$ angles which look Lipschitz to a point sitting above them higher up in the domain. This is essentially a modification of Peter Jones's algorithm for turning chord arc domains composed of Whitney boxes in the disk into Lipschitz graph domains in his proof of the Analyst's Traveling Salesman Theorem in the complex plane (see pg. 8 of [Jon90]). We now construct Σ_W.

Fix T and $W \in m(T)$. By translating and dilating, we can without loss of generality assume $W = [-1,1]^d \times [2,4]$. For any function $f : \mathbb{R}^d \to \mathbb{R}$, we let Graph$(f)$ denote the graph of f in $\mathbb{R}$

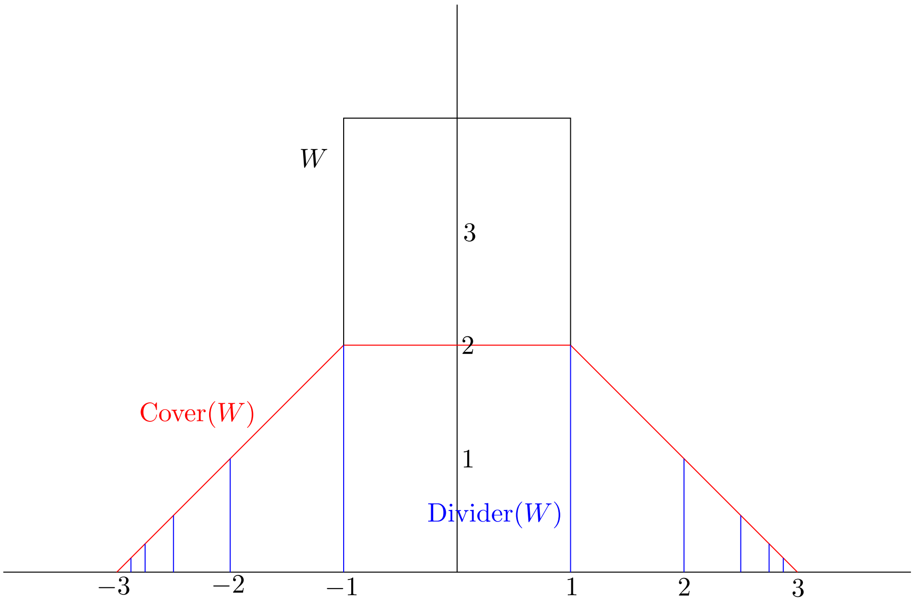

Figure 1: A representation of W, Cover(W), and Divider(W) in $\mathbb{R}^2$.

over $\mathbb{R}^d \times \{0\}$. We begin by defining, for $1 \le j \le d$,

$$\begin{aligned} H_0(x) &= 2, \\ H_{2j-1}(x) &= 3 + x_j, \\ H_{2j}(x) &= 3 - x_j. \end{aligned}$$

The graphs of these functions (except H_0) over $\mathbb{R}^d$ are planes which make an angle of $\frac{\pi}{4}$ with $\mathbb{R}^d$ and contain the edges of $\mathrm{Bot}(W)$ with $x_j = -1$ and $x_j = 1$ respectively. We define

$$\begin{aligned} H_W(x) &= \min_{0 \le i \le 2d} H_i(x), \\ \mathrm{Cover}(W) &= \mathrm{Graph}(H_W) \cap \mathbb{H}^{d+1}. \end{aligned}$$

Cover(W) is the lower envelope of the collection of planes given by the graphs of the H_i. In $\mathbb{R}^3$, Cover(W) forms the sides of a square pyramid minus its tip with base $[-3,3]^2 \times \{0\}$. In general, Cover(W) divides $\mathbb{H}^{d+1}$ into two components: a bounded component C_W with boundary Cover$(W) \cup [-3,3]^d \times \{0\}$ and the unbounded complimentary component. It also follows that

$$\mathcal{H}^d(\mathrm{Cover}(W)) \lesssim_d \mathcal{H}^d(\mathrm{Bot}(W)) = \ell(W)^d. \tag{5.2}$$

Cover(W) is one of two parts of Σ_W. The second part will be called Divider(W) because its purpose will be to ensure that all future domains beneath Cover(W) look similar to the top domain by separating future domains from one another with vertical plane extensions of the sides of cubes sliced by Cover(W). See Figure 1.

We begin by defining $t_n = 1 + \sum_{j=0}^{n-1} 2^{-j}$ and

$$\mathscr{Q}_n = \Big\{ Q \in \Delta^d([-3,3]^d \times \{0\}) : \ell(Q) = t_{n+1} - t_n = 2^{-n}, \\ \exists j,\ 1 \le j \le d,\ a_j = \pm t_n,\ Q = \prod_{j=1}^{d} [a_j, b_j] \Big\}$$

where $\Delta^d([-3,3]^d \times \{0\})$ is the set of d-dimensional dyadic cubes contained in $[-3,3]^d \times \{0\}$. Intuitively, we think of t_n as the radii of growing balls in the ℓ_∞ metric centered at 0, and the cubes

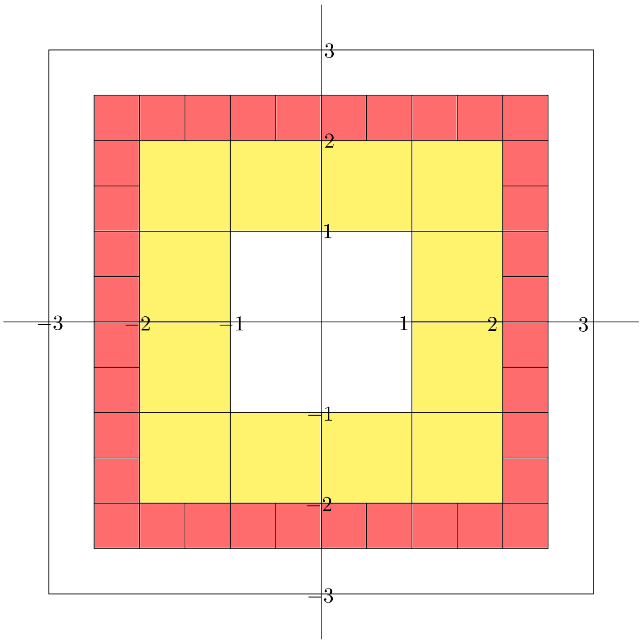

Figure 2: A representation of $[-3,3]^2 \times \{0\}$ split into $\mathscr{Q}_1$ in yellow, $\mathscr{Q}_2$ in red, and $\cup_{n=3}^{\infty}\mathscr{Q}_n$ left uncolored at the edge of $\mathscr{Q}_2$ (The white square in the middle sits below the cube $W \in m(T)$, hence nothing above it lies in $\mathcal{D}_T$). The set Divider(W) shoots out of the page as a union of extensions of the sides of the squares up to the points at which they hit the slanting top of Cover(W).

inside $\mathscr{Q}_n$ as the natural collection of dyadic cubes tiling the set difference between successive balls with side length exactly equal to the gap between the two square rings forming the boundaries of the ℓ_∞ balls (See Figure 2). Set $\mathscr{Q} = \bigcup_{n=1}^{\infty} \mathscr{Q}_n$ and define

$$\mathrm{Divider}(W) = C_W \cap \bigcup \{F_j \times [0, 2\ell(Q)] : F_j \in \mathrm{Faces}(Q),\ Q \in \mathscr{Q}\}.$$

Because $\sum_{j=1}^{2d} \mathcal{H}^d(F_j \times [0, 2\ell(Q)]) \lesssim_d \mathcal{H}^d(Q)$ and $[-3,3]^d \times \{0\} = \bigcup_{Q \in \mathscr{Q}} Q$ is a disjoint union, it follows immediately that

$$\mathcal{H}^d(\mathrm{Divider}(W)) \lesssim_d \mathcal{H}^d(\mathrm{Bot}(W)) = \ell(W)^d. \tag{5.3}$$

Now, we define

$$\Sigma_W = \mathrm{Cover}(W) \cup \mathrm{Divider}(W),$$

$$\Sigma_T = \bigcup_{W \in m(T)} \Sigma_W \cap \mathcal{D}_T.$$

We first prove the upper regularity claim of Proposition 5.1.

Lemma 5.3. *Σ_T is upper d-Ahlfors upper regular with constant $C \lesssim_d 1$.*

Proof. Fix $R > 0$ and $x \in \Sigma_W \subseteq \Sigma_T$ for some $W \in m(T)$. We write

$$\mathcal{H}^d(\Sigma_T \cap B(x,R)) = \sum_{\substack{W \in m(T) \\ h(W) < 10R}} \mathcal{H}^d(\Sigma_W \cap B(x,R)) + \sum_{\substack{W \in m(T) \\ h(W) \geq 10R}} \mathcal{H}^d(\Sigma_W \cap B(x,R)).$$

We note that $\pi(W)$ and $\pi(W')$ have disjoint interiors for any $W, W' \in m(T)$ with $W \neq W'$, so that

$$\sum_{\substack{W \in m(T) \\ h(W) < 10R}} \mathcal{H}^d(\Sigma_W \cap B(x,R)) \lesssim_d \sum_{\substack{W \in m(T) \\ h(W) < 10R}} \mathcal{H}^d(\mathrm{Bot}(W)) \leq (20R)^d.$$

On the other hand, there are a uniformly bounded number of minimal cubes $N(d)$ with $h(W) \geq 10R$ such that $B(x,R) \cap \Sigma_W \neq \varnothing$ so that

$$\sum_{\substack{W \in m(T) \\ h(W) \geq 10R}} \mathcal{H}^d(\Sigma_W \cap B(x,R)) \leq N(d) \cdot c(d) R^d \lesssim_d R^d$$

because $\mathcal{H}^d(\Sigma_W \cap B(x,R)) \leq c(d)R^d$ for any particular W by construction. Therefore, Σ_T is upper regular. ∎

We now finish the proof of Proposition 5.1.

Proof of Proposition 5.1. It follows from (5.2) and (5.3) that

$$\mathcal{H}^d(\Sigma_T) \leq \sum_{W \in m(T)} \mathcal{H}^d(\Sigma_W) \lesssim_d \sum_{W \in m(T)} \mathcal{H}^d(\mathrm{Bot}(W)) \leq \mathcal{H}^d(\mathrm{Bot}(W(T))) \lesssim_d \mathcal{H}^d(\partial \mathcal{D}_T)$$

which proves (5.1). We now need to show that the resulting domains $\mathcal{D}_T^j$ are Lipschitz-graphical. If $\mathcal{D}_T^j$ is the domain containing $W(T)$, then the claim follows with the choice of central point $c_{W(T)}$. Indeed, the cube $W(T)$ is clearly Lipschitz-graphical with respect to $c_{W(T)}$, and any boundary point of $\mathcal{D}_T^j$ not in $\partial W(T)$ is either in a vertical plane containing one of the vertical faces of $W(T)$, or is part of the Lipschitz graph consisting of the horizontally planar faces $\mathrm{Bot}(W)$ for $W \in m(T)$ and the planes of $\mathrm{Cover}(W)$ making $\frac{\pi}{4}$ angles with the bottom faces.

Now, suppose $\mathcal{D}_T^j \cap W(T) = \varnothing$. We have set up the construction such that this will not differ too much from the top cube case. Let $W \in m(T)$ be a cube of minimal height such that $\mathcal{D}_T^j \subseteq C_W$ and $\mathcal{H}^d(\partial \mathcal{D}_T^j \cap \mathrm{Cover}(W)) > 0$. Such W exists because its minimality implies that for any $W' \in m(T)$ of smaller side length than W, $\mathrm{Cover}(W')$ can only be part of the "lower" boundary of $\mathcal{D}_T^j$ while the only non-vertical planar pieces in Σ_T are bottoms and covers of minimal cubes. Then the cube R of maximal height such that $R \cap \mathcal{D}_T^j \neq \varnothing$ is exactly the cube of length $\ell(Q)$ sitting above $Q \subseteq \mathbb{R}^d \times \{0\}, Q \in \mathscr{Q}$ used in the definition of $\mathrm{Divider}(W)$.

Therefore, $R \cap \mathcal{D}_T^j$ is a cube sliced by finitely many d-planes passing through its sides and corners at $\frac{\pi}{4}$ angles. By the geometry described above, $\mathcal{D}_T^j$ contains the convex hull of c_R and $\mathrm{Bot}(R)$, so we have that $\mathcal{D}_T^j$ is Lipschitz-graphical with respect to $\frac{1}{2}(c_R + c_{\mathrm{Bot}(R)})$. Indeed, Lipschitz-graphicality follows for points in $R \cap \mathcal{D}_T^j$ immediately, and follows for the rest of $\mathcal{D}_T^j$ by the same argument as for the region containing $W(T)$ because the definition of $\mathrm{Divider}(W)$ ensures that all cubes which make up $\mathcal{D}_T^j$ are children of R. Indeed, the boundary outside of ∂R consists of vertical planes containing one of the vertical faces of R or is part of a Lipschitz graph consisting of horizontally planar faces $\mathrm{Bot}(W')$ for $W' \in m(T)$ with $W' \leq R$ and the planes of $\mathrm{Cover}(W')$ making $\frac{\pi}{4}$ angles with the bottom faces. ∎

5.2 Images of Lipschitz graph domains

We now show that the Lipschitz graph domain property is preserved under images of maps whose derivatives are nearly constant. We begin by observing that linear transformations preserve Lipschitz graph domains

Lemma 5.4. *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be an L_0-Lipschitz graph domain and let $A : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ be an L'-bi-Lipschitz affine map. Then there exists a constant $L_1(L_0, L')$ such that $A(\Omega)$ is an L_1-Lipschitz graph domain.*

Proof. Without loss of generality, assume $A(0) = 0$ and set $\Omega' = A(\Omega)$. Then since $\Omega = \{t\theta : 0 \leq t \leq r(\theta),\ \theta \in \mathbb{S}^d\}$, we know that $\Omega' = \{tA(\theta) : 0 \leq t \leq A(\theta),\ \theta \in \mathbb{S}^d\}$ so that Ω' is star-shaped and $r_{\Omega'}$ is well-defined. We have

$$\partial \Omega' = A(\partial \Omega) = \{A(r(\theta)\theta) = r(\theta)A(\theta) : \theta \in \mathbb{S}^d\}.$$

Therefore, given $\psi \in \mathbb{S}^d$, we see that

$$r_{\Omega'}(\psi) = r_\Omega\left(\frac{A^{-1}(\psi)}{|A^{-1}(\psi)|}\right)\frac{1}{|A^{-1}(\psi)|}.$$

Because A^{-1} is L'-bi-Lipschitz and r_Ω is L_0-Lipschitz on $\mathbb{S}^d$, $r_{\Omega'}$ is composed of products and compositions of bounded Lipschitz functions and it follows that there exists $L_1(L_0, L')$ such that $r_{\Omega'}$ satisfies the requirements of Definition 1.2 after scaling. ∎

We now move from affine maps to maps whose derivative is sufficiently close to the identity. In preparation, define ℓ_z for any $z \in \mathbb{R}^{d+1}$ to be the line passing through 0 and z and let $P_z = \ell_z^\perp + z$. Define the radial cone at x of aperture α and radius R as

$$C_x(\alpha, R) = \left\{y \in B(x,R) : \frac{\operatorname{dist}(y, \ell_x)}{\operatorname{dist}(y, P_x)} < \tan(\alpha)\right\} \setminus \{x\}.$$

Lemma 5.5. *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be an L_0-Lipschitz graph domain. There exists a constant $\delta_0(L_0,d) > 0$ such that if $\delta < \delta_0$ and $\varphi : \overline{\Omega} \to \varphi(\overline{\Omega})$ is a $(1+\delta)$-bi-Lipschitz C^1 map satisfying*

$$|D\varphi(z) - I| \le \delta \tag{5.4}$$

for all $z \in \overline{\Omega}$, then there exists $L_1 \lesssim_{L_0,d} 1$ such that $\varphi(\Omega)$ is a L_1-Lipschitz graph domain.

Proof. Assume without loss of generality that Ω is Lipschitz graphical with respect to 0 and $\varphi(0) = 0$. We first verify that $r_\Omega : \mathbb{S}^d \to \mathbb{R}^+$ is well-defined, i.e., the domain is star-shaped with respect to 0. Let $\varphi(x) \in \partial\Omega$ and let $\gamma(t) = t\varphi(x)$. We want to show $\gamma \cap \partial\varphi(\Omega) = \{\varphi(x)\}$. Set $\tilde{\gamma}(t) = \varphi^{-1}(\gamma(t))$. We would like to prove

$$|\tilde{\gamma}'(t) - x| \le 5\delta|x| \tag{5.5}$$

for all $t \in [0,1]$. First note that

$$|D\varphi(z)^{-1} - I| = |D\varphi(z)^{-1} \cdot \left[I - D\varphi(z)^{-1}\right]| \le 2\delta||D\varphi(z)^{-1}| \le 3\delta$$

using the bound $|D\varphi(z)^{-1}| \le \frac{1}{\sigma_{\min}(D\varphi(z))} \le (1+2\delta)$ where $\sigma_{\min}(D\varphi(z))$ is the smallest singular value of $D\varphi(z)$. This means

$$\begin{aligned}|\tilde{\gamma}'(t) - x| &= |D\varphi^{-1}(\gamma(t)) \cdot \gamma'(t) - x| = |\left[D\varphi(\tilde{\gamma}(t))^{-1} - I\right] \cdot \gamma'(t) + \gamma'(t) - x| \\ &\le 3\delta|\varphi(x)| + |\varphi(x) - x| \le 5\delta|x|\end{aligned}$$

where the final line follows from the fact that $\varphi(x) = \int_0^1 D\varphi(tx)\cdot x\, dt = x + \int_0^1 (D\varphi(tx) - I)\cdot x\, dt$ so that $|\varphi(x) - x| \le \delta|x|$. It follows from the mean value theorem that $\tilde{\gamma} \subseteq C_x(10\delta, |x|)$. Since $C_x(10\delta, |x|) \cap \partial\Omega = \varnothing$ for δ sufficiently small in terms of L_0, it follows that choosing δ_0 small enough gives $\tilde{\gamma} \cap \partial\Omega = \{x\}$ so that $\gamma \cap \partial\varphi(\Omega) = \{\varphi(x)\}$ as desired.

Set $\Omega' = \varphi(\Omega)$. Now, $r_{\Omega'}$ is well-defined and (5.4) implies

$$\frac{1}{2(L_0+1)} \le r_{\Omega'}(\theta) \le 2$$

so that we only need to show that $r_{\Omega'}$ satisfies the Lipschitz bound in Definition 1.2 for some constant $L_1(L_0,d)$. Let $a, b \in \partial\Omega'$ with $a = |a|\psi_1$ and $b = |b|\psi_2$. Let $\psi = |\psi_1 - \psi_2|$. If $\psi \ge \frac{\pi}{4}$, then the result follows directly from the fact that φ is $(1+\delta)$-bi-Lipschitz. If instead $\psi < \frac{\pi}{4}$, then there exist unique $x, y \in \partial\Omega$ such that $a = \varphi(x)$ and $b = \varphi(y)$ and we assume without loss of generality that $|x| \ge |y|$. Let $x = r_\Omega(\theta_1)\theta_1 = |x|\theta_1,\ y = r_\Omega(\theta_2)\theta_2 = |y|\theta_2$ and set $\theta = |\theta_1 - \theta_2|$.

We first claim that it suffices to show

$$|\theta_1 - \theta_2| \lesssim_{L_0,d} |\psi_1 - \psi_2|. \tag{5.6}$$

Indeed, if (5.6) holds, then

$$\begin{aligned}|r_{\Omega'}(\psi_1) - r_{\Omega'}(\psi_2)| &= ||a| - |b|| \le |a-b| = |\varphi(x) - \varphi(y)| \le (1+\delta)|x-y| \\ &\le (1+\delta)(|x - \theta_1|y|| + |\theta_1|y| - y|) \\ &= (1+\delta)(r_\Omega(\theta_1) - r_\Omega(\theta_2) + |y||\theta_1 - \theta_2|) \\ &\le (1+\delta)(L_0+1)|\theta_1 - \theta_2| \lesssim_{L_0,d} |\psi_1 - \psi_2|.\end{aligned}$$

Now, we concentrate on proving 5.6.

Put $z = (1 - |x-y|)x$ and $c = (1 - |a-b|)a$ and define

$$\alpha = \angle zxy, \ \alpha' = \angle \varphi(z)ab, \ \beta = \angle cab.$$

By the law of cosines,

$$\begin{aligned}\cos\alpha &= \frac{|z-x|^2 + |x-y|^2 - |z-y|^2}{2|z-x||x-y|} = 1 - \frac{|z-y|^2}{2|z-x|^2}, \\ \cos\alpha' &= \frac{|\varphi(z) - \varphi(x)|^2 + |\varphi(x) - \varphi(y)|^2 - |\varphi(z) - \varphi(y)|^2}{2|\varphi(x) - \varphi(z)||\varphi(x) - \varphi(y)|} \\ &\le \frac{2(1+\delta)^2|z-x|^2 - (1-\delta)^2|z-y|^2}{2(1-\delta)^2|z-x|^2} \le 1 - \frac{|z-y|^2}{2|z-x|^2} + 5\delta = \cos\alpha + 5\delta.\end{aligned}$$

Because Ω is L_0-Lipschitz-graphical, $\alpha \gtrsim_{L_0} 1$ so that if δ is sufficiently small, then $\alpha' \ge \frac{\alpha}{2}$. In addition, (5.5) implies that $\varphi([x,z]) \subseteq C_{\varphi(x)}(10\delta, 2(|\varphi(x)| - |c|)$ so that $|\beta - \alpha'| \le 20\delta$, meaning $\beta \ge \frac{\alpha}{4}$ as long as δ is small enough. To complete the proof, observe that $|\psi_1 - \psi_2| \simeq \angle a0b$, $|\theta_1 - \theta_2| \simeq \angle x0y$, $\beta = \angle 0ab$, and $\alpha = \angle 0xy$ so that $\beta \ge \frac{\alpha}{4}$ implies (5.6) using the fact that φ is $(1+\delta)$-bi-Lipschitz. ■

Finally, by chaining Lemmas 5.4 and 5.5, we can prove the following desired proposition:

Proposition 5.6. *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be an L_0-Lipschitz graph domain and suppose $g : \Omega \to g(\Omega) \subseteq \mathbb{R}^{d+1}$ is C^1 and L-bi-Lipschitz. There exist constants $L_1, \delta_0(L_0, L) > 0$ such that if $\delta < \delta_0$ and*

$$|Dg(z) \cdot Dg(w)^{-1} - I| \le \delta \tag{5.7}$$

for all $z, w \in \Omega$, then $g(\Omega)$ is an L_1-Lipschitz graph domain.

Proof. Suppose Ω is Lipschitz graphical around 0 and set

$$L(z) = Dg(0) \cdot z.$$

By Lemma 5.4, $L(\Omega)$ is $L_0'(L, L_0)$-Lipschitz graphical. The map $\varphi : L(\Omega) \to g(\Omega)$ given by

$$\varphi = g \circ L^{-1}$$

satisfies

$$D\varphi(z)(L(w)) = Dg(L^{-1}(L(w))) \cdot DL^{-1}(L(w)) = Dg(w) \cdot Dg(0)^{-1} \cdot w$$

so that

$$|D\varphi - I| \le \delta.$$

By taking δ_0 sufficiently small in terms of L_0', Lemma 5.5 implies that there exists $L_1(L_0')$ such that $g(\Omega)$ is L_1-Lipschitz graphical. ■

6 Controlling the change in the derivative of Reifenberg parameterizations

The goal of this appendix is to give conditions under which we can say that the change in the derivative of a Reifenberg parameterization $g : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ is small. This is specified exactly in Proposition 6.7 below.

6.1 Preliminary derivative estimates and regularity

In this section, we review some properties of the maps used in the construction of a Reifenberg parameterization g that we need to make specific estimates on the change in Dg. First, the surface Σ_k has a nice local Lipschitz representation:

Lemma 6.1 ([DT12] Lemma 6.12). *For $k \geq 0$ and $y \in \Sigma_k$, there is an affine d-plane P through y and a $C\varepsilon$-Lipschitz and C^2 function $A : P \to P^\perp$ such that $|A(x)| \leq C\epsilon r_k$ for all $x \in B(y, 19r_k)$ and*

$$\Sigma_k \cap B(y, 19r_k) = \Gamma \cap B(y, 19r_k).$$

where Γ denotes the graph of A over P.

Now, we record distortion estimates for $D\sigma_k$ as in [DT12] chapter 7. Importantly, $D\sigma_k$ is very close to the identity in the following sense:

Lemma 6.2 ([DT12] Lemma 7.1). *For $k \geq 0$, σ_k is a C^2-diffeomorphism from Σ_k to Σ_{k+1} and, for $y \in \Sigma_k$,*

$$D\sigma_k(y) : T\Sigma_k(y) \to T\Sigma_{k+1}(\sigma_k(y)) \text{ is bijective and } (1 + C\varepsilon)\text{-bi-Lipschitz.}$$

In addition,

$$|D\sigma_k(y) \cdot v - v| \leq C\varepsilon|v| \quad \text{for } y \in \Sigma_k \text{ and } v \in T\Sigma_k(y)$$
$$|\sigma_k(y) - \sigma_k(y') - y + y'| \leq C\varepsilon|y - y'| \quad \text{for } y, y' \in \Sigma_k.$$

More precise estimates can be obtained when restricting $D\sigma_k$ to its action on vectors tangent to Σ_k. The best way to capture this is to define quantities which take into account exactly how close the nearby planes of appropriate scale in the CCBP are. These are the ϵ'_k numbers, defined by

$$\epsilon'_k(y) = \sup\big\{d_{x_{i,l},100r_l}(P_{j,k}, P_{i,l})\,;\, j \in J_k,\, l \in \{k-1, k\}, \\ i \in J_l, \text{ and } y \in 10B_{j,k} \cap 11B_{i,l}\big\} \tag{6.1}$$

The following lemma gives estimates in terms of these numbers

Lemma 6.3 ([DT12] Lemma 7.32). *For $k \geq 1$ and $y \in \Sigma_k \cap V_k^8$, choose $i \in J_k$ such that $|y - x_{i,k}| \leq 10r_k$. Then*

$$|D\pi_{i,k} \circ D\sigma_k(y) \circ D\pi_{i,k} - D\pi_{i,k}| \leq C\varepsilon'_k(y)^2, \tag{6.2}$$

and

$$\big||D\sigma_k(y) \cdot v| - 1\big| \leq C\varepsilon'_k(y)^2 \quad \text{for every unit vector } v \in T\Sigma_k(y). \tag{6.3}$$

Similarly, these numbers also control the distance between tangent planes to the surface and nearby $P_{j,k}$. For any $k \geq 0$ and $y \in \Sigma_k \cap V_k^8$ and $i \in J_k$ such that $|y - x_{i,k}| \leq 10r_k$, we have ([DT12] (7.22))

$$\text{Angle}(T\Sigma_k(y), P_{i,k}) \leq C\epsilon'_k(y). \tag{6.4}$$

Finally, we also use an estimate on $D^2\sigma_k$ obtained in by Ghinassi in [Ghi20] in work on constructing $C^{1,\alpha}$ parametrizations.

Lemma 6.4 ([Ghi20] Lemma 3.16). *For $k \geq 0$, $y \in \Sigma_k \cap V_k^8$,*

$$|D^2\sigma_k(y)| \leq C\frac{\epsilon_k(y)}{r_k} \leq C\frac{\epsilon}{r_k}$$

where we interpret the norm on the tensor $D^2\sigma_k$ as the Euclidean norm on $\mathbb{R}^{n^3}$. We also provide the following lemma and proof adapted from a proof of [DT12] to fit our needs.

Lemma 6.5 (cf. [DT12] (11.22)). *Suppose Σ_0 is such that for any $x, x' \in \Sigma_0$, there exists a curve γ_0 connecting x and x' with $\ell(\gamma_0) \le (1 + C\epsilon)|x - x'|$. Let $1 \le M^3\epsilon < c(d) < 1$ with $c(d)$ sufficiently small and $k \ge 0$ be such that $|f_k(x) - f_k(x')| < Mr_k$. Then there is a curve $\gamma : I \to \Sigma_k$ such that $\ell(\gamma) \le 2|f_k(x) - f_k(x')|$.*

Proof. We first prove the following claim:
Claim: For any $0 \le p \le k$,

$$|f_{k-p}(x) - f_{k-p}(x')| < \frac{Mr_{k-p}}{5^p}. \tag{6.5}$$

Proof: We prove this by induction. Indeed, observe that

$$|f_{k-p-1}(x) - f_{k-p-1}(x')| = |\sigma_{k-p-1}^{-1}(f_{k-p}(x)) - \sigma_{k-p-1}^{-1}(f_{k-p}(x'))| \le (1 + C\epsilon)|f_{k-p}(x) - f_{k-p}(x')|.$$

by (2.10). Applying this for $p = 1$ gives

$$|f_{k-1}(x) - f_{k-1}(x')| < (1 + C\epsilon)(Mr_k) < \frac{Mr_{k-1}}{5}$$

This proves the base case. Assuming the claim holds for some p, we get

$$|f_{k-p-1}(x) - f_{k-p-1}(x')| \le (1 + C\epsilon)\frac{Mr_{k-p}}{5^p} < \frac{Mr_{k-p-1}}{5^{p+1}}.$$

To continue the proof of the lemma, we modify the proof of [DT12] (11.22). If $|f_k(x) - f_k(x')| < 18r_k$, then the claim follows immediately from the local Lipschitz graph description of Σ_k in Lemma 6.1. So, assume $|f_k(x) - f_k(x')| > 18r_k$ and suppose first that there exists an integer $0 \le m \le k$ such that $|f_m(x) - f_m(x')| < 5r_m$. We calculate

$$\frac{Mr_m}{5^{k-m}} < 5r_m \iff \log_5 M - 1 < k - m$$

so that by the above claim we can assume $k - m < \log_5 M < \log M$. Applying the Lipschitz graph lemma for $B(f_m(x), 19r_m)$, we see that there exists a path $\gamma_m \subseteq \Sigma_m$ such that

$$\begin{aligned} \ell(\gamma_m) \le (1 + C\epsilon)|f_m(x) - f_m(x')| &\le (1 + C\epsilon)(C\epsilon r_m + |f_k(x) - f_k(x')|) \\ &\le (1 + C\epsilon)|f_k(x) - f_k(x')| + C\epsilon r_k \log M. \end{aligned}$$

On the other hand, since $|f_m(x) - f_m(x')| < 5r_m$, we get $\ell(\gamma_m) \le 10r_m$ and so we can choose a chain of $N \le \frac{10r_m}{10r_k} = 10^{k-m} \le M^2$ points contained in γ_m with consecutive points separated by a distance of at least $11r_k$ beginning at $f_m(x)$ and ending at $f_m(x')$. Call this collection of points $\{f_m(x_l)\}_{l=1}^N$ for $x_l \in \Sigma_0$. This implies the total length of the string of points $\{f_k(x_l)\}$ is

$$\begin{aligned} L' = \sum_{l=1}^{N} |f_k(x_l) - f_k(x_{l+1})| \le \sum_{l=1}^{N} [C\epsilon r_m + |f_m(x_l) - f_m(x_{l+1})|] &\le C\epsilon r_k M^2 \log M + \ell(\gamma_m) \\ &\le C\epsilon r_k M^2 \log M + (1 + C\epsilon)|f_k(x) - f_k(x')|. \end{aligned}$$

In addition, for any admissible l we can calculate

$$|f_k(x_l) - f_k(x_{l+1})| \le C\epsilon r_m + |f_m(x_l) - f_m(x_{l+1})| \le C\epsilon r_k \log M + 11r_k < 12r_k. \tag{6.6}$$

Using (6.6) and Lemma 6.1 once again, we connect each pair $(f_k(x_l), f_k(x_{l+1}))$ by a curve γ_l of length $\ell(\gamma_l) \le (1 + C\epsilon)|f_k(x_l) - f_k(x_{l+1})|$ to get a curve γ with

$$\ell(\gamma) \le (1 + C\epsilon)L' \le (1 + C\epsilon)|f_k(x) - f_k(x)| + C\epsilon r_k M^2 \log M \le 2|f_k(x) - f_k(x')|$$

using the fact that $|f_k(x) - f_k(x')| > 18r_k$ and $M \ll \epsilon^{-1}$, i.e. ϵ is sufficiently small compared to M. This completes the proof if there exists such an m where $f_k(x)$ and $f_k(x')$ pull back to a Lipschitz neighborhood in Σ_m. If there does not exist such an m (i.e., k is too small), then we instead use the assumed curve $\gamma_0 \subseteq \Sigma_0$ in place of γ_m and argue as in the previous case. ∎

We also recall a reverse triangle inequality:

Lemma 6.6. *(Reverse Triangle Inequality) Let $u, v \in \mathbb{R}^{d+1}$ with $\langle u, v\rangle \geq -\frac{1}{2}|u||v|$. Then*

$$|u| + |v| \leq 2|u + v|. \tag{6.7}$$

6.2 Controlling the change in Dg

Proposition 6.7 follows from a series of computations involving the derivative of the map g produced by Theorem 2.4 for a given CCBP. Proposition 6.7 says that given a "central" point $z \in \mathbb{R}^{d+1}$ and an inflation factor $1 \leq M_0$ such that $M_0\epsilon$ is sufficiently small, we can get a set $G_z^{M_0}$ such that $w \in G_z^{M_0}$ means $Dg(w)$ is very close to $Dg(z)$ in the sense of (6.8).

Proposition 6.7 follows from Lemmas 6.8 and 6.10 which give separate horizontal and vertical estimates respectively. Define the sets of horizontal and vertical vectors by

$$H = \mathbb{R}^d \times \{0\},\ V = \{0\}^d \times \mathbb{R}.$$

These lemmas show how to appropriately bound the individual pieces of the difference $Dg(x, y) - Dg(x', y)$ and $Dg(x', y) - Dg(x', y')$ between points $z = (x, y), z' = (x', y')$ respectively when acting on $v \in\ H \cup V$. Corollaries 6.9 and 6.11 put these pieces together to get the requisite Dg estimates, from which we prove Proposition 6.7.

Proposition 6.7. *Let $0 < \epsilon < \delta < 1$ and $M_0 > 0$ such that $1 \leq M_0^3\epsilon < c(d)$ with $c(d)$ sufficiently small. Fix a CCBP $(\Sigma_0, \{B_{j,k}\}, \{P_{j,k}\})$. Let $z, z' \in \mathbb{R}^d \times \mathbb{R}$ with $z = (x, y),\ z' = (x', y')$ where $|y'| \leq |y|$ and assume $f_{n(y)}(x) \in V^8_{n(y)}$ and $f_{n(y')}(x') \in V^8_{n(y')}$ (see (2.13)). Define*

$$G_z^{M_0} = \Big\{z' = (x', y') \in \mathbb{R}^{d+1} : |f_{n(y)}(x) - f_{n(y)}(x')| < M_0 r_{n(y)},\ \sum_{k=n(y)}^{n(y')} \epsilon'_k(f_k(x'))^2 < \epsilon,$$
$$\operatorname{Angle}(T_k(x'), T_{n(y)}(x') \leq \delta\Big\}.$$

Then there exists $C(d) > 0$ such that for any $w \in G_z^{M_0}$, we have

$$|Dg(w) \cdot Dg(z)^{-1} - I| \leq C(d)\delta. \tag{6.8}$$

Lemma 6.8 (Horizontal Estimates)**.** *Let z, z', M_0, ϵ be as in Proposition 6.7 and let $v \in H \cup V$. Let k be such that $\rho_k(y) > 0$. If $|f_k(x) - f_k(x')| < M_0 r_k$, then*

$$|Df_k(x) \cdot v - Df_k(x') \cdot v| \leq C\epsilon|Df_k(x) \cdot v|, \tag{6.9}$$

$$\big|(D_x(R_k(x) \cdot y)) \cdot v - (D_x(R_k(x') \cdot y)) \cdot v\big| \leq C\epsilon|Df_k(x) \cdot v|. \tag{6.10}$$

In any case,

$$\left|\frac{\partial g}{\partial y}(x, y) - \frac{\partial g}{\partial y}(x', y)\right| \leq C\epsilon\left|\frac{\partial g}{\partial y}(x, y)\right| \tag{6.11}$$

where the constant C depends on M.

Proof. We begin with proving (6.9). We have

$$\begin{aligned}|Df_k(x) \cdot v - Df_k(x') \cdot v| &= \big|\big[D\sigma_{k-1}(f_{k-1}(x)) - D\sigma_{k-1}(f_{k-1}(x'))\big]Df_{k-1}(x) \cdot v \\ &\qquad + D\sigma_{k-1}(f_{k-1}(x'))\big[Df_{k-1}(x) - Df_{k-1}(x')\big] \cdot v\big| \\ &\leq |Df_{k-1}(x) \cdot v||D\sigma_{k-1}(f_{k-1}(x)) - D\sigma_{k-1}(f_{k-1}(x'))| \\ &\qquad + |D\sigma_{k-1}(f_{k-1}(x'))||Df_{k-1}(x) \cdot v - Df_{k-1}(x') \cdot v|\end{aligned}$$

Recursively applying this inequality for decreasing values of k gives

$$
\begin{aligned}
&|Df_k(x)\cdot v - Df_k(x')\cdot v| \\
&\le |Df_{k-1}(x)\cdot v||D\sigma_{k-1}(f_{k-1}(x)) - D\sigma_{k-1}(f_{k-1}(x'))| \\
&\qquad + |D\sigma_{k-1}(f_{k-1}(x'))|\big(|Df_{k-2}(x)\cdot v||D\sigma_{k-2}(f_{k-2}(x)) - D\sigma_{k-2}(f_{k-2}(x'))| \\
&\qquad + |D\sigma_{k-2}(f_{k-2}(x'))||Df_{k-2}(x)\cdot v - Df_{k-2}(x')\cdot v|\big) \\
&\le |Df_{k-1}(x)\cdot v||D\sigma_{k-1}(f_{k-1}(x)) - D\sigma_{k-1}(f_{k-1}(x'))| \\
&+ \sum_{p=1}^{k}\left(\prod_{m=1}^{p}|D\sigma_{k-m}(f_{k-m}(x'))|\right)|Df_{k-p-1}(x)\cdot v|\cdot|D\sigma_{k-p-1}(f_{k-p-1}(x)) - D\sigma_{k-p-1}(f_{k-p-1}(x')|.
\end{aligned}
\tag{6.12}
$$

Now, Lemma 6.2 implies

$$
\prod_{m=1}^{p}|D\sigma_{k-m}(f_{k-m}(x'))| \le (1+C\epsilon)^p, \tag{6.13}
$$

and

$$
|Df_{k-p-1}(x)\cdot v| = \left|\prod_{m=1}^{p+1} D\sigma_{k-m}^{-1}(f_{k-m+1}(x))Df_k(x)\cdot v\right| \le (1+C\epsilon)^{p+1}|Df_k(x)\cdot v|. \tag{6.14}
$$

Using Lemma 6.5, we see that (6.5) implies $|f_{k-p-1}(x) - f_{k-p-1}(x')| < \frac{M_0 r_{k-p-1}}{5^{p+1}} \le M_0 r_{k-p-1}$ so that we get a rectifiable curve γ_{k-p-1} connecting $f_{k-p-1}(x)$ and $f_{k-p-1}(x')$ such that $\ell(\gamma_{k-p-1}) \le 2|f_{k-p-1}(x) - f_{k-p-1}(x')|$. Lemma 6.4 gives

$$
\begin{aligned}
|D\sigma_{k-p-1}(f_{k-p-1}(x)) - D\sigma_{k-p-1}(f_{k-p-1}(x'))| &= \left|\int_I D^2\sigma_{k-p-1}(\gamma_{k-p-1}(t))\cdot\gamma'_{k-p-1}(t)\ dt\right| \\
&\le \int_I |D^2\sigma_{k-p-1}(\gamma_{k-p-1}(t))||\gamma'_{k-p-1}(t)|dt \\
&\le C\frac{\epsilon}{r_{k-p-1}}\cdot 2|f_{k-p-1}(x) - f_{k-p-1}(x')| \\
&\le C\frac{\epsilon}{r_{k-p-1}}\frac{M_0 r_{k-p-1}}{5^{p+1}} \\
&\le CM_0\frac{\epsilon}{5^{p+1}}.
\end{aligned}
\tag{6.15}
$$

Applying (6.13), (6.14), and (6.15) to (6.12) gives

$$
\begin{aligned}
|Df_k(x)\cdot v - Df_k(x')\cdot v| &\le (1+C\epsilon)C\frac{\epsilon}{5}|Df_k(x)\cdot v| + \sum_{p=1}^{k}(1+C\epsilon)^p(1+C\epsilon)^{p+1}|Df_k(x)\cdot v|CM_0\frac{\epsilon}{5^{p+1}} \\
&\le C\epsilon|Df_k(x)\cdot v| + C\epsilon M_0|Df_k(x)\cdot v|\sum_{p=1}^{k}\frac{(1+C\epsilon)^{2p}}{5^{p+1}} \\
&\le C\epsilon|Df_k(x)\cdot v|.
\end{aligned}
$$

We now prove (6.10). For any $t>0$, Proposition 2.5 implies that the quantity $R_k(x+tv)\cdot e_{d+1} - R_k(x)\cdot e_{d+1}$ is the difference between the unit normal vectors to the linear subspaces $T\Sigma_k(f_k(x+tv))$ and $T\Sigma_k(f_k(x))$. But by Lemma 2.6, we have

$$
|R_k(x+tv)\cdot e_{d+1} - R_k(x)\cdot e_{d+1}| \le D(T\Sigma_k(f_k(x+tv)), T\Sigma_k(f_k(x))) \le C\frac{\epsilon}{r_k}|f_k(x+tv) - f_k(x)|. \tag{6.16}
$$

Hence, we can write

$$
\begin{aligned}
|(D_x(R_k(x)\cdot y))\cdot v| &\le |y|\lim_{t\to 0}\frac{|R_k(x+tv)\cdot e_{d+1} - R_k(x)\cdot e_{d+1}|}{|t|} \le C\epsilon\frac{|y|}{r_k}\lim_{t\to 0}\frac{|f_k(x+tv) - f_k(x)|}{|t|} \\
&\le C\epsilon|Df_k(x)\cdot v|
\end{aligned}
\tag{6.17}
$$

where $|y| \lesssim r_k$ since $\rho_k(y) > 0$. We then have

$$\left|(D_x(R_k(x)\cdot y))\cdot v - (D_x(R_k(x')\cdot y))\cdot v\right| \le C\epsilon(|Df_k(x)\cdot v| + |Df_k(x')\cdot v|) \le C\epsilon|Df_k(x)\cdot v|$$

using (6.9).

Finally, we prove (6.11). First, we compute

$$\begin{aligned}\frac{\partial g}{\partial y}(x,y) - \frac{\partial g}{\partial y}(x',y) &= \sum_{k\ge 0}\frac{\partial \rho_k}{\partial y}(y)\left\{f_k(x) - f_k(x') + R_k(x)\cdot y - R_k(x')\cdot y\right\}\\ &\quad + \sum_{k\ge 0}\rho_k(y)(R_k(x)\cdot e_{d+1} - R_k(x')\cdot e_{d+1})\\ &=: I + II\end{aligned}$$

Let $p, p+1$ be the values of k such that $\rho_k(y) > 0$. Since $\rho_p(y) + \rho_{p+1}(y) = 1$, we have $\frac{\partial \rho_p}{\partial y}(y) + \frac{\partial \rho_{p+1}}{\partial y}(y) = 0$. This implies

$$\begin{aligned}I &= \frac{\partial \rho_p}{\partial y}(y)\left(f_p(x) - f_{p+1}(x) + R_p(x)\cdot y - R_{p+1}(x)\cdot y\right)\\ &\quad + \frac{\partial \rho_p}{\partial y}(y)\left(f_p(x') - f_{p+1}(x') + R_p(x')\cdot y - R_{p+1}(x')\cdot y\right)\end{aligned}$$

But using (2.10) and (2.11), we have

$$|I| \le \frac{C}{r_p}(C\epsilon r_p + C\epsilon|y|) \le C\epsilon.$$

By (6.16) we have

$$\begin{aligned}|II| &\le |\rho_p(y)|\frac{C\epsilon}{r_p}|f_p(x) - f_p(x')| + |\rho_{p+1}(y)|\frac{C\epsilon}{r_{p+1}}|f_{p+1}(x) - f_{p+1}(x')|\\ &\le CM_0\epsilon(|\rho_p(y)| + |\rho_{p+1}(y)|) \le C\epsilon.\end{aligned}$$

We've proven that $\left|\frac{\partial g}{\partial y}(x,y) - \frac{\partial g}{\partial y}(x',y)\right| \le C\epsilon$. We will complete the proof of (6.11) by showing that $\left|\frac{\partial g}{\partial y}(x,y)\right| \gtrsim 1$. Indeed,

$$\begin{aligned}\frac{\partial g}{\partial y}(x,y) = \Bigg|&\left[\frac{\partial \rho_p}{\partial y}(y)\left(f_p(x) - f_{p+1}(x) + R_p(x)\cdot y - R_{p+1}(x)\cdot y\right)\right]\\ &+ \left[\rho_p(y)R_p(x)\cdot e_{d+1} + \rho_{p+1}(y)R_{p+1}(x)\cdot e_{d+1}\right]\Bigg|.\end{aligned} \tag{6.18}$$

But the previous computation shows that the first expression has norm $\le C\epsilon$, while the second expression is a convex combination of two nearly parallel unit vectors because $R_p(x)$ and $R_{p+1}(x)$ are orthogonal matrices which are $C\epsilon$ close. Hence, we get

$$\left|\frac{\partial g}{\partial y}\right| \gtrsim 1. \tag{6.19}$$

■

Corollary 6.9. *Let z, z' be as in Lemma 6.8 and set $p = n(y'), m = n(y)$. Then for any vector $v \in H \cup V$, we have*

$$|Dg(x,y)\cdot v - Dg(x',y)\cdot v| \le C\epsilon|Dg(x,y)\cdot v|$$

Proof. First, suppose $v = v_x \in H$. Since $v_x \cdot e_{d+1} = 0$, we have

$$Dg(x,y) \cdot v_x = \sum_{k\geq 0} \rho_k(y) \left\{ Df_k(x) \cdot v_x + D(R_k(x) \cdot y) \cdot v_x \right\}. \tag{6.20}$$

Therefore, we get

$$\begin{aligned} &|Dg(x,y) \cdot v_x - Dg(x',y) \cdot v_x| \\ &\quad \leq \sum_{k\geq 0} \rho_k(y) \big\{ |Df_k(x) \cdot v_x - Df_k(x') \cdot v_x| + |D(R_k(x) \cdot y) \cdot v_x - D(R_k(x')) \cdot v_x| \big\} \\ &\quad \leq CM_0\epsilon \sum_{k\geq 0} \rho_k(y) \left\{ |Df_k(x) \cdot v_x| \right\}. \end{aligned} \tag{6.21}$$

using (6.9) and (6.10). We now want to bound $\sum_{k\geq 0} \rho_k(y) \{|Df_k(x) \cdot v_x|\}$ by $|Dg(x,y) \cdot v_x|$. In order to do so, we first simplify notation by setting

$$\begin{gathered} s = \rho_p(y),\ t = \rho_{p+1}(y), \\ v_1 = Df_p(x) \cdot v_x,\ u_1 = Df_{p+1}(x) \cdot v_x, \\ v_2 = D(R_p(x) \cdot y) \cdot v_x,\ u_2 = D(R_{p+1}(x) \cdot y) \cdot v_x. \end{gathered}$$

Putting $v = v_1 + v_2,\ u = u_1 + u_2$, we have $Dg(x,y) \cdot v_x = sv + tu$. In this notation,

$$|v_1 - u_1| \leq C\epsilon |v_1|, \tag{6.22}$$

$$|v_2|, |u_2| \leq C\epsilon |v_1|, \tag{6.23}$$

by Lemma 6.2 and (6.17). We then want to prove the following claim:

Claim: $s|v_1| + t|u_1| \lesssim |sv + tu|$.

Proof: Using (6.23), we get $s|v_1| \leq s|v_1| + s|v_2| \leq s|v|$ and similarly $t|u_1| \leq t|u|$. We now just need to show that $|sv| + |tu| \lesssim |sv + tu|$. By Lemma 6.6, this follows if we can show $\langle sv, tu \rangle \geq -\frac{1}{2}|sv||tu|$. Indeed, we have

$$\begin{aligned} \langle sv, tu \rangle &= st \left(\langle v_1, u_1 \rangle + \langle v_1, u_2 \rangle + \langle v_2, u_1 \rangle + \langle v_2, u_2 \rangle \right), \\ &\geq st(|v_1|^2 - \langle v_1, u_1 - v_1 \rangle - C\epsilon |v_1|^2) \geq st(1 - C\epsilon)|v_1|^2 \geq 0. \end{aligned}$$

This completes the proof for $v = v_x$. If instead $v = v_y \in V$, then $Dg(z) \cdot v_y = v_y \cdot \frac{\partial g}{\partial y}(z)$ and the result follows from (6.11) in Lemma 6.8. ∎

Lemma 6.10 (Vertical Estimates)**.** *Let z, z', v, p, m be as in Corollary 6.9. If $\sum_{k=p}^{m} \epsilon_k'(f_k(x'))^2 \leq C\epsilon$ and* $\mathrm{Angle}(T\Sigma_p(f_p(x')), T\Sigma_m(f_m(x'))) \leq C\delta$*, we have*

$$\left| \sum_{k\geq 0} (\rho_k(y) - \rho_k(y')) Df_k(x') \cdot v \right| \leq C\delta |Df_p(x') \cdot v|, \tag{6.24}$$

$$\left| \sum_{k\geq 0} \rho_k(y) D(R_k(x') \cdot y) \cdot v - \rho_k(y') D(R_k(x') \cdot y') \cdot v \right| \leq C\delta |Df_p(x') \cdot v|, \tag{6.25}$$

$$\left| \frac{\partial g}{\partial y}(x', y) - \frac{\partial g}{\partial y}(x', y') \right| \leq C\delta \left| \frac{\partial g}{\partial y}(x', y) \right|. \tag{6.26}$$

Proof. We being by proving (6.24). First, since $D\sigma_k$ is $(1+C\epsilon)$-bi-Lipschitz for any k, we have

$$|Df_p(x')\cdot v - Df_{p+1}(x')\cdot v| \le C\epsilon |Df_p(x')\cdot v|.$$

This implies

$$\begin{aligned}\left|\sum_{k\ge 0}\rho_k(y)Df_k(x')\cdot v - Df_p(x')\cdot v\right| &\le \sum_{k\ge 0}\rho_k(y)|Df_k(x')\cdot v - Df_p(x')\cdot v| \\ &\le C\epsilon|Df_p(x')|\end{aligned} \tag{6.27}$$

because $\rho_k(y)\neq 0$ only for $k=p,p+1$. An identical argument gives (6.27) with y replaced by y' and p replaced by m. We now want a similar bound for $|Df_m(x')\cdot v - Df_p(x')\cdot v|$. For ease of notation, define $u = Df_p(x')\cdot v$ and $w = \prod_{k=p}^{m-1} D\sigma_k(f_k(x'))\cdot u$. We can then write

$$|Df_m(x')\cdot v - Df_p(x')\cdot v| = \left|\left[\prod_{k=p}^{m-1} D\sigma_k(f_k(x'))\right]\cdot u - u\right| = |w-u|.$$

The fact that $\sum_{k=p}^{m}\epsilon_k'(f_k(x'))^2 \le C\epsilon^2$ means

$$\left|\prod_{k=p}^{m-1} D\sigma_k(f_k(x'))\right| \le \prod_{k=p}^{m-1} 1 + CM_0^2\epsilon_k'(f_k(x'))^2 \le 1 + CM_0^2\epsilon^2. \tag{6.28}$$

Hence, $||w|-|u|| \le CM_0^2\epsilon^2|u|$. Since $w\in T\Sigma_m(f_m(x'))$, $u\in T\Sigma_p(f_p(x'))$, and we've assumed that $\text{Angle}(T\Sigma_p(f_p(x')), T\Sigma_m(f_m(x'))) \le C\delta$, we have $\text{Angle}(w,u)\le C\delta$ and it follows that

$$|w-u| \lesssim_{M_0} \delta|u| \tag{6.29}$$

as long as δ and ϵ are sufficiently small. Finally, using (6.27) and (6.29), we see

$$\begin{aligned}&\left|\sum_{k\ge 0}(\rho_k(y)-\rho_k(y'))Df_k(x')\cdot v\right| \\ &\qquad \le \left|\sum_{k\ge 0}\rho_k(y)Df_k(x')\cdot v - Df_p(x')\cdot v\right| + \left|\sum_{k\ge 0}\rho_k(y')Df_k(x')\cdot v - Df_m(x')\cdot v\right| \\ &\qquad\quad + |Df_p(x')\cdot v - Df_m(x')\cdot v| \\ &\qquad \le C\epsilon|Df_p(x')\cdot v| + C\epsilon|Df_m(x')\cdot v| + C\delta|Df_p(x')\cdot v| \\ &\qquad \le C\delta|Df_p(x')\cdot v|.\end{aligned}$$

The proof of (6.25) follows from (6.17) and (6.24). Indeed,

$$\begin{aligned}&\left|\sum_{k\ge 0}\rho_k(y)D(R_k(x')\cdot y)\cdot v - \rho_k(y')D(R_k(x')\cdot y')\cdot v\right| \\ &\qquad\qquad \le C\epsilon|Df_p(x')\cdot v| + C\delta|Df_p(x')\cdot v| + C\epsilon|Df_m(x')\cdot v| \\ &\qquad\qquad \le C\delta|Df_p(x')\cdot v|.\end{aligned}$$

Finally, we prove (6.26). We have

$$\begin{aligned}\left|\frac{\partial g}{\partial y}(x',y) - \frac{\partial g}{\partial y}(x',y')\right| &\le \sum_{k\ge 0}\left|\frac{\partial\rho_k}{\partial y}(y)\left\{f_k(x') + R_k(x')\cdot y\right\}\right| + \left|\frac{\partial\rho_k}{\partial y}(y')\left\{f_k(x') + R_k(x')\cdot y'\right\}\right| \\ &\quad + |(\rho_k(y)-\rho_k(y'))R_k(x')\cdot e_{d+1}| \\ &=: \delta_1+\delta_2+\delta_3.\end{aligned}$$

We first handle δ_1 and δ_2. We have

$$\delta_1 \leq \left|\frac{\partial \rho_p}{\partial y}(y)\right| \left(|f_p(x') - f_{p+1}(x')| + |R_p(x') - R_{p+1}(x')||y|\right) \leq \frac{C}{r_p}(C\epsilon r_p + C\epsilon r_p) \leq C\epsilon$$

by (2.10) and (2.11). A nearly identical calculation gives the same bound for δ_2. We now handle δ_3. First, notice that

$$\left|\sum_{k\geq 0} \rho_k(y) R_k(x') \cdot e_{d+1} - R_p(x') \cdot e_{d+1}\right| \leq \sum_{k\geq 0} |\rho_k(y)||R_k(x') \cdot e_{d+1} - R_p(x') \cdot e_{d+1}| \leq C\epsilon \tag{6.30}$$

by (2.11). Because $R_k(x')$ is an isometry such that $R_k(x')(T\Sigma_0(x')) = T\Sigma_k(f_k(x))$, $R_k(x') \cdot e_{d+1}$ is the unit normal to $T\Sigma_k(f_k(x'))$ so that

$$|R_p(x') \cdot e_{d+1} - R_m(x') \cdot e_{d+1}| \leq C \operatorname{Angle}(T\Sigma_p(f_p(x')), T\Sigma_m(f_m(x'))) \leq C\delta. \tag{6.31}$$

Finally, (6.30) and (6.31) imply

$$\begin{aligned}\delta_3 &\leq \left|\sum_{k\geq 0} \rho_k(y) R_k(x') \cdot e_{d+1} - R_p(x') \cdot e_{d+1}\right| + \left|\sum_{k\geq 0} \rho_k(y') R_k(x') \cdot e_{d+1} - R_m(x') \cdot e_{d+1}\right| \\ &\quad + |R_p(x') \cdot e_{d+1} - R_m(x') \cdot e_{d+1}| \\ &\leq C\epsilon + C\epsilon + C\delta \leq C\delta \left|\frac{\partial g}{\partial y}(y')\right|.\end{aligned}$$

where the final inequality uses (6.19). ■

Corollary 6.11. *Let z, z' be as in Lemma 6.10. Then for any vector $v \in H \cup V$, we have*

$$|Dg(x', y) \cdot v - Dg(x', y') \cdot v| \leq C\delta |Dg(x', y) \cdot v|$$

Proof. Suppose first that $v = v_x \in H$. Then using (6.20), we compute

$$\begin{aligned}&|Dg(x', y) \cdot v_x - Dg(x', y') \cdot v_x| \qquad (6.32)\\ &\quad = \left|\sum_{k\geq 0} (\rho_k(y) - \rho_k(y')) Df_k(x') \cdot v_x + \rho_k(y) D(R_k(x') \cdot y) \cdot v_x - \rho_k(y') D(R_k(x') \cdot y') \cdot v_x\right| \qquad (6.33)\\ &\quad \leq C\delta |Df_p(x') \cdot v_x| \leq C\delta |Dg(x', y) \cdot v_x| \\ &\quad \leq C\delta(1 + C\delta)|Dg(x, y) \cdot v_x| \leq C\delta |Dg(x, y) \cdot v_x| \qquad (6.34)\end{aligned}$$

using (6.24) and (6.25) in the first inequality, (6.17) in the second, and (6.21) in the third. If instead $v = v_y \in V$, then $Dg(x', y) = v_y \cdot \frac{\partial g}{\partial y}(x', y)$ and the result follows from (6.26) and (6.11). ■

Using Corollaries 6.9 and 6.11, we can prove Proposition 6.7.

Proof of Proposition 6.7. Let $z' = (x', y') \in G_z^{M_0}$. We will show that for any vector $v \in H \cup V$,

$$|Dg(x, y) \cdot v - Dg(x', y) \cdot v| \leq C\delta |Dg(x, y) \cdot v|. \tag{6.35}$$

The set $G_z^{M_0}$ is designed exactly so that $z' \in G_z^{M_0}$ implies that the hypotheses of Lemmas 6.8 and 6.10 are satisfied. Hence, we can apply Corollaries 6.9 and 6.11 so that

$$\begin{aligned}|Dg(x, y) \cdot v - &Dg(x', y') \cdot v| \\ &\leq |Dg(x, y) \cdot v - Dg(x', y) \cdot v| + |Dg(x', y) \cdot v - Dg(x', y') \cdot v| \\ &\leq C\delta |Dg(x, y) \cdot v| + C\delta |Dg(x', y) \cdot v| \\ &\leq C\delta |Dg(x, y) \cdot v|.\end{aligned}$$

By decomposing an arbitrary $v' \in \mathbb{R}^{d+1}$ as $v' = v_x + v_y$ where $v_x \in H$ and $v_y \in V$, we write

$$
\begin{aligned}
|Dg(x,y)\cdot v' - &Dg(x',y')\cdot v'| \\
&\leq |Dg(x,y)\cdot v_x - Dg(x',y')\cdot v_x| + |Dg(x,y)\cdot v_y - Dg(x',y')\cdot v_y| \\
&\leq C\delta(|Dg(x,y)\cdot v_x| + |Dg(x,y)\cdot v_y|) \\
&\leq C\delta|Dg(x,y)\cdot v'|,
\end{aligned} \tag{6.36}
$$

Where the final inequality follows from an application of the reverse triangle inequality in Lemma 6.6. We justify the application of the lemma by looking at the equations (6.20) and (6.18). These imply that the vector $Dg(x,y)\cdot v_x$ is nearly parallel to $T\Sigma_k(x)$ while the vector $Dg(x,y)\cdot v_y$ is nearly perpendicular to $T\Sigma_k(x)$ for some value of k where the deviations described are on the order of ϵ. This implies $|\langle Dg(x,y)\cdot v_x, Dg(x,y)\cdot v_y\rangle| \leq \frac{1}{2}|Dg(x,y)\cdot v_x|\cdot|Dg(x,y)\cdot v_y|$ so that the lemma applies. With this, we now compute,

$$
\begin{aligned}
|Dg(z')\cdot Dg(z)^{-1}\cdot v' - v'| = |[Dg(z') - Dg(z)]\cdot Dg(z)^{-1}\cdot v'| &\leq C\delta|Dg(z)\cdot Dg(z)^{-1}\cdot v'| \\
&= C\delta|v'|. \qquad \blacksquare
\end{aligned}
$$

This concludes the computations we need to bound the change in Dg. By integrating Dg over paths in a quasiconvex domain Ω, we get a companion result to Proposition 6.7 which roughly states that the map $g|_\Omega$ is a $(1 + C\delta)$-bi-Lipschitz perturbation of $Dg(z_0)$ for any $z_0 \in \Omega$. More precisely, for any $z \in \mathbb{R}^{d+1}$ define

$$
L_{z_0}(z) = z_0 + Dg(z_0)(z - z_0). \tag{6.37}
$$

This is the affine transformation which approximates g near z_0. Define

$$
\varphi_{z_0} = g \circ L_{z_0}^{-1} \tag{6.38}
$$

Proposition 6.12. *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a quasiconvex domain with constant M_0 such that $\Omega \subseteq G_{z_0}^{M_0}$ for some $z_0 \in \Omega$ and M_0, ϵ be as in Proposition 6.7. Then the map $\varphi_{z_0} : L_{z_0}(\Omega) \to g(\Omega)$ is $(1 + C\delta)$-bi-Lipschitz and*

$$
|D\varphi_{z_0}(w) - I| \leq C\delta \tag{6.39}
$$

for all $w \in L_{z_0}(\Omega)$.

Proof. Because $w \in L_{z_0}(G_{z_0}^{M_0})$ by assumption, we get

$$
D\varphi_{z_0}(w) = Dg(L_{z_0}^{-1}(w))\cdot DL_{z_0}^{-1}(w) = Dg(z)\cdot Dg(z_0)^{-1}
$$

for $z' = L_{z_0}^{-1}(w) \in G_{z_0}^{M_0}$. Equation (6.39) follows from (6.7).

To prove that φ_{z_0} is $(1 + C\delta)$-bi-Lipschitz, let $\gamma : [0,1] \to \mathbb{R}^{d+1}$ be a path with $\gamma(0) = z_0$, $\gamma(1) = z$, and $\ell(\gamma) \lesssim_{M_0} |z_0 - z|$. Put $\tilde{\gamma}(t) = L_{z_0}(\gamma(t))$ and $w_0 = L_{z_0}(z) = z$. Observe that

$$
L_{z_0}^{-1}(w) = z_0 + Dg(z_0)^{-1}(w - z_0).
$$

We estimate

$$
\begin{aligned}
|\varphi_{z_0}(w) - \varphi_{z_0}(w_0)| &= \left|\int_0^1 D(g \circ L_{z_0}^{-1})(\tilde{\gamma}(t))\cdot\tilde{\gamma}'(t)dt\right| \\
&= \left|\int_0^1 Dg(\gamma(t))\cdot DL_{z_0}^{-1}(\tilde{\gamma}(t))\cdot\tilde{\gamma}'(t)dt\right| \\
&= \left|\int_0^1 Dg(\gamma(t))\cdot Dg(z_0)^{-1}\cdot\tilde{\gamma}'(t)dt\right| \\
&= \left|w - w_0 + \int_0^1 \left[Dg(\gamma(t))\cdot Dg(z_0)^{-1} - I\right]\cdot\tilde{\gamma}'(t)dt\right|.
\end{aligned}
$$

Using the fact that $\gamma(t) \in G_{z_0}^{M_0}$ for all t, Proposition 6.7 implies, on one hand

$$\begin{aligned}|\varphi_{z_0}(w) - \varphi_{z_0}(w_0)| &\leq |w - w_0| + \int_0^1 \left|Dg(\gamma(t)) \cdot Dg(z_0)^{-1} - I\right| \cdot |\tilde{\gamma}'(t)| dt \\ &\leq |w - w_0| + C\delta |Dg(z_0)| \cdot \ell(\gamma) \\ &\leq (1 + C\delta)|w - w_0|.\end{aligned}$$

On the other,

$$\begin{aligned}|\varphi_{z_0}(w) - \varphi_{z_0}(w_0)| &\geq |w - w_0| - \int_0^1 \left|Dg(\gamma(t)) \cdot Dg(z_0)^{-1} - I\right| \cdot |\tilde{\gamma}'(t)| dt \\ &\geq |w - w_0| - C\delta |Dg(z_0)| \cdot \ell(\gamma) \\ &\geq (1 - C\delta)|w - w_0|\end{aligned}$$

where the final inequality on both hands comes from the fact that $|w' - w| = |Dg(z) \cdot (z' - z)| \leq |Dg(z)| \cdot |z - z'|$ and our assumption that $\ell(\gamma) \lesssim_{M_0} |z - z'|$. ■

7 Further questions

Question 7.1. Can one find a disjoint decomposition rather than one of bounded overlap in Theorems B and C?

In the construction given in this paper for Theorems B and C, the only overlap between domains occurs in intersections between $(1 + C\delta)$-bi-Lipschitz images of Whitney cubes of comparable side length inside the “buffer zones”. It seems plausible that one could devise a different scheme to divide the space between the disjoint “core” stopping time domains into Lipschitz graph domains.

Similarly, it seems plausible that one could obtain a similar result by modifying the methods of the proof to prove a version of Theorem A with assumption (ii) removed (see Remark 3.6).

Question 7.2. Can Theorem B be extended to general lower-content d regular sets?

It seems possible that the necessary tools to handle this extension to non-Reifenberg flat sets are present in the β-number estimates in [AS18] and [Hyd22b]. The technical disconnect between this paper and those ones is caused by the “smoothing” procedure needed there in defining the stopping time regions makes this generalization non-obvious to the author. It seems that a proof would require new ideas.

A Graph coronizations for Reifenberg flat sets

The goal of this section is to provide a proof of Proposition 2.18 which states that there exist (sufficiently small in terms of d) constants $\epsilon, \delta > 0$ such that Reifenberg flat sets admit (M, ϵ, δ)-graph coronizations.

Reifenberg flat sets are a subset of a more general class of sets called *lower content d-regular sets* studied by Azzam and Schul [AS18] and later Hyde [Hyd22a] as a class of objects for d-dimensional traveling salesman results.

Definition A.1 (lower content d-regularity)**.** A set $E \subseteq \mathbb{R}^{d+1}$ is said to be *lower content d-regular* in a ball $B(x, r)$ if there exists a constant $c > 0$ and $r_B > 0$ such that

$$\mathscr{H}^d_\infty(E \cap B(x, r)) \geq cr^d \text{ for all } x \in E \cap B \text{ and } r \in (0, r_B).$$

A set E is *lower content d-regular* if there exists a constant c such that E is lower content regular with constant c in every ball centered on E.

Since a Reifenberg flat set Σ satisfies $b\beta_\Sigma(B) \le \epsilon$ in every ball by definition, the only remaining requirements for the existence of a graph coronization are control over $\beta_\Sigma^{d,1}$-squared sums and control over the frequency of angle turning of well-approximating planes. The necessary control over β-sums is contained in the following traveling salesman theorems formulated for general lower content regular sets:

Theorem A.2 ([Hyd22a] Theorem 1.6)**.** *Let H be a Hilbert space and $1 \le d < \dim(H)$, $1 \le p < p(d)$, $C_0 > 1$, and $A > 10^5$. Let $E \subseteq H$ be a lower content d-regular set with regularity constant c and Christ-David cubes $\mathscr{D}$. There exists $\epsilon > 0$ small enough so that the following holds. Let $Q_0 \in \mathscr{D}$ and*

$$\beta_{E,C_0,d,p}(Q_0) = \ell(Q_0)^d + \sum_{Q \subseteq Q_0} \beta_E^{d,p}(C_0 B_Q)^2 \ell(Q)^d.$$

Then

$$\beta_{E,C_0,d,p}(Q_0) \lesssim_{A,d,c,p,C_0,\epsilon} \mathcal{H}^d(Q_0) + \mathrm{BWGL}(Q_0, A, \epsilon). \tag{A.1}$$

Theorem A.3 ([Hyd22a] Theorem 1.7)**.** *Let H be a Hilbert space, $1 \le d < \dim(H)$, $1 \le p < \infty$, $A > 1$, $\epsilon > 0$, and $C_0 > 2\rho^{-1}$ where ρ is as in the construction of the Christ-David lattice $\mathscr{D}$. Let $E \subseteq H$ be lower content d-regular with regularity constant c and Christ-David cubes $\mathscr{D}$. For $Q_0 \in \mathscr{D}$, we have*

$$\mathcal{H}^d(Q_0) + \mathrm{BWGL}(Q_0, A, \epsilon) \lesssim_{A,d,c,C_0,\epsilon} \beta_{E,C_0,d,p}(Q_0).$$

If E is (ϵ, d)-Reifenberg flat, then the BWGL terms above vanish and (A.1) gives a Carleson packing condition for the content beta number sum reminiscent of the strong geometric lemma for uniformly rectifiable sets from which we will conclude the desired β^2 sum control.

We will require small technical tweaks of the stopping time machinery of Azzam and Schul on Reifenberg flat sets. We review the necessary definitions here, but refer to [AS18] sections 5-8 for a full treatment of the construction.

Definition A.4 (d-dimensional traveling salesman stopping time)**.** We fix constants $0 < \epsilon \ll \alpha^4$ with $\alpha(d), \epsilon(d)$ to be chosen sufficiently small in terms of δ as required in [AS18] . For any cube $Q \in \mathscr{D}$, we define a stopping time region S_Q^α by adding cubes $R \subseteq Q$ to S_Q if

(i) $R^{(1)} \in S_Q^\alpha$,

(ii) $\mathrm{Angle}(P_U, P_Q) < \alpha$ for any sibling U of R (including R itself).

For any collection of cubes $\mathscr{Q}$, define a distance function

$$d_{\mathscr{Q}}(x) = \inf\{\ell(Q) + \mathrm{dist}(x, Q) : Q \in \mathscr{Q}\}.$$

For any $Q \in \mathscr{D}$, define

$$d_{\mathscr{Q}}(Q) = \inf_{x \in Q} d_{\mathscr{Q}}(x) = \inf\{\ell(R) + \mathrm{dist}(Q, R) : R \in \mathscr{Q}\}.$$

We let $m(S)$ be the set of minimal cubes of S, those which have no children contained in S and define

$$z(S) = Q(S) \setminus \bigcup_{Q \in m(S)} Q.$$

Let

$$\mathrm{Stop}(-1) = \mathscr{D}_0$$

and fix a small constant $\tau \in (0,1)$. Suppose we have defined $\mathrm{Stop}(N-1)$ for some integer $N \ge 0$ and define

$$\mathrm{Layer}(N) = \bigcup \{S_Q^\alpha : Q \in \mathrm{Stop}(N-1)\}.$$

We then set $\mathrm{Up}(-1) = \varnothing$ and put

$$\begin{aligned}\mathrm{Stop}(N) &= \{Q \in \mathscr{D} : Q \text{ maximal such that } Q \text{ has a sibling } Q' \text{ with } \ell(Q') < \tau d_{\mathrm{Layer}(N)}(Q')\},\\ \mathrm{Up}(N) &= \mathrm{Up}(N-1) \cup \{Q \in \mathscr{D} : Q \supset R \text{ for some } R \in \mathrm{Stop}(N) \cup \mathrm{Layer}(N)\}\end{aligned}$$

[AS18] Lemma 5.5 says that, in fact

$$\mathrm{Up}(N) = \{Q \in \mathscr{D} : Q \not\subset R \text{ for any } R \in \mathrm{Stop}(N)\}.$$

Essentially, $\mathrm{Layer}(N)$ is a layer of stopping time regions S_Q^α beginning at the stopped cubes of the previous generation and continuing until reaching a cube R with a child R' such that $\mathrm{Angle}(P_Q, P_{R'}) > \alpha$. $\mathrm{Stop}(N)$ is formed by taking a "smoothing" of $\mathrm{Layer}(N)$ that ensures that nearby minimal cubes in $\mathrm{Stop}(N)$ are of similar size. One forms a CCBP from the centers and $b\beta$-minimizing planes of cubes in $\mathrm{Up}(N)$ which gives a surface Σ_N for any $N \geq 0$ which converges to Σ as $N \to \infty$. Azzam and Schul give tools for proving bounds on the degree of stopping in this construction in the following lemma

Lemma A.5 ([Hyd22a] Lemma 4.4 (5))**.** *Let Σ be (ϵ, d)-Reifenberg flat and $\mathscr{D}$ a Christ-David lattice for Σ. Let $N \geq 0$. For any $Q_0 \in \mathscr{D}$,*

$$\sum_{N \geq 0} \sum_{\substack{Q \in \mathrm{Stop}(N) \\ Q \subseteq Q_0}} \ell(Q)^d \lesssim_{d,\alpha,\epsilon} \mathcal{H}^d(Q_0)$$

Proof of Proposition 2.18. Fix $M \geq 1$, and $\epsilon, \alpha > 0$ sufficiently small in terms of M, d, n determined by Lemma A.5 and Theorem A.2 and let $\delta = 100\alpha$. Let $\mathscr{D}$ be a Christ-David lattice for Σ and let $\{P_Q\}_{Q \in \mathscr{D}}$ be a family of d-planes such that $x_Q \in P_Q$ and $\beta_\Sigma^{d,1}(MB_Q, P_Q) \leq 2\beta_\Sigma^{d,1}(MB_Q)$. Fix $Q_0 \in \mathscr{D}$ and form a collection of stopping time regions $\mathscr{F} = \{S_Q\}$ contained within Q_0 satisfying the stopping conditions Items (ii) and (iii) of Definition 2.15. We set $\mathscr{G} = \mathscr{D}$, $\mathscr{B} = \varnothing$. To prove that $\mathscr{C} = (\mathscr{G}, \mathscr{B}, \mathscr{F})$ is an (M, ϵ, δ)-graph coronization, we only need to show that $\mathscr{C}$ is a coronization, i.e., there exists a constant $C(M, \epsilon, \delta, d)$ such that

$$\sum_{S \in \mathscr{F}} \ell(Q(S))^d \leq C\mathcal{H}^d(Q_0).$$

Define

$$\begin{aligned}S_\delta &= \{Q \in \mathscr{D} : \exists S \in \mathscr{F},\ Q \in \mathrm{Stop}(S),\ \mathrm{Angle}(P_Q, P_{Q(S)}) > \delta\},\\ S_\beta &= \left\{Q \in \mathscr{D} : \exists S \in \mathscr{F},\ Q \in \mathrm{Stop}(S), \sum_{Q \subseteq R \subseteq Q(S)} \beta_\Sigma^{d,1}(MB_R)^2 > \eta\right\}.\end{aligned}$$

It suffices to show that $\sum_{Q \in S_\delta \cup S_\beta} \ell(Q)^d \leq C\mathcal{H}^d(Q_0)$. We define

$$\mathrm{Stop}(-1, \delta) = \{Q_0\},$$

and, given $\mathrm{Stop}(N-1, \delta)$ for some integer $N \geq 0$, we define

$$\mathrm{Stop}(N, \delta) = \{R \in S_\delta : R \text{ maximal such that } R \subseteq Q \in \mathrm{Stop}(N-1, \delta)\}.$$

With this, we have

$$S_\delta = \bigcup_{N \geq 0} \mathrm{Stop}(N, \delta).$$

We will use this to show that $\sum_{Q \in S_\delta} \ell(Q)^d \leq C\mathscr{H}^d(Q_0)$.

Fix $Q \in \mathrm{Stop}(N,\delta)$ and let $x \in Q \setminus z(S)$. Then there exists $R \in S_\delta,\ R \subset Q$ such that $x \in R$ and, since $\delta \geq 100\alpha$, there exists a cube $R' \in \mathrm{Stop}(K)$ for some $K \geq 0$ such that $R \subseteq R' \subseteq Q$. Set

$$\mathrm{Stop}(Q) = \left\{ R \in \mathscr{D} : R \text{ maximal such that } R \in \bigcup_{N \geq 0} \mathrm{Stop}(N) \text{ and } R \subseteq Q \right\}.$$

The above argument has shown that $Q \setminus z(S_Q) \subseteq \cup_{R \in \mathrm{Stop}(Q)} R$. We see

$$\ell(Q)^d \lesssim_d \sum_{R \in \mathrm{Stop}(Q)} \ell(R)^d + \mathscr{H}^d(z(S_Q)).$$

This means

$$\begin{aligned}
\sum_{Q \in S_\delta} \ell(Q)^d &= \sum_{N \geq 0} \sum_{Q \in \mathrm{Stop}(N,\delta)} \ell(Q)^d \\
&\lesssim_d \sum_{N \geq 0} \sum_{Q \in \mathrm{Stop}(N,\delta)} \left(\sum_{R \in \mathrm{Stop}(Q)} \ell(R)^d + \mathcal{H}^d(z(S_Q)) \right) \\
&\lesssim \mathcal{H}^d(Q_0) + \sum_{K \geq 0} \sum_{R \in \mathrm{Stop}(K)} \ell(R)^d \\
&\lesssim_{d,\delta,\epsilon} \mathcal{H}^d(Q_0)
\end{aligned}$$

where the penultimate line follows from the fact that $\mathrm{Stop}(Q) \cap \mathrm{Stop}(Q') = \varnothing$ for $Q, Q' \in S_\delta,\ Q \neq Q'$, and the final line follows from Lemma A.5.

We now show that $\sum_{Q \in S_\beta} \ell(Q)^d \leq C \mathcal{H}^d(Q_0)$. We have

$$\begin{aligned}
\sum_{Q \in S_\beta} \ell(Q)^d &\leq \sum_{Q \in S_\beta} \ell(Q)^d \left[\epsilon^{-2} \sum_{\substack{Q \subseteq R \subseteq Q(S) \\ Q \in S \in \mathscr{F}}} \beta_\Sigma^{d,1}(MB_R)^2 \right] = \epsilon^{-2} \sum_{R \in \mathscr{D}} \beta_\Sigma^{d,1}(MB_R)^2 \sum_{\substack{Q \text{ maximal } \subseteq R \\ Q \in S_\beta}} \ell(Q)^d, \\
&\lesssim \epsilon^{-2} \sum_{R \in \mathscr{D}} \beta_\Sigma^{d,1}(MB_R)^2 \ell(R)^d \lesssim_{d,\eta} \mathcal{H}^d(Q_0) \qquad \blacksquare
\end{aligned}$$

using Theorem A.2 in the last line.

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