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Empiricism, Probability, and Knowledge of Arithmetic: A Preliminary Defense

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Abstract

The topic of this paper is our knowledge of the natural numbers, and in particular, our knowledge of the basic axioms for the natural numbers, namely the Peano axioms. The thesis defended in this paper is that knowledge of these axioms may be gained by recourse to judgements of probability. While considerations of probability have come to the forefront in recent epistemology, it seems safe to say that the thesis defended here is heterodox from the vantage point of traditional philosophy of mathematics. So this paper focuses on providing a preliminary defense of this thesis, in that it focuses on responding to several objections. Some of these objections are from the classical literature, such as Frege's concern about indiscernibility and circularity (§ 2.1), while other are more recent, such as Baker's concern about the unreliability of small samplings in the setting of arithmetic (§ 2.2). Another family of objections suggests that we simply do not have access to probability assignments in the setting of arithmetic, either due to issues related to the ω-rule (§ 3.1) or to the non-computability and non-continuity of probability assignments (§ 3.2). Articulating these objections and the responses to them involves developing some non-trivial results on probability assignments (Appendix A-Appendix C), such as a forcing argument to establish the existence of continuous probability assignments that may be computably approximated (Theorem 4 Appendix B). In the concluding section, two problems for future work are discussed: developing the source of arithmetical confirmation and responding to the probabilistic liar.

Keywords: Philosophy of Mathematics, Inductive Logic
03A, 03C62, 03B48

Contents

1. Introduction

The topic of this paper is the tenability of a certain type of empiricism about our knowledge of the Peano axioms. The Peano axioms constitute the standard contemporary axiomatization of arithmetic, and they consist of two parts, a set of eight axioms called *Robinson's Q*, which ensure the correctness of the addition and multiplication tables, and the principle of *mathematical induction*, which says that if zero has a given property and $n + 1$ has it whenever n has it, then all natural numbers have this property.[1] The type of empiricism about the Peano axioms which I want to consider holds that arithmetical knowledge is akin to the knowledge by which we infer from the past to the future, or from the observed to the unobserved. It is not uncommon today to hold that such inductive inferences can be rationally sustained by appeal to informed judgments of probability. The goal of this paper is to defend an empiricism which contends that judgements of probability can help us to secure knowledge of the Peano axioms.

This empiricism merits our attention primarily because standard accounts

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[1] More formally, the axioms of Robinson's Q are the following:

(Q1) $Sx \neq 0$ (Q2) $Sx = Sy \rightarrow x = y$ (Q3) $x \neq 0 \rightarrow \exists w\, x = Sw$
(Q4) $x + 0 = x$ (Q5) $x + Sy = S(x + y)$ (Q6) $x \cdot 0 = 0$
(Q7) $x \cdot Sy = x \cdot y + x$ (Q8) $x \leq y \leftrightarrow \exists z\, x + z = y$

For the formal result that indicates that Robinson's Q ensures the correctness of the addition and multiplication tables (among other things), see Proposition 1 in § 2.2. The Peano axioms then consist of Robinson's Q along with each instance of the mathematical induction schema, wherein $\varphi(x)$ ranges over first-order formulas with one free variable x:

$$[\varphi(0) \,\&\, \forall\, y\, \varphi(y) \rightarrow \varphi(S(y))] \rightarrow [\forall\, x\, \varphi(x)] \tag{1}$$

Hence, what I am describing in this paper as "the Peano axioms" is first-order Peano arithmetic, as described and studied in e.g. [1]. This is to be distinguished from second-order Peano arithmetic as studied in e.g. [2], wherein the mathematical induction schema is replaced by single induction axiom and in which one additionally adds the comprehension schema, which says that every formula with a free first-order variable determines a second-order entity. Against the background of second-order logic with the comprehension schema, the mathematical induction axiom is equivalent to the version of the mathematical induction schema (1) wherein $\varphi(x)$ is permitted to range over second-order formulas with one free object variable x. Hence it makes no difference to the arguments presented here whether one works in a first- or second-order setting, and so for the sake of simplicity I keep here to the first-order setting.

of our knowledge of the Peano axioms face difficult problems, problems going above and beyond skepticism about knowledge of abstract objects. For instance, logicism suggests that knowledge of the Peano axioms may be based on knowledge of ostensibly logical principles– such as Hume's Principle– and the knowledge that the Peano axioms are representable within these logical principles (cf. [3] p. xiv, p. 131). The success of logicism thus hinges upon identifying a concept of representation that can sustain this inference, and as I have argued elsewhere, it seems that we presently possess no such concept ([4]). Alternatively, some structuralists have suggested that knowledge of the Peano axioms may be based on our knowledge of the class of finite structures. However, this account then owes us an explanation of why the analogues of the Peano axioms hold on the class of finite structures. For example, this account must tell us something about how we know that there's no finite structure that is larger than all the other finite structures (cf. [5] p. 112, [6] p. 159).

The second reason that this kind of empiricism about the Peano axioms merits our attention is that it has been suggested in different ways by both historical and contemporary sources. For instance, prior to Frege, a not uncommon view seems to have been that mathematical induction was an empirical truth akin to enumerative induction. This is why Kästner thought that mathematical induction was not fit to be an axiom ([7] pp. 426-428), and this is part of the background to Reid's begrudging concession that "necessary truths may sometimes have probable evidence" ([8] VII.ii.1).[2] However, some contemporary authors writing on the epistemology of arithmetic and arithmetical cognition have also suggested views related to this. For instance, Rips and Asmuth– two cognitive scientists who work on mathematical cognition– have recently considered the suggestion that "the theoretical distinction between math[ematical] induction and empirical induction" is not as clear as has been claimed, and that "even if the theoretical difference were secure, it wouldn't follow that the psychological counterparts of these operations are distinct" ([10] p. 205). Finally, in the course of their work on the epistemic propriety of randomized algorithms, Gaifman and Easwaran have both suggested the possibility of extending the notion of probability which they

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[2]Mill, by contrast, thought that proofs related to mathematical induction ought not be conceived of as instances of enumerative induction (cf. [9] vol. 7 p. 288 ff, Book III, Chapter 2). The history of this topic obviously deserves more discussion than I am able to give here.

employ to broader issues in the epistemology of arithmetic ([11] pp. 107-108, [12] p. 347).

The empiricism about arithmetical knowledge which I want to consider is centered around the notion of a probability assignment and the associated confirmation relation. A *probability assignment* is a mapping P from sentences in a fixed formal language to real numbers that satisfies the following three axioms (cf. [13] pp. 20 ff, [14] pp. 35 ff):

(P1) $P(\varphi) \geq 0$
(P2) $P(\varphi) = 1$ if $\models \varphi$
(P3) $P(\varphi \vee \psi) = P(\varphi) + P(\psi)$ if $\models \neg(\varphi \,\&\, \psi)$

In what follows, all the probability assignments under consideration shall be assumed to have a domain that includes all the sentences in the language of the Peano axioms. Further, it shall be assumed that the consequence relation $\models$ in axioms P2-P3 is the logical consequence relation from first-order logic, so that $\models \varphi$ holds if and only if φ is true on all models. The notion of confirmation is then defined as an increase of the probability of a hypothesis conditional on evidence relative to the background knowledge. That is, hypothesis h is said to be *confirmed* by evidence e *relative* to background knowledge K if $P(h|e \,\&\, K) > P(h|K)$, assuming that the conditional probabilities $P(h|e \,\&\, K), P(h|K)$ are defined, where these conditional probabilities are given by the usual equation $P(h'|e') = \frac{P(h' \,\&\, e')}{P(e')}$. Further, in the case where confirmation occurs, the quantity $P(h|e \,\&\, K) - P(h|K)$ is said to be the *degree of confirmation*.[3]

Since there are two parts to the Peano axioms– namely Robinson's Q and mathematical induction– so there are two complementary forms of empiricism which I want to consider here, which I call inceptive empiricism and amplificatory empiricism. *Amplificatory empiricism* contends that one is justified in inferring from the antecedent of an instance of mathematical induction to its consequent, relative to the background knowledge consisting of the conjunction of the eight axioms of Robinson's Q, because the consequent is confirmed by the antecedent relative to this background knowledge. Since in conjunction with the eight axioms of Robinson's Q, the consequent of such an instance (the claim that all numbers have a given property) logically

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[3]There are many alternative measures of degree of confirmation (cf. [15]). The important differences between these measures do not, as far as I can discern, affect the arguments presented in this paper.

implies its antecedent (the claim that zero has this property and that $n + 1$ does whenever n does), it then follows that the consequent is confirmed by the antecedent (relative to the background knowledge consisting of the conjunction of the eight axioms of Robinson's Q) if the conjunction of these eight axioms and the antecedent is assigned a non-zero probability strictly less than the probability assigned to the conjunction of these eight axioms. Hence, were one to accept amplificatory empiricism, then there would be a straightforward connection between justification and probability, according to which one would be justified in inferring from the antecedent of an instance of mathematical induction to its consequent, against the background knowledge of the eight axioms of Robinson's Q, because of the probabilities assigned to these sentences.

Whereas amplificatory empiricism is a claim about how one may rationally proceed from Robinson's Q to mathematical induction, inceptive empiricism is a claim about how one may rationally arrive at Robinson's Q in the first place. In particular, *inceptive empiricism* is the contention that one is justified in inferring from several instances of the axioms of Robinson's Q to these axioms themselves because the axioms are confirmed by the conjunction of these several instances. For instance, Robinson's Q includes the axiom $\forall\, x, y\, [x \cdot S(y) = xy + x]$ wherein S denotes the successor function, and inceptive empiricism claims that confirmation justifies one in inferring *to* this axiom *from* several of its instances, such as $6 \cdot S(7) = 6 \cdot 7 + 6$. Let us call this type of confirmation, wherein a universal claim is confirmed by several of its instances, *instance confirmation*. Further, in the case where the claims in question are arithmetical in character (resp. physical in character) let us call this type of confirmation *arithmetical instance confirmation* (resp. *physical instance confirmation*). So inceptive empiricism contends that the axioms of Robinson's Q can be justified by means of arithmetical instance confirmation.

The goal of this chapter is to defend these two forms of empiricism against several challenges, and in doing so to defend the tenability of the probabilistic account of the justification of the Peano axioms that is jointly provided by these two forms of empiricism. In § 2, I focus on direct challenges to these types of empiricism that are extant in the philosophy of mathematics literature, focusing on those from Frege's *Grundlagen* (§ 2.1) and Baker's recent article "Is there a Problem of Induction for Mathematics?" ([16]) (§ 2.2). In § 3, I turn to various challenges to the effect that we simply don't have access to probability assignments in the setting of arithmetic, either because of issues related to probabilistic analogues of the ω-rule (§ 3.1) or to

the non-computability and non-continuity of various probability assignments (§ 3.2). By articulating and responding to the most pressing objections to inceptive and amplificatory empiricism– objections which, if unanswerable, would render such empiricism unworthy of further investigation– this paper constitutes a prolegomenon to a thoroughgoing empiricism about the epistemology of arithmetic. In the concluding section § 4, I briefly describe two primary tasks to which a complete defense of this empiricism must attend, namely, the sources of arithmetical confirmation and the probabilistic liar.

2. Challenges from the Literature: Frege and Baker

2.1. Frege on Indiscernibility and Circularity

In the *Grundlagen*, Frege articulates two distinct objections to empirical foundations of arithmetic: an objection related to indiscernibility and an objection related to circularity. Given the signature importance of Frege within the philosophy of mathematics in general and the philosophy of arithmetic in particular, it is crucially important to adequately and fully respond to Frege's objections. Further, doing so will serve to highlight two distinctive features of the type of empiricism defended here: first, that it is more applicable to arithmetic than geometry; and second, that it must countenance a dissimilarity between knowledge of natural numbers and knowledge of real numbers.

In § 10 of the *Grundlagen*, Frege writes that "In ordinary inductions we often make good use of the proposition that every position in space and every moment in time is as good as every other," but that each "[...] number is formed in its own special way and has its own peculiarities [...]" ([17] § 10 p. 15). Later, Frege notes that geometrical points are "[...] not really particular at all, which is what enables them to stand as representatives of the whole of their kind" ([17] § 13 p. 20). It's natural to describe Frege's distinction in terms of indiscernibility. In a structure M, two elements a, b of M are said to be *indiscernible or homogenous with respect to a first-order formula* $\theta(x)$ which has only x free and no parameters, if one has that M models $\theta(a)$ if and only if M models $\theta(b)$. Further, a, b are said to be *indiscernible* or *homogenous* if they are so with respect to all such first-order formulas $\theta(x)$ which have only x free and no parameters (cf. Marker [18] § 4.1 pp. 115 ff). Further, it's obvious that any two natural numbers are not indiscernible since e.g. two is the second successor of zero whilst no other number has

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this property. But if one casts Euclidean geometry as a two-sorted structure consisting of points and lines, then it turns out that any two points are indiscernible. This is because there is an automorphism of the structure taking the one point to the other, and first-order properties are preserved under such automorphisms.

However, Frege's point might be thought to be weak since indiscernibility is a mere artifact of the particular languages employed to describe the structure. After all, if one gives every object in the structure a specific name by working in an expanded language with constants for every individual, then any structure can be made to possess no indiscernibility. In spite of this, the association of geometric structures with high levels of indiscernibility is surprisingly resilient. For instance, in apparent independence from Frege, Weyl too suggests that indiscernibility accounts for the intuitive character of geometry in contradistinction from arithmetic ([19] p. 7). In more recent work, Manders has suggested that part of what makes diagrammatic reasoning rigorous is that the features traditionally inferred from the diagram– the so-called *coexact* features– are invariant under slight perturbations of the diagram, and another way of putting this same point is that figures and their slight perturbations are indiscernible with respect to coexact features ([20] § 4.2.2 pp. 91 ff). Hence, even though indiscernibility is indeed language-dependent, many authors agree with Frege that the language in which we reason about geometric structures does admit high levels of indiscernibility.

So it seems wisest to concede Frege's claim that some mathematical structures admit high levels of indiscernibility, and rather to question Frege's claim that high levels of indiscernibility are necessary for "ordinary inductions" to be suasive. The first thing to note is that Frege seems to be operating under a conception of inductive inference as a species of deductive inference (cf. [17] § 3 footnote). So indiscernibility would be relevant because one would be using it as a premise to infer from the truth of a particular $\theta(a)$ to the truth of a general claim $\forall\ x\ \theta(x)$. So indiscernibility would then be the mathematical analogue of the "uniformity" premises that feature in traditional conceptions of inductive inference stemming from Hume and Mill. It's obvious that the probabilistic conception of inductive inference adopted here is simply *not* deductive in character, and hence does not rely on any uniformity or indiscernibility premise because it does not proceed deductively from premises.

However, not only is indiscernibility *not a precondition* for successful inductive inference, but on the probabilistic conception, known indiscernibility

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is *inimical* to successful inductive inference. For, traditionally, an important part of successful inductive inference is the principle that, all else being equal, the observation of further instances should lead to a higher degree of confirmation. But this principle is formally incompatible with known indiscernibility. For, suppose that the elements of some non-empty domain D are indiscernible with respect to a first-order formula $\theta(x)$. Assuming all this is part of the background knowledge K, one will have that $e_n \equiv \bigwedge_{i=1}^{n} D(a_i) \wedge \theta(a_i)$ and $h \equiv \forall\ x\ (D(x) \rightarrow \theta(x))$ are pairwise equivalent across K. Hence, for all $n > 1$ one will have that

$$P(h|e_n\ \&\ K) - P(h|K) = P(h|e_1\ \&\ K) - P(h|K) \qquad (2)$$

So, in the presence of known indiscernibility, the degree of confirmation conferred upon the universal hypothesis h by observing one positive instance e_1 is the same as that had by observing e_n for $n = 10,000$.

However, note that even though known indiscernibility is incompatible with "further observed instances confirm to a higher degree," known indiscernibility is not incompatible with "a universal hypothesis may be highly confirmed by the observation of several instances." For, if the prior plausibility $P(h|K)$ of the hypothesis is low, where e_n and h are as in the previous paragraph, then the degree of confirmation will be comparatively high, as we can see from the below calculation, using the fact that e_n and h are pairwise equivalent across K:

$$P(h|e_n\ \&\ K) - P(h|K) = 1 - P(h|K) \qquad (3)$$

So while there may be confirmation engendered by the observation of several instances, as equation (3) shows, there is no *further* degree of confirmation engendered by the observation of *further* instances, as equation (2) shows. Since this is such a basic part of successful inductive practice, it seems that the conclusion to draw is that the type of probabilistic empiricism considered here is inapplicable to mathematical settings like geometry where there are high levels of known indiscernibility. This of course is not to say that some other form of non-probabilistic empiricism about geometrical knowledge is not possible, but merely to say that there's an incompatibility between known indiscernibility, the probabilistic model of confirmation discussed here, and the thought that further observed instances engender higher degree of confirmation.

The second concern voiced in the *Grundlagen* by Frege is that a probabilistic foundation of arithmetic would be circular, since "how probability

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theory could possibly be developed without presupposing arithmetical laws is beyond comprehension" ([17] § 10 pp. 16-17). Presumably the concern is that probability assignments take real numbers as values, and hence in operating with probability assignments one will end up doing calculations like $P(h|e) = \frac{5}{12} > \frac{3}{12} = P(h'|e)$, which at first glance seems to involve just as much arithmetic as does $5 \cdot 12 > 3 \cdot 12$ and other consequences of Robinson's Q. So the thought would be that to secure knowledge of the latter via considerations of probability is patently circular: the only agent to whom knowledge of probability assignments would be available would be an agent who already possessed knowledge of Robinson's Q and other basic arithmetical truths. One response to this circularity objection would be to suggest that the agent need not be aware of these probabilistic calculations in order to thereby gain knowledge via these calculations– the probability calculations might be just another reliable mechanism by which agents acquire knowledge. However, while such broadly externalist considerations are entirely mainstream in epistemology, it seems that internalist intuitions dominate in philosophy of mathematics. Hence, it would be good if there were a response to the circularity objection that is compatible with the idea that agents acquire knowledge through probability by actively reasoning with probability assignments.

One such response would begin with the observation that even though real numbers are traditionally defined in terms of natural numbers and sequences and sets thereof, a fundamental lesson of mathematical logic in the 20th Century is that the real numbers are a much more tractable structure than the natural numbers. In particular, Tarski showed that the real field has a complete and decidable first-order theory and that it consequently has a computable model like the natural numbers ([18] Corollary 3.2.3 p. 85 and Corollary 3.3.16 p. 97). In particular, the smallest model of the first-order theory of the real numbers, namely the real algebraic numbers, has a computable presentation, just like the natural numbers (cf. [2] Theorem II.9.7 p. 98 or [21] Theorem 4.1 p. 18). Further, additional work shows that the field of real numbers is mutually interpretable with the Euclidean plane, which means that each can be recovered from the other– at least up to isomorphism– using only first-order resources ([22] p. 21 ff, [23] Theorem 21.1 p. 187, Corollary 21.2 p. 191). This work thus suggests a multitude of sources for the knowledge of real numbers that does not presuppose the natural numbers: this knowledge might be based on a knowledge of the axioms of real numbers regarded as an implicit definition thereof, or a knowledge

of the computable presentation of the real algebraic integers, or it might be based on visual knowledge of the Euclidean plane. To be sure, each of these sources of knowledge – implicit definition, algorithms, visualization– deserves further study and development. However, here it merely suffices to stress that none of these sources seems to patently presuppose the type of arithmetical knowledge codified in the axioms of Robinson's Q and the Peano axioms.[4]

Of course, this response presumes that the operations on probability assignments which one employs are recoverable in terms of the first-order field structure of the real numbers, i.e. in terms of addition and multiplication on real numbers. This suffices of course for the types of operations featuring in P1-P3 and other elementary probability calculations. Further, this response obviously presumes that knowledge of the properties and laws of addition and multiplication on real numbers does not presuppose knowledge of the properties and laws of addition and multiplication on natural numbers. One might object that this is implausible since each of the natural numbers *is* a real number, or at least can be canonically identified with a specific real number. However, any structure of cardinality less than or equal to the real numbers can be identified with a collection of real numbers, and no one thinks that knowledge of real numbers presupposes knowledge of all these other structures. The relevant difference between the natural numbers and such arbitrary structures is that the natural numbers have a second-order definition within the real numbers. And I concede for the sake of argument that knowledge of the second-order properties of the real numbers would presuppose a fairly extensive knowledge of the natural numbers, or at least of structures similar to the natural numbers. However, the response proffered above was merely related to the first-order structure of the real numbers, and hence this response incurs an additional obligation of distinguishing the source of knowledge of first-order properties of the real numbers from the source of knowledge of second-order properties of the real numbers.

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[4]That said, it might well be *more* contentious to say that knowledge of algorithms does not presuppose knowledge of the Peano axioms than it is to say that implicit definitions and visualization does not so presuppose knowledge of the Peano axioms. But this might depend on whether the knowledge of algorithms was necessarily viewed as being implemented, or was rather viewed more as knowledge of an abstract procedure.

2.2. Baker and the Exigencies of Arithmetical Sampling

In his recent essay "Is there a Problem of Induction for Mathematics?", Baker argues that arithmetical instance confirmation is *biased* in a way in which physical instance confirmation is not, and that this is due to the samplings in arithmetical instance confirmation being small (cf. [16] pp. 67-68). Recall from § 1 that *arithmetical instance confirmation* is our term for when a universal arithmetical hypothesis is confirmed by several of its instances, just as *physical instance confirmation* is our term for when a universal physical hypothesis is confirmed by several of its instances. Now there are at least two natural senses in which such samplings may be said to be small, which we might call *setwise-small* and *pointwise-small*. These two notions of setwise-smallness and pointwise-smallness apply to finite *sets* of natural numbers, and they both are defined in terms of a third notion of smallness which applies to *individual* natural numbers. To illustrate the latter, 100 is a small natural number, but 100^{1000} is not a small natural number, and if one natural number is small and another number is less than it, then that second natural number is small as well.[5] Then a set X is said to be *setwise-small* if its cardinality is a small natural number, and a set X is said to be *pointwise-small* if each element of X is a small natural number. Hence, with the exception of the set of all small natural numbers, any pointwise-small set is itself setwise-small. However, the converse is not in general true. For instance, while $Y = \{2, 3, 4\}$ is both setwise- and pointwise-small, the set $X = \{200^{1000}, 300^{1000}, 400^{1000}\}$ is setwise-small but not pointwise-small.

One version of Baker's thesis would thus contend that arithmetical instance confirmation is biased in a way in which physical instance confirmation is not, and that this is due to the samplings in arithmetical instance confirmation being *setwise-small*. It is presumably indisputable that arithmetical instance confirmation is in fact based on setwise-small samplings of natural numbers. For, even with the aid of computers, one can only look at so many natural numbers, and in comparison with the set of *all* natural numbers, the cardinality of such samplings will inevitably appear diminutive. However, presumably physical instance confirmation relies on setwise-small samplings in exactly the same manner: indeed, the same sorts of constraints

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[5]It might also well be the case that the notion of "being a small natural number" is a vague term, or is context-sensitive. As far as I can see, this issue does not materially affect the present discussion.

that prevent us from doing innumerable calculations also prevent us from taking innumerable measurements. Hence, a difference between the levels of bias in arithmetical instance confirmation and physical instance confirmation cannot be attributable to a difference in the manner in which they rely upon setwise-small samplings, simply because they so rely on setwise-small samplings in exactly the same way. Hence, versions of Baker's thesis centered around setwise-smallness seem plainly untenable.

Thus, it seems that Baker's thesis might be more profitably understood in terms of pointwise-smallness. This version of the thesis holds that arithmetical instance confirmation is unreliable in a way that physical instance confirmation is not, and that this is due to the samplings in arithmetical instance confirmation being *pointwise-small*. Here, the unreliability of instance confirmation is understood in a standard manner, so that a relative increase in unreliability is concomitant with a relative increase in the number of false universal hypotheses which are confirmed by several true instances. Now, it seems hard to dispute that samplings in arithmetical instance confirmation are drawn exclusively from pointwise-small samples: for, given constraints of time and space, even the best computers can only calculate with numbers of so large a size, and mathematicians doing calculations likewise face similar sampling constraints and limitations.

Hence, it seems that the only contentious point in this version of Baker's thesis is the claim that such a reliance upon pointwise-small samples renders arithmetical instance confirmation unreliable in a way in which physical instance confirmation is not. However, there is an obvious analogue of this reliance upon pointwise-small sampling in the case of physical instance confirmation, an analogue suggested by Baker himself. In particular, say that a sampling of physical data is *timewise-small* if each data point in the sampling was measured (or otherwise observed) at a point in time that is relatively close to the present. Just as it seems indisputable that samplings of natural numbers are pointwise-small, so it seems indisputable that samplings of physical data are timewise-small. However, it is generally conceded that physical instance confirmation is sufficiently reliable. Hence, if the inference from the dependence of physical instance confirmation on timewise-small samplings to the unreliability of physical instance confirmation is *rejected*, but at the same time the inference from the dependence of arithmetical instance confirmation on pointwise-small samplings to the unreliability of arithmetical instance confirmation is *accepted*, then one should be able to point out some relevant difference between the two cases.

Baker suggests that the relevant difference between the two consists in the fact that "there are no [. . .] systematic differences between the past and the future [. . .]" ([16] p. 68). It may indeed be the case that *many* of the properties that interest scientists are in fact temporally invariant in this sense, so that what is true of timewise-small samplings will likewise be true in general. However, it is also the case that many of the properties that interest mathematicians are such that what is true of pointwise-small samplings is likewise true in general. For instance, this is the case with respect to the properties that feature in the instance confirmation of the axioms of Robinson's Q. So if there is to be a disanalogy between the setting of arithmetic and the setting of the physical sciences here, it has to be with regard to something more than the fact that *many* of the properties which interest scientists (resp. mathematicians) are projectable from timewise-small samplings (resp. pointwise-small samplings).

Hence, one might try to suggest that the relevant difference between arithmetical and physical instance confirmation consists in the fact that *all* physical properties are projectable from timewise-small samplings, whereas *not* all arithmetical properties are projectable from pointwise-small samplings. But if the key term of *physical properties* is understood as a naturalistic term that simply picks out spatio-temporal properties describing portions of the external world, then it is simply false that all physical properties are so projectable. Indeed, were this the case, then knowledge of the future would be much easier to come by than it actually is. Likewise, if the key term of *physical properties* is understood historically, as picking out those properties that have interested certain intellectual communities, then again it is false that all these properties are temporally-projectable: for, were this the case, then science would be endowed with a kind of infallibility which is definitively vitiated by the historical record.

It might then be suggested that the important difference between arithmetical instance confirmation and physical instance confirmation is the success of the extant practice: most of the physical properties picked out by the community of scientists have in fact turned out to be temporally projectable, whereas there is no similar track record of success of mathematicians projecting from the pointwise-small. So part of the idea here would be that natural scientists somehow learned to discern the properties of physical objects that do not depend on their temporal location, whereas mathematicians have yet to learn to discern the properties of numbers that do not depend on their location in the ordering of greater-than and less-than on the natural num-

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bers. I am willing to grant all this for the sake of argument: however, what I want to emphasize is that this does not establish an entailment from the reliance upon pointwise-small samplings to the unreliability of arithmetical instance confirmation, any more than pointing to the failures of pre-scientific communities to project from the temporally-small would establish that there is an entailment from the reliance upon temporally-small samplings to the unreliability of physical instance confirmation.

Hence, it seems that no relevant difference has been identified which would justify us in *accepting* the inference from the reliance upon pointwise-small samplings to the unreliability of arithmetical instance confirmation while simultaneously *rejecting* the inference from the reliance upon temporally-small samplings to the unreliability of physical instance confirmation. However, it is important to emphasize that one could legitimately reject the need to identify such a relevant difference. For instance, one might legitimately reject such a need by producing a valid premise-conclusion argument (with plausible premises) for the inference from the reliance upon pointwise-small samplings to the unreliability of arithmetical instance confirmation. However, neither Baker nor anyone else that I know of has produced such an argument. Absent such an argument for this inference, it does not seem to be unreasonable philosophical methodology to withhold assent from this inference until some relevant difference between it and a clearly dubious albeit similar inference has been identified.

Even if one withholds assent from Baker's thesis, nonetheless his work raises a pressing question for the types of empiricism considered in this paper. In particular, Baker centers his discussion around the Goldbach conjecture, a conjecture which at the time of writing is unresolved but of which multitudes of cases have been manually and computer verified. It's not obvious that this observational evidence has raised the collective credence assigned to the conjecture by the community, and hence this casts a level of doubt over the plausibility of the types of empiricism considered here. Baker has in effect set the task of explaining why this evidence has not raised our credence in the conjecture, and even if there is reason to withhold assent from Baker's own explanation, his work calls on one who would find a place for confirmation in mathematics to explain why this evidence has not raised our credence in this conjecture.

But it turns out that the probabilistic model of confirmation explains this fact, and the explanation concerns the comparative syntactic simplicity of the Goldbach conjecture. Recall that a Σ_1^0-formula is a formula which

begins with one existential quantifier and all of whose other quantifiers are bounded, i.e. are of the form $\exists\, y < z$ or $\forall\, y < z$. Then we have the following important proposition (cf. [1] Theorem 1.8 p. 30, [24] Corollary 2.9 p. 25):

Proposition 1. (Σ_1^0-Completeness of Robinson's Q): Suppose that φ is a Σ_1^0-sentence. Then φ is true on the standard model if and only if it is provable from Robinson's Q.

Here the standard model is simply the structure $(\omega, 0, S, +, \times, \leq)$, where of course $\omega = \{0, 1, 2, \ldots\}$ is the set of natural numbers. Now, suppose that P is a probability assignment that gives non-zero probability to the conjunction of the eight axioms of Robinson's Q. Suppose that $h \equiv \forall\, x\, \varphi(x)$ where φ is Σ_1^0 and suppose that $e_n \equiv \bigwedge_{i=1}^{n} \varphi(S^i(0))$ where $S^0(0) = 0$ and $S^{i+1} = S(S^i(0))$. Then if h is true on the standard model, then each e_n is true on the standard model, and hence by the above proposition are provable from Robinson's Q, so that $P(e_n\ \&\ K) = P(K)$, where K is the conjunction of the eight axioms of Robinson's Q. Then

$$P(h|e_n\ \&\ K) - P(h|K) = 0 \tag{4}$$

and hence evidence e_n does not confirm the hypothesis h relative to background knowledge K. So in sum, any true arithmetical statement of the form $\forall\, x\, \varphi(x)$ where $\varphi(x)$ is Σ_1^0 cannot be confirmed by its instances relative to any probability assignment that gives non-zero probability to the conjunction of the eight axioms of Robinson's Q.

It turns out that the Goldbach conjecture and many other number theoretic statements such as Fermat's Last Theorem and Goodstein's theorem have this simple syntactic form. Hence, the probabilistic model of confirmation indicates that these hypotheses will incur no confirmation by the observation of many instances. One might view this as a weakness of the theory, since one might require that (a) the observation of several instances of a universal hypothesis *always* confers some limited measure of confirmation on the universal hypothesis. However, one might alternatively begin with the thought that we should privilege the judgements of practitioners and hence hold that (b) our confidence has not been substantially raised in classical number theoretic conjectures by virtue of the verification of many instances. My own sense is that much of the suspicion about confirmation in mathematics is brought about by the conjunction of (a) and (b). Obviously, the probabilistic model proffered here removes the tension by rejecting (a), and it while

it is compatible with (b), it is also compatible with holding that arithmetical instance confirmation can and should raise our credence in other cases, and in particular, either when Robinson's Q is not yet in the background knowledge (as in the case of inceptive empiricism) or when the number-theoretic hypotheses display more syntactic complexity (as in the case of the instances of mathematical induction featuring in amplificatory empiricism).

So having said this, one might like some explicit assurance that *there actually are* probability assignments on which either (i) the axioms of Robinson's Q are confirmed to a high degree by evidence consisting of various of its instances, or (ii) the hypothesis consisting of the consequent of an instance of mathematical induction is confirmed to a high degree by the evidence consisting of the antecedent of this instance of mathematical induction against background knowledge which includes Robinson's Q. With respect to (i), one can prove the following elementary proposition:

Proposition 2. Let K be a finitely axiomatized L-theory. Suppose that h, e are L-sentences such that $K + h \vdash e$ and $K \nvdash e$ and $K + h$ is consistent. Let $\epsilon > 0$ with $\epsilon < 1$. Then there is a probability assignment such that

$$1 - \epsilon \leq P(h|e \,\&\, K) - P(h|K) < 1 \tag{5}$$

For the proof of this proposition, see Appendix B. This proposition is applicable to (i) since if one lets h be an axiom of Robinson's Q and e be the conjunction of several instances of this universal hypothesis and one lets K be the empty theory, then the triple h, e, K will satisfy the antecedents of the proposition. But this is of course not to say that the probability assignment thereby acquired will necessarily be natural or anything like an intended model of the pre-theoretic probabilistic notions. Regarding (ii), it is helpful to first introduce some terminology to talk about the antecedent and consequent of an instance of mathematical induction. As is standard, let us abbreviate an instance of mathematical induction as:

$$I_\varphi \equiv [\varphi(0) \,\&\, \forall\, n\, (\varphi(n) \rightarrow \varphi(n+1))] \rightarrow [\forall\, n\, \varphi(n)] \tag{6}$$

wherein $\varphi(x)$ is an L-formula with one distinguished free variable x, and wherein L is the signature of first-order arithmetic, and so consists just of zero, successor, addition, multiplication, and less-than. Let us then define:

$$e(I_\varphi) \;=\; \varphi(0) \,\&\, \forall\, n\, \varphi(n) \rightarrow \varphi(n+1) \tag{7}$$
$$h(I_\varphi) \;=\; \forall\, n\, \varphi(n) \tag{8}$$

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Expressed in terms of these abbreviations, we have that $I_\varphi \equiv e(I_\varphi) \to h(I_\varphi)$. Since $h(I_\varphi)$ trivially implies $e(I_\varphi)$, we have that the above proposition applies whenever $K \nvdash e(I_\varphi)$ and $K + h(I_\varphi)$ is consistent.

But this is still yet merely a sufficient condition, and so it's useful to point out that there actually are several examples wherein the degree of confirmation is high. In the case where the background information K is true on the standard model of arithmetic, we have the following proposition:

Proposition 3. Let K be a finitely axiomatized extension of Robinson's Q that is true on the standard model of arithmetic. Then there are infinitely many L-formulas $\varphi(x)$ such that for all $\epsilon > 0$ with $\epsilon < 1$ there is a probability assignment P such that

$$1 - \epsilon \leq P(h(I_\varphi)|e(I_\varphi) \,\&\, K) - P(h(I_\varphi)|K) < 1 \qquad (9)$$

As with the earlier proposition, we defer the proof until Appendix B. So this proposition tells us that there are many instances I_φ of mathematical induction such that its consequent $h(I_\varphi)$ can be confirmed to a high degree relative to its antecedent $e(I_\varphi)$ against the background of theories like Robinson's Q. Obviously we cannot demand that *all* instances of mathematical induction are like this. For instance, consider the instances of mathematical induction corresponding to the formulas $\varphi_0(x) \equiv x = x$ and $\varphi_1(x) \equiv x \neq x$. There is no confirmation in either of these two cases: in the first case because both $e(I_{\varphi_0})$ and $h(I_{\varphi_0})$ are tautologies, and in the second case because both $\neg e(I_{\varphi_1})$ and $\neg h(I_{\varphi_1})$ are tautologies. So in short, we have a sufficient condition for when the consequent of an instance of mathematical induction is confirmed by its antecedent, and we further have that there are infinitely many instances wherein there is actually a high degree of confirmation. So while the discussion of Baker has lead to an explanation of why there is never confirmation of e.g. the Goldbach conjecture by its instances, nonetheless there is sometimes high degree of confirmation of the axioms of Robinson's Q by its instances, and there is sometimes high degree of confirmation of an instance of mathematical induction by its antecedent relative to background knowledge.

Earlier we stated that: any true arithmetical statement of the form $\forall x \, \varphi(x)$ where $\varphi(x)$ is Σ_1^0 cannot be confirmed by its instances relative to any probability assignment that gives non-zero probability to the conjunction of the eight axioms of Robinson's Q. One might be of a mind that this fact

is simply a reductio of the view on offer here, since it shows that comparatively simple universal hypotheses cannot be confirmed by their instances once Robinson's Q is in the background knowledge. However, the view on offer here primarily seeks to articulate a viable candidate for the source of our knowledge of arithmetical axioms. If this source is a source of knowledge for these axioms but for comparatively few other arithmetical truths, then it's not obvious that this should be a strike against the view on offer here. To provide an account of knowledge of axioms of an area of mathematics need not be to provide a mechanism by which to assay all mathematical truths in that area.

3. Challenges from Access to Probability Assignments

Both inceptive and amplificatory empiricism presuppose that confirmation is a source of justification in the setting of arithmetic, and the challenges to be considered in this section suggest in different ways that we do not have access to this source of justification in the setting of arithmetic, due to the fact that probability in this setting is quite different in character from probability in the setting of the natural sciences. In particular, the challenges considered here adduce reasons for thinking that access to probability assignments in the setting of arithmetic is no less difficult than access to arithmetical truth itself.

3.1. Countable Additivity: Aligning the True and the Probable

The first such challenge relates to countable additivity. There are several different versions of countable additivity, but their common impetus lies in the thought that the probability axioms P1-P3 articulate rules of probability only for the *propositional* connectives. For instance, it is straightforward to derive from P1-P3 the following rules which relate probabilities to disjunctions, conjunctions, and negations:

(P4) $P(\varphi \lor \psi) + P(\varphi \mathbin{\&} \psi) = P(\varphi) + P(\psi)$

(P5) $P(\neg\varphi) = 1 - P(\varphi)$

The basic motivation behind countable additivity is to exhibit analogous rules for the quantifiers. In particular, suppose that the formal language or signature under consideration is the signature L of the Peano axioms. One can then articulate the following version of countable additivity, which for the sake of disambiguation can be referred to as *ω-additivity*, wherein $\varphi(x)$

is an L-formula with free variable x, and wherein $S^0(0) = 0$ and $S^{i+1} = S(S^i(0))$:

(Pω) $P(\forall\, x\; \varphi(x)) = \lim_N P(\bigwedge_{n=0}^N \varphi(S^n(0)))$

Hence, the idea of ω-additivity is that the probability of a universal arithmetical hypothesis may be approximated arbitrarily closely by the probabilities assigned to the sentences expressing that further and further arithmetical terms satisfy this hypothesis.

To obtain a different version of countable additivity, one can consider an extension to a setting where one can form new sentences by taking conjunctions and disjunctions over countable sets of sentences. These operations of conjunction and disjunction are respectively written as $\bigwedge_n \varphi_n$ and $\bigvee_n \varphi_n$, and the resulting class of sentences are called $L_{\omega_1\omega}$-*sentences*. Relative to a natural semantics and deductive system for these sentences, there is a completeness theorem for $L_{\omega_1\omega}$-sentences ([25] Theorem 3 p. 16, [26] Theorem 3.2.1 p. 280), and hence the notion of a probability assignment on these sentences can be defined. In particular, an $L_{\omega_1\omega}$-*probability assignment* is an assignment of real numbers to $L_{\omega_1\omega}$-sentences which satisfies P1-P3 (relative to the consequence relation on $L_{\omega_1\omega}$-sentences for which the completeness theorem holds). One can then consider the following version of countable additivity, which for the sake of disambiguation can be referred to as ω_1-*additivity*, where $\varphi_0, \varphi_1, \ldots$ is a countable sequence of $L_{\omega_1\omega}$-sentences:

(Pω_1) $P(\bigwedge_n \varphi_n) = \lim_N P(\bigwedge_{n=0}^N \varphi_n)$

Outside of the difference between the universal quantifier and infinite conjunction, the primary difference Pω and Pω_1 is that on the right-hand side of Pω the natural number n is employed to make a number-theoretic statement about the n-th successor of zero, whereas on the right-hand side of Pω_1 it is only employed as an index for the sentence φ_n, which may or may not be a statement about numbers at all.

While this difference between ω-additivity and ω_1-additivity may seem innocuous, it is not difficult to see that ω-additivity and only ω-additivity requires that "having a high probability" align with arithmetical truth. For, suppose that the conjunction of the eight axioms of Robinson's Q is assigned a high probability, say greater than $1-\epsilon$, where ϵ is some small non-zero error threshold. Under these circumstances, it follows from the Σ_1^0-completeness of Robinson's Q (cf. Proposition 1 in § 2.2) that if a probability assignment satisfies ω-additivity, then an arithmetical sentence is true of the standard model if and only if it has probability greater than $1 - \epsilon$. More formally, we have the following result, whose proof we defer until Appendix A:

Proposition 4. *Proposition on the Alignment of the True and the Probable*: Suppose that L is the signature of Peano arithmetic and that K is the conjunction of the eight axioms of Robinson's Q. Suppose that $0 < \epsilon < \frac{1}{2}$. Suppose that P is an ω-additive probability assignment such that $P(K) > 1 - \epsilon$. Then (a) $(\omega, 0, S, +, \times, \leq) \models \varphi$ if and only if $P(\varphi) > 1 - \epsilon$, and (b) $(\omega, 0, S, +, \times, \leq) \models \varphi$ implies $P(\varphi \,\&\, K) = P(K)$.

Here the standard model is the structure $(\omega, 0, S, +, \times, \leq)$, where of course $\omega = \{0, 1, 2, \ldots\}$ is the set of natural numbers. Hence, if a probability assignment satisfies ω-additivity, then registering a high probability by reference to this assignment is coextensive with truth for arithmetical sentences. However, the same is not the case with respect to ω_1-additivity. In particular, it is not difficult to see that for any sentence of first-order predicate logic which is not a consequence of the axioms of Robinson's Q, there is an ω_1-additive probability assignment that assigns this sentence probability zero and that still gives the conjunction of the axioms of Robinson's Q a high probability. This simple fact shows that unlike ω-additivity, it is not the case that the satisfaction of ω_1-additivity forces the alignment of the arithmetically true and the arithmetically probable.

There are at least two reasons why it is important that we not be committed to ω-additivity. First, if we were so committed, then the alignment in part (a) of Proposition 4 would cast doubt on our access to judgements about confirmation and probability in the setting of arithmetic. To see this, consider an analogous scenario centered not around probability but around perception. Should someone posit perception as a source of justification about arithmetic, but then inform us that this sort of perception happened to be infallible, it seems that the proper response would be to question whether this type of perception is something which we actually possess, given that it is so manifestly different from our normal modes of perception. The second problem caused by ω-additivity is that part (b) of Proposition 4 implies that a true hypothesis h cannot be confirmed by true evidence e. In particular, if h and e are true on the standard model and K is as in Proposition 4, then we have

$$P(h|e \,\&\, K) - P(h|K) = \frac{P(h \,\&\, e \,\&\, K)}{P(e \,\&\, K)} - \frac{P(h \,\&\, K)}{P(K)} = 1 - 1 = 0 \quad (10)$$

So since such confirmation is vital to the ultimate tenability of inceptive empiricism and amplificatory empiricism, it is necessary to say why these forms of empiricism are not committed to ω-additivity.

My response to this challenge is to argue that the reasons which commit inceptive empiricism and amplificatory empiricism to the probability axioms P1-P3 do not extend to ω-additivity, even though they do extend to ω_1-additivity. For, it is common today to justify commitment to P1-P3 by taking recourse to Dutch Book arguments, and just as it is demonstrable that ω_1-additivity is justifiable by recourse to such arguments, so it is likewise demonstrable that ω-additivity is not so justifiable. Let me first describe the relevant theorems and non-theorems before turning to the relation of the theorems to the justification of probability axioms. The theorems and non-theorems in question here concern complete consistent theories T in the signature L of the Peano axioms, and in what follows it will be convenient to regard such complete extensions as zero-one valued functions on the set of L-sentences, so that $T(\varphi) = 1$ if $T \models \varphi$ and $T(\varphi) = 0$ otherwise. Having this convention in place, the standard version of the Dutch Book Theorem reads as follows (cf. [13] p. 79, [14] p. 38):

Theorem 1. *Dutch Book Theorem, Standard Version*: Suppose that P is a function from L-sentences to real numbers. Then P is a probability assignment if for every finite sequence of real numbers $s_1, \ldots, s_N$ and every finite sequence of L-sentences $\varphi_1, \ldots, \varphi_N$, there is a complete consistent L-theory T such that $\sum_{n=1}^{N} s_n(T(\varphi_n) - P(\varphi_n)) \geq 0$.

The situation described in the antecedent of the theorem is commonly vivified as follows. Suppose that a bookie offers stakes s_n of units of currency on sentence φ_n and that a bettor provides the bookie with $s_n P(\varphi_n)$ units. Suppose further that there is an agreement in place that if φ_n turns out false, then the bettor wins nothing (for a net total of $-s_n P(\varphi_n)$ units), and that if φ_n turns out true, then the bettor wins s_n (for net total of $s_n - s_n P(\varphi_n)$ units). Finally, say that the bettor is *invulnerable to a Dutch book* if for any finite sequence of bets there is always some situation– representable in terms of a complete, consistent theory– in which the net total due to the bettor across all bets is *not* strictly negative. Hence, cast in these terms, the Dutch Book theorem says that invulnerability to a Dutch book is a sufficient condition for an assignment to be a probability assignment.

The technical point which I view as relevant here is that while the analogous theorem holds for ω_1-additivity, it does not hold for ω-additivity. In particular, it is well-known that by appropriately augmenting the proof of the standard version of the Dutch Book Theorem, one can establish the following (cf. [27] pp. 411-412):

Theorem 2. *Dutch Book Theorem, ω_1-additive Version*: Suppose that P is a function from $L_{\omega_1\omega}$-sentences to real numbers. Then P is an ω_1-additive $L_{\omega_1\omega}$-probability assignment if for every infinite sequence of real numbers s_n and every infinite sequence of $L_{\omega_1\omega}$-sentences φ_n such that the sequence $s_n P(\varphi_n)$ is absolutely convergent, there is a complete consistent $L_{\omega_1\omega}$-theory T such that $\sum_n s_n(T(\varphi_n) - P(\varphi_n)) \geq 0$.

In developing the analogous version for ω_1-additivity, it turns out that it is important to include the stipulation about absolute convergence. Here, absolute convergence means that $\sum_{n=1}^{\infty} |s_n P(\varphi_n)| < \infty$, i.e. that the sequence of partial sums $\sum_{n=1}^{N} |s_n P(\varphi_n)|$ approaches a finite limit in the real numbers. In terms of the betting scenario described above, this corresponds to the requirement that the units of currency potentially exchanged between the bookie and the bettor be finite. However, when we turn from ω_1-additivity to ω-additivity, what we find is that the analogous version of the Dutch Book Theorem is false:

Proposition 5. *Counterexample to Dutch Book Theorem, ω-additive Version*: There is a function P from L-sentences to real numbers such that (i) P is *not* an ω-additive probability assignment, and such that (ii) P has the following property: for every infinite sequence of real numbers s_n and every infinite sequence of L-sentences φ_n such that the sequence $s_n P(\varphi_n)$ is absolutely convergent, there is a complete consistent L-theory T such that $\sum_n s_n(T(\varphi_n) - P(\varphi_n)) \geq 0$.

The proof of this proposition is deferred until Appendix A, so that we may focus here on discussing the import of these results.

The philosophical significance of Dutch Book Theorem resides in the fact that invulnerability to a Dutch book is indicative of a certain type of rationality when the assignment in question is reflective of degrees of belief, so that the theorems show that conformity to the probability axioms P1-P3 is a necessary condition of a certain type of rationality. The type of rationality implicated here is of course minimally thought to require a disposition to arrange degrees of belief in such a way that *were* one to bet units of currency on these degrees of belief, then there would be at least one situation in which a loss would not be suffered. There are thus at least two presuppositions to the contention that this type of rationality constitutes an epistemic virtue. The first presupposition is that some virtues are revealed purely in terms of counterfactual behavior, since it is obviously not envisioned here that one actually

engages in such betting scenarios. But while such "dormant virtues" may be a rarity in the practical sphere, they are commonplace in the theoretical sphere. For instance, there is a virtue related to consistency which consists in a disposition to retract previously endorsed axioms were they to exhibit a demonstrable inconsistency, and it seems reasonable to say that this virtue is present in our reasoning even if it turns out that the axioms in question (say the set-theoretic axioms) are in fact consistent. The second presupposition is that there is a suitable abundance of potential situations across which gains or losses may be incurred, since were the number or variety of these situations to be highly curtailed, then the demands of invulnerability would become quite severe. However, since we are identifying potential situations with complete consistent theories in a given formal language or signature, the fact that there are continuum many of these in the setting of arithmetic (cf. [28] Theorem 6.2 (iii), (v) p. 76) would seem sufficient to allay concerns about the severity of the demands of invulnerability.

Hence, my response to the challenge of ω-additivity is to suggest that inceptive and amplificatory empiricism be conceived as justifying their appeal to confirmation and probability by means of Dutch Book Theorems, so that the fact that there is no ω-additive Dutch Book Theorem may be taken as evidence that these forms of empiricism are not committed to ω-additivity. While this response clearly meets the challenge of ω-additivity, it has at least two drawbacks. The first is that if these forms of empiricism are tied to the philosophical interpretation of the Dutch Book Theorems described above, then all the concerns voiced in the literature about this interpretation automatically become concerns for these forms of empiricism. The second drawback is that if inceptive and amplificatory empiricism are going to operate *only* with those rules of probability that are licensed by Dutch Book Theorems, then these forms of empiricism cannot justify the contention that various kinds of confirmation *actually* occur by recourse to probabilistic rules. For instance, inceptive empiricism turns on the supposition that several instances of a universal arithmetical hypothesis are assigned a non-zero probability strictly less than one, and this supposition by no means follows from the probability axioms P1-P3 alone. Hence, if these forms of empiricism are *only* allowed to operate with these probabilistic rules, then for their ultimate success they must provide *other* reasons for giving such assignments. This concern is related to the future task, described in the concluding § 4, of articulating further the sources of arithmetical probabilities.

Before turning to the challenge from computability, it is helpful to briefly

compare this response to ω-additivity to Isaacson' well-known response to the ω-rule ([29] § III). The ω-rule is a proof-theoretic rule which licenses the inference to the claim that $\forall\ x\ \varphi(x)$ from the totality of all claims of the form $\varphi(S^n(0))$, where n ranges over natural numbers. For the very same reasons that the arithmetically true and the highly probable become aligned under ω-additive probability assignments that assign high probability to Robinson's Q, so the arithmetically true is aligned with that which is derivable from Robinson's Q in deductive systems augmented by the ω-rule. Isaacson was concerned with this because he had previously argued that the Peano axioms in conjunction with the standard rules of inference were effectively complete and completely determined our concept of number ([30]). Thus Isaacson was concerned to show that the ω-rule was not part of our concept of number, since otherwise the collapse of truth and proof engendered by the ω-rule would make this concept vastly outstrip that which is given to us by the Peano axioms and standard rules of inference.

One of Isaacson's basic strategies was to point out that standard defenses of the ω-rule appealed to truth about the natural numbers, and such an appeal to truth about the natural numbers is not part of our concept of number, but goes above and beyond this concept, and is essentially a second-order or higher-order concept ([29] p. 108). There are many differences between Isaacson's strategy for handling the ω-rule and the above strategy for handling ω-additivity, but the one difference that perhaps bears mentioning is that the discussion of ω-additivity did not at any place appeal to points specific to our concept of number. Rather, this discussion focused entirely on what did and did not follow from a standard justification of the probability axioms. The analogue of this strategy in Isaacson's setting would be to argue the ω-rule did not follow from some standard justification of the other accepted rules of inference such as modus ponens.

3.2. Non-Computability and Non-Continuity of Assignments

The aim of this section is to describe two related challenges to our access to probability assignments in the setting of arithmetic. The *first* challenge simply asks whether probability assignments in this setting may be computable, and suggests that if they are not, then there is no reason to think that this is something to which we are capable of implicitly or explicitly appealing. The second challenge begins from the observation that one basic class of probability assignments are the finite counting assignments. Let us

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say that P is a *finite counting assignment* if it may be written in the form:

$$P(\varphi) = a_1 \cdot T_1(\varphi) + \cdots + a_n \cdot T_n(\varphi) \tag{11}$$

where the positive real numbers a_i sum to 1, and where the T_i are distinct complete consistent theories in the language of the Peano axioms (where we view such theories as zero-one valued functions on sentences). The *second* challenge then asks whether probability assignments in this setting are always finite counting assignments, and suggests that if they are, then a basic dilemma presents itself relating to true arithmetic $Th(\mathbb{N})$, i.e. the set of sentences true on the standard model $(\omega, 0, S, +, \times, \leq)$ of the Peano axioms. On the one hand, if one of the T_i is equal $Th(\mathbb{N})$, then the question is: why do we think that access to P is any easier than access to T_i? On the other hand, if none of the T_i are equal to $Th(\mathbb{N})$, then the question is: why do we think that these are a reliable guide to arithmetical truth?

In regard to the first challenge, it turns out that if Robinson's Q is given non-zero probability, then the probability assignment must be non-computable. This is encapsulated in the following theorem, whose proof we defer until Appendix B:

Theorem 3. *Theorem on Non-Computability of Probability Assignments*: Suppose that $P : \mathrm{Sent}(L) \to \mathbb{R}$ is a probability assignment that assigns the eight axioms of Robinson's Q non-zero probability. Then the set of pairs $\{(\varphi, r) \in \mathrm{Sent}(L) \times \mathbb{Q} : P(\varphi) > r\}$ is non-computable.

Hence, in regard to this challenge, it is merely a question of how the advocate of the types of empiricism articulated here is going to respond to the suggestion that we do not have access to the non-computable. In regard to the second challenge, it turns out that there are many probability assignments that are not finite counting assignments. In particular, if K is a theory, then let's say that P is K-*sound* if $K \models \varphi$ implies $P(\varphi) = 1$ and let's say that P is K-*regular* if the converse holds, i.e. $P(\varphi) = 1$ implies $K \models \varphi$. Then using a forcing argument, the details of which we present in Appendix B, we can establish the following:

Theorem 4. *Theorem on Continuous Probability Assignments*: Suppose that K is a consistent computable extension of Robinson's Q. Then there is a K-sound K-regular probability assignment P such that P is not a finite counting assignment. Further, P may be chosen so that the set of pairs $\{(\varphi, r) \in \mathrm{Sent}(L) \times \mathbb{Q} : P(\varphi) > r\}$ is computable in the halting set.

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Since there are many different consistent computable extensions of Robinson's Q, it thus follows that there are many probability assignments that are not finite counting assignments. Hence, the dilemma associated to the second challenge can be avoided simply by suggesting that the probability assignments which we avail ourselves of are not finite counting assignments.

So how should the advocate of the types of empiricism considered here respond to the concern that we can't access the non-computable? It would be a mistake to dismiss this concern out of hand, since there is a long tradition in the foundations of mathematics and the philosophy of mathematics of privileging computable theories. These discussions often appear in the context of Gödel's incompleteness theorems, which apply only to computable theories. For instance, in their textbook presentation of these theorems, Cori and Lascar write: "To be recursive is a reasonable requirement for a theory; we might even say that non-recursive theories are artificial: how could we hope to deal with derivations if we do not have effective knowledge of axioms?" ([31] p. 87). To avoid incompleteness, Gödel himself articulated (but did not endorse) the view that "[...] the human mind [...] infinitely surpasses the powers of any finite machine" ([32] p. 310). Boolos, speaking of authors who went on to argue for this view, says: "It is fair to say that the arguments of these writers have as yet obtained little credence" ([32] p. 295). Hence, if one is appealing to the non-computable in an epistemic context, then one is deviating non-trivially from the apparent consensus in the foundations of mathematics and the philosophy of mathematics.

Another response would suggest that the distinction between the computable and the non-computable is more of a continuum than a binary opposition. For instance, there are a large class of non-computable sets X that can be *computably approximated*, in that there is a computable sequence of sets X_s such that for each n one has:

$$\lim_s X_s(n) = X(n) \tag{12}$$

where the limit notation merely indicates that there's some stage s_n such that $X_s(n) = X(n)$ for all $s \geq s_n$, or in other words: for each input n there's some stage past which the approximating sequence X_s agrees with the true value X on this input.[6] Appealing to computable approximations might be persuasive depending on the type of arguments put forward for the require-

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[6]The term "computably approximable" is my own. This notion does not have its own

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ment of computability. For instance, to answer Cori and Lascar's question about how to deal with derivations in the setting of the non-computable, one could simply point to small but established literature on proof predicates for theories given in computably approximated manner (cf. [34]). However, one vulnerability of this line of argument is that there are computably approximated theories that avoid the incompleteness theorems (cf. [28] Theorem 3.6 p. 54, Theorem 6.1 p. 75). So one might worry that any dissatisfaction with an appeal to computable approximation in the setting of the incompleteness theorems would transfer over to the setting of access to probability assignments.

In my view, a better response is to suggest that the probability assignment should be represented not as a single probability assignment, but rather as a large family of probability assignments.[7] One could then view the agent as being committed to and working off of a formal or informal description of this class probability assignments. This description would differ from formal descriptions codified in the Peano axioms in that it would describe not the notion of natural number *per se*, but rather the notion of a probability assignment defined on sentences in the signature of the Peano axioms. This would meet the concern about access to the non-computable since this formal theory of probability would itself be computable, just like the formal theory of the Peano axioms is computable even though it has no computable consistent extensions.[8] Further, this approach is also compatible with the requirement that the probability assignment not be a finite counting assignment, since there are various axioms on probability assignments that preclude their being finite counting assignments.[9]

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name in the computability theory literature simply because The Limit Lemma and Post's Theorem together imply that a set of natural numbers is computably approximable if and only if it is Δ_2^0 (cf. [33] Lemma III.3.3 p. 57 and Theorem IV.2.2 p. 64).

[7]This view is not new but has been advocated by many others for various different reasons. See Joyce [35] p. 171 for references and discussion.

[8]There's a certain similarity between this point and Kelly's use of computably approximable sets of natural numbers. Kelly suggests that we view scientific hypotheses not as specifying a single stream of empirical data but rather as specifying a large class of such streams, so that whilst each single stream may be computationally complex, the description of the class itself may be comparatively simple (cf. [36]).

[9]In particular, equation (B.7) from Appendix B will suffice to preclude the probability assignment from being a finite counting assignment.

4. Conclusions

So this essay has been a preliminary defense of an empiricism according to which our knowledge of arithmetic is of a piece with the knowledge by which we infer from the past to the future or from the observed to the unobserved. It is a *preliminary* defense in that I have *not* adduced positive arguments for the types of empiricism which I consider here, but rather have defended the tenability of these forms of empiricism against various challenges and objections. These objections are by no means the only objections which one could mount against this type of empiricism, but in my view they are the objections that are the most pressing, precisely because they concern various apparent difficulties that emerge when one begins to take seriously the idea that just as enumerative induction can be justified by recourse to informed judgments of probability, so too can mathematical induction and the other Peano axioms.

The primary task for future work on these forms of empiricism lies in developing and articulating an account of the sources of arithmetical confirmation. This task really splits into two parts. First, it is necessary to present a positive account of our access to probability assignments, and whether these are merely reflective of credences or whether notions like exchangeability (cf. [37] § 8, [38]) can be used to facilitate this access. Second, it is necessary to say something about the perceptual or other evidential sources which we may draw on in updating probabilities. In the ordinary empirical setting, the idea is usually that we update on observations. Hence, a positive account of how this might possibly work in the setting of arithmetic is required, and here it will be necessary to discuss explicitly how this relates to other treatments of non-deductive reasoning in mathematics by authors such as Kitcher [39], Maddy [40], and Giaquinto [41], [42].

Another important task resides in assaying the extent to which the problems that affect formal treatments of truth in the setting of arithmetic may or may not affect formal treatments of probability in the setting of arithmetic. In particular, a pressing problem deserving of independent discussion is the problem of what I call the *probabilistic liar*. This is a sentence λ which is equivalent to the claim that $P(\lambda) \leq \epsilon$ for some small value of $\epsilon \geq 0$. This presents an acute problem for our concept of confirmation because one can show, under reasonable hypotheses, that the hypothesis λ is confirmed by the evidence that $P(\lambda) \leq \epsilon$ (cf. Proposition 12 in Appendix C). Hence, the probabilistic account of confirmation presented here would suggest that

one could infer from the *improbability* of λ to the *truth* of λ, a result that seems intuitively incorrect. Hence, in future work it is necessary to discuss the extent to which this seeming incorrectness may be contained or assuaged. However, just as work on formal theories of truth is merely technical if not properly motivated by the tenability of deflationism or other minimal theories of truth, so it has been necessary to focus in this essay on first defending the tenability of a probabilistic conception of justification of arithmetical axioms.

Appendix A. Proofs of Results Stated in § 3.1

The aim of this brief appendix is to present proofs of two results from § 3.1, namely the proposition on the alignment of the true and the probable (Proposition 4) and the counterexample to the ω-additive Dutch Book Theorem (Proposition 5).

Proposition 4. *Proposition on the Alignment of the True and the Probable*: Suppose that L is the signature of Peano arithmetic and that K is the conjunction of the eight axioms of Robinson's Q. Suppose that $0 < \epsilon < \frac{1}{2}$. Suppose that P is an ω-additive probability assignment such that $P(K) > 1 - \epsilon$. Then (a) $(\omega, 0, S, +, \times, \leq) \models \varphi$ if and only if $P(\varphi) > 1 - \epsilon$, and (b) $(\omega, 0, S, +, \times, \leq) \models \varphi$ implies $P(\varphi \mathbin{\&} K) = P(K)$.

Proof. Let us first establish (b), and then show how (a) follows from (b). Hence, we now show by induction on the syntactic complexity of φ that $(\omega, 0, S, +, \times, \leq) \models \varphi$ implies $P(\varphi \mathbin{\&} K) = P(K)$. (i) First, by the Σ^0_1-completeness of Robinson's Q (cf. Proposition 1 in § 2.2) and by P1-P3, it follows that if φ is Σ^0_1 and $(\omega, 0, S, +, \times, \leq) \models \varphi$ then $P(\varphi \mathbin{\&} K) = P(K)$. (ii) Second, if $\varphi(x)$ is Δ^0_0 and $(\omega, 0, S, +, \times, \leq) \models \forall\, x\; \varphi(x)$, then by ω-additivity and (i) it follows that

$$P([\forall\, x\; \varphi(x)] \mathbin{\&} K) = \lim_n P([\bigwedge_{i=0}^{n} \varphi(S^i(0))] \mathbin{\&} K) = P(K) \qquad (\text{A}.1)$$

(iii) Suppose that the statement is true for all Σ^0_n and Π^0_n-formulas for $n \geq 1$. Suppose that $\varphi(x)$ is Σ^0_n or Π^0_n. Suppose that $(\omega, 0, S, +, \times, \leq) \models \exists\, x\; \varphi(x)$. Then $(\omega, 0, S, +, \times, \leq) \models \varphi(S^m(0))$ for some $m \geq 0$. Then by the induction

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hypothesis and ω-additivity it follows that

$$P([\exists\ x\ \varphi(x)]\ \&\ K) = \lim_n P(\bigvee_{i=0}^{n}[\varphi(S^i(0))\ \&\ K]) \geq P(\varphi(S^m(0))\ \&\ K) = P(K)$$

(A.2)

But it also follows from P1-P3 that $P([\exists\ x\ \varphi(x)]\ \&\ K) \leq P(K)$, so that in fact we have $P([\exists\ x\ \varphi(x)]\ \&\ K) = P(K)$. Now, suppose that the structure $(\omega, 0, S, +, \times, \leq) \models \forall\ x\ \varphi(x)$. Then by the induction hypothesis and ω-additivity it follows that

$$P([\forall\ x\ \varphi(x)]\ \&\ K) = \lim_n P([\bigwedge_{i=0}^{n}\varphi(S^i(0))]\ \&\ K) = P(K) \qquad (A.3)$$

Hence, we have finished verifying (b). Now let us explain how this implies (a). For (a), note that it suffices to prove the left-to-right direction. For suppose that we knew the left-to-right direction, i.e. suppose we knew that $(\omega, 0, S, +, \times, \leq) \models \varphi$ implies $P(\varphi) > 1 - \epsilon$. To prove the right-to-left direction, suppose for the sake of contradiction that we are given sentence φ such that $P(\varphi) > 1 - \epsilon$ and $(\omega, 0, S, +, \times, \leq) \models \neg\varphi$. Then by the left-to-right direction, it follows that $P(\neg\varphi) > 1 - \epsilon$ and from P1-P3 it follows that $1 - P(\varphi) > 1 - \epsilon$ and hence $P(\varphi) < \epsilon < \frac{1}{2} < 1 - \epsilon < P(\varphi)$, which is a contradiction. Hence, in fact, for (a) it suffices to prove the left-to-right direction. Now, note that the left-to-right direction of (a) follows almost automatically from (b). For, it follows from P1-P3 and (b) that $P(\varphi) \geq P(\varphi\ \&\ K) = P(K) > 1 - \epsilon$. $\qquad \square$

Proposition 5. *Counterexample to Dutch Book Theorem, ω-additive Version*: There is a function P from L-sentences to real numbers such that (i) P is *not* an ω-additive probability assignment, and such that (ii) P has the following property: for every infinite sequence of real numbers s_n and every infinite sequence of L-sentences φ_n such that the sequence $s_n P(\varphi_n)$ is absolutely convergent, there is a complete consistent L-theory T such that $\sum_n s_n(T(\varphi_n) - P(\varphi_n)) \geq 0$.

Proof. Choose a complete consistent L-theory T_0 such that T_0 implies Robinson's Q and such that T_0 proves $\neg\psi$, where ψ is true on the standard model and where $\psi \equiv \forall\ x\ \psi_0(x)$ begins with a universal quantifier followed by a quantifier-free formula $\psi_0(x)$ or by a formula $\psi_0(x)$ whose quantifiers are bounded. For instance, the claim that x is always strictly less than $2x$

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for non-zero values of x can be expressed in this way, as well as the consistency statement for the Peano axioms. Given this sentence ψ and this theory T_0, then define a function P from L-sentences to real numbers by setting $P(\varphi) = T_0(\varphi)$. By the Σ_1^0-completeness of Robinson's Q (cf. Proposition 1 in § 2.2), it follows that

$$P(\forall\, x\, \psi_0(x)) = T_0(\psi) = 0 \neq 1 = \lim_N T_0(\bigwedge_{n=0}^{N} \psi_0(S^n(0))) = \lim_N P(\bigwedge_{n=0}^{N} \psi_0(S^n(0)))$$
$$\text{(A.4)}$$

This implies (i), namely that P is *not* an ω-additive probability assignment. For (ii), note that this follows trivially, since we may choose $T = T_0$. $\qquad\square$

Appendix B. Proofs of Results Stated in § 3.2 and § 2.2

In this section we present proofs of results from § 3.2 and § 2.2, namely the theorem on the non-computability of probability assignments (Theorem 3 from § 3.2), the theorem on continuous probability assignments (Theorem 4 from § 3.2), and the propositions on high degree of confirmation (Proposition 2 and 3 from § 2.2).

Theorem 3. *Theorem on Non-Computability of Probability Assignments*: Suppose that $P : \mathrm{Sent}(L) \to \mathbb{R}$ is a probability assignment that assigns the eight axioms of Robinson's Q non-zero probability. Then the set of pairs $\{(\varphi, r) \in \mathrm{Sent}(L) \times \mathbb{Q} : P(\varphi) > r\}$ is non-computable.

Proof. For ease of reference, let us abbreviate

$$\widetilde{P} = \{(\varphi, r) \in \mathrm{Sent}(L) \times \mathbb{Q} : P(\varphi) > r\} \qquad\qquad \text{(B.1)}$$

Here we are viewing both sentences in the formal language of arithmetic, as well as rational numbers, as coded by natural numbers in the usual way, so that $\widetilde{P}$ can be viewed as a subset of natural numbers. Hence, it suffices to show that $\widetilde{P}$ is non-computable. For this, it suffices to show $\widetilde{P}$ computes a complete consistent extension of Robinson's Q, which is known to be non-computable by work of Tarski (cf. [43] Theorem 9 p. 60, [44] Theorem 2 p. 19). To this end, enumerate the sentences $\mathrm{Sent}(L)$ in the signature L of Robinson's Q as $\varphi_1, \ldots, \varphi_n, \ldots$ in such a way that φ_1 is the conjunction of the eight axioms of Robinson's Q. Supposing that P is a probability assignment such that $P(\varphi_1) > 0$, it must be shown that P computes a complete consistent

extension T_P of Robinson's Q. Let $T_P(\varphi_1) = 1$, and suppose that for all $i < n$ it has already been decided whether to set $T_P(\varphi_i) = 0$ or $T_P(\varphi_i) = 1$ in such a way that for $i < n$:

$$0 < P(\bigwedge_{T_P(\varphi_i)=1} \varphi_i \ \& \ \bigwedge_{T_P(\varphi_i)=0} \neg\varphi_i) \tag{B.2}$$

Then it follows from P1-P3 that

$$0 < P([\bigwedge_{T_P(\varphi_i)=1} \varphi_i] \ \& \ \varphi_n \ \& \ [\bigwedge_{T_P(\varphi_i)=0} \neg\varphi_i]) + P([\bigwedge_{T_P(\varphi_i)=1} \varphi_i] \ \& \ [\bigwedge_{T_P(\varphi_i)=0} \neg\varphi_i] \ \& \ \neg\varphi_n)$$
$$\tag{B.3}$$

so that at least one of the two quantities featured in this sum is strictly positive. If one computes from $\widetilde{P}$ that the first quantity is strictly positive, then set $T(\varphi_n) = 1$ and $T(\neg\varphi_n) = 0$, and otherwise do the converse. This construction results in complete theory T_P which extends Robinson's Q and which is computable from $\widetilde{P}$. Further, this theory is consistent, since if not, then there is some finite fragment of the theory which proves a contradiction. Since the axioms P1-P3 imply that contradictions are assigned probability zero, and since they likewise imply that equivalent sentences are assigned the same probability, and since anything which proves a contradiction is equivalent to a contradiction, it follows from P1-P3 that the conjunction of some finite fragment of T_P would be assigned probability zero. This contradicts our construction, in which all the finite fragments of T_P were assigned non-zero probability. $\qquad\square$

So now we turn to developing the materials needed to prove the theorem on continuous probability assignments (Theorem 4). Recall from § 3.2 that a *finite counting assignment* is a probability assignment P of the following form:

$$P(\varphi) = a_1 \cdot T_1(\varphi_1) + \cdots + a_n \cdot T(\varphi_n) \tag{B.4}$$

where $T_1, \ldots, T_n$ are distinct complete theories and $a_1 + \cdots + a_n = 1$ and $a_i > 0$. Note that any function P satisfying equation (B.4) is in fact a probability assignment. For, to see that P3 is satisfied, note that if $\models \neg(\varphi \wedge \psi)$ then since T_i is complete we have $T_i(\varphi \vee \psi) = T_i(\varphi) + T_i(\psi)$.

The theorem on continuous probability shows that there are many probability assignments that are not finite counting assignments. It does this by showing that we can build such assignments that interact well with consistent computable extensions of Robinson's Q, of which there are known to

be many. Hence, let us now introduce some terminology on how theories and probability assignments may relate to one another. Regrettably, even though these notions are absolutely basic, there seems to be no established terminology for these notions. So letting K be a theory in a language L, let us stipulate that a probability assignment $P : \mathrm{Sent}(L) \to \mathbb{R}$ is *K-sound* if $K \vdash \varphi$ implies $P(\varphi) = 1$; and that P is *K-regular* if $P(\varphi) = 1$ implies $K \vdash \varphi$. It's perhaps helpful to explicitly point out that P is K-regular iff $P(\varphi) = 0$ implies $K \vdash \neg\varphi$.

The notion of a finite counting assignment was motivated in § 3.2 by way of a philosophical dilemma for the kinds of empiricism considered in this paper. However, it is important to stress that it is connected to the well-studied mathematical notion of a continuous probability measure. To see this connection, let us briefly recall some basic concepts from descriptive set theory and measure theory. Suppose that K is a theory in a countable language L. Then let $[K]$ be the topological space of complete consistent extensions of K, where the topology is generated by the classes $[\varphi] = \{T \in [K] : T \models \varphi\}$ (cf. [28] p. 41). Suppose that P is a K-sound probability assignment. Then the Carathéodory Extension Theorem ([45] Theorem 1.14 p. 31, [46] Proposition 17.6 p. 106) implies that there is a unique probability assignment $\widehat{P} : \mathrm{Borel}([K]) \to \mathbb{R}$ such that $\widehat{P}([\varphi]) = P(\varphi)$, where $\mathrm{Borel}([K])$ designates the Borel subsets of $[K]$. Finally, we say that an *atom* of $\widehat{P}$ is a theory $T \in [K]$ such that $\widehat{P}(\{T\}) > 0$. Further $\widehat{P}$ is said to be *continuous* if it has no atoms (cf. [46] p. 117).

To see the connection with finite counting measures let us prove the following proposition, which describes exactly the atoms of $\widehat{P}$ when P is a finite counting measure. First let us briefly review some notation on finite strings used in this proof and some subsequent proofs. The set of finite strings $\sigma, \tau, \rho, \ldots$ of zeros and one is designated as $2^{<\omega}$, and elements of this set are written with lower-case Greek letters, perhaps subscripted. One writes $\sigma \preceq \tau$ if τ extends σ as a string, and one writes $|\sigma|$ for the length of σ, and one views σ as a function $\sigma : \{0, \ldots, |\sigma| - 1\} \to \{0, 1\}$. Finally, we write $\sigma \perp \tau$ if σ and τ have no common extension. Now, let $\varphi_1, \ldots, \varphi_n, \ldots$ be a fixed enumeration of $\mathrm{Sent}(L)$ and for $\sigma \in 2^{<\omega}$ define the following sentence, wherein i range over numbers $< |\sigma|$:

$$\chi_\sigma \equiv (\bigwedge_{\sigma(i)=1} \varphi_i) \wedge (\bigwedge_{\sigma(i)=0} \neg\varphi_i) \tag{B.5}$$

These sentences will be useful in what follows since they allow us to define

finite initial segments of complete theories in terms of finite strings of ze-
ros and ones. With this terminology in place, let us prove the proposition
describing the atoms of $\widehat{P}$ when P is a finite counting measure:

Proposition 6. Suppose that P is a finite counting assignment as in equa-
tion (B.4), and further suppose that P is K-sound. Then $\widehat{P}(\{T_i\}) > 0$.
Further, if $\widehat{P}(\{T\}) > 0$ then $T = T_i$ for some i.

Proof. For $\ell > 0$ define $T_i \upharpoonright \ell \in 2^{<\omega}$ by $(T_i \upharpoonright \ell)(j) = 1$ iff $T_i(\varphi_j) = 1$. Note
that $\{T_i\} = \bigcap_{\ell>0}[\chi_{T_i \upharpoonright \ell}]$. Hence by continuity from below (cf. [45] Theorem
1.8 p. 25) we have obtain the first identity in the following equation:

$$\widehat{P}(\{T_i\}) = \lim_\ell \widehat{P}([\chi_{T_i \upharpoonright \ell}]) = a_i > 0 \tag{B.6}$$

The second identity in the above equation follows from the fact that since
there are only finitely many complete distinct theories $T_1, \ldots, T_n$, there is
some sentence φ_{n_i} such that $T_j(\varphi_{n_i}) = 1$ iff $j = i$. Now suppose, for the sake
of contradiction, that $\widehat{P}(\{T\}) > 0$ but that $T \neq T_i$ for all i. Choose ψ_i such
that $T(\psi_i) = 1$ but $T_i(\psi_i) = 0$. Then $P(\bigwedge_{i=1}^n \psi_i) = 0$ and hence $\widehat{P}(\{T\}) = \lim_\ell \widehat{P}([\chi_{T \upharpoonright \ell}]) \leq P(\bigwedge_{i=1}^n \psi_i) = 0$. $\qquad \square$

Now we show how to characterize the continuity of $\widehat{P}$ in terms of proper-
ties of P. This is important both for the proof of Theorem 4, as well as for
a philosophical point made at the end of § 3.2, namely that we can isolate
an axiomatic condition on P that guarantees that it is not a finite counting
assignment. In particular, if P satisfies equation (B.7) in the below proposi-
tion, then it cannot be a finite counting assignment. For, if it were a finite
counting assignment, then by the previous proposition $\widehat{P}$ would have atoms.

Proposition 7. Suppose that K is a countable theory and that P is K-
sound. Then $\widehat{P}$ is continuous iff

$$\forall \, \epsilon > 0 \, \exists \, \ell > 0 \, \forall \, \sigma \, (|\sigma| = \ell \to P(\chi_\sigma) < \epsilon) \tag{B.7}$$

Proof. First suppose that $\widehat{P}$ is continuous but that

$$\exists \, \epsilon > 0 \, \forall \, \ell > 0 \, \exists \, \sigma \, (|\sigma| = \ell \wedge P(\chi_\sigma) \geq \epsilon) \tag{B.8}$$

Consider the following set of strings $S = \{\sigma \in 2^{<\omega} : P(\chi_\sigma) \geq \epsilon\}$. Then
if $\sigma \in 2^{<\omega}$ and $\tau \preceq \sigma$ we have that $\models (\chi_\sigma \to \chi_\tau)$, so that $P(\chi_\tau) \geq P(\chi_\sigma) \geq \epsilon$.

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So S is a tree, and further the hypothesis in equation (B.8) implies that S is a infinite tree, so that by König's Lemma it follows that it has a path T. Then we have $\widehat{P}(\{T\}) = \lim_\ell P(\chi_{T\restriction\ell}) \geq \epsilon > 0$, which contradicts our assumption that $\widehat{P}$ is continuous. Now, conversely, suppose for the sake of contradiction that equation (B.7) holds but that $\widehat{P}$ is not continuous. Then $\widehat{P}(\{T\}) > 0$ for some $T \in [K]$. Choose $\epsilon > 0$ such that $\widehat{P}(\{T\}) \geq \epsilon > 0$. Then by hypothesis in equation (B.7), we have $\widehat{P}(\chi_{T\restriction\ell_0}) < \epsilon$ for some $\ell_0 > 0$. Then we have $\widehat{P}(\{T\}) = \lim_\ell \widehat{P}(\chi_{T\restriction\ell}) < \epsilon$. $\qquad\square$

Now we focus on establishing Theorem 4. The proof of this theorem is a recursion-theoretic forcing argument (cf. [47] Chapter 13 pp. 273 ff, [48] Chapter 12 pp. 737 ff). Hence, the idea is to first define a partial order in Definition 5 below, then to show in the next two propositions that certain sets are dense in this partial order. Then, as usual, the theorem (Theorem 4) follows from showing that we can effectively construct (in an oracle) a set that is generic for these dense sets. To define this partial order, let us first introduce some further notation. Suppose that $s = \langle\varphi_0, \ldots, \varphi_{\ell-1}\rangle$ is a finite sequence of L-sentences. If $t = \langle\psi_0, \ldots, \psi_{m-1}\rangle$ is another finite sequence, then we say that $t \subseteq s$ if $\{\psi_0, \ldots, \psi_{m-1}\} \subseteq \{\varphi_0, \ldots, \varphi_{\ell-1}\}$. For $\sigma \in 2^{<\omega}$ with $|\sigma| = \ell$ we define the following, where the conjunctions are over $i < \ell$:

$$\chi_\sigma^s = \bigwedge_{\sigma(i)=1} \varphi_i \wedge \bigwedge_{\sigma(i)=0} \neg\varphi_i \tag{B.9}$$

Further, we define:

$$\widehat{s} = \{\bigvee_{j=1}^m \chi_{\sigma_j}^s : |\sigma_1| = \cdots = |\sigma_m| = \ell\} \tag{B.10}$$

Note that for $\sigma \neq \tau$ we have that the χ_σ^s and χ_τ^s are incompatible. Also, note that $\widehat{s}$ is closed under the propositional connectives, in the sense that if φ, ψ is in $\widehat{s}$ then there is ξ in $\widehat{s}$ such that ξ is equivalent to $\neg\varphi$ (resp. to $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$). Finally, note that each element φ_i of $s = \langle\varphi_0, \ldots, \varphi_{\ell-1}\rangle$ is equivalent to an element of $\widehat{s}$, namely the disjunction over all $\chi_{\sigma_j}^s$ such that $\sigma_j(i) = 1$. Hence, by abuse of notation, we may assume that $s \subseteq \widehat{s}$. Now we can present the definition of the partial order:

Definition 5. Suppose that K is a countable theory. If $s = \langle\varphi_0, \ldots, \varphi_{\ell-1}\rangle$ is a finite sequence of L-sentences, then we say that $P : \widehat{s} \to \mathbb{Q}$ is a K-*sound s-probability-assignment* if it satisfies P1-P3 and K-soundness with respect to

the sentences in $\widehat{s}$. The collection of all K-sound s-probability assignments for all s is designated as $\mathbb{P}[K]$. There is a natural partial order on $\mathbb{P}[K]$ defined by extension. In particular, if P' is an K-sound s'-probability-assignment and P is a K-sound s-probability assignment, then we write $P' \leq P$ iff $s' \supseteq s$ and $P' \upharpoonright \widehat{s} = P$. Finally, if $P \in \mathbb{P}$ and P is a K-sound s-probability assignment, then we write $\mathrm{dom}_0(P) = s$ and $\mathrm{dom}(P) = \widehat{s}$. Finally, we define $\mathbb{P}_{\mathrm{reg}}[K]$ to be the subset of elements P of $\mathbb{P}[K]$ satisfying the additional condition of K-regularity with respect to the sentences in the set $\widehat{s} = \mathrm{dom}(P)$.

Note that the condition $P' \upharpoonright \widehat{s} = P$ in the definition of extension makes good sense since each element of $\widehat{s}$ is equivalent to an element of $\widehat{s'}$. In particular, if s has length ℓ and s' has length $\ell' \geq \ell$ and $|\sigma| = \ell$, then

$$\chi_\sigma^s \equiv \bigvee_{\tau \succeq \sigma \ \& \ |\tau|=\ell'} \chi_\tau^{s'} \tag{B.11}$$

Finally, if P_0 is a K-sound $\widehat{s}$-probability assignment and P is a K-sound probability assignment, then we may define $P \leq P_0$ to simply mean that $P \upharpoonright \widehat{s} = P_0$.

Note that P being a K-sound s-probability-assignment is a $\Pi_1^{0,K}$-condition since provability occurs in the antecedent of conditions in the P1-P3 conditions and K-soundness, and that K-regularity is correspondingly a $\Sigma_1^{0,K}$-condition. Thus, both $\mathbb{P}[K]$ and $\mathbb{P}_{\mathrm{reg}}[K]$ are computable in the halting set when K is computable. Now we turn to showing that various sets are dense in our partial order. Recall that D is *dense* in a partial order $\mathbb{P}$ if for every $P \in \mathbb{P}$ there is $P' \in D$ such that $P' \leq P$. The basic idea of the below proposition is very clear: we're trying to find a way to extend a probability assignment defined on n sentences to a probability assignment defined on $n+1$ sentences. The only thing that makes this harder than the obvious proof in the propositional case is that the $(n+1)$-st sentence might not be independent of the earlier sentences.

Proposition 8. Suppose that ψ is an L-sentence. Then the set $D_\psi = \{P \in \mathbb{P}[K] : \mathrm{dom}_0(P) \ni \psi\}$ is dense in $\mathbb{P}[K]$ and $\mathbb{P}_{\mathrm{reg}}[K]$. In particular, if P is a K-sound s-probability assignment in $\mathbb{P}[K]$ (resp. in $\mathbb{P}_{\mathrm{reg}}[K]$), and $s' = s^\frown \langle \psi \rangle$, then there is a K-sound s'-probability assignment P' extending P and in $\mathbb{P}[K]$

(resp. in $\mathbb{P}_{\mathrm{reg}}[K]$) such that

$$
P'(\chi_\sigma^s \wedge \psi) = \begin{cases} P(\chi_\sigma^s) & \text{if } K \wedge \chi_\sigma^s \models \psi, \\ 0 & \text{if } K \wedge \chi_\sigma^s \models \neg\psi, \\ \frac{1}{2}P(\chi_\sigma^s) & \text{otherwise,} \end{cases} \tag{B.12}
$$

$$
P'(\chi_\sigma^s \wedge \neg\psi) = \begin{cases} 0 & \text{if } K \wedge \chi_\sigma^s \models \psi, \\ P(\chi_\sigma^s) & \text{if } K \wedge \chi_\sigma^s \models \neg\psi, \\ \frac{1}{2}P(\chi_\sigma^s) & \text{otherwise,} \end{cases} \tag{B.13}
$$

Proof. It suffices to show that if we define P' by using equations (B.12)-(B.13) and extend additively, then we satisfy P1-P3 and K-soundness (resp. K-regularity). From this definition, P1 and P3 are trivial. Note that it is also trivial that $P'(\chi_\sigma^s) = P(\chi_\sigma^s)$ since P' is defined by extending additively:

$$
P'(\chi_\sigma^s) = P'((\chi_\sigma^s \wedge \psi) \vee (\chi_\sigma^s \wedge \neg\psi)) = P'(\chi_\sigma^s \wedge \psi) + P'(\chi_\sigma^s \wedge \neg\psi) = P(\chi_\sigma^s) \tag{B.14}
$$

So we focus on verifying K-soundness, from which P2 trivially follows. So suppose that $K \models \xi$ where

$$
\xi \equiv \bigvee_{i=1}^{n}(\chi_{\sigma_i}^s \wedge \psi) \vee \bigvee_{j=1}^{m}(\chi_{\tau_j}^s \wedge \neg\psi) \tag{B.15}
$$

Let us define $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ and $\Theta = \{\tau_1, \ldots, \tau_m\}$. First consider the case where $\Sigma = \emptyset$ or $\Theta = \emptyset$. Without loss of generality, we consider the case where $\Theta = \emptyset$. Then since $K \models \psi$, we have $K \wedge \chi_{\sigma_i}^s \models \psi$. So we may argue as follows:

$$
P'(\bigvee_{i=1}^{n} \chi_{\sigma_i}^s \wedge \psi) = \sum_{i=1}^{n} P'(\chi_{\sigma_i}^s \wedge \psi) = \sum_{i=1}^{n} P(\chi_{\sigma_i}^s) = P(\bigvee_{i=1}^{n} \chi_{\sigma_i}^s) = 1 \tag{B.16}
$$

So now we move onto the case where $\Sigma \neq \emptyset$ and $\Theta \neq \emptyset$. Then we define: $R = \Sigma \cap \Theta$ and $\Sigma' = \Sigma \setminus R$ and $\Theta' = \Theta \setminus R$. Now note that if $\rho \notin \Sigma \cup \Theta$ then $K \models \neg\chi_\rho^s$. Further, we have that $K \wedge \chi_{\sigma_i}^s \models \psi$ for $\sigma_i \in \Sigma'$ and $K \wedge \chi_{\tau_j}^s \models \neg\psi$ for

$\tau_j \in \Theta'$. Then we may argue

$$
\begin{aligned}
1 &= P(\bigvee_{\rho \notin \Sigma \cup \Theta} \chi_\rho^s) + P(\bigvee_{\rho \in R} \chi_\rho^s) + P(\bigvee_{\sigma \in \Sigma'} \chi_\sigma^s) + P(\bigvee_{\tau \in \Theta'} \chi_\tau^s) \\
&= \sum_{\rho \notin \Sigma \cup \Theta} P(\chi_\rho^s) + \sum_{\rho \in R} P(\chi_\rho^s) + \sum_{\sigma \in \Sigma'} P(\chi_\sigma^s) + \sum_{\tau \in \Theta'} P(\chi_\tau^s) \\
&= 0 + \sum_{\rho \in R}[P'(\chi_\rho^s \wedge \psi) + P'(\chi_\rho^s \wedge \neg\psi)] + \sum_{\sigma \in \Sigma'} P'(\chi_\sigma^s \wedge \psi) + \sum_{\tau \in \Theta'} P'(\chi_\tau^s \wedge \neg\psi) \\
&= P'(\xi) \tag{B.17}
\end{aligned}
$$

Now, in the case where P satisfied K-regularity, we also want to verify that P' is regular. So suppose that $P(\xi) = 0$ where ξ is from $\widehat{s'}$. Since ξ is from $\widehat{s'}$, it is a disjunction as in equation (B.15). So each of these disjuncts also has probability zero under P'.

Consider first a disjunct of the form $\chi_{\sigma_i}^s \wedge \psi$. Since $P'(\chi_{\sigma_i}^s \wedge \psi) = 0$, we claim that the definition in equation (B.12) and K-regularity of P requires that $K \models \neg(\chi_{\sigma_i}^s \wedge \psi)$, or what is the same $K \models \neg\chi_{\sigma_i}^s \vee \neg\psi$. For, there are three case to consider, depending on the three cases in the definition in equation (B.12). First suppose that $K \wedge \chi_{\sigma_i}^s \models \psi$. Then $0 = P'(\chi_{\sigma_i}^s \wedge \psi) = P(\chi_{\sigma_i}^s)$, which by the K-regularity of P implies that $K \models \neg\chi_{\sigma_i}^s$. Second suppose that $K \wedge \chi_{\sigma_i}^s \models \neg\psi$. Then we have $K \models \chi_{\sigma_i}^s \to \neg\psi$ or what is the same: $K \models \neg\chi_{\sigma_i}^s \vee \neg\psi$. Third suppose that neither $K \wedge \chi_{\sigma_i}^s \models \psi$ nor $K \wedge \chi_{\sigma_i}^s \models \neg\psi$. Then $0 = P'(\chi_{\sigma_i}^s \wedge \psi) = \frac{1}{2}P(\chi_{\sigma_i}^s)$, which by the K-regularity of P requires that $K \models \neg\chi_{\sigma_i}^s$.

Consider second a disjunct of the form $\chi_{\tau_j}^s \wedge \neg\psi$. Now we argue that since $P'(\chi_{\tau_j}^s \wedge \neg\psi) = 0$, the definition in equation (B.13) and K-regularity of P requires that $K \models \neg(\chi_{\tau_j}^s \wedge \neg\psi)$, or what is the same $K \models \neg\chi_{\tau_j}^s \vee \psi$. For, there are three case to consider, depending on the three cases in the definition in equation (B.13). First suppose that $K \wedge \chi_{\tau_j}^s \models \psi$. Then trivially we have $K \models \chi_{\tau_j}^s \to \psi$ or what is the same $K \models \neg\chi_{\tau_j}^s \vee \psi$. Second, suppose that $K \wedge \chi_{\tau_j}^s \models \neg\psi$. Then we have $0 = P'(\chi_{\tau_j}^s \wedge \neg\psi) = P(\chi_{\tau_j}^s)$, which by the K-regularity of P requires that $K \models \neg\chi_{\tau_j}^s$. Third suppose that neither $K \wedge \chi_{\tau_j}^s \models \psi$ nor $K \wedge \chi_{\tau_j}^s \models \neg\psi$. Then $0 = P'(\chi_{\tau_j}^s \wedge \neg\psi) = \frac{1}{2}P(\chi_{\tau_j}^s)$, which by the K-regularity of P requires that $K \models \neg\chi_{\tau_j}^s$.

Putting all this information together, we have that $K \models \neg(\chi_{\sigma_i}^s \wedge \psi)$ and $K \models \neg(\chi_{\tau_j}^s \wedge \neg\psi)$. Since the conjunction of all these is equivalent to $\neg\xi$, we have that $K \models \neg\xi$, which is what we wanted to show. $\square$

Now we focus on our second density proposition, which by Proposition 7 is obviously what we need if we want our generic to be continuous and hence not a finite counting assignment:

Proposition 9. Suppose that K is a consistent computable extension of Robinson's Q. Suppose that $\epsilon > 0$. Then the following set is dense in $\mathbb{P}[K]$ and $\mathbb{P}_{\mathrm{reg}}[K]$:

$$D_\epsilon = \{P \in \mathbb{P}[K] : \mathrm{dom}_0(P) = s = \langle \varphi_0, \ldots, \varphi_{\ell-1} \rangle \ \& \ [\forall \sigma \ |\sigma| = \ell \Longrightarrow P(\chi_\sigma^s) < \epsilon]\} \tag{B.18}$$

Proof. It suffices to show how to extend P with $\mathrm{dom}(P) = s = \langle \varphi_0, \ldots, \varphi_{\ell-1} \rangle$ to P' in $\mathbb{P}[K]$ (resp. in $\mathbb{P}_{\mathrm{reg}}[K]$) where $\mathrm{dom}(P) = s' = s^\frown \langle \psi \rangle$ in such a way that we have $P'(\chi_\sigma^s \wedge \psi) = P'(\chi_\sigma^s \wedge \neg\psi) = \frac{1}{2}P(\chi_\sigma^s)$. For by repeatedly applying this, we may successively "halve" the probabilities until we get below ϵ. But by the previous proposition, and in particular equations (B.12)-(B.13), it suffices to find ψ that is independent of all $K \wedge \chi_\sigma^s$ with $P(\chi_\sigma^s) > 0$. So let

$$C = \{\sigma \in 2^{<\omega} : |\sigma| = \ell \ \& \ P(\chi_\sigma^s) > 0\} \tag{B.19}$$

Hence for each $\sigma \in C$ we have that $K \wedge \chi_\sigma^s$ is a consistent computable extension of Robinson's Q. For each $\sigma \in C$ choose φ_σ which is independent of $K \wedge \chi_\sigma^s$ (cf. [1] Theorem 2.10 p. 161, [44] Theorem 2 p. 26). Define the sentence $\psi = \bigwedge_{\sigma \in C}(\chi_\sigma^s \to \varphi_\sigma)$. Then $K \wedge \chi_\sigma^s \nvDash \psi$ for all $\sigma \in C$. For, if $\sigma_0 \in C$ and $K \wedge \chi_{\sigma_0}^s \vDash \psi$, then we have $K \wedge \chi_{\sigma_0}^s \vDash \varphi_{\sigma_0}$. Likewise, $K \wedge \chi_\sigma^s \nvDash \neg\psi$ for all $\sigma \in C$. For, suppose $\sigma_0 \in C$ and $K \wedge \chi_{\sigma_0}^s \vDash \neg\psi$. Since $K \wedge \chi_{\sigma_0}^s \nvDash \neg\varphi_{\sigma_0}$, choose a model M of K and $\chi_{\sigma_0}^s$ and φ_{σ_0} and hence also of $\neg\psi$. Since the χ_ρ^s's for $\rho \in C$ are all inequivalent, the only disjunct of $\bigvee_{\sigma \in C}(\chi_\sigma^s \wedge \neg\varphi_\sigma)$ which can be true on M is the disjunct corresponding to σ_0. But this of course implies that $M \vDash \neg\varphi_{\sigma_0}$. Hence, in fact we have found ψ. $\qquad\square$

Finally, putting all of these things together, we may prove our last theorem of this appendix. Note that there's an obvious connection to the notion of computable approximations described at the end of § 3.2 since anything computable in the halting set is computably approximable.

Theorem 4. *Theorem on Continuous Probability Assignments*: Suppose that K is a consistent computable extension of Robinson's Q. Then there is a K-sound K-regular probability assignment P such that P is not a finite counting assignment. Further, P may be chosen so that the set of pairs $\{(\varphi, r) \in \mathrm{Sent}(L) \times \mathbb{Q} : P(\varphi) > r\}$ is computable in the halting set. Further, for any $P_0 \in \mathbb{P}_{\mathrm{reg}}[K]$, P can be chosen so that $P \leq P_0$.

Proof. Define a subset $G \subseteq \mathbb{P}_{\mathrm{reg}}[K]$ as follows computably in the halting set. Let P_0 be some arbitrary element of $\mathbb{P}_{\mathrm{reg}}[K]$. Suppose $P_n \leq P_{n-1} \leq \cdots \leq P_0$ has been defined. If n is even, choose $P_{n+1} \leq P_n$ from $\mathbb{P}_{\mathrm{reg}}[K]$ such that P_{n+1} is in D_φ where $\frac{n}{2}$ is the Gödel number of φ. If n is odd, choose $P_{n+1} \leq P_n$ such that P_{n+1} is in D_ϵ where $\epsilon = 2^{-n}$. Now define a K-sound K-regular probability assignment P by setting $P(\varphi) = P'(\varphi)$ for any $P' \in G$ with $\varphi \in \mathrm{dom}_0(P')$. We can do this since G intersects D_φ. Further, by Proposition 7, P is continuous since it intersects each D_ϵ for $\epsilon > 0$. Further, clearly this definition is recursive in the halting set since the sets D_φ and D_ϵ are computable in the halting set for rational ϵ. $\qquad\square$

Now we turn to the proofs of Propositions 2 and 3 from § 2.2. Given the aims of this paper, it's natural to ask whether the condition of "not being a finite counting assignment" is compatible with their actually being instances of mathematical induction whose consequent is strongly confirmed by its antecedent against the background of theories such as Robinson's Q. An affirmative answer to this question is established by Proposition 3. However, let us begin with some preliminary observations related to a preliminary proposition (namely, Proposition 2 below). As mentioned in § 1, for a hypothesis h to be confirmed by evidence e against background knowledge K means that

$$0 < P(h|e \ \& \ K) - P(h|K) \qquad\qquad (\mathrm{B.20})$$

Since we are viewing probability assignments as functions $P : \mathrm{Sent}(L) \to \mathbb{R}$, the condition in equation (B.20) only makes sense if K is a finitely axiomatized theory so that we can view it as the conjunction of its members. Further, if $K \vdash e$, then the condition in equation (B.20) cannot occur, since in this circumstance we will have $P(h|e \ \& \ K) = P(h|K)$, at least if the quotients are defined. Likewise, if $K + h$ is inconsistent, then similarly the condition in equation (B.20) cannot occur, since in this circumstance we will have $P(h|e \ \& \ K) = 0 = P(h|K)$, assuming the quotients are defined. These elementary observations thus motivate restricting attention to when K is finitely axiomatized, $K \nvdash e$, and $K + h$ is consistent. Finally, in many of our canonical applications discussed in this paper, we also have that $K + h \vdash e$. When all these conditions are met, the following proposition indicates that one may construct probability assignments wherein the degree of confirmation is high. This proposition was first stated at the close of § 2.2:

Proposition 2. Let K be a finitely axiomatized L-theory. Suppose that h, e are L-sentences such that $K + h \vdash e$ and $K \nvdash e$ and $K + h$ is consistent.

Let $\epsilon > 0$ with $\epsilon < 1$. Then there is a probability assignment such that

$$1 - \epsilon \leq P(h|e \ \& \ K) - P(h|K) < 1 \tag{B.21}$$

Moreover, P may be taken to be K-sound and K-regular.

Proof. Enumerate the L-sentences φ_n such that $K + e \nvdash \varphi_n$, where without loss of generality we may assume that $\varphi_1 \equiv \neg h$. This is because if $K + e \vdash \neg h$ then $K + h \vdash \neg e$, so that $K + h$ proves both e and $\neg e$, and hence $K + h$ would not be consistent, contrary to hypothesis. Enumerate the L-sentences φ_n^* such that $K \nvdash \varphi_n^*$ and $K + e \vdash \varphi_n^*$, where without loss of generality we may assume that $\varphi_1^* = e$. Choose witnessing L-structures $M_n \models K + e + \neg\varphi_n$ and $M_n^* \models K + \neg\varphi_n^* + \neg e$, noting that $M_1 \models h$ and $M_n^* \models \neg e \wedge \neg h$ and $M_n \models e$. Let T_n be the complete theory of M_n and let T_n^* be the complete theory of M_n^*. Let $\eta = \frac{\epsilon}{2}$ and let $\delta = 1 - \eta$. Define two sequences of real numbers a_n and a_n^* by $a_1 = \delta\eta$ and $a_n = \frac{(1-\delta)\eta}{2^{n-1}}$ for $n > 1$ and $a_n^* = \frac{1-\eta}{2^n}$ for $n > 0$, noting that $\sum_{n=1}^{\infty} a_n = \delta\eta + (1 - \delta)\eta = \eta$ and $\sum_{n=1}^{\infty} a_n^* = 1 - \eta$. Define a K-sound, K-regular probability assignment $P : \text{Sent}(L) \to \mathbb{R}$ as follows:

$$P(\psi) = \sum_{n=1}^{\infty} a_n T_n(\psi) + \sum_{n=1}^{\infty} a_n^* T_n^*(\psi) \tag{B.22}$$

Then since $T_1(h) = 1$ and $T_n^*(h) = 0$, we have that $\delta\eta \leq P(h) \leq \eta$. Further, since $T_n(e) = 1$ and $T_n^*(e) = 0$, we have that $P(e) = \eta$. Then since P is K-sound we have $P(K) = 1$, and hence we may calculate:

$$P(h|e \ \& \ K) - P(h|K) = \frac{P(h)}{P(e)} - P(h) \geq \frac{\delta\eta}{\eta} - \eta = \delta - \eta = 1 - \epsilon \quad \text{(B.23)}$$

Further, since $K + h \vdash e$, we have that $P(h) \leq P(e)$ and hence $P(h|e \ \& \ K) \leq 1$. Since $P(h|K) \geq \delta\eta > 0$, we have that $P(h|e \ \& \ K) - P(h|K) < 1$. $\qquad\square$

The above proposition applies directly to the setting where K is empty and h is an axiom of Robinson's Q and e is several instances of this universal arithmetical hypothesis h. To apply this proposition to the example of mathematical induction and to combine it with the the probability assignment not being a finite counting assignment, let us first introduce some notation. As is standard, let us abbreviate an instance of mathematical induction as:

$$I_\varphi \equiv [\varphi(0) \ \& \ \forall \, n \ \varphi(n) \to \varphi(n+1)] \to [\forall \, n \ \varphi(n)] \tag{B.24}$$

wherein $\varphi(x)$ is an L-formula with one distinguished free variable x. (The set of such formulas are henceforth abbreviated as $\mathrm{Form}(L)$). One of the ideas of this paper is that the consequent of I_φ may be confirmed by the antecedent of I_φ against the background of Robinson's Q. Let us further introduce some abbreviations for the antecedent and consequent of I_φ:

$$
\begin{align}
e(I_\varphi) &= \varphi(0) \,\&\, \forall\, n \; \varphi(n) \to \varphi(n+1) \tag{B.25}\\
h(I_\varphi) &= \forall\, n \; \varphi(n) \tag{B.26}
\end{align}
$$

Expressed in terms of these abbreviations, we have that $I_\varphi \equiv e(I_\varphi) \to h(I_\varphi)$. Using these abbreviations, we can now state and prove the following propositions, which says that so long as the background knowledge does not prove the antecedent of mathematical induction and is consistent with the consequent of the instance of mathematical induction, a high degree confirmation of the consequent by the antecedent is compatible with the probability assignment not being a finite counting assignment:

Proposition 10. Let K be a finitely axiomatized extension of Robinson's Q. Suppose that $\varphi(x)$ is an L-formula such that $K \nvdash e(I_\varphi)$ and $K + h(I_\varphi)$ is consistent. Let $\epsilon > 0$ such that $\epsilon < 1$. Then there is a K-sound, K-regular probability assignment which is not a finite counting assignment such that

$$
1 - \epsilon \le P(h(I_\varphi)|e(I_\varphi) \,\&\, K) - P(h(I_\varphi)|K) < 1 \tag{B.27}
$$

Proof. Since $K, h(I_\varphi), e(I_\varphi)$ satisfy the antecedents of the preceding proposition, we have that there is a K-sound, K-regular probability assignment P satisfying equation (B.27). Let $s = \langle K, e(I_\varphi), h(I_\varphi)\rangle$ and let $P_0 = P \restriction \widehat{s}$. Then by Theorem 4, there is a K-sound K-regular probability assignment P' which is not a finite counting assignment such that $P' \le P_0$. Since P', P_0, and P agree on the values of boolean combinations of $K, e(I_\varphi), h(I_\varphi)$, we have the equation (B.27) continues to hold when we replace P by P'. $\square$

So it's thus natural to ask whether there are any examples of L-formulas $\varphi(x)$ with $K \nvdash e(I_\varphi)$ and with $K + h(I_\varphi)$ being consistent. The following proposition directly implies that there are infinitely many of them, at least when K is true on the standard model. But, at the same time, the proposition says that there is no computable method for detecting them:

Proposition 11. Let K be a finitely axiomatized extension of Robinson's Q that is true on the standard model. Then following set C_K computes the halting set:

$$C_K = \{\varphi(x) \in \mathrm{Form}(L) : K \nvdash e(I_\varphi) \ \& \ K + h(I_\varphi) \text{ is consistent}\} \qquad \text{(B.28)}$$

Further, suppose that P is a K-sound K-regular probability assignment. Then the following set $C_{K,P}$ is equal to C_K and thus also computes the halting set:

$$C_{K,P} = \{\varphi(x) \in \mathrm{Form}(L) \ : \ 0 < P(h(I_\varphi)|e(I_\varphi) \ \& \ K) - P(h(I_\varphi)|K) < 1$$
$$\& \ P(e(I_\varphi) \ \& \ K), P(h(I_\varphi) \ \& \ K) > 0\} \ \text{(B.29)}$$

Proof. For C_K, it suffices to show that C_K can compute whether an arbitrary Π^0_1-sentence $\forall x \ \psi(x)$ is true on the standard model. First note that $h(I_\psi)$ is identical to $\forall x \ \psi(x)$. Second note that by the Σ^0_1-completeness of Robinson's Q, a Π^0_1-sentence is true on the standard model if and only if it is consistent with K. (For the Σ^0_1-completeness of Robinson's Q, see Proposition 1 in § 2.2).

Now, let $\theta(n)$ express that "there is no proof of $0 = 1$ of length $\leq n$ steps from the axioms of $K + PA$." Then $h(I_\theta)$ is identical to $\mathrm{Con}(K + PA)$. Let $\varphi(x) \equiv \psi(x) \wedge \theta(x)$, so that $h(I_\varphi) \equiv h(I_\psi) \wedge h(I_\theta)$. Hence $K + PA \nvdash h(I_\varphi)$ and thus $K \nvdash e(I_\varphi)$. Hence $\varphi(x) \in C_K$ if and only if $K + h(I_\varphi)$ is consistent, which happens if and only if $h(I_\psi) \wedge h(I_\theta)$ is true on the standard model. Since we know that $h(I_\theta)$ is true on the standard model, we thus have that $\varphi(x) \in C_K$ if and only if $h(I_\psi)$ is true on the standard model. Since φ was produced computably from ψ, we thus can thus compute from C_K whether an arbitrary Π^0_1-sentence $\forall x \ \psi(x)$ is true on the standard model.

Now, suppose that P is a K-sound K-regular probability assignment. In this paragraph, we show that

$$\varphi(x) \in C_K \iff \varphi(x) \in C_{K,P} \qquad \text{(B.30)}$$

For the sake of readability, in this paragraph we write $h = h(I_\varphi)$ and $e = e(I_\varphi)$ and we omit the K inside the P-operator, which we can do since P is K-sound. For the left-to-right direction of equation (B.30), suppose that $\varphi(x) \in C_K$. Then since $K \nvdash \neg h$, we have by K-regularity that $P(\neg h) < 1$ and hence $1 - P(h) < 1$ so $P(h) > 0$. Since $K \nvdash e$, by K-regularity we have $P(e) < 1$, so $0 < P(h) \leq P(e) < 1$, so $0 < \frac{P(h)}{P(e)} < 1$, so $\frac{P(h)}{P(e)} - P(h) < 1 - P(h) < 1$.

Further, since $P(e) < 1$, we have $\frac{1}{P(e)} > 1$ and hence $\frac{P(h)}{P(e)} > P(h)$ and thus $\frac{P(h)}{P(e)} - P(h) > 0$. For the right-to-left direction of equation (B.30), suppose that $\varphi(x) \in C_{K,P}$. We claim that $K \nvDash e$. For, suppose not. Then the quantity $P(h|e) - P(h) = 0$, contrary to hypothesis. Likewise, we claim that $K + h$ is consistent. For, suppose not. Then $K \models \neg h$, and by K-soundness, we have that $P(h) = 0$, contrary to hypothesis. $\qquad \square$

Finally, we can thus deduce the following proposition, which was first stated at the close of § 2.2. The statement given here differs from the original statement in § 2.2 in that it is additionally noted here that the probability assignment can be taken to have properties such as K-soundness, K-regularity, and to not be a finite counting assignment.

Proposition 3. Let K be a finitely axiomatized extension of Robinson's Q that is true on the standard model of arithmetic. There are infinitely many L-formulas $\varphi(x)$ such that for all $\epsilon > 0$ with $\epsilon < 1$ there is a probability assignment P such that

$$1 - \epsilon \leq P(h(I_\varphi)|e(I_\varphi) \,\&\, K) - P(h(I_\varphi)|K) < 1 \qquad \text{(B.31)}$$

Moreover, the probability assignments P may be taken to be K-sound, K-regular, and not finite counting assignments.

Proof. This follows directly from the two previous propositions since the C_K is non-computable and hence infinite. $\qquad \square$

Appendix C. Proofs of Results Stated in § 4

The aim of this last appendix is to prove a result mentioned in § 4 on the probabilistic liar. This proposition needs only one definition. If L is a countable signature, then let us say that probability assignment $P : \mathrm{Sent}(L) \to \mathbb{R}$ is *arithmetically definable* if the following set is arithmetically definable:

$$\widetilde{P} = \{(\varphi, r) \in \mathrm{Sent}(L) \times \mathbb{Q} : P(\varphi) > r\} \qquad \text{(C.1)}$$

Now we may prove the following proposition, where recall from Appendix B that K-soundness means $K \vdash \varphi$ implies $P(\varphi) = 1$; and that K-regularity means $P(\varphi) = 1$ implies $K \vdash \varphi$; and where recall from § 3.2 that true arithmetic or $Th(\mathbb{N})$ is the set of sentences that are true on the standard model

$(\omega, 0, S, +, \times, \leq)$ of the Peano axioms. Finally, note that by the last theorem of Appendix B, namely, Theorem 4, there are many K-sound, K-regular arithmetically definable probability assignments, since the halting set is arithmetically definable and since being arithmetically definable is preserved downwards under relative computability.

Proposition 12. (*The Probabilistic Liar*) Suppose that K is a consistent extension of Robinson's Q which is a subtheory of true arithmetic. Suppose that P is a K-sound K-regular probability assignment that is arithmetically definable. Suppose finally that $0 \leq \epsilon < 1$ is a rational number. Then there is a sentence λ_ϵ such that

$$Q \vdash (\lambda_\epsilon \leftrightarrow P(\lambda_\epsilon) \leq \epsilon) \tag{C.2}$$

$$0 < P(\lambda_\epsilon) < P(\lambda_\epsilon | P(\lambda_\epsilon) \leq \epsilon) = 1 \tag{C.3}$$

$$0 < P(\lambda_\epsilon | P(\lambda_\epsilon) \leq \epsilon) - P(\lambda_\epsilon) < 1 \tag{C.4}$$

Proof. Of course equation (C.4) follows directly from equation (C.3). So it suffices to prove equations (C.2)-(C.3). By diagonalization (cf. [44] Lemma 1 p. 17, [1] Theorem III.2.1 p. 158), choose a sentence $\lambda = \lambda_\epsilon$ such that $Q \vdash (\lambda \leftrightarrow P(\lambda) \leq \epsilon)$. Since K is an extension of Robinson's Q and P is K-sound, we have $P(\lambda) = P(P(\lambda) \leq \epsilon)$. Suppose that $P(\lambda) = 0$. Since P is K-regular, we then have that $K \vdash \neg\lambda$. Since K is a subtheory of true arithmetic, we have that $\neg\lambda$ is true and hence also that $\neg[P(\lambda) \leq \epsilon]$ is true, which implies $P(\lambda) > \epsilon$, a contradiction. Clearly we also have that $P(\lambda | P(\lambda) \leq \epsilon) = 1$. Finally, to see that $P(\lambda) < 1$, suppose that $P(\lambda) = 1$. Since P is K-regular, we have that $K \models \lambda$. Since K is a subtheory of true arithmetic, we then have that $P(\lambda) \leq \epsilon < 1$, a contradiction. This is the one place in the proof where we use the hypothesis that $\epsilon < 1$. $\square$

Appendix D. Acknowledgements

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