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Agent Heterogeneity and the Real Exchange Rate

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Agent Heterogeneity and the Real Exchange Rate

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by

Julian A Batista

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ABSTRACT OF THE DISSERTATION

Agent Heterogeneity and the Real Exchange Rate

by

Julian A Batista

Doctor of Philosophy in Economics

University of California, Los Angeles, 2023

Professor Pierre-Olivier Weill, Co-Chair

Professor William R. Zame, Co-Chair

While the impact of agent heterogeneity has long been recognized in the Economic literature, the link between agent heterogeneity and international asset pricing is yet to be fully understood. In this dissertation I use an overlapping generations framework to study the impact that agent heterogeneity in risk aversion has on the real exchange rate determination.

Chapter 1 presents and develops the theoretical model used to study the implications of agent heterogeneity in risk aversion on the real exchange rate. I introduce a two-country model that features heterogeneous risk aversion profiles for agents, both within and between countries. Furthermore, it is shown that the model can explain the Cyclicality puzzle documented in Backus and Smith (1993), which highlights the empirical disconnect between the exchange rate and relative consumption growth. This chapter also presents the numerical outcome of the model.

Chapter 2 explains the quantitative methodology used to code and find the numerical solution of the model presented in chapter 1. The model does not admit a closed form solution and thus the presented outcome relies on the application of Monte Carlo Methods, the Feynman-Kac Theorem and the Piccard Iteration Theorem.

Finally, Chapter 3 presents recent empirical evidence on the Cyclicality puzzle between the US and 4 OECD countries: UK, France, Germany and Italy.

The dissertation of Julian A Batista is approved.

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2023

Para Patricia —el amor de mi vida—,

para mis padres, Alberto y María Adela,

para mi hermano, Fernando,

para mis abuelos, Albertina, Alberto, Juan y María Laura,

para los amigos de toda la vida,

para el resto de mi familia,

para los amigos del Doctorado,

para mis advisors Andy, Bill, Pierre y Stavros,

hicieron posible este sueño.

¡Gracias, gracias, gracias!

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CHAPTER 1

Agent Heterogeneity and the Real Exchange Rate

This paper studies the implications of risk-aversion heterogeneity for the real exchange rate determination. We propose a two-country model that features heterogeneous risk aversion profiles for agents, both within and between countries. We show that the model can explain the Cyclicality puzzle documented in Backus and Smith (1993), which highlights the empirical disconnect between the exchange rate and relative consumption growth. The key mechanism is a combination of home bias and heterogeneity in risk aversion. Real exchange rate changes, which reflect stochastic discount factor (SDF) heterogeneity across countries, are driven both by differences in risk aversion adjusted consumption growth rates and by differences in consumption weighted average risk aversion growth rates. The introduction of risk aversion heterogeneity, both between and within countries, generates a new risk sharing dynamic across agents and countries compared to a homogeneous representative agent framework: The less risk averse country will be leveraged in equilibrium, providing insurance to the most risk averse country. Thus, a negative aggregate endowment level shock affecting both countries alike, generates a consumption redistribution from the less risk averse country to the most risk averse country and from the less risk averse agent to the most risk averse agents, generating an asymmetric redistribution of national consumption share across countries and therefore significant differences in consumption weighted average risk aversion changes across countries.

1.1 Introduction

The Cyclicality puzzle established in Backus and Smith (1993) arises because the international risk-sharing condition that relative consumption change across countries should be strongly positively correlated with the real exchange rate growth is significantly violated in the data. This paper departs from the standard homogeneous representative agent assumption and studies the implications of heterogeneity in risk aversion on the real exchange rate determination and its role in explaining the Cyclicality puzzle. Under this new setup, the real exchange rate change can be decomposed in a relative risk aversion adjusted country consumption growth component and a relative consumption weighted average country risk aversion change component. If there are significant differences in the agents' risk aversion, both between and within countries, the latter becomes the main driver of the real exchange rate changes. In this case, the model features a significant degree of cross-country insurance but a very asymmetric degree of intra-country insurance.

I propose a two-country, two-goods, two-agents per country model expanding Panageas (2020) framework. The agents have standard CRRA with home-bias preferences but heterogeneous risk aversion, both within and between countries. Agents in each country are endowed with a stream of a single differentiated perishable good. There are no frictions, neither in the international trade of goods or in the financial markets. This implies that the two countries are able to achieve optimal international risk sharing.

The key attribute of the model is the interaction between home bias and heterogeneity in risk aversion, both between and within countries. The full explanation of the puzzle in an heterogeneous agent model includes the three elements.

In the absence of home bias, the real exchange has no volatility and thus the model cannot reproduce the Cyclicality correlation. On the other hand, the introduction of risk aversion heterogeneity deeply impacts the risk sharing dynamic across agents. The less risk averse agents will be leveraged in equilibrium providing insurance to the more risk averse agents. This conclusion is also extended to the aggregate country level: the less risk averse country provides insurance to the most risk averse country. Thus, positive endowment shocks temporarily decrease the consumption share of the more risk averse agents and countries, while negative endowment shocks temporarily increase the consumption share of the more risk averse agents and countries.

This mechanism explains the Cyclicality exchange rate disconnect results. Real exchange rate changes, which reflect stochastic discount factor (SDF) heterogeneity across countries, are driven by differences in country consumption growth rates adjusted by risk aversion as well as by cross-country differences in consumption weighted average risk aversion growth rates. Thus, in this model, the unconditional correlation between real exchange changes and consumption growth rate differentials can be low or even negative. To achieve this, there should be a significant degree of cross country insurance and, at the same time, significant differences in intra-country insurance across countries.

A negative aggregate endowment level shock that affects both countries alike, will generate an appreciation of the real exchange rate, defined as the quotient between the SDF of the foreign country (most risk averse) to the SDF of the domestic country (less risk averse). This appreciation is a result of the interaction of both components, the relative risk aversion adjusted country consumption growth and the relative consumption weighted average risk aversion change. Under a setup where there is a significant degree of risk aversion heterogeneity, the less risk averse agent (located in the domestic country), will transfer resources to the most risk averse agents, located both at home and abroad. As a result, the consumption of the domestic country will experience a sharper fall compared to the consumption of the foreign country. The appreciation of the real exchange rate is thus driven by the difference in risk aversion changes across countries. The insured country will experience little intra-country consumption reallocation as both of its agents are insured in equilibrium. On the other hand, the domestic country will experience a high degree of intra-country consumption reallocation, as one of the agents is being insured by the other agent.

The model is analytically tractable: the equilibrium dynamics can be fully characterized by a system of four second order stochastic partial differential equations (SPDE) that are solved numerically, relying on the Feynman–Kac theorem and on Picard's fixed point theorem. The model is calibrated to match the empirical moment for the correlation between the real exchange rate and relative country consumption growth between the US and UK for the years 1990 and 2020.

My paper contributes to two strands of literature. On the one hand, this work expands the extensive literature on the asset pricing implications of investor heterogeneity and portfolio constraints: Dumas (1989); Wang (1996); Chan and Kogan (2002); Longstaff and Wang (2012); Gârleanu and Panageas (2015); Santos and Veronesi (2010); Panageas (2020); Kargar (2021) study economies with heterogeneous agents featuring different preference assumptions. In particular, I expand the one economy model developed in Panageas (2020) to a two economies model and study the implications for the determination of the real exchange rate.

Also, this paper contributes to the proposed solutions to the Cyclicality puzzle. Colacito and Croce (2013) feature recursive preferences, exogenous output subject to short-run and long-run shocks, and international trade. In their model, the Cyclicality puzzle is addressed due to the effect of long-run news, which generates a negative correlation between real exchange rate changes and consumption growth rate differentials, partly offsetting the positive correlation generated by short-run news.

Bai and Ríos-Rull (2015) develop a standard two-country, two-good economy with frictions in the goods market. Good market frictions make demand shocks work like productivity shocks. When exerting effort, households contribute to measured productivity by extracting more output out of the economy. As a result, total factor productivity responds positively to increases in expenditures. In this model, an increase in domestic demand leads to a rise in domestic consumption and consumer prices. The real exchange rate, of home consumption in terms of foreign consumption thus appreciates.

Stathopoulos (2017) proposes a two-country external habit formation model that features time-varying heterogeneity in conditional risk aversion across countries, endogenously arising from the interaction between external habit formation and preference home bias. In this model, the disconnect between the real exchange growth and the relative country consumption growth is a consequence of the interaction between home biased preferences and relative risk aversion of countries featuring habit formation preferences.

Lustig et al. (2019) consider an extreme case of incomplete markets, where the domestic (foreign) agents, when investing abroad, can only trade in the foreign (domestic) risk-free rates. They find that even in this extreme departure from complete spanning has limits: it can help match quantitatively the volatility of exchange rates in the data and the currency risk premium, but it has no impact on exchange rate cyclicality.

I contribute to this second strand of literature by building a transparent and straightforward model capable of explaining a complex phenomenon: the key mechanism relies just on home bias and heterogeneity in risk aversion, both between and within countries.

In Section 1.2 I provide a description of the model. Section 1.3 describes the proposed solution. Section 1.4 describes the data and the calibration. Section 1.5 presents the results. Section 1.6 concludes.

1.2 Model

1.2.1 Countries and Agents

The global economy is composed of two countries: the domestic country and the foreign country. Each country is populated by two type of agents, A and B in the domestic country and A^* and B^* in the foreign country. For a specific country, type A agents are always less risk averse than type B agents. Agents of the same type (i.e. A and A^*) but from different countries do not necessarily have the same risk aversion.

1.2.2 Endowments and Overlapping Generations Framework

The agents of each country are endowed with long-lived assets that are referred to as "trees" from their own country. The country trees from the domestic country produce domestic good X as dividends, whereas the country tree from the foreign economy produces foreign good Y as dividends. To ensure stationarity, the model assumes the arrival of new agents who are endowed with new trees.

A mass of π agents is born per unit of time. At the same time, existing agents face a constant hazard rate of death π. According to the law of large numbers,at time t the surviving population of agents who were born at time $s \leq t$ is $\pi e^{-\pi(t-s)}$. The total world population is constant and normalized to $\int_{-\infty}^{t} \pi e^{-(t-s)} ds = 1$. The proportion of agents of type $i \in \{A, B, A^*, B^*\}$ is denoted by μ^i.

At time t, newly born agents from the domestic country are equally endowed with shares to a domestic tree (good X) born at time t, while newly born agents from the foreign country are equally endowed with shares to a foreign tree (good Y) born at time t. If $s \leq t$ denotes the birthtime of a tree, its time t dividends are given by $D_{t,s} = \delta e^{-\delta(t-s)} D_t$, where $D \in \{X, Y\}$, $\delta \in (0, 1)$ represents the depreciation rate of the country trees and D_t is the aggregate country dividend or aggregate endowment of the respective country good. Thus, the total endowment of each economy is $\int_{-\infty}^{t} D_{t,s} ds = \left(\int_{-\infty}^{t} \delta e^{-\delta(t-s)} ds \right) \times D_t = D_t$.

Since agents have no bequest motives, we assume that they enter contracts that require them to surrender their wealth upon death in exchange for an income stream of πdt while alive. The (competitive) insurance company offering this contract breaks even as it collects a fraction π of aggregate wealth from the dying agents and distributes it as an income stream to the surviving agents.

The aggregate endowment for the domestic good X follows a geometric Brownian motion with mean μ and volatility σ^X:

$$d\log X_t = \mu^X dt + \sigma^X dB_t^{(1)},$$

where $dB_t^{(1)}$ is a Brownian increment. We will refer to this Brownian increment as the aggregate endowment level shock.

To ensure stationarity in the ratio of the good endowments, the law of motion for the aggregate endowment for the foreign country Y is derived from a Wright-Fisher[1] process. We define the ratio of the aggregate endowment of the goods as $\omega_t = \frac{Y_t}{X_t}$, $\omega_t \in [\underline{\omega}, \overline{\omega}]$. The increment in ω_t is given by:

$$d\omega_t = S \times \left(\omega^M - \omega_t\right) dt + \sigma^\omega \sqrt{\left(\omega_t - \underline{\omega}\right)\left(\overline{\omega} - \omega_t\right)} dB_t^{(2)},$$

where $dB_t^{(2)}$ is a Brownian with instantaneous correlation ρ^{XY} with $dB_t^{(1)}$, ω^M is the mean reverting value for ω, S is the speed of convergence, σ^ω is a volatility parameter fot the Wright-Fisher process and $\underline{\omega}$ and $\overline{\omega}$ are the lower and upper bound for the endowment ratio ω respectively. We will refer to $dB_t^{(2)}$ as the aggregate endowment ratio shock.

We can find the law of motion for the aggregate endowment of the foreign good Y through Ito's lemma.The endowment of the foreign good follows a geometric Brownian motion:

[1]See Karlin and Taylor (1981)

$$d \log Y_t = \left[S \frac{\left(\omega^M - \omega_t\right)}{\omega_t} + \mu^X + \frac{\rho^{XY} \sigma^X \sigma^\omega \sqrt{\left(\omega_t - \underline{\omega}\right)\left(\overline{\omega} - \omega_t\right)}}{\omega_t} \right] dt$$
$$+ \sigma^X dB_t^{(1)} + \sigma^\omega \frac{\sqrt{\left(\omega_t - \underline{\omega}\right)\left(\overline{\omega} - \omega_t\right)}}{\omega_t} dB_t^{(2)}.$$

Hence, the growth rate of Y is affected by the endowment level shock, $dB_t^{(1)}$, in the same magnitude as the aggregate endowment of good X. On the other hand, we can appreciate that the rate of growth of the aggregate endowment of good Y is affected by the the endowment ratio shock $dB_t^{(2)}$, while the aggregate endowment of good X is not.

Both, financial and goods markets are complete and frictionless. Agents can trade goods and a complete set of Arrow-Debreu securities in a costless way with every living agent.

1.2.3 Preferences

We will describe the problem for a domestic agent. The problem for a foreign agent is symmetric. A domestic agent $i \in \{A, B\}$ born in t maximizes its life-time constant relative risk aversion (CRRA) utility:

$$E_t \int_t^\infty e^{-(\rho+\pi)(u-t)} U\left(X_u, Y_u\right) du = E_t \int_t^\infty e^{-(\rho+\pi)(u-t)} \frac{C_{u,t}^{1-\gamma^i}}{1-\gamma^i} du, \qquad (1.1)$$

where $\gamma^i > 0$ and $\rho > 0$ denote, respectively, the risk aversion and the discount factor of agent $i \in \{A, B\}$; X and Y are the quantity of the domestic and the quantity of the foreign good that the agent consumes; C is the country consumption basket defined as

$$C \equiv X^{\alpha} Y^{1-\alpha},$$

where $\alpha \in [0,1]$ denotes relative preference for good X. The country consumption basket is characterized by a Cobb-Douglas aggregator, so the elasticity of substitutions between the goods is equal to 1. An agent's relative preferences are assumed homogeneous within a country and characterized by Home Bias. So, $\alpha > 1/2$.

1.2.4 Optimization

The complete market assumption leads to a straightforward objective for the agent to solve. A domestic agent $i \in \{A, B\}$ born at time u, maximizes the intertemporal utility function subject to the intertemporal budget constraint:

$$E_t \int_t^{\infty} e^{-\pi(u-t)} \left[\frac{H_u^X}{H_t^X} X_{u,t}^i + \frac{H_u^Y}{H_t^X} Y_{u,t}^i \right] du = \frac{1}{\pi\left(\mu^A + \mu^B\right)} P_{t,t}^X, \tag{1.2}$$

where H_u^D is the discount factor for good $D \in \{X, Y\}$ in period u. $\frac{1}{\pi(\mu^A+\mu^B)} P_{t,t}^X$ is the domestic country cohort member value of the shares of the domestic good X trees born at time t. $P_{t,t}^X$ is denominated in units of good X in period t. $\pi\left(\mu^A + \mu^B\right)$ the amount of new agents that are born at time t in the domestic country, who will get an equal amount of the shares of the newly born domestic trees.

We can write the Lagrangian:

$$\max_{X^i_{u,t},Y^i_{u,t}} E_t\left\{\int_t^\infty e^{-\pi(u-t)}\left[e^{-\rho(u-t)}\frac{\left[\left(X^i_{u,t}\right)^\alpha\left(Y^i_{u,t}\right)^{1-\alpha}\right]^{1-\gamma^i}}{1-\gamma^i}\right.\right.$$
$$\left.\left. -\lambda^i_t\left(\frac{H^X_u}{H^X_t}X^i_{u,t}+\frac{H^Y_u}{H^X_t}Y^i_{u,t}\right)\right]du\right\}+\lambda^i_t\frac{1}{\pi\left(\mu^A+\mu^B\right)}P^X_{t,t}. \quad (1.3)$$

This results in the first order conditions:

$$\frac{e^{-\rho(u-t)}\alpha C^{-\gamma^i}_{u,t}}{X^i_{u,t}}=\lambda^i_t\frac{H^X_u}{H^X_t}, \quad (1.4)$$

$$\frac{e^{-\rho(u-t)}\left(1-\alpha\right)C^{-\gamma^i}_{u,t}}{Y^i_{u,t}}=\lambda^i_t\frac{H^Y_u}{H^X_t}. \quad (1.5)$$

These equations show the standard result that marginal utility must be proportional to the stochastic discount factors as well as the fact that marginal utility is increasing in risk aversion.

If we take together equations (1.4) and (1.5) and we evaluate them at time t, we get that:

$$\left(\frac{C^i_{u,t}}{C^i_{t,t}}\right)=e^{-\frac{\rho}{\gamma^i}(u-t)}\left[\left(\frac{H^X_u}{H^X_t}\right)^\alpha\left(\frac{H^Y_u}{H^Y_t}\right)^{(1-\alpha)}\right]^{-\frac{1}{\gamma^i}}. \quad (1.6)$$

Equation (6) shows that all consumers of type i that belong to the cohort born at time u experience the same growth in their consumption of the country consumption basket between times t and u. This is a reflection of the complete market assumption, which allows perfect risk sharing within a cohort.

1.2.5 The Real Exchange Rate

The price of each consumption basket is defined as the minimum expenditure required to buy a unit of the basket and is derived by minimizing the corresponding expenditure function. Then the price of the domestic consumption basket is given by:

$$M_t = \left(\frac{Q_t^X}{\alpha}\right)^{\alpha} \left(\frac{Q_t^Y}{1-\alpha}\right)^{1-\alpha},$$

where Q_t^X is defined as the price for good X at time t, while Q_t^Y is the price of good Y at time t.

We define the real exchange rate as the ratio of the price of the foreign consumption basket to the price of the domestic consumption basket.

$$E_t = \frac{M_t^*}{M_t} = \frac{\alpha^{\alpha}}{(\alpha^*)^{\alpha^*}} \frac{(1-\alpha)^{1-\alpha}}{(1-\alpha^*)^{1-\alpha^*}} \left(\frac{Q_t^Y}{Q_t^X}\right)^{\alpha-\alpha^*}.$$

The price of the two consumption baskets may be different because they have a different composition. In particular, when $\alpha \neq \alpha^*$ the real exchange rate is time varying and purchasing power parity fails. In that case, the real exchange rate volatility is increasing in the volatility of the terms of trade.

1.2.5.1 Real Exchange Rate Decomposition and the Cyclicality Puzzle

We can also consider an alternative decomposition of the real exchange rate, which will be useful for the discussion of the Cyclicality Puzzle. We can rewrite the real exchange rate as:

$$E_t = \frac{\frac{e^{-\rho^{i^*} t}}{\lambda_t^{i^*}\left(C_t^{i^*}\right)^{\gamma^{i^*}}}}{\frac{e^{-\rho^{i} t}}{\lambda_t^{i}\left(C_t^{i}\right)^{\gamma^{i}}}} = \frac{\frac{e^{-\rho^{i^*} t}}{\lambda_u^{i^*}\left(\frac{C_t^{i^*}}{C_t^*}C_t^*\right)^{\gamma^{i^*}}}}{\frac{e^{-\rho^{i} t}}{\lambda_t^{i}\left(\frac{C_t^{i}}{C_t}C_t\right)^{\gamma^{i}}}},$$

where C_t^i is the amount of the domestic consumption basket consumed by agent $i \in \{A, B\}$ at time t, C_t is the total amount of the domestic consumption basket consumed by both domestic agents, $C_t^{i^*}$ is the amount of the foreign consumption basket consumed by agent $i^* \in \{A^*, B^*\}$ is the total amount of the foreign consumption basket consumed by both foreign agents, λ_t^i is the marginal utility of newly born agents in t for agent i and $\lambda_t^{i^*}$ is the marginal utility of newly born agents in t for agent i^*.

We can decompose the growth rate of the real exchange rate E_t in the following way:

$$\frac{dE_t}{E_t} = \left(\gamma^i \frac{dC_t}{C_t} - \gamma^{i^*}\frac{dC_t^*}{C_t^*}\right) + \left(\gamma^i \frac{d\frac{C_t^i}{C_t}}{\frac{C_t^i}{C_t}} - \gamma^{i^*}\frac{d\frac{C_t^{i^*}}{C_t^*}}{\frac{C_t^{i^*}}{C_t^*}}\right) + o(dt).$$

We can see that the change in the real exchange rate can be explained not only by the relative risk aversion country consumption good growth but also by the cross country differences in within country consumption reallocation.

Let's take for example $i = B$ and $i^* = B^*$. Under my baseline calibration we have that $\gamma^{B^*} > \gamma^B > \gamma^{A^*} > \gamma^A$, thus the domestic country is the less risk averse country. A negative endowment level shock that decrease the level of the endowment of both goods alike will result in a temporarily reallocation of consumption from the less risk averse agent, A to the more risk averse agents A^*, B and B^*. This will generate 2 effects. On the one hand, consumption of the domestic country will fall at a higher pace than the consumption of the foreign country, as the former provides insurance to the latter. On the other hand, there will be a significant reallocation of consumption within the domestic country, as agent A provides insurance to agent B, but a mild reallocation of consumption in the foreign country, as both agent are insured in equilibrium by agent A. If the latter phenomenon prevails over the former one, then the real exchange rate will experience an appreciation while the relative consumption growth, $d\log C - d\log C^{*}$, will have a negative sign, reproducing the empirical disconnect between these variables.

1.3 Model Solution

To solve the model, we need to determine how prices, SDFs and consumption processes for all agents depend on the historical paths of the aggregate shocks $dB_t^{(1)}$ and $dB_t^{(2)}$. The equilibrium can be characterized in a recursive form where all equilibrium objects are a function of four endogenous state variables.

The computation of the equilibrium requires solving four second order partial stochastic differential equations (PSDE). But the system of non-linear stochastic partial differential equations does not admit a closed-form solution, and thus the model is solved using numerical techniques based on monte carlo simulations, the Feynman-Kac theorem and the Picard fixed point theorem.

1.3.1 Endogenous state variables

All the variables in the model at time t are determined by four state variables. The state variables consist of three consumption shares (out of eight) and the aggregate endowment ratio ω_t. The remaining consumption shares are determined through Pareto conditions and market clearing conditions. In this case, we chose as state variables the consumption shares of good X for agents A and A^* (x^A and x^{A^*}) and the consumption share of good Y for agent A (y^A). These variables can be defined, respectively, in the following way:

$$x_t^A \equiv \frac{\mu_A \int_{-\infty}^t \pi e^{-\pi(t-s)} X_{t,s}^A ds}{X_t},$$
$$x_t^{A^*} \equiv \frac{\mu_{A^*} \int_{-\infty}^t \pi e^{-\pi(t-s)} X_{t,s}^{A^*} ds}{X_t},$$
$$y_t^A \equiv \frac{\mu_A \int_{-\infty}^t \pi e^{-\pi(t-s)} Y_{t,s}^A ds}{Y_t}.$$

So, we will restrict our attention to a Markovian equilibrium that is defined below, in the state space $\left(\omega, x^A, x^{A^*}, y^A\right) \in [\underline{\omega}, \overline{\omega}] \times (0,1) \times (0,1) \times (0,1)$ where all processes are functions of $\left(\omega, x_t^A, x_t^{A^*}, y_t^A\right)$ only.

Propositions 1, 2, and 3 characterize the dynamics of the state variables $\left(x^A, x^{A^*}\right)$, as well as the dynamics of the remaining consumption shares that were not chosen as state variables[2].

Proposition 1. *The dynamics of the SDFs for country goods X and Y are given by:*

$$\begin{aligned}\frac{dH_t^X}{H_t^X} &= -r_t^X dt - \kappa_t^{X,(1)} dB_t^{(1)} - \kappa_t^{X,(2)} dB_t^{(2)},\\ \frac{dH_t^Y}{H_t^Y} &= -r_t^Y dt - \kappa_t^{Y,(1)} dB_t^{(1)} - \kappa_t^{Y,(2)} dB_t^{(2)},\end{aligned} \tag{1.7}$$

where r_t^D is the real interest rate denominated in good $D \in \{X, Y\}$ and $\kappa_t^{D,(j)}$ is the market price of risk $j \in \{1, 2\}$ in terms of good $D \in \{X, Y\}$.

Proposition 2. *The law of motion for the consumption share of domestic good X*

[2]The dynamics for the state variable y^A can be found in the Appendix

for agent $i \in \{A, B, A^*, B^*\}$ *is given by*[3]*:*

$$dx_t^i = \mu_t^{x^i} dt + \sigma_t^{x^i,(1)} dB_t^{(1)} + \sigma_t^{x^i,(2)} dB_t^{(2)}$$

1. *where the drift of the law of motion of* x^i *is given by:*
$\mu_t^{x^i} = \left[\frac{1-(1-\alpha^i)(1-\gamma^i)}{\gamma^i} r_t^X + \frac{(1-\alpha^i)(1-\gamma^i)}{\gamma^i} r_t^Y + -\mu_{\widetilde{X}} - \frac{\rho_i}{\gamma^i} - \pi + \Omega_t^{x^i} \right] x_t^i + \pi \mu_i \frac{X_{t,t}^i}{X_t},$
where r^X *is the real interest rate of a local good* X *denominated bond and* r^Y *is the real interest rate of a foreign good* Y *denominated bond and* $\Omega_t^{x^i}$ *is a term that captures second order elements*[4]*.*

2. *The diffusions of the law of motion of* x^i *are given by:*
$\sigma_t^{x^i,(1)} = \left[\frac{1-(1-\alpha_i)(1-\gamma^i)}{\gamma^i} \kappa_t^{X,(1)} + \frac{(1-\alpha^i)(1-\gamma^i)}{\gamma^i} \kappa_t^{Y,(1)} - \sigma_t^X \right] x_t^i,$
$\sigma_t^{x^i,(2)} = \left[\frac{1-(1-\alpha^i)(1-\gamma^i)}{\gamma^i} \kappa_t^{X,(2)} + \frac{(1-\alpha^i)(1-\gamma^i)}{\gamma^i} \kappa_t^{Y,(2)} \right] x_t^i,$
where $\kappa_t^{D,(j)}$ *represent the market price of risk for risk* $j \in \{(1), (2)\}$ *(where* (1) *refers to the endowment level risk and* (2) *refers to the endowment ratio risk) denominated in good* $D \in \{X, Y\}$*.*

The following proposition provides the boundary conditions that the state variables diffusions satisfy and ensure the survival of every type of agent in the long run.

Proposition 3. *The boundary conditions for the consumption shares of the domestic good* X *are given by:*[5]

[3]The law of motion for the consumption share of the foreign good is derived in the Appendix

[4]An expression for this term can be found in the Appendix.

[5]The proof of boundary conditions can be found in the Appendix

1. $\lim_{x_t^i \to 0} \sigma_t^{x^i,(j)} = \lim_{y_t^i \to 0} \sigma_t^{x^i,(j)} = \lim_{x_t^i \to 1} \sigma_t^{x^i,(j)} = \lim_{y_t^i \to 1} \sigma_t^{x^i,(j)} = 0,$

2. $\lim_{x_t^i \to 0} \mu_t^{x^i} = \lim_{y_t^A \to 0} \mu_t^{x^i} = \pi \mu_i \frac{X_{t,t}^i}{X_t} > 0,$

3. $\lim_{x_t^i \to 1} \mu_t^{x^i} = \lim_{y_t^A \to 1} \mu_t^{x^i} = -\pi \sum_{l \neq i} \mu_l \frac{X_{t,t}^l}{X_t} < 0,$

where $i, l \in \{A, A^*, B, B^*\}$ *and* $j \in \{1, 2\}$.

These boundary conditions ensure the survival of

1.3.2 Completing the Construction of the Equilibrium

The determination of the law of motion for the consumption shares of good X requires the calculation of $\frac{X_{t,t}^i}{X_t}$, which can be obtained from the agents' budget constraint at time t[6]. The budget constraint for local agents at time t requires that:

$$E_t \int_t^\infty e^{-\pi(u-t)} \left[\left(\frac{H_u^X}{H_t^X} \right) X_{u,t}^i + \left(\frac{H_u^Y}{H_t^X} \right) Y_{u,t}^i \right] du = \frac{1}{\pi \left(\mu_A + \mu_B \right)} P_{t,t}^X =$$
$$= \frac{1}{\pi \left(\mu_A + \mu_B \right)} E_t \int_t^\infty e^{-\delta(u-t)} \left(\frac{H_u^X}{H_t^X} \right) X_{u,t} du.$$

The budget constraint for foreign agents at time t requires that:

$$E_t \int_t^\infty e^{-\pi(u-t)} \left[\left(\frac{H_u^X}{H_t^X} \right) X_{u,t}^i + \left(\frac{H_u^Y}{H_t^X} \right) Y_{u,t}^i \right] du = \frac{1}{\pi \left(\mu_{A^*} + \mu_{B^*} \right)} P_{t,t}^Y =$$
$$= \frac{1}{\pi \left(\mu_{A^*} + \mu_{B^*} \right)} E_t \int_t^\infty e^{-\delta(u-t)} \left(\frac{H_u^Y}{H_t^X} \right) Y_{u,t} du.$$

So, using some algebra we can rewrite the terms that appear in the drift of the law

[6] The conditions for the determination of the law of motion for the consumption shares of good Y are shown in the Appendix

of motion of the the consumption shares of good X, $\frac{X_{t,t}^A}{X_t}$ and $\frac{X_{t,t}^{A^*}}{X_t}$, in the following way:

$$\frac{X_{t,t}^A}{X_t} = \frac{\delta}{\pi(\mu_A+\mu_B)}\frac{p_t^X}{g_t^{A,X}},$$

$$\frac{X_{t,t}^{A^*}}{X_t} = \frac{\delta}{\pi(\ \mu_{A^*}+\mu_{B^*})}\left(\frac{H_t^Y}{H_t^X}\right)\omega_t\frac{p_t^Y}{g_t^{A^*,X}},$$

where p_t^D is the price dividend ratio for good $D \in \{X, Y\}$ and $g_t^{i,X}$ is the wealth to consumption of good X ratio for agent i.

1.3.3 Markovian equilirbium

We derive a Markov equilibrium in state variables ω^A, x^A, x^{A^*} and y^A. All the equilibrium objects (consumption shares and prices) can be written as functions of these three states variables.

Definition 1. *A Markov equilibrium in state variables ω^A, x^A, x^{A^*} and y^A is the set of dividend price ratios $p_t^X\left(\omega^A, x^A, x^{A^*}, y^A\right)$ and $p_t^Y\left(\omega^A, x^A, x^{A^*}, y^A\right)$, wealth to consumption of good X ratios $g_t^{A,X}\left(\omega^A, x^A, x^{A^*}, y^A\right)$ and $g_t^{A^*,X}\left(\omega^A, x^A, x^{A^*}, y^A\right)$, real interest rates $r_t^X\left(\omega^A, x^A, x^{A^*}, y^A\right)$ and $r_t^Y\left(\omega^A, x^A, x^{A^*}, y^A\right)$, stochastic discount factors $H_t^X\left(\omega^A, x^A, x^{A^*}, y^A\right)$ and $H_t^Y\left(\omega^A, x^A, x^{A^*}, y^A\right)$, policy functions $X_t^i\left(\omega^A, x^A, x^{A^*}, y^A\right)$ and $Y_t^i\left(\omega^A, x^A, x^{A^*}, y^A\right)$ for $i \in \{A, B, A^*, B^*\}$ and law of motion for endogenous state variables $\mu_t^{x^A}\left(\omega^A, x^A, x^{A^*}, y^A\right)$, $\mu_t^{x^{A^*}}\left(\omega^A, x^A, x^{A^*}, y^A\right)$, $\mu_t^{y^A}\left(x^A, x^{A^*}, y^A\right)$, $\sigma_t^{x^A,(1)}\left(\omega^A, x^A, x^{A^*}, y^A\right)$, $\sigma_t^{x^{A^*},(1)}\left(\omega^A, x^A, x^{A^*}, y^A\right)$, $\sigma_t^{y^A,(1)}\left(\omega^A, x^A, x^{A^*}, y^A\right)$ and $\sigma_t^{x^A,(2)}\left(\omega^A, x^A, x^{A^*}, y^A\right)$, $\sigma_t^{x^{A^*},(2)}\left(\omega^A, x^A, x^{A^*}, y^A\right)$, $\sigma_t^{y^A,(2)}\left(\omega^A, x^A, x^{A^*}, y^A\right)$ such that:*

i) $X_t^i\left(\omega^A, x^A, x^{A^*}, y^A\right)$ and $Y_t^i\left(\omega^A, x^A, x^{A^*}, y^A\right)$ for $i \in \{A, B, A^*, B^*\}$ solve the consumer problem.

ii) Markets clear:

$$\sum_{i \in \{A,B,A^*,B^*\}} x_t^i = \sum_{i \in \{A,B,A^*,B^*\}} y_t^i = 1.$$

1.3.4 Numerical Solution

Although the model does not have a closed form solution, it is analytically tractable. Its dynamics can be fully characterized by a system of second order stochastic partial differential equations that are solved numerically appealing to the Feynman-Kac Theorem, the Picard Fixed point theorem[7] and Monte Carlo simulations. The computation of the equilibrium requires solving the law of motion for the state variables, whose drifts depend on p_t^X, p_t^Y, $g_t^{A,X}$ and $g_t^{A^*,X}$.

1.4 Data and Calibration

1.4.1 Data

Data for consumption, gross domestic product (GDP), inflation and exchange rate for the United States is retrieved from the Federal Reserve Bank of St. Louis Database. Data for consumption, GDP and inflation for the United Kingdom is retrieved from the Office of National Statistics from the United Kingdom.

All data are seasonally adjusted; any time series not initially adjusted undergoes seasonal adjustment using the U.S. Census Bureau's X12 seasonal adjustment method.

[7]See Alfuraidan and Ansari (2016)

Table 1.1: Calibrated Parameters

Parameters	Symbol	Value
Panel A: Agent parameters		
Risk aversion of type A agents	γ^A	1.5
Risk aversion of type B agents	γ^B	8
Risk aversion of type A^* agents	γ^{A^*}	3.5
Risk aversion of type B^* agents	γ^{B^*}	10
Discount factor	ρ	0.001
Domestic preference for the domestic good	α	0.9
Foreign preference for the domestic good	α^*	0.1
Population size of agent A	μ_A	0.01
Population size of agent B	μ_A	0.49
Population size of agent A^*	μ_{A^*}	0.01
Population size of agent B^*	μ_{B^*}	0.49
Death-born rate	π	0.008
Panel B: Endowment parameters		
Endowment of the domestic good growth mean	μ^X	0.023
Endowment of the domestic good growth volatility	σ^X	0.017
Speed of convergence	S	0.77
Ratio of endowment of goods growth volatility	σ^ω	0.14
Mean reverting ratio value	ω^M	1
Correlation between brownian motions	ρ^{XY}	0.26

1.4.2 Calibration

The paramaters chosen to calibrate the model are listed in Table 1.1. As the model is set in continuous time, said parameters are annual values rather than quarter values.

The drift and the diffusion of the aggregate endowment level process are chosen to match the 2 pertinent moments of the US annual GDP growth for the 1990-2020 period. μ^X is set to 0.023 and σ^X is set to be equal to 0.017.

The parameters for the Wright-Fisher aggregate endowment ratio process are chosen to match the pertinent models for the process characterizing the relationship between the nominal GDP for the US and the nominal GDP for the UK for the 1990-2020 period. The speed of convergence matches the empirical autocorrelation of 0.75. The mean reverting value for the endowment ratio ω^M is normalized to 1 to avoid any size effects on the results. The upper and lower bounds for the ratio are normalized to $\underline{\omega} = 0.5$ and $\overline{\omega} = 2$. Under this normalization the value for σ^ω consistent with the empirical data is 0.15. The empirical value correlation between the Brownian increments is found to be 0.26.

The subjective discount rate ρ is set to be equal to 0.001 for every agent while the death/birth ratio π is set to match the US mortality rate, 0.008. The agents effective discount rate is equal to $\rho + \pi = 0.009$.

The home bias parameters are set to be equal to $\alpha = 1 - \alpha^* = 0.9$.

The risk aversion parameters are set to allow for significant differences in risk aversion across countries: $\gamma^A = 1.5$, $\gamma^{A^*} = 3.5$, $\gamma^B = 9$, $\gamma^{B^*} = 11$.

1.5 Results

1.5.1 Moments

I simulate fifty different random paths of the economy, each one starting from a different initial set of consumption shares. These initial consumption shares satisfy both market clearing and Pareto conditions. Each path has 7,000 years. The first 1,000 years are dropped to allow the simulations to converge to their stationary distribution. Each 6,000 year path is divided in two hundreds 30-year windows that are counted as one observation. Each 30-year windows has 120 quarter observations.

Table 1.2: Results

Outcome	Heterogeneous Agents Model	Homogeneous Agents Model	Data
Panel A: Exchange rate and Cyclicality			
$\text{Corr}(\text{dlog}E, \text{dlog}C - \text{dlog}C^*)$	-0.2522	0.9995	-0.2456
	$[-0.2540, -0.2503]$	$[0.9995, 0.9995]$	
$\text{Corr}(\text{dlog}E, \gamma^B \text{dlog}C - \gamma^{B^*} \text{dlog}C^*)$	0.0273		
	$[0.0252, 0.0294]$		
$\text{Corr}(\text{dlog}E, \gamma^B \text{dlog}\frac{C^B}{C} - \gamma^{B^*} \text{dlog}\frac{C^{B^*}}{C^*})$	0.7440		
	$[0.7431, 0.7449]$		
$\text{SD}(\text{dlog}E)$	0.0958	0.1402	0.0895
	$[0.0955, 0.0961]$	$[0.1397, 0.1407]$	$[0.1148, 0.1157]$
$\frac{\text{SD}(\text{dlog}E)}{\text{SD}(\text{dlog}X)}$	5.6353	8.2471	5.2647
	$[5.6176, 5.6529]$	$[8.2176, 8.2764]$	
Panel B: Consumption			
$\text{SD}(\text{dlog}C)$	0.0238	0.0319	0.0162
	$[0.0237, 0.0238]$	$[0.0318, 0.0320]$	
$\text{SD}(\text{dlog}C^*)$	0.01503	0.0671	0.022
	$[0.0152, 0.0154]$	$[0.0669, 0.0673]$	

Note: All confidence intervals are calculated at a 95% confidence level.

To explore the quantitative effects of intra and inter country heterogeneity, I also perform one additional simulation exercise: I simulate two economies inhabited by only one type of agent each. Both agents have the same risk aversion and are thus homogeneous.

Table 1.2 presents the key simulated moments as well as the corresponding empirical moments for the United States and the United Kingdom for the period 1990-2020.

Panel A presents the moments related to the Cyclicality puzzle and its components. In the first row of the panel, we can appreciate the extent to which each version of the model is able to account for the Cyclicality puzzle. The heterogeneous agent model is able to reproduce the empirical value for the correlation between the real exchange rate and the relative consumption growth rates almost perfectly. On the other hand, the homogeneous agent model is not able to explain this puzzles and features a nearly perfect correlation between the relevant variables.

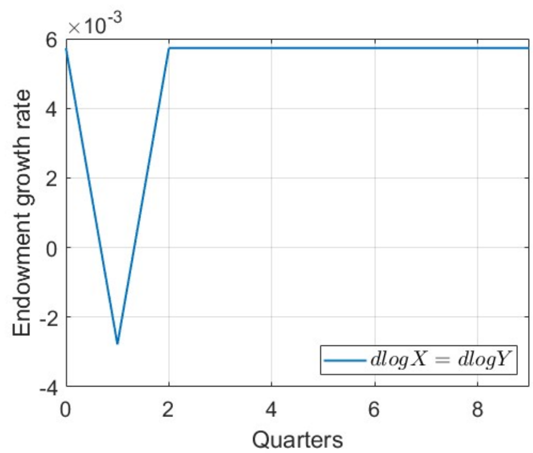
Rows two and three of Panel A show that the main driver of the explanation of the puzzle in the heterogeneous agent model is the relative change in risk aversion component, that has a strong and positive correlation with the real exchange rate. The relative risk aversion adjusted country consumption growth component is weakly and negatively correlated with the real exchange rate.

Panel B shows the main moments for country consumption. The standard deviation for country consumption in the heterogeneous agents model is close to the empirical counterpart. On the other hand, the standard deviation for country consumption in the homogeneous agents model is significantly larger than the empirical benchmark.

1.5.1.1 Negative Aggregate Endowment Level Shock

In this subsection we analyze the impact of a negative aggregate endowment level shock on the main variables of the economy, which contributes to the clarification of the mechanism behind the disconnect between the real exchange rate change and relative country consumption growth. The aggregate endowment level shock can be seen in Figure 1.1. This negative shock generates a temporary fall in the rate of growth the endowment of both goods.

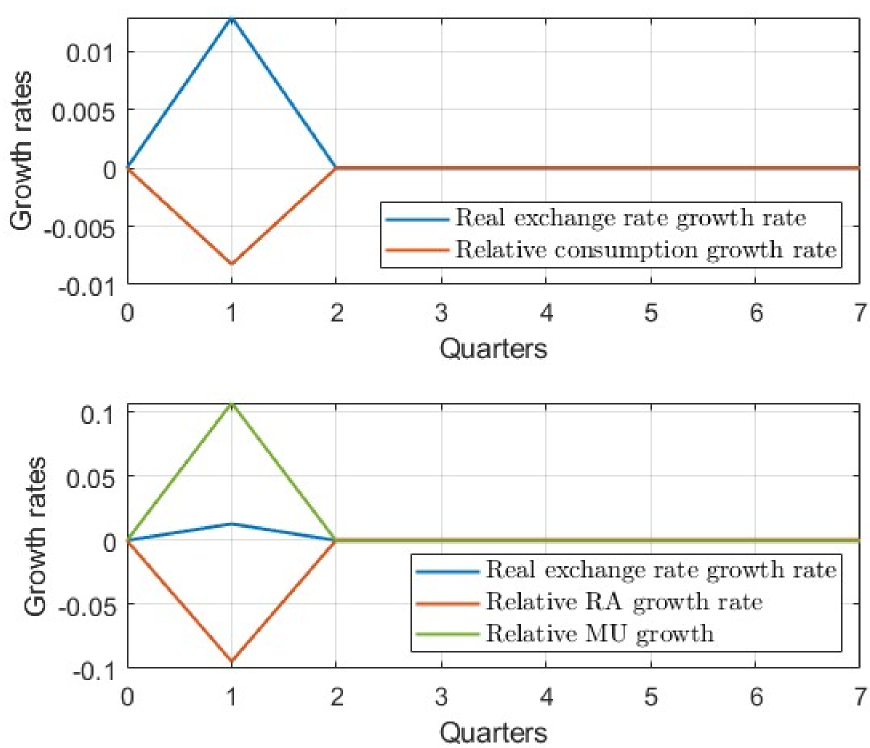
Figure 1.1: Endowment growth rate



In figure 1.2 we can appreciate the nature of the negative correlation between the real exchange rate change and relative country consumption growth. In Panel A, we can observe that the negative aggregate endowment level shock results in a jump in the growth rate of the real exchange rate while, at the same time, it generates a fall in the relative country consumption growth. The reason for this is that the consumption of the local country, the less risk averse country, is falling at a larger pace than the consumption of the foreign country, the most risk averse country. In equilibrium the domestic country provides insurance the foreign country.

In Panel B we decompose $d\log E$ into $\gamma^B d\log C - \gamma^{B^*} d\log C^*$ and $\gamma^B d\log\frac{C^B}{C} - \gamma^{B^*} d\log\frac{C^{B^*}}{C^*}$. We can observe that the behavior of $d\log E$ is driven mainly by the changes in the relative risk aversion of the countries given by $\gamma^B d\log\frac{C^B}{C} - \gamma^{B^*} d\log\frac{C^{B^*}}{C^*}$: a significant degree of consumption redistribution is taking place inside the domestic country, while there is no significant consumption redistribution within the foreign country.

Figure 1.2: Cyclicality decomposition



A negative aggregate endowment level shock temporarily decreases the consumption of the domestic country as a share of the total available domestic consumption basket. On the other hand, the shock temporarily increases the consumption of the foreign country as a share of the total available foreign consumption basket. The less risk averse domestic country absorbs the negative shock and provides insurance to the more risk averse foreign country. This is portrayed in figure 1.3.

Figure 1.3: Country consumption of domestic and foreign consumption basket as a share of all the available domestic and foreign consumption goods

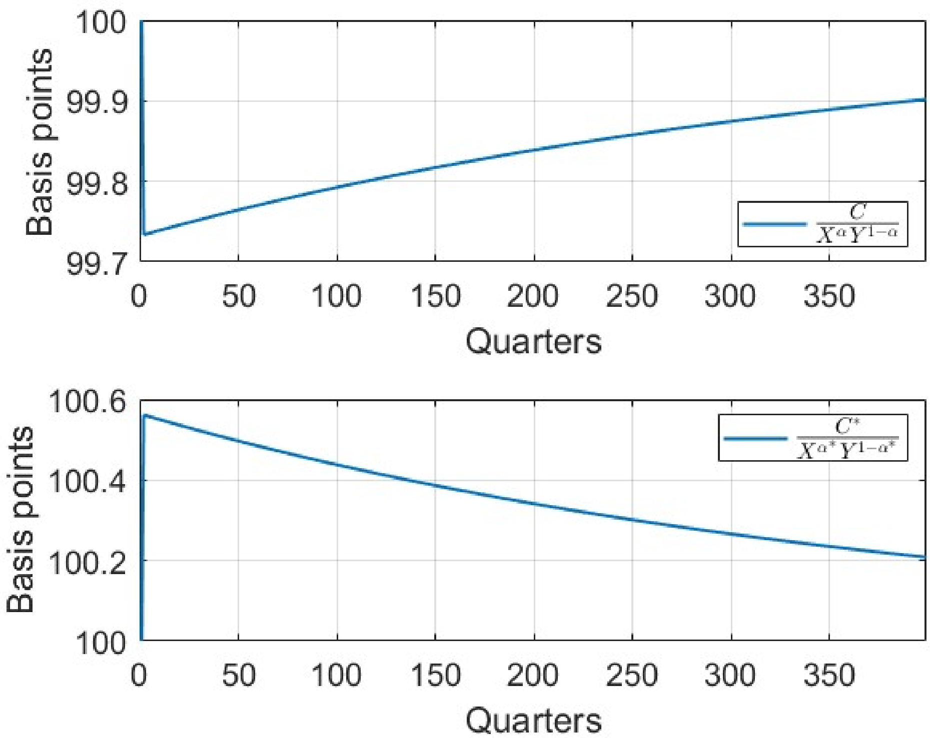
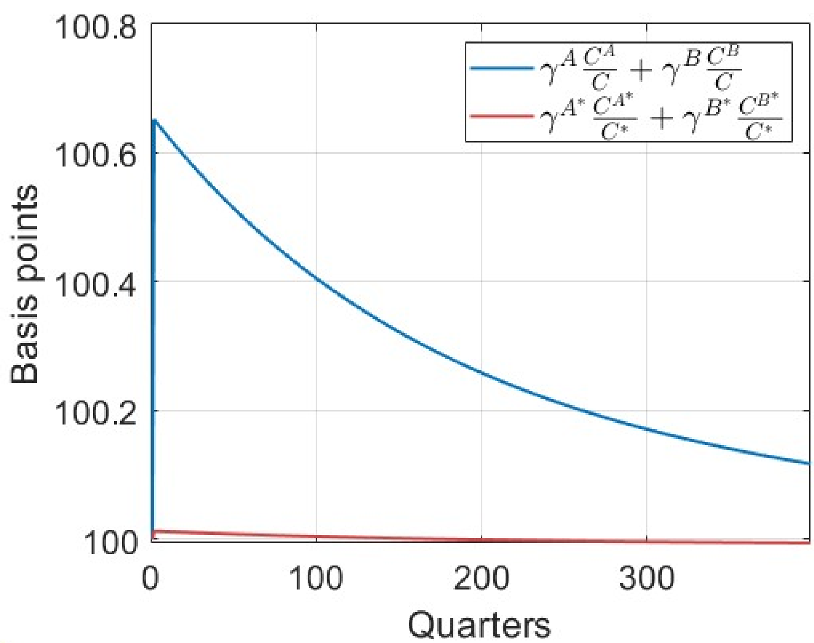


Figure 1.4 shows the change in the consumption weighted average risk aversion of

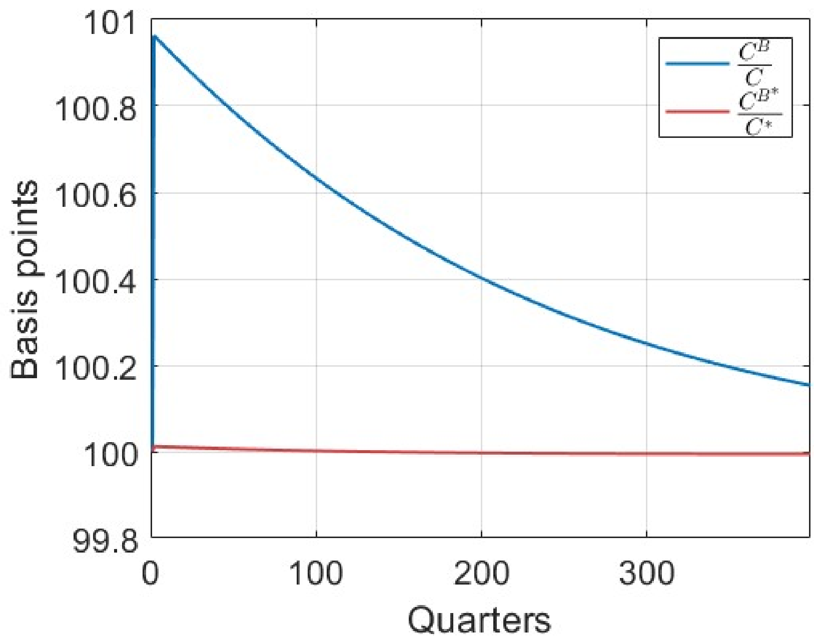
each country after the negative aggregate endowment level shock. Although there is an increment in risk aversion for both countries, the jump is larger for the domestic country, facing a larger within country consumption redistribution.

Figure 1.4: Country consumption weighted risk aversion



The reason behind the asymmetric change in risk aversion across countries shown in Figure 4 is explained in Figure 1.5. Here, we can see that although there is a consumption reallocation towards the less risk averse agents in both countries, the redistribution is stronger in the domestic country.

Figure 1.5: Consumption share of the less risk averse agents in each country's consumption



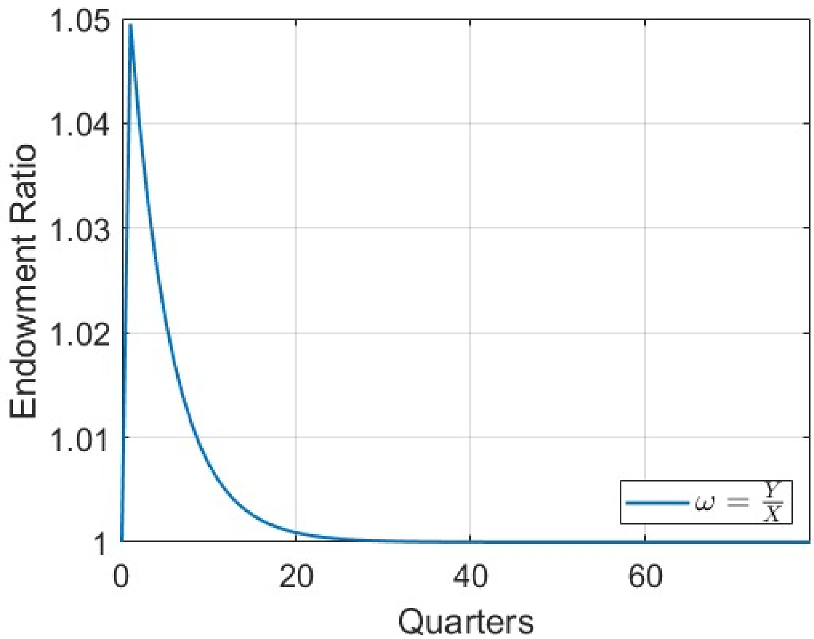
1.5.2 Impulse Response Functions

1.5.2.1 Positive Ratio Shock

In this section we analyze the impact of a positive aggregate endowment ratio shock on the main variables of the Economy. Although this shock is not symmetric as the aggregate endowment level shock, the economy behaves in a similar way under the chosen parametrization.

The positive endowment ratio shock can be appreciated on the value of $\omega_t = \frac{Y_t}{X_t}$ in Figure 1.6. A positive aggregate endowment ratio shock increases temporary the value of ω_t.

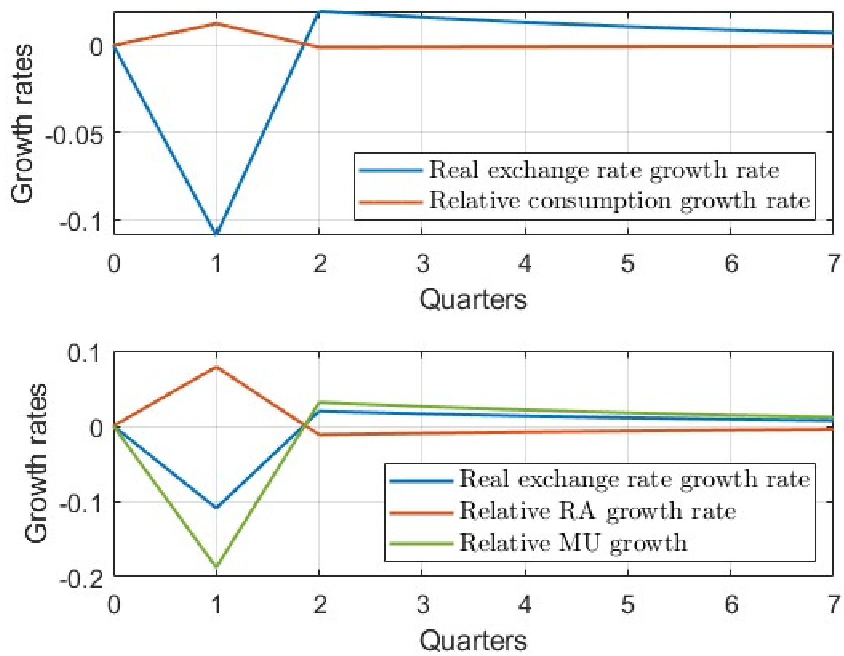
Figure 1.6: Endowent Ratio Shock



In figure 1.7 we can appreciate the the nature of the negative correlation between the real exchange rate change and relative country consumption growth. In Panel A, we can observe that the positive aggregate endowment ratio shock results in a fall in the growth rate of the real exchange rate while, at the same time, it generates a jump in the relative country consumption growth.The reason for this is that the consumption of the local country, the less risk averse country, is increasing at a larger pace than the consumption of the foreign country, the most risk averse country.

In Panel B we decompose $d\log E$ into $\gamma^B d\log C - \gamma^{B^*} d\log C^*$ and $\gamma^B d\log\frac{C^B}{C} - \gamma^{B^*} d\log\frac{C^{B^*}}{C^*}$. We can observe that the behavior of $d\log E$ is driven mainly by the changes in the relative risk aversion of the countries given by $\gamma^B d\log\frac{C^B}{C} - \gamma^{B^*} d\log\frac{C^{B^*}}{C^*}$: a significant degree of consumption redistribution is taking place inside the domestic country, while there is no significant consumption redistribution within the foreign country.

Figure 1.7: Backus-Smith Decomposition



A positive aggregate endowment ratio shock temporarily increase the consumption of the domestic country as a share of the total available domestic consumption basket. On the other hand, the shock temporarily decreases the consumption of the foreign country as a share of the total available foreign consumption basket. This is portrayed in figure 1.8.

Figure 1.8: Country consumption of domestic and foreign consumption baskets as a share of all the available domestic and foreign consumption goods

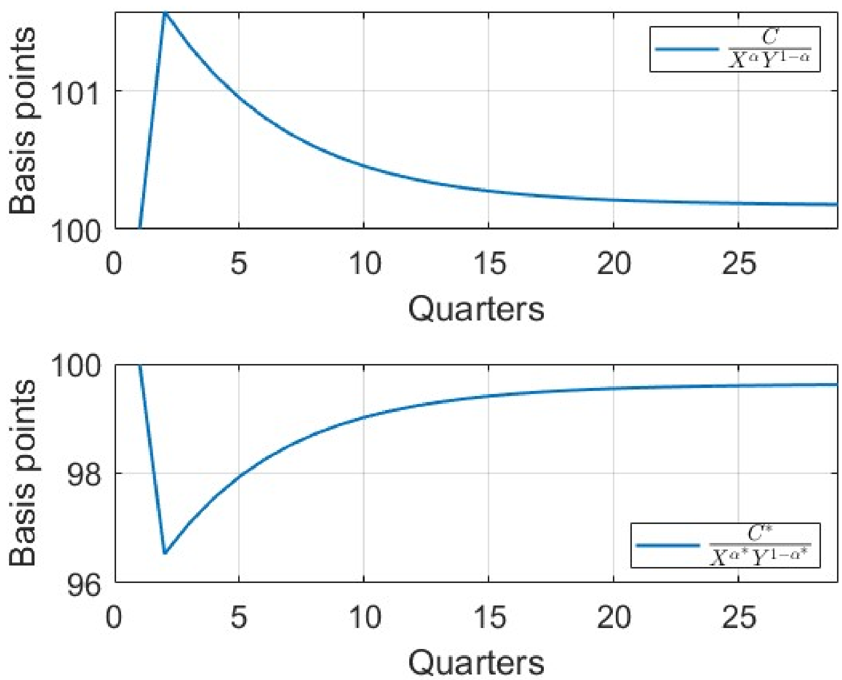
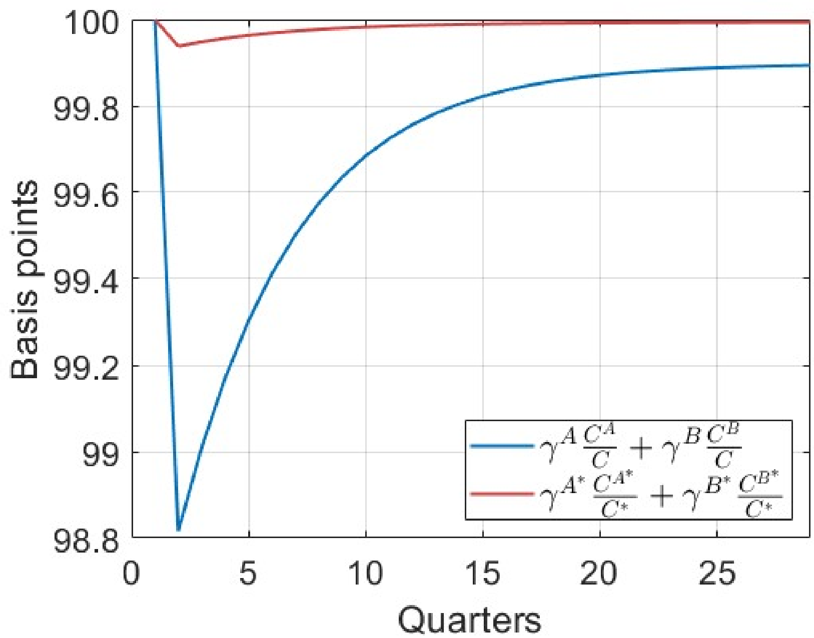


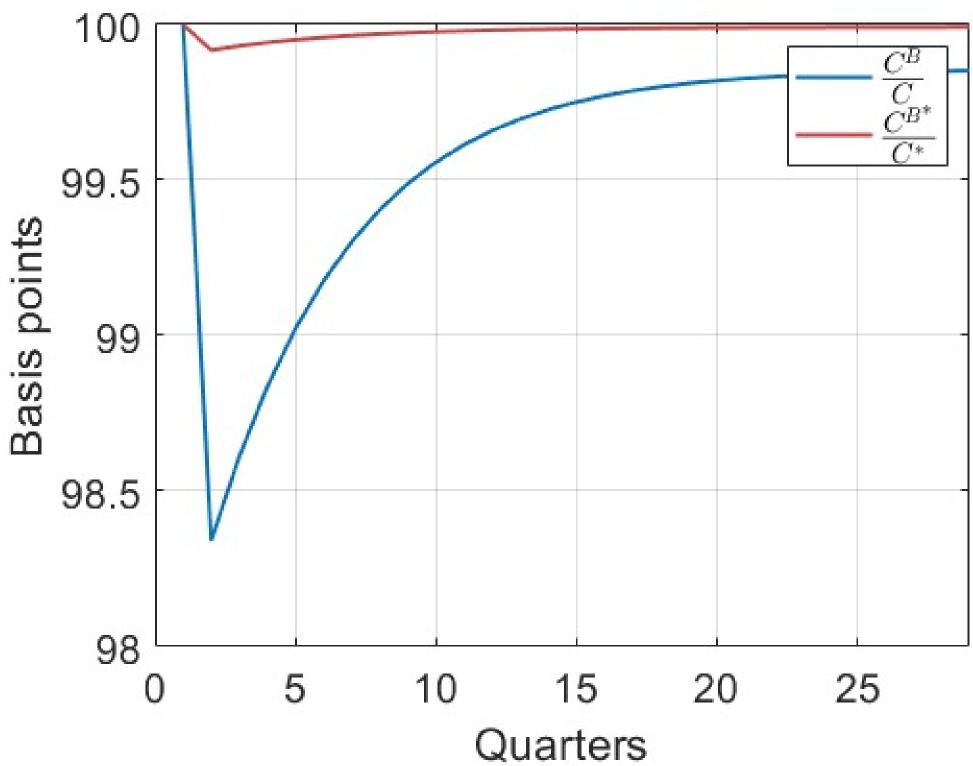
Figure 1.9 shows the change in the consumption weighted average risk aversion of each country after the positive aggregate endowment ratio shock. Although there is an fall in risk aversion for both countries, the change is larger for the domestic country, facing a larger within country consumption redistribution

Figure 1.9: Country consumption weighted risk aversion



The reason behind the asymmetric change in risk aversion across countries shown in Figure 9 is explained in Figure 1.10. Here, we can see that although there is a consumption reallocation towards the more risk averse agents in both countries, the effect is stronger in the domestic country.

Figure 1.10: Country Consumption weighted risk aversion



1.6 Conclusion

In this paper it is shown that a two-country, two-goods, two-agents per country model that features risk aversion heterogeneity both within and between countries is able to explain the Cyclicality puzzle established by Backus and Smith (1993).

The model proposed in this paper assumes a highly stylized environment that exhibits frictionless international trade in goods and assets and can be further expanded to tackle other relevant international finance puzzle in future research, as the Volatility puzzle of Brandt et al. (2006) and the Uncovered Interest Rates puzzle of Fama (1984).

Appendix

A.1 Model Solution

The Appendix presents the detailed solution of the model and proofs omitted in the main body of the paper.

The endowment growth of the local country follows a geometric Brownian motion:

$$d\log X_t = \mu^X dt + \sigma^X dB_t^{(1)}.$$

The ratio of the endowments of the foreign to local goods $\omega_t \in [\underline{\omega}, \overline{\omega}]$, where $\omega_t = \frac{Y_t}{X_t}$ follows a Wright-Fisher process:

$$d\omega_t = C\left(\omega^M - \omega_t\right)dt + \sigma^\omega \sqrt{(\omega_t - \underline{\omega})(\overline{\omega} - \omega_t)}dB_t^{(2)}.$$

Where ω^M is the mean reverting value for ω_t and $dB_t^{(1)} \times dB_t^{(2)} = \rho^{XY} \times dt$

We can derive the law of motion for the endowment of the foreign country $Y_t = \omega_t X_t$ through the application of Ito's lemma and conclude that the endowment growth of the foreign country follows a geometric Brownian motion:

$$d\log Y_t = \mu^Y dt + \sigma^{Y,(1)} dB_t^{(1)} + \sigma_t^{Y,(2)} dB_t^{(1)}.$$

Where:

$\mu_t^Y = C\frac{(\omega^* - \omega_t)}{\omega_t} + \mu^X + corr \times \sigma^X \times \sigma^\omega \times \frac{\sqrt{(\omega_t - \underline{\omega})(\overline{\omega} - \omega_t)}}{\omega_t},$

$\sigma^{Y,(1)} = \sigma^X,$

$\sigma_t^{Y,(2)} = \frac{1}{\omega}\sigma^\omega \sqrt{\left(\omega_t - \underline{\omega}\right)\left(\overline{\omega} - \omega_t\right)},$

Now, we will look into the Agent's optimization problem. Agent $i \in \{A, B, A^*, B^*\}$ born at time t intertemporal budget constraint is given by:

$$\max_{X^i_{u,t}, Y^i_{u,t}} E_t \Bigg\{ \int_t^\infty e^{-\pi(u-t)} \Bigg[e^{-\rho^i(u-t)} \frac{\left[\left(X^i_{u,t}\right)^{\alpha^i}\left(Y^i_{u,t}\right)^{1-\alpha^i}\right]^{1-\gamma^i}}{1-\gamma^i} - \lambda^i_t \left(\frac{H^X_u}{H^X_t} X^i_{u,t} + \frac{H^Y_u}{H^X_t} Y^i_{u,t} \right) \Bigg] du \Bigg\} + \lambda^i_t \frac{P^D_{t,t}}{\pi \times \text{Population}}.$$

Where $\frac{1}{\pi \times \text{Population}} P^D_{t,t}$ is the per–country j-cohort member value of the shares of the country good D trees born at time t. Note that $P^D_{t,t}$ is denominated in units of good X in period t. Also note that for the domestic agents $i \in \{A, B\}$ we have Population $= \mu_A + \mu_B$ and $D = X$. For the foreign agents $i \in \{A^*, B^*\}$ we have Population $= \mu_{A^*} + \mu_{B^*}$ and $D = Y$. The associated stochastic discount factor for good $D \in \{X, Y\}$ in period t is H^D_t.

The first order conditions of this optimization problem are given by:

$$e^{-\rho^i(u-t)} \alpha^i \left(C^{X,i}_{u,t}\right)^{\alpha^i\left(1-\gamma^i\right)-1} \left(C^{Y,i}_{u,t}\right)^{\left(1-\alpha^i\right)\left(1-\gamma^i\right)} = \lambda^i_t \frac{H^X_u}{H^X_t}, \tag{A.1}$$

$$e^{-\rho^i(u-t)} \left(1-\alpha^i\right) \left(C^{X,i}_{u,t}\right)^{\alpha^i\left(1-\gamma^i\right)} \left(C^{Y,i}_{u,t}\right)^{\left(1-\alpha^i\right)\left(1-\gamma^i\right)-1} = \lambda^i_t \frac{H^Y_u}{H^X_t}. \tag{A.2}$$

These equations taken together imply that the consumption growth of a consumer of type $i \in \{A, B, A^*, B^*\}$ for each good is given by:

$$\frac{X^i_{u,t}}{X^i_{t,t}} = e^{-\frac{\rho^i}{\gamma^i}(u-t)} \left(\frac{H^X_u}{H^X_t}\right)^{\frac{\left(1-\alpha^i\right)\left(1-\gamma^i\right)-1}{\gamma^i}} \left(\frac{H^Y_u}{H^Y_t}\right)^{-\frac{\left(1-\alpha^i\right)\left(1-\gamma^i\right)}{\gamma^i}}, \tag{A.3}$$

$$\frac{Y^i_{u,t}}{Y^i_{t,t}} = e^{-\frac{\rho^i}{\gamma^i}(u-t)} \left(\frac{H^X_u}{H^X_t}\right)^{-\frac{\alpha^i\left(1-\gamma^i\right)}{\gamma^i}} \left(\frac{H^Y_u}{H^Y_t}\right)^{\frac{\alpha^i\left(1-\gamma^i\right)-1}{\gamma^i}}. \tag{A.4}$$

In equilibrium the consumption market needs to clear at each time t. To analyze the implications of market clearing we define the consumption shares of good X and good Y for agent $i \in \{A, B, A^*, B^*\}$ as:

$$x^i_t \equiv \frac{\mu_i \int_{-\infty}^t \pi e^{-\pi(t-s)} X^i_{t,s} ds}{X_t}, \tag{A.5}$$

$$y^i_t \equiv \frac{\mu_i \int_{-\infty}^t \pi e^{-\pi(t-s)} Y^i_{t,s} ds}{Y_t}. \tag{A.6}$$

Substituting (A.3) into (A.5) leads after some re-arranging to

$$\begin{aligned} x^i_t \equiv \Bigg[& \left(H^X_t\right)^{\frac{\left(1-\alpha^i\right)\left(1-\gamma^i\right)-1}{\gamma^i}} \left(H^Y_t\right)^{-\frac{\left(1-\alpha^i\right)\left(1-\gamma^i\right)}{\gamma^i}} \mu^i \pi e^{\left(-\frac{\rho^i}{\gamma^i}-\pi\right)t} \\ & \times \int_{-\infty}^t e^{\left(\pi+\frac{\rho^i}{\gamma^i}\right)s} \left(H^X_s\right)^{\frac{1-\left(1-\alpha^i\right)\left(1-\gamma^i\right)}{\gamma^i}} \left(H^Y_s\right)^{\frac{\left(1-\alpha^i\right)\left(1-\gamma^i\right)}{\gamma^i}} X^i_{s,s} ds \Bigg] \left(X_t\right)^{-1} \end{aligned} \tag{A.7}$$

Substitution (A.4) into (A.6) leads after some re-arrenging to

$$y_t^i \equiv \Bigg[\left(H_t^X\right)^{-\frac{\alpha^i\left(1-\gamma^i\right)}{\gamma^i}} \left(H_t^Y\right)^{\frac{\alpha^i\left(1-\gamma^i\right)-1}{\gamma^i}} \mu^i \pi e^{\left(-\frac{\rho^i}{\gamma^i}-\pi\right)t}$$

$$\times \int_{-\infty}^{t} e^{\left(\pi+\frac{\rho^i}{\gamma^i}\right)s} \left(H_s^X\right)^{\frac{\alpha^i\left(1-\gamma^i\right)}{\gamma^i}} \left(H_s^Y\right)^{\frac{1-\alpha^i\left(1-\gamma^i\right)}{\gamma^i}} Y_{s,s}^i ds \Bigg] \left(Y_t\right)^{-1} \quad \text{(A.8)}$$

To start, is useful to recall a standard result in asset pricing, namely that the dynamics of the stochastic discount factors for goods X and Y are given by:

$$\frac{dH_t^X}{H_t^X} = -r_t^X dt - \kappa_t^{X,(1)} dB_t^{(1)} - \kappa_t^{X,(2)} dB_t^{(2)},$$

$$\frac{dH_t^Y}{H_t^Y} = -r_t^Y dt - \kappa_t^{Y,(1)} dB_t^{(1)} - \kappa_t^{Y,(2)} dB_t^{(2)}.$$

Where r_t^D is the real interest rate denominated in good $D \in \{X, Y\}$ and $\kappa_t^{D,(j)}$ is the market price of risk $j \in \{1, 2\}$ in terms of good $D \in \{X, Y\}$.

Applying Ito's lemma to the right hand side of (A.7), we conclude that the consumption share increment of good X for agent $i \in \{A, B, A^*, B^*\}$ is an arithmetic Brownian motion process:

$$dx_t^i = \mu_t^{x^i} dt + \sigma_t^{x^i,(1)} dB_t^{(1)} + \sigma_t^{x^i,(2)} dB_t^{(2)}. \quad \text{(A.9)}$$

Where:

$$\mu_t^{x^i} = \left[-\frac{\left(1-\alpha^i\right)\left(1-\gamma^i\right)}{\gamma^i} r_t^Y - \frac{\left(1-\alpha^i\right)\left(1-\gamma^i\right)-1}{\gamma^i} r_t^X - \frac{\rho^i}{\gamma^i} - \pi - \mu^X + \Omega_t^{x^i} \right] x_t^i + \frac{X_{t,t}^i}{X_t} \pi \mu^i,$$

$$\sigma_t^{x^i,(1)} = \left[-\frac{\left(1-\alpha^i\right)\left(1-\gamma^i\right)-1}{\gamma^i} \kappa_t^{X,(1)} + \frac{\left(1-\alpha^i\right)\left(1-\gamma^i\right)}{\gamma^i} \kappa_t^{Y,(1)} - \sigma^X \right] x_t^i,$$
$$\sigma_t^{x^i,(2)} = \left[-\frac{\left(1-\alpha^i\right)\left(1-\gamma^i\right)-1}{\gamma^i} \kappa_t^{X,(2)} + \frac{\left(1-\alpha^i\right)\left(1-\gamma^i\right)}{\gamma^i} \kappa_t^{Y,(2)} \right] x_t^i,$$

where $\Omega_t^{x^i}$ encompasses all the second order terms:

$$\begin{aligned}
\Omega_t^{x^i} = & \frac{\left[1-\left(1-\alpha^i\right)\left(1-\gamma^i\right)\right]\alpha^i\left(1-\gamma^i\right)}{2(\gamma^i)^2}\left(\frac{dH_t^X}{H_t^X}\right)^2 \\
& + \frac{\left(1-\alpha^i\right)\left(1-\gamma^i\right)\left[\left(1-\alpha^i\right)\left(1-\gamma^i\right)+\gamma^i\right]}{2(\gamma^i)^2}\left(\frac{dH_t^Y}{H_t^Y}\right)^2 \\
& - \frac{\left(1-\alpha^i\right)\left(1-\gamma^i\right)\left[\left(1-\alpha^i\right)\left(1-\gamma^i\right)-1\right]}{(\gamma^i)^2}\left(\frac{dH_t^X}{H_t^X}\right)\left(\frac{dH_t^Y}{H_t^Y}\right) + \left(\frac{dX_t}{X_t}\right)^2 \\
& - \left[\frac{\left(1-\alpha^i\right)\left(1-\gamma^i\right)-1}{\gamma^i}\right]\left(\frac{dH_t^X}{H_t^X}\right)\left(\frac{dX_t}{X_t}\right) + \frac{\left(1-\alpha^i\right)\left(1-\gamma^i\right)}{\gamma^i}\left(\frac{dH_t^Y}{H_t^Y}\right)\left(\frac{dX_t}{X_t}\right).
\end{aligned}$$

Where

$$\left(\frac{dH_t^X}{H_t^X}\right)^2 = \left(\kappa_t^{X,(1)}\right)^2 + \left(\kappa_t^{X,(2)}\right)^2 + 2\rho^{XY}\kappa_t^{X,(1)}\kappa_t^{X,(2)},$$
$$\left(\frac{dH_t^Y}{H_t^Y}\right)^2 = \left(\kappa_t^{Y,(1)}\right)^2 + \left(\kappa_t^{Y,(2)}\right)^2 + 2\rho^{XY}\kappa_t^{Y,(1)}\kappa_t^{Y,(2)},$$
$$\left(\frac{dH_t^X}{H_t^X}\right)\left(\frac{dH_t^Y}{H_t^Y}\right) = \kappa_t^{X,(1)}\kappa_t^{Y,(1)} + \kappa_t^{X,(2)}\kappa_t^{Y,(2)} + \rho^{XY}\left[\kappa_t^{X,(1)}\kappa_t^{Y,(2)} + \kappa_t^{X,(2)}\kappa_t^{Y,(1)}\right],$$
$$\left(\frac{dX_t}{X_t}\right)^2 = \left(\sigma^X\right)^2,$$
$$\left(\frac{dH_t^X}{H_t^X}\right)\left(\frac{dX_t}{X_t}\right) = \kappa_t^{X,(1)}\sigma^X + \rho^{XY}\kappa_t^{X,(2)}\sigma^X,$$
$$\left(\frac{dH_t^Y}{H_t^Y}\right)\left(\frac{dX_t}{X_t}\right) = \kappa_t^{Y,(1)}\sigma^X + \rho^{XY}\kappa_t^{Y,(2)}\sigma^X.$$

The law of motion for the consumption shares of good Y increments for agent $i \in \{A, B, A^*, B^*\}$ can be obtained alike by applying Ito's Lemma to the right hand side of equation (A.8):

$$dy_t^i = \mu_t^{y^i} dt + \sigma_t^{y^i,(1)} dB_t^{(1)} + \sigma_t^{y^i,(2)} dB_t^{(2)}. \tag{A.10}$$

Where:

$$\mu_t^{y^i} = \left[\tfrac{1-\alpha^i\left(1-\gamma^i\right)}{\gamma^i} r_t^Y + \tfrac{\alpha^i\left(1-\gamma^i\right)}{\gamma^i} r_t^X - \tfrac{\rho^i}{\gamma^i} - \pi - \mu^Y + \Omega_t^{y^i} \right] y_t^i + \tfrac{Y_{t,t}^i}{Y_t} \pi \mu^i,$$

$$\sigma_t^{y^i,(1)} = \left[\tfrac{\alpha^i\left(1-\gamma^i\right)}{\gamma^i} \kappa_t^{X,(1)} + \tfrac{1-\alpha^i\left(1-\gamma^i\right)}{\gamma^i} \kappa_t^{Y,(1)} - \sigma^X \right] y_t^i,$$

$$\sigma_t^{y^i,(2)} = \left[\tfrac{\alpha^i\left(1-\gamma^i\right)}{\gamma^i} \kappa_t^{X,(2)} + \tfrac{1-\alpha^i\left(1-\gamma^i\right)}{\gamma^i} \kappa_t^{Y,(2)} - \sigma_t^{Y,(2)} \right] y_t^i,$$

where $\Omega_t^{y^i}$ encompasses all the second order terms:

$$\begin{aligned}\Omega_t^{y^i} = &\frac{\left[\gamma^i + \alpha^i\left(1-\gamma^i\right)\right]\alpha^i\left(1-\gamma^i\right)}{2(\gamma^i)^2}\left(\frac{dH_t^X}{H_t^X}\right)^2 + \frac{\alpha^i\left(1-\gamma^i\right)\left[\alpha^i\left(1-\gamma^i\right)+\gamma^i\right]}{2(\gamma^i)^2}\left(\frac{dH_t^Y}{H_t^Y}\right)^2 \\ &+ \frac{\alpha^i\left(1-\gamma^i\right)\left[1-\alpha^i\left(1-\gamma^i\right)\right]}{(\gamma^i)^2}\left(\frac{dH_t^X}{H_t^X}\right)\left(\frac{dH_t^Y}{H_t^Y}\right) + \left(\frac{dY_t}{Y_t}\right)^2 \\ &+ \left[\frac{\alpha^i\left(1-\gamma^i\right)}{\gamma^i}\right]\left(\frac{dH_t^X}{H_t^X}\right)\left(\frac{dY_t}{Y_t}\right) + \frac{1-\alpha^i\left(1-\gamma^i\right)}{\gamma^i}\left(\frac{dH_t^Y}{H_t^Y}\right)\left(\frac{dY_t}{Y_t}\right).\end{aligned}$$

$$\left(\tfrac{dY_t}{Y_t}\right)^2 = \left(\sigma^{Y,(1)}\right)^2 + \left(\sigma_t^{Y,(2)}\right)^2 + 2\rho^{XY}\sigma^{Y,(1)}\sigma_t^{Y,(2)},$$

$$\left(\tfrac{dY_t}{Y_t}\right)\left(\tfrac{dH_t^Y}{H_t^Y}\right) = \kappa_t^{Y,(1)}\sigma^{Y,(1)} + \kappa_t^{Y,(2)}\sigma_t^{Y,(2)} + \rho^{XY}\left[\kappa_t^{Y(1)}\sigma_t^{Y,(2)} + \kappa_t^{Y,(2)}\sigma^{Y,(1)}\right].$$

The expressions for r_t^X and r_t^Y can be derived from the fact that $\sum_{i\in\{A,B,A^*,B^*\}} \mu_t^{x^i} = 0$ and $\sum_{i\in\{A,B,A^*,B^*\}} \mu_t^{y^i} = 0$ respectively for every t. This is derived from the application of Ito's Lemma to $\sum_{i\in\{A,B,A^*,B^*\}} x_t^i = 1$ and $\sum_{i\in\{A,B,A^*,B^*\}} y_t^i = 1$:

$$r_t^X = \frac{\pi\left(1-\sum_{i\in\{A,B,A^*,B^*\}}\frac{X_{t,t}^i}{X_t}\mu_i\right)+\sum_{i\in\{A,B,A^*,B^*\}}x_t^i\frac{\rho^i}{\gamma^i}+\mu^X-\Omega_t^x}{\left\{\frac{\Phi}{\sum_{i\in\{A,B,A^*,B^*\}}y_t^i\frac{\alpha^i(1-\gamma^i)-1}{\gamma^i}}\right\}}$$

$$+\frac{\pi\left(1-\sum_{i\in\{A,B,A^*,B^*\}}\frac{Y_{t,t}^i}{Y_t}\mu_i\right)+\sum_{i\in\{A,B,A^*,B^*\}}y_t^i\frac{\rho^i}{\gamma^i}+\mu_t^Y-\Omega_t^y}{\left\{\frac{\Phi}{\sum_{i\in\{A,B,A^*,B^*\}}x_t^i\frac{(1-\alpha^i)(1-\gamma^i)}{\gamma^i}},\right\}}$$

$$r_t^Y = \frac{\pi\left(1-\sum_{i\in\{A,B,A^*,B^*\}}\frac{X_{t,t}^i}{X_t}\mu_i\right)+\sum_{i\in\{A,B,A^*,B^*\}}x_t^i\frac{\rho^i}{\gamma^i}+\mu^X-\Omega_t^x}{\left\{\frac{\Phi}{\sum_{i\in\{A,B,A^*,B^*\}}y_t^i\frac{(1-\alpha^i)(1-\gamma^i)-1}{\gamma^i}}\right\}}$$

$$+\frac{\pi\left(1-\sum_{i\in\{A,B,A^*,B^*\}}\frac{Y_{t,t}^i}{Y_t}\mu_i\right)+\sum_{i\in\{A,B,A^*,B^*\}}y_t^i\frac{\rho^i}{\gamma^i}+\mu_t^Y-\Omega_t^y}{\left\{\frac{\Phi}{\sum_{i\in\{A,B,A^*,B^*\}}y_t^i\frac{\alpha^i(1-\gamma^i)}{\gamma^i}}\right\}}$$

where Ω_t^x and Ω_t^y are the weighted precautionary savings terms

$$\Omega_t^x = \sum_{i\in\{A,B,A^*,B^*\}} x_t^i\Omega_t^{x^i},$$

$$\Omega_t^y = \sum_{i\in\{A,B,A^*,B^*\}} y_t^i\Omega_t^{y^i},$$

$$\Phi = \sum_{i\in\{A,B,A^*,B^*\}} x_t^i \frac{(1-\alpha^i)(1-\gamma^i)}{\gamma^i} \sum_{i\in\{A,B,A^*,B^*\}} \frac{y_t^i}{\gamma^i} + \sum_{i\in\{A,B,A^*,B^*\}} y_t^i \frac{\alpha^i(1-\gamma^i)}{\gamma^i} \sum_{i\in\{A,B,A^*,B^*\}} \frac{x_t^i}{\gamma^i} - \sum_{i\in\{A,B,A^*,B^*\}} \frac{y_t^i}{\gamma^i} \sum_{i\in\{A,B,A^*,B^*\}} \frac{x_t^i}{\gamma^i}.$$

The expression for $\kappa_t^{X,(1)}$ and $\kappa_t^{X,(2)}$ can be derived from $\sum_{i\in\{A,B,A^*,B^*\}} \sigma_t^{x^i,(1)} = 0$ and $\sum_{i\in\{A,B,A^*,B^*\}} \sigma_t^{x^i,(2)} = 0$ for every t. The expression for $\kappa_t^{Y,(1)}$ and $\kappa_t^{Y,(2)}$ can be derived from that fact that $\sum_{i\in\{A,B,A^*,B^*\}} \sigma_t^{y^i,(1)} = 0$ and $\sum_{i\in\{A,B,A^*,B^*\}} \sigma_t^{y^i,(2)} = 0$ for every t. This is derived from the application of Ito's Lemma to $\sum_{i\in\{A,B,A^*,B^*\}} x_t^i = 1$ and $\sum_{i\in\{A,B,A^*,B^*\}} y_t^i = 1$:

$$\kappa_t^{X,(1)} = \frac{\sigma_t^X \left[\sum_{i\in\{A,B,A^*,B^*\}} y_t^i \frac{\alpha^i\left(1-\gamma^i\right)-1}{\gamma^i} + \sum_{i\in\{A,B,A^*,B^*\}} x_t^i \frac{\left(1-\alpha^i\right)\left(1-\gamma^i\right)}{\gamma^i} \right]}{\Phi}$$

$$\kappa_t^{X,(2)} = \frac{\sigma_t^{Y,(2)} \sum_{i\in\{A,B,A^*,B^*\}} x_t^i \frac{\left(1-\alpha^i\right)\left(1-\gamma^i\right)}{\gamma^i}}{\Phi}$$

$$\kappa_t^{Y,(1)} = \frac{\sigma_t^X \left[\sum_{i\in\{A,B,A^*,B^*\}} x_t^i \frac{\left(1-\alpha^i\right)\left(1-\gamma^i\right)-1}{\gamma^i} + \sum_{i\in\{A,B,A^*,B^*\}} y_t^i \frac{\alpha^i\left(1-\gamma^i\right)}{\gamma^i} \right]}{\Phi}$$

$$\kappa_t^{Y,(2)} = \frac{\sigma_t^{Y,(2)} \sum_{i\in\{A,B,A^*,B^*\}} x_t^i \frac{\left(1-\alpha^i\right)\left(1-\gamma^i\right)-1}{\gamma^i}}{\Phi}$$

A.2 The real exchange rate

The price of each consumption basket is defined as the minimum expenditure required to buy a unit of the basket and is derived by minimizing the corresponding expenditure function. So, we can define the price of the domestic consumption basket at time t:

$$M_t = \left(\frac{Q_t^X}{\alpha}\right)^{\alpha} \left(\frac{Q_t^Y}{1-\alpha}\right)^{1-\alpha}, \tag{A.11}$$

where Q_t^X and Q_t^Y are the prices of goods X and Y) of the goods in the corresponding period.

We can also get the price for the foreign consumption basket:

$$M_t^* = \left(\frac{Q_t^X}{\alpha^*}\right)^{\alpha^*} \left(\frac{Q_t^Y}{1-\alpha^*}\right)^{1-\alpha^*}, \tag{A.12}$$

Applying Ito's Lemma to (A.11) we get the law of motion for the domestic pricing kernel:

$$\frac{dM_t}{M_t} = -R_t dt - K_t^{(1)} dB_t^{(1)} - K_t^{(2)} dB_t^{(2)},$$

where

$$R_t = \alpha r_t^X + (1-\alpha) r_t^Y + \tfrac{1}{2}\alpha(1-\alpha)\left[\left(\tfrac{dH_t^X}{H_t^X}\right)^2 + \left(\tfrac{dH_t^Y}{H_t^Y}\right)^2\right] - \alpha(1-\alpha)\left(\tfrac{dH_t^X}{H_t^X}\right)\left(\tfrac{dH_t^Y}{H_t^Y}\right),$$

$$K_t^{(1)} = \alpha\kappa_t^{X,(1)} + (1-\alpha)\kappa_t^{Y,(1)},$$

$$K_t^{(2)} = \alpha\kappa_t^{X,(2)} + (1-\alpha)\kappa_t^{Y,(2)}.$$

Applying Ito's Lemma to (A.12) we get the law of motion for the foreign pricing kernel:

$$\frac{dM_t^*}{M_t^*} = -R_t^* dt - K_t^{*,(1)} dB_t^{(1)} - K_t^{*,(2)} dB_t^{(2)},$$

where

$$R_t^* = \alpha^* r_t^X + (1-\alpha^*)\, r_t^Y + \alpha^* (1-\alpha^*) \left\{ \frac{1}{2} \left[\left(\frac{dH_t^X}{H_t^X} \right)^2 + \left(\frac{dH_t^Y}{H_t^Y} \right)^2 \right] - \left(\frac{dH_t^X}{H_t^X} \right) \left(\frac{dH_t^Y}{H_t^Y} \right) \right\},$$

$$K_t^{*,(1)} = \alpha^* \kappa_t^{X,(1)} + (1-\alpha^*)\, \kappa_t^{Y,(1)},$$

$$K_t^{*,(2)} = \alpha^* \kappa_t^{X,(2)} + (1-\alpha^*)\, \kappa_t^{Y,(2)}.$$

The real exchange rate is defined as the quotient between the price of the foreign consumption basket to the price of the domestic consumption basket:

$$E_t = \frac{M_t^*}{M_t} = \frac{\alpha^\alpha}{(\alpha^*)^{\alpha^*}} \frac{(1-\alpha)^{1-\alpha}}{(1-\alpha^*)^{1-\alpha^*}} \left(\frac{Q_t^Y}{Q_t^X} \right)^{\alpha-\alpha^*},$$

where Q_t^X and Q_t^Y are the prices of good X and good Y in period t respectively.

Applying Ito's Lemma we get the low of motion for the real exchange rate:

$$\frac{dE_t}{E_t} = \mu_t^E dt + \sigma_t^{E,(1)} dB_t^{(1)} + \sigma_t^{E,(2)} dB_t^{(2)},$$

where

$$\mu_t^E = R_t - R_t^* + \left(\frac{dP_t}{P_t} \right)^2 - \left(\frac{dP_t}{P_t} \right) \left(\frac{dP_t^*}{P_t^*} \right),$$

$$\sigma_t^{E,(1)} = K_t^{(1)} - K_t^{*(1)},$$

$$\sigma_t^{E,(2)} = K_t^{(2)} - K_t^{*(2)},$$

$$\left(\frac{dP_t}{P_t} \right)^2 = \left(K_t^{(1)} \right)^2 + \left(K_t^{(2)} \right)^2 + 2\rho^{XY} K_t^{(1)} K_t^{(2)}$$

$$\left(\frac{dP_t}{P_t} \right) \left(\frac{dP_t^*}{P_t^*} \right) = K_t^{(1)} K_t^{*,(1)} + K_t^{(2)} K_t^{*,(2)} + \rho^{XY} \left(K_t^{(1)} K_t^{*,(2)} + K_t^{(2)} K_t^{*,(1)} \right)$$

A.3 Country Consumption

The domestic country's consumption of the domestic consumption basket is given by: $C_t = \left(x_t^A X_t\right)^{\alpha}\left(y_t^A Y_t\right)^{1-\alpha} + \left(x_t^B X_t\right)^{\alpha}\left(y_t^B Y_t\right)^{1-\alpha}$.

Applying Ito's lemma we get the law of motion for the domestic country consumption:

$$\frac{dC_t}{C_t} = \mu_t^C dt + \sigma_t^{C,(1)} dB_t^{(1)} + \sigma_t^{C,(2)} dB_t^{(2)},$$

where the diffusion on the aggregate endowment level shock is given by:
$\sigma_t^{C,(1)} = \left(\frac{\frac{C_t^A}{C_t}}{\gamma^A} + \frac{\frac{C_t^B}{C_t}}{\gamma^B}\right)\Xi_t^{(1)}\sigma^X,$
and the diffusion on the aggregate endowment ratio shock is given by:
$\sigma_t^{C,(2)} = \left(\frac{\frac{C_t^A}{C_t}}{\gamma^A} + \frac{\frac{C_t^B}{C_t}}{\gamma^B}\right)\Xi_t^{(2)}\frac{\sqrt{(\omega_t-\underline{\omega})(\overline{\omega}-\omega_t)}}{\omega_t}\sigma^{\omega},$
where

$$\Xi_t^{(1)} = \Bigg[\sum_{i\in\{A,B,A^*,B^*\}} x_t^i \frac{(1-\alpha^i)(1-\gamma^i)}{\gamma^i} + \sum_{i\in\{A,B,A^*,B^*\}} y_t^i \frac{\alpha^i(1-\gamma^i)}{\gamma^i} - \alpha \sum_{i\in\{A,B,A^*,B^*\}} \frac{y_t^i}{\gamma^i} - (1-\alpha)\sum_{i\in\{A,B,A^*,B^*\}} \frac{x_t^i}{\gamma^i}\Bigg]\Phi^{-1}$$

$$\Xi_t^{(2)} = \left[\sum_{i\in\{A,B,A^*,B^*\}} x_t^i \frac{(1-\alpha^i)(1-\gamma^i)}{\gamma^i} - (1-\alpha)\sum_{i\in\{A,B,A^*,B^*\}} \frac{x_t^i}{\gamma^i}\right]\Phi^{-1}$$

The foreign country's consumption of the foreign consumption basket is given by: $C_t^* = \left(x_t^{A^*} X_t\right)^{\alpha^*} \left(y_t^{A^*} Y_t\right)^{1-\alpha^*} + \left(x_t^{B^*} X_t\right)^{\alpha^*} \left(y_t^{B^*} Y_t\right)^{1-\alpha^*}$.

Applying Ito's lemma we get the law of motion for the foreign country consumption:

$$\frac{dC_t^*}{C_t^*} = \mu_t^{C^*} dt + \sigma_t^{C^*,(1)} dB_t^{(1)} + \sigma_t^{C^*,(2)} dB_t^{(2)},$$

where the diffusion on the aggregate endowment level shock is given by: $\sigma_t^{C^*,(1)} = \left(\frac{\frac{C_t^{A^*}}{C_t^*}}{\gamma^A} + \frac{\frac{C_t^{B^*}}{C_t^*}}{\gamma^B} \right) \Xi_t^{*,(1)} \sigma^X$,

and the diffusion on the aggregate endowment ratio shock is given by: $\sigma_t^{C^*,(2)} = \left(\frac{\frac{C_t^{A^*}}{C_t^*}}{\gamma^{A^*}} + \frac{\frac{C_t^{B^*}}{C_t^*}}{\gamma^{B^*}} \right) \Xi_t^{*,(2)} \frac{\sqrt{(\omega_t - \underline{\omega})(\overline{\omega} - \omega_t)}}{\omega_t} \sigma^\omega$,

where

$$\begin{aligned}\Xi_t^{*,(1)} = & \Bigg[\sum_{i \in \{A,B,A^*,B^*\}} x_t^i \frac{(1-\alpha)(1-\gamma^i)}{\gamma^i} \\ & + \sum_{i \in \{A,B,A^*,B^*\}} y_t^i \frac{\alpha (1-\gamma^i)}{\gamma^i} - \alpha^* \sum_{i \in \{A,B,A^*,B^*\}} \frac{y_t^i}{\gamma^i} - (1-\alpha^*) \sum_{i \in \{A,B,A^*,B^*\}} \frac{x_t^i}{\gamma^i} \Bigg] \Phi^{-1}\end{aligned}$$

$$\Xi_t^{(2)} = \left[\sum_{i \in \{A,B,A^*,B^*\}} x_t^i \frac{(1-\alpha^i)(1-\gamma^i)}{\gamma^i} - (1-\alpha^*) \sum_{i \in \{A,B,A^*,B^*\}} \frac{x_t^i}{\gamma^i} \right] \Phi^{-1}$$

A.4 Exchange rate decomposition

Combining the equations (A.1), (A.2) and (A.9) we can rewrite the pricing kernel as:

$$P_t = \left(\frac{e^{-\rho^i t} \left(C_t^i\right)^{\left(1-\gamma^i\right)}}{\lambda_t^i X_t^i} \right)^{\alpha} \left(\frac{e^{-\rho^i t} \left(C_t^i\right)^{\left(1-\gamma^i\right)}}{\lambda_t^i Y_t^i} \right)^{1-\alpha} = \frac{e^{-\rho^i t} \left(C_t^i\right)^{\left(1-\gamma^i\right)}}{\lambda_t^i \left(X_t^i\right)^{\alpha} \left(Y_t^i\right)^{1-\alpha}}$$
$$= \frac{e^{-\rho^i t}}{\lambda_t^i \left(C_t^i\right)^{\gamma^i}} = \frac{e^{-\rho^i t}}{\lambda_t^i \left(S_t^i \times C_t\right)^{\gamma^i}},$$

where S_t^i is the share of agent i in its own country consumption. Then we can rewrite the real exchange rate int he following way:

$$E_t = \frac{\frac{e^{-\rho^j (u-t)}}{\lambda_t^i \left(S_t^{j,*} \times C_{u,t}^*\right)^{\gamma_j}}}{\frac{e^{-\rho^i (u-t)}}{\lambda_t^i \left(S_{country}^i \times C_{u,t}\right)^{\gamma^i}}}.$$

If we choose type B agents for each country, the real exchange rate can be decomposed into:

$$\frac{dE_t}{E_t} = \left(\gamma^B \frac{dC_t}{C_t} - \gamma^{B^*} \frac{dC_t^*}{C_t^*} \right) + \left(\gamma^B \frac{dS_t^B}{S_t^B} - \gamma^{B^*} \frac{dS_t^{B^*}}{S_t^{B^*}} \right) + \left(\frac{d\lambda_t^B}{\lambda_t^B} - \frac{d\lambda_t^{B^*}}{\lambda_t^{B^*}} \right).$$

A.5 Completing the Construction of Equilibrium

The determination of the drift of the law of motion for the consumption shares of good X and of good Y requires the calculation of $\frac{X^i_{t,t}}{X_t}$ and $\frac{Y^i_{t,t}}{Y_t}$ respectively, which can be obtained from the agents' budget constraint at time t. The budget constraint for local agents at time t requires that:

$$E_t \int_t^{\infty} e^{-\pi(u-t)} \left[\left(\frac{H^X_u}{H^X_t} \right) X^i_{u,t} + \left(\frac{H^Y_u}{H^X_t} \right) Y^i_{u,t} \right] du = \frac{1}{\pi\left(\mu_A + \mu_B\right)} P^X_{t,t} =$$
$$= \frac{1}{\pi\left(\mu_A + \mu_B\right)} E_t \int_t^{\infty} e^{-\delta(u-t)} \left(\frac{H^X_u}{H^X_t} \right) X_{u,t} du.$$

The budget constraint for foreign agents at time t requires that:

$$E_t \int_t^{\infty} e^{-\pi(u-t)} \left[\left(\frac{H^X_u}{H^X_t} \right) X^i_{u,t} + \left(\frac{H^Y_u}{H^X_t} \right) Y^i_{u,t} \right] du = \frac{1}{\pi\left(\mu_{A^*} + \mu_{B^*}\right)} P^Y_{t,t} =$$
$$= \frac{1}{\pi\left(\mu_{A^*} + \mu_{B^*}\right)} E_t \int_t^{\infty} e^{-\delta(u-t)} \left(\frac{H^Y_u}{H^X_t} \right) Y_{u,t} du.$$

We can define the Price Dividend ratios p^X_t and p^Y_t for good X and good Y respectively:

$$p^X_t \equiv \mathrm{E}_t \int_t^{\infty} e^{-\delta(u-t)} \left(\frac{H^X_u}{H^X_t} \right) \left(\frac{X_u}{X_t} \right) du \tag{A.13}$$

$$p^Y_t \equiv \mathrm{E}_t \int_t^{\infty} e^{-\delta(u-t)} \left(\frac{H^Y_u}{H^Y_t} \right) \left(\frac{Y_u}{Y_t} \right) du \tag{A.14}$$

Also, we can define he Wealth to Consumption of Good X ratios $g^{X,A}_t$ and g^{X,A^*}_t

for agents A and A^* respectively:

$$g_t^{X,A} \equiv E_t \int_t^{\infty} e^{-\pi(u-t)} \left[\left(\frac{H_u^X}{H_t^X}\right) \left(\frac{X_{u,t}^A}{X_{t,t}^A}\right) + \left(\frac{H_u^Y}{H_t^X}\right) \left(\frac{Y_{u,t}^A}{Y_{t,t}^A}\right) \right] du \qquad \text{(A.15)}$$

$$g_t^{X,A^*} \equiv E_t \int_t^{\infty} e^{-\pi(u-t)} \left[\left(\frac{H_u^X}{H_t^X}\right) \left(\frac{X_{u,t}^{A^*}}{X_{t,t}^{A^*}}\right) + \left(\frac{H_u^Y}{H_t^X}\right) \left(\frac{Y_{u,t}^{A^*}}{Y_{t,t}^{A^*}}\right) \right] du \qquad \text{(A.16)}$$

So, using some algebra we can rewrite $\frac{X_{t,t}^A}{X_t}$, $\frac{X_{t,t}^{A^*}}{X_t}$ and $\frac{Y_{t,t}^A}{Y_t}$ in the following way:

$$\frac{X_{t,t}^A}{X_t} = \frac{\delta}{\pi(\mu_A+\mu_B)} \frac{p_t^X}{g_t^{A,X}},$$

$$\frac{X_{t,t}^{A^*}}{X_t} = \frac{\delta}{\pi(\ \mu_{A^*}+\mu_{B^*})} \left(\frac{H_t^Y}{H_t^X}\right) \omega_t \frac{p_t^Y}{g_t^{A^*,X}},$$

$$\frac{Y_{t,t}^A}{Y_t} = \frac{\delta}{\pi(\mu_A+\mu_B)} \frac{1-\alpha}{\alpha} \frac{1}{\left(\frac{H_t^Y}{H_t^X}\right)} \frac{1}{\omega_t} \frac{p_t^X}{g_t^{A,X}},$$

A.6 Stochastic Partial Differential Equations

This section derive the stochastic partial differential equations that characterize p_t^X, p_t^Y, $g_t^{A,X}$ and $g_t^{A^*,X}$.

From (A.13) - (A.15), we can write:

$$p_t^X e^{-\delta t} H_t^X X_t + \int_s^t e^{-\delta u} H_u^X X_u du \quad = E_t \int_s^{\infty} e^{-\delta u} H_u^X X_u du \qquad \text{(A.17)}$$

$$p_t^Y e^{-\delta t} H_t^Y Y_t + \int_s^t e^{-\delta u} H_u^Y Y_u du \quad = E_t \int_s^{\infty} e^{-\delta u} H_u^Y Y_u du \qquad \text{(A.18)}$$

$$
\begin{aligned}
g_t^{A,X} e^{\left(-\frac{\rho^A}{\gamma^A}-\pi\right)t} \left(H_t^X\right)^{\frac{\left(1-\alpha^A\right)\left(1-\gamma^A\right)-1}{\gamma^A}} \left(H_t^Y\right)^{-\frac{\left(1-\alpha^A\right)\left(1-\gamma^A\right)}{\gamma^A}} &- \frac{H_t^Y}{H_t^X}\frac{\left(1-\alpha^A\right)}{\alpha^A} \\
\times \int_t^\infty e^{\left(-\frac{\rho^A}{\gamma^A}-\pi\right)u} \left(H_u^X\right)^{-\frac{\alpha^A\left(1-\gamma^A\right)}{\gamma^A}} \left(H_u^Y\right)^{\frac{\alpha^A\left(1-\gamma^A\right)-1}{\gamma^A}+1} du \\
+ \int_s^t e^{\left(-\frac{\rho^A}{\gamma^A}-\pi\right)u} \left(H_u^X\right)^{-\frac{\alpha^A\left(1-\gamma^A\right)}{\gamma^A}} \left(H_u^Y\right)^{-\frac{\left(1-\alpha^A\right)\left(1-\gamma^A\right)}{\gamma^A}} du \\
= E_t \int_s^\infty e^{\left(-\frac{\rho^A}{\gamma^A}-\pi\right)u} \left(H_u^X\right)^{-\frac{\alpha^A\left(1-\gamma^A\right)}{\gamma^A}} \left(H_u^Y\right)^{-\frac{\left(1-\alpha^A\right)\left(1-\gamma^A\right)}{\gamma^A}} du
\end{aligned}
\tag{A.19}
$$

$$
\begin{aligned}
g_t^{A^*,X} e^{\left(-\frac{\rho^{A^*}}{\gamma^{A^*}}-\pi\right)t} \left(H_t^X\right)^{\frac{\left(1-\alpha^{A^*}\right)\left(1-\gamma^{A^*}\right)-1}{\gamma^{A^*}}} \left(H_t^Y\right)^{-\frac{\left(1-\alpha^{A^*}\right)\left(1-\gamma^{A^*}\right)}{\gamma^{A^*}}} &- \frac{H_t^Y}{H_t^X}\frac{\left(1-\alpha^{A^*}\right)}{\alpha^{A^*}} \\
\times \int_t^\infty e^{\left(-\frac{\rho^{A^*}}{\gamma^{A^*}}-\pi\right)u} \left(H_u^X\right)^{-\frac{\alpha^{A^*}\left(1-\gamma^{A^*}\right)}{\gamma^{A^*}}} \left(H_u^Y\right)^{\frac{\alpha^{A^*}\left(1-\gamma^{A^*}\right)-1}{\gamma^{A^*}}+1} du \\
+ \int_s^t e^{\left(-\frac{\rho^{A^*}}{\gamma^{A^*}}-\pi\right)u} \left(H_u^X\right)^{-\frac{\alpha^{A^*}\left(1-\gamma^{A^*}\right)}{\gamma^{A^*}}} \left(H_u^Y\right)^{-\frac{\left(1-\alpha^{A^*}\right)\left(1-\gamma^{A^*}\right)}{\gamma^{A^*}}} du \\
= E_t \int_s^\infty e^{\left(-\frac{\rho^{A^*}}{\gamma^{A^*}}-\pi\right)u} \left(H_u^X\right)^{-\frac{\alpha^{A^*}\left(1-\gamma^{A^*}\right)}{\gamma^{A^*}}} \left(H_u^Y\right)^{-\frac{\left(1-\alpha^{A^*}\right)\left(1-\gamma^{A^*}\right)}{\gamma^{A^*}}} du
\end{aligned}
\tag{A.20}
$$

Observe that the right hand side of equations (A.17), (A.18), (A.19) and (A.20) are conditional expectations and hence martingales. This means that the left hand sides must be martingales as well.

To proceed, conjecture that the equilibrium is Markovian in ω_t, x_t^A,$x_t^{A^*}$ and y_t^A. This conjecture implies that p_t^X, p_t^Y, $g_t^{A,X}$ and $g_t^{A^*,X}$ can be written exclusively as functions of ω_t, x_t^A,$x_t^{A^*}$ and y_t^A as well.

Applying Ito's Lemma to compute the drift of the left-hand side of (A.17) and setting the resulting expression to zero leads after some simplifications to the following differential equation:

$$\begin{aligned}
\sum_{i \in \left\{\omega, x^{A}, x^{A^*}, y^{A^*}\right\}} & \Bigg\{ e^{-\delta t} H_t^X X_t \left(p^X\right)^{i\,\prime} \mu_t^i \\
& + \frac{1}{2} \left(p^X\right)''_{ii} e^{-\delta t} H_t^X X_t \left[\left(\sigma_t^{i,(1)}\right)^2 + \left(\sigma_t^{i,(2)}\right)^2 + 2\rho^{XY} \sigma_t^{i,(1)} \sigma_t^{i,(2)} \right] \\
& - X_t \left(p^X\right)^{i\,\prime} e^{-\delta t} \\
& \times \left[\sigma_t^{i,(1)} \kappa_t^{X,(1)} + \sigma_t^{i,(2)} \kappa_t^{X,(2)} + \rho^{XY} \left(\sigma_t^{i,(1)} \kappa_t^{X,(2)} + \sigma_t^{i,(2)} \kappa_t^{X,(1)} \right) \right] \\
& + e^{-\delta t} H_t^X \left(p^X\right)^{i\,\prime} \left[\sigma_t^{i,(1)} \sigma^X + \rho^{XY} \sigma_t^{i,(2)} \sigma^X \right] \\
& + \sum_{j \neq i \in \left\{\omega, x^{A}, x^{A^*}, y^{A^*}\right\}} e^{-\delta t} H_t^X X_t \left(p^X\right)^{j,i\,\prime\prime} \\
& \times \left[\sigma_t^{i,(1)} \sigma_t^{j,(1)} + \sigma_t^{i,(2)} \sigma_t^{j,(2)} + 2\rho^{XY} \left(\sigma_t^{i,(1)} \sigma_t^{j,(2)} + \sigma_t^{i,(2)} \sigma_t^{j,(1)} \right) \right] \Bigg\} \\
& - \left[\sigma^X \kappa_t^{X,(1)} + \rho^{XY} \sigma_t^X \kappa_t^{X,(2)} \right] + \left(p^X\right) \delta e^{-\delta t} H_t^X X_t dt \\
& + e^{-\delta t} H_t^X X_t \left(1 + \mu^X - r_t^X\right) = 0
\end{aligned}$$

This stochastic partial differential equation characterizes the solution for p^X. The stochastic partial differential equations that characterize the solutions for p^Y, $g^{A,X}$ and $g^{A^*,X}$ can be found by applying Ito's lemma to the left hand side of equations (A.18), (A.19) and (A.20) respectively.

A.7 Boundary Conditions

This section provides a proof for Proposition 3.

Proposition 3.1:

$\lim_{x_t^i \to 0} \sigma_t^{x^i,(j)} = \lim_{y_t^i \to 0} \sigma_t^{x^i,(j)} = \lim_{x_t^i \to 1} \sigma_t^{x^i,(j)} = \lim_{y_t^i \to 1} \sigma_t^{x^i,(j)} = 0,$

where $i \in \{A, A^*, B, B^*\}$ and $j \in \{1, 2\}$.

Proof:

$\lim_{x_t^i \to 0} \sigma_t^{x^i,(1)} = \left[-\frac{(1-\alpha^i)(1-\gamma^i)-1}{\gamma^i} \kappa_t^{X,(1)} + \frac{(1-\alpha^i)(1-\gamma^i)}{\gamma^i} \kappa_t^{Y,(1)} - \sigma_t^X \right] \times 0 = 0.$

By Pareto conditions, if $y_t^i \to 0 \Rightarrow x_t^i \to 0 \Rightarrow \lim_{y_t^i \to 0} \sigma_t^{x^i,(1)} = 0.$

If $x_t^i \to 1$, then $x_t^h \to 0$ for every agent $h \neq i \Rightarrow \sigma_t^{x^h,(1)} = 0$ for every agent $h \neq i$.

But $\sum_{i \in \{A,B,A^*,B^*\}} \sigma_t^{x^i,(1)} = 0 \Rightarrow \lim_{x_t^i \to 1} \sigma_t^{x^i,(j)} = 0.$

If $y_t^i \to 1 \Rightarrow$, then $y_t^h \to 0$ for every agent $h \neq i$. Then by Pareto conditions $x_t^h \to 0$ for every agent $h \neq i$. So, $x_t^i \to 1$. Then, $\lim_{y_t^i \to 1} \sigma_t^{x^i,(j)} = 0$

Proposition 3.2

$\lim_{x_t^i \to 0} \mu_t^{x^i} = \lim_{y_t^i \to 0} \mu_t^{x^i} = \pi \mu^i \frac{X_{t,t}^i}{X_t} > 0.$

Proof:

$\lim_{x_t^i \to 0} \mu_t^{x^i} = \left[-\frac{(1-\alpha^i)(1-\gamma^i)}{\gamma^i} r_t^Y - r_t^X \frac{(1-\alpha^i)(1-\gamma^i)-1}{\gamma^i} - \frac{\rho^i}{\gamma^i} - \pi + \Omega_t^{i,x} \right] \times 0 + \frac{X_{t,t}^i}{X_t} \pi \mu^i = \frac{X_{t,t}^i}{X_t} \pi \mu^i > 0.$

By Pareto conditions, if $y_t^i \to 0 \Rightarrow x_t^i \to 0 \Rightarrow \lim_{y_t^i \to 0} \mu_t^{x^i} = \frac{X_{t,t}^i}{X_t} \pi \mu^i > 0.$

Proposition 3.3

$\lim_{x_t^i \to 1} \mu_t^{x^i} = \lim_{y_t^i \to 1} \mu_t^{x^i} = -\pi \sum_{h \neq i} \mu_h \frac{X_{t,t}^h}{X_t} < 0.$

Proof:

If $x_t^i \to 1$, then $x_t^h \to 0$ for every agent $h \neq i \Rightarrow \mu_t^{x^h} = \pi \mu_h \frac{X_{t,t}^h}{X_t}$ for every agent $h \neq i$. But $\sum_{i \in \{A,B,A^*,B^*\}} \mu_t^{x^i} = 0 \Rightarrow \lim_{x_t^i \to 1} \mu_t^{x^i} = -\pi \sum_{h \neq i} \mu_h \frac{X_{t,t}^h}{X_t} < 0.$

If $y_t^i \to 1 \Rightarrow$, then $y_t^h \to 0$ for every agent $h \neq i$. Then by Pareto conditions $x_t^h \to 0$ for every agent $h \neq i$. So, $x_t^i \to 1. \Rightarrow \lim_{y_t^i \to 1} \mu_t^{x^i} = -\pi \sum_{h \neq i} \mu_h \frac{X_{t,t}^h}{X_t}$.

CHAPTER 2

Quantitative Methodology

This chapter presents the quantitative methodology implemented to find a solution to the model presented in Chapter 1. The model does not admit a closed form solution so the resolution of the model relies on the application of Monte Carlo Methods, the Feynman-Kac theorem and Piccard's fixed-point theorem.

2.1 Introduction

The drifts of the 3 consumption shares that constitute state variables of the model described in Chapter 1 depend on 2 wealth to consumption ratios and 2 price dividend ratios. The solution for these variables boils down to a second order partial stochastic differential equation (PSDE) each. Unfortunately, these PSDE do not admit a closed form solution. In order to present an outcome to the Asset Pricing model described in Chapter 1, the model was coded and simulated: a numerical solution to the aforementioned PDSE was found by leveraging on the Feynman-Kac Theorem, the Piccard Fixed Point Theorem and Monte Carlo Simulations. The code is written in Python and is designed to be effective and efficient and to be able to run under different initial conditions, offering insights and understanding of the underlying processes that drive the system's dynamics. This chapter describes the quantitative methodology used to obtain the outcome of the model described in Chapter 1.

2.2 Python Libraries and Modules

2.2.1 NumPy

NumPy is a fundamental library for numerical computing in Python. It provides support for working with large, multi-dimensional arrays and matrices, along with a variety of mathematical functions to operate on these arrays.

Key functionalities in the code:

- Creating and manipulating arrays: The code relies on NumPy's array functionalities to create and manipulate the data structures used in the calculations.
- Mathematical operations: NumPy provides a wide range of mathematical functions, including element-wise operations, linear algebra, and statistical functions. The code uses these functions to perform calculations on the arrays.
- Random number generation: The Monte Carlo simulations require random number generation, and NumPy's random module is used to generate random numbers with specific distributions.

2.2.2 Pandas

Pandas is a widely used library for data manipulation and analysis in Python. It provides data structures like Series and DataFrame, which make it easy to work with structured data. Key functionalities in the code:

- Reading input data: The code uses Pandas to read input data from CSV files and store it in DataFrame objects for easy access and manipulation.
- Creating DataFrames: During the calculations, DataFrames are used to store intermediate results and organize the data in a structured format.
- Saving results: Once the calculations are complete, the results are saved to CSV files using Pandas' DataFrame.to_csv() function.

2.2.3 Concurrent.futures

The concurrent.futures module provides a high-level interface for asynchronously executing callables. It is used to parallelize the Monte Carlo simulations, taking advantage of multi-core processing capabilities to speed up the calculations. Key functionalities in the code:

- ProcessPoolExecutor: The code uses ProcessPoolExecutor to create a pool of worker processes that can execute the simulations in parallel. This helps distribute the computational workload across multiple CPU cores, resulting in faster execution times.
- ThreadPoolExecutor: In addition to ProcessPoolExecutor, ThreadPoolExecutor is used for saving results in parallel to improve the overall performance of the code.

2.3 Functions

In this section, I will present the utility functions[1] that are used throughout the code to carry out the calculations that will be described in the next section

- Initialization Utility Function: This function initializes the multiprocessing pool with the necessary global variables, which allows for parallel execution of the Simulation Utility Function.

- Simulation Utility Function: This function takes the initial sets of consumption shares and performs the Monte Carlo simulation for the given set of parameters.

- Picard Iteration Utility Function: This function represents the implementation of the Picard fixed-point iteration method. It takes the current guess for the 2 price dividend ratios and 2 wealth to consumption ratios, and computes the next estimate using the Simulation Utility Function, and checks for convergence. The process is repeated until the solution converges or a maximum number of iterations is reached.

- Outcome Utility Functions: The code includes utility functions to calculate various statistics, such as mean, correlation and standard deviation.

[1]A utility function in coding refers to a self-contained piece of code that performs a specific task. It is not related to the concept of Utility Function in Economics.

2.4 Execution Flow and Code Implementation

In this section I present the code implementation details that are essential to solving the mathematical dynamic model using Monte Carlo simulations and iterative methods based on the Feynman-Kac Theorem and the Picard fixed-point theorem. There are 3 versions of the code. Although their structure is mostly the same, they present some differences:

- The code that solves the 4 heterogeneous agents model
- The code that generates the impulse response functions for the 4 heterogeneous model
- The code that solves the 2 homogeneous agents model that is used as benchmark for the standard result of perfect correlation between real exchange rate changes and relative country consumption growth.

I will start by the describing the code that solves the 4 heterogeneous agents model and then describe how the other 2 codes differ from this one. The code follows a sequential execution flow, beginning with the initialization of variables and data structures, followed by running the Monte Carlo simulations, Picard Iterations and finally saving and plotting the results. The main steps in the execution flow for the 4 heterogeneous agents model is the following:

1. Set up the relevant all the relevant parameters of the model in the corresponding Jason files. These parameters are:

- Agent parameters: risk aversion, discount factor, population share, death born-rate and cobb-douglas preferences parameters.
- Endowment parameters: endowment level growth mean, endowment level growth volatility, speed of convergence of the endowment ratio, endowment ratio change volatility, mean reverting endowment ratio value and correlation between the brownian motions.
- Model simulation parameters: number of years that will be simulated, size of the relevant period, number of random generated initial sets of state variables to be simulated, and maximum number of iterations.

2. Read input data and set up initial conditions: The code reads the input data from Jason files sets up the initial conditions in arrays. The initial condition includes de initial guess for the 2 price dividend ratios and the 2 wealth to consumption ratios as well as the initial sets of initial values for the consumption shares of the agents.

3. Initialize multiprocessing pool: The ProcessPoolExecutor is used to create a multiprocessing pool, which allows for parallel execution of the Simulation Utility Function across multiple initial set of initial consumption shares in step 5.

4. Generate random initial values for the state variables. The initial value of the endowment ratio is set to be always equal to 1. The consumption shares must satisfy pareto conditions and market clearing conditions.

5. Iterate through the Picard steps using the Picard Iteration Utility Function, running Monte Carlo simulations at each step to update the system's variables.

(a) If this is the first iteration, take the initial guess for the 2 price dividend ratios and 2 wealth to consumption ratios described in step 4. Else use values calculated in the previous iterations as the initial guess, stored in CSV files described in step 5.(d).

(b) Monte Carlo methods: Several random paths are ran per each initial set of consumption shares using the Simulation Utility Function. For each random path, the value of the 2 Price Dividend ratio and the 2 Wealth to Consumption ratio are calculated.

The variables that are calculated here, which are constructed as objects are:

- The aggregate endowment of the goods X_t and Y_t.

  The expressions used to calculate these variables for $t \in [\Delta t, T]$ are:

$$X_t = X_{t-1} \exp \left\{ \left(\mu^X - \frac{\sigma_X^2}{2} \right) \Delta t + \sigma_X dB_t^{(1)} \sqrt{\Delta t} \right\}$$

$$Y_t = \omega_t X_t$$

  Where $X_0 = \omega_0 = Y_0 = 1$, T is the last period of the simulation and Δt is the discrete period length.

- The state variables ω_t, x_t^A, $x_t^{A^*}$ and y_t^A

  The expressions used to calculate these variables for $t \in [\Delta t, T]$ are:

$$\omega_t = \omega_{t-1} + S \left(\omega^M - \omega_{t-1} \right) \Delta t + \sigma^\omega \sqrt{\left(\overline{\omega} - \omega_{t-1} \right) \left(\omega_{t-1} - \underline{\omega} \right)} dB_t^{(1)} \sqrt{\Delta t}$$

$$x_t^A = x_{t-1}^A + \mu_{t-1}^{x^A} \Delta t + \sigma_{t-1}^{x^A,(1)} dB_t^{(1)} \sqrt{\Delta t} + \sigma_{t-1}^{x^A,(2)} dB_t^{(2)} \sqrt{\Delta t}$$

$$x_t^{A^*} = x_{t-1}^{A^*} + \mu_{t-1}^{x^{A^*}} \Delta t + \sigma_{t-1}^{x^{A^*},(1)} dB_t^{(1)} \sqrt{\Delta t} + \sigma_{t-1}^{x^{A^*},(2)} dB_t^{(2)} \sqrt{\Delta t}$$

$$y_t^{A} = y_{t-1}^{A} + \mu_{t-1}^{y^{A}} \Delta t + \sigma_{t-1}^{y^{A},(1)} dB_t^{(1)} \sqrt{\Delta t} + \sigma_{t-1}^{y^{A},(2)} dB_t^{(2)} \sqrt{\Delta t}$$

- The stochastic discount factors H_t^X and H_t^Y

  The expressions used to calculate these variables for $t \in [\Delta t, T]$ are:

$$H_t^X = H_{t-1}^X \exp\left\{ \left(-r_{t-1}^X - \frac{(\kappa_{t-1}^{X,(1)})^2}{2} - \frac{(\kappa_{t-1}^{X,(2)})^2}{2} - \rho^{XY} \kappa_{t-1}^{X,(1)} \kappa_{t-1}^{X,(2)} \right) \Delta t - \kappa_{t-1}^{X,(1)} dB_t^{(1)} \sqrt{\Delta t} - \kappa_{t-1}^{X,(2)} dB_t^{(2)} \sqrt{\Delta t} \right\}$$

$$H_t^Y = H_t^X \frac{\left(1 - \alpha^A\right)}{\alpha^A} \frac{y_t^A Y_t}{x_t^A X_t}$$

  Where H_0^X=1

The aforementioned variables are used to calculate quarterly discrete version of the 2 price ratios and 2 wealth to consumption ratios:

$$p_t^X = \frac{1}{2\delta H_0^X X_0} \sum_{t=\Delta t}^{T} H_t^X X_t \left(e^{-2\delta \times (t-\Delta t)} - e^{-2\delta \times t} \right)$$

$$p_t^Y = \frac{1}{2\delta H_0^Y Y_0} \sum_{t=\Delta t}^{T} H_t^Y Y_t \left(e^{-2\delta \times (t-\Delta t)} - e^{-2\delta \times t} \right)$$

$$g_t^{X,A} = \frac{1}{2\pi H_0^X x_0^A X_0} \sum_{t=\Delta t}^{T} \left(H_t^X x_t^A X_t + H_t^Y y_t^A Y_t \right) \left(e^{-2\pi \times (t-\Delta t)} - e^{-2\pi \times t} \right)$$

$$g_t^{X,A^*} = \frac{1}{2\pi H_0^X x_0^{A^*} X_0} \sum_{t=\Delta t}^{T} \left(H_t^X x_t^{A^*} X_t + H_t^Y y_t^{A^*} Y_t \right) \left(e^{-2\pi \times (t-\Delta t)} - e^{-2\pi \times t} \right)$$

.

(c) The values of each one of the 4 variables is averaged out across the random simulations for a given set of initial consumption shares and this value is used as the initial guess in the next Picard iteration for that same given set of initial consumption shares.

(d) The results of the calculations for the 4 variables are saved to a CSV file.

6. Repeat step 4 until the desired level of convergence for the 2 price dividend ratios and for the 2 wealth to consumption ratios is achieved or until the maximum number of iterations is reached.

7. After convergence in step 5, the code runs one last Monte Carlo simulation per initial set of consumption shares to carry out the calculations for the output of the model, which are stored in a dictionary. The first 1,000 years of data are dropped to ensure that the model converges to its mean reverting values. Data is divided in 30 years trenches to match the empirical counterpart used in Chapter 1. In this step the code uses the Outcome Utility Functions, as well as Numpy library functions to calculate correlations between the relevant variables and standard deviations. The variables that are calculated here, which are constructed as objects are:

   - The aggregate endowment of the goods X_t and Y_t.
   - The state variables ω_t, x_t^A, $x_t^{A^*}$ and y_t^A.
   - The stochastic discount factors H_t^X and H_t^Y.
   - The pricing kernels for the countries M_t and M_t^*.
   - The real exchange rate E_t
   - Agents' consumption C_t^A, C_t^B, $C_t^{A^*}$ and $C_t^{B^*}$

- Countries' consumption C_t and C_t^*

8. Output is exported to a Jason file

This structured approach ensures that the code can efficiently solve the mathematical dynamic model and provide insights into the system's behavior under various conditions.

The execution flow for the code that generates the impulse response functions for the 4 heterogeneous agent model, matches the aforementioned execution flow with the exception of step 7. In this case, in step 7, all sources of randomness are shut down. 4 paths per initial set of consumption shares are ran for 1,000 years. After that, a different 1 time 1 standard deviation random shock is generated in each one of the 4 paths: one path is shocked with a positive endowment level shock, one path is shocked with a negative endowment level shock, one path is shocked with a positive endowment ratio shock and one path is shocked with a negative endowment ratio shock. The paths of the variables that are relevant for the description of the mechanism of the model are saved to CSV files.

The execution flow for the code that generates the 2 homogeneous agent model presents some differences with respect to the code for the 4 heterogeneous agents model. First, in step 1, the population shares for agents of type B in both countries must be equal to 0. Second, in step 4, the initial consumption shares for agents of type B must be set up to be equal to 0. Third, this model only has 2 state variables: x_t^A and ω_t, as $x_t^{A^*}$, y_t^A and $y_t^{A^*}$ can be obtained from Pareto and market clearing conditions given the aforementioned 2 state variables. So, we will only have to solve the SPDEs for p_t^X and $g_t^{X,A}$. This demands the corresponding modifications in steps 2, 4, 5 and 6.

2.5 Conclusion

This section presented the code used to provide a numerical solution for the model described in chapter 1. This structured approach efficiently solves the mathematical dynamic model and provide insights into the system's behavior under various conditions.

CHAPTER 3

Emprical Data

This chapter documents the recent years empirical disconnect between between real exchange rate changes and relative country consumption growth that was first documented in Backus and Smith (1993) for the US and 4 OECD countries. The model presented in Chapter 1 was successful in reproducing this puzzle.

3.1 Introduction

This chapter presents the recent empirical evidence that gives rise to what is commonly known in the international asset pricing literature as the Cyclicality puzzle. This puzzle was first documented by Backus and Smith (1993). The aforementioned authors found a negative correlation between real exchange rate changes and relative country consumption growth using data for eight OECD countries. The period that they span goes from the first quarter of 1971 to the last quarter of 1990.

The startling fact that gives rise to the puzzle is that under identical constant relative risk aversion preferences, relative country consumption growth and real exchange rate must be positively and perfectly correlated. This result was reproduced in chapter 1 with the 2 identical agents model. Notwithstanding, the empirical evidence shows a persistent mildly negative or weakly positive correlation between these 2 variables. The 4 heterogeneous agent model presented in Chapter 1 was able to generate a theoretical mechanism that reproduced this empirical fact.

In this chapter, I span data for the Cyclicality puzzle for the US and 4 OECD countries: UK, France, Italy and Germany for different periods between the 1st quarter of 1975 and the 1st quarter of 2020.

3.2 Data

In Table 3.1 we can observe the values for the correlation between real exchange rate changes and relative country consumption growth for the US with respect to the UK, France, Germany and Italy and for different periods ranging from the first quarter of the year 1975 to the first quarter of the year 2020.

Data for consumption inflation and exchange rate for the United States, France, Italy and Germany is retrieved from the Federal Reserve Bank of St. Louis Database. Data for consumption, GDP and inflation for the United Kingdom is retrieved from the Office of National Statistics from the United Kingdom.

All data are seasonally adjusted; any time series not initially adjusted undergoes seasonal adjustment using the U.S. Census Bureau's X12 seasonal adjustment method.

We can observe that the aforementioned stylized fact pointed out by Backus and smith (1993) can be observed for different periods of time and also for different countries.

Table 3.1: Correlation between real exchange rate changes and relative country consumption growth for the US and the UK, France, Germany and Italy

Backus-Smith Correlation				
Period	UK	France	Germany	Italy
1975 - 2020	-	-	-0.0814	-
1980 - 2020	-0.1965	-0.3349	0.0231	-
1985 - 2020	-0.2469	-0.3889	0.0876	-
1990 - 2020	-0.2456	-0.1973	0.1384	-
1995 - 2020	-0.2004	-0.2074	0.2966	-0.0747
2000 - 2020	-0.1943	-0.1928	0.1384	-0.0122
2005 - 2020	-0.2111	0.0320	0.1242	-0.0118
2010 - 2020	-0.0481	0.0643	0.1967	-0.1637

3.3 Conclusion

This chapter presented recent data that gives rise to what is commonly know in the asset pricing literature as the Cyclicality puzzle. The empirical data presented here shows the prevalence of this puzzle both across time, and across countries for the US.

Bibliography

Alfuraidan, M. and Ansari, Q. (2016). *Fixed point theory and graph theory: Foundations and integrative approaches*. Academic Press.

Alvarez, F., Atkeson, A., and Kehoe, P. J. (2009). Time-varying risk, interest rates, and exchange rates in general equilibrium. *The Review of Economic Studies*, 76(3):851–878.

Asparouhova, E., Bossaerts, P., Roy, N., and Zame, W. (2016). "lucas" in the laboratory. *The Journal of Finance*, 71(6):2727–2780.

Backus, D. K. and Smith, G. W. (1993). Consumption and real exchange rates in dynamic economies with non-traded goods. *Journal of International Economics*, 35(3-4):297–316.

Bai, Y. and Ríos-Rull, J.-V. (2015). Demand shocks and open economy puzzles. *American Economic Review*, 105(5):644–49.

Bansal, R. and Shaliastovich, I. (2013). A long-run risks explanation of predictability puzzles in bond and currency markets. *The Review of Financial Studies*, 26(1):1–33.

Bansal, R. and Yaron, A. (2004). Risks for the long run: A potential resolution of asset pricing puzzles. *The journal of Finance*, 59(4):1481–1509.

Brandt, M. W., Cochrane, J. H., and Santa-Clara, P. (2006). International risk sharing is better than you think, or exchange rates are too smooth. *Journal of monetary economics*, 53(4):671–698.

Burnside, A. C. and Graveline, J. J. (2020). On the asset market view of exchange rates. *The Review of Financial Studies*, 33(1):239–260.

Campbell, J. Y. and Cochrane, J. H. (1999). By force of habit: A consumption-based explanation of aggregate stock market behavior. *Journal of political Economy*, 107(2):205–251.

Chan, Y. L. and Kogan, L. (2002). Catching up with the joneses: Heterogeneous preferences and the dynamics of asset prices. *Journal of Political Economy*, 110(6):1255–1285.

Colacito, R. and Croce, M. M. (2011). Risks for the long run and the real exchange rate. *Journal of Political economy*, 119(1):153–181.

Colacito, R. and Croce, M. M. (2013). International asset pricing with recursive preferences. *The Journal of Finance*, 68(6):2651–2686.

Corsetti, G., Dedola, L., and Leduc, S. (2008). International risk sharing and the transmission of productivity shocks. *The Review of Economic Studies*, 75(2):443–473.

Devereux, M. B., Smith, G. W., and Yetman, J. (2012). Consumption and real exchange rates in professional forecasts. *Journal of international Economics*, 86(1):33–42.

Dumas, B. (1989). Two-person dynamic equilibrium in the capital market. *The Review of Financial Studies*, 2(2):157–188.

Epstein, L. G. and Zin, S. E. (1991). Substitution, risk aversion, and the temporal behavior of consumption and asset returns: An empirical analysis. *Journal of political Economy*, 99(2):263–286.

Fama, E. F. (1984). Forward and spot exchange rates. *Journal of monetary economics*, 14(3):319–338.

Farhi, E. and Gabaix, X. (2016). Rare disasters and exchange rates. *The Quarterly Journal of Economics*, 131(1):1–52.

Gabaix, X. and Maggiori, M. (2015). International liquidity and exchange rate dynamics. *The Quarterly Journal of Economics*, 130(3):1369–1420.

Gârleanu, N. and Panageas, S. (2015). Young, old, conservative, and bold: The implications of heterogeneity and finite lives for asset pricing. *Journal of Political Economy*, 123(3):670–685.

Guttag, J. V. (2013). *Introduction to computation and programming using Python*. Mit Press.

Hall, R. E. (2016). The role of the growth of risk-averse wealth in the decline of the safe real interest rate. Technical report, National Bureau of Economic Research.

Hansen, L. P. and Hodrick, R. J. (1980). Forward exchange rates as optimal predictors of future spot rates: An econometric analysis. *Journal of political economy*, 88(5):829–853.

Hess, G. D. and Shin, K. (2010). Understanding the backus–smith puzzle: It's the (nominal) exchange rate, stupid. *Journal of International Money and Finance*, 29(1):169–180.

Itskhoki, O. and Mukhin, D. (2021). Exchange rate disconnect in general equilibrium. *Journal of Political Economy*, 129(8):2183–2232.

Jiang, Z., Krishnamurthy, A., Lustig, H. N., and Sun, J. (2021). Beyond incomplete spanning: Convenience yields and exchange rate disconnect.

Kargar, M. (2021). Heterogeneous intermediary asset pricing. *Journal of Financial Economics*, 141(2):505–532.

Karlin, S. and Taylor, H. E. (1981). *A second course in stochastic processes*. Elsevier.

Kollmann, R. (1995). Consumption, real exchange rates and the structure of international asset markets. *Journal of International money and finance*, 14(2):191–211.

Longstaff, F. A. and Wang, J. (2012). Asset pricing and the credit market. *The Review of Financial Studies*, 25(11):3169–3215.

Lucas Jr, R. E. (1982). Interest rates and currency prices in a two-country world. *Journal of Monetary Economics*, 10(3):335–359.

Lustig, H., Stathopoulos, A., and Verdelhan, A. (2019). The term structure of currency carry trade risk premia. *American Economic Review*, 109(12):4142–77.

Lustig, H. and Verdelhan, A. (2012). Exchange rates in a stochastic discount factor framework. *Handbook of exchange rates*, pages 391–420.

Martin, I. (2011). The forward premium puzzle in a two-country world. Technical report, National Bureau of Economic Research.

Nagel, S. (2021). *Machine learning in asset pricing*, volume 1. Princeton University Press.

Obstfeld, M. and Rogoff, K. (2000). The six major puzzles in international macroeconomics: is there a common cause? *NBER macroeconomics annual*, 15:339–390.

Opazo, L. A. (2006). The backus-smith puzzle: the role of expectations. *Documentos de Trabajo (Banco Central de Chile)*, (395):1.

Panageas, S. (2020). The implications of heterogeneity and inequality for asset pricing. *Foundations and Trends in Finance*, 12(3):199–275.

Rietz, T. A. (1988). The equity risk premium a solution. *Journal of monetary Economics*, 22(1):117–131.

Santos, T. and Veronesi, P. (2010). Habit formation, the cross section of stock returns and the cash-flow risk puzzle. *Journal of Financial Economics*, 98(2):385–413.

Schneider, A. (2022). Risk-sharing and the term structure of interest rates. *The Journal of Finance*, 77(4):2331–2374.

Stathopoulos, A. (2017). Asset prices and risk sharing in open economies. *The Review of Financial Studies*, 30(2):363–415.

Verdelhan, A. (2010). A habit-based explanation of the exchange rate risk premium. *The Journal of Finance*, 65(1):123–146.

Wang, J. (1996). The term structure of interest rates in a pure exchange economy with heterogeneous investors. *Journal of Financial Economics*, 41(1):75–110.

White, R. and Downs, T. E. (2002). *How computers work*. Que.

Zivot, E. (2017). *Introduction to computational finance and financial econometrics*. Chapman & Hall Crc.