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Random Inscribed Polytopes

A dissertation submitted in partial satisfaction of the
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by

Lei Wu

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2006

The dissertation of Lei Wu is approved, and it is acceptable in quality and form for publication on microfilm:

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To My Grandfather, Wu, Jian

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ABSTRACT OF THE DISSERTATION

Random Inscribed Polytopes

by

Lei Wu
Doctor of Philosophy in Mathematics

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For convex bodies K with $\mathcal{C}^2$ boundary and everywhere positive Gauß-Kronecker curvature in $\mathbb{R}^d$, we explore random polytopes K_n with the n vertices chosen uniformly along the boundary of K. In particular, we determine the asymptotic properties of the volume of these random polytopes when n is large.

We provide results concerning the variance and higher moments of this functional. Previously, these results are considered very difficult to obtain due to the high technicalities in the existing integral methods. We will demonstrate here a different method for obtaining such estimates, namely the so-called divide-and-conquer martingale technique. We first give a concentration result for $\mathrm{Vol_d}(K_n)$ which indicates the behavior of exponential decay of the deviation of volume from its mean. This result not only implies the upper bound on the variance of $\mathrm{Vol_d}(K_n)$ previously obtained by Reitzner [54] via refinement of integral methods, it also gives us an upper bound on any k-th moments of the volume for $k \geq 2$ expressed in terms of the variance. Then we give a matching lower bound on the variance, which is tight up to a multiplicative constant factor that depends only on the fixed dimension d and the convex body K.

Lastly, we show that central limit theorem holds asymptotically for the volume functional of our inscribing model provided that the random polytopes are constructed

with vertices chosen on the boundary of K according to the Poisson Process.

1

History of Random Polytopes

We let $\mathcal{K}$ be the set of convex bodies in $\mathbb{R}^d$, i.e. compact convex sets with nonempty interior in $\mathbb{R}^d$. Assume $K \in \mathcal{K}$ and let $t_1, \ldots, t_n$ be independent random points chosen according to some distribution μ in K. Here "independent" means that the joint distribution $\mathcal{P}$ of $t_1, \ldots, t_n$ is given by the product measure $\mathcal{P}_{t_1} \otimes \ldots \otimes \mathcal{P}_{t_n}$ of the distribution $\mathcal{P}_{t_i}$ of t_i. The convex hull of the t_i's is called a *random polytope*.

Random polytope find its applications in many scientific areas, including mathematical programming [19, 42], and algorithm research [31, 53], etc. In particular, random polytope has long been widely used in computational geometry and its related fields, e.g. image processing [58] and pattern recognition [4]. Many algorithms that run on polytopes have bad worst-case running time. Hence, one is interested in running them on random polytopes in hope that the average running time is better than the worst-case scenario.

The study of random polytopes has been an active area of research which links together combinatorics, geometry and probability since the middle of the nineteenth century. It traces its root to Sylvester's famous "four-point question" [67] (also see Bárány [11] and [12] for recent results). This question asked for the probability of four random points in the plane forming a convex quadrangle. It has been generalized to many forms in the following centuries. Another milestone was established by Rényi and Sulanke in [59, 60] in 1960's, where they studied the asymptotic behavior of random polytopes, which has become a mainstream research area ever since. Out of the large number of contributions, we only mention the work of Blaschke [18], Dalla and Larman [30], Giannopoulos [35], Buchta [20], and Buchta and Reitzner [26] where the expectations of different functionals of random polytopes are dealt with in the case $d = 2$, and for $d \geq 3$, Groemer [36, 37], Kingman [46], Affentranger [1], and Buchta and Reitzner [27].

Throughout this paper, we always assume n is sufficiently large. We use the usual asymptotic notations Ω, O, Θ, o etc. with respect to $n \to \infty$. All constants are assumed to depend on at most the dimension d, the body K, and the probability measure we use. We also write $f \approx g$ when $f = (1 + o(1))g$.

One popular model for random polytopes is the following. Let $K \in \mathcal{K}$ be fixed and we choose random points $t_1, \ldots, t_n$ independently, uniformly in K. "Uniformly" here means the random points all have the same distribution $\mathcal{P}_{t_i} = \Phi_d(K, \cdot)$ where $\Phi_d(K, \cdot)$ is the Lebesgue measure restricted to K. This coincides with the d-dimensional Hausdorff measure on K as well as the "uniform" measure in the usual sense. We denote this random polytope by P_n, then $P_n = [t_1, \ldots, t_n]$ where $[S]$ stands for the convex hull of the set S.

Most of the work done in random polytopes since 1960's has been focused on the expectation of various functionals associated with P_n. These functionals are, for instance, the number of vertices, $f_0(P_n)$, or more generally, the number of i-dimensional faces, $f_i(P_n)$; the i-th intrinsic volume of P_n, $\mathrm{Vol}_i(P_n)$, in particular, the d-dimensional volume $\mathrm{Vol}_{\mathrm{d}}(P_n)$, the surface area $2\mathrm{Vol}_{d-1}(P_n)$, and the mean width, which is a multiple of $\mathrm{Vol}_1(P_n)$. In most cases, the explicit calculation of $\mathbb{E}\, g$ is complicated, where g is any functional mentioned above, even for simple convex bodies K. In fact, these calculations all follow from Rényi and Sulanke [59, 60] (and its extensions): since P_n is simplicial with probability one, each facet is of the form $[t_{i_1}, \ldots, t_{i_d}]$, we let $\mathcal{F}$ be the collection of these facets. Let $\mathbf{1}\{A\}$ (sometimes also written as $\mathbf{1}_A$) be the indicator of event A, then

$$\mathbb{E}\, g(P_n) = \binom{n}{d} \int_K \cdots \int_K \mathbf{1}\{[t_1, \ldots, t_d] \in \mathcal{F}\} g([t_1, \ldots, t_d]) dt_1 \ldots dt_d \tag{1.1}$$

This formula alone is not easy to evaluate in most instances. In the case of volume functional, Rényi and Sulanke's gave an estimate of $\mathrm{Vol_d}(K) - \mathbb{E}\,\mathrm{Vol_d}(P_n)$ in the case of $d = 2$ and smooth K. This was extended to all d-dimensional Euclidean ball by Wieacker [71]. Bárány [10] generalized this to all convex bodies with $\mathcal{C}^3$-boundary and everywhere positive curvature, which is then further extended to arbitrary convex bodies by Schütt [65]:

Theorem 1.1. *Let K be a convex body in $\mathbb{R}^d$. Then*

$$\mathbb{E}\,\mathrm{Vol_d}(P_n) = \mathrm{Vol_d}(K) - (c + o(1))n^{-\frac{2}{d+1}},$$

where the constant c here only depends on K and d.

For analogous results in the case of polytope K, we refer to, e.g. [14, 16]. Concerning expectation of key functionals, we would also like to point out a few related results including Bárány [6, 9], Buchta [21, 24], Efron [32], Groeneboom [38], Gruber [39], Müller [24, 50], Schneider [64]. We also refer readers to the excellent surveys [6], [22], [40], [63], and [70]. These surveys encompass the subjects of random polytopes and stochastic geometry and provide a complete history and comprehensive background of these exciting fields.

Given the difficulties in computing the first-order estimates of these functionals, it is perhaps not surprising that the higher-order information remained open for a long time, as coined by Weil and Wieacker's survey from the Handbook of Convex Geometry (see the concluding paragraph of [70]):

> "We finally emphasize that the results described so far give mean values hence first-order information on random sets and point processes. This is due to the geometric nature of the underlying integral geometric results. There are also some less geometric methods to obtain higher-order information or distributions, but generally the determination of variance, e.g., is a major open problem."

Already to establish limit laws seems highly nontrivial and out of reach in many cases. In the planar case, Schneider [62] proved a strong law of large numbers for $\mathrm{Vol_d}(P_n)$ if K is smooth, and Cabo and Groeneboom [28] determined the asymptotic behavior of $\mathrm{Var}\,\mathrm{Vol_d}(P_n)$ for convex polygons and proved a central limit theorem, but the stated asymptotic value of the variance appears incorrect (see [44],p.111) and a corrected version is given by Buchta [23]. Only in the special case that K is the unit ball further results are available: for all $d \geq 1$ Kuefer [47] gave an upper bound on $\mathrm{Var}\,\mathrm{Vol_d}(P_n)$. And Hsing [43] used an analogous estimate to prove a central limit theorem in the case $d = 2$.

The last few years have seen several developments in this direction, thanks to new methods and tools from modern probability. In a series of remarkable papers, Reitzner [56, 54, 55] established bounds on the variance of the volume and number of vertices in the case of smooth convex bodies K:

$$\mathrm{Var}\,\mathrm{Vol_d}(P_n) = O\left(n^{-\frac{d+3}{d+1}}\right),$$

$$\mathrm{Var}\,f_i(P_n) = O\left(n^{\frac{d-1}{d+1}}\right).$$

Relating to martingale techniques, Vu [68] proved the following tail estimate

$$\mathbb{P}\left(|\,\mathrm{Vol_d}(P_n) - \mathbb{E}\,\mathrm{Vol_d}(P_n)| \geq \sqrt{\lambda n^{-\frac{d+3}{d+1}}}\right) \leq \exp(-c\lambda)$$

for any $0 < \lambda < n^{\alpha}$, where c, c' and α are positive constants for smooth K. A similar bound also holds for f_i with the same proof. Vu [69] also confirmed that the volume of random polytope P_n with vertices chosen inside a smooth convex body K satisfies central limit theorem asymptotically, improving Reitzner [56].

The results mentioned in this section together provide a fairly comprehensive picture about P_n when we choose points randomly inside K. Another model that is of great interest, in fact, the main object of study in this dissertation, is the inscribing model of random polytopes, which we will discuss in the rest of the paper.

2

Random Inscribed Polytopes

The main goal of this dissertation is to provide a comprehensive picture for the asymptotic behavior of random inscribed polytopes.

Approximating a fixed convex body is a basic question in computational geometry that generates interest in many scientific areas. Heuristically, the advantage of choosing points only on the boundary of the convex body is that the resulting polytope approximates the convex body better than choosing points inside. Moreover, every point we choose now is forced to be a vertex, which is useful in many instances.

2.1 Introduction

Throughout this paper, we fix a smooth convex body $K \in \mathcal{K}_+^2$, where $\mathcal{K}_+^2$ is the set of compact, convex bodies in $\mathbb{R}^d$ which have non-empty interior and whose boundaries belong to differentiability class $\mathcal{C}^2$ and have everywhere positive Gauß-Kronecker curvature. (Note: the reader who is interested in the case of general K, e.g. when K is a polytope, is referred to [6, 17, 68, 69]. It is noteworthy that it is not clear how this case should be dealt with given the methods known today.) Without loss of generality, we also assume K has volume 1. As before, we will choose n random points to construct a random polytope, except that we will restrict the points to be chosen only on the boundary of K.

Before we may speak about selecting points on the boundary ∂K, we need to specify the probability measure on ∂K. One wants the random polytope to approximate the original convex body K in the sense that the symmetric difference of the volume of K and K_n is as small as possible. Hence, intuitively, a measure that puts more weight on regions of higher curvature is desired. A good discussion on this can be found in [66]. Let μ_{d-1} be the $(d-1)$-dimensional Hausdorff measure restricted to ∂K. We let μ_ρ be a probability measure on ∂K such that

$$d\mu_\rho = \rho d\mu_{d-1}, \tag{2.1}$$

where $\rho : \partial K \to \mathbb{R}_+$ is a positive, continuous function with $\int_{\partial K} \rho d\mu_{d-1} = 1$.

Note that the assumption $\rho > 0$ is essential, as otherwise we might have a measure that causes K_n to always lie in at most half (or any portion) of K with probability 1.

With the boundary measure properly defined, we can choose n random points on the boundary of K independently according to μ_ρ on ∂K. Denote the convex hull of these n points by K_n and we call it *random inscribed polytope*. For this model, the volume is perhaps the most interesting functional (as the number of vertices is always n), and it will be the focus of the present work. For notational convenience, we denote Z for $\mathrm{Vol_d}(K_n)$ throughout this paper.

2.2 Expectation of volume

The inscribing model is somewhat more difficult to analyze than the model where points are chosen inside K. Buchta, Müller and Tichy [25] and Müller [51] determined the asymptotic behavior of $\mathrm{Vol}_1(K_n)$, later generalized by Gruber [41] to all $K \in \mathcal{K}^k_+$ for all $k \geq 2$. In the special case that K is a ball and $\rho = 1$, Müller [50] determined $\mathrm{Vol}_i(K_n)$ for all $i = d-1, d$ and Affentranger [2] for all i. For the case $d = 2$, Schneider gave the rate of convergence for the expectation of $\mathrm{Vol_d}(K_n)$ in [62]. For the general case, sharp estimates of the expectation were obtained only a few years ago, thanks to the tremendous effort of Schütt and Werner, in a long (over one hundred pages) and highly technical paper [66]:

Theorem 2.1. *Let $K \in \mathcal{K}_+^2$ and $Z = \mathrm{Vol_d}(K_n)$ when K_n is a random inscribed polytope with vertices chosen on the boundary ∂K of K. Also, let $\rho : \partial K \to \mathbb{R}_+$ be a positive, continuous function with $\int_{\partial K} \rho d\mu_{d-1} = 1$. then*

$$\mathbb{E}_\rho Z = \mathrm{Vol_d}(K) - (c_K + o(1))n^{-\frac{2}{d-1}}, \tag{2.2}$$

where c_K is a constant depending on K and ρ. Moreover, the constant is minimized when the normalized affine surface area measure is used.

Here we write $\mathbb{E}_\rho$ for the expectation to emphasize the dependence on ρ. In fact, Schütt and Werner [66] proved that any probability measure μ_ρ defined as in (2.1) yields the same asymptotics in the estimate of expectation up to a constant factor. Hence, in the rest of the paper, we will simply write $\mathbb{E}$ and μ instead of $\mathbb{E}_\rho$ and μ_ρ. If we normalize the volume of K, then we have

$$\mathbb{E}\, Z = 1 - (c_K + o(1))n^{-\frac{2}{d-1}}.$$

Remark 2.2. It is worth recalling from Theorem 1.1 that in the model where points are chosen uniformly inside K we have $\mathbb{E}\,\mathrm{Vol_d}(K \backslash P_n) = O\left(n^{-\frac{2}{d+1}}\right)$. Observe that by inserting $n^{\frac{d+1}{d-1}}$ for n in this result we obtain a function $O\left(n^{-\frac{2}{d-1}}\right)$, which is the correct growth rate found in (2.2). We can explain this (at least intuitively) by noting that in the uniform model, the expected number of vertices is $\mathbb{E}\, f_0(P_n) = \Theta\left(n^{\frac{d-1}{d+1}}\right)$. However, in the inscribing model all points are vertices. Thus we may view the uniform model on n points as yielding the same type of behavior as the inscribing model on $n^{\frac{d-1}{d+1}}$ points. Further evidence for this behavior is given by Reitzner in [57] where he obtains estimates of expectation (which are sharp up to a constant factor) for all intrinsic volumes.

2.3 Results

Reitzner [54] gave an upper bound on the variance of the d-dimensional volume:

$$\operatorname{Var} Z = O\left(n^{-\frac{d+3}{d-1}}\right).$$

The first result we show in this paper is that the variance estimate is sharp, up to a constant factor.

Theorem 2.3 (Variance)**.** *Given $K \in \mathcal{K}^2_+$,*

$$\operatorname{Var} Z = \Omega\left(n^{-\frac{d+3}{d-1}}\right),$$

where the implicit constant depends on dimension d and the convex body K only.

The argument for this theorem is similar to the approach of that of Reitzner [56] which analyzes the volume change near the boundary induced by the change of position of a random point in a small region.

On the other hand, Reitzner obtained the upper bound on variance via Efron-Stein jackknife inequality (see [34]) which implies

$$\operatorname{Var} g(K_n) \le (n+1)\,\mathbb{E}\left(g(K_n) - g(K_{n+1})\right)^2,$$

for any functional g of the random polytope. This suggests that deviation can be estimated through a "one-point-at-a-time" approach. This process reminds us of the martingale technique which has gained much attention in recent years due to heavy use of probabilistic methods in combinatorics. In fact, we use this technique to obtain the following concentration result, which can be considered an alternative to integral methods for obtaining higher moment estimates of functionals of random polytopes. We show that the deviation of volume from its mean has exponential tail.

Theorem 2.4 (Concentration)**.** *For a given convex body $K \in \mathcal{K}^2_+$, there are constants α and c such that the following holds. For any constant $0 < \eta < \frac{d-1}{3d+1}$ and $0 < \lambda \le \frac{\alpha}{4} n^{\frac{d-1}{3d+1} + \frac{2(d+1)\eta}{d-1}} < \frac{\alpha}{4} n$, we have*

$$\mathbb{P}\left(|Z - \mathbb{E}\, Z| \ge \sqrt{\lambda V_0}\right) \le 2\exp(-\lambda/4) + \exp(-cn^{\frac{d-1}{3d+1} - \eta}), \tag{2.3}$$

where $V_0 = \alpha n^{-\frac{d+3}{d-1}}$.

From this tail estimate, one can not only deduce the upper bound on variance bound given by Reitzner but also obtain bounds for any fixed moments:

Corollary 2.5 (Moments)**.** *For any given convex body* K *and* $k \geq 2$*, the* k*-th moments of* Z *satisfies*

$$M_k = O\left(\left(n^{-\frac{d+3}{d-1}}\right)^{k/2}\right).$$

To emphasize the dependence of $Z = \mathrm{Vol_d}\, K_n$ on n, we write Z_n instead of Z in the following result. Equipped with the concentration result, we not only confirm limit law of the form

$$\lim_{n\to\infty} \frac{Z_n}{\mathbb{E}\, Z_n} = 1,$$

but also determine the rate of convergence:

Corollary 2.6 (Rate of Convergence)**.** *There is a constant* α *such that the following holds.*

$$\lim_{n\to\infty} \left| \left(\frac{Z_n}{\mathbb{E}\, Z_n} - 1 \right) f(n) \right| = 0$$

almost surely, for

$$f(n) = \delta(n) \left(n^{-\frac{d+3}{d-1}} \ln n \right)^{-1/2}$$

where $\delta(n)$ *is a function tending to zero arbitrarily slowly as* $n \to \infty$.

Since we determine the upper bounds on all moments, the next natural question to ask is what the asymptotic distribution of the functionals are. The concentration result suggests distribution of exponential tail, hence we ask whether the tail is the same as normal tail asymptotically. This is another topic that has gained significant development in recent years: the central limit theorem. We have the following conjecture:

Conjecture. (Central Limit Theorem Conjecture) *Fix a* $K \in \mathcal{K}^2_+$ *with volume one. Let* K_n *be the random polytope determined by* n *random points chosen on the* *boundary of K. Further, we write $Z = \mathrm{Vol_d}(K_n)$. Then there is a function $\epsilon(n)$ tending to zero with n such that for every x*

$$\left|\mathbb{P}\left(\frac{Z - \mathbb{E}\, Z}{\sqrt{\mathrm{Var}\, Z}} \leq x\right) - \Phi(x)\right| \leq \epsilon(n),$$

where Φ denotes the distribution function of the standard normal distribution.

Although we were unable to prove the complete result, we obtain the central limit theorem for the so-called Poisson model as follows. We let $\mathrm{Pois}(n)$ be a Poisson point process with intensity n. Then the intersection of $\mathrm{Pois}(n)$ and ∂K consists of random points $\{t_1, \ldots, t_N\}$ where the number of points N is Poisson distributed with mean $n\mu(\partial K) = n$. We write $\Pi_n = [t_1, \ldots, t_N]$. We show that the distribution of $\mathrm{Vol_d}(\Pi_n)$ converges to normal distribution asymptotically.

Theorem 2.7. *Given $K \in \mathcal{K}^2_+$, we have*

$$\left|\mathbb{P}\left(\frac{\mathrm{Vol_d}(\Pi_n) - \mathbb{E}\,\mathrm{Vol_d}(\Pi_n)}{\sqrt{\mathrm{Var}\,\mathrm{Vol_d}(\Pi_n)}} \leq x\right) - \Phi(x)\right| = o(1),$$

where the $o(1)$ term is of order $O\left(n^{-\frac{1}{4}} \ln^{\frac{d+2}{d-1}} n\right)$ as $n \to \infty$.

We hope this result will infer central limit theorem for K_n. This is the case for P_n, random polytope where the points are chosen inside K, as confirmed by Vu [69]. However, for random inscribed polytopes, some difficulties remain. Our computations show that the two models are very close in the sense that the expectations of volume for the two models are asymptotically equivalent, and the variances are only off by constant multiplicative factor (see Theorem 6.1).

3

Boundary Structure

3.1 Notations

Before we go on further with our discussion, it is necessary to introduce the notations in this paper.

The vectors $e_1, \ldots, e_d$ always represent a fixed orthonormal basis of $\mathbb{R}^d$. The discussions in this paper, unless otherwise specified, are all based on this basis. For a vector x, we denote its coordinate by $x^1, \ldots, x^d$, i.e. $x = (x^1, \ldots, x^d)$. By $B^i(x, r)$ we indicate the i-dimensional Euclidean closed ball of radius r centered at x, i.e.

$$B^i(x, r) = \left\{ y \in \mathbb{R}^i \mid ||x - y|| = r \right\}.$$

The norm $||\cdot||$ is the Euclidean norm. When the dimension is d, we sometimes simply write $B(x, r)$.

For points $t_1, \ldots, t_n \in \mathbb{R}^d$, the convex hull of them is defined by

$$[t_1, \ldots, t_n] = \{\lambda_1 t_1 + \cdots + \lambda_n t_n | 0 \leq \lambda_i \leq 1, 1 \leq i \leq n, \sum_{i=1}^{n} \lambda_i = 1\}.$$

In particular, the closed line segment between two points x and y is

$$[x, y] = \{\lambda x + (1 - \lambda) y | 0 \leq \lambda \leq 1\}.$$

To analyze the geometry, it is necessary to introduce the following. For any $y \in \mathbb{R}^d$ write $y = (y^1, \ldots, y^d)$ for the coordinates with respect to some fixed basis $e_1, \ldots, e_d$. For unit vector $u \in \mathbb{R}^d$, let $H(u,h) = \{x \in \mathbb{R}^d \mid \langle x, u \rangle = h\}$, where here $\langle , \rangle$ denotes the standard inner product on $\mathbb{R}^d$. Further, the halfspace associated to this hyperplane we denote by $H^+(u,h) = \{x \in \mathbb{R}^d \mid \langle x, u \rangle \geq h\}$. Since K is smooth, for each point $y \in \partial K$, there is some unique outward normal u_y. We thus may define the *cap* $C = C(y,h)$ of K to be $H^+(u_y, h_K(y) - h) \cap K$, where $h_K(y)$ is the support function such that $H^+(u_y, h_K(y))$ intersects K in the point y only. In general, one should think of a cap as $K \cap H^+$ where H^+ is some closed half space. Throughout this paper, we also use the notion of ϵ-*cap* to emphasize that $\mathrm{Vol_d}(C) = \mathrm{Vol_d}(K \cap H^+) = \epsilon$. Similarly, we call $C = K \cap H^+$ an ϵ-*boundary* cap to emphasize that $\mu(\partial K \cap H^+) = \epsilon$.

We define the ϵ-*wet part* of K to be the union of all caps that are ϵ-boundary caps of K and we denote it by F_ϵ^c. The complement of the ϵ-wet part in K is said to be the ϵ-*floating body* of K, which we denote by F_ϵ. This notion comes from the mental picture that when K is a three dimensional convex body containing ϵ units of water, the floating body is the part that floats above water (see [16] and [52]). Finally, consider the floating body F_ϵ and a point $x \in F_\epsilon^c$. We say that x *sees* y if the chord $[x,y]$ does not intersect F_ϵ. Set $S_{x,\epsilon}$ to be the set of those y seen by x. We then define

$$g(\epsilon) = \sup_{x \in F_\epsilon^c} \mathrm{Vol_d}(S_{x,\epsilon}).$$

In particular, we note that $S_{x,\epsilon}$ is the union of all ϵ-boundary caps containing x.

Since K is smooth, it is well known that $g(\epsilon) = \Theta(\mathrm{Vol_d}(\epsilon - \text{boundary cap}))$ (see [16]).

3.2 Cap-covering

In 1963, Rényi and Sulanke [59, 60] found that even in the planar case, surprisingly, the expectation of the number of vertices, $f_0(P_n)$, of random polytopes with points chosen inside K depends heavily on the boundary structure of K. It is of order $\ln n$ when K is a convex polygon, and of order $n^{1/3}$ when K is a circle (or more generally, any smooth enough bodies). Extensions of Rényi and Sulanke revealed that for smooth bodies, the vertices of P_n are distributed evenly near boundary of K, while for polygons, they are concentrated near the vertices of the original polygon. This somewhat explains the different behavior of $\mathbb{E}\, f_0(P_n)$. Bárány and Larman [16] (also see [6]) proved that the same kind of extreme behaviors hold for the volume functionals for these two extreme classes of convex bodies, namely smooth convex bodies and polytopes. They showed this through two steps; first, they give an estimate of the expectation through floating body:

Theorem 3.1. *For any convex body $K \in \mathbb{R}^d$ with volume one, there are constants c_1 and c_2, such that*

$$c_1 \operatorname{Vol_d}((F^I_{1/n})^c) \leq 1 - \mathbb{E} \operatorname{Vol_d}(P_n) \leq c_2 \operatorname{Vol_d}((F^I_{1/n})^c).$$

where the $F^I_{1/n}$ is the floating body constructed through $(1/n)$-caps, instead of $(1/n)$-boundary caps according to our definitions in the previous section. But we note that these are essentially the same construction except that the related parameter $1/n$ stand for different volumes (see Lemma 3.12 for their relations).

Then they showed

Theorem 3.2. *Let K be a polytope. There is a constant c_3 such that*

$$\operatorname{Vol_d}((F^I_{1/n})^c) \approx c_3 \frac{1}{n} (\ln n)^{d-1}.$$

And,

Theorem 3.3. *Let K be a convex body in $\mathcal{K}^2_+$. There is a constant c_4 such that*

$$\operatorname{Vol_d}((F^I_{1/n})^c) \approx c_4 n^{-2/(d+1)}.$$

Moreover, they discovered that for a convex body K in between these two cases, the expectation of volume fluctuates with n between the two extreme estimates:

Theorem 3.4. *Assume $\omega(n) \to 0$ and $\Omega(n) \to \infty$. Then for most (in the Baire category sense) convex bodies with volume one, we have, for infinitely many n,*

$$1 - \mathbb{E}\, P_n \geq \omega(n) n^{-\frac{2}{d+1}},$$

and for infinitely many n,

$$1 - \mathbb{E}\, P_n \leq \Omega(n) \frac{(\ln n)^{d-1}}{n}.$$

The key idea in Bárány and Larman [16] is that one can cover the boundary of K with a series of caps which capture all actions of random polytopes approximating a convex body, this is known as the Cap-covering Lemma. The proof of this lemma is given in slightly different forms in [33], [6], [16], [56], and [68]. For the case of smooth convex body, the main idea of the proof is that its boundary locally looks like that of a ball (after certain affine transformation) in the sense that one can find paraboloids which approximate ∂K uniformly for all $x \in \partial K$ (see Section 3.3 for details). This enables one to find a minimal covering of the boundary with balls that are "uniform" in size. These results are very well known, hence we do not provide complete proofs here. The formulation we present mostly follow Reitzner [56], [57] and Vu [68].

In the following, we assume ϵ is sufficiently small whenever necessary.

Lemma 3.5. *Given $K \in \mathcal{K}^2_+$, there exist constants d_1, d_2 such that for each cap $C(x, h)$ with $h \leq h_0$, we have*

$$\partial K \cap B(x, d_1 h^{\frac{1}{2}}) \subset C(x, h) \subset B(x, d_2 h^{\frac{1}{2}}).$$

Lemma 3.6. *Given $K \in \mathcal{K}^2_+$, there exists constants d_3 and d_4 such that for each cap $C(x, h)$ with $h \leq h_0$, we have*

$$d_3 h^{\frac{d+1}{2}} \leq \mathrm{Vol_d}(C(x, h)) \leq d_4 h^{\frac{d+1}{2}}.$$

That is, for sufficiently small ϵ, an ϵ-cap has height $\frac{1}{d_4}\epsilon^{\frac{2}{d+1}} \leq h \leq \frac{1}{d_3}\epsilon^{\frac{2}{d+1}}$.

Lemma 3.7 (Cap Covering). *Let m be sufficiently large, i.e. $m \geq m_0$ for some constant m_0. Given $K \in \mathcal{K}_+^2$, there are points $y_1, \dots, y_m \in \partial K$, and caps $C_i = C(y_i, h_m)$ and $\overline{C}_i = C(y_i, (2d_2/d_1)^2 h_m)$ with*

$$C_i \subset B(y_i, d_2 h_m^{1/2}) \subset \mathrm{Vor}(y_i)$$

$$\mathrm{Vor}(y_i) \cap \partial K \subset B(y_i, 2d_2 h_m^{1/2}) \cap \partial K \subset \overline{C}_i,$$

and

$$h_m = \Theta(m^{-\frac{2}{d-1}}).$$

Here $\mathrm{Vor}(y_i)$ *is the Voronoi cell of y_i in K defined by:*

$$\mathrm{Vor}(y_i) = \{x \in K : \| x - y_i \| \leq \| x - y_k \| \;\; \textit{for all } k \neq i\},$$

and we have

$$\mathrm{Vol_d}(C_i) = \Theta(m^{-\frac{d+1}{d-1}}),$$

for all $i = 1, \dots, m$.

Proof. The proof follows from the fact that given $m > m_0$, for a suitable r_m, we can find balls $B(y_i, r_m), i = 1, \dots, m$ such that they form a maximal packing of ∂K, hence $B(y_i, 2r_m)$ form a covering of ∂K. By Lemma 3.5, we can find C_i's such that $C_i \subseteq B(y_i, r_m)$, and $\overline{C}_i$'s such that $\partial K \cap B(y_i, 2r_m) \subseteq \overline{C}_i$. Also note that $\sum_m \kappa_{d-1} r_m^{d-1}$ is approximately the surface area of K where κ_{d-1} is the $(d-1)$-volume of the $(d-1)$-dimensional unit ball. Obviously $B(y_i, r_m) \subseteq \mathrm{Vor}(y_i)$ and $\mathrm{Vor}(y_i) \cap \partial K \subseteq \partial K \cap B(y_i, 2r_m)$. Thus, use Lemma 3.5, one can convert among the parameter m, the height of cap h_m and radius of the ball r_m to obtain the estimates above. □

Corollary 3.8. *Given K, there are constants d_6, d_7, d_8, d_9 and a system of pairwise disjoint $d_6\epsilon$-caps $C_i, i = 1, \dots, m$ with $m \leq d_7 \epsilon^{-\frac{d-1}{d+1}}$, such that any ϵ-cap contains at least one of the C_i's. Also, there are $d_8\epsilon$-caps $C_i', i = 1, \dots, m'$ with $m' \leq d_9 \epsilon^{-\frac{d-1}{d+1}}$ such that any cap with volume at most ϵ is contained in at least one of the C_i'.*

Lemma 3.9. *Let K, m be given, and $y_i, i = 1, \ldots, m$ be chosen as in Lemma 3.7. The number of Voronoi cells* $\mathrm{Vor}(y_j)$ *intersecting a cap* $C(y_i, h)$ *is* $O((h^{\frac{1}{2}} m^{\frac{1}{d-1}} + 1)^{d+1})$, $i = 1, \ldots, m$.

Lemma 3.10. *Let m, K and $y_i, C_i, i = 1, \ldots, m$ be chosen as in Lemma 3.7. Choose on the boundary within each cap C_i an arbitrary point x_i (i.e. $x_i \in C_i \cap \partial K$), then*

$$\delta^H(K, [x_1, \ldots, x_m]) = O(m^{-\frac{2}{d-1}}),$$

where $\delta(K, K')$ stands for the Hausdorff distance between convex bodies K and K'. And there is a constant c such that for any $y \in \partial K$ with $y \notin C(y_i, cm^{-\frac{2}{d-1}})$, the line segment $[y, x_i]$ intersects the interior of the convex hull $[x_1, \ldots, x_m]$.

In what follows, we always assume the parameters are sufficiently small or large whenever needed for the lemmas to hold. This can be achieved without exceptions.

3.3 Boundary approximation

In order to carry out detailed calculation of the variance of volume, we need to know what the boundary of K is like. In fact, for $K \in \mathcal{K}_+^2$, at each point $x \in \partial K$ there is a unique paraboloid Q_x, given by a quadratic form b_x, osculating ∂K at x. We may describe Q_x and b_x by identifying the tangent hyperplane of ∂K at x with $\mathbb{R}^{d-1}$ and x with the origin. This is a well known fact, see e.g.[56]. In a neighborhood of x, we can represent ∂K as the graph of a $\mathcal{C}^2$, convex function $f : \mathbb{R}^{d-1} \to \mathbb{R}$, i.e. each point in ∂K near x can be written in the form $(y, f_x(y))$, where $y \in R^{d-1}$ the form $(y^1, \ldots, y^{d-1})$. Thus, we may write

$$b_x(y) = \frac{1}{2} \sum_{1 \le i,j \le d-1} \frac{\partial f_x}{\partial y^i \partial y^j}(0) y^i y^j,$$

and

$$Q_x = \left\{ (y, z) \mid z \ge b_x(y), y \in \mathbb{R}^{d-1}, z \in \mathbb{R} \right\},$$

here $\frac{\partial f_x}{\partial y^i \partial y^j}(0)$ denote the second partial derivative of f_x at the origin with respect to y^i and y^j. The main thrust of the above is that these paraboloids approximate the boundary structure. The formulation given here is due to Reitzner, who provides a proof in [57].

Lemma 3.11. *Let $K \in \mathcal{K}_+^2$ and choose $\delta > 0$ sufficiently small. Then there exists a $\lambda > 0$, depending only on δ and K, such that for each point $x \in \partial K$ the following holds: If we identify the tangent hyperplane to ∂K at x with $\mathbb{R}^{d-1}$ and x with the origin, then we may define the $\lambda-$neighborhood U^λ of $x \in \partial K$ by* $\operatorname{proj} U^\lambda = B^{d-1}(0, \lambda)$. *$U^\lambda$ can be represented by a convex function $f_x(y) \in \mathcal{C}^2$, for $y \in B^{d-1}(0, \lambda)$. Furthermore,*

$$(1+\delta)^{-1} b_x(y) \le f_x(y) \le (1+\delta) b_x(y), \tag{3.1}$$

and

$$\sqrt{1 + |\nabla f_x(y)|^2} \le (1+\delta). \tag{3.2}$$

for $y \in B^{d-1}(0, \lambda)$, where b_x is defined as above and $\nabla f_x(y)$ stands for the gradient of $f_x(y)$.

This lemma proves that at each point $x \in \partial K$, the deviation of the boundary of the approximating paraboloid ∂Q_x from ∂K is uniformly bounded in a small neighborhood of x.

We use this lemma to show how one can relate ϵ-caps to ϵ-boundary caps. This relationship is used repeatedly throughout the paper as it allows us to work with volumes of different dimensions.

Lemma 3.12. *For a given $K \in \mathcal{K}_+^2$, there exists constants $\epsilon_0, c, c' > 0$ such that for all $0 < \epsilon < \epsilon_0$ we have that for any ϵ-cap C of K,*

$$c^{-1} \epsilon^{(d-1)/(d+1)} \le \mu(C \cap \partial K) \le c\epsilon^{(d-1)/(d+1)}$$

and for any ϵ-boundary cap C' of K,

$$c'^{-1} \epsilon^{(d+1)/(d-1)} \le \mathrm{Vol_d}(C') \le c' \epsilon^{(d+1)/(d-1)}.$$

Proof. We shall prove the first statement. Fix some $\delta > 0$ for lemma 3.11.

Consider in $\mathbb{R}^d$ the paraboloid given by the equation

$$z^d \geq (z^1)^2 + (z^2)^2 + \ldots + (z^{d-1})^2.$$

Intersecting this paraboloid with the halfspace defined by the equation $z^d \leq 1$ gives an object which we shall call the standard cap, E. We form $(1+\delta)^{-1}E$ and $(1+\delta)E$ similarly by the equations $z^d \geq (1+\delta)^{-1}((z^1)^2 + (z^2)^2 + \ldots, (z^{d-1})^2)$ and $z^d \geq (1+\delta)((z^1)^2 + (z^2)^2 + \ldots, (z^{d-1})^2)$, using the same halfspace as before. We note the inclusions

$$(1+\delta)^{-1}E \supset E \supset (1+\delta)E.$$

Let $c_1 = \mathrm{Vol_d}((1+\delta)^{-1}E)$ and $c_2 = \mathrm{Vol_d}((1+\delta)E)$, and further set $c_3 = \mu(\mathrm{proj}((1+\delta)^{-1}E))$ and $c_4 = \mu(\mathrm{proj}((1+\delta)E))$ where here proj is the orthogonal projection onto the hyperplane spanned by the first $(d-1)$ coordinates.

Now, let C be our ϵ-cap. Let x be the unique point in ∂K whose tangent hyperplane is parallel to the hyperplane defining C. Assuming that Lemma 3.11 applies, we may equate the tangent hyperplane of ∂K at x with $\mathbb{R}^{d-1}$, and view $C \cap \partial K$ as being given by some convex function $f : \mathbb{R}^{d-1} \to \mathbb{R}$. Further, let Q_x be the unique paraboloid osculating ∂K at x. Let A be a linear transform that takes E to Q_x. We observe that Q_x is the paraboloid defined by the set $z^d \geq b_x(z^1, \ldots, z^{d-1})$ intersected with the halfspace $z^d \leq h$, for some $h > 0$. We can define $(1+\delta)^{-1}Q_x$ (resp. $(1+\delta)Q_x$) to be the set defined by the intersection of this same half space and the points given by $z^d \geq (1+\delta)^{-1}b_x(z^1, \ldots, z^{d-1})$ (resp. $z^d \geq (1+\delta)b_x(z^1, \ldots, z^{d-1})$). Observe that $A((1+\delta)^{-1}E) = (1+\delta)^{-1}Q_x$ and $A((1+\delta)E) = (1+\delta)Q_x$.

Appealing to Lemma 3.11, we see that

$$(1+\delta)^{-1}Q_x \supset C \supset (1+\delta)Q_x.$$

This gives

$$c_1|\det A| \geq \epsilon \geq c_2|\det A|. \tag{3.3}$$

Let $\tilde{f} : \mathbb{R}^{d-1} \to \partial K$ be the function induced by f, i.e. $\tilde{f}(y) = (y, f_x(y))$.

Using the inclusion

$$\tilde{f}(\mathrm{proj}((1+\delta)^{-1}Q_x)) \supset C \cap \partial K \supset \tilde{f}(\mathrm{proj}((1+\delta)Q_x))$$

and the bound

$$(1+\delta) \geq \sqrt{1+|\nabla f|^2} \geq 1$$

furnished by Lemma 3.11, if A' represents the restriction of A to the first $(d-1)$ coordinates, we obtain

$$c_3|\det A'|(1+\delta) \geq \mu(C \cap \partial K) \geq c_4|\det A'|. \tag{3.4}$$

A simple computation shows $|\det A| = 2^{(d-1)/2}\kappa^{-1/2}h^{(d+1)/2}$ and $|\det A'| = 2^{(d-1)/2}$ $\kappa^{-1/2}h^{(d-1)/2}$, where κ is the Gauß-Kronecker curvature of ∂K at x. Using this and (3.3) gives upper and lower bounds on h, and this bound with (3.4) gives

$$c_5\epsilon^{(d-1)/(d+1)} \geq \mu(C \cap \partial K) \geq c_6\epsilon^{(d-1)/(d+1)},$$

where here c_5, c_6 are constants depending only on κ. As K is compact and κ is always positive we can assume we can change c_5 and c_6 to be independent of κ, and hence x.

Finally, we return to the issue of values of ϵ (hence h) for which Lemma 3.11 applies. We note that in general every quadratic form b_x can be given by

$$b_x(y) = \frac{1}{2}\sum_i k_i(y^i)^2,$$

where k_i are the principal curvatures. We observe that as the Gauß-Kronecker curvature is positive then there are positive constants k' and k'' depending only on K such that $0 < k' < k_i < k''$. This bounds the possible geometry of Q_x, and implies the existence of an ϵ_0 such that for $0 < \epsilon < \epsilon_0$, such that $\mathrm{proj}((1+\delta)^{-1}Q_x) \subset B(0,\lambda)$ (λ as given in Lemma 3.11), allowing us to apply Lemma 3.11. This completes the proof of the first statement. The second statement is similar. Relaxing constants allows the statement as given.

$\square$

Remark 3.13. It is important to note that the above is *not* true for general convex bodies. In particular, any polytope P provides an example of a convex body with caps C such that the quantities $\mathrm{Vol_d}(C)$ and $\mu(C \cap \partial P)$ are unrelated.

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4

Variance

In this section, we provide a proof of Theorem 2.3. It follows an argument first used by Reitzner in [56], which has also been utilized by Bárány and Reitzner [8] to prove a lower bound of the variance in the case where the convex body is a polytope. Essentially, we condition on arrangements of our polytope where vertices can be perturbed in such a way that the resulting change in volume is independent for each vertex in question.

Choosing the vertices along the boundary according to a given distribution, as opposed to uniformly in the body adds technical complication and requires greater use of the boundary structure. The key to the study is the boundary approximation mentioned in the previous section.

4.1 Small local perturbations

We begin by establishing some notation. Define the standard paraboloid E to be

$$E = \left\{ z \in \mathbb{R}^d \mid z^d \geq (z^1)^2 + \ldots + (z^{d-1})^2 \right\}.$$

We similarly define $2E = \left\{ z \in \mathbb{R}^d \mid z^d \geq \frac{1}{2}((z^1)^2 + \ldots + (z^{d-1})^2) \right\}$ and observe that we have the inclusion

$$E \subset 2E.$$

We now choose a simplex S in the cap $C(0,1)$ of E. Choose the base of the simplex to be a regular simplex with vertices in $\partial E \cap H(e_d, h_d)$ and the origin (h_d to be determined later). We shall denote by $v_0, v_1, \ldots, v_d$ the vertices of this simplex, singling out v_0 to be the apex of S (i.e. the origin). The important point here is that for sufficiently small h_d, the cone $\left\{\lambda x \in \mathbb{R}^d \mid \lambda \geq 0, x \in S\right\}$ contains $2E \cap H(e_d, 1)$. Indeed, as the radius of $E \cap H(e_d, h_d)$ is $\sqrt{h_d}$, the inradius of base of the simplex is $\sqrt{h_d/d^2}$, hence for $h_d < 1/2d^2$ our above inclusion holds.

Now, look at the orthogonal projection of the vertices of the simplex to the plane spanned by $\{e_1, \ldots, e_{d-1}\}$, which we think of as $\mathbb{R}^{d-1}$ and denote the relevant operator as

$$\mathrm{proj} : \mathbb{R}^d \to \mathbb{R}^{d-1}.$$

Around the origin we center a ball B_0 of radius r, and around each projected point (except the origin) we can center a ball in $\mathbb{R}^{d-1}$ of radius r', both to be chosen later. We label these balls $B_1, \ldots, B_d$, where B_i is the ball about $\mathrm{proj}(v_i)$. We can form the corresponding sets B_i' to be the inverse image of these sets on ∂E under the projection operator. In other words, if $b : \mathbb{R}^{d-1} \to \mathbb{R}$ is the quadratic form whose graph defines E, $\tilde{b} : \mathbb{R}^{d-1} \to \partial E$ the map induced by b, then

$$B_i' = \tilde{b}(B_i), \qquad i = 0, \ldots, d.$$

We note that if we choose r sufficiently small, then for any choice of random points $Y \in B_0'$ and $x_i \in B_i', i = 1, \ldots, d$ the cone on these points is close to the cone on the simplex in the sense that

$$\{\lambda x \mid x \in [Y, x_1, \ldots, x_d], \lambda \geq 0\} \supset 2E \cap H(e_d, 1).$$

We may also think of Y being chosen randomly, according to the distribution induced from the $(d-1)$-dimensional Hausdorff measure on E, say. Then, passing to a smaller r if necessary, we see that for any choice of $x_i \in B_i', i = 1, \ldots, d$, we have

$$\mathrm{Var}_Y(\mathrm{Vol_d}([Y, x_1, \ldots, x_d])) \geq c_0 > 0.$$

All the above follows from continuity. We hope results of this type to be true for arbitrary caps of ∂K, and indeed our current construction will serve both model and computational tool for similar constructions on arbitrary caps.

We now consider the general paraboloid

$$Q = \left\{ z \in \mathbb{R}^d \mid z^d \geq \frac{1}{2}(k_1(z^1)^2 + \ldots, +k_{d-1}(z^{d-1})^2) \right\},$$

where here $k_i > 0$ for all i and let the curvature be $\kappa = \prod k_i$. We now transform the cap $C(0,1)$ of E to the cap $C(0,h)$ of Q by the (unique) linear map A which preserves the coordinate axis. Let D_i be the image of B_i under this affinity. We find that the volume of the D_i scales to give

$$\mu(D_i) = c_1 h^{\frac{d-1}{2}}, \qquad i = 1, \ldots, d, \tag{4.1}$$

where here c_1 is some positive constant only depending on the curvature $\kappa = \prod k_i$ and our choice of r and r'.

Next, for each point $x \in \partial K$ we identify our general paraboloid Q with the approximating paraboloid Q_x of K at x (in particular, we identify $\mathbb{R}^{d-1}$ with the tangent hyperplane at x and the origin with x). We thus write $D_i(x)$ to indicate the set $D_i, i = 1, \ldots, d$, corresponding to Q_x. Analogously to the construction of the $\{B_i'\}$ we can construct the $\{D_i'(x)\}$ as follows. Let $f^x : \mathbb{R}^{d-1} \to \mathbb{R}$ be the function whose graph locally defines ∂K at x (this exists for h sufficiently small, see Lemma 3.11), $\tilde{f} : \mathbb{R}^{d-1} \to \partial K$ the induced function. Let

$$D_i'(x) = \tilde{f}(D_i(x)).$$

We note here that in general the sets $D_i'(x)$ are *not* the images of B_i' under A as $A(B_i')$ may not lie on the boundary ∂K in general.

Because the curvature is bounded above and below by positive constants, as is ρ, we see that the volume of $D_i(x)$ is given by

$$c_3 h^{\frac{d-1}{2}} \leq \mu(D_i(x)) \leq c_4 h^{\frac{d-1}{2}}, \tag{4.2}$$

where c_3, c_4 are constants depending only on K.

We now wish to get bounds for $\mathrm{Var}_Y(\mathrm{Vol_d}([Y, x_1, \ldots, x_d]))$ where $x_i \in D_i'(x), i = 1, \ldots, d$ and we choose Y randomly in $D_0'(x)$ according to the distribution on the boundary. To begin with, we'll need the following technical lemma.

Lemma 4.1. *There exists a $r_0 > 0$ and r_0' such that for all $r_0 > r > 0$ and $r_0' > r' > 0$ we have an $h_r > 0$ such that for any choice of $x_i \in D_i'(x), i = 1, \ldots, d$, and $h_r > h > 0$:*

$$c_5 h^{d+1} \leq \mathrm{Var}_Y([Y, x_1, \ldots, x_d]) \leq c_6 h^{d+1}, \tag{4.3}$$

where c_5, c_6 are positive constants depending only on K and r.

The proof of this lemma is given in Section 4.3. Assuming this lemma is true, we proceed with our analysis as follows.

Fix some choice for $h_d < 1/2d^2$. Let $v_0, \ldots, v_d$ denote the vertices of the simplex S. Then by continuity we know that there is some $\eta > 0$ such that choosing x_i in η-balls $B(v_i, \eta)$ centered at the vertices preserves our desired inclusion, namely

$$\{\lambda x \mid x \in [x_0, x_1, \ldots, x_d], \lambda \geq 0\} \supset 2E \cap H(e_d, 1). \tag{4.4}$$

We now desire to set $r' > 0$ such that $D_i'(x) \subset A(B(v_i, \eta))$ for all $x \in \partial K$. As a consequence, we will obtain the inclusion, for $x_i \in D_i'(x)$,

$$\{\lambda x \mid x \in [x_0, x_1, \ldots, x_d], \lambda \geq 0\} \supset 2Q_x \cap H(u_x, h) \supset K \cap H(u_x, h).$$

Choose $\epsilon > 0$ such that

$$U_i = \\ \left\{(x, y) \in \mathbb{R}^d \mid x \in B(\operatorname{proj} v_i, \eta/2) \subset \mathbb{R}^{d-1} \text{ and } (1+\epsilon)^{-1} b_E(x) \leq y \leq (1+\epsilon) b_E(x)\right\} \\ \subset B(v_i, \eta) \tag{4.5}$$

for each i, where b_E is the quadratic form defining our standard paraboloid E. Appealing to Lemma 3.11 we take h sufficiently small such that for all $x \in \partial K$,

$$(1+\epsilon)^{-1} b_x(y) \leq f_x(y) \leq (1+\epsilon) b_x(y).$$

Choosing $r' < \eta/2$ forces the B_i to be balls of radius r' about $\operatorname{proj} v_i$, which by the above causes $D_i'(x) \subset A(U_i) \subset A(B(v_i, \eta))$.

With these choices for r, r' and some constant $h_0 > 0$ to enforce the condition that h is sufficiently small above, we now proceed to the body of our argument.

4.2 Proof of lower bound on variance

Choose n points $t_1, \ldots, t_n$ randomly in ∂K according to the probability induced by the distribution. Choose n points $y_1, \ldots, y_n \in \partial K$ and corresponding disjoint caps according to Lemma 3.7. In each cap $C(y_j, h_n)$ (of K) establish sets $\{D_i(y_j)\}$ and $\{D_i'(y_j)\}$ for $i = 0, \ldots, d$ and $j = 1, \ldots, n$ as in the above discussion.

We let $A_j, j = 1, \ldots, n$ be the event that exactly one random point is contained in each of the $D_i(y_j), i = 0, \ldots, d$ and every other point is outside $C(y_j, h_n) \cap \partial K$. We calculate the probability as

$$\begin{aligned} P(A_j) = n(n-1)\cdots(n-d)\,&\mathbb{P}(t_i \in D_i'(y_j), i = 0, \ldots, d) \\ &\cdot \mathbb{P}(t_i \notin C(y_j, h_n) \cap \partial K, i \geq d+1) \\ &= n(n-1)\cdots(n-d)\prod_{i=0}^{d} \mu(D_i'(y_j)) \prod_{k=d+1}^{n} (1 - \mu(C(y_j, h_n) \cap \partial K)) \end{aligned}$$

We can give a lower bound for this quantity with (4.2) and Lemma 3.7 , and noting specifically that $h_n = \Theta(n^{-2/(d-1)})$:

$$\mathbb{P}(A_j) \geq c_7 n^{d+1} n^{-d-1} (1 - c_8 n^{-1})^{n-d-1} \geq c_9 > 0, \tag{4.6}$$

where c_7, c_8, c_9 are positive constants. In particular, denoting by $\mathbf{1}_A$ the indicator function of event A. We obtain that

$$\mathbb{E}(\sum_{j=1}^{n} \mathbf{1}_{A_j}) = \sum_{j=1}^{n} \mathbb{P}(A_j) \geq c_9 n. \tag{4.7}$$

Now we denote by $\mathcal{F}$ the position of all points of $\{t_1, \ldots, t_n\}$ except those which are contained in $D_0'(y_j)$ with $\mathbf{1}_{A_j} = 1$. We then use the conditional variance formula to obtain a lower bound:

$$\begin{aligned}\operatorname{Var} Z &= \mathbb{E}\operatorname{Var}(Z|\mathcal{F}) + \operatorname{Var}\mathbb{E}(Z|\mathcal{F})\\ &\geq \mathbb{E}\operatorname{Var}(Z|\mathcal{F}).\end{aligned}$$

Now we look at the case where $\mathbf{1}_{A_j}$ and $\mathbf{1}_{A_k}$ are both 1 for some $j, k \in \{1, \ldots, n\}$. Assume without loss of generality that t_j and t_k are the points in $D_0'(y_j)$ and $D_0'(y_k)$, respectively. We note that by construction there can be no edge between t_j and t_k, so the volume change affected by moving t_j within $D_0'(y_j)$ is independent of the volume change of moving t_k within $D_0'(y_k)$. This independence allows us to write the conditional variance as the sum

$$\operatorname{Var}(Z|\mathcal{F}) = \sum_{j=1}^{n} \operatorname{Var}_{t_j}(Z)\mathbf{1}_{A_j},$$

where here each variance is taken over $t_j \in D_0'(y_j)$. We now invoke Lemma 4.1, equation (4.7), and the bound $h_n \approx n^{-2/(d-1)}$ to compute

$$\begin{aligned}\mathbb{E}\operatorname{Var}(Z|\mathcal{F}) &= \mathbb{E}\left(\sum_{j=1}^{n} \operatorname{Var}_{t_j}(Z)\mathbf{1}_{A_j}\right)\\ &\geq c_5 h^{d+1}\,\mathbb{E}\left(\sum_{j=1}^{n}\mathbf{1}_{A_j}\right)\\ &\geq c_{10}(n^{-2/(d-1)})^{d+1} c_6 n\\ &= c_{11} n^{-(d+3)/(d-1)}.\end{aligned}$$

Thus, the above provides the promised lower bound on $\operatorname{Var} Z$.

4.3 Proof of Lemma 4.1

We first prove the following claim. The notation follows that found in Section 4.1

Claim 4.2. *Let $x \in \partial K$. There is some $h(K) > 0$ such that for $h(K) > h > 0$ there exists a constant $c(r) > 0$ depending only on r and K such that*

$$\frac{1}{2}|\det A|^2 c(r) \leq \operatorname{Var}_Y(\mathrm{Vol_d}([Y, Av_1, \ldots, Av_d])) \leq 2|\det A|^2 c(r),$$

and Y is a random point chosen in $D_0'(x)$ according to the distribution on ∂K.

Proof of Claim. To prove this claim, we compute. Recall that A is the linear map which takes E to the paraboloid Q_x. We shall denote by A' the map A restricted to $\mathbb{R}^{d-1}$. We shall denote by $f : T_x(\partial K) \approx \mathbb{R}^{d-1} \to \mathbb{R}$ the function whose graph defines ∂K locally, and $\tilde{f} : \mathbb{R}^{d-1} \to \partial K$ the function induced by f.Thus, we have:

$$\begin{aligned}
&\mathbb{E}_Y(\mathrm{Vol_d}([Y, Av_1, \dots, Av_d])) \\
&= \frac{\int_{D_0} \mathrm{Vol_d}([\tilde{f}(Y), Av_1, \dots, Av_d])\rho(\tilde{f}(Y))\sqrt{1 + f^2_{Y^1} + \dots f^2_{Y^{d-1}}}dY}{\int_{A'(C_0)} \rho(f'(Y))\sqrt{1 + f^2_{Y^1} + \dots f^2_{Y^{d-1}}}dY} \\
&= \frac{|\det A'| \int_{C_0} \mathrm{Vol_d}([\tilde{f}(AX), Av_1, \dots, Av_d])\rho(\tilde{f}(AX))\sqrt{1 + f^2_{Y^1} + \dots f^2_{Y^{d-1}}}(AX)dX}{|\det A'| \int_{C_0} \rho(f'(AX))\sqrt{1 + f^2_{Y^1} + \dots f^2_{Y^{d-1}}}(AX)dY}.
\end{aligned} \tag{4.8}$$

Observe that if we set $A^{-1} \circ \tilde{f}(AX) = f^*$ to be the pullback of $\tilde{f}$ under A then $\mathrm{Vol_d}([\tilde{f}(AX), Av_1, \dots, Av_d]) = |\det A| \cdot \mathrm{Vol_d}([f^*(X), v_1, \dots, v_d])$. Letting $b : \mathbb{R}^{d-1} \to \mathbb{R}$ denote the quadratic form defining E, $\tilde{b} : \mathbb{R}^{d-1} \to \partial E$ the induced function, we then use Lemma 3.11 to get the bound

$$2^{-1}b \leq f \circ A' \leq 2b, \tag{4.9}$$

when h is sufficiently small. Thus, we get the bound

$$\mathrm{Vol_d}([2^{-1}b(X), v_1, \dots, v_d]) \geq \mathrm{Vol_d}([f^*(X), v_1, \dots, v_d]) \geq \mathrm{Vol_d}([2b(X), v_1, \dots, v_d]),$$

which follows from the geometry. Now, since $v_1, \dots, v_d$ form a $(d-1)$ simplex parallel to the plane $\mathbb{R}^{d-1}$ we can write $\mathrm{Vol_d}([b(X), v_1, \dots, v_d]) = c_d(1 - b(X))$, where c_d is some positive constant depending only on dimension. We may write $b(X) = |X|^2$, and this allows us to see that

$$\begin{aligned}
&\mathrm{Vol_d}([2^{-1}\tilde{b}(X), v_1, \ldots, v_d]) \\
&= \mathrm{Vol_d}([\tilde{b}(X), v_1, \ldots, v_d])(1 - 2^{-1}|X|^2)/(1 - |X|^2) \\
&= \mathrm{Vol_d}([\tilde{b}(X), v_1, \ldots, v_d])(1 - 2^{-1}|X|^2)(1 + |X|^2 + |X|^4 + \ldots) \\
&= \mathrm{Vol_d}([\tilde{b}(X), v_1, \ldots, v_d])(1 + o_r(1)),
\end{aligned}$$

Here, $o_r(1)$ indicates a function which goes to 0 as r goes to 0. Similarly, we have

$$\mathrm{Vol_d}([2\tilde{b}(X), v_1, \ldots, v_d]) = \mathrm{Vol_d}([\tilde{b}(X), v_1, \ldots, v_d])(1 + o_r(1)).$$

Thus, we may write

$$\begin{aligned}
&\frac{\int_{C_0} \mathrm{Vol_d}([f^*(X), v_1, \ldots, v_d])\rho(\tilde{f}(AX))\sqrt{1 + f_{Y^1}^2 + \ldots f_{Y^{d-1}}^2}(AX)dX}{\int_{C_0} \rho(\tilde{f}(AX))\sqrt{1 + f_{Y^1}^2 + \ldots f_{Y^{d-1}}^2}(AX)dY} \\
&\geq (1 + o_r(1)) \cdot \frac{\int_{C_0} \mathrm{Vol_d}([\tilde{b}(X), v_1, \ldots, v_d])\rho(\tilde{f}(AX))\sqrt{1 + f_{Y^1}^2 + \ldots f_{Y^{d-1}}^2}(AX)dX}{\int_{C_0} \rho(\tilde{f}(AX))\sqrt{1 + f_{Y^1}^2 + \ldots f_{Y^{d-1}}^2}(AX)dY}.
\end{aligned} \tag{4.10}$$

Setting $F(X) = \rho(\tilde{f}(AX))\sqrt{1 + f_{Y^1}^2 + \ldots f_{Y^{d-1}}^2}(AX)$ the above is thus

$$\geq (1 + o_r(1)) \cdot \frac{\min_{C_0} F(X)}{\max_{C_0} F(X)} \cdot \frac{\int_{C_0} \mathrm{Vol_d}([b(X), v_1, \ldots, v_d])dX}{\int_{C_0} dX}.$$

Now, if we can show that the term $\frac{\min_{C_0} F(X)}{\max_{C_0} F(X)} \geq (1 + o_{r,h}(1))$, only depending on r and h, then from our earlier observation we can conclude that (4.8) is bounded below by

$$|\det A| \cdot (1 + o_{r,h}(1)) \cdot \frac{\int_{C_0} \mathrm{Vol_d}([b(X), v_1, \ldots, v_d])dX}{\int_{C_0} dX}.$$

Note $o_{r,h}(1)$ denotes a function which goes to 0 as both r and h go to 0.

Invoking Lemma 3.11, we observe that we may make the term

$$\sqrt{1 + f_{Y^1}^2 + \ldots f_{Y^{d-1}}^2}(AX)$$

sufficiently less than $(1+\delta)$, for any $\delta > 0$, by choosing r, h both sufficiently small (independent of f). Thus, we may write $\sqrt{1+f_{Y^1}^2+\dots f_{Y^{d-1}}^2}(AX) = (1+o_{r,h}(1))$.

Next, we note that ρ is a uniformly continuous function on K. It is not too hard to see that the function $\min_{C_0}\rho(f'(AX))/\max_{C_0}\rho(f'(AX)) = (1+o_{r,h}(1))$, where again the $o(1)$ function is independent of the basepoint. Using the fact that $\min\rho(f'(AX))\sqrt{1+f_{Y^1}^2+\dots f_{Y^{d-1}}^2}(AX) \geq (\min\rho(f'(AX)))\cdot \left(\min\sqrt{1+f_{Y^1}^2+\dots f_{Y^{d-1}}^2}(AX)\right)$ (similarly for max) we thus find that

$$(1+o_{r,h}(1)) \geq \frac{\min_{C_0} F(X)}{\max_{C_0} F(X)} \geq (1+o_{r,h}(1)),$$

where the functions in question are independent of basepoint.

If we let $\phi_1(r) = \frac{\int_{C_0}\mathrm{Vol_d}([\tilde{b}(X),v_1,\dots,v_d])dX}{\int_{C_0}dX}$ then we can summarize our findings as, independent of basepoint,

$$\lim_{h\to 0}\frac{\mathbb{E}_Y(\mathrm{Vol_d}([Y,Av_1,\dots,Av_d]))}{|\det A|\phi_1(r)} = (1+o_r(1)). \tag{4.11}$$

By an identical argument, if we set $\phi_2(r) = \frac{\int_{C_0}\mathrm{Vol_d}^2([\tilde{b}(X),v_1,\dots,v_d])dX}{\int_{C_0}dX}$ then we have

$$\lim_{h\to 0}\frac{\mathbb{E}_Y(\mathrm{Vol_d}^2([Y,Av_1,\dots,Av_d]))}{|\det A|^2\phi_2(r)} = (1+o_r(1)). \tag{4.12}$$

Using (4.11) and (4.12) we can compute:

$$\begin{aligned}\lim_{h\to 0}\mathrm{Var}_Y([Y,Av_1,\dots,Av_d])/|\det A|^2 &= \lim_{h\to 0}\mathbb{E}_Y(\mathrm{Vol_d}^2([Y,Av_1,\dots,Av_d]))/|\det A|^2\\ &\quad -\lim_{h\to 0}\mathbb{E}_Y^2(\mathrm{Vol_d}([Y,Av_1,\dots,Av_d]))/|\det A|^2\\ &= \phi_2(r)(1+o_r(1)) - \phi_1^2(r)(1+o_r(1))^2\\ &= (\phi_2(r)-\phi_1^2(r))(1+o_r(1)). \end{aligned} \tag{4.13}$$

Thus, by letting r become sufficiently small so that the final $(1+o_r(1)) > 0$ we note that (4.13) is positive, since this quantity $\phi_2(r)-\phi_1^2(r)$ is just the variance of $\mathrm{Vol_d}([b(X),v_1,\dots,v_d])$ where X is taken over C_0, thus always positive. This proves there exists $c_1 > 0$ such that for h sufficiently small,

$$\mathrm{Var}_Y([Y,Av_1,\dots,Av_d]) \geq c_1|\det A|^2.$$

By the same arguments we also get

$$\mathrm{Var}_Y([Y, Av_1, \ldots, Av_d]) \leq c_2 |\det A|^2.$$

So the claim is proved. $\square$

With the preceding claim, we now prove Lemma 4.1. Instead of the convex hull of $[Y, Av_1, \ldots, Av_d]$ we shall study the convex hull $[Y, x_1, \ldots, x_d]$, where $x_i \in D_i'$, using the fact that the x_i are close to the Av_i when h is small. To do this, we'll need a second claim.

Claim 4.3. *There exists a $\delta > 0$ such that If for each i, $x_i \in B(v_i, d)$, then*

$$\mathrm{Vol_d}([2^{-1}b(X), x_1, \ldots, x_d]) = \mathrm{Vol_d}([b(X), x_1, \ldots, x_d])(1 + o_r(1))$$

and

$$\mathrm{Vol_d}([2b(X), x_1, \ldots, x_d]) = \mathrm{Vol_d}([b(X), x_1, \ldots, x_d])(1 + o_r(1)),$$

where the hidden functions depend only on r (i.e. they are not functions of the x_i).

Proof. We simply note that there exists a $\delta > 0$ such that for any fixed choice of x_i,

$$\frac{\mathrm{Vol_d}([2^{-1}b(X), x_1, \ldots, x_d])}{\mathrm{Vol_d}([b(X), x_1, \ldots, x_d])} \to 1 \text{ as } X \to 0.$$

We also note that $X, x_1, \ldots, x_d$ lie in $C_0 \times B(v_1, \delta) \times \cdots \times B(v_d, \delta)$, a compact set. These two conditions guarantee that the maximum of the ratio, taken over all $x_1, \ldots, x_d$, converges to 1 as $X \to 0$. Thus, the ratio converges to 1 independently of the choice of $x_1, \ldots, x_d$, and hence the claimed result.

The statement for $\mathrm{Vol_d}([2b(X), x_1, \ldots, x_d])$ is analogous. $\square$

With this claim, we can adapt claim 4.2 to work for any $x_i \in B(v_i, \delta)$, by using the above claim in place of (4.10). With this we can show that for h sufficiently small we can choose r sufficiently small such that

$$\begin{aligned}\frac{1}{2}|\det A|^2 \operatorname{Var}_X(\mathrm{Vol_d}([b(X), x_1, \ldots, x_d])) &\leq \operatorname{Var}_Y(\mathrm{Vol_d}([Y, Ax_1, \ldots, Ax_d])) \\ &\leq 2|\det A|^2 \operatorname{Var}_X(\mathrm{Vol_d}([b(X), x_1, \ldots, x_d])), \quad (4.14)\end{aligned}$$

where here the quantity $\operatorname{Var}_X(\mathrm{Vol_d}([b(X), x_1, \ldots, x_d]))$ is the variance taken over C_0. But as $\operatorname{Var}_X(\mathrm{Vol_d}([b(X), v_1, \ldots, v_d]))$ is positive, continuity guarantees that

$$c' > \operatorname{Var}_X(\mathrm{Vol_d}([b(X), x_1, \ldots, x_d])) > c > 0$$

if the x_i are sufficiently close to the v_i, say $x_i \in B(v_i, \eta)$ for all i, for some $\eta > 0$. Then,

$$\frac{1}{2}|\det A|^2 c' \leq \operatorname{Var}_Y(\mathrm{Vol_d}([Y, Ax_1, \ldots, Ax_d])) \leq 2|\det A|^2 c, \qquad (4.15)$$

if $x_i \in B(v_i, \eta)$ for all i.

Now, we need to verify that we can choose C_i sufficiently small such that points in D_i' always map into $B(v_i, \eta)$, which will complete the lemma. To do this, note that if we set $r' < \eta/2$, then we can choose $\epsilon > 0$ such that

$$\begin{aligned}U_i = \\ &\left\{(x, y) \in \mathbb{R}^d \mid x \in B(\operatorname{proj} v_i, \eta/2) \subset \mathbb{R}^{d-1} \text{ and } (1+\epsilon)^{-1} b(x) \leq y \leq (1+\epsilon) b(x)\right\} \\ &\subset B(v_i, \eta) \quad (4.16)\end{aligned}$$

for each i. By Lemma 3.11 we can take h to be sufficiently small such that for all $x \in \partial K$

$$(1+\epsilon)^{-1} b_x(y) \leq f_x(y) \leq (1+\epsilon) b_x(y)$$

in all caps of height h. So if we thus choose C_i to be the $\eta/2$ ball about $\operatorname{proj} v_i$, then we note that $D_i' \subset A(U_i)$. Thus, any $y_i \in D_i'$ can be written as Ax_i for some $x_i \in U_i \subset B(v_i, \eta)$, and thus (4.15) holds. Hence, the lemma.

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5

Concentration

Our concentration result shows that $\mathrm{Vol_d}(K_n)$ is highly concentrated about its mean. Namely, we obtain a bound of the form

$$\mathbb{P}(|Z - \mathbb{E}\, Z| \geq \sqrt{\lambda \operatorname{Var} Z}) \leq c_1 \exp(-c_2 \lambda) \tag{5.1}$$

for positive constants c_1, c_2. Such an inequality indicates that Z has an exponential tail, which proves sufficient to provide information about the higher moments of Z and the rate of convergence of Z to its mean.

5.1 Discrete geometry

We now set up some basic geometry which will be the subject of our analysis. Let L be a finite collection of points. For a point $x \in K$, define

$$\Delta_{x,L} = \mathrm{Vol_d}([L \cup x]) - \mathrm{Vol_d}([L]).$$

A key property is the following observation.

Lemma 5.1. *Let L be a set whose convex hull contains the floating body F_ϵ. Then for any x,*

$$\Delta_{x,L} \leq g(\epsilon).$$

The major geometry result which allows for our analysis is the following lemma quantifying the fact that K_n contains the floating body F_ϵ with high probability.

Lemma 5.2. *There are positive constants* c *and* c' *such that the following holds for every sufficiently large* n*. For any* $\epsilon \geq c' \ln n/n$*, the probability that* K_n *does not contain* F_ϵ *is at most* $\exp(-c\epsilon n)$*.*

Bárány and Dala [15] proved a similar lemma to this in which the floating body is defined slightly differently, namely, as the union of all ϵ-caps instead of ϵ-boundary caps. They also assume that the distribution on K is uniform. Vu [68] used a surprisingly different methods from discrete geometry, namely VC-dimension, to generalize their result to any distribution on K. We will give a similar proof here to our lemma. First, we need some definitions:

Definition 5.3. Let X be a set and $\mathcal{F}$ be a family of subset of X. For a subset $A \subset X$, the restriction of $\mathcal{F}$ on A is

$$\mathcal{F}|_A = \{S \cap A | S \in \mathcal{F}\}.$$

We call a subset A is shattered by $\mathcal{F}$ if each subset of A can be obtained as the intersection of some $S \in \mathcal{F}$ with A, i.e. if

$$2^A = \mathcal{F}|_A.$$

We define the VC-dimension of $\mathcal{F}$ to be

$$\dim_{\mathrm{VC}}(\mathcal{F}) = \sup_{\substack{A \subseteq X \\ \text{A is shattered by } \mathcal{F}}} \{|A|\}$$

It is easy to observe the following fact:

Lemma 5.4. *Let* X *and* $\mathcal{F}$ *as in Definition 5.3, if* $X' \subseteq X$ *and* $\mathcal{F}' = \mathcal{F}|_{X'}$*, then* $\dim_{\mathrm{VC}}(\mathcal{F}') \leq \dim_{\mathrm{VC}}(\mathcal{F})$*.*

Proof. For any $A \subseteq X' \subseteq X$ such that A is shattered by $\mathcal{F}'$ and $|A| = \dim_{\mathrm{VC}}(\mathcal{F}')$, we have for any $B \subseteq A$, there exists some $S' \in \mathcal{F}'$ such that $B = S' \cap A$. That is, there exists $S \in \mathcal{F}$ such that $S' = S \cap X'$. But $B = S' \cap A = S \cap A$ (otherwise, if there is $x \in S \cap A$ such that $x \notin B$, then $x \in S \backslash S'$, i.e. $x \notin X'$, a contradiction). So A is shattered by $\mathcal{F}$. $\square$

The following fact is well known in discrete geometry (see e.g. Lemma 10.3.1. in [49]):

Lemma 5.5. *The VC-dimension of the system of all (closed) half-spaces in* $\mathbb{R}^d$ *is* $d + 1$.

Immediately by Lemma 5.4, we have

Corollary 5.6. *The VC-dimension of the family of all (closed) half-spaces restricted to* ∂K *in* $\mathbb{R}^d$ *is at most* $d + 1$.

Definition 5.7. If X is equipped with a probability measure μ and let $\mathcal{F}$ be a family of measurable subsets of X, then we call a subset $N \subseteq X$ an ϵ-net of $\mathcal{F}$ if N intersects all "big" subsets of X in $\mathcal{F}$. Precisely, this means

$$N \cap S \neq \emptyset$$

for any $S \in \mathcal{F}$ with $\mu(S) \geq \epsilon$.

Now by a famous theorem of Haussler and Welzl [45]:

Theorem 5.8. *There is a constant* c'' *such that the following holds. If* X *and* $\mathcal{F}$ *are defined as in Definition 5.7 and* $\dim_{\mathrm{VC}}(\mathcal{F}) = d$*, then* $\mathcal{F}$ *has an* ϵ*-net of size* $c'' d \epsilon^{-1} \ln \frac{1}{\epsilon}$.

We refer readers to [45] or p.239-241 in [50] for a complete treatment of this theorem and the following corollary. The proof of this theorem uses a probabilistic argument. One shows that with positive probability, a random set of size $c'' d \epsilon^{-1} \ln \frac{1}{\epsilon}$ intersect all elements $S \in \mathcal{F}$ with $\mu(S) \geq \epsilon$. In fact, if one examines the proof closely, one finds that this probability is fairly large:

Corollary 5.9. *There is a constant c'' and α such that the following holds. If X is a set with a probability measure mu, $\mathcal{F}$ is a family of measurable subsets of X, and $\dim_{\mathrm{VC}}(\mathcal{F}) = d$, then the probability that a random set of size $c''d\epsilon^{-1}\ln\frac{1}{\epsilon}$ fails to hit all $S \in \mathcal{F}$ such that $\mu(S) \geq \epsilon$ is at most*

$$\alpha^d(\epsilon^{c''/4}\ln\frac{1}{\epsilon})^d.$$

Now we are ready to prove Lemma 5.2:

Proof. Let $N = c''(d+1)\epsilon^{-1}\ln\frac{1}{\epsilon}$ and $l = n/N$. Choose c' such that $\epsilon \geq c'\ln n/n$, then $l \geq 1$. Without loss of generality, assume l is an integer. We will sample n random points on ∂K in l rounds. In each round, we sample N points, and the probability that these points fail to hit all the ϵ-boundary cap is at most

$$\alpha^{(d+1)}(\epsilon^{c''/4}\ln\frac{1}{\epsilon})^{(d+1)} \leq \epsilon^{\beta} = \exp(-\beta\ln\frac{1}{\epsilon}),$$

for some positive constant β, since c'', d are constants and ϵ is sufficiently small. Hence in l rounds, the probability that the n points fail to hit all ϵ-boundary caps is at most

$$\exp(-l\beta\ln\frac{1}{\epsilon}) = \exp(-c\frac{n}{\epsilon^{-1}\ln\frac{1}{\epsilon}}\ln\frac{1}{\epsilon}) = \exp(-c\epsilon n),$$

for some constant c depending on d, c', c'' and α but not on ϵ or n. Note that repetition of points in the sample will only increase the probability in concern, so the above statement is valid. □

5.2 A slightly weaker result

The proof of Theorem 2.4 is rather technical. So we will first attempt a simpler one of a slightly weaker result, which represents one of the main methodology used in this paper.

Put $G_0 = 3g(\epsilon)$ and $V_0 = 36ng(\epsilon)^2$, where $g(\epsilon)$ is as defined in the previous section. We show:

Theorem 5.10. *For a given $K \in \mathcal{K}_+^2$ there are positive constants α, c, and ϵ_0 such that the following holds: for any $\alpha \ln n/n < \epsilon \leq \epsilon_0$ and $0 < \lambda \leq V_0/4G_0^2$, we have*

$$\mathbb{P}(|Z - \mathbb{E}\,Z| \geq \sqrt{\lambda V_0}) \leq 2\exp(-\lambda/4) + \exp(-c\epsilon n).$$

We note that the constants used in the definition of G_0 and V_0 are chosen for convenience and can be optimized, though we make no effort to do so.

To compare Theorem 5.10 with Theorem 2.4, we first compute V_0. $\Theta(\epsilon^{(d+1)/(d-1)})$, by Lemma 3.12. So, setting $\epsilon = \alpha \ln n/n$ for some positive constant c greater than α from our theorem gives

$$\begin{aligned} V_0 &= 36ng(\epsilon)^2 \\ &= 36n\Theta(\epsilon^{(d+1)/(d-1)})^2 \\ &= \Theta(nn^{-2(d+1)/(d-1)}(\ln n)^{(d+1)/(d-1)}) \\ &= \Theta(n^{-(d+3)/(d-1)}(\ln n)^{2(d+1)/(d-1)}). \end{aligned} \tag{5.2}$$

So, up to a logarithmic factor V_0 is comparable to $\operatorname{Var} Z$.

To obtain Theorem 2.4 we utilize a martingale inequality (Lemma 5.11). This inequality, which is a generalization of an earlier result of Kim and Vu [48], appears to be a new and powerful tool in the study of random polytopes. It was first used by Vu in [68], and seems to provide a very general framework for the study of key functionals. The reader who is familiar with other martingale inequalities, most notably that of Azuma [5], will be familiar with the general technique (see also [3]).

Letting $t_i, i = 1, \ldots, n$ be independent random points in ∂K, the sample space be $\Omega = \{t | t = (t_1, \ldots, t_n), t_i \in \partial K\}$, and $Z = Z(t_1, \ldots, t_n) = \mathrm{Vol_d}(K_n)$ a function of these points, we may define the *(absolute) martingale difference sequence*

$$G_i(t) = \left|\mathbb{E}(Z \mid t_1, \ldots, t_{i-1}, t_i) - \mathbb{E}(Z \mid t_1, \ldots, t_{i-1})\right|.$$

Thus, $G_i(t)$ is a function of $t = (t_1, \ldots, t_n)$ that only depends on the first i points.

We then set

$$V_i(t) = \int G_i^2(t) \partial t_i$$
$$V(t) = \sum_{i=1}^{n} V_i(t)$$
$$G_i'(t) = \sup_{t_i} G_i(t)$$

and

$$G(t) = \max_i G_i'(t).$$

Note also that $|Z - \mathbb{E}\, Z| \leq \sum_i G_i$. The key to our proof is the following concentration lemma, which was derived using the so-called divide-and-conquer martingale technique (see [68]).

Lemma 5.11. *For any positive λ, G_0 and V_0 satisfying $\lambda \leq V_0/4G_0^2$, we have*

$$\mathbb{P}(|Z - \mathbb{E}\, Z| \geq \sqrt{\lambda V_0}) \leq 2\exp(-\lambda/4) + \mathbb{P}(V(t) \geq V_0 \text{ or } G(t) \geq G_0). \tag{5.3}$$

The proof of this lemma can be found in [68].

Comparing Lemma 5.11 to Theorem 5.10 we find that the technical difficulty comes in bounding the term $\mathbb{P}(V(t) \geq V_0$ or $G(t) \geq G_0)$, which corresponds to the error term p_{NT}.

Set $V' = n^{-1}V_0 = 36g(\epsilon)^2$. We find that we can replace $\exp(-c\epsilon n)$ with $n\exp(-c'\epsilon n)$ by adjusting the relevant constant c' so that $n\exp(-c'\epsilon n) < \exp(-c\epsilon n)$. Thus, we're going to prove that

$$\mathbb{P}(G(t) \geq G_0 \text{ or } V(t) \geq V_0) \leq n\exp(-c\epsilon n)$$

for some positive constant c.

To do this, we'll prove the following claim.

Claim 5.12. *There is a positive constant c such that for any $1 \leq i \leq n$,*

$$\mathbb{P}(G_i'(t) \geq G_0 \text{ or } V_i(t) \geq V') \leq \exp(-c\epsilon n).$$

From this claim the trivial union bound gives

$$\mathbb{P}(G(t) \geq G_0 \textit{ or } V(t) \geq V_0) \leq n \exp(-c\epsilon n),$$

hence quoting Lemma 5.11 finishes our proof of Theorem 5.10.

5.3 Proof of Claim 5.12

Proof. Recall that $Z = Z(t_1, \ldots, t_n) = \mathrm{Vol}_{\mathrm{d}}(K_n)$ for points $t_i \in \partial K$.

The triangle inequality gives us

$$\begin{aligned} G_i(t) &= |\,\mathbb{E}(Z|t_1, \ldots, t_{i-1}, t_i) - \mathbb{E}(Z|t_1, \ldots, t_{i-1})| \\ &\leq \mathbb{E}_x\,|\,\mathbb{E}(Z|t_1, \ldots, t_{i-1}, t_i) - \mathbb{E}(Z|t_1, \ldots, t_{i-1}, x)|, \end{aligned}$$

where $\mathbb{E}_x$ denotes the expectation over a random point x. The analysis for the two terms in the last inequality is similar, so we will estimate the first one. Let us fix (arbitrarily) $t_1, \ldots, t_{i-1}$. Let L be the union of $\{t_1, \ldots, t_{i-1}\}$ and the random set of points $\{t_{i+1}, \ldots, t_n\}$. Since

$$\mathrm{Vol}_{\mathrm{d}}([L \cup t_i]) = \mathrm{Vol}_{\mathrm{d}}([L]) + \Delta_{t_i,L},$$

we have

$$\mathbb{E}(Z|t_1, \ldots, t_{i-1}, t_i) = \mathbb{E}(\mathrm{Vol}_{\mathrm{d}}([L])|t_1, \ldots, t_{i-1}) + \mathbb{E}(\Delta_{t_i,L}|t_1, \ldots, t_{i-1}).$$

The key inequality of the analysis is the following:

$$\mathbb{E}(\Delta_{t_i,L}|t_1, \ldots, t_{i-1}) \leq \mathbb{P}(F_\epsilon \not\subseteq [L]|t_1, \ldots, t_{i-1}) + g(\epsilon). \tag{5.4}$$

The inequality (5.4) follows from two observations:

- If $F_\epsilon \not\subseteq [L]$, $\Delta_{t_i,L}$ is at most 1.
- If $[L]$ contains F_ϵ, $\Delta_{t_i,L} \leq g(\epsilon)$ by the definition of $g(\epsilon)$.

We denote by $\Omega_{(j)}$ and $\Omega^{<j>}$ the product spaces spanned by $\{t_1, \ldots, t_j\}$ and $\{t_j, \ldots, t_n\}$, respectively.

Set $\delta = n^{-4}$. We say that the set $\{t_1, \ldots, t_{i-1}\}$ is *typical* if

$$\mathbb{P}_{\Omega^{<i+1>}}(F_\epsilon \subseteq [L] | t_1, \ldots, t_{i-1}) \geq 1 - \delta.$$

The rest of the proof has two steps. In the first step, we show that if $\{t_1, \ldots, t_{i-1}\}$ is typical then $G_i'(t) \leq G_0$ and $V_i(t) \leq V'$. In the second step, we bound the probability that $\{t_1, \ldots, t_{i-1}\}$ is not typical.

First step. Assume that $\{t_1, \ldots, t_{i-1}\}$ is typical, so $\mathbb{P}_{\Omega^{<i+1>}}(F_\epsilon \not\subseteq [L] | t_1, \ldots, t_{i-1}) \leq \delta = n^{-4}$. We first bound $G_i'(t)$. Observe that

$$\begin{aligned}
G_i(t) &\leq \mathbb{E}_x \left| \mathbb{E}(Z|t_1, \ldots, t_{i-1}, t_i) - \mathbb{E}(Z|t_1, \ldots, t_{i-1}, x) \right| \\
&\leq \mathbb{E}_x \left| \mathbb{E}(\Delta_{t_i,L}|t_1, \ldots, t_{i-1}) - \mathbb{E}(\Delta_{x,L}|t_1, \ldots, t_{i-1}) \right| \\
&\leq \mathbb{E}(\Delta_{t_i,L}|t_1, \ldots, t_{i-1}) + \mathbb{E}_x \mathbb{E}(\Delta_{x,L}|t_1, \ldots, t_{i-1}) \\
\text{(by (5.4))} \quad &\leq 2g(\epsilon) + 2n^{-4} \leq 3g(\epsilon) = G_0
\end{aligned}$$

In the last inequality we use the fact that $\epsilon = \Omega(\ln n/n), g(\epsilon) = \Omega(\epsilon^{(d+1)/(d-1)}) \gg n^{-4}$. Thus it follows that

$$G_i'(t) = \max_{t_i} G_i(t) \leq G_0.$$

Calculating $V_i(t)$ using the above bound on $G_i(t)$ it follows that

$$\begin{aligned}
V_i(t) &= \int G_i(t)^2 d\mu(t_i) \\
&\leq \int 9g(\epsilon)^2 d\mu(t_i) \\
&= 9g(\epsilon)^2 < V'.
\end{aligned}$$

Second step. In this step, we bound the probability that $\{t_1, \ldots, t_{i-1}\}$ is not typical. First of all, we will need a technical lemma as follows. Let Ω' and Ω'' be probability spaces and set Ω''' to be their product. Let A be an event in Ω''' which occurs with probability at least $1 - \delta'$, for some $0 < \delta' < 1$.

Lemma 5.13. *For any* $1 > \delta > \delta'$

$$\mathbb{P}_{\Omega'}\left(\mathbb{P}_{\Omega''}(A \mid x) \leq 1 - \delta\right) \leq \delta'/\delta,$$

where x *is a random point in* Ω' *and* $\mathbb{P}_{\Omega'}$ *and* $\mathbb{P}_{\Omega''}$ *are the probabilities over* Ω' *and* Ω''*, respectively.*

Proof. Recall that $\mathbb{P}_{\Omega'''}(A) \geq 1 - \delta'$. However,

$$\mathbb{P}_{\Omega'''}(A) = \int_{\Omega'} \mathbb{P}_{\Omega''}(A \mid x)\partial x \leq 1 - \delta\, \mathbb{P}_{\Omega'}\left(\mathbb{P}_{\Omega''}(A \mid x) \leq 1 - \delta\right).$$

The claim follows. □

Recall that $L = \{t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n\}$. Lemma 5.2 yields

$$\mathbb{P}(F_\epsilon \nsubseteq [L]) \leq \exp(-c_0 \epsilon n),$$

for some positive constant c_0 depending only on K. Applying lemma 5.13 with $\Omega' = \Omega_{(i-1)}, \Omega'' = \Omega^{<i+1>}, \delta' = \exp(-c\epsilon n)$ and $\delta = n^{-4}$, we have

$$\begin{aligned}
\mathbb{P}_{\Omega_{(i-1)}}(\{t_1, \ldots, t_{i-1}\} \text{ is not typical}) &= \mathbb{P}_{\Omega_{(i-1)}}(\mathbb{P}_{\Omega^{<i+1>}}(F_\epsilon \nsubseteq [L] | t_1, \ldots, t_{i-1}) \leq 1 - \delta) \\
&\leq \delta'/\delta \\
&= n^4 \exp(-c_0 \epsilon n) \\
&\leq \exp(-c\epsilon n)
\end{aligned}$$

for $c = c_0/2$, given $c_0 \epsilon n \geq 8 \ln n$. This final condition can be satisfied by setting the α involved in the lower bound of ϵ to be sufficiently large. Thus, our proof is complete. □

5.4 A better bound on deviation

By using more of the smooth boundary structure, we can obtain a better result. As we shall see at the end of the proof, this result implies Theorem 2.4.

Theorem 5.14. *For any smooth convex body K with distribution μ along the boundary, there are constants $c, c', \alpha, \epsilon_0$ such that the following holds. For any $V_0 \geq \alpha n^{-(d+3)/(d-1)}, \epsilon_0 \geq \epsilon > \alpha \ln n/n$, $G_0 \geq 3\epsilon^{(d+1)/(d-1)}$, and $0 < \lambda \leq V_0/4G_0^2$, we have*

$$\mathbb{P}(|Z - \mathbb{E}\, Z| \geq \sqrt{\lambda V_0}) \leq 2\exp(-\lambda/4) + p_{NT}$$

where

$$p_{NT} = \exp(-c\epsilon n) + \exp(-c' n^{\frac{d-1}{3d+1} - \eta}),$$

and η is any small positive constant less than $\frac{d-1}{3d+1}$.

The proof of Theorem 5.14 follows from more careful estimates concerning $\Delta_{x,L}$. First, we will introduce a few technical lemmas: Let L be a finite set of points.

Lemma 5.15. *For any cap ϵ-boundary cap C, the boundary measure of the intersection of ∂K with the union of all ϵ-boundary cap intersecting C is at most $d_5\epsilon$ for some constant d_5, i.e.*

$$\mu(\partial K \cap (\cup_{C' \cap C \neq \emptyset} C')) \leq d_5 \epsilon.$$

Proof. By Lemma 3.12, $\mathrm{Vol_d}(C) = c\epsilon^{\frac{d+1}{d-1}}$, for some c. Suppose $C = C(x, h)$ where $x (\in \partial K)$ is the unique point where the tangent hyperplane is parallel to the hyperplane defining C and h is the height, then $(\frac{c}{d_4})^{2/(d+1)} \epsilon^{\frac{2}{d-1}} \leq h \leq (\frac{c}{d_3})^{2/(d+1)} \epsilon^{\frac{2}{d-1}}$ by Lemma 3.6. For any point $y \in C'$, where C' is any ϵ-boundary cap intersecting C, we have $||y - x|| \leq ||y - z|| + ||z - x||$, where $z \in C \cap C'$. By Lemma 3.5, $C \subseteq B(x, d_2 h^{1/2})$, and $C' \subseteq B(x', d_2 h^{1/2})$ for some $x' \in C' \cap \partial K$. Hence, $||y - z|| \leq 2d_2 h^{1/2}$, and $||z - x|| \leq 2d_2 h^{1/2}$. So $y \in B(x, 4d_2 h^{1/2})$, i.e. $\cup_{C' \cap C \neq \emptyset} C' \subseteq B(x, 4d_2 h^{1/2})$. Since ϵ is sufficiently small, so is h, by Lemma 3.5 again, $\partial K \cap B(x, 4d_2 h^{1/2}) \subseteq C(x, c'h)$ for some constant c'. $\mathrm{Vol_d}(C(x, c'h)) = \Theta(h^{(d+1)/2}) = \Theta(\epsilon^{(d+1)/(d-1)})$. Hence $\mu(C(x, c'h)) = \Theta(\epsilon)$. Therefore, $\mu(\partial K \cap (\cup_{C' \cap C \neq \emptyset} C')) \leq d_5 \epsilon$. for some constant d_5. $\square$

Let L be a set of points on the boundary of K. If a cap does not intersect $[L]$, the convex hull of L, then we say that it avoids L. Let $E_{\delta,L}$ be the intersection of ∂K and the union of all δ-boundary caps that avoid L. Then

Lemma 5.16. *There is a positive constants d_{10} such that for any set L, the set $E_{\delta,L}$ contains the intersection of ∂K with at least $\lfloor d_{10}\delta^{-1}\mu(E_{\delta,L})\rfloor$ pairwise disjoint δ-boundary caps.*

Proof. Note that if $E_{\delta,L} = \emptyset$, then the conclusion is trivial. Suppose $E_{\delta,L}$ contains at least the intersection of one δ-boundary cap and ∂K. Let $C_1, \ldots, C_l$ be a maximal system of pairwise disjoint δ-boundary caps such that their intersections with ∂K lie in $E_{\delta,L}$. By maximality, we have

$$E_{\delta,L} = \cup_{i=1}^{l} \mathcal{C}_i \cap \partial K,$$

where $\mathcal{C}_i$ is the union of all δ-boundary caps intersecting C_i whose intersection with ∂K is in $E_{\delta,L}$, for $i = 1, \ldots, l$. By Corollary 5.15, $\mu(\mathcal{C}_i \cap \partial K) \leq d_5\delta$. So

$$\mu(E_{\delta,L}) \leq \sum_{i=1}^{l} \mu(\mathcal{C}_i \cap \partial K) = O(l\delta),$$

which implies $l = \Omega(\delta^{-1}\mu(E_{\delta,L}))$. □

Lemma 5.17. *Let $x \in \partial K$, L be a set of points on ∂K and assume that $\Delta_{x,L}$ is at least ϵ. There is a positive constant d_{11} such that the boundary measure of one of the caps determined by the facets containing x is at least $d_{11}\epsilon^{(d-1)/(d+1)}$ and so $x \in E_{d_{11}\epsilon^{(d-1)/(d+1)},L}$.*

Proof. Suppose all the caps determined by the facets containing x has boundary measure at most $\epsilon^{(d-1)/(d+1)}$. Hence they have d-dimensional volume at most ϵ by Lemma 3.12. Since all these caps intersect (at x), the convex hull of them has volume at most $c\epsilon$ for some constant c (a one-line proof can be deduced much the same way as that of Lemma 5.15). But this convex hull contains $[L \cup x]\backslash[L]$, so $\Delta_{x,L} \leq c\epsilon$. □

For a finite set L of points in ∂K, we call a point $x \in \partial K$ δ-large with respect to L if $\Delta_{x,L} \geq \delta$. And we let

$$X_{\delta,L} = \{x | x \text{ is } \delta\text{-large with respect to } L\}.$$

If there are n points in L, the following lemma states that with high probability, there are not many points that are very "large" with respect to L.

Lemma 5.18. *Let L be a set of n random points on the boundary. There are positive constants c, c', c'' and c''' such that the following holds: For any $\delta > c' n^{-(d+1)/(d-1)}$ and any $T \geq \max\{c''\delta^{(d-1)/(d+1)}, \exp(-c'''\delta^{(d-1)/(d+1)}n)\}$, we have*

$$\mathbb{P}(\mu(X_{\delta,L}) \geq T) \leq \exp(-cnT)$$

Proof. Due to Lemma 5.17, $\mu(X_{\delta,L}) \leq \mu(E_{d_{11}\delta^{(d-1)/(d+1)},L})$. So it suffices to give an upper bound on $\mathbb{P}(\mu(E_{d_{11}\delta^{(d-1)/(d+1)},L}) \geq T)$.

Assume $\mu(E_{d_{11}\delta^{(d-1)/(d+1)},L}) \geq T$, we will upper-bound the probability of the consequent event hence give a bound of the above probability. By Lemma 5.16, there are at least $l = \Omega(\delta^{-(d-1)/(d+1)}T)$ many disjoint $d_{11}\delta^{(d-1)/(d+1)}$-boundary caps C_i's whose intersections with ∂K are contained in $E_{\delta^{(d-1)/(d+1)},L}$ and each C_i' avoids L. Note that by the cap-conversion lemma, $\mathrm{Vol_d}(C_i') = \Theta(\delta)$. With the right choice of constant, $T \geq c''\delta^{(d-1)/(d+1)}$ guarantees that l is at least one. Now by Lemma 3.8, for some small constant d_{12}, there is a fixed system of disjoint $d_{12}\delta$-caps (hence they are $d_{13}\delta^{(d-1)/(d+1)}$-boundary caps by Lemma 3.12 for some constant d_{13}), C_i, $i = 1, \ldots, m$ where $m = O(\delta^{-(d-1)/(d+1)})$, such that each C_i' contains at least one of the C_j. These C_j's then must also avoid L, and there are at least l many of them.

Since L has n points on the boundary of K, the probability that a fixed $d_{13}\delta^{(d-1)/(d+1)}$-boundary cap avoids L is at most

$$(1 - d_{13}\delta^{(d-1)/(d+1)})^n \leq \exp(-d_{13}\delta^{(d-1)/(d+1)}n).$$

The probability that the system $C_1, \ldots, C_m$ contains a subsystem of l elements avoiding L is at most

$$\begin{aligned}\binom{m}{l}\exp(-d_{13}\delta^{(d-1)/(d+1)}n)^l &\leq (\frac{em}{l})^l \exp(-d_{13}\delta^{(d-1)/(d+1)}nl) \\ &\leq \exp\left((-d_{13}\delta^{(d-1)/(d+1)}n + \ln\frac{em}{l})l\right). \quad (5.5)\end{aligned}$$

Now $m = O(\delta^{-(d-1)/(d+1)})$ and $l = \Omega(\delta^{-(d-1)/(d+1)}T)$. Thus,

$$\ln \frac{em}{l} \leq \ln d_{14} T^{-1}$$

for some positive constant d_{14}. Next, we note that choosing the right constants c' and c''' so that $T \geq \exp(-c'''\delta^{(d-1)/(d+1)}n)$ and $\delta \geq c' n^{-(d+1)/(d-1)}$ gives us

$$\ln d_{14} T^{-1} \leq \frac{1}{2} d_{13} \delta^{(d-1)/(d+1)} n.$$

Thus we obtain

$$\exp\left((-d_{13}\delta^{(d-1)/(d+1)}n + \ln \frac{em}{l})l\right) \leq \exp(-\frac{d_{13}}{2}\delta^{(d-1)/(d+1)}nl) \leq \exp(-cnT).$$

□

Recall the proof of Theorem 5.10, we see that to prove Theorem 5.14, we only need to show that under the assumptions of Theorem 5.14, there are positive constants c, c' such that for any $1 \leq i \leq n$,

$$\mathbb{P}(G_i'(t) \geq G_0 \textit{ or } V_i(t) \geq n^{-1}V_0) \leq \exp(-c\epsilon n) + \exp(-c' n^{\frac{d-1}{3d+1}-\eta}).$$

We can show $\mathbb{P}(G_i'(t) \geq G_0) \leq \exp(-c\epsilon n)$ as before. The key to the improvement here is the following claim:

Claim 5.19. *There is constant c' such that for any $1 \leq i \leq n$ and $0 < \eta < \frac{d-1}{3d+1}$,*

$$\mathbb{P}(V_i(t) \geq n^{-1}V_0) \leq \exp(-c' n^{\frac{d-1}{3d+1}-\eta}).$$

Proof. First, recall that

$$G_i(t) \leq \mathbb{E}(\Delta_{t_i,L}|t_1,\ldots,t_{i-1}) + \mathbb{E}_x\,\mathbb{E}(\Delta_{x,L}|t_1,\ldots,t_{i-1}).$$

So

$$\begin{aligned} V_i(t) &= \int_{\partial K} G_i^2(t)dt_i \\ &\leq \int_{\partial K} (\mathbb{E}(\Delta_{t_i,L}|t_1,\ldots,t_{i-1}) + \mathbb{E}_x\,\mathbb{E}(\Delta_{x,L}|t_1,\ldots,t_{i-1}))^2 dt_i \\ &\leq 4\int_{\partial K} \mathbb{E}^2(\Delta_{t_i,L}|t_1,\ldots,t_{i-1})dt_i. \end{aligned}$$

Pick $0 < \eta < \frac{d-1}{3d+1}$, and set $\epsilon_0 = n^{-\frac{2d+2}{3d+1}-\eta}$. Let $\delta_0 = n^{-(d+1)/(d-1)}$ and $T_0 = 1$. Also, set $\delta_j = \delta_0 2^j$ and $T_j = (j+1)^{-2}4^{-j}T_0$. Since K is smooth, there is a constant a such that $g(\epsilon_0) \leq a\epsilon_0^{(d+1)/(d-1)}$. Let j_0 be the smallest positive integer such that $\delta_{j_0} \geq a\epsilon_0^{(d+1)/(d-1)}$. One can check that for $j \leq j_0$ the condition of Lemma 5.18 is satisfied so we can apply it.

We say the point set $L = \{t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n\}$ is *nice* if the followings hold:

- $[L]$ contains F_{ϵ_0}
- $\mu(X_{\delta_j, L}) \leq T_j$ for all $j = 0, 1, 2, \dots$.

Following the proof of Theorem 5.10, we call a set $\{t_1, \dots, t_{i-1}\}$ *typical* if

$$\mathbb{P}_{\Omega^{<i+1>}}(L \text{ is not nice}|t_1, \dots, t_{i-1}) \leq n^{-6}.$$

Similar to the proof of Theorem 5.10, we will show the claim in two steps. First, we will show that if $\{t_1, \dots, t_{i-1}\}$ is typical, then the claim holds. Then we give an upper bound on the probability that the above point set is not typical.

First step. Assume $\{t_1, \dots, t_{i-1}\}$ is typical, notice that

$$\int_{\partial K} \mathbb{E}^2(\Delta_{t_i,L}|t_1, \dots, t_{i-1})dt_i = \int_{\Omega^{<i+1>}} (\int_{\partial K} \Delta^2_{t_i,L} dt_i) dt^{<i+1>}$$

Let $\Omega_1^{<i+1>}$ be the set of those $(t_{i+1}, \dots, t_n)$ such that the set L is nice and let $\Omega_2^{<i+1>}$ be the rest.

Note that $\int_{\partial K} \Delta^2_{t_i,L} dt_i \leq 1$ since $\Delta_{t_i,L} \leq 1$. Hence

$$\int_{\Omega_2^{<i+1>}} \int_{\partial K} \Delta^2_{t_i,L} dt_i dt^{<i+1>} \leq \mathbb{P}_{\Omega^{<i+1>}}(L \text{ is not nice}|t_1, \dots, t_{i-1}) \leq n^{-6}.$$

Now, when L is nice i.e. in the space of $\Omega_2^{<i+1>}$, we have

$$\begin{aligned}
\int_{\partial K} \Delta_{t_i,L}^2 dt_i &\leq \delta_0^2 + \sum_{j=0}^{\infty} \delta_{j+1}^2 \mu(X_{\delta_j,L}) \\
&\leq \delta_0^2 + \sum_{j=0}^{j_0} \delta_{j+1}^2 \mu(X_{\delta_j,L}) \\
&\leq \delta_0^2 + \sum_{j=0}^{j_0} \delta_{j+1}^2 T_j \\
&\leq O(n^{-2(d+1)/(d-1)}).
\end{aligned}$$

Here, we use the fact that when $j > j_0$, $\delta_j \geq a\epsilon_0^{(d+1)/(d-1)} = g(\epsilon_0)$ so $X_{\delta_j,L}$ is empty.

Combining the analysis of $\Omega_1^{<i+1>}$ and $\Omega_2^{<i+1>}$, we have the upper bound for $V_i(t)$

$$n^{-6} + O(n^{-2(d+1)/(d-1)}) \leq O(n^{-2(d+1)/(d-1)}).$$

So one has

$$nO(n^{-2(d+1)/(d-1)}) = O(n^{-(d+3)/(d-1)}),$$

that is,

$$V_i \leq n^{-1} V_0$$

for our choice of V_0.

Second step. Now we analyze our error term. We will do this using Lemma 5.13. From our definition of "niceness", we see that

$$\begin{aligned}
\mathbb{P}(L \text{ is not nice}) &\leq \mathbb{P}(F_{\epsilon_0} \nsubseteq [L]) + \sum_{j=0}^{j_0} \mathbb{P}(\mu(X_{\delta_j,L} \geq T_j)) \\
&\leq \exp(-\Omega(n\epsilon_0)) + \sum_{j=0}^{j_0} \exp(-\Omega(nT_j)).
\end{aligned}$$

Note that

$$nT_j \geq nT_{j_0} \gg n\delta_{j_0}^{(d-1)/(d+1)} \geq a\epsilon_0$$

and there are at most $j_0 \le O(\log_2 n)$ many terms in the sum. So the first term in (5.4) is dominating. Thus, we have the error term

$$\exp(-cn^{\frac{d-1}{3d+1}-\eta}), \quad 0 < \eta < \frac{d-1}{3d+1}.$$

Now by Lemma 5.13,

$$\begin{aligned}
&\mathbb{P}_{\Omega_{(i-1)}}(\{t_1, \ldots, t_{i-1} \text{ is not typical }\}) \\
&= \mathbb{P}_{\Omega_{(i-1)}}(\mathbb{P}_{\Omega^{<i+1>}}(L \text{ is nice } | t_1, \ldots, t_{i-1}) \ge 1 - n^{-6}) \\
&\le n^6 \exp(-cn^{\frac{d-1}{3d+1}-\eta}) \\
&= \exp(-c' n^{\frac{d-1}{3d+1}-\eta}),
\end{aligned}$$

for some constant c'. □

The key difference between this result and Theorem 5.10 is that here V_0 is independent of ϵ, so we can set $V_0 = \alpha n^{-(d+3)/(d-1)}$ without affecting the tail estimate. If we also set $\epsilon = n^{-\frac{2d+2}{3d+1}-\eta}$, then the two error terms in p_{NT} are the same (up to a constant factor). Since $G_0 = 3g(\epsilon) = 3\Theta(\epsilon^{(d+1)/(d-1)})$, we have $\lambda < V_0/4G_0^2 \le c'' n^{\frac{d-1}{3d+1}+\frac{2(d+1)\eta}{d-1}}$ for some constant c''. Hence Theorem 2.4 is proved.

The key idea used in the proof of Theorem 5.14 can also be used to prove the following important lemma, which we will use in Section 6.

Lemma 5.20. *For large n,*

$$\mathbb{E}\,\mathrm{Vol_d}(K_{n+1}) - \mathbb{E}\,\mathrm{Vol_d}(K_n) = O(n^{-(d+1)/(d-1)}).$$

Proof. Set ϵ_0, δ_j and $T_j, j = 0, \ldots, j_0$ just as in the proof of Claim 5.19.

Following the notations found in the concentration proof, we set $\Omega' = \{t = (t_1, \ldots, t_n) \mid t_i \in \partial K\}$, and put $L = \{t_1, \ldots, t_n\}$. Again, we call L "nice" as before in the proof of Claim 5.19. Let Ω'_1 be the family of all "nice" L, and Ω'_2 the rest.

Observe that we can write

$$\begin{aligned}\mathbb{E}\,\mathrm{Vol_d}(K_{n+1}) - \mathbb{E}\,\mathrm{Vol_d}(K_n) &= \int_{\Omega'}\int_{\partial K} \mathrm{Vol_d}([t_1,\ldots,t_n,t_{n+1}]) - \mathrm{Vol_d}([t_1,\ldots,t_n])dt_{n+1}dt \\ &= \int_{\Omega'}\int_{\partial K} \Delta_{t_{n+1},L}dt_{n+1}dt \\ &= \int_{\Omega_1'}\int_{\partial K} \Delta_{t_{n+1},L}dt_{n+1}dt + \int_{\Omega_2'}\int_{\partial K} \Delta_{t_{n+1},L}dt_{n+1}dt \end{aligned} \tag{5.6}$$

The first integrand can be estimated by

$$\begin{aligned}\int_{\partial K} \Delta_{t_{n+1},L}dt_{n+1} &\le \delta_0 + \sum_{j=0}^{j_0} \delta_{j+1}\mu(X_{\delta_j,L}) \\ &\le \delta_0 + \sum_{j=0}^{j_0} \delta_{j+1}T_j \\ &\le O(n^{-(d+1)/(d-1)}),\end{aligned}$$

Now $\int_{\Omega_1'} dt = 1$, so the first term in (5.6) is bounded by $O(n^{-(d+1)/(d-1)})$. Since $\int_{\partial K} \Delta_{t_{n+1},L}dt_{n+1} \le 1$ and

$$\mathbb{P}(L \text{ is not nice}) \le \exp(-c\epsilon_0 n) \ll O(n^{-(d+1)/(d-1)}),$$

with the appropriate constant c for sufficiently large n, we prove the lemma. □

5.5 Higher moments and rate of convergence

Proof of Corollary 2.5: Let $\lambda_0 = \frac{\alpha}{4} n^{\frac{d-1}{3d+1}+\frac{2(d+1)\eta}{d-1}}$ be the upper bound for λ given in Theorem 2.4. So for $\lambda > \lambda_0$, by (2.4)

$$\begin{aligned}\mathbb{P}(|Z - \mathbb{E}\,Z| \ge \sqrt{\lambda V_0}) &\le \mathbb{P}(|Z - \mathbb{E}\,Z| \ge \sqrt{\lambda_0 V_0}) \\ &\le 2\exp(-\lambda_0/4) + \exp(-cn^{\frac{d-1}{3d+1}-\eta}).\end{aligned}$$

Combining (2.4) and the above, we get for any $\lambda > 0$,

$$\mathbb{P}(|Z - \mathbb{E}\,Z| \ge \sqrt{\lambda V_0}) \le 2\exp(-\lambda/4) + 2\exp(-\lambda_0/4) + \exp(-cn^{\frac{d-1}{3d+1}-\eta}). \tag{5.7}$$

We then compute the kth moment M_k of Z, beginning with the definition:

$$M_k = \int_0^\infty t^k d\,\mathbb{P}(|Z - \mathbb{E}\,Z| < t).$$

If we set $\gamma(t) = \mathbb{P}(|Z - \mathbb{E}\,Z| \geq t)$ then we can write

$$\begin{aligned} M_k &= \int_0^\infty t^k d\,\mathbb{P}(|Z - \mathbb{E}\,Z| < t) \\ &= -\int_0^\infty t^k d\gamma(t) \\ &= \left((-t^k\gamma(t))\big|_0^\infty + \int_0^\infty kt^{k-1}\gamma(t)dt \right) \\ &= \int_0^1 kt^{k-1}\gamma(t)dt. \end{aligned}$$

Note that the limits of integration can be limited to $[0,1]$ because we've assumed the volume of K is normalized to 1.

Setting $t = \sqrt{\lambda V_0}$ we get

$$\begin{aligned} \int_0^1 kt^{k-1}\gamma(t)dt &= \int_0^{1/V_0} k(\sqrt{\lambda V_0})^{k-1}\,\mathbb{P}(|Z - \mathbb{E}\,Z| \geq \sqrt{\lambda V_0})\frac{\sqrt{V_0}}{2\sqrt{\lambda}}d\lambda \\ \text{by (5.7)} &\leq \frac{k}{2}V_0^{k/2}\int_0^{1/V_0} \lambda^{\frac{k}{2}-1}2\exp(-\lambda/4) + 2\exp(-\lambda_0/4) + \exp(-cn^{\frac{d-1}{3d+1}-\eta})d\lambda. \end{aligned}$$

We may now evaluate each term separately.

For the first term we observe that

$$\int_0^{1/V_0} 2\lambda^{\frac{k}{2}-1}\exp(-\lambda/4)d\lambda \leq \int_0^\infty 2\lambda^{\frac{k}{2}-1}\exp(-\lambda/4)d\lambda = c_k,$$

where c_k is a constant depending only on k.

Since

$$V_0 = \alpha n^{-\frac{d+3}{d-1}} \gg n^{-5},$$

we can compute the second term:

$$\begin{aligned} \int_0^{1/V_0} \lambda^{\frac{k}{2}-1}2\exp(-\lambda_0/4)d\lambda &\leq \frac{2}{k}2\exp(-\lambda_0/4)V_0^{-\frac{k}{2}} \\ &\leq \frac{2}{k}2\exp(-\frac{\alpha}{16}n^{\frac{d-1}{3d+1}+\frac{2(d+1)\eta}{d-1}})n^{\frac{5k}{2}} \\ &= o(1). \end{aligned}$$

The last term can be computed similarly and gives $o(1)$ again. Hence,

$$M_k \leq (c_k + o(1))kV_0^{k/2} = O(V_0^{k/2}).$$

Proof of Corollary 2.6:

$$\begin{aligned}
\mathbb{P}\left(\left|\frac{Z_n}{\mathbb{E}\,Z_n} - 1\right| f(n) \geq \delta(n)\right) &\leq \mathbb{P}(|Z_n - \mathbb{E}\,Z_n| \geq \mathbb{E}\,Z_n\sqrt{32n^{-\frac{d+3}{d-1}}\ln n}) \\
&\leq \mathbb{P}(|Z_n - \mathbb{E}\,Z_n| \geq \sqrt{8\ln nV_0}) \\
&\leq 2\exp(-8\ln n/4) + \exp(-cn^{\frac{d-1}{3d+1}-\eta}) \\
&\leq 3\exp(-2\ln n) \\
&\leq 3n^{-2},
\end{aligned}$$

by Theorem 2.4. The second inequality above is due to the fact that $\mathbb{E}\,Z_n = 1 - c_K n^{-\frac{2}{d-1}} > 1/2$ when n is large. Since $\sum n^{-2}$ is convergent, by Borel-Cantelli, $\left|\frac{Z_n}{\mathbb{E}\,Z_n} - 1\right| f(n)$ converges to 0 almost surely, hence the corollary.

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6

Central Limit Theorem

6.1 Approximating K_n by Π_n

Before we prove the central limit theorem for Poisson model, we should first give a brief review of the Poisson point process.

Let $K \in \mathcal{K}^2_+$, and let Pois(n) be a Poisson point process with intensity n. Then the intersection of Pois(n) and ∂K consists of random points $\{x_1, \dots, x_N\}$ where the number of points N is Poisson distributed with mean $n\mu(\partial K) = n$. We write $\Pi_n = [x_1, \dots, x_N]$. Conditioning on N, the points $x_1, \dots, x_N$ are independently uniformly distributed in ∂K. For two disjoint subsets A and B of ∂K, their intersections with Pois(n), i.e. the point sets $A \cap \text{Pois}(n) = \{x_1, \dots, x_N\}$ and $B \cap \text{Pois}(n) = \{y_1, \dots, y_M\}$, are independent. This means N and M are independently Poisson distributed with intensity $n\mu(A)$ and $n\mu(B)$ respectively, and x_i and y_j are chosen independently.

The following standard estimates of the tail of Poisson distribution will be used repeatedly throughout this section. Let X be a Poisson random variable with mean

λ. Then

$$\begin{aligned}\mathbb{P}(X \leq \frac{\lambda}{2}) &= \sum_{k=0}^{\lambda/2} e^{-\lambda}\frac{\lambda^k}{k!} \leq e^{-\lambda} + \sum_{k=1}^{\lambda/2} e^{-\lambda}(\frac{e\lambda}{k})^k \\ &\leq \frac{\lambda+1}{2}e^{-\lambda}(2e)^{\lambda/2} \leq \frac{\lambda+1}{2}(\frac{e}{2})^{-\lambda/2} \\ &= \Theta((\frac{e}{2})^{-\lambda/2}),\end{aligned} \tag{6.1}$$

where the last equality holds when λ is large. Similarly,

$$\mathbb{P}(X \geq 3\lambda) \leq \sum_{k=3\lambda}^{\infty} e^{-\lambda}(\frac{e\lambda}{k})^k \leq \sum_{k=0}^{\infty} e^{-\lambda}(\frac{e}{3})^k = ce^{-\lambda}, \tag{6.2}$$

where c is a small constant.

As is pointed out in the introduction, Π_n approximates K_n quite well, as one might expect.

Theorem 6.1. *Let Π_n be the convex hull of points chosen on ∂K according to the Poisson point process* Pois(n). *Then,*

$$\mathbb{E}\,\mathrm{Vol_d}(\Pi_n) \approx \mathbb{E}\,\mathrm{Vol_d}(K_n) = 1 - c(K,d)n^{-\frac{2}{d-1}},$$

as $n \to \infty$, and

$$\mathrm{Var}\,\mathrm{Vol_d}(\Pi_n) = \Theta(\mathrm{Var}\,\mathrm{Vol_d}(K_n)) = \Theta(n^{-\frac{d+3}{d-1}}).$$

Proof. Due to the conditioning property of Poisson point process, we have

$$\mathbb{E}\,\mathrm{Vol_d}(\Pi_n) = \sum_{|k-n|\leq n^{7/8}} \mathbb{E}\,\mathrm{Vol_d}(K_k)e^{-n}\frac{n^k}{k!} + \sum_{|k-n|\geq n^{7/8}} \mathbb{E}\,\mathrm{Vol_d}(K_k)e^{-n}\frac{n^k}{k!}.$$

For Poisson distribution, the Chebyschev's inequality gives $\mathbb{P}(|k-n| \geq n^{7/8}) \leq n^{-3/4}$. Hence the second summand is bounded above by $n^{-3/4}$ since $\mathbb{E}\,\mathrm{Vol_d}(K_k)$ is at most 1. By (2.2), $\mathbb{E}\,\mathrm{Vol_d}(K_k) = 1 - k^{-\frac{2}{d-1}} = 1 - (1+o(1))n^{-\frac{2}{d-1}}$, when $|k-n| \leq n^{7/8}$.

For the variance, we can rewrite $\mathrm{Var}\,\mathrm{Vol_d}(\Pi_n)$ as follows:

$$\mathrm{Var}\,\mathrm{Vol_d}(\Pi_n) = \mathbb{E}_N\,\mathrm{Var}(\mathrm{Vol_d}(\Pi_n)|N) + \mathrm{Var}_N\,\mathbb{E}(\mathrm{Vol_d}(\Pi_n)|N).$$

By (6.1), the second term in the above equation becomes:

$$\begin{aligned}
\operatorname{Var}\mathbb{E}(\mathrm{Vol_d}(\Pi_n)|N) &= \mathbb{E}_N \mathbb{E}^2 \mathrm{Vol_d}(K_N) - (\mathbb{E}_N \mathbb{E}\,\mathrm{Vol_d}(K_N))^2 \\
&= \sum_{j=\frac{n}{2}}^{\infty}\sum_{k=\frac{n}{2}}^{\infty}(\mathbb{E}^2\,\mathrm{Vol_d}(K_k) - \mathbb{E}\,\mathrm{Vol_d}(K_k)\,\mathbb{E}\,\mathrm{Vol_d}(K_j))e^{-2n}\frac{n^{k+j}}{k!j!} \\
&\quad + O((\frac{e}{2})^{-n/2}) \\
&= \sum_{j=\frac{n}{2}}^{\infty}\sum_{k=j}^{\infty}(\mathbb{E}\,\mathrm{Vol_d}(K_k) - \mathbb{E}\,\mathrm{Vol_d}(K_j))^2 e^{-2n}\frac{n^{k+j}}{k!j!} + O((\frac{e}{2})^{-n/2}),
\end{aligned}$$

where the third equality is due to (6.1). By Lemma 5.20, $\mathbb{E}\,\mathrm{Vol_d}(K_{j+1}) - \mathbb{E}\,\mathrm{Vol_d}(K_j) = c(K,d)j^{-\frac{d+1}{d-1}}$ when $j \to \infty$, hence

$$\mathbb{E}\,\mathrm{Vol_d}(K_k) - \mathbb{E}\,\mathrm{Vol_d}(K_j) = \sum_{i=j}^{k-1}\mathbb{E}\,\mathrm{Vol_d}(K_{i+1}) - \mathbb{E}\,\mathrm{Vol_d}(K_i) \le c(K,d)(k-j)j^{-\frac{d+1}{d-1}},$$

and

$$\begin{aligned}
\operatorname{Var}\mathbb{E}(\mathrm{Vol_d}(\Pi_n)|N) &\le c(K,d)\sum_{j=\frac{n}{2}}^{\infty}\sum_{k=j}^{\infty}(k-j)^2 j^{-\frac{2d+2}{d-1}}e^{-2n}\frac{n^{k+j}}{k!j!} + O((\frac{e}{2})^{-n/2}) \\
&\le cn^{-\frac{2d+2}{d-1}}\operatorname{Var} N + O((\frac{e}{2})^{-n/2}) \\
&= O(n^{-\frac{d+3}{d-1}}).
\end{aligned}$$

Now, $\operatorname{Var}\mathrm{Vol_d}(K_n) = \Theta(n^{-\frac{d+3}{d-1}})$, so by (6.1) and (6.2), we have

$$\begin{aligned}
\mathbb{E}\operatorname{Var}\mathrm{Vol_d}(\Pi_n|N) &= \mathbb{E}(\Theta(N^{-\frac{d+3}{d-1}})) \\
&= O(\mathbb{P}(N \le \frac{n}{2})) + \mathbb{E}(N^{-\frac{d+3}{d-1}}\mathbf{1}\{\frac{n}{2} < N \le 3n\}) + O(\mathbb{P}(3n < N)) \\
&= \Theta(n^{-\frac{d+3}{d-1}}).
\end{aligned}$$

$\square$

6.2 Poisson Central Limit Theorem

The key ingredient of the proof is the following theorem:

Theorem 6.2 (Baldi-Rinott[7]). *Let G be the dependency graph of random variables Y_i's, $i = 1, \dots, m$, and let $Y = \sum_i Y_i$. Suppose the maximal degree of G is D and $|Y_i| \le B$ a.s., then*

$$\left| \mathbb{P}\left(\frac{Y - \mathbb{E}\, Y}{\sqrt{\operatorname{Var} Y}} \le x \right) - \Phi(x) \right| = O(\sqrt{S}),$$

where $\Phi(x)$ is the standard normal distribution and $S = \frac{mD^2B^3}{(\sqrt{\operatorname{Var} Y})^3}$.

Here the dependency graph of random variables Y_i's is a graph on m vertices such that there is no edge between any two disjoint subsets, A_1 and A_2, of $\{Y_i\}_{i=1}^m$ if these two sets of random variables are independent.

Because we can dissect the convex body K into Voronoi cells according to the cap covering Lemma 3.7, we will study $\mathrm{Vol_d}(\Pi_n)$ as a sum of random variables which are volumes of the intersection of Π_n with each of the Voronoi cell. And the theorem above allows us to prove central limit theorem for sums of random variables that may have small dependency on each other.

First we let

$$m = \left\lfloor \frac{n}{4d \ln n} \right\rfloor .$$

By Lemma 3.7, given $K \in \mathcal{K}_+^2$, we can choose m points, namely $y_1, \dots, y_m$, on ∂K. And the Voronoi Cells $\mathrm{Vor}(y_i)$ of these points dissect K into m parts. Let

$$Y_i = \mathrm{Vol_d}(\mathrm{Vor}(y_i)) - \mathrm{Vol_d}(\mathrm{Vor}(y_i) \cap \Pi_n),$$

$i = 1, \dots, m$. So

$$Y = \sum_i Y_i = \mathrm{Vol_d}(K) - \mathrm{Vol_d}(\Pi_n), \tag{6.3}$$

Moreover, these Voronoi cells also dissect the boundary of K into m parts, and each contains a cap C_i with d-dimensional volume

$$\mathrm{Vol_d}(C_i) = \Theta(m^{-\frac{d+1}{d-1}}),$$

by Lemma 3.7. Now by Lemma 3.12 it is a boundary cap with $(d-1)$-dimensional volume

$$\mu(C_i \cap \partial K) = \Theta(m^{-1}) = \Theta(\frac{4d \ln n}{n}).$$

Denote by $A_i (i = 1, \ldots, m)$ the number of points generated by the Poisson point process of intensity n contained in $C_i \cap \partial K$, hence A_i is Poisson distributed with mean $\lambda = n\mu(C_i \cap \partial K) = \Theta(4d \ln n)$. Then

$$\mathbb{P}(A_i = 0) = e^{-\lambda} = O(n^{-4d}).$$

And by (6.1),

$$\mathbb{P}(A_i \geq 3\lambda) = \mathbb{P}(A_i \geq 12d \ln n) = O(n^{-4d}).$$

Now let A^m be the event that there is at least one point and at most $12d \ln n$ points in every A_i for $i = 1, \ldots, m$. Then

$$1 \geq \mathbb{P}(A^m) = \mathbb{P}(\cap_i \{1 \leq A_i \leq 12d \ln n\}) \geq 1 - \Omega(n^{-4d+1}). \tag{6.4}$$

The rest of the proof is organized as follows. We first prove the central limit theorem for $\mathrm{Vol_d}(\Pi_n)$ when we condition on A^m, then we show removing the condition doesn't affect the estimate much, as A^m holds almost surely. Let $\widetilde{\mathbb{P}}$ denote the conditional probability measure induced by the Poisson point process $X(n)$ on ∂K given A^m, i.e.

$$\widetilde{\mathbb{P}}(\mathrm{Vol_d}(\Pi_n) \leq x) = \mathbb{P}(\mathrm{Vol_d}(\Pi_n) \leq x | A^m).$$

Similarly, we define the corresponding conditional expectation and variance to be $\widetilde{\mathbb{E}}$ and $\widetilde{\mathrm{Var}}$, then

Lemma 6.3.

$$\left| \widetilde{\mathbb{P}} \left(\frac{\mathrm{Vol_d}(\Pi_n) - \widetilde{\mathbb{E}} \, \mathrm{Vol_d}(\Pi_n)}{\sqrt{\widetilde{\mathrm{Var}} \, \mathrm{Vol_d}(\Pi_n)}} \leq x \right) - \Phi(x) \right| = O(n^{-\frac{1}{4}} \ln^{\frac{d+2}{d-1}} n). \tag{6.5}$$

Proof. Note that by (6.3), $\mathrm{Vol_d}(\Pi_n) - \widetilde{\mathbb{E}}\,\mathrm{Vol_d}(\Pi_n) = \widetilde{\mathbb{E}}\,Y - Y$, and $\widetilde{\mathrm{Var}}\,Y = \widetilde{\mathrm{Var}}\,\mathrm{Vol_d}(\Pi_n) = \Theta(n^{-\frac{d+3}{d-1}})$, by Theorem 6.1. Hence it suffices to show Y satisfies the Central Limit Theorem under $\widetilde{\mathbb{P}}$.

Given A^m, we define the dependency graph on random variables $Y_i, i = 1, \dots, m$ as follows: we connect Y_i and Y_j if $\mathrm{Vor}(y_i) \cap C(y_j, cm^{-\frac{2}{d-1}}) \neq \emptyset$ for some constant c which satisfies Lemma 3.10. To check dependency, we see that if $Y_i \nsim Y_j$, then $\mathrm{Vor}(y_i) \cap C(y_j, cm^{-\frac{2}{d-1}}) = \emptyset$. Thus, for any point $P_1 \in \mathrm{Vor}(y_i) \cap \partial K$, $P_2 \in \mathrm{Vor}(y_j) \cap \partial K$ such that P_1, P_2 are vertices of Π_n, the line segment $[P_1, P_2]$ cannot be contained in the boundary of Π_n. Otherwise, it would be a contradiction to Lemma 3.10. Therefore, there is no edge of Π_n between vertices in $\mathrm{Vor}(y_i)$ and $\mathrm{Vor}(y_j)$, hence Y_i and Y_j are independent given A^m.

To apply Theorem 6.2 to Y, we are left to estimate parameters D and B.

By Lemma 3.9, $C(y_j, cm^{-\frac{2}{d-1}})$ $(j = 1, \dots, m)$ can intersect at most $O(1)$ many $\mathrm{Vor}(y_i)$'s. Hence $D = O(1)$.

By Lemma 3.10, for any point x_i in C_i, $i = 1, \dots, m$,

$$\delta^H(K, \Pi_n) \leq \delta^H(K, [x_1, \dots, x_m]) = O(m^{-\frac{2}{d-1}}).$$

So

$$\mathrm{Vor}(y_i) \backslash \Pi_n \subseteq C(y_i, h'), \tag{6.6}$$

where $h' = O(m^{-\frac{2}{d-1}})$. By Lemma 3.6 and (6.6),

$$Y_i \leq \mathrm{Vol_d}(C(y_i, h')) = O(m^{-\frac{d+1}{d-1}}) = O((\frac{4d \ln n}{n})^{\frac{d+1}{d-1}}) := B.$$

Hence by the Baldi-Rinott Theorem, the rate of convergence in (6.5) is $\Theta(n^{-\frac{1}{4}}(\ln n)^{\frac{d+2}{d-1}})$, and we finish the proof. $\square$

Now, we will remove the condition A^m. First observe an easy fact

Proposition 6.4. *For any events A and B,*

$$|\,\mathbb{P}(B|A) - \mathbb{P}(B)| \leq \mathbb{P}(A^c).$$

Proof. Since

$$\mathbb{P}(B) - (1 - \mathbb{P}(A)) = \mathbb{P}(B) - \mathbb{P}(A^c) \leq \mathbb{P}(B \backslash A^c) \leq \mathbb{P}(B),$$

we have $|\,\mathbb{P}(B|A) - \mathbb{P}(B)| \leq 1 - \mathbb{P}(A)$.

$\square$

Hence we can deduce

Lemma 6.5.

$$\left|\widetilde{\mathbb{P}}(\mathrm{Vol_d}(\Pi_n) \leq x) - \mathbb{P}(\mathrm{Vol_d}(\Pi_n) \leq x)\right| = O(n^{-4d+1}), \tag{6.7}$$

$$\left|\widetilde{\mathbb{E}}\,\mathrm{Vol_d}^k(\Pi_n) - \mathbb{E}\,\mathrm{Vol_d}^k(\Pi_n)\right| = O(n^{-4d+1}), \tag{6.8}$$

for $k = 1, 2$, *and*

$$\left|\widetilde{\mathrm{Var}}\,\mathrm{Vol_d}(\Pi_n)) - \mathrm{Var}\,\mathrm{Vol_d}(\Pi_n)\right| = O(n^{-4d+1}). \tag{6.9}$$

Proof. Equation (6.7) follows immediately from Proposition 6.4. Now when $k = 1, 2$, since $\mathrm{Vol_d}(\Pi_n) \leq 1$,

$$\widetilde{\mathbb{E}}\,\mathrm{Vol_d}^k(\Pi_n) - \mathbb{E}\,\mathrm{Vol_d}^k(\Pi_n) \leq \mathbb{E}\,\mathrm{Vol_d}^k(\Pi_n)\left(\frac{1}{\mathbb{P}(A^m)} - 1\right) = O(n^{-4d+1}),$$

and

$$\mathbb{E}\,\mathrm{Vol_d}^k(\Pi_n) - \widetilde{\mathbb{E}}\,\mathrm{Vol_d}^k(\Pi_n) \leq \mathbb{E}\left(\mathrm{Vol_d}^k(\Pi_n)\,(1 - \mathbf{1}(A^m))\right) = O(n^{-4d+1}).$$

Hence the variance follows from the moments estimate above. $\square$

As a result of Lemma 6.5, we can remove the condition A^m and obtain Theorem 2.7 as follows. For notational convenience, we denote $\mathrm{Vol_d}(\Pi_n)$ by X temporarily. For each x, let $\widetilde{\mathrm{x}}$ be such that

$$\mathbb{E}\,X + x\sqrt{\mathrm{Var}\,X} = \widetilde{\mathbb{E}}\,X + \widetilde{\mathrm{x}}\,\sqrt{\widetilde{\mathrm{Var}}\,X},$$

then

$$|x - \widetilde{\mathrm{x}}\,| = O(n^{-4d+1+\frac{d+3}{2(d-1)}}) + |x|O(n^{-4d+1+\frac{d+3}{d-1}}), \tag{6.10}$$

by (6.7) and Lemma 6.3. We have

$$\begin{aligned} F_X(x) = \mathbb{P}(X \le \mathbb{E}\,X + x\sqrt{\operatorname{Var} X}) &= \widetilde{\mathbb{P}}(X \le \widetilde{\mathbb{E}}\,X + \widetilde{\mathrm{x}}\,\sqrt{\widetilde{\operatorname{Var}}}) + O(n^{-4d+1}) \\ &= \Phi(\widetilde{\mathrm{x}}) + O(n^{-\frac{1}{4}}\ln^{\frac{d+2}{d-1}} n) + O(n^{-4d+1}). \end{aligned}$$

But $|\Phi(x) - \Phi(\widetilde{\mathrm{x}})| = O(n^{-1})$, since $|\Phi(x) - \Phi(\widetilde{\mathrm{x}})| \le |x - \widetilde{\mathrm{x}}\,| \le O(n^{-1})$ when $|x| \le n$ and by (6.10) $|\,\widetilde{\mathrm{x}}\,| \ge cn$ when $|x| \ge n$ which implies $|\Phi(x) - \Phi(\widetilde{\mathrm{x}})| \le \exp(-\Omega(n))$. So

$$|F_X(x) - \Phi(x)| = |\,\mathbb{P}(X \le \mathbb{E}\,X + x\sqrt{\operatorname{Var} X}) - \Phi(x)| = O(n^{-\frac{1}{4}}\ln^{\frac{d+2}{d-1}} n).$$

Hence finishes the proof of Theorem 2.7.

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Bibliography

[1] F. Affentranger, The expected volume of a random polytope in a ball, J. Microscopy **151**, 277-287, 1988.

[2] F. Affentranger, The convex hull of random points with spherically symmetric distributions, Rend. Sem. Mat. Univ. Politec. Torino **49**, 359-383, 1991.

[3] N. Alon and J. Spencer, *Probabilistic Method*, John Wiley & Sons,Inc., 2000.

[4] S. G. Akl and G. T. Toussaint, Efficient convex hull algorithms for pattern recognition applications, Proc. 4th Int'l Joint Conf. on Pattern Recognition, Kyoto, Japan, 483-487, 1978.

[5] K. Azuma, Weighted sums of certain dependent random variables, Tohoku Math. J., **19**, 357-367, 1967.

[6] I. Bárány, Random polytopes, convex bodies, and approximation, *Stochastic Geometry*, ed. W. Weil, Springer, in press.

[7] P. Baldi and Y. Rinott, On normal approximations of distributions in terms of dependency graphs, The Annals of Probability, Vol.17, No.4, 1646-1650, 1989.

[8] I. Bárány, Personal conversations, UCSD, 2005.

[9] I. Bárány, Intrinsic volumes and f-vectors of random polytopes, Math. Ann. **285**, 671-699, 1989.

[10] I. Bárány, Random polytopes in smooth convex bodies, Mathematika **39**, 81-92, 1992.

[11] I. Bárány, Sylvester's question: the probability that n points are in convex position. Annals of Prob. **27**, 2020-2034, 2000.

[12] I. Bárány, A note on Sylvester's four-point problem, Studia Math. Hung.**38**, 2020-2034, 2001.

[13] I. Bárány and C. Buchta, On the convex hull of uniform random points in an arbitrary d-polytope, Anz. Österreich. Akad. Wiss. Math.-Natur. Kl. **127**, 25-27, 1990.

[14] I. Bárány and C. Buchta, Random polytopes in a convex polytope, independence of shape, and concentration of vertices, Math. Ann. **297**, 467-497, 1993.

[15] I. Bárány and L. Dalla, Few points to generate a random polytope, Mathematika **44** no.2, 325-331, 1997.

[16] I. Bárány and D. G. Larman, Convex bodies, economic cap coverings, random polytopes, Mathematika, **35** no. 2, 274-291, 1988.

[17] I. Bárány and M. Reitzner, Central limit theorem for random polytopes in convex polytopes, manuscript, 2005.

[18] W. Blaschke, Über affine Geometrie XI: Lösung des "Vierpunktproblems" von Sylvester aus der Theorie der geometrischen Wahrscheinlichkeiten, Ber. Verh. Sächs. Ges. Wiss. Leipzig, Math.-Phys. Kl. **69**, 436-453, 1917. [Reprinted in: *Wilhelm Blaschke. Gesammelte Werke 3 (Konvexgeometrie)*, ed. W. Burau et al., 284-301, Thales, Essen, 1985.]

[19] K. H. Borgwardt, *The Simplex Method: a probabilistic analysis, algorithms and combinatorics*, Study and Research Texts 1., Springer-Verlag, Berlin, 1987.

[20] C. Buchta, Zufallspolygone in konvexen Vielecken, J. Reine Angew. Math. **347**, 212-220, 1984.

[21] C. Buchta, Stochastische Appproximation konvexer Polygone, Z. Wahrscheinlichkeitsth. Verw. Geb. **67**, 283-304, 1984.

[22] C. Buchta, Zufällige Polyeder - Eine Übersicht, *Zahlentheoretische Analysis.* ed. E. Hlawka, Lecture Notes in Matehamtics, Vol.1114, Springer, Berlin, 1-13, 1985.

[23] C. Buchta, An identity relating moments of functionals of convex hulls, unpublished manuscript, 2002.

[24] C. Buchta and J. Müller, Random Polytopes in a ball, J. Appl. Probab. **21**, 753-762, 1984.

[25] C. Buchta, J. Müller and R. F. Tichy, Stochastical approximation of convex bodies, Math. Ann. **271**, 225-235, 1985.

[26] C. Buchta and M. Reitzner, Equiaffine inner parallel curves of a plane convex body and the convex hulls of randomly chosen points, Probab. Theory Related Fields **108**, 385-415, 1997.

[27] C. Buchta and M. Reitzner, The convex hull of random points in a tetrahedron: Solution of Blaschke's problem and more general results, J. Reine Angew. Math. **536**, 1-29, 2001.

[28] A. J. Cabo and P. Groeneboom, Limit theorems for functionals of convex hulls, Probab. Theory Related Fields **100**, 31-55, 1994.

[29] F. Chung and L. Lu, Concentration inequalities and martingale inequalities - a survey, Internet Math, to appear, 2006.

[30] L. Dalla and D. G. Larman, Volumes of a random polytope in a convex set, *Applied Geometry and Discrete Mathematics: The Victor Klee Festschrift*, ed. P. Gritzmann and B. Sturmfels, Amer. Math. Soc., Providence, RI, 175-180, 1991.

[31] H. Edelsbrunner, *Algorithms in Combinatorial Geometry*, Springer-Verlag, Heidelberg, 1987.

[32] B. Efron, The convex hull of a random set of points. Biometrika, **52**, 331-343, 1965.

[33] G. Ewald, L. G. Larman and C. A. Rogers, The directions of the line segments and of the r-dimensional balls on the boundary of a convex body in Euclidean space, Mathematika **17**, 1-20, 1970.

[34] B. Efron and C. Stein, The jackknife estimate of variance, Ann. Statist. **9**, 586-596, 1981.

[35] A. A. Giannopoulos, On the mean value of the area of a random polygon in a plane convex body, Mathematika **39**, 279-290, 1992.

[36] H. Groemer, On some mean values associated with a randomly selected simplex in a convex set, Pacific J. Math. **45**, 525-533, 1973.

[37] H. Groemer, On the mean value of the volume of a random polytope in a convex set, Arch. Math. **25**, 86-90, 1974.

[38] P. Groeneboom, Limit theorems for convex hulls, Probability Theory Related Fields **79**, 327-368.

[39] P. M. Gruber, In most cases approximation is irregular, Rend. Sem. Mat. Univ. Politec. Torino **41**, 19-33, 1983.

[40] P. M. Gruber, Aspects of approximation of convex bodies, *Handbook of Convex Geometry, Vol A*, North-Holland/Elsevier Science Publishers, 319-346, 1993.

[41] P. M. Gruber, Expectation of random polytopes, Manuscripta Math. **91**, 393-419, 1996.

[42] P. Gritzmann and V. Klee, Mathematical programming and convex geometry, *Handbook of Convex Geometry, Vol A*, North-Holland/Elsevier Science Publishers, 627-674, 1993.

[43] T. Hsing, On the asymptotic distribution of the area outside a random convex hll in a disk, Ann. Appl. Probab. **4**, 478-493, 1994.

[44] J. Hüsler, On the convex hull of dependent random vectors, Rend. Circ. Mat. Palermo II Ser. Suppl. **41**, 109-117, 1997.

[45] D. Haussler and E. Welzl, ϵ-nets and simplex range queries, Discrete and Computational Geometry, **2** no.2, 127-151, 1987.

[46] J. F. C. Kingman, Random secants of a convex body, J. Appl. Probab. **6**, 660-672, 1969.

[47] K. -H. Küfer, On the approximation of a ball by random polytopes, Adv. in Appl. Probab. **26**, 876-892, 1994.

[48] J. H. Kim and V. H. Vu, Concentration of multi-variate polynomials and its applications, Combinatorica, **20** (3) 417-434, 2000.

[49] J. Matoušek, *Lectures on Discrete Geometry*, Graduate Texts in Mathematics, 212. Springer-Verlag, New York, 2002.

[50] J. Müller, Approximation of a ball by random polytopes, J. Approx. Theory **63**, 198-209, 1990.

[51] J. Müller, On the mean width of random polytopes, J. Approx. Theory **63**, 198-209, 1990.

[52] K. Leichtweiss, Affine geometry of convex bodies. Barth, Heidelberg, 1998.

[53] F. Preparata and I. Shamos, *Computational Geometry: an introduction*, Texts and Monographs in Computer Science, Springer-Verlag, New York, 1985.

[54] M. Reitzner, Random polytopes and the Efron-Stein Jacknife inequality, Ann. Probab. **31**, 2136-2166, 2003.

[55] M. Reitzner, The combinatorial structure of random polytopes. Adv. Math., **191**, 178-208, 2005.

[56] M. Reitzner, Central Limit Theorems for Random Polytopes, Probab. Theory Relat. Fields, to appear.

[57] M. Reitzner, Random points on the boundary of smooth convex bodies, Transactions of the American Mathematical Society, **354** No. 6, 2243-2278, 2002.

[58] A. Rosenfeld, *Picture Processing by Computers*, Academic Press, New York, 1969.

[59] A. Rényi and R. Sulanke, Über die konvexe Hülle von n zufällig gewählten Punkten. Z. Wahrscheinlichkeitsth. Verw. Geb., **2**, 75-84, 1963.

[60] A. Rényi and R. Sulanke, Über die konvexe Hülle von n zufällig gewählten Punkten. II, Z. Wahrscheinlichkeitsth. Verw. Geb., **3**, 138-147, 1964.

[61] Y. Rinott, On normal approximation rates for certain sums of random vairables, J. Comput. Appl. Math. 55, 135-143, 1994.

[62] R. Schneider, Random approximation of convex sets, J. Microscopy **151**, 211-227,1988.

[63] R. Schneider, Discrete aspects of stochastic geometry, *Handbook of Discrete and Computational Geometry*, 255-278, CRC Press, 2004.

[64] R. Schneider, Approximation of convex bodies by random polytopes, Aequationes Math. **32**, 304-310, 1987.

[65] C. Schütt, Random polytopes and affine surface areas, Mathematische Nachrichten **170**, 227-249, 1994.

[66] C. Schütt and E. Werner, Polytopes with vertices chosen randomly from the boundary of a convex body, *Geometric Aspects of Functional Analysis 2001-2002*, Springer Lecture Notes in Mathematics 1807, Spring-Verlag, 241-422, 2003.

[67] J. J. Sylvester, Question 1491, Educational Times, London, April, 1864.

[68] V. H. Vu, Sharp concentration of random polytopes, to appear, GAFA, 2005.

[69] V. H. Vu, Central limit theorems for random polytopes in a smooth convex set, Advances in Mathematics, 2005.

[70] W. Weil and J. Wieacker, Stochastic geometry, Handbook of convex geometry, Vol. B, North-Holland, Amsterdam, 1391-1438, 1993.

[71] J. Wieacker, Einige Probleme der polyedrischen Approximation, Diplomarbeit, Univ. Freiburg, 1978.