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Chain enumeration, partition lattices and polynomials with only real roots

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Abstract. The coefficients of the chain polynomial of a finite poset enumerate chains in the poset by their number of elements. The chain polynomials of the partition lattices and their standard type B analogues are shown to have only real roots. The real-rootedness of the chain polynomial is conjectured for all geometric lattices and is shown to be preserved by the pyramid and the prism operations on Cohen–Macaulay posets. As a result, new families of convex polytopes whose barycentric subdivisions have real-rooted f-polynomials are presented. An application to the face enumeration of the second barycentric subdivision of the boundary complex of the simplex is also included.

Keywords. Chain polynomial, geometric lattice, partition lattice, real-rooted polynomial, flag h-vector, convex polytope, barycentric subdivision

Mathematics Subject Classifications. 05A05, 05A18, 05E45, 06A07, 26C10

1. Introduction

The chain polynomial of a finite partially ordered set (poset, for short) $\mathcal{L}$ is defined as $p_{\mathcal{L}}(x) := \sum_{k\geqslant 0} c_k(\mathcal{L})x^k$, where $c_k(\mathcal{L})$ stands for the number of k-element chains in $\mathcal{L}$ (thus, $p_{\mathcal{L}}(x)$ is the f-polynomial of the order complex of $\mathcal{L}$). The general question which motivates this paper is as follows.

Question 1.1. For which finite posets does the chain polynomial have only real roots?

This question has been studied and proven to be very interesting and challenging for specific classes of posets. For finite distributive lattices, it is known to be equivalent to the poset conjecture for natural labelings, posed in the seventies by Neggers [Neg78] (see also [Sta89,

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Conjecture 1]) and finally disproved by Stembridge [Ste07], after counterexamples to a more general conjecture were found by Brändén [Brä04]. For face lattices of convex polytopes, it was raised by Brenti and Welker in their study of f-vectors of barycentric subdivisions [BW08]. The question is currently open in this case and is known to have an affirmative answer for face lattices of simplicial (equivalently, simple) polytopes [BW08], cubical polytopes [Ath21] and, by the results of [Gal05], polytopes of dimension at most five. Another notable result [Sta98, Corollary 2.9] asserts that the chain polynomial $p_{\mathcal{L}}(x)$ has only real roots for every poset $\mathcal{L}$ which does not contain the disjoint union of a three-element chain and a one-element chain as an induced subposet.

This paper partly aims to show that Question 1.1 is very interesting for other classes of posets as well, especially for geometric lattices (the lattices of flats of matroids). The following statement is the main conjecture posed in this paper (a more precise conjecture appears in Section 5).

Conjecture 1.2. The chain polynomial $p_{\mathcal{L}}(x)$ has only real roots for every geometric lattice $\mathcal{L}$.

Our first main result verifies this conjecture for some important geometric lattices, such as the subspace lattice $\mathcal{L}_n(q)$ of all linear subspaces of an n-dimensional vector space over the field with q elements, the partition lattice Π_n [Sta12, Section 3.1] and its standard type B analogue Π_n^B [Wac07, Section 1.3]. The statement about uniform matroids follows from the main result of [BW08] and is included for the sake of completeness.

Theorem 1.3. *Conjecture 1.2 is true for*

(a) the subspace lattices $\mathcal{L}_n(q)$,

(b) the partition lattices Π_n and Π_n^B,

(c) the lattices of flats of near-pencils and uniform matroids.

Conjecture 1.2 has also been verified computationally for all geometric lattices with at most nine atoms.

Our second main result gives constructions of posets which preserve the real-rootedness of the chain polynomial. More specifically, there are natural operations Pyr and Prism on posets (called pyramid and prism, see Section 2), such that if $\mathcal{L}$ is the face lattice of a convex polytope $\mathcal{P}$, then $\mathrm{Pyr}(\mathcal{L})$ and $\mathrm{Prism}(\mathcal{L})$ are the face lattices of the pyramid and the prism over $\mathcal{P}$, respectively. Among other applications, the following statement implies the existence of large families of nonsimplicial, nonsimple and noncubical polytopes in any dimension, the face lattices of which have real-rooted chain polynomials.

Theorem 1.4. *If the chain polynomial of a bounded Cohen–Macaulay poset $\mathcal{L}$ has only real roots, then so do the chain polynomials of the pyramid and the prism over $\mathcal{L}$.*

The content, methods and structure of this paper may be described as follows. Section 2 provides basic definitions and useful background from algebraic, enumerative and geometric combinatorics (mainly on chain enumeration in posets) and the theory of real-rooted polynomials. Sections 3 and 4 prove parts (a) and (b) of Theorem 1.3, respectively. The proofs depend on specific combinatorial features of these posets and do not seem to extend easily to other geometric lattices. They proceed by exploiting explicit combinatorial interpretations of the flag

h-vectors and the h-polynomials of the order complexes of the posets in question. To the best of our knowledge, such combinatorial interpretations have not appeared in the literature before for the partition lattices Π_n and Π_n^B. Part (a) of Theorem 1.3 is easier to prove and serves as an introductory example.

Section 5 discusses Question 1.1 for some general classes of Cohen–Macaulay posets, including that of geometric lattices, proves Theorem 1.4 and deduces from that part (c) of Theorem 1.3 (see Proposition 5.8). Proposition 5.3 suggests that the real-rootedness of chain polynomials of geometric lattices should perhaps be studied in connection to that of chain polynomials of face lattices of zonotopes and oriented matroids. The proofs in Section 5 rely heavily on results of Ehrenborg and Readdy [ER98] on the chain enumeration of pyramids and prisms and of Billera, Ehrenborg and Readdy [BER97] on the chain enumeration of big face lattices of oriented matroids (reviewed in Section 2). Section 6 applies Proposition 5.3 to give an unexpected combinatorial interpretation of the h-polynomial of the second barycentric subdivision of the boundary complex of a simplex and of its associated γ-polynomial, thus solving a problem posed in [Ath18].

As noted already, the chain polynomial $p_{\mathcal{L}}(x)$ coincides with the f-polynomial of the order complex $\Delta(\mathcal{L})$ of a poset $\mathcal{L}$. The results of Sections 3, 4 and 5 are phrased in terms of the h-polynomial of $\Delta(\mathcal{L})$ instead, which is more natural from an algebraic combinatorics point of view for the classes of posets we are interested in.

2. Preliminaries

This section includes preliminaries on notation and definitions and discusses some of the key tools and results from chain enumeration and the theory of real-rooted polynomials which will be used in this paper. We assume familiarity with main objects of study in algebraic, enumerative and geometric combinatorics, such as posets, matroids, simplicial complexes, convex polytopes and oriented matroids; standard references are [Bjö92, BLVS+93, Oxl11, Sta96, Sta12, Wac07, Zie95]. Any undefined terminology can be found there.

We will denote by $\mathfrak{S}_n$ the symmetric group of permutations of $[n] := \{1, 2, \dots, n\}$ and by $|S|$ the cardinality of a finite set S.

2.1. Simplicial complexes and face enumeration

Given an $(n-1)$-dimensional (finite, abstract) simplicial complex Δ, the *f-polynomial* and the *h-polynomial* are defined as

$$f(\Delta, x) = \sum_{i=0}^{n} f_{i-1}(\Delta) x^i$$

$$h(\Delta, x) = (1-x)^n f(\Delta, \frac{x}{1-x}) = \sum_{i=0}^{n} f_{i-1}(\Delta)\, x^i (1-x)^{n-i},$$

where $f_{i-1}(\Delta)$ stands for the number of $(i-1)$-dimensional faces of Δ. The h-polynomial has nonnegative coefficients for all Cohen–Macaulay simplicial complexes [Sta96, Section II.3] (this property is meant to be considered here over the field $\mathbb{Q}$ of rational numbers) and, in particular, for all simplicial complexes of interest in this paper. Morever, $h(\Delta, x)$ can be considered as an x-analogue of $f_{n-1}(\Delta)$, to which its coefficients sum up. Since $f(\Delta, x)$ has only real roots if and only if so does $h(\Delta, x)$, the former can be replaced by the latter as far as real-rootedness is concerned.

2.2. Order complexes and chain enumeration

We will be mostly interested in order complexes of posets. The *order complex* of a finite poset $(\mathcal{L}, \preceq)$ is defined [Sta12, Section 3.8] as the simplicial complex $\Delta(\mathcal{L})$ which consists of all chains in $\mathcal{L}$. We have $f(\Delta(\mathcal{L}), x) = p_{\mathcal{L}}(x)$, where $p_{\mathcal{L}}(x)$ is the chain polynomial defined in the introduction. To simplify notation, we set $h_{\mathcal{L}}(x) = h(\Delta(\mathcal{L}), x)$ throughout this paper. Thus, $p_{\mathcal{L}}(x)$ and $h_{\mathcal{L}}(x)$ are related by the equation

$$h_{\mathcal{L}}(x) = (1-x)^n \, p_{\mathcal{L}}\left(\frac{x}{1-x}\right),$$

where n is the largest cardinality of a chain in $\mathcal{L}$. The polynomial $h_{\mathcal{L}}(x)$ is an x-analogue of the number of n-element chains of $\mathcal{L}$, which are exactly the $(n-1)$-dimensional faces of $\Delta(\mathcal{L})$.

Suppose now that $\mathcal{L}$ has a minimum element $\hat{0}$, a maximum element $\hat{1}$ and that it is graded of rank n, say with rank function $\rho : P \to \{0, 1, \dots, n\}$. Thus, all maximal chains in $\mathcal{L}$ have exactly $n+1$ elements. Following [Sta12, Section 3.13], for $S \subseteq [n-1]$ we denote by $\alpha_{\mathcal{L}}(S)$ the number of maximal chains of the subposet $\{t \in \mathcal{L} : \rho(t) \in S\} \cup \{\hat{0}, \hat{1}\}$ of $\mathcal{L}$ and set

$$\beta_{\mathcal{L}}(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \, \alpha_{\mathcal{L}}(T).$$

The numbers $\beta_{\mathcal{L}}(S)$ for $S \subseteq [n-1]$ are the entries of the *flag h-vector* of $\mathcal{L}$. They are nonnegative if $\mathcal{L}$ is Cohen–Macaulay (see, for instance, [Sta96, Theorem 4.4]) and refine the coefficients of $h_{\mathcal{L}}(x)$, in the sense that

$$h_{\mathcal{L}}(x) = \sum_{S \subseteq [n-1]} \beta_{\mathcal{L}}(S) x^{|S|}. \tag{2.1}$$

We also set $\overline{\mathcal{L}} = \mathcal{L} \smallsetminus \{\hat{0}, \hat{1}\}$ and recall that $h_{\mathcal{L}}(x) = h_{\mathcal{L} \smallsetminus \{\hat{0}\}}(x) = h_{\mathcal{L} \smallsetminus \{\hat{1}\}}(x) = h_{\overline{\mathcal{L}}}(x)$.

The theory of edge labelings can provide useful combinatorial interpretations of the numbers $\beta_{\mathcal{L}}(S)$. We denote by $\mathcal{E}(\mathcal{L})$ and $\mathcal{M}(\mathcal{L})$ the set of covering relations and maximal chains of $\mathcal{L}$, respectively. An edge labeling of $\mathcal{L}$ for us will be a map $\lambda : \mathcal{E}(\mathcal{L}) \to \mathbb{Z}$. Given a maximal chain $C : \hat{0} = c_0 \prec c_1 \prec \cdots \prec c_n = \hat{1}$ of $\mathcal{L}$, such a map induces the sequence of labels

$$\lambda(C) = (\lambda(c_0, c_1), \lambda(c_1, c_2), \dots, \lambda(c_{n-1}, c_n)).$$

We denote by $\mathrm{Des}_\lambda(C)$ the set of indices $i \in [n-1]$ for which $\lambda(c_{i-1}, c_i) \geqslant \lambda(c_i, c_{i+1})$ and call C *strictly increasing* with respect to λ if no such index exists. By restricting λ, all these definitions apply to the closed intervals in $\mathcal{L}$. We say that λ is a *strict R-labeling* if every closed interval in $\mathcal{L}$ has a unique strictly increasing maximal chain with respect to λ. Under this assumption on λ, for every $S \subseteq [n-1]$, $\beta_{\mathcal{L}}(S)$ is equal to the number of maximal chains $C \in \mathcal{M}(\mathcal{L})$ such that $\mathrm{Des}_\lambda(C) = S$; see [Sta12, Section 3.14] [Wac07, Section 3.2] for examples and more information on R-labelings.

The flag h-vector of $\mathcal{L}$ is nicely encoded by the $\mathbf{ab}$-index $\Psi_{\mathcal{L}}$. Given noncommuting variables $\mathbf{a}$ and $\mathbf{b}$, this may be defined as

$$\Psi_{\mathcal{L}} = \Psi_{\mathcal{L}}(\mathbf{a}, \mathbf{b}) = \sum_{S \subseteq [n-1]} \beta_{\mathcal{L}}(S) \mathbf{u}_S,$$

where $\mathbf{u}_S = u_1 u_2 \cdots u_{n-1}$, with

$$u_i = \begin{cases} \mathbf{b}, & \text{if } i \in S \\ \mathbf{a}, & \text{if } i \notin S, \end{cases}$$

is the $\mathbf{ab}$-monomial associated to S in the standard way. We note that Equation (2.1) may be rewritten as $h_{\mathcal{L}}(x) = \Psi_{\mathcal{L}}(1, x)$.

2.3. Face lattices of polytopes

We will denote by $\mathcal{F}(\mathcal{P})$ the face lattice of a polytope $\mathcal{P}$. The *barycentric subdivisions* $\mathrm{sd}(\mathcal{P})$ and $\mathrm{sd}(\partial\mathcal{P})$ of $\mathcal{P}$ and its boundary complex $\partial\mathcal{P}$ are defined as the order complexes $\Delta(\mathcal{L} \smallsetminus \{\hat{0}\})$ and $\Delta(\overline{\mathcal{L}})$, respectively, where $\mathcal{L} = \mathcal{F}(\mathcal{P})$. In particular, the f-polynomials of $\mathrm{sd}(\mathcal{P})$ and $\mathrm{sd}(\partial\mathcal{P})$ are the chain polynomials of $\mathcal{L} \smallsetminus \{\hat{0}\}$ and $\overline{\mathcal{L}}$ and their real-rootedness is equivalent to that of $h_{\mathcal{L}}(x) = h_{\mathcal{L} \smallsetminus \{\hat{0}\}}(x) = h(\mathrm{sd}(\mathcal{P}), x) = h_{\overline{\mathcal{L}}}(x) = h(\mathrm{sd}(\partial\mathcal{P}), x)$.

There is another lattice associated to any zonotope $\mathcal{Z}$, namely the geometric lattice of flats of the matroid defined by the generators of $\mathcal{Z}$. This lattice, say $\mathcal{L}(\mathcal{Z})$, is isomorphic to the intersection lattice of the linear hyperplane arrangement $\mathcal{H}_{\mathcal{Z}}$ corresponding to $\mathcal{Z}$; the face poset $\mathcal{F}(\mathcal{Z}) \smallsetminus \{\hat{0}\}$ is anti-isomorphic to the face poset of $\mathcal{H}_{\mathcal{Z}}$ [Zie95, Section 7.3]. The main result of [BER97] expresses the flag h-vector of $\mathcal{F}(\mathcal{Z})$ in terms of that of $\mathcal{L}(\mathcal{Z})$. Consider the linear function $\omega : \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \to \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ defined as follows: if v is any $\mathbf{ab}$-word, then $\omega(v)$ is obtained from v by first replacing each occurrence of $\mathbf{ab}$ with $2(\mathbf{ab} + \mathbf{ba})$ and then replacing each of the remaining letters with $\mathbf{a} + \mathbf{b}$.

Theorem 2.1. ([BER97, Corollary 3.2]) *We have* $\Psi_{\mathcal{F}(\mathcal{Z})} = \omega(\mathbf{a} \cdot \Psi_{\mathcal{L}})$ *for every zonotope* $\mathcal{Z}$, *where* $\mathcal{L}$ *is the lattice of flats of the matroid associated to* $\mathcal{Z}$.

Given a bounded, finite poset $\mathcal{L}$, the *pyramid* over $\mathcal{L}$ is defined as $\mathrm{Pyr}(\mathcal{L}) = \mathcal{L} \times \mathcal{L}_1$, where $\mathcal{L}_1$ is the 2-element chain. The *prism* over $\mathcal{L}$, denoted $\mathrm{Prism}(\mathcal{L})$, is defined as the poset obtained by adding a minimum element to $(\mathcal{L} \smallsetminus \{\hat{0}\}) \times (\mathcal{L}_2 \smallsetminus \{\hat{0}\})$, where $\mathcal{L}_2 = \mathcal{L}_1 \times \mathcal{L}_1$. Then, for every polytope $\mathcal{P}$, the posets $\mathrm{Pyr}(\mathcal{F}(\mathcal{P}))$ and $\mathrm{Prism}(\mathcal{F}(\mathcal{P}))$ are isomorphic to the face lattices of the pyramid and the prism over $\mathcal{P}$, respectively. Following [ER98], we consider the linear derivation $D : \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \to \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ defined by setting $D(\mathbf{a}) = D(\mathbf{b}) = \mathbf{ab} + \mathbf{ba}$.

Theorem 2.2. ([ER98, Theorem 4.4]) *We have*

$$2\,\Psi_{\mathrm{Pyr}(\mathcal{L})} = \Psi_{\mathcal{L}} \cdot (\mathbf{a} + \mathbf{b}) + (\mathbf{a} + \mathbf{b}) \cdot \Psi_{\mathcal{L}} + D(\Psi_{\mathcal{L}}), \tag{2.2}$$

$$\Psi_{\mathrm{Prism}(\mathcal{L})} = \Psi_{\mathcal{L}} \cdot (\mathbf{a} + \mathbf{b}) + D(\Psi_{\mathcal{L}}) \tag{2.3}$$

for every bounded, graded finite poset $\mathcal{L}$.

2.4. Real-rooted polynomials

A polynomial $f(x)$ with real coefficients is called *real-rooted* if every root of $f(x)$ is real, or $f(x) \equiv 0$.

A real-rooted polynomial $f(x)$, with roots $\alpha_1 \geqslant \alpha_2 \geqslant \cdots$, is said to *interlace* a real-rooted polynomial $g(x)$, with roots $\beta_1 \geqslant \beta_2 \geqslant \cdots$, if

$$\cdots \leqslant \alpha_2 \leqslant \beta_2 \leqslant \alpha_1 \leqslant \beta_1.$$

By convention, the zero polynomial interlaces and is interlaced by every real-rooted polynomial. A sequence $(f_0(x), f_1(x), \dots, f_m(x))$ of real-rooted polynomials is called *interlacing* if $f_i(x)$ interlaces $f_j(x)$ for $0 \leqslant i < j \leqslant m$. The following standard lemma (see, for instance, [Brä15, Section 7.8]) will be applied several times in this paper.

Lemma 2.3. *Let $(f_0(x), f_1(x), \dots, f_m(x))$ be an interlacing sequence of real-rooted polynomials with positive leading coefficients.*

(a) Every nonnegative linear combination $f(x)$ of $f_0(x), f_1(x), \dots, f_m(x)$ is real-rooted. Moreover, $f(x)$ interlaces $f_m(x)$ and it is interlaced by $f_0(x)$.

(b) The sequence $(g_0(x), g_1(x), \dots, g_{m+1}(x))$ defined by

$$g_k(x) = x \sum_{i=0}^{k-1} f_i(x) \, + \, \sum_{i=k}^{m} f_i(x)$$

for $k \in \{0, 1, \dots, m+1\}$ is also interlacing.

A polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ with real coefficients is called *symmetric*, with center of symmetry $n/2$, if $a_i = a_{n-i}$ for all $0 \leqslant i \leqslant n$. Then, $f(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i x^i (1+x)^{n-2i}$ for some uniquely defined real numbers $\gamma_0, \gamma_1, \dots, \gamma_{\lfloor n/2 \rfloor}$ and $\gamma(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i x^i$ is the γ*-polynomial* associated to $f(x)$. The latter is called γ*-positive* if $\gamma_i \geqslant 0$ for every i; see [Ath18] [Brä15, Section 7.3] for more information on this concept.

3. Subspace lattices

This section confirms Theorem 1.3 for subspace lattices. The proof essentially follows by combining [Sta12, Theorem 3.13.3] with [SV15, Theorem 5.4] but serves as a paradigm for the proofs of the corresponding statements for the lattices Π_n and Π_n^B in the following section, which are more involved.

Given a positive integer n and a prime power q, let $V_n(q)$ be an n-dimensional vector space over the field with q elements. The *subset lattice* $\mathcal{L}_n$, known as the Boolean algebra of rank n, and the *subspace lattice* $\mathcal{L}_n(q)$ are defined [Sta12, Example 3.1.1] as the set of all subsets of the set $[n]$ and as the set of all linear subspaces of $V_n(q)$, respectively, partially ordered by inclusion. The posets $\mathcal{L}_n$ and $\mathcal{L}_n(q)$ are geometric lattices of rank n, the latter being considered as a q-analogue of the former. We recall that a *descent* of a permutation $w \in \mathfrak{S}_n$ is an index $i \in [n-1]$ such that $w(i) > w(i+1)$ and denote by $\mathrm{des}(w)$ and $\mathrm{Des}(w)$ the number and the set of all descents of w, respectively.

The order complex $\Delta(\overline{\mathcal{L}}_n)$ is isomorphic to the first barycentric subdivision of the boundary complex of the $(n-1)$-dimensional simplex and its h-polynomial $h_{\mathcal{L}_n}(x) = h(\Delta(\overline{\mathcal{L}}_n), x)$ is known [Pet15, Theorem 9.1] to be equal to the nth Eulerian polynomial

$$A_n(x) := \sum_{w \in \mathfrak{S}_n} x^{\mathrm{des}(w)}.$$

The latter is well known [Brä15, Section 7.8.1] to have only real roots for every n. As already discussed, the following statement is equivalent to part (a) of Theorem 1.3.

Proposition 3.1. *The polynomial $h_{\mathcal{L}}(x)$ is real-rooted for every subspace lattice $\mathcal{L} = \mathcal{L}_n(q)$.*

Proof. Setting $\mathcal{L} := \mathcal{L}_n(q)$, we have [Sta12, Theorem 3.13.3]

$$\beta_{\mathcal{L}}(S) = \sum_{w \in \mathfrak{S}_n :\, \mathrm{Des}(w) = S} q^{\mathrm{inv}(w)},$$

where $\mathrm{inv}(w)$ is the number of pairs of indices $1 \leqslant i < j \leqslant n$ for which $w(i) > w(j)$. This formula and Equation (2.1) imply that $h_{\mathcal{L}}(x) = A_n(x;q)$, where

$$A_n(x;q) := \sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)} x^{\mathrm{des}(w)}$$

is one of the well studied q-analogues of $A_n(x)$ [Sta12, Section 3.19]. The fact that $A_n(x;q)$ is real-rooted for every positive real number q was shown in [SV15, Theorem 5.4]. To give a self-contained proof, we set

$$A_{n,k}(x;q) = \sum_{w \in \mathfrak{S}_n :\, w(n) = k} q^{\mathrm{inv}(w)} x^{\mathrm{des}(w)}$$

for $k \in [n]$ and note that $A_n(x;q) = A_{n+1,n+1}(x;q)$ for every n and that

$$A_{n+1,k}(x;q) = q^{n+1-k} \left(\sum_{i=1}^{k-1} A_{n,i}(x;q) \, + \, x \sum_{i=k}^{n} A_{n,i}(x;q) \right) \tag{3.1}$$

for $k \in [n+1]$. An application of Lemma 2.3 shows by induction on n that the sequence $(A_{n,n}(x;q), A_{n,n-1}(x;q), \ldots, A_{n,1}(x;q))$ is interlacing for all n and positive q. As a result, $A_n(x;q) = A_{n+1,n+1}(x;q)$ is real-rooted for all n and positive q. □

The previous argument also shows that $A_n(x;q)$ interlaces $A_{n+1}(x;q)$ for all n and all positive q. This follows from part (a) of Lemma 2.3 since, by (3.1), $A_n(x;q) = A_{n+1,n+1}(x;q)$ is a positive linear combination of the $A_{n,k}(x;q)$ for $1 \leqslant k \leqslant n$ and $A_{n,n}(x;q) = A_{n-1}(x;q)$. For the real-rootedness of other q-analogues of $A_n(x)$, see [SV15, Section 5].

4. Partition lattices

This section proves part (b) of Theorem 1.3. As in Section 3, we will show the equivalent statement that $h_{\Pi_n}(x)$ and $h_{\Pi_n^B}(x)$ are real-rooted for every $n \geqslant 1$.

4.1. The partition lattice of type *A*

We recall that Π_n consists of all partitions of the set $[n]$, partially ordered by reverse refinement. It is isomorphic to the intersection lattice of the Coxeter hyperplane arrangement of type A_{n-1} and, as such, it is a geometric lattice of rank $n-1$. We will first give an explicit combinatorial interpretation of the flag h-vector of Π_n and will deduce one for $h_{\Pi_n}(x)$. A recurrence for the entries of the former was found by Sundaram [Sun94, Proposition 2.16]; an additional formula appears as [Sun94, Proposition 2.18]. We consider the multiset

$$\mathcal{A}_n := \{1\} \times \{1,1,2\} \times \{1,1,1,2,2,3\} \times \cdots \times \{1,1,\ldots,1,\ldots,n-2,n-2,n-1\},$$

e.g., $\mathcal{A}_3 = \{(1,1),(1,1),(1,2)\}$. We define the *descent set* of $\sigma = (\sigma_1,\sigma_2,\ldots,\sigma_{n-1}) \in \mathcal{A}_n$ as $\mathrm{Des}(\sigma) = \{i \in [n-2] : \sigma_i \geqslant \sigma_{i+1}\}$ and denote its cardinality by $\mathrm{des}(\sigma)$. The multiset $\mathcal{A}_n$ has $\prod_{k=2}^{n} \binom{k}{2} = \frac{n!(n-1)!}{2^{n-1}}$ elements, as many as the maximal chains of Π_n. Moreover, one element of $\mathcal{A}_n$ has empty descent set and $(n-1)!$ of them have descent set equal to $[n-2]$. These facts agree with the following statement.

Proposition 4.1. *For every $n \geqslant 2$ and every $S \subseteq [n-2]$, the number $\beta_{\Pi_n}(S)$ is equal to the number of elements of the multiset $\mathcal{A}_n$ with descent set equal to $n-1-S := \{n-1-x : x \in S\}$. In particular,*

$$h_{\Pi_n}(x) = \sum_{\sigma \in \mathcal{A}_n} x^{\mathrm{des}(\sigma)} \tag{4.1}$$

for every $n \geqslant 2$.

To prepare for the proof, we recall [Wac07, Section 3.2.2] the following edge labeling of Π_n, due to Gessel. For a covering relation $(x,y) \in \mathcal{E}(\Pi_n)$ we define $\lambda(x,y)$ as the maximum of $\min(B)$ and $\min(B')$, where y is obtained from x by merging the blocks B and B' of x. The labeling $\lambda : \mathcal{E}(\Pi_n) \to \{2,3,\ldots,n\}$ is a strict R-labeling (even a strict EL-labeling). In particular, $\beta_{\Pi_n}(S)$ is equal to the number of maximal chains C of Π_n such that $\mathrm{Des}_\lambda(C) = S$, for every $S \subseteq [n-2]$.

Proof of Proposition 4.1. We consider the multiset

$$\mathcal{A}_n^* := \{2,3,3,4,4,4,\ldots,n,n,\ldots,n\} \times \cdots \times \{2,3,3,4,4,4\} \times \{2,3,3\} \times \{2\}$$

and set $\mathrm{Des}^*(\sigma) = \{i \in [n-2] : \sigma_i > \sigma_{i+1}\}$ for $\sigma = (\sigma_1,\sigma_2,\ldots,\sigma_{n-1}) \in \mathcal{A}_n^*$. The bijection $\mathcal{A}_n \mapsto \mathcal{A}_n^*$ defined by

$$(\sigma_1,\sigma_2,\ldots,\sigma_{n-1}) \ \mapsto \ (n+1-\sigma_{n-1}, n-\sigma_{n-2},\ldots,3-\sigma_1)$$

shows that for every $S \subseteq [n-2]$, the number of elements $\sigma \in \mathcal{A}_n^*$ with $\mathrm{Des}^*(\sigma) = S$ is equal to the number of elements $\sigma \in \mathcal{A}_n$ with $\mathrm{Des}(\sigma) = n-1-S$. Thus, we need to show that the former is equal to $\beta_{\Pi_n}(S)$.

For that, we employ Gessel's labeling $\lambda : \mathcal{E}(\Pi_n) \to \{2, 3, \dots, n\}$. Given a covering relation $(x, y) \in \mathcal{E}(\Pi_n)$, we list the blocks of $x = \{B_1, B_2, \dots, B_k\}$ so that $\min(B_1) < \min(B_2) < \cdots < \min(B_k)$ and set $\varphi(x, y) = (i, j)$ and $\psi(x, y) = j$, where $i < j$ and y is obtained from x by merging B_i with B_j. For a maximal chain $C : \hat{0} = c_1 \prec c_2 \prec \cdots \prec c_n = \hat{1}$ of Π_n we set

$$\begin{aligned} \widetilde{\varphi}(C) &= (\varphi(c_1, c_2), \varphi(c_2, c_3), \dots, \varphi(c_{n-1}, c_n)), \\ \widetilde{\psi}(C) &= (\psi(c_1, c_2), \psi(c_2, c_3), \dots, \psi(c_{n-1}, c_n)) \end{aligned}$$

and consider the resulting maps

$$\begin{aligned} \widetilde{\varphi} : \ \mathcal{M}(\Pi_n) &\to \binom{[n]}{2} \times \binom{[n-1]}{2} \times \cdots \times \binom{[2]}{2}, \\ \widetilde{\psi} : \ \mathcal{M}(\Pi_n) &\to \{2, 3, \dots, n\} \times \{2, 3, \dots, n-1\} \times \cdots \times \{2\}, \end{aligned}$$

where $\binom{T}{2}$ stands for the set of 2-element subsets of T.

Clearly, $\widetilde{\varphi}$ is a bijection. We claim that $\mathrm{Des}_\lambda(C) = \mathrm{Des}^*(\widetilde{\psi}(C))$ for every $C \in \mathcal{M}(\Pi_n)$. This would imply that $\beta_{\Pi_n}(S)$, which is equal to the number of chains $C \in \mathcal{M}(\Pi_n)$ with $\mathrm{Des}_\lambda(C) = S$, is also equal to the number of chains $C \in \mathcal{M}(\Pi_n)$ with $\mathrm{Des}^*(\widetilde{\psi}(C)) = S$. Since $\widetilde{\varphi}$ is a bijection, the latter is equal to the number of $\sigma \in \mathcal{A}_n^*$ with $\mathrm{Des}^*(\sigma) = S$.

Thus, it remains to verify the claim. Equivalently, for $(x, y), (y, z) \in \mathcal{E}(\Pi_n)$, we need to verify that $\lambda(x, y) \geqslant \lambda(y, z) \Leftrightarrow \psi(x, y) > \psi(y, z)$. Indeed, let $x = \{B_1, B_2, \dots, B_k\}$ with $\min(B_1) < \min(B_2) < \cdots < \min(B_k)$ and suppose that y is obtained from x by merging B_i with B_j, where $i < j$. Then, $\lambda(x, y) = \min(B_j)$ and $\psi(x, y) = j$. Each one of the inequalities $\lambda(y, z) \leqslant \min(B_j)$ and $\psi(y, z) < j$ is equivalent to the statement that z is obtained from y by merging two blocks other than $B_{j+1}, \dots, B_k$ and the proof follows. □

For small values of n,

$$h_{\Pi_n}(x) = \begin{cases} 1, & \text{if } n = 1 \\ 1, & \text{if } n = 2 \\ 1 + 2x, & \text{if } n = 3 \\ 1 + 11x + 6x^2, & \text{if } n = 4 \\ 1 + 47x + 108x^2 + 24x^3, & \text{if } n = 5 \\ 1 + 197x + 1268x^2 + 1114x^3 + 120x^4, & \text{if } n = 6 \\ 1 + 870x + 13184x^2 + 29383x^3 + 12542x^4 + 720x^5, & \text{if } n = 7. \end{cases}$$

The following corollary proves and strengthens Theorem 1.3.

Corollary 4.2. *The polynomial $h_{\Pi_n}(x)$ is real-rooted and it interlaces $h_{\Pi_{n+1}}(x)$ for every $n \geqslant 1$.*

Proof. For $k \in [n-1]$ we set

$$h_{n,k}(x) = \sum_{\sigma \in \mathcal{A}_{n,k}} x^{\mathrm{des}(\sigma)},$$

where $\mathcal{A}_{n,k}$ is the multiset consisting of all words $(\sigma_1, \sigma_2, \dots, \sigma_{n-1}) \in \mathcal{A}_n$ with $\sigma_{n-1} = k$. By Proposition 4.1,

$$h_{\Pi_n}(x) = h_{n+1,n}(x) = \sum_{k=1}^{n-1} h_{n,k}(x)$$

for every $n \geqslant 2$ and

$$h_{n+1,k}(x) = (n+1-k) \left(\sum_{i=1}^{k-1} h_{n,i}(x) \, + \, x \sum_{i=k}^{n-1} h_{n,i}(x) \right)$$

for $k \in [n]$. Since $(h_{n+1,n}(x), h_{n+1,n-1}(x), \dots, h_{n+1,1}(x))$ is an interlacing sequence if and only if $(h_{n+1,n}(x), h_{n+1,n-1}(x)/2, \dots, h_{n+1,1}(x)/n)$ has the same property, an application of Lemma 2.3 shows by induction on n that $(h_{n,n-1}(x), h_{n,n-2}(x), \dots, h_{n,1}(x))$ is interlacing for every $n \geqslant 2$ and that $h_{\Pi_n}(x) = \sum_{k=1}^{n-1} h_{n,k}(x)$ is interlaced by $h_{n,n-1}(x) = h_{\Pi_{n-1}}(x)$. □

The following conjecture has been verified for $n \leqslant 20$.

Conjecture 4.3. The polynomial $h_{\Pi_n}(x)$ is interlaced by the Eulerian polynomial $A_{n-1}(x)$ for every $n \geqslant 2$.

4.2. The partition lattice of type *B*

We find it convenient to define Π_n^B as the set of all partitions π of $\{-n, -n+1, \dots, n\}$ with the following properties:

(i) $B \in \pi \Rightarrow (-B) \in \pi$,

(ii) if $\{i, -i\} \subseteq B$ for some $i \in [n]$ and some block $B \in \pi$, then $0 \in B$.

For example, $\{\{0, 2, -2\}, \{1, -3, 5\}, \{-1, 3, -5\}, \{4\}, \{-4\}\} \in \Pi_5^B$. The partial order on Π_n^B is again reverse refinement. The unique block of $\pi \in \Pi_n^B$ containing 0 is called the *zero block*. The poset Π_n^B is isomorphic to the intersection lattice of the Coxeter hyperplane arrangement of type B_n and therefore it is a geometric lattice of rank n.

The proof of Theorem 1.3 for the lattice Π_n^B parallels that for Π_n. One can easily verify that Π_n^B has exactly $(n!)^2$ maximal chains. We consider the multiset

$$\mathcal{B}_n := \{1\} \times \{1,1,1,2\} \times \{1,1,1,1,1,2,2,2,3\} \times \cdots \times \{1,1,\dots,1,\dots,n-1,n-1,n-1,n\},$$

where the kth factor has $2k - 2i + 1$ elements equal to i, for $1 \leqslant i \leqslant k$. We define the descent set $\mathrm{Des}(\sigma) \subseteq [n-1]$ and its cardinality $\mathrm{des}(\sigma)$ for $\sigma \in \mathcal{B}_n$ just as for elements of $\mathcal{A}_n$. We note that $\mathcal{B}_n$ has $(n!)^2$ elements, one of which has empty descent set and $(2n-1)!!$ of which have descent set equal to $[n-1]$. These facts agree with the following analogue of Proposition 4.1.

Proposition 4.4. *For every $n \geqslant 1$ and every $S \subseteq [n-1]$, the number $\beta_{\Pi_n^B}(S)$ is equal to the number of elements of the multiset $\mathcal{B}_n$ with descent set equal to $n - S$. In particular,*

$$h_{\Pi_n^B}(x) = \sum_{\sigma \in \mathcal{B}_n} x^{\mathrm{des}(\sigma)} \tag{4.2}$$

for every $n \geqslant 1$.

The proof uses the following analogue of Gessel's edge labeling for Π_n. Let us denote by $|B|$ the set of absolute values of the elements of a set $B \subseteq \mathbb{Z}$. Given a covering relation $(x, y) \in \mathcal{E}(\Pi_n^B)$, there exists a unique pair $\{B, B'\}$ of distinct blocks of x such that

(i) $\min(|B|) \in B$ and $\min(|B'|) \in B'$,

(ii) either B and B' are nonzero and $B \cup B'$ or $(-B) \cup B'$ is a nonzero block of y, or one of B and B' is the zero block of x and $B \cup B'$ is contained in the zero block of y.

We then define $\lambda(x, y)$ as the maximum of $\min(|B|)$ and $\min(|B'|)$ and leave to the reader to verify that the resulting map $\lambda : \mathcal{E}(\Pi_n^B) \to \{1, 2, \dots, n\}$ is a strict R-labeling. In particular, $\beta_{\Pi_n^B}(S)$ is equal to the number of maximal chains C of Π_n^B such that $\mathrm{Des}_\lambda(C) = S$, for every $S \subseteq [n-1]$.

Proof of Proposition 4.4. We adapt the proof of Proposition 4.1 as follows. We consider the multiset

$$\mathcal{B}_n^* := \{1, 2, 2, 2, 3, 3, 3, 3, 3, \dots, n, n, \dots, n\} \times \cdots \times \{1, 2, 2, 2, 3, 3, 3, 3, 3\} \times \{1, 2, 2, 2\} \times \{1\},$$

where the kth factor has $2i - 1$ elements equal to i, for $i \in [n-k+1]$. We set $\mathrm{Des}^*(\sigma) = \{i \in [n-1] : \sigma_i > \sigma_{i+1}\}$ for $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n) \in \mathcal{B}_n^*$. The bijection $\mathcal{B}_n \mapsto \mathcal{B}_n^*$ defined by

$$(\sigma_1, \sigma_2, \dots, \sigma_n) \;\mapsto\; (n + 1 - \sigma_n, n - \sigma_{n-1}, \dots, 2 - \sigma_1)$$

shows that, for every $S \subseteq [n-1]$, the number of elements $\sigma \in \mathcal{B}_n^*$ with $\mathrm{Des}^*(\sigma) = S$ is equal to the number of elements $\sigma \in \mathcal{B}_n$ with $\mathrm{Des}(\sigma) = n - S$. Thus, it suffices to show that the former equals $\beta_{\Pi_n^B}(S)$.

Let us call a nonzero block B of a partition $x \in \Pi_n^B$ *positive* if $\min(|B|) \in B$. Given a covering relation $(x, y) \in \mathcal{E}(\Pi_n^B)$, we let B_0 be the zero block of x and list its positive blocks, say $B_1, B_2, \dots, B_k$, so that $\min(|B_1|) < \min(|B_2|) < \cdots < \min(|B_k|)$. Then, some positive block B_j merges in y either with B_0, or with B_i or $-B_i$ for some $1 \leqslant i < j$. We set $\varphi(x, y) = (j, j)$ or (i, j) or (j, i) in these cases, respectively, and $\psi(x, y) = j$. For a maximal chain $C : \hat{0} = c_0 \prec c_1 \prec \cdots \prec c_n = \hat{1}$ of Π_n^B we set

$$\begin{aligned} \widetilde{\varphi}(C) &= (\varphi(c_0, c_1), \varphi(c_1, c_2), \dots, \varphi(c_{n-1}, c_n)), \\ \widetilde{\psi}(C) &= (\psi(c_0, c_1), \psi(c_1, c_2), \dots, \psi(c_{n-1}, c_n)) \end{aligned}$$

and consider the resulting maps

$$\begin{aligned} \widetilde{\varphi} :\; \mathcal{M}(\Pi_n^B) &\to [n]^2 \times [n-1]^2 \times \cdots \times [1]^2, \\ \widetilde{\psi} :\; \mathcal{M}(\Pi_n^B) &\to [n] \times [n-1] \times \cdots \times [1]. \end{aligned}$$

We note that $\widetilde{\varphi}$ is a bijection, verify that $\mathrm{Des}_\lambda(C) = \mathrm{Des}^*(\widetilde{\psi}(C))$ for every $C \in \mathcal{M}(\Pi_n^B)$, just as in the proof of Proposition 4.1, and conclude that $\beta_{\Pi_n^B}(S)$ is equal to the number of elements $\sigma \in \mathcal{B}_n^*$ with $\mathrm{Des}^*(\sigma) = S$ for every $S \subseteq [n-1]$. □

For small values of n,

$$h_{\Pi_n^B}(x) = \begin{cases} 1, & \text{if } n = 1 \\ 1 + 3x, & \text{if } n = 2 \\ 1 + 20x + 15x^2, & \text{if } n = 3 \\ 1 + 111x + 359x^2 + 105x^3, & \text{if } n = 4 \\ 1 + 642x + 5978x^2 + 6834x^3 + 945x^4, & \text{if } n = 5 \\ 1 + 4081x + 92476x^2 + 268236x^3 + 143211x^4 + 10395x^5, & \text{if } n = 6. \end{cases}$$

Corollary 4.5. *The polynomial $h_{\Pi_n^B}(x)$ is real-rooted and it interlaces $h_{\Pi_{n+1}^B}(x)$ for every $n \geqslant 1$.*

Proof. For $1 \leqslant k \leqslant n$ we set

$$h_{n,k}^B(x) = \sum_{\sigma \in \mathcal{B}_{n,k}} x^{\mathrm{des}(\sigma)},$$

where $\mathcal{B}_{n,k}$ is the multiset consisting of all words $(\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathcal{B}_n$ with $\sigma_n = k$. By Proposition 4.4,

$$h_{\Pi_n^B}(x) = h_{n+1,n+1}^B(x) = \sum_{k=1}^{n} h_{n,k}^B(x)$$

for every $n \geqslant 1$ and

$$h_{n+1,k}^B(x) = (2n - 2k + 3) \left(\sum_{i=1}^{k-1} h_{n,i}^B(x) \; + \; x \sum_{i=k}^{n} h_{n,i}^B(x) \right)$$

for $k \in [n+1]$. The result follows from these formulas as in the proof of Corollary 4.2. □

The intersection lattice of the Coxeter hyperplane arrangement of type D_n is isomorphic to the subposet Π_n^D of Π_n^B which consists of all elements of the latter with zero block not of the form $\{0, i, -i\}$ for any $i \in [n]$. The number of maximal chains of Π_n^D can be shown to equal $(n!)^2/2$ for every $n \geqslant 2$. For small values of n,

$$h_{\Pi_n^D}(x) = \begin{cases} 1 + x, & \text{if } n = 2 \\ 1 + 11x + 6x^2, & \text{if } n = 3 \\ 1 + 67x + 175x^2 + 45x^3, & \text{if } n = 4 \\ 1 + 397x + 3143x^2 + 3239x^3 + 420x^4, & \text{if } n = 5 \\ 1 + 2539x + 50272x^2 + 134160x^3 + 67503x^4 + 4725x^5, & \text{if } n = 6. \end{cases}$$

We leave the analogue of Propositions 4.1 and 4.4 for Π_n^D open in this paper. Computational data suggest that the polynomial $h_{\Pi_n^D}(x)$ is real-rooted for every $n \geqslant 2$ and that the sequence $(h_{\Pi_{n+1}}(x), h_{\Pi_n^D}(x), h_{\Pi_n^B}(x))$ is interlacing for every $n \geqslant 3$.

5. Geometric lattices and face lattices of polytopes

This section addresses the question of real-rootedness of $h_{\mathcal{L}}(x)$ for geometric lattices, face lattices of polytopes and other classes of posets. Moreover, it discusses connections among these questions, proves Theorem 1.4 and deduces some partial results from that.

We recall that a finite lattice is called geometric if it is atomic and semimodular (see [Sta12, Section 3.3] for explanations and more information) or, equivalently, if it is isomorphic to the lattice of flats of a matroid. For a geometric lattice $\mathcal{L}$, the possible inequalities among the coefficients of $h_{\mathcal{L}}(x)$ were studied by Nyman and Swartz [NS04], who showed that

- $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_{\lfloor (n-1)/2 \rfloor}$, and
- $a_i \leqslant a_{n-1-i}$ for $0 \leqslant i \leqslant (n-1)/2$

for every geometric lattice $\mathcal{L}$ of rank n, where $h_{\mathcal{L}}(x) = \sum_{i=0}^{n-1} a_i x^i$ (later, these inequalities were extended to the h-polynomials of $(n-1)$-dimensional simplicial complexes having a convex ear decomposition in [Swa06] and, more recently, to the h-polynomials of all $(n-1)$-dimensional doubly Cohen–Macaulay simplicial complexes in [APP21, Section 6]). To the best of our knowledge, the unimodality of $h_{\mathcal{L}}(x)$ is open. The following conjecture, which has been verified computationally for all geometric lattices with at most nine atoms, is a much stronger statement.

Conjecture 5.1. The polynomial $h_{\mathcal{L}}(x)$ has only real roots and is interlaced by the Eulerian polynomial $A_n(x)$ for every geometric lattice $\mathcal{L}$ of rank n.

The question of real-rootedness of $h_{\mathcal{L}}(x)$ was raised by Brenti and Welker [BW08, Question 1] for face lattices of polytopes and (in a stronger form) by Athanasiadis and Tzanaki [AT21, Question 7.4] for face lattices of more general classes of polyhedral complexes, including polyhedral balls and doubly Cohen–Macaulay polyhedral complexes. It seems natural to pose the following even more general question. We recall that a finite poset $\mathcal{L}$, having a minimum element $\hat{0}$, is said to be *lower Eulerian* if the closed interval $[\hat{0}, x]$ in $\mathcal{L}$ is Eulerian (see [Sta12, Section 3.16] for the definition and information about Eulerian posets) for every $x \in \mathcal{L}$.

Question 5.2. Does $h_{\mathcal{L}}(x)$ have only real roots for every lower Eulerian Cohen–Macaulay poset $\mathcal{L}$?

Special classes of lower Eulerian Cohen–Macaulay posets for which it would be interesting to investigate this question include:

- lower Eulerian Cohen–Macaulay meet semi-lattices,
- face posets of Cohen–Macaulay regular cell complexes,
- face posets of Cohen–Macaulay polyhedral complexes,
- Gorenstein* posets,
- Gorenstein* lattices,
- face lattices of convex polytopes [BW08, Question 1],
- face lattices of zonotopes.

Given Equation (2.1), the following statement suggests that Conjecture 5.1 may be closely related to the special case of Question 5.2 concerning face lattices of zonotopes (more generally, of oriented matroids). We recall that $\mathcal{F}(\mathcal{P})$ denotes the face lattice of a polytope $\mathcal{P}$. For $S \subseteq [n-1]$, we denote by $\mathrm{lpeak}(S)$ the number of elements $i \in S$ for which $i-1 \notin S$, known [Pet15, p. 298] as the *left peaks* of the permutation $w \in \mathfrak{S}_n$ when $S = \mathrm{Des}(w)$.

Proposition 5.3. *For every n-dimensional zonotope $\mathcal{Z}$*

$$h_{\mathcal{F}(\mathcal{Z})}(x) = \sum_{S \subseteq [n-1]} \beta_{\mathcal{L}}(S)\, (4x)^{\mathrm{lpeak}(S)} (1+x)^{n-2\mathrm{lpeak}(S)}, \tag{5.1}$$

where $\mathcal{L}$ is the lattice of flats of the matroid associated to $\mathcal{Z}$. In particular, the polynomial $h_{\mathcal{F}(\mathcal{Z})}(x)$ has only real roots if and only if so does $\sum_{S \subseteq [n-1]} \beta_{\mathcal{L}}(S) x^{\mathrm{lpeak}(S)}$.

Proof. The number of occurrences of $\mathbf{ab}$ in a word $\mathbf{a} \cdot w$ of degree n in $\mathbf{a}$ and $\mathbf{b}$ is equal to $\mathrm{lpeak}(S)$, where S is the subset of $[n-1]$ associated to w. Thus, the first statement follows by substituting $\mathbf{a} = 1$ and $\mathbf{b} = x$ in the formula $\Psi_{\mathcal{F}(\mathcal{Z})} = \omega(\mathbf{a} \cdot \Psi_{\mathcal{L}})$ of Theorem 2.1. The second statement follows from the first and the fact (see [Gal05, Remark 3.1.1]) that a γ-positive polynomial $\sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i x^i (1+x)^{n-2i}$ is real-rooted if and only if so is the associated γ-polynomial $\sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i x^i$. □

Remark 5.4. Let $\mathcal{Z}$ and $\mathcal{L}$ be as in Proposition 5.3. Setting $x = 1$ in Equation (5.1) shows that the number of facets of the barycentric subdivision of $\mathcal{Z}$ is equal to 2^n times the number of maximal chains of $\mathcal{L}$. □

The main result of this section, which is a stronger version of Theorem 1.4, allows one to construct new posets with real-rooted chain polynomials from posets known to have this property.

Theorem 5.5. *Let $\mathcal{L}$ be a bounded, graded poset of positive rank n.*

(a) We have

$$h_{\mathrm{Pyr}(\mathcal{L})}(x) = (1+nx)h_{\mathcal{L}}(x) + (x-x^2)h'_{\mathcal{L}}(x), \tag{5.2}$$

$$\begin{aligned} h_{\mathrm{Prism}(\mathcal{L})}(x) &= (1+(2n-1)x)h_{\mathcal{L}}(x) + 2(x-x^2)h'_{\mathcal{L}}(x) \\ &= 2h_{\mathrm{Pyr}(\mathcal{L})}(x) - (1+x)h_{\mathcal{L}}(x). \end{aligned} \tag{5.3}$$

(b) Assume that $\mathcal{L}$ is Cohen–Macaulay. If the polynomial $h_{\mathcal{L}}(x)$ is real-rooted, then $h_{\mathrm{Pyr}(\mathcal{L})}(x)$ and $h_{\mathrm{Prism}(\mathcal{L})}(x)$ are also real-rooted and each of them is interlaced by $h_{\mathcal{L}}(x)$.

Proof. Part (a) follows by substituting $\mathbf{a} = 1$ and $\mathbf{b} = x$ in the formulas of Theorem 2.2. Indeed, if w is any word of degree $n-1$ in $\mathbf{a}$ and $\mathbf{b}$ having k letters equal to $\mathbf{b}$, then $D(w)$ can be expressed as a sum of $2n-2$ words of degree n in $\mathbf{a}$ and $\mathbf{b}$, of which $2k$ have k letters equal to $\mathbf{b}$ and $2n-2k-2$ have $k+1$ letters equal to $\mathbf{b}$. As a result, Equation (2.2) implies that

$$h_{\mathrm{Pyr}(\mathcal{L})}(x) = (1+x)h_{\mathcal{L}}(x) + \delta_n(h_{\mathcal{L}}(x)),$$

where $\delta_n : \mathbb{R}_{n-1}[x] \to \mathbb{R}_n[x]$ is the linear map defined by $\delta_n(x^k) = kx^k + (n-k-1)x^{k+1}$ for $0 \leqslant k \leqslant n-1$. Clearly, $\delta_n(h(x)) = (n-1)xh(x) + (x-x^2)h'(x)$ for every $h(x) \in \mathbb{R}_{n-1}[x]$ and Equation (5.2) follows. The same argument shows that

$$h_{\mathrm{Prism}(\mathcal{L})}(x) = (1+x)h_{\mathcal{L}}(x) + 2\,\delta_n(h_{\mathcal{L}}(x))$$

and yields Equation (5.3).

For part (b) we rewrite Equations (5.2) and (5.3) as

$$\frac{h_{\mathrm{Pyr}(\mathcal{L})}(x)}{(1-x)^{n+2}} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{xh_{\mathcal{L}}(x)}{(1-x)^{n+1}}\right), \tag{5.4}$$

$$\frac{h_{\mathcal{L}}(x)h_{\mathrm{Prism}(\mathcal{L})}(x)}{(1-x)^{2n+3}} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x(h_{\mathcal{L}}(x))^2}{(1-x)^{2n+2}}\right). \tag{5.5}$$

Since $\mathcal{L}$ is Cohen–Macaulay, $h_{\mathcal{L}}(x)$ has nonnegative coefficients. Since the latter is assumed to be real-rooted and has constant term equal to 1, all its roots are negative. Let d be the degree of $h_{\mathcal{L}}(x)$. Then, $d \leqslant n-1$ and, as it follows from part (a) and its proof, $h_{\mathrm{Pyr}(\mathcal{L})}(x)$ and $h_{\mathrm{Prism}(\mathcal{L})}(x)$ have nonnegative coefficients and degree $d+1$. Applying Rolle's theorem and taking into account that

$$\lim_{x\to-\infty}\frac{xh_{\mathcal{L}}(x)}{(1-x)^{n+1}} = \lim_{x\to-\infty}\frac{x(h_{\mathcal{L}}(x))^2}{(1-x)^{2n+2}} = 0$$

we conclude from Equations (5.4) and (5.5) that each of $h_{\mathrm{Pyr}(\mathcal{L})}(x)$ and $h_{\mathrm{Prism}(\mathcal{L})}(x)$ has $d+1$ negative roots which are interlaced by those of $h_{\mathcal{L}}(x)$ and the proof follows. □

Corollary 5.6. *Let $\mathcal{M}$ and $\mathcal{M}'$ be matroids with lattices of flats $\mathcal{L}$ and $\mathcal{L}'$, respectively. If $h_{\mathcal{L}}(x)$ is real-rooted and $\mathcal{M}'$ is obtained by successively adding coloops to $\mathcal{M}$, then $h_{\mathcal{L}'}(x)$ is real-rooted as well.*

Proof. This follows from part (a) of Theorem 5.5, since $\mathcal{L}'$ is isomorphic to $\mathrm{Pyr}(\mathcal{L})$ for every matroid $\mathcal{M}'$ which can be obtained by adding one coloop to $\mathcal{M}$. □

The following corollary of Theorem 5.5 provides classes of nonsimplicial, nonsimple and noncubical polytopes in any dimension, the barycentric subdivisions of which have real-rooted f-polynomials. For instance, it implies that any polytope which is obtained from one of dimension at most 5 by applying successively the pyramid or the prism construction has this property.

Corollary 5.7. *Let $\mathcal{P}$ be a convex polytope. If $h_{\mathcal{F}(\mathcal{P})}(x)$ is real-rooted, then $h_{\mathcal{F}(\mathrm{Pyr}(\mathcal{P}))}(x)$ and $h_{\mathcal{F}(\mathrm{Prism}(\mathcal{P}))}(x)$ have the same property and each of them is interlaced by $h_{\mathcal{F}(\mathcal{P})}(x)$.*

The near-pencil of rank n on m elements can be obtained from the rank two uniform matroid on $m-n+2$ elements by adding $n-2$ coloops. Near-pencils and uniform matroids are known [NS04, Section 3] to minimize and maximize, respectively, the entries of the flag h-vector of the lattice of flats among all matroids of given rank and number of elements. The following statement confirms Conjecture 1.2 for the lattices of flats of these matroids and completes the proof of Theorem 1.3. Let us denote by $\mathcal{L}(\mathcal{M})$ the lattice of flats of a matroid $\mathcal{M}$. The Eulerian polynomial $B_n(x)$ can be defined (see, for instance, [Ste92, Sections 3.1 and 3.4]) as $h_{\mathcal{F}(\mathcal{P})}(x)$, where $\mathcal{P}$ is the n-dimensional cube, or as

$$B_n(x) = \sum_{w \in \mathfrak{S}_n^{\pm}} x^{\mathrm{des}_B(w)},$$

where $\mathfrak{S}_n^{\pm}$ denotes (this is nonstandard notation) the set of signed permutations of $[n]$, meaning sequences $w = (w_1, w_2, \dots, w_n)$ for which $(|w_1|, |w_2|, \dots, |w_n|) \in \mathfrak{S}_n$, and $\mathrm{des}_B(w)$ is the number of indices $i \in \{0, 1, \dots, n-1\}$ such that $w_i > w_{i+1}$, where $w_0 := 0$.

Proposition 5.8. *Let $\mathcal{M}_{m,n}$ and $\mathcal{U}_{m,n}$ denote the near-pencil and the uniform matroid, respectively, of rank n on m elements and let $\mathcal{F}(\mathcal{M}_{m,n})$ and $\mathcal{F}(\mathcal{U}_{m,n})$ be the face lattices of any zonotopes with associated matroids $\mathcal{M}_{m,n}$ and $\mathcal{U}_{m,n}$, respectively.*

- *(a) The polynomials $h_{\mathcal{L}(\mathcal{M}_{m,n})}(x)$ and $h_{\mathcal{L}(\mathcal{U}_{m,n})}(x)$ have only real roots. Moreover, the latter is interlaced by the Eulerian polynomial $A_n(x)$.*
- *(b) The polynomials $h_{\mathcal{F}(\mathcal{M}_{m,n})}(x)$ and $h_{\mathcal{F}(\mathcal{U}_{m,n})}(x)$ have only real roots. Moreover, the latter is interlaced by the Eulerian polynomial $B_{n-1}(x)$.*

Proof. By definition, the near-pencil $\mathcal{M}_{m,n}$ is obtained by successively adding coloops to a rank two matroid. Thus, the real-rootedness of $h_{\mathcal{L}(\mathcal{M}_{m,n})}(x)$ follows from Corollary 5.6. Similarly, since adding a coloop to a linear matroid $\mathcal{M}$ with associated zonotope $\mathcal{Z}$ yields a matroid whose associated zonotope is combinatorially isomorphic to the prism over $\mathcal{Z}$, the real-rootedness of $h_{\mathcal{F}(\mathcal{M}_{m,n})}(x)$ follows from Corollary 5.7.

Since the geometric lattice $\mathcal{L}(\mathcal{U}_{m,n})$, with its maximum element removed, is a simplicial poset with nonnegative h-vector, the real-rootedness of $h_{\mathcal{L}(\mathcal{U}_{m,n})}(x)$ is a special case of [BW08, Theorem 2]. Similarly, since the zonotope associated to the uniform matroid $\mathcal{U}_{m,n}$ is an n-dimensional cubical polytope, the statement that $h_{\mathcal{F}(\mathcal{U}_{m,n})}(x)$ is real-rooted and interlaced by $B_{n-1}(x)$ is a special case of [Ath21, Corollary 3.5]. Thus, it remains to show that $h_{\mathcal{L}(\mathcal{U}_{m,n})}(x)$ is interlaced by $A_n(x)$.

To simplify the notation, we set $\mathcal{L} := \mathcal{L}(\mathcal{U}_{m,n})$ and recall that $\overline{\mathcal{L}}$ is combinatorially isomorphic to the poset of nonempty faces of the $(n-2)$-dimensional skeleton, say $\Delta_{m,n-1}$, of the $(m-1)$-dimensional simplex $2^{[m]}$. As a result, $\Delta(\overline{\mathcal{L}})$ is combinatorially isomorphic to the barycentric subdivision $\mathrm{sd}(\Delta_{m,n-1})$ and $h_{\mathcal{L}}(x) = h(\mathrm{sd}(\Delta_{m,n-1}), x)$. As explained in [Ath22] [Ath21, Section 1], this expression implies that

$$h_{\mathcal{L}}(x) = \sum_{k=0}^{n-1} c_k p_{n-1,k}(x),$$

where $h(\Delta_{m,n-1}, x) = \sum_{k=0}^{n-1} c_k x^k$ and $(p_{n-1,0}(x), p_{n-1,1}(x), \dots, p_{n-1,n-1}(x))$ is an interlacing sequence of real-rooted polynomials with nonnegative coefficients, originally defined in

[BW08], which sum to $A_n(x)$. Since $\Delta_{m,n-1}$ is the one-coskeleton of the Cohen–Macaulay simplicial complex $\Delta_{m,n}$, as explained in the proof of [AT21, Theorem 6.1] we must have $1 = c_0 \leqslant c_1 \leqslant \cdots \leqslant c_{n-1}$. An application of [ABJK22, Lemma 2.2 (c)] [HOW99, Lemma 8] then shows that $\sum_{k=0}^{n-1} p_{n-1,k}(x) = A_n(x)$ interlaces $\sum_{k=0}^{n-1} c_k p_{n-1,k}(x) = h_{\mathcal{L}}(x)$ and the proof follows. □

Computational data, along with part (b) of Proposition 5.8, suggest the following question.

Question 5.9. Is $h_{\mathcal{F}(\mathcal{Z})}(x)$ interlaced by $B_{n-1}(x)$ for every n-dimensional zonotope $\mathcal{Z}$?

6. An application to the second barycentric subdivision

As mentioned in Section 3, the nth Eulerian polynomial $A_n(x)$ is equal to the h-polynomial of the first barycentric subdivision of the boundary complex $\partial\Delta_n$ of the $(n-1)$-dimensional simplex Δ_n. As an application of results of previous sections, we now give explicit combinatorial interpretations of the h-polynomial of the second barycentric subdivision of $\partial\Delta_n$ and of its associated γ-polynomial, thus answering a question raised in [Ath18, Example 4.4].

Let us write $\mathrm{sd}^2(\Delta) = \mathrm{sd}(\mathrm{sd}(\Delta))$ for the second barycentric subdivision of a simplicial complex Δ. The polynomial $h(\mathrm{sd}^2(\partial\Delta_n), x)$ is an x-analogue of $(n-1)!n!$, which is the number of $(n-2)$-dimensional faces (facets) of $\mathrm{sd}^2(\Delta_n)$. For small values of n,

$$h(\mathrm{sd}^2(\partial\Delta_n), x) = \begin{cases} 1+x, & \text{if } n=2 \\ 1+10x+x^2, & \text{if } n=3 \\ 1+71x+71x^2+x^3, & \text{if } n=4 \\ 1+536x+1806x^2+536x^3+x^4, & \text{if } n=5 \\ 1+4677x+38522x^2+38522x^3+4677x^4+x^5, & \text{if } n=6. \end{cases}$$

Since $(n-1)!n!$ is equal to 2^{n-1} times the number of maximal chains of the partition lattice Π_n, and the latter is equal to the number of elements of the multiset $\mathcal{A}_n$, it is not unreasonable to expect that the coefficients of $h(\mathrm{sd}^2(\partial\Delta_n), x)$ count signed elements of $\mathcal{A}_n$ by some descent-type statistic. Indeed, let us denote by $\mathcal{A}_n^{\pm}$ the multiset of all signed elements of $\mathcal{A}_n$, meaning sequences $\tau = (\tau_1, \tau_2, \dots, \tau_{n-1})$ such that $(|\tau_1|, |\tau_2|, \dots, |\tau_{n-1}|) \in \mathcal{A}_n$. For such $\tau \in \mathcal{A}_n^{\pm}$, let us denote by $\mathrm{eDes}_B(\tau)$ the set of indices $i \in \{0, 1, \dots, n-2\}$ for which

- $\tau_i > \tau_{i+1}$, or
- $\tau_i = \tau_{i+1} > 0$,

where $\tau_0 := 0$, and by $\mathrm{edes}_B(\tau)$ the cardinality of $\mathrm{eDes}_B(\tau)$. For example, the multiset $\mathcal{A}_3^{\pm}$ consists of the twelve signed words $(\pm 1, \pm 1)$, $(\pm 1, \pm 1)$ and $(\pm 1, \pm 2)$. There is one such word τ with $\mathrm{edes}_B(\tau) = 0$, ten with $\mathrm{edes}_B(\tau) = 1$ and one with $\mathrm{edes}_B(\tau) = 2$.

The combinatorial interpretation provided for the coefficients $\gamma_{n,2,i}$ in the following statement is analogous to the one provided by Petersen [Pet07, Proposition 4.15] [Pet15, Section 13.2] for the coefficients of the γ-polynomial associated to the Eulerian polynomial $B_n(x)$. For $\sigma \in \mathcal{A}_n$, we denote by $\mathrm{lpeak}(\sigma)$ the number of descents $i \in [n-2]$ of σ for which either $i-1$ is an ascent of σ, or $i = 1$.

Proposition 6.1. *For every $n \geqslant 2$*

$$h(\mathrm{sd}^2(\partial\Delta_n), x) = \sum_{\tau \in \mathcal{A}_n^{\pm}} x^{\mathrm{edes}_B(\tau)} \tag{6.1}$$

$$= \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,2,i}\, x^i (1+x)^{n-1-2i}, \tag{6.2}$$

where $\gamma_{n,2,i}$ is equal to 4^i times the number of words $\sigma \in \mathcal{A}_n$ with $\mathrm{lpeak}(\sigma) = i$.

Proof. The poset of faces of $\mathrm{sd}(\partial\Delta_n)$ is combinatorially isomorphic to $\mathcal{F}(\mathcal{H}_n)$, where $\mathcal{H}_n$ is the Coxeter hyperplane arrangement of type A_{n-1}. As a result, we have $h(\mathrm{sd}^2(\partial\Delta_n), x) = h(\mathrm{sd}(\mathcal{Z}_n), x) = h_{\mathcal{F}(\mathcal{Z}_n)}(x)$, where $\mathcal{Z}_n$ is the zonotope associated to $\mathcal{H}_n$ (known as the $(n-1)$-dimensional permutohedron). Since the geometric lattice $\mathcal{L}(\mathcal{H}_n)$ is combinatorially isomorphic to Π_n, applying Proposition 5.3 to $\mathcal{Z}_n$ we get

$$h(\mathrm{sd}^2(\partial\Delta_n), x) = \sum_{S \subseteq [n-2]} \beta_{\Pi_n}(S)\, (4x)^{\mathrm{lpeak}(S)} (1+x)^{n-1-2\mathrm{lpeak}(S)}.$$

Combined with Proposition 4.1, this expression yields Equation (6.2). To deduce Equation (6.1) from that, it suffices to show that for every $\sigma \in \mathcal{A}_n$

$$\sum_{\tau \in \Sigma(\sigma)} x^{\mathrm{edes}_B(\tau)} = (4x)^{\mathrm{lpeak}(\sigma)} (1+x)^{n-1-2\mathrm{lpeak}(\sigma)}, \tag{6.3}$$

where $\Sigma(\sigma)$ stands for the set of all words $(\tau_1, \tau_2, \dots, \tau_{n-1})$ such that $(|\tau_1|, |\tau_2|, \dots, |\tau_{n-1}|) = \sigma$. Indeed, given $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_{n-1}) \in \mathcal{A}_n$ and $\tau = (\tau_1, \tau_2, \dots, \tau_{n-1}) \in \Sigma(\sigma)$ such that $\tau_i = \varepsilon_i \sigma_i$ for every $i \in [n-1]$, with $\varepsilon_i \in \{-1, 1\}$, one can verify that

- $0 \in \mathrm{eDes}_B(\tau)$ if and only if $\varepsilon_1 = -1$,
- for $\sigma_i < \sigma_{i+1}$, we have $i \in \mathrm{eDes}_B(\tau)$ if and only if $\varepsilon_{i+1} = -1$,
- for $\sigma_i \geqslant \sigma_{i+1}$, we have $i \in \mathrm{eDes}_B(\tau)$ if and only if $\varepsilon_i = 1$.

Then, the argument given for permutations $\sigma \in \mathfrak{S}_n$ in [Pet15, Section 13.2] applies verbatim to our situation and proves (6.3); the details are left to the interested reader. □

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