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Designing Automated Market Makers with Adaptive Liquidity

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Designing Automated Market Makers with Adaptive Liquidity

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by

Xiaolong Li

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ABSTRACT OF THE THESIS

Designing Automated Market Makers
with Adaptive Liquidity

by

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Master of Science in Computer Science
University of California, Los Angeles, 2012
Professor Jennifer W. Vaughan, Chair

There is a recent surge on the design of automated market makers for securities markets over a finite outcome space, among which is a growing interest in finding automated market makers that have sensitive liquidity. That is, the class of market makers that could adjust their liquidity levels according to market activities. Towards this goal, we first give a formal definition of *liquidity adaptation* that captures this intuition. Then, building on the ideas from Abernethy et al. [2011, 2012] and Othman and Sandholm [2011], we introduce a framework for the design of duality-based marker makers that enjoy liquidity adaptation as well as other desired properties. We will show that in return for liquidity adaptation, we must trade off information accuracy. A detailed discussion about this trade-off then follows. We will also define *asymptotic profit rate* that represent the ability of the market to make a profit in a long run. Finally we discuss the design of *optimal market* that maximize worst case liquidity level given information accuracy and profit rate requirements.

The thesis of Xiaolong Li is approved.

Lieven Vandenberghe

Adnan Y. Darwiche

Jennifer W. Vaughan, Committee Chair

University of California, Los Angeles

2012

To my mother . . .

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CHAPTER 1

Introduction

A securities market offers a set of securities with payoffs linked to future states of the world. Securities markets can be used to effectively aggregate the private information of many traders and produce unified predictions about the likelihood of future events.

Recently there has been a surge of research on the design and use of automated market makers for securities markets. An automated market maker is an algorithmic agent that adaptively sets prices for each security and is always willing to buy or sell securities at those prices. In his influential work, Hanson [2003, 2007] introduced *market scoring rules*, a class of market makers for complete markets (i.e., markets with one security associated with each possible state of the world) based on proper scoring rules. Any market scoring rule can be implemented as a cost function based market, in which the cost of any bundle of securities is determined by a potential function C [Chen and Pennock, 2007, Chen and Vaughan, 2010]. Abernethy et al. [2011, 2012] generalized this idea to move beyond complete markets, providing an optimization-based framework for market design, which has received attention in the machine learning community due to its mathematical similarities to online linear optimization.

These market makers all share a common property referred to by Othman and Sandholm [2011] as *liquidity insensitivity*: the prices set by the market maker depend only on the *differences* between the quantities of various securities that have been purchased. For example, consider a market offering one security that is worth \$1 if and only if a Democrat is elected in the 2012 US Presidential election, and a second worth \$1 if and only if a Republican is elected. A trader who entered the market after 10 units of each security had been purchased would pay the same price as a trader who entered the market after ten million of each security had been purchased, and his trade would have the same impact on future market prices.

Othman et al. [2010] introduced a *liquidity-sensitive* market maker based on Hanson's logarithmic market scoring rule. The key property of this market maker is that price movement slows down as the quantity of trades in the market grows. Othman and Sandholm [2011] introduced a broader class of liquidity-sensitive markets which they call *homogeneous risk measures* due to their relation to risk measures in the finance literature.

Combining ideas from the duality-based cost function market makers of Abernethy et al. [2012] and the homogeneous risk measures of Othman and Sandholm [2011], we introduce a generalized class of automated market makers with adaptive liquidity. Following Abernethy et al., we take an axiomatic approach and give a unified framework for liquidity adaptive market design. We first provide a formal definition of liquidity adaptivity, which captures the idea that price movement slows down as the quantity of securities traded grows. Then we give necessary and sufficient conditions for a market maker to be liquidity adaptive as well as for other properties. The markets in our framework behave like the duality-based markets of Abernethy et al. initially, but transition to behaving like homogeneous risk measures in the limit as the number of trades grows large. Thus we get the benefits of both; early traders have more incentive to participate in our market compared with homogeneous risk measures, but price movement slows over time.

We also define and discuss a notion of *information loss* for the class of liquidity adaptive market makers. We provide an information loss bound, which can be decomposed into a sum of "soft" loss (similar to the notion of loss discussed in Chen and Pennock [2007] and Abernethy et al. [2012]) and "hard" loss, both of which have implications on the design of optimal markets.

Finally, we talk about profitability and optimal market design using tools from differential geometry.

CHAPTER 2

Background

In this chapter, we introduce some background of our work, including notations, terms, and conventions used in subsequent chapters.

2.1 List of Notations and Naming Conventions

n

The number of securities in the market.

$\mathbb{R}^n$

The n-dimensional Euclidean space.

$\mathbf{x}, \mathbf{y}, \ldots$

Vectors in $\mathbb{R}^n$ for general use.

$\mathbf{0}, \mathbf{1}, \ldots$

n-dimensional vector with all component having the same value.

$\mathbf{e}_i$

The ith standard basis in $\mathbb{R}^n$.

x_i

ith component of a vector.

$\succeq$, $\preceq$, $\succ$, $\prec$

Component-wise comparison. For example, $\mathbf{x} \succeq \mathbf{y}$ if (and only if) $x_i \geq y_i$ for all i.

$\mathbb{R}^n_+$

The positive orthant, i.e., $\{\mathbf{x} \in \mathbb{R}^n | \mathbf{x} \succeq \mathbf{0}\}$

Δ_n

The $(n-1)$-dimensional probability simplex, i.e., $\{\mathbf{x} \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = 1,\ \mathbf{x} \succeq \mathbf{0}\}$

∂Y

Boundary of a set.

int$Y$

Interior of a set.

ri$Y$

Relative interior of a set.

cl$Y$**,** cl$f$

Closure of a set or a function[1].

C

The cost function.

R

The conjugate function.

dom(R)

The effective domain of a function, i.e., the set where the function takes finite value.

$\mathbf{q}$

A vector in $\mathbb{R}^n$ representing the total quantities of securities sold to traders.

$\mathbf{p}$

A vector in $\mathbb{R}^n$ representing the instantaneous prices of securities.

[1]The closure of a function f is defined as the greatest lower semi-continuous function majored by f, which is a very useful notion in convex analysis. See Rockafellar [1997] for details.

r

A vector in $\mathbb{R}^n$ representing trade.

b

A vector in Δ_n representing belief.

2.2 Cost Function Based Market

We consider the design of automated market makers for complete markets over n mutually exclusive and exhaustive outcomes.[2] A complete market offers a security associated with each outcome i, worth \$1 if outcome i occurs and \$0 otherwise.

Let's first examine some example. The simplest example is binary market with two mutually exclusive outcomes. For instance, whether the Republican or the Democratic will win the election. For a combinatorial market, such as market for the outcome of a horse race, there are multiple ways to design a complete market, i.e., multiple ways to partition the outcome space. For instance, one can offer a security for each horse ending up with first place, or a security for each possible ranking of all horses. In markets based on market makers, it is possible to price the securities in many ways, such as using market scoring rule [Hanson, 2003] or utility function [Chen and Pennock, 2007]. However, for markets with large set of outcomes, it is believed that cost function based approach is most intuitive and is easiest to implement.

A cost function based market prices the securities using a potential function C called the *cost function*. To specify a market, one must only specify the function C. Let $\mathbf{q}$ be a vector of length n, with q_i denoting the number of securities associated with outcome i that have been purchased by traders in the market so far. If the market is path independent (a concept introduced later), then $\mathbf{q}$ tells us everything about the current state of the market, and we sometimes refer to it as *market state* or simply *state*. Suppose a trader would like to purchase a bundle of securities $\mathbf{r}$, with r_i denoting the number of securities

[2] These ideas can be extended to *complex markets* [Abernethy et al., 2012], but sticking with complete markets simplifies presentation.

associated with outcome i that he would like to purchase. (Here r_i can be 0, or in some cases, negative, representing a sale.) Then the amount of money that the trader must pay to the market maker for this bundle is $C(\mathbf{q}+\mathbf{r}) - C(\mathbf{q})$. If C is differentiable, then the instantaneous price is well defined. The vector of prices of the n securities is denoted as $\mathbf{p}$, or $\mathbf{p}(\mathbf{q})$ when we want to emphasize its relationship with market state. The price for security i is than p_i or $p_i(\mathbf{q})$. The instantaneous price, by definition, can be calculated as $\mathbf{p}(\mathbf{q}) = \nabla C(\mathbf{q})$.

Abernethy et al. [2011, 2012] introduced the class of duality-based cost functions, which can be written as convex optimization problems of the form

$$C(\mathbf{q}) = \max_{\mathbf{p}\in\Delta_n} \mathbf{p}\cdot\mathbf{q} - R(\mathbf{p}), \tag{2.1}$$

where $R(\mathbf{p})$ is a strictly convex function known in convex analysis as the conjugate of C. They showed that markets with cost functions of this form are the unique markets to satisfy a set of desirable properties, discussed in Chapter 4. We will refer to functions defined as Equation 2.1 as translation invariant cost functions for reasons that will become clear later. We call a market defined by such cost function simply as a translation invariant market.

The above class of cost functions are liquidity-insensitive. Formally, for any $\mathbf{q}$ and any scalar α, $\mathbf{p}(\mathbf{q}) = \mathbf{p}(\mathbf{q}+\alpha\mathbf{1})$. As an alternative, Othman and Sandholm [2011] proposed a class of cost functions called homogeneous risk measures, (they use the term cost function and risk measure interchangeably, a convention we will follow as well) and showed that it can be represented as conjugate of indicator function of a compact and convex set $\mathbb{Y}$ in the positive orthant, i.e.,

$$C(\mathbf{q}) = \max_{\mathbf{p}\in\mathbb{Y}} \mathbf{p}\cdot\mathbf{q} \tag{2.2}$$

It is called homogeneous since the cost function satisfies positive first order homogeneity: $C(\lambda\mathbf{q}) = \lambda C(\mathbf{q})$ for all $\lambda \geq 0$. The epigraph of a homogeneous cost function is actually a cone with its tip at the origin, thus $C(\mathbf{q})$ is rarely differentiable at $\mathbf{0}$. To avoid the non-differentiable problem, Othman and Sandholm require the market to start with a non-zero $\mathbf{q}$. To remove this unintuitive requirement, we define homogeneous cost function in a slightly different way:

$$C(\mathbf{q}) = \max_{\mathbf{p}\in\mathbb{Y}} \mathbf{p}\cdot(\mathbf{q}+\mathbf{q}_0) = \max_{\mathbf{p}\in\mathbb{Y}} \mathbf{p}\cdot\mathbf{q} + \mathbf{p}\cdot\mathbf{q}_0 \tag{2.3}$$

for some $\mathbf{q}_0 \in \mathbb{R}^n_+$, $\mathbf{q}_0 \neq \mathbf{0}$, and some $\mathbb{Y} \subset \mathbb{R}^n_+$ with a strictly non-linear positive boundary (the reason for the requirement on the boundary will become clear in the next chapter). Now starting the market at $\mathbf{q} = \mathbf{0}$ is the same as starting with $\mathbf{q} = \mathbf{q}_0$ in original definition. The homogeneity is skewed as a consequence, but we will still call functions in this form as homogeneous risk measures or homogeneous cost functions, and markets defined by it as homogeneous markets.

The function R used to define a duality-based cost function plays a role similar to a regularizer in machine learning, adding stability. A similar role is played by $\mathbb{Y}$ in defining the homogeneous cost function. (The linear term $\mathbf{p}\cdot\mathbf{q}_0$ does not add stability, but is necessary only to make the function differentiable at $\mathbf{q} = \mathbf{0}$.) The markets we consider incorporate both simultaneously. In particular, we consider cost functions of the form

$$C(\mathbf{q}) = \max_{\mathbf{p}\in\mathbb{Y}} \mathbf{p}\cdot\mathbf{q} - R(\mathbf{p}). \tag{2.4}$$

The effect of R dominates when $\|\mathbf{q}\|_2$ is small, whereas the effect of $\mathbb{Y}$ dominates when $\|\mathbf{q}\|_2$ is large. We show that if R is strictly convex and bounded, and if $\mathbb{Y}$ has a strictly non-linear positive boundary, then the cost function defined in Equation 2.4 satisfies a property similar to homogeneity as the size of $\mathbf{q}$ grows. For this reason, we refer to functions of this form as *asymptotic homogeneous cost functions*, and related markets as *asymptotic homogeneous markets*.

Our results for asymptotic homogeneous markets, except when it comes to differentiability, also apply to the case where $R(\mathbf{p})$ is bounded but not strictly convex. Therefore, homogeneous markets defined in (2.3) are for most of the time included as a special case where $R(\mathbf{p})$ is affine.

In the following discussions, if not explicitly stated otherwise, we are always talking about asymptotic homogeneous markets.

2.3 A Discussion about Price Range

We use the convex analysis fact throughout our discussion that if C is differentiable at some point $\mathbf{q}$, the instantaneous price $\mathbf{p}$ is the unique maximizer of the optimization, i.e.,

$$\mathbf{p}(\mathbf{q}) = \nabla C(\mathbf{q}) = \arg\max_{\mathbf{p}\in\mathbb{Y}} \mathbf{p}\cdot\mathbf{q} - R(\mathbf{p}).$$

As a consequence, price at any time is taken from the set $\mathbb{Y}$. We thus refer to $\mathbb{Y}$ as *price range.*

First of all, we assume $\mathbb{Y}$ to be bounded since it is unreasonable to have unbounded price for securities that pay off at most one dollar.

Although $R(\mathbf{p})$ can take value from extended real line $[-\infty, +\infty]$ on $\mathbb{Y}$, we require it to be finite on $\mathbb{Y}$ just to make things simpler. This can be easily done without loss of generality by shrinking $\mathbb{Y}$ to the subset where $R(\mathbf{p})$ is finite. On one hand, $R(\mathbf{p})$ is always bounded below on bounded set $\mathbb{Y}$ by convexity. On the other hand, removing points from $\mathbb{Y}$ with $R(\mathbf{p}) = +\infty$ will not affect the maximization. Moreover, we can assume that $\text{dom}(R)$ and $\mathbb{Y}$ are identical. If there are points out of $\mathbb{Y}$ where R takes finite value, then we can simply set R to be $+\infty$ on those points instead. This will not affect the convexity of R since $\mathbb{Y}$ is a convex set by assumption.

To summarize, we will assume in following discussions that $\mathbb{Y}$ is bounded, and $\text{dom}(R) = \mathbb{Y}$.

CHAPTER 3

Liquidity Adaptation

While Othman and Sandholm [2011] argue that liquidity sensitivity is desirable, they do not formally define what it means. We have informally defined liquidity adaptation as the property that prices change more slowly with trades as the quantity $\|\mathbf{q}\|_2$ grows and we are going to formalize this idea in the current chapter. We will then derive an important property of liquidity adaptive market, the *buy-only* principle, without which the idea of liquidity adaptation will not make sense. But before that, we would like to re-examine types of market makers in previous work, some of which we have already seen in the introductory sections.

3.1 Translation Invariant Market Makers

Nearly all previous market makers proposed has the property that the prices of securities always sum up to unity. This property is frequently listed as a desired one and researchers refer to it with different names. Chen and Vaughan [2010] call a cost function *valid* if the sum of prices is always one, in addition that the prices are always nonnegative. They also showed an equivalent condition for this: for any $\mathbf{q}$ and constant k, $C(\mathbf{q} + k\mathbf{1}) = C(\mathbf{q}) + k$, and they call it *positive translation invariance.* This term is adopted by some other researchers as well and we will then refer to this kind of market makers as *translation invariant.*

Translation invariance is a natural assumption to make since prices that sums up to unity can be directly related to trader's belief, i.e., the chance of each outcome as estimated by a trader. In fact, all cost function based market makers derived directly from scoring rules are translation invariant, since probability distribution is explicitly modeled by scoring rule market makers. As a concrete example, the price for LMSR [Hanson, 2003] is

$$p_i(\mathbf{q}) = \frac{\exp(q_i/b)}{\sum_{j=1}^{n} \exp(q_i/b)}$$

where $b > 0$ is some constant that controls the worst case monetary loss as well as liquidity. The sum of prices is obviously one. Machine learning society often call this the *softmax* function, which can be seen as a smoothed differentiable version of max function:

$$max_i(\mathbf{q}) = \begin{cases} 1 & q_i = \max_j q_j \\ 0 & \text{otherwise} \end{cases}$$

The max function, which can be viewed as a special case of LMSR with $b \to 0$, can in turn be used to define a translation invariant cost function. However, that cost function is not usually used since the price response is not continuous.

For translation invariant markets, instantaneous price is directly connected to the underlying belief of traders. In fact, it is shown that a myopic trader can maximize his expected payoff by setting the price to his belief (by trading in the market). This observation is essential, as it reveals the reason why such market works regarding information integration.

3.2 Homogeneous Risk Measure

One disadvantage of LMSR, noticed by many researchers, is that the market's liquidity is set in advance, before the market maker knows how active the market will be and how large the bets of traders are. This has several problems. On one hand, too little liquidity will cause the market's price to fluctuate wildly, leading to large information inaccuracy. On the other hand, market with too large liquidity will need very large bet to move the prices and traders might not have enough fund to reach prices that reflect their beliefs.

Another disadvantage, which applies to all translation invariant market makers, is that the liquidity will not self-adjust regarding activity level. Here is an exaggerated example:

suppose we have two securities corresponding to two mutually exclusive outcomes, then the two billion and first dollar, after one billion is bet on each outcome, will move the price as much as the first dollar bet into the market, which is unintuitive.

Othman et al. [2010] proposed the OSPR cost function, a modification of LMSR cost function to address the above liquidity issue, written as

$$C(\mathbf{q}) = \alpha b(\mathbf{q}) \log \left(\sum_i \exp(q_i/b(\mathbf{q})) \right),$$

where $b(\mathbf{q}) = \sum_i q_i$. The idea is to replace the constant "liquidity parameter" b by one that increases with market activity level. They showed that to avoid translation invariance, the prices must not always sum up to unity. For the case of OSPR cost function, the prices sums up to at least one. The difference between the sum of prices and unity is viewed as the profit cut of the market maker, which grants potential profitability, another advantage of OSPR cost function.

Othman and Sandholm [2011] extended the OSPR cost function and proposed the class of market makers called homogeneous risk measures. Homogeneous risk measures inherit the advantages of OSPR cost function mentioned above.

It will be quite important for the following discussions to understand how homogeneous risk measures introduce liquidity sensitivity into the market. A cost function C is positive homogeneous if

$$C(\lambda \mathbf{q}) = \lambda C(\mathbf{q}) \quad \forall \lambda \geq 0.$$

We will measure liquidity by how slow price vector $\mathbf{p}$ changes regarding to the change of quantity vector $\mathbf{q}$, which is best captured by the inverse of the partial derivative $(\frac{\partial p_i}{\partial q_i})^{-1}$. Notice that the price vector is just the gradient of the cost function, we have

$$p_i(\lambda \mathbf{q}) = \frac{\partial C(\lambda \mathbf{q})}{\partial \lambda q_i} = \frac{1}{\lambda} \cdot \frac{\partial C(\lambda \mathbf{q})}{\partial q_i} = \frac{1}{\lambda} \cdot \frac{\partial (\lambda C(\mathbf{q}))}{\partial q_i} = \frac{\partial C(\mathbf{q})}{\partial q_i} = p_i(\mathbf{q})$$

Moreover

$$\frac{\partial p_i}{\partial q_i}(\lambda \mathbf{q}) = \frac{\partial p_i(\lambda \mathbf{q})}{\partial \lambda q_i} = \frac{1}{\lambda} \cdot \frac{\partial p_i(\mathbf{q})}{\partial q_i}$$

The above equation means that liquidity increases linearly with the size of quantity vector.

Before moving on, we list a result that will be useful in later discussions. Monotonicity and convexity mentioned in the next proposition should be temporarily accepted as criteria for any reasonable cost function, which we will discuss in detail later.

Proposition 1. *[Othman and Sandholm, 2011] A cost function satisfying monotonicity, convexity and positive homogeneity, could be written as*

$$C(\mathbf{q}) = \sup_{\mathbf{p} \in \mathbb{Y}} \mathbf{p} \cdot \mathbf{q}$$

where $\mathbb{Y}$ *is a convex and compact set in* $\mathbb{R}^n_+$.

3.3 A Formal Definition of Liquidity Adaptation

Othman and Sandholm [2011] called translation invariant market makers as liquidity insensitive but didn't give a formal definition of liquidity sensitivity except informally describing it as "more muted price responses" at higher level of activity. Their work centered on cost functions that are homogeneous, but they also pointed out another direction of exploring cost functions that display some characteristics of liquidity sensitivity without necessarily being homogeneous, the direction which inspired our work in the first place. But before trying to find other functions that are liquidity sensitive, we must define it rigorously. To distinguish our definition from that of Othman and Sandholm, we will call it liquidity adaptation, instead of liquidity sensitivity,

Following our derivation in the previous section, we can define liquidity adaptive cost functions as those whose Hessian matrix goes to zero as market state goes to infinity, i.e.,

$$\lim_{\|\mathbf{q}\|_2 \to \infty} \nabla^2 C(\mathbf{q}) = \mathbf{0}$$

Note that $(\nabla^2 C(\mathbf{q}))_{ij}$ is $\frac{\partial p_i}{\partial q_i}$. The problem of this definition is that it requires C to be twice differentiable, which is not necessary. In fact, we are not interested in the "instantaneous liquidity", but the price change when some bundle $\mathbf{r}$ is purchased. With this in mind, we now provide our formal definition of liquidity adaptation that only requires C to be once differentiable (which is necessary for price to be well defined).

Definition 1 (Liquidity Adaptation)**.** *A cost function C is liquidity adaptive if for any* $\mathbf{r} \in \mathbb{R}^n_+$*,*

$$\lim_{\|\mathbf{q}\|_2 \to \infty} \mathbf{p}(\mathbf{q} + \mathbf{r}) - \mathbf{p}(\mathbf{q}) = 0. \tag{3.1}$$

The limit in our definition is taken with respect to the size of $\mathbf{q}$, regardless of its direction. In other words, the convergence needs to be uniform along all direction. We will expand on this later.

3.4 The Buy-only Principle

Othman et al. [2010] showed that selling should be disallowed in order to get around the obvious arbitrage opportunity to sell equal amount of each securities if the sum of prices is at least one. Here, we justify this argument in a different way, starting from the definition of liquidity adaptation.

Definition 1 says that price movement slows down when market state $\mathbf{q}$ increases. This only makes sense if $\mathbf{q}$ can reflect the activity level of the market, which unfortunately is not true in general case. If we allow both selling and buying in the market, then $\mathbf{q} = \mathbf{0}$ doesn't mean the market is extremely inactive. It is well possible that some trader just purchased a large bundle and then another trader sold the same bundle back to the market maker. One way to make $\mathbf{q}$ a sound signal of market activity is to disallow selling.

If fact, we can show that if selling is allowed, then $\mathbf{q}$ will not always grow with trades in the market. The proof of the following two theorems used some result from next chapter.

Theorem 1. *If selling is allowed in homogeneous market defined by Equation 2.3, and if the market has bounded loss, then a myopic, risk-neutral trader will always maximize his expected payoff by setting* $\mathbf{q}$ *to* $-\mathbf{q}_0$*.*

Proof. It is the same to proof that in market defined by Equation 2.2, a myopic risk-neutral trader will always want to set $\mathbf{q}$ to $\mathbf{0}$. Suppose $\mathbf{q}$ is the current market state and $\mathbf{b} \in \Delta_n$

is the trader's belief. By the assumption of bounded loss and Proposition 4, $\mathbf{b} \in \Delta_n \subseteq \mathbb{Y}$. Thus $\mathbf{b} \cdot \mathbf{q} \leq \max_{\mathbf{p} \in \mathbb{Y}} \mathbf{p} \cdot \mathbf{q} = C(\mathbf{q})$, i.e., $C(\mathbf{q}) - \mathbf{b} \cdot \mathbf{q} \geq 0$. Notice that by selling the amount $\mathbf{q}$ back to the market, the trader first earns $C(\mathbf{q}) - C(\mathbf{0}) = C(\mathbf{q})$, then pays $\mathbf{b} \cdot \mathbf{q}$ in expectation. This means that the expected payoff is nonnegative regardless of $\mathbf{b}$ and $\mathbf{q}$, hence proves the theorem. □

Theorem 2. *If selling is allowed in asymptotic homogeneous market defined by Equation 2.4 with liquidity adaptation, and there exists $\epsilon > 0$ such that the belief $\mathbf{b}$ of any trader satisfies $\mathbf{b} \succeq \epsilon\mathbf{1}$, then there exists a bound M such that no myopic, risk-neutral trader is willing to set $\|\mathbf{q}\|_2 > M$. In other words, $\mathbf{q}$ is always bounded.*

Proof. We claim from Proposition 6 that if an asymptotic homogeneous market has liquidity adaptation, then the price range $\mathbb{Y}$ must have strictly non-linear positive boundary. Also, price range $\mathbb{Y}$ must include the probability simplex. Otherwise, there is some belief that is not in $\mathbb{Y}$ and a myopic trader with that belief will always want to sell or purchase (since there must be some price that is either greater than or less than his corresponding belief) and will never stop.

By assumption, there exists $\epsilon > 0$ such that the belief $\mathbf{b}$ of any trader satisfies $\mathbf{b} \succeq \epsilon\mathbf{1}$. Put it another way, $\mathbf{b} \in B = \{\mathbf{x} \in \Delta_n \mid \mathbf{x} \succeq \epsilon\mathbf{1}\}$. A useful fact that appears in many analysis books is the distance between disjoint compact and closed set is positive. Since B is compact, $\mathrm{cl}(\partial\mathbb{Y})$ is closed, and it easily follows from the claim that they are disjoint, therefore they have positive distance, say $d > 0$. Therefore, the closed set $E = \{\mathbf{x} \mid d(\mathbf{x}, B) \leq d/2\}$ is contained in $\mathbb{Y}$. Also using some analysis, it can be shown that if a convex function is finite on some closed set E, it is also bounded on E. $R(\mathbf{p})$ is indeed finite on E since E is contained in $\mathbb{Y}$, the effective domain of R, as we pointed out in previous chapter. The boundedness of $R(\mathbf{p})$ on E implies that the subgradient of R on B is bounded as well.

Finally, for a trader with belief $\mathbf{b} \in B$ to maximize his expected payoff, he needs to set $\mathbf{p}$ to $\mathbf{b}$, which by duality theory is equivalent to setting $\mathbf{q}$ to the subgradient of R at $\mathbf{b}$, which is just proven to be bounded. □

As a summary, the first theorem proves the necessity of disallowing selling in homogeneous market. The second theorem says allowing selling in asymptotic homogeneous market will prevent $\|\mathbf{q}\|_2$ from growing large unless traders have arbitrarily extreme beliefs. With all these arguments, we therefore disallow selling throughout. To avoid confusion, we will refer to markets that allow both purchases and sales as *buy-sell markets* and markets that allow only purchases as *buy-only markets*. If unquantified, the word market refers to buy-only market.

CHAPTER 4

An Axiomatic Approach

Several papers have suggested natural criteria that one might like an automated market maker to satisfy [Agrawal et al., 2011, Chen and Vaughan, 2010, Abernethy et al., 2011, 2012, Othman and Sandholm, 2011]. Most of the criteria were defined for buy-sell markets, but can be adapted easily to buy-only markets. Section 4.1 is mostly review. The more interesting results follow.

4.1 Basic Conditions

We begin by reviewing previously studied conditions.

Condition 1 (Path Independence)**.** $\forall \mathbf{r}, \mathbf{r}', \mathbf{r}'' \in \mathbb{R}^n_+$ *s.t.* $\mathbf{r} = \mathbf{r}' + \mathbf{r}''$*, given any fixed history of previous market transactions, the cost to a trader of purchasing bundle* $\mathbf{r}$ *is the same as the cost of purchasing* $\mathbf{r}'$ *and then immediately purchasing* $\mathbf{r}''$*.*

Abernethy et al. [2012] showed that path independence holds if and only if costs can be calculated via a cost function C, which remains true for buy-only markets. This is why we restricted our attention to cost function based markets from the beginning.

Condition 2 (Information Incorporation)**.** $\forall \mathbf{q}, \mathbf{r} \in \mathbb{R}^n_+$, $C(\mathbf{q} + 2\mathbf{r}) - C(\mathbf{q} + \mathbf{r}) \geq C(\mathbf{q} + \mathbf{r}) - C(\mathbf{q})$.

Abernethy et al. [2012] showed that information incorporation holds if and only if C is convex. This also remains true in buy-only markets.

If C is additionally closed, then there exists a closed and convex function R and a set $\mathbb{Y} \subseteq \mathbb{R}^n$ such that $C(\mathbf{q}) = \max_{\mathbf{p} \in \mathbb{Y}} \mathbf{p} \cdot \mathbf{q} - R(\mathbf{p})$ as in Equation 2.4 Rockafellar [1997]. R

is called the convex conjugate of C. For the rest of the paper, we restrict attention to cost functions of this form. We additionally require R to be bounded, which is essential for several of our results. In fact, some of the following theorem will otherwise have easily constructable counter examples if R is not bounded. If R is closed and bounded, then $\mathrm{dom}(R)$ is closed by simple analysis arguments. Recall from Section 2.3 that $\mathbb{Y}$ is bounded, and $\mathrm{dom}(R) = \mathbb{Y}$, therefore $\mathbb{Y}$ is closed and bounded, thus compact.

Condition 3 (Instantaneous Prices). *C is continuous and differentiable everywhere on $\mathbb{R}^n_+$.*

Essentially, cost function of the form in Equation 2.4 is differentiable on $\mathbb{R}^n_+$ if and only if the maximization has unique optimal solution $\mathbf{p}^*$ on $\mathbb{Y}$ for every $\mathbf{q} \in \mathbb{R}^n_+$. More meaningful equivalence conditions about $\mathbb{Y}$ and $R(\mathbf{p})$ could be derived based on conjugate duality. However, for our purpose of market design, we give two sufficient conditions. Before that, let us introduce a notation that will save us a lot of words in the discussion.

Definition 2. *The* positive boundary *of a set $\mathbb{Y}$ is defined as*

$$\partial^+\mathbb{Y} = \{\mathbf{y} \in \mathrm{cl}\mathbb{Y} \mid \nexists \mathbf{x} \in \mathbb{Y},\ \mathbf{x} \neq \mathbf{y}\ s.t.\ \mathbf{x} \succeq \mathbf{y}\}.$$

Pictorially, the positive boundary of a set is the subset of its boundary that points to the positive orthant. We say $\mathbf{x}$ dominates $\mathbf{y}$ if $\mathbf{x} \succeq \mathbf{y}$, then the positive boundary of a set consists of those points that are not dominated by any other point in the set. An observation is that $\partial^+\mathbb{Y} \subseteq \partial\mathbb{Y}$, since any ball centered at $\mathbf{x} \in \partial^+\mathbb{Y}$ contains a point that dominate $\mathbf{x}$.

Moreover, we say a set to be *strictly non-linear* if it contains no line segment.

Proposition 2. *Homogeneous cost function, as defined by Equation 2.3, satisfies Condition 3 if $\mathbb{Y}$ has strictly non-linear positive boundary and $\mathbf{p}_0 \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$.*

Proof. For $\mathbf{q} \in \mathbb{R}^n_+$, $\mathbf{q} + \mathbf{q}_0 \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$. Suppose a point $\mathbf{x}$ is the maximizer of $\mathbf{p} \cdot (\mathbf{q} + \mathbf{q}_0)$ on $\mathbb{Y}$, then $\mathbf{x}$ can not be dominated by another point $\mathbf{y} \in \mathbb{Y}$, otherwise $\mathbf{y}$ will give larger objective value. Therefore, $\mathbf{x}$ must be in $\partial^+\mathbb{Y}$. Notice for a concave objective function, the maximizers must form a convex set. If the maximizer is not unique, say $\mathbf{x} \neq \mathbf{y}$ are both maximizers, then the line segment connecting $\mathbf{x}$ and $\mathbf{y}$ is on $\partial^+\mathbb{Y}$, contradicting our assumption that $\partial^+\mathbb{Y}$ is strictly non-linear. Therefore, the maximizer must be unique for all $\mathbf{q} \in \mathbb{R}^n_+$ and Condition 3 is satisfied. □

Proposition 3. *Asymptotic homogeneous cost function, as defined by Equation 2.4, satisfies Condition 3 if R is strictly convex.*

Proof. The strict convexity of R means the objective function as a whole is strictly convex, thus guarantees the uniqueness of maximizer. □

Condition 4 (Bounded Loss)**.** *The worst-case loss of the market maker,* $\sup_{\mathbf{q}\in\mathbb{R}^n_+}(\max_{i\in\{1,\cdots,n\}} q_i - C(\mathbf{q}) + C(\mathbf{0}))$*, is bounded.*

Abernethy et al. [2012] showed that a complete buy-sell market with a cost function as in Equation 2.4 has bounded loss if and only if $\mathbb{Y}$ contains the probability simplex. This idea extends to buy-only markets with slight modification.

Definition 3 (Extension)**.** *The extension of a set* $\mathbb{Y}$ *is defined as extend*$(\mathbb{Y}) = \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{Y},\ y \succeq x\}$

In other words, the extension of a set is all the points dominated by some point in the set.

Proposition 4. *If R is bounded, then the market has bounded loss if and only if extend*$(\mathbb{Y})$ *contains the probability simplex.*

Proof. "⇐". Suppose $\Delta_n \subseteq \textit{extend}(\mathbb{Y})$, then $\mathbf{e}_i \in \textit{extend}(\mathbb{Y})$ for all i. We have

$$\begin{aligned} C(\mathbf{q}) &= \max_{\mathbf{p}\in\mathbb{Y}} \mathbf{p}\cdot\mathbf{q} - R(\mathbf{p}) \\ &\geq \mathbf{e}_i \cdot \mathbf{q} - R(\mathbf{e}_i) \\ &= q_i - R(\mathbf{e}_i) \end{aligned}$$

Since R is bounded, let $M = \max_i R(\mathbf{e}_i) < \infty$, then $C(\mathbf{q}) \geq q_i - M$ for all $\mathbf{q}$ and i. Also by boundedness of R, $C(\mathbf{0}) = \max_{\mathbf{p}\in\mathbb{Y}} -R(\mathbf{p}) = -\min R(\mathbf{p}) = m < \infty$. Therefore the worst case loss $\sup_{\mathbf{q}}(\max_i q_i - C(\mathbf{q}) + C(\mathbf{0})) \leq M + m < \infty$.

"$\Rightarrow$". Suppose $\exists \mathbf{x} \in \Delta_n$ such that $\mathbf{x} \notin \mathit{extend}(\mathbb{Y})$. Notice that $\mathit{extend}(\mathbb{Y})$ is convex, $\mathit{extend}(\mathbb{Y}) = \bigcup$ half spaces containing $\mathit{extend}(\mathbb{Y})$. Since $\mathbf{x} \notin \mathit{extend}(\mathbb{Y})$, there is some half space $H = \{\mathbf{y} \mid \mathbf{y}\cdot\mathbf{b} < \beta\}$, such that extend$(\mathbb{Y}) \subseteq H$ and $\mathbf{x} \notin H$. It is easy to show that $\mathbf{b} \succeq \mathbf{0}$ from the definition of $\mathit{extend}(\mathbb{Y})$. Since $\mathbb{Y}$ is compact, it is easy to show that $\mathit{extend}(\mathbb{Y})$ is closed. $\mathbf{x}$ is closed as well, thus we can assume that the hyperplane $P = \{\mathbf{y} \mid \mathbf{y}\cdot\mathbf{b} = \beta\}$ separates $\mathit{extend}(\mathbb{Y})$ and $\mathbf{x}$ strongly with possibly modifying β a little bit [Rockafellar, 1997], i.e., $\mathbf{x}\cdot\mathbf{b} > \beta > \max_{\mathbf{p}\in\mathbb{Y}} \mathbf{p}\cdot\mathbf{b}$. Let $\delta = \mathbf{x}\cdot\mathbf{b} - \max_{\mathbf{p}\in\mathbb{Y}} \mathbf{p}\cdot\mathbf{b} > 0$ then

$$
\begin{aligned}
C(k\mathbf{b}) &= \max_{\mathbf{p}\in\mathbb{Y}} k(\mathbf{p}\cdot\mathbf{b}) - R(\mathbf{p}) \\
&\leq k(\mathbf{x}\cdot\mathbf{b} - \delta) + \sup_{\mathbf{p}\in\mathbb{Y}} |R(\mathbf{p})| \\
&\leq k(\mathbf{x}\cdot(\max_i b_i)\mathbf{1}) - k\delta + \sup_{\mathbf{p}\in\mathbb{Y}} |R(\mathbf{p})| \\
&= k\max_i b_i - k\delta + \sup_{\mathbf{p}\in\mathbb{Y}} |R(\mathbf{p})|
\end{aligned}
$$

Thus, the loss

$$
\begin{aligned}
&\sup_{\mathbf{q}\in\mathbb{R}^n_+} (\max_i q_i - C(\mathbf{q}) + C(\mathbf{0})) \\
\geq &\sup_{\mathbf{q}=k\mathbf{b},k\in\mathbb{R}^+} (\max_i q_i - C(\mathbf{q}) + C(\mathbf{0})) \\
= &\sup_{k\in\mathbb{R}^+} (k\max_i b_i - C(k\mathbf{b}) + C(\mathbf{0})) \\
\geq &\sup_{k\in\mathbb{R}^+} k\delta - \sup_{\mathbf{p}\in\mathbb{Y}} |R(\mathbf{p})| - \inf R(\mathbf{p}) \\
\geq &\sup_{k\in\mathbb{R}^+} k\delta - 2M \\
= &+\infty,
\end{aligned}
$$

where M is the bound of $|R(\mathbf{p})|$. Thus the worst case loss is unbounded. □

Several literature such as Abernethy et al. [2012] also proposed as desired property the *no-arbitrage* condition, which means there is no opportunity at any time that gives the trader guaranteed payoff regardless of the outcome. This condition can be easily weakened to prevent only arbitrage opportunities that arrive from purchases.

Condition 5 (No Buy-Only Arbitrage)**.** $\forall \mathbf{q}, \mathbf{r} \in \mathbb{R}^n_+$, $\exists i \in \{1, \cdots, n\}$ *s.t.* $C(\mathbf{q}+\mathbf{r}) - C(\mathbf{q}) \geq r_i$.

Define $S = \{x \in \mathbb{R}^n_+ \mid \sum_{i=1}^n x_i < 1\}$ and call it the sub-probability simplex, then one can show the following result.

Proposition 5. *If C is differentiable, then there is no buy-only arbitrage if and only if $\mathbb{Y} \subseteq \mathbb{R}^n_+ \setminus S$. More precisely, there exist $\mathbb{Y}' \subseteq \mathbb{R}^n_+ \setminus S$ convex such that replacing $\mathbb{Y}$ by $\mathbb{Y}'$ will not change C.*

Proof. "$\Leftarrow$". Suppose there is no buy-only arbitrage.

From simple analysis, we know if a convex function is differentiable, then it is continuously differentiable, thus the price mapping $\mathbf{q} \to \mathbf{p}(\mathbf{q})$ is continuous. Suppose there exists $\mathbf{q}_0$ such that $\mathbf{p}(\mathbf{q}_0)$ has negative component p_i, then by continuity of price, there exists some small purchase $\delta\mathbf{e}_i$ such that p_i remains negative for $\mathbf{q} = \mathbf{q}_0 + \theta\mathbf{e}_i$, $0 \leq \theta \leq \delta$. The purchase of $\delta\mathbf{e}_i$ at market state $\mathbf{q}$ is an arbitrage. Therefore, any $\mathbf{q} \notin \mathbb{R}^n_+$ should never be the price, or maximizer in the optimization. Thus, replacing $\mathbb{Y}$ with $\mathbb{Y} \cap \mathbb{R}^n_+$ will not change C.

Similarly, if $\exists \mathbf{q}_0 \in \mathbb{R}^n_+$, such that $\mathbf{p}(\mathbf{q}_0) \in S$, then $\mathbf{p}(\mathbf{q}_0) \cdot \mathbf{1} < 1$. By continuity of price, $\exists \delta > 0$ such that $\mathbf{p}(\mathbf{q}_0 + \theta\mathbf{1}) \cdot \mathbf{1} < 1$ for $0 \leq \theta \leq \delta$. Let $f(t) = C(\mathbf{q}_0 + t\mathbf{1})$, then $f'(t) = \mathbf{p}(\mathbf{q}_0 + t\mathbf{1}) \cdot \mathbf{1}$.

$$C(\mathbf{q}_0 + \delta 1) - C(\mathbf{q}_0) = f(\delta) - f(0) = \int_0^\delta \mathbf{p}(\mathbf{q}_0 + \theta\mathbf{1}) \cdot \mathbf{1} \mathrm{d}\theta < \int_0^\delta 1 \mathrm{d}\theta = \delta.$$

The purchase of $\delta\mathbf{1}$ at market state $\mathbf{q}_0$ is an arbitrage, since the trader pays less than δ but is guaranteed to earn δ regardless of the outcome.

Therefore, we can replace $\mathbb{Y}$ with $\mathbb{Y} \cap (\mathbb{R}^n_+ \setminus S)$ without changing C. Notice that the convexity of $\mathbb{Y}$ is kept since $\mathbb{R}^n_+ \setminus S$ is convex and intersection of convex sets is still convex.

"$\Rightarrow$". Suppose $\mathbb{Y} \subseteq \mathbb{R}^n_+ \setminus S$.

For any $\mathbf{q}$, $\mathbf{r} \in \mathbb{R}^n_+$, let $\mathbf{p}^* = \mathbf{p}(\mathbf{q}) = \arg\max_{\mathbf{p}\in\mathbb{Y}} \mathbf{p}\cdot\mathbf{q} - R(\mathbf{p})$. Since $\mathbf{p}^* \in \mathbb{Y}$, $\mathbf{p}^*\cdot\mathbf{1} \geq 1$.

$$
\begin{aligned}
& C(\mathbf{q}+\mathbf{r}) - C(\mathbf{q}) \\
&= (\max_{\mathbf{p}\in\mathbb{Y}} \mathbf{p}\cdot(\mathbf{q}+\mathbf{r}) - R(\mathbf{p})) - (\mathbf{p}^*\cdot\mathbf{q} - R(\mathbf{p}^*)) \\
&\geq (\mathbf{p}^*\cdot(\mathbf{q}+\mathbf{r}) - R(\mathbf{p}^*)) - (\mathbf{p}^*\cdot\mathbf{q} - R(\mathbf{p}^*)) \\
&= \mathbf{p}^*\cdot\mathbf{r} \\
&\geq \mathbf{p}^*\cdot(\min_i r_i)\mathbf{1} \\
&\geq \min_i r_i
\end{aligned}
$$

□

In most of the figures about the shape of $\mathbb{Y}$ in following chapters, we are only interested in the positive boundary of $\mathbb{Y}$ and we may not explicitly exclude the sub-probability simplex from the drawing. But we should always implicitly agree that the sub-probability simplex is not included in $\mathbb{Y}$.

4.2 Adding Liquidity Adaptation

The primary condition of interest in our work is liquidity adaptation, as defined in Section 3.

Condition 6 (Liquidity Adaptation)**.** $\forall \mathbf{r} \in \mathbb{R}^n_+$, $\lim_{\|\mathbf{q}\|_2\to\infty} \mathbf{p}(\mathbf{q}+\mathbf{r}) - \mathbf{p}(\mathbf{q}) = 0$.

Surprisingly enough, we can show that that strict non-linearity of $\partial^+\mathbb{Y}$, which is imposed in Othman and Sandholm [2011] to add differentiability to homogeneous risk measure, is actually a sufficient and necessary condition for liquidity adaptation. Since the whole proof is rather long, we split it into several steps. First, we show that given strictly non-linear positive boundary, the price response of asymptotic homogeneous market converges to constant along each direction (of $\mathbf{q}$) uniformly. Notice that in the following proofs, we use the fact that $\lim_{\|\mathbf{q}\|_2\to\infty} \mathbf{p}(\mathbf{q}+\mathbf{r}) - \mathbf{p}(\mathbf{q}) = 0$ is equivalent to $\lim_{\lambda\to\infty} \mathbf{p}(\lambda\mathbf{q}+\mathbf{r}) - \mathbf{p}(\lambda\mathbf{q}) = 0$ uniformly for all $\|\mathbf{q}\|_2 = 1$.

Lemma 1. *For asymptotic homogeneous market defined in Equation 2.4. If* $R(\mathbf{p})$ *bounded on* $\mathbb{Y}$ *and* $\partial^+\mathbb{Y}$ *is strictly non-linear, then* $\lim_{\lambda\to\infty}\mathbf{p}(\lambda\mathbf{q}) = \hat{\mathbf{p}}(\mathbf{q})$ *uniformly for all* $\|\mathbf{q}\|_2 = 1$, $\mathbf{q} \in \mathbb{R}^n_+$. *Where* $\hat{\mathbf{p}}(\mathbf{q}) = \arg\max_{\mathbf{p}\in\mathbb{Y}} \mathbf{p}\cdot\mathbf{q}$ *is the price function for* $C(\mathbf{q})$*'s homogeneous counterpart.*

Proof. For any $\epsilon > 0$, make a ball centered at $\hat{\mathbf{p}}$ with radius ϵ.We denote the ball as $B(\hat{\mathbf{p}}, \epsilon)$ Suppose $\mathbf{p}'$ and $\mathbf{p}''$ are the points when $\partial\mathbb{Y}$ intersect with $B(\hat{\mathbf{p}}, \epsilon)$, as shown in Figure 4.1. Since $\hat{\mathbf{p}}\cdot\mathbf{q}$ maximizes $\mathbf{p}\cdot\mathbf{q}$ on $\mathbb{Y}$, we know that $\mathbf{p}'\cdot\mathbf{q} \le \hat{\mathbf{p}}\cdot\mathbf{q}$ and $\mathbf{p}''\cdot\mathbf{q} \le \hat{\mathbf{p}}\cdot\mathbf{q}$. In fact, we must have strict inequality, otherwise the arc $\overset{\frown}{\mathbf{p}'\mathbf{p}}$ or $\overset{\frown}{\mathbf{p}''\mathbf{p}}$ on $\partial\mathbb{Y}$ will degenerate to line segment, contradicting the strict non-linearality of $\partial^+\mathbb{Y}$. Therefore, we have

$$\max_{\mathbf{p}\in\mathbb{Y}\setminus B(\hat{\mathbf{p}},\epsilon)} \mathbf{p}\cdot\mathbf{q} \le \max\{\mathbf{p}''\cdot\mathbf{q}, \mathbf{p}'\cdot\mathbf{q}\} < \hat{\mathbf{p}}\cdot\mathbf{q}$$

Let $\delta = (\hat{\mathbf{p}}\cdot\mathbf{q} - \max\{\mathbf{p}''\cdot\mathbf{q}, \mathbf{p}'\cdot\mathbf{q}\})/2$, then $\mathbf{p}\cdot\mathbf{q} < \hat{\mathbf{p}}\cdot\mathbf{q} - \delta$ for all $\mathbf{p} \in \mathbb{Y}\setminus B(\hat{\mathbf{p}}, \epsilon)$. Let $M = \sup_{\mathbf{p}\in\mathbb{Y}} R(\mathbf{p})$ and pick $\lambda > \frac{2M}{\delta}$. If $\mathbf{p}(\lambda\mathbf{q}) \in \mathbb{Y}\setminus B(\hat{\mathbf{p}}, \epsilon)$, then

$$\begin{aligned}
&\lambda\mathbf{q}\cdot\mathbf{p}(\lambda\mathbf{q}) - R(\mathbf{p}(\lambda\mathbf{q})) \\
&< \lambda(\hat{\mathbf{p}}\cdot\mathbf{q} - \delta) - (R(\hat{\mathbf{p}}) - 2M) \\
&< \lambda\hat{\mathbf{p}}\cdot\mathbf{q} - 2M - R(\hat{\mathbf{p}}) + 2M \\
&= \lambda\hat{\mathbf{p}}\cdot\mathbf{q} - R(\hat{\mathbf{p}}),
\end{aligned}$$

which contradict the fact that $\mathbf{p}(\lambda\mathbf{q})$ is the maximizer. Therefore $\mathbf{p}(\lambda\mathbf{q}) \in B(\hat{\mathbf{p}}, \epsilon)$, i.e., $\|\mathbf{p}(\lambda\mathbf{q}) - \hat{\mathbf{p}}\|_2 < \epsilon$ if $\lambda > \frac{2M}{\delta}$, thus $\lim_{\lambda\to\infty}\mathbf{p}(\lambda\mathbf{q}) = \hat{\mathbf{p}}$.

To prove that the limit process is uniform for all $\|\mathbf{q}\|_2 = 1$, $\mathbf{q} \in \mathbb{R}^n_+$, it is enough to show we can find the δ above uniformly for all such $\mathbf{q}$. Notice that by our construction, δ is a continuous function of $\hat{\mathbf{p}}$. Notice also that $\hat{\mathbf{p}}$ is a continuous function of $\mathbf{q}$ by strict non-linearity of $\partial^+\mathbb{Y}$, thus $\delta(\mathbf{q})$ is continuous which attains minimum value on the closed set $\{\|\mathbf{q}\|_2 = 1, \mathbf{q} \in \mathbb{R}^n_+\}$. The minimum must be positive as argued above and this positive minimum is the uniform δ we are looking for. □

The next lemma extends the above one and says that for a fixed purchase unit, the price change after the purchase goes to $\mathbf{0}$ as the market state goes to infinity.

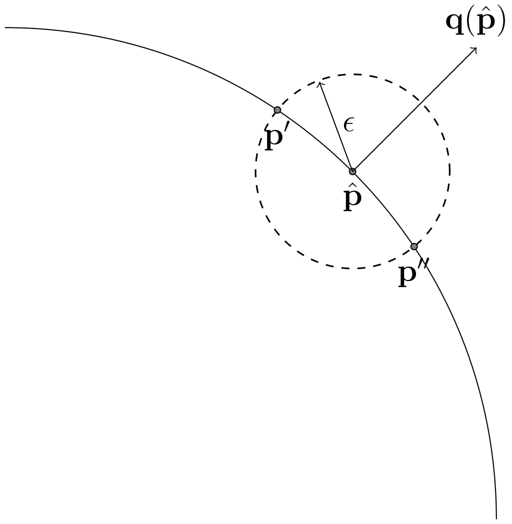


Figure 4.1: Lemma 1

Lemma 2. *Under the same condition of previous lemma, and for any i,*

$$lim_{\lambda\to\infty}\mathbf{p}(\lambda\mathbf{q}+\mathbf{e}_i)-\mathbf{p}(\lambda\mathbf{q})=0$$

uniformly for all $\|\mathbf{q}\|_2=1$, $\mathbf{q}\in\mathbb{R}^n_+$.

Proof. We already shown that $\lim_{\lambda\to\infty}\mathbf{p}(\lambda\mathbf{q})=\hat{\mathbf{p}}(\mathbf{q})$ uniformly so it is enough to show that $\lim_{\lambda\to\infty}\mathbf{p}(\lambda\mathbf{q}+\mathbf{e}_i)=\hat{\mathbf{p}}(\mathbf{q})$ uniformly as well. Let δ and M be defined as in the previous proof, let $P=\sup_{\mathbf{p}\in\mathbb{Y},i}|\mathbf{p}_i|$, pick $\lambda>\frac{2M+P}{\delta}$. We know P is finite since $\mathbb{Y}$ is bounded. Suppose $\mathbf{p}(\lambda\mathbf{q}+\mathbf{e}_i)\in\mathbb{Y}\setminus B(\hat{\mathbf{p}},\epsilon)$, then

$$\begin{aligned}
&(\lambda\mathbf{q}+\mathbf{e}_i)\cdot\mathbf{p}(\lambda\mathbf{q}+\mathbf{e}_i)-R(\mathbf{p}(\lambda\mathbf{q}+\mathbf{e}_i))\\
&=\lambda\mathbf{q}\cdot\mathbf{p}(\lambda\mathbf{q}+\mathbf{e}_i)+\mathbf{p}_i(\lambda\mathbf{q}+\mathbf{e}_i)-R(\mathbf{p}(\lambda\mathbf{q}+\mathbf{e}_i))\\
&<\lambda\mathbf{q}\cdot\hat{\mathbf{p}}-\lambda\delta+P-R(\hat{\mathbf{p}})+2M\\
&<\lambda\mathbf{q}\cdot\hat{\mathbf{p}}-R(\hat{\mathbf{p}})\\
&\leq\lambda\mathbf{q}\cdot\hat{\mathbf{p}}+\hat{\mathbf{p}}_i-R(\hat{\mathbf{p}})\\
&=(\lambda\mathbf{q}+\mathbf{e}_i)\hat{\mathbf{p}}-R(\hat{\mathbf{p}}),
\end{aligned}$$

which contradict the fact that $\mathbf{p}(\lambda\mathbf{q}+\mathbf{e}_i)$ is the maximizer. It follows that $\lim_{\lambda\to\infty}\mathbf{p}(\lambda\mathbf{q}+\mathbf{e}_i)=\hat{\mathbf{p}}$.

The uniformity proof is the same as previous lemma. □

Applying the above lemma for all direction $\mathbf{e}_i$ yields Equation 3.1. The next lemma shows the necessity of strictly non-linear positive boundary.

Lemma 3. *If $\partial^+\mathbb{Y}$ contains a line segment, then Equation 3.1 does not hold.*

Proof. Let $M=\sup_{\mathbf{p}\in\mathbb{Y}}|R(\mathbf{p})|$. Let $\overrightarrow{ab}$ be a line segment on $\partial^+\mathbb{Y}$. Let o be the mid-point of $\overrightarrow{ab}$, $\mathbf{q}$ be the norm of a supporting hyperplane at o about $\mathbb{Y}$ pointing outwards. Denote the length of $\overrightarrow{ob}$ as l. For $\lambda>0$, the price $\mathbf{p}(\lambda\mathbf{q})\in\mathbb{Y}$. Let s be the projection of $\mathbf{p}(\lambda\mathbf{q})$ onto $\overrightarrow{ab}$. This is shown in Figure 4.2.

Suppose s is on the ray of $\overrightarrow{ob}$, as is the case in the figure. Let $\mathbf{k}$ be a unit vector on the direction of $\overrightarrow{oa}$. Make a ball centered at $\mathbf{p}(\lambda\mathbf{q})$ with radius $l/2$, denote it as B. Use $\mathbf{a}$ to denote the position vector of a. For any point $\mathbf{p}'\in\mathbb{Y}\cap B$, $\mathbf{p}'\cdot\lambda\mathbf{q}\le\mathbf{a}\cdot\lambda\mathbf{q}$, $\mathbf{p}'\cdot\mathbf{k}\le\mathbf{a}\cdot\mathbf{k}-l/2$. Therefore

$$\begin{aligned}&\mathbf{p}'\cdot(\lambda\mathbf{q}+\frac{5M}{l}\mathbf{k})-R(\mathbf{p}')\\ \le\ &\mathbf{a}\cdot\lambda\mathbf{q}+\frac{5M}{l}(\mathbf{a}\cdot\mathbf{k}-l/2)+2M-R(\mathbf{a})\\ =\ &\mathbf{a}\cdot(\lambda\mathbf{q}+\frac{5M}{l}\mathbf{k})-M/2-R(\mathbf{a})\\ <\ &\mathbf{a}\cdot(\lambda\mathbf{q}+\frac{5M}{l}\mathbf{k})-R(\mathbf{a})\end{aligned}$$

This means $\mathbf{p}'$ can not maximize $\mathbf{p}\cdot(\lambda\mathbf{q}+\frac{5M}{l}\cdot\mathbf{k})-R(\mathbf{p})$, i.e., $\mathbf{p}'\neq\mathbf{p}(\lambda\mathbf{q}+\frac{5M}{l}\mathbf{k})$, thus

$$\|\mathbf{p}(\lambda\mathbf{q}+\frac{5M}{l}\mathbf{k})-\mathbf{p}(\lambda\mathbf{q})\|_2>l/2. \tag{4.1}$$

If s is on the ray of $\overrightarrow{oa}$, we can derive similarly that

$$\|\mathbf{p}(\lambda\mathbf{q}-\frac{5M}{l}\mathbf{k})-\mathbf{p}(\lambda\mathbf{q})\|_2>l/2. \tag{4.2}$$

Therefore for any $\lambda>0$, either (4.1) or (4.2) holds. As a result, $\lim_{\lambda\to\infty}\|\mathbf{p}(\lambda\mathbf{q}+\mathbf{r})-\mathbf{p}(\lambda\mathbf{q})\|_2=0$ can not be true for both $\mathbf{r}=\frac{5M}{l}\mathbf{k}$ and $\mathbf{r}=-\frac{5M}{l}\mathbf{k}$, thus Equation 3.1 can not hold. □

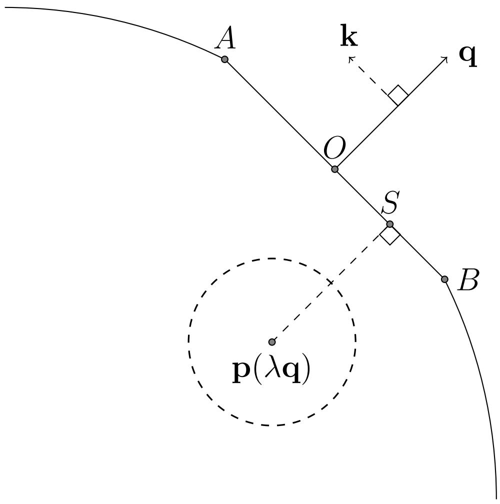


Figure 4.2: Lemma 3

Proposition 6 (Liquidity Adaptation)**.** *Consider a market with a cost function as in Equation 2.4, with R bounded. The market is liquidity adaptive if and only if* $\partial^+\mathbb{Y}$ *is strictly non-linear.*

Proof. This is a direct corollary of previous lemmas. □

Now we are ready to define a broad class of automated market makers that satisfy Conditions1-6 by concluding the propositions above. The next theorem serves this purpose.

Theorem 3. *Any buy-only market maker satisfying Conditions 1 and 2 must use a cost function as in Equation 2.4 for some* R *and* $\mathbb{Y}$*. Assuming* C *is closed and* R *is bounded, this market maker satisfies Conditions 4-6 if and only if (1)* $\mathbb{Y}$ *is a compact set in* $\mathbb{R}_+^n \setminus \{\mathbf{x} \in \mathbb{R}_+^n \mid \sum_{i=1}^n x_i < 1\}$*; (2)* $\Delta_n \subseteq \mathrm{extend}(\mathbb{Y})$*; (3)* $\partial^+\mathbb{Y}$ *contains no line segment. To satisfy Condition 3, it is sufficient to require additionally that* $R(\mathbf{p})$ *is strictly convex or affine with coefficient* $\mathbf{q}_0 \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$*.*

4.3 Conflicts with Expressiveness

Unfortunately, in order to gain liquidity adaptation, it is necessary to make sacrifices in terms of *expressiveness*, another desirable property defined by Abernethy et al. [2012] that can easily be adapted to buy-only markets. Here $\mathbf{b}$ represents a probability distribution over outcomes reflecting the belief of a trader.

Condition 7 (Buy-Only Expressiveness)**.** *$\forall \mathbf{p} \in \mathbb{R}^n_+$, $\forall \mathbf{b} \in \Delta_n$, $\exists \mathbf{r} \in \mathbb{R}^n_+$ s.t. $\mathbf{p}(\mathbf{q}+\mathbf{r}) = \mathbf{b}$.*

Theorem 4. *A market cannot be both liquidity adaptive and buy-only expressive.*

Proof. Suppose the market is buy-only expressive, then $\Delta_n \subseteq \mathbb{Y}$. Pick any $\mathbf{x} \in \mathrm{ri}\Delta_n$, i.e., $\mathbf{x}$ is a distribution without zero component.

If $\mathbf{x} \in \partial\mathbb{Y}$, then the supporting hyperplane of $\mathbb{Y}$ at $\mathbf{x}$ must be $\{\mathbf{x} \cdot \mathbf{1} = 1\}$, then every point of Δ_n is in $\partial\mathbb{Y}$, conflicting the strict non-linearality of $\partial^+\mathbb{Y}$, which is necessary for liquidity adaptation as shown in Proposition 6.

If $\mathbf{x} \in \mathrm{ri}\mathbb{Y}$, then there is a closed ball $B(\mathbf{x}, \epsilon)$ in $\mathrm{ri}\mathbb{Y}$ centered at $\mathbf{x}$ with radius $\epsilon > 0$. Recall from Chapter 2 that $\mathbb{Y}$ is the effective domain of R and notice that a convex function must be finite on a closed set contained in the relative interior of its effective domain, hence R must be bounded on $B(\mathbf{x}, \epsilon)$. Let $M = \sup_{\mathbf{p} \in B(\mathbf{x},\epsilon)} |R(\mathbf{p})|$, choose any $\mathbf{q} \in \mathbb{R}^n_+$ such that $\|\mathbf{q}\|_2 > \frac{2M}{\epsilon}$. Then for any $\mathbf{r} \in \mathbb{R}^n_+$, $\mathbf{x}$ can not maximize $\mathbf{p} \cdot (\mathbf{q} + \mathbf{r}) - R(\mathbf{p})$ on $\mathbb{Y}$, even only on $B(\mathbf{x}, \epsilon)$, since some point on the boundary of $B(\mathbf{x}, \epsilon)$ gives larger objective value. Therefore $\mathbf{x} \neq \mathbf{p}(\mathbf{q} + \mathbf{r})$ for all $\mathbf{r} \in \mathbb{R}^n_+$, thus the market is not buy-only expressive. □

CHAPTER 5

Information Loss

Without expressiveness, there may exist a market state $\mathbf{q}$ such that traders with distinct beliefs $\mathbf{b}$ and $\mathbf{b}'$ both will not trade, making them impossible to distinguish. The inability of expressing exactly a trader's belief result in loss of information. Define the *Belief Set* $B(\mathbf{q})$ given the current market state $\mathbf{q}$ as the set of underlying beliefs with which a trader is not willing to trade. For perfect market where traders can express their belief uniquely, the belief set should be at most a single point. Smaller $B(\mathbf{q})$ indicate more accurate market. In this sense, we define *information loss* as 1-norm diameter of $B(\mathbf{q})$.

Definition 4 (Information Loss)**.** *Information loss at market state* $\mathbf{q}$ *is defined as* $Loss(\mathbf{q}) = \sup\{\|\mathbf{b}_1 - \mathbf{b}_2\|_1 \mid \mathbf{b}_1, \mathbf{b}_2 \in B(\mathbf{q})\}$.

We derive two information loss bound. At first, we assume that a trader can purchase arbitrary non-negative bundle, then we discuss a more complicated case where only purchase of fixed unit is allowed.

5.1 Information Loss With Arbitrarily Small Purchase

Let us first assume that traders can purchase arbitrarily small bundle, a condition to simplify the problem that will be removed in the next section.

Lemma 4. *If arbitrarily small purchase is allowed, then* $\mathbf{b} \in B(\mathbf{q})$ *implies* $\mathbf{b} \preceq \mathbf{p}(\mathbf{q})$. *In other words, the current prices forms a component-wise upper bound of the underlying belief.*

Proof. When arbitrary small purchase is allowed in buy-only market, a myopic trader with belief $\mathbf{b}$ who wants to maximize expected payoff should calculate his purchase $\mathbf{r}$ by the following optimization problem (We use the equivalent problem of minimizing minus expected payoff):

$$\text{minimize}_{\mathbf{r}} \quad C(\mathbf{q}+\mathbf{r}) - \mathbf{b}\cdot(\mathbf{q}+\mathbf{r})$$

$$\text{s.t.} \quad \mathbf{r} \in \mathbb{R}^n_+$$

We already assumed that the cost function is differentiable, then the partial derivative of the objective function is $\frac{\partial(C(\mathbf{q}+\mathbf{r})-\mathbf{b}\cdot(\mathbf{q}+\mathbf{r}))}{\partial r_i} = p_i(\mathbf{q}+\mathbf{r}) - b_i$. By optimal condition for optimization in nonnegative orthant [Boyd and Vandenberghe, 2004], we have

$$\mathbf{r}\text{ optimal} \iff r_i \geq 0,\ p_i(\mathbf{q}+\mathbf{r}) - b_i \geq 0,\ r_i\left(p_i(\mathbf{q}+\mathbf{r}) - b_i\right) = 0 \quad \text{for all } i$$

The second condition $p_i(\mathbf{q}+\mathbf{r}) - b_i \geq 0$ proves the lemma. □

The meaning of the previous lemma is clear: a trader is always willing to buy a security whose price is less than the estimated probability that event will happen. Therefore, the trader is not willing to buy any security if and only if his belief is component-wisely upper bounded by the price. Now we give the first information loss bound.

Theorem 5. *If arbitrary small purchase is allowed, then the information loss is bounded above by* $2\left(\max_{\mathbf{p}\in\mathbb{Y}} \sum p_i - 1\right)$.

Proof. The l_1 distance between belief $\mathbf{b} \in B(\mathbf{q})$ and the price $\mathbf{p}(\mathbf{q})$ can be bounded above as follows, notice that $p_i \geq b_i$ for all i by Lemma 4:

$$\|\mathbf{b}-\mathbf{p}\|_1 = \sum |b_i - p_i| = \sum (p_i - b_i) = \sum p_i - 1 \leq \max_{\mathbf{p}\in\mathbb{Y}} \sum p_i - 1$$

By triangular inequality, for any two belief in $\mathbf{b}_1$ and $\mathbf{b}_2$ in $B(\mathbf{q})$

$$\|\mathbf{b}_1-\mathbf{b}_2\|_1 \leq \|\mathbf{b}_1-\mathbf{p}\|_1 + \|\mathbf{b}_2-\mathbf{p}\|_1 \leq 2\left(\max_{\mathbf{p}\in\mathbb{Y}} \sum p_i - 1\right)$$

The theorem follows by the definition of information loss. □

5.2 Information Loss With Minimum Purchase Unit

In real world securities market, purchase are based on some smallest allowable unit ϵ. In this more refined scenario, the information loss comes in two ways, by the nature of the market and by the inability to accurately express belief via discretized purchase. A myopic trader with belief $\mathbf{b}$ will not trade if any allowable purchase costs more than his expected payoff, i.e.,

$$\mathbf{b} \cdot \mathbf{r} < C(\mathbf{q} + \mathbf{r}) - C(\mathbf{q}) \;\; \forall \text{ legal } \mathbf{r}, \tag{5.1}$$

where a purchase bundle $\mathbf{r}$ is "legal" if it is not $\mathbf{0}$ and its components are non-negative multiples of ϵ. The belief set give $\mathbf{q}$ is then

$$B(\mathbf{q}) = \{\mathbf{b} \mid \mathbf{b} \cdot \mathbf{r} < C(\mathbf{q} + \mathbf{r}) - C(\mathbf{q}) \;\; \forall \text{ legal } \mathbf{r}\}.$$

We can derive the following upper bound.

Theorem 6. *Assume C is twice differentiable and ϵ is small, then for all $\mathbf{q} \in \mathbb{R}^n_+$,*

$$\text{Loss}(\mathbf{q}) \leq 2\big(\max_{\mathbf{p} \in \mathbb{Y}} \sum_{i=1}^{n} \mathbf{p}_i - 1\big) + 2\epsilon n \lambda_{max}(\nabla^2 C(\mathbf{q})). \tag{5.2}$$

Proof. Suppose $\mathbf{b} \in B(\mathbf{q})$, then for all legal $\mathbf{r}$

$$\mathbf{b} \cdot \mathbf{r} < C(\mathbf{q} + \mathbf{r}) - C(\mathbf{q}).$$

Since $\mathbf{r} = \epsilon \mathbf{e}_i$ is legal, $i = 1, 2, \ldots, n$, we have

$$\mathbf{b} \cdot \epsilon \mathbf{e}_i = \epsilon b_i < C(\mathbf{q} + \epsilon \mathbf{e}_i) - C(\mathbf{q}).$$

Under the assumption that C is twice differentiable and ϵ is small, by Taylor's theorem

$$\begin{aligned} & C(\mathbf{q} + \epsilon \mathbf{e}_i) \\ \approx\; & C(\mathbf{q}) + \nabla C(\mathbf{q}) \cdot \epsilon \mathbf{e}_i + \epsilon \mathbf{e}_i^T \nabla^2 C(\mathbf{q}) \epsilon \mathbf{e}_i \\ =\; & C(\mathbf{q}) + \epsilon p_i + \epsilon^2 (\nabla^2 C(\mathbf{q}))_{ii} \\ \leq\; & C(\mathbf{q}) + \epsilon p_i + \epsilon^2 \lambda_{max} \nabla^2 (C(\mathbf{q})) \end{aligned}$$

Plug into above inequality, we have

$$b_i \leq p_i + \epsilon\lambda_{max}\nabla^2(C(\mathbf{q})).$$

Let $\epsilon = 0$, which means we allow arbitrarily small purchase, the above inequality become $b_i \leq p_i$, which is exactly Lemma 4. However, with non-zero ϵ, the price upper bound is relaxed, leading to larger belief set and thus greater information loss.

Let $s_i = \max\{0, b_i - p_i\}$ as the slackness of price upper bound, then $0 \leq s_i \leq \epsilon\lambda_{max}\nabla^2(C(\mathbf{q}))$. $\mathbf{p}+\mathbf{s}$ is now a componentwise upper bound of $\mathbf{b}$, follow the same steps in the proof of Theorem 5, we have

$$\begin{aligned}
&\|\mathbf{b}_1 - \mathbf{b}_2\|_1 \\
\leq\ &\|\mathbf{b}_1 - (\mathbf{p}+\mathbf{s}_1)\|_1 + \|\mathbf{b}_2 - (\mathbf{p}+\mathbf{s}_2)\|_1 \\
\leq\ &2\left(\max_{\mathbf{p}\in\mathbb{Y}} \sum p_i + \sum s_i - 1\right) \\
\leq\ &2\left[\left(\max_{\mathbf{p}\in\mathbb{Y}} \sum p_i - 1\right) + n\epsilon\lambda_{max}\nabla^2(C(\mathbf{q}))\right]
\end{aligned}$$

□

The above bound consists of two terms. The first term, $\max_{\mathbf{p}\in\mathbb{Y}} \sum p_i - 1$, measures how much the price range $\mathbb{Y}$ deviates from the probability simplex. We call this the *hard information loss*, or simply hard loss, since it is invariant against the instantaneous liquidity. As we have seen in the proof of Theorem 4, it is necessary for $\mathbb{Y}$ to deviate from Δ_n in order to gain liquidity adaptation. Thus, we conclude that the hard loss is due to the nature of liquidity adaptive market. The hard loss still present even if we allow arbitrarily small purchase, as shown in Theorem 5.

We call the second term, $n\epsilon\lambda_{max}\nabla^2(C(\mathbf{q}))$, as *soft information loss* or simply soft loss. Soft loss is due to market liquidity, it emerges with the introduction of minimum purchase unit ϵ. Notice that the soft loss will diminish if $\lambda_{max}\nabla^2(C(\mathbf{q}))$ goes to zero, meaning infinite liquidity. Notice also that the soft loss is controlled by n, the dimension of the complete market, since the inaccuracy of the market, measured by l_1 norm, grows linearly with the number of beliefs it tries to estimate. One thing to notice is that for asymptotic homogeneous market, the soft loss goes to zero eventually since the market liquidity grows infinitely with market state $\mathbf{q}$. This means try to reduce hard loss is more meaningful than reduce soft loss in a long run, an important consideration when designing optimal market in Chapter 8. The soft loss is similar to the *bid-ask spread* examined in previous work [Chen and Pennock, 2007, Abernethy et al., 2012].

CHAPTER 6

Estimating Underlying Belief

The ultimate purpose of automated market maker is to gather information from traders and estimate the collaborated belief. It will be meaningless to design fancy market makers without the ability to recover the belief of traders. For translation invariant market satisfying expressiveness, people never worry about how to recover belief using market state, since the market price just does the work. For asymptotic homogeneous market, traders of different beliefs could maximize their expected payoff with the same market state, as we just saw in previous chapter. Specifically, the results from last chapter says that when arbitrarily small purchase is allowed, any belief that is upper bounded by current price is a candidate; and when we consider a minimum purchase unit, the belief is not exactly upper bounded by price, but with some slackness $\epsilon \cdot \text{diag}(\nabla^2 C(\mathbf{q}))$. In general, our task of estimating underlying belief is to find a single belief in the belief set $B(\mathbf{q})$ that is best in the sense of minimizing some error measure. Note that the belief set is always in the following form, here we use $\mathbf{p}$ to represent either the price or the price plus slackness.

$$B(\mathbf{q}) = \{\mathbf{b} \mid \mathbf{b} \preceq \mathbf{p},\ \mathbf{b} \in \Delta_n\} \tag{6.1}$$

6.1 Expectation of Beliefs

It may appear that the expectation of underlying belief may be a good choice, which by definition, will minimize the mean square error. However, we can not calculate the expectation unless we have some prior distribution about beliefs of traders, which in not generally available.

6.2 Chebyshev Center

A better choice when we have no prior knowledge about beliefs is the worst case center, i.e., the point in the set that has shortest worst case distance from any other points. If we use Euclidean distance, then this point is usually called the Chebyshev center, as defined below, where we use $\mathbf{b}^*$ to denote this optimal choice:

$$\mathbf{b}^* = \arg\min_{\mathbf{b}} \max_{\mathbf{x}\in B(\mathbf{q})} \|\mathbf{b} - \mathbf{x}\|_2$$

Chebyshev center could also be characterized as the center of smallest ball that encloses a set. However, finding the Chebyshev center of arbitrary set is not an easy task. First of all, let us notice that the belief set $B(\mathbf{q})$ as given in Equation 6.1 is just a $n-1$ dimensional polyhedron defined by $2n$ linear inequality constraints and one linear equality constraint. It is quite obvious that the Chebyshev center of a convex polyhedron is the same as the Chebyshev center of its extreme points. Moreover, finding the Chebyshev center of a finite set of points in $\mathbb{R}^n$ is easy. Suppose the set of extreme points of $B(\mathbf{q})$ is $S = (\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_m)$, then the Chebyshev center of S is given by the following unconstrained convex optimization problem:

$$minimize \quad \max_{i} \|\mathbf{x} - \mathbf{s}_i\|_2 \tag{6.2}$$

Notice that we don't need the explicit constraint that $x \in \Delta_n$ in our case. This is because $B(\mathbf{q})$ is contained in Δ_n, so does its extreme points, hence optimal solution must lie in Δ_n as well. This problem could be easily formulated as a standard SOCP (Second Order Cone Programming) [Boyd and Vandenberghe, 2004], which is readily solvable:

$$\begin{aligned} &minimize \quad t \\ &s.t. \quad \|\mathbf{x} - \mathbf{s}_i\|_2 \le t, \; i = 1, 2, \ldots, m \end{aligned}$$

Notice that if we use L_1 or L_∞ norm instead of Euclidean norm, the above problem could be solved as LP (Linear Programming). Also, there exists randomized methods to find Chebyshev center of finite set in linear expected time [Botkin and Turova-Botkina, 1994]

The hardness in our case comes from the number of extreme points m, which could be exponential to the $2n$ linear inequality constraints that defines the belief set. The following section shows that m could actually achieve an exponential of n, therefore provides a hardness result.

6.3 Hardness of finding Chebyshev center

Definition 5 (Extreme Point). *A point $\mathbf{x}$ is an extreme point of a convex set E if it is in E and there is no points $\mathbf{y}_1$ and $\mathbf{y}_2$ in E, such that $\mathbf{x} \neq \mathbf{y}_1$, $\mathbf{x} \neq \mathbf{y}_2$ and $\mathbf{x}$ is a convex combination of $\mathbf{y}_1$ and $\mathbf{y}_2$.*

Lemma 5. *$B(\mathbf{q})$ as defined in (6.1), then x is and extreme point of $B(p)$, if and only if there is a subset of indices $I \subsetneq \{1, 2, \ldots, n\}$ and another index $j \notin I$, such that the following three conditions hold:*

1. $\sum_{i \in I} p_i \leq 1$
2. $\left(\sum_{i \in I} p_i\right) + p_j \geq 1$
3. $x_i = \begin{cases} p_i & \text{if } i \in I \\ 1 - \sum_{i \in I} p_i & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$

Proof. Notice that only condition 3 is essential, which says that x_i takes the extreme value 0 or p_i except for $i = j$. The first two conditions simply make sure that the values of x_i are valid.

"⇒". Suppose condition 3 doesn't hold, then there are two indices $j \neq k$, such that neither x_j nor x_k takes extreme value. Let $\mathbf{x}'$ and $\mathbf{x}''$ be defined as

$$x'_i = \begin{cases} x_i + \epsilon & \text{if } i = j, \\ x_i - \epsilon & \text{if } i = k, \\ x_i & \text{otherwise.} \end{cases} \qquad x''_i = \begin{cases} x_i - \epsilon & \text{if } i = j, \\ x_i + \epsilon & \text{if } i = k, \\ x_i & \text{otherwise.} \end{cases}$$

Then there exists some $\epsilon > 0$, such that both $\mathbf{x}'$ and $\mathbf{x}''$ are in $B(\mathbf{q})$. On the other hand, $x = (x' + x'')/2$, thus x is not an extreme point by definition.

"$\Leftarrow$". Suppose x satisfies all the conditions and x is the convex combination of $\mathbf{x}' \in B(\mathbf{q})$ and $\mathbf{x}'' \in B(\mathbf{q})$, $\mathbf{x} \neq \mathbf{x}'$, $\mathbf{x} \neq \mathbf{x}''$. Then $\mathbf{x}'$ must have at least two components that are different from that of $\mathbf{x}$, due to the fact that the components must sum up to one. By condition 3, there is at most one index j such that x_j is does not take extreme value, thus there must be some $k \neq j$ such that $x_k \neq x'_k$ and x_k takes extreme value. WLOG, suppose $x_k > x'_k$, then $x_k < x''_k$. This contradicts the fact that x_k takes extreme value. □

In the following discussion, we call i a full index if $x_i = p_i$, a zero index if $x_i = 0$ and a slack index otherwise. The above lemma says that for $\mathbf{x}$ to be an extreme point, it can has at most one slack index. It is possible that all components of $\mathbf{x}$ are extreme values, in which case we pick any index as the slack index and we can conveniently say that all extreme points have exactly one slack index. Now we are ready to give the hardness theorem.

Theorem 7. *$B(\mathbf{q})$ can have up to $n \cdot \binom{\lceil \frac{n-1}{2} \rceil}{n-1} > 2^{n-1}$ extreme points.*

Proof. If n is even, pick price $\mathbf{p}$ such that $p_i = \frac{2}{n+1}$ for all i.Then by Lemma 5, $\mathbf{x}$ is an extreme point if and only if $x_i = p_i = \frac{2}{n+1}$ for $\frac{n}{2}$ indices, $x_i = 0$ for $\frac{n}{2} - 1$ indices and $x_i = \frac{1}{n+1}$ for the slack index. Therefore, the number of extreme points is $n \cdot \binom{\frac{n}{2}}{n-1}$.

If n is odd, pick p such that $p_i = \frac{2}{n}$ for all i. Similarly, we have the number of extreme points is $n \cdot \binom{\frac{n-1}{2}}{n-1}$. In general, we have the number of extreme points as $n \cdot \binom{\lceil \frac{n-1}{2} \rceil}{n-1} \geq n \cdot \frac{2^{n-1}}{n-1} > 2^{n-1}$. The first inequality follows from binomial expansion. □

The next theorem proves that $n \cdot \binom{\lceil \frac{n-1}{2} \rceil}{n-1}$ is actually a tight upper bound of the number of extreme points. This theorem is of less significance, but might be a useful guide if we want to find the Chebyshev center by solving (6.2) for small n. Notice that there are many algorithms for enumerating extreme points, such as [Dyer and Proll, 1977].

Theorem 8. *$B(\mathbf{q})$ can have at most $n \cdot \binom{\lceil \frac{n-1}{2} \rceil}{n-1}$ extreme points.*

Proof. We assume that n is odd, the proof is very similar for n even. Also for simplicity, we assume that $p_i > 0$ for all i. If there are some $p_i = 0$ we can just ignore that component and decrease n accordingly.

Let m_k be the number of extreme points with k zero indices, $0 \le k \le n-1$. Then the number of extreme points is

$$m = \sum_{k=0}^{n-1} m_k$$

Let X be the set of extreme points, let Π be the set of all permutation of $[n]$. For every $\pi \in \Pi$, there is an unique j, such that $\sum_{i \le j} p_{\pi(i)} \le 1$ but $\sum_{i \le j+1} p_{\pi(i)} \ge 1$. Thus, there is a unique $\mathbf{x} \in X$, such that $\{\pi(i) \mid i \le j\}$ is the set of full indices, $\{\pi(i) \mid i \ge j+2\}$ is the set of zero indices and $\pi(j+1)$ is the slack index. We denote this as a mapping $f : \Pi \to X$. For any $\mathbf{x}$ that has k zero indices, then the size of its pre-image $|f^{-1}(\mathbf{x})| = k! \cdot (n-1-k)!$, since the order of all full indices and of all zero indices in the permutation is arbitrary. Thus we have

$$n! = |\Pi| = \sum_{\mathbf{x} \in X} |f^{-1}(\mathbf{x})| = \sum_{k=0}^{n-1} m_k \cdot k!(n-1-k)!$$

Since $k!(n-1-k)! \ge \left[(\frac{n-1}{2})!\right]^2$, we have

$$n! = \sum_{k=0}^{n-1} m_k \cdot k!(n-1-k)! \ge \sum_{k=0}^{n-1} m_k \left[(\frac{n-1}{2})!\right]^2 = m \left[(\frac{n-1}{2})!\right]^2$$

Thus $m \le \frac{n!}{\left[(\frac{n-1}{2})!\right]^2} = n \cdot \frac{(n-1)!}{\left[(\frac{n-1}{2})!\right]^2} = n \cdot \binom{n-1}{\frac{n-1}{2}}$. □

6.4 A Special Case Where Chebyshev Center is Easy to Find

In spite of the hardness result, there are some cases where the Chebyshev center is efficiently computable. We will now give such a case, which, when n is small or sum of prices is close to 1, occurs quite often.

From (6.1), we notice that the structure of $B(\mathbf{q})$ is not complicated and can be represented as $B(\mathbf{q}) = \Delta_1 \cap \Delta_2$, where $\Delta_1 = \{\mathbf{x} \mid x_i \ge 0,\ \sum x_i = 1\}$ is the probability simplex, $\Delta_2 = \{\mathbf{x} \mid x_i \le p_i,\ \sum x_i = 1\}$ is another simplex. Notice that if $\Delta_2 \subset \Delta_1$, then

$B(\mathbf{q}) = \Delta_2$. Notice also that Δ_2 is symmetric and its Chebyshv center is just the mean of its n extreme points. The next theorem formalize this idea.

Theorem 9. *If there is no proper subset of indices $I \subsetneq \{1, 2, \ldots, n\}$ such that $\sum_{i \in I} p_i > 1$, then the Chebyshev center of $B(\mathbf{q})$ can be calculated as $x_i^* = p_i - \delta$, where $\delta = \frac{\sum_1^n p_i - 1}{n}$.*

Proof. Suppose $\mathbf{x} \in \Delta_2$, then $x_i \le p_i$ and $\sum x_i = 1$. We have

$$x_i = 1 - \sum_{j \neq i} x_j \ge 1 - \sum_{j \neq i} p_i \ge 0.$$

thus $x \in \Delta_1$. The last step follows from the assumption that no proper subset of p_i's sum up to more than 1.

Now we can calculate the Chebyshev center of $B(\mathbf{q})$ as the mean of the extreme points of δ_2. Notice that each extreme point of Δ_2 has one slack index and $n - 1$ full indices, the result follows after simple calculation. □

The above procedure could be described as "normalizing by sum", rather than the usually used normalization which is "normalizing by ratio". As a by product, we can show that the usual normalization is suboptimal regarding worst case distance. Consider the case where $n = 2$, and $\mathbf{p} = (0.65, 0.45)$. Normalized it by sum, we have $\mathbf{b}^* = (0.6, 0.4)$, but the usual normalization does not give the same result.

6.5 Approximation of Chebyshev Center and Other Centers

Now that the Chebyshev center is hard to find in general, we may want to consider approximation of Chebyshev center or other definition of centers that are efficiently computable.

Theorem 10. *Any point in belief set $B(\mathbf{q})$ is a 2-approximation of Chebyshev center, with respect to worst case Euclidean distance.*

Proof. Notice that the distance from any two points in $B(\mathbf{q})$ is at most the diameter of its smallest enclosing ball, which is twice the radius of the ball. □

An easy way to find a point in $B(\mathbf{q})$ is simply normalizing $\mathbf{p}$ in the usual way since we will only decrease each component. This is a justification for normalization as proposed by Othman et al. [2010].

We can also consider the center of largest inscribed ball or the center of largest volume inscribed ellipsoid of $B(\mathbf{q})$. Finding these two centers could be formulated as convex optimization problems with constraints polynomial of n, as discussed in Boyd and Vandenberghe [2004] (the authors call the center of largest inscribed ball also as Chebyshev center, but we will use this name only for the center of smallest enclosing ball). The meaning of these two centers is not as clear as the Chebyshev center, but intuitively the center of largest volume inscribed ellipsoid seems to be a good approximation. Notice that the linear constraints that defines $B(\mathbf{q})$ have very regular directions. Also notice that if $B(\mathbf{q})$ is parallelohedron or symmetric simplex, then the Chebyshev center coincide with the center of largest volume inscribed ellipsoid.

CHAPTER 7

Making A Profit

7.1 Asymptotic Profit Rate

By deviating the price range from the probability simplex, liquidity adaptive markets have the potential to earn a profit. When analyzing their market, Othman et al. [2010] defined the profit region as the set of final states $\mathbf{q}$ at which the market maker earns a guaranteed profit, regardless of the outcome. We take a different approach and consider a market's *asymptotic profit rate*, or simply profit rate, as the ratio between the market maker's total profit cut and the total amount of money accumulated by the market given large $\mathbf{q}$. An asymptotic profit rate of greater than 0 guarantees that the market will make money in the case that $\mathbf{q}$ grows sufficiently large, while a profit rate less than 0 implies unbounded loss. When the profit rate is exactly 0, the market can make a proof, have bounded loss or have unbounded sublinear loss, and we can not say much about it.

Definition 6 (Asymptotic Profit Rate)**.** *The asymptotic profit rate of a market maker with cost function C, denoted by ρ, is defined as*

$$\rho = \lim_{\lambda \to \infty} \inf_{\|\mathbf{q}\|_2 = 1, \mathbf{q} \in \mathbb{R}_+^n} \frac{C(\lambda \mathbf{q}) - C(\mathbf{0}) - \lambda \max_i q_i}{C(\lambda \mathbf{q}) - C(\mathbf{0})}.$$

Just like when calculating the monetary loss, we calculate the profit rate with respect to the worst possible $\mathbf{q}$ and outcome i for the market maker. We can interpret it as the worst case guaranteed profit: given any $\epsilon > 0$, there is some $\lambda > 0$ such that if the market ends up with $\|\mathbf{q}\|_2 > \lambda$, then no matter what specific value $\mathbf{q}$ it takes and what the actual outcome i is, the market maker earns at least $(\rho - \epsilon)(C(\mathbf{q}) - C(\mathbf{0}))$, or loses at most the negative.

The next theorem shows that there is close relationship between the asymptotic profit rate and the shape of $\mathbb{Y}$.

Theorem 11. *For asymptotic homogeneous market, if R is bounded on $\mathbb{Y}$, then the asymptotic profit rate is*

$$\rho = 1 - 1/(\min_k \max_{\mathbf{p}\in extend(\mathbb{Y})} p_k) \tag{7.1}$$

Proof. Use k to denote the index where q_k is greatest and let $M = \sup_{\mathbf{p}\in\mathbb{Y}} |R(\mathbf{p})| < \infty$, we have

$$\begin{aligned}
&\frac{C(\lambda\mathbf{q}) - C(\mathbf{0}) - \lambda \max_i q_i}{C(\lambda\mathbf{q}) - C(\mathbf{0})}\\
&= 1 - \frac{\lambda q_k}{C(\lambda\mathbf{q}) - C(\mathbf{0})}\\
&= 1 - \frac{\lambda q_k}{\max_{\mathbf{p}\in\mathbb{Y}}(\lambda\mathbf{p}\cdot\mathbf{q} - R(\mathbf{p})) - \min_{\mathbf{p}\in\mathbb{Y}} R(\mathbf{p})}\\
&\geq 1 - \frac{\lambda q_k}{\max_{\mathbf{p}\in\mathbb{Y}}(\lambda\mathbf{p}\cdot\mathbf{q} - M) - M}\\
&= 1 - \frac{q_k}{\max_{\mathbf{p}\in\mathbb{Y}} \mathbf{p}\cdot\mathbf{q} - 2M/\lambda}
\end{aligned}$$

Consider $\hat{\mathbf{p}} \in \mathbb{Y}$ that maximized its kth component, i.e., $\hat{p}_k = \max_{\mathbf{p}\in\mathbb{Y}} p_k$. Then $\max_{\mathbf{p}\in\mathbb{Y}} \mathbf{p}\cdot\mathbf{q} \geq \hat{\mathbf{p}}\cdot\mathbf{q} = (\max_{\mathbf{p}\in\mathbb{Y}} p_k)\cdot q_k$. Plug into above equation, we have

$$\begin{aligned}
&\frac{C(\lambda\mathbf{q}) - C(\mathbf{0}) - \lambda \max_i q_i}{C(\lambda\mathbf{q}) - C(\mathbf{0})}\\
&\geq 1 - \frac{q_k}{\max_{\mathbf{p}\in\mathbb{Y}} \mathbf{p}\cdot\mathbf{q} - 2M/\lambda}\\
&\geq 1 - \frac{q_k}{(\max_{\mathbf{p}\in\mathbb{Y}} p_k)\cdot q_k - 2M/\lambda}\\
&= 1 - \frac{1}{\max_{\mathbf{p}\in\mathbb{Y}} p_k - 2M/(\lambda q_k)}\\
&\geq 1 - \frac{1}{\max_{\mathbf{p}\in\mathbb{Y}} p_k - 2\sqrt{n}M/\lambda}
\end{aligned}$$

To get the last inequality, notice $\|\mathbf{q}\|_2 = 1$, $\mathbf{q}\in\mathbb{R}^n_+$ and q_k is the largest component, then $q_k \geq \frac{1}{\sqrt{n}}$. The above inequality holds for any $\|\mathbf{q}\|_2 = 1$, $\mathbf{q}\in\mathbb{R}^n_+$, thus the following holds for the infimum over all $\mathbf{q}$:

$$\frac{C(\lambda\mathbf{q}) - C(\mathbf{0}) - \lambda \max_i q_i}{C(\lambda\mathbf{q}) - C(\mathbf{0})} \geq 1 - \frac{1}{\min_k \max_{\mathbf{p}\in\mathbb{Y}} p_k - 2\sqrt{n}M/\lambda}$$

Take λ to infinity, we get $\rho \geq 1 - 1/\min_k \max_{\mathbf{p}\in\mathbb{Y}} p_k$.

To see why this is actually an equality, take $\mathbf{q} = \lambda\mathbf{e}_i$, where $i = \arg\min_k \max_{\mathbf{p}\in\mathbb{Y}} p_k$, and suppose outcome i actually happens, then it is easy to verify that the profit rate is exactly $1 - 1/\min_k \max_{\mathbf{p}\in\mathbb{Y}} p_k$.

The last step is to notice that $\max_{\mathbf{p}\in\mathbb{Y}} p_k = \max_{\mathbf{p}\in extend(\mathbb{Y})} p_k$. □

From the above theorem, we see that if R is bounded, then the asymptotic profitability of the market is only controlled by the shape of $\mathbb{Y}$, coincide with our earlier intuition that asymptotic homogeneous market behave like its homogeneous counterpart given large $\mathbf{q}$.

7.2 Examples

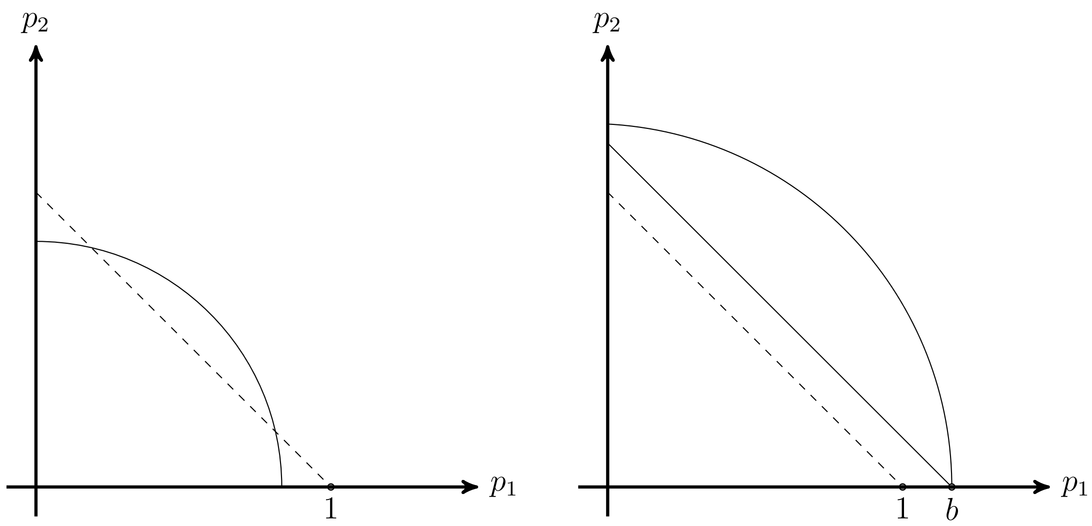
It might be useful to think about several examples to understand the relationship between the shape of $\mathbb{Y}$ and profitability. We will always assume R to be bounded in these examples.

Let us first consider a translation invariant market, where $\mathbb{Y} = \Delta_n$. In this case $\max_{\mathbf{p}\in\mathbb{Y}} p_k = 1$ for all k, thus $\rho = 0$ exactly. From the above theorem, we are not sure whether the market is profitable or suffering from loss. But more careful inspection tells us the loss is bounded [Abernethy et al., 2011]. In fact, potential profit is also bounded by symmetry.

In Proposition 4, we showed that the market has bounded loss if and only if $\Delta_n \subseteq extend(\mathbb{Y})$. It is not hard to see that this condition is equivalent to $\mathbf{e}_i \in \text{extend}(\mathbb{Y})$ for all i. If the condition doesn't hold, as shown in Figure 7.1(a), then there is some $\mathbf{e}_k \notin \text{extend}(\mathbb{Y})$, thus $\min_k \max_{\mathbf{p}\in\mathbb{Y}} p_k < 1$. We therefore have $\rho < 0$ by Equation 7.1, indicating unbounded loss.

The more desirable case that $\rho > 0$ happens when Δ_n is contained in the interior of $extend(\mathbb{Y})$, as shown in Figure 7.1(b). To calculate ρ pictorially, let us draw a simplex

$\{\mathbf{x}|x_i \geq 0, \sum x_i = b\}$ contained in $extend(\mathbb{Y})$ and make b as large as possible, then it is easy to show $b = \min_k \max_{\mathbf{p}\in\mathbb{Y}} p_k$ and the profit rate is therefore $1 - 1/b$. This observation will be useful when we design optimal market.



(a) Market with unbounded loss

(b) Market with guaranteed profit

Figure 7.1: Example of markets with different profitability, the dashed line represents probability simplex

CHAPTER 8

Optimal Market Design

In this chapter, we will discuss the design of optimal market, in the sense of minimizing information loss while maximizing profit, via results we established in previous chapters. In our current discussion, we will try to achieve our goal by designing the optimal shape of $\mathbb{Y}$ without worrying too much about how to define R. We will just assume that R is some bounded function on $\mathbb{Y}$. The reason is that $\mathbb{Y}$ controls the asymptotic behavior of the market such as hard loss and asymptotic profit rate while R is in charge of modifying the market's initial behavior, and we are much more interested in those asymptotic behaviors. If the market runs long enough, how much will the initial performance contributes to the whole? Othman and Sandholm [2011] proposed a class of homogeneous market called ball market maker, with $\mathbb{Y}$ being a l_p ball centered at origin with radius 1, and p is chosen properly according to maximum profit cut. Although this sounds like a very natural choice, we will show as a main result in this chapter that the optimal market, in our previous sense, should have $\mathbb{Y}$ being the Euclidean(l_2) ball and the radius and center need not to be 1 and origin respectively. Just for convenience, when we say $\mathbb{Y}$ is a ball, we actually means $\mathbb{Y}$ is the intersection of a ball and positive orthant.

8.1 Complete Price Range

We say the price range $\mathbb{Y}$ is complete, if $extend(\mathbb{Y}) = \mathbb{Y}$. It is complete in the sense that it can not be extended further, as given in the next proposition.

Proposition 7. *Let $\mathbb{Y}$ be any subset of $\mathbb{R}^n$, then $extend(extend(\mathbb{Y})) = extend(\mathbb{Y})$.*

Proof. By definition, $extend(\mathbb{Y})$ is the set of all points dominated by some point in $\mathbb{Y}$, recall that we say $\mathbf{x}$ dominates $\mathbf{y}$ if $\mathbf{x} \succeq \mathbf{y}$. The transitivity of domination proofs the proposition. □

We have already shown that the asymptotic profit rate is decided only by $extend(\mathbb{Y})$, we will now show that hard information loss is decided only by $extend(\mathbb{Y})$ as well. Recall that the hard information loss is defined as twice the value $\max_{\mathbf{p}\in\mathbb{Y}} \sum_i p_i - 1$.

Lemma 6. *Let* $\mathbb{Y}$ *be any subset of* $\mathbb{R}^n$*, then* $\max_{\mathbf{p}\in\mathbb{Y}} \sum_i p_i = \max_{\mathbf{p}\in extend(\mathbb{Y})} \sum_i p_i$.

Proof. First of all $\max_{\mathbf{p}\in\mathbb{Y}} \sum_i p_i \leq \max_{\mathbf{p}\in extend(\mathbb{Y})} \sum_i p_i$ since $\mathbb{Y} \subseteq extend(\mathbb{Y})$. On the other hand, for any point $\mathbf{p} \in \text{extend}(\mathbb{Y})$, there is some point $\mathbf{p}' \in \mathbb{Y}$ that dominates $\mathbf{p}$, thus $\max_{\mathbf{p}\in\mathbb{Y}} \sum_i p_i \geq \max_{\mathbf{p}\in extend(\mathbb{Y})} \sum_i p_i$ □

Now, we can conclude that extending the price range from $\mathbb{Y}$ to $extend(\mathbb{Y})$ will not affect asymptotic behavior of the market, therefore we can always assume that the price range is complete, i.e., $\mathbb{Y} = extend(\mathbb{Y})$. We make this assumption in the rest of the chapter.

8.2 Euclidean Ball Market Maker

According to our previous discussion, we should consider three design criterion, asymptotic profit rate, hard information loss and soft information loss. To design optimal market, we will first meet the requirement of asymptotic profit rate and hard information loss, then optimize the soft information loss under those requirement. The reason why we don't prioritize the soft information loss is two-fold. First of all, soft information loss only occurs when minimum purchase unit ϵ is enforced, and it will be negligible if ϵ is very small. Secondly, the soft information loss is not an asymptotic behavior, in fact, the soft information loss, $n\epsilon\lambda_{max}\nabla^2(C(\mathbf{q}))$, goes to zero asymptotically for liquidity adaptive market.

We already showed in the end of Chapter 7 that asymptotic profit rate could be characterized by a dilated probability simplex $\{\mathbf{x}|x_i \geq 0, \sum x_i = b\}$ contained in $\mathbb{Y}$. Similarly, we can characterize the hard information loss as another dilated probability simplex $\{\mathbf{x}|x_i \geq 0, \sum x_i = a\}$ "containing" $\mathbb{Y}$. This is showed in Figure 8.2. Conversely, given minimum profit requirement ρ and maximum hard information loss requirement L, we have $b = 1/(1-\rho)$, $a = 1 + L/2$, and our goal now is the find $\partial^+\mathbb{Y}$ sandwiched between these two simplices that minimize the soft information loss.

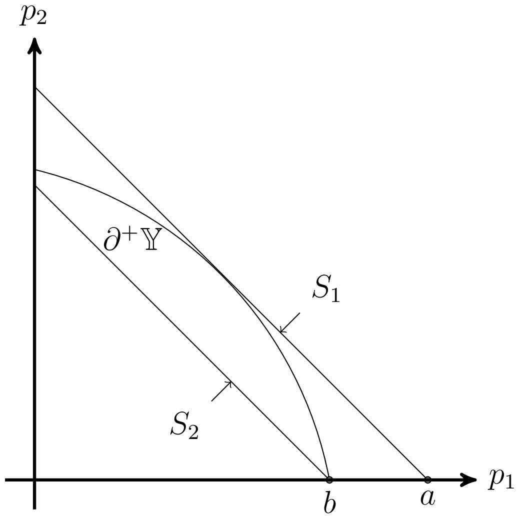


Figure 8.1: The positive boundary of $\mathbb{Y}$ sandwiched between two simplex

To minimize the soft information loss $n\epsilon\lambda_{max}\nabla^2(C(\mathbf{q}))$, we want to minimize $\lambda_{max}\nabla^2(C(\mathbf{q}))$. However, calculate $\nabla^2(C(\mathbf{q}))$ is very hard in general case if $\mathbf{p}(\mathbf{q})$ is not in the relative interior of $\mathbb{Y}$. However, in the special case of homogeneous market, i.e., when $R = 0$, we can characterize it directly by the curvature of $\mathbb{Y}$. In fact, approximating a general market with its homogeneous counterpart is not that inaccurate, as the effect of any bounded R will diminish with large $\mathbf{q}$. Note also that instead of minimizing $\lambda_{max}\nabla^2(C(\mathbf{q}))$ for specific $\mathbf{q}$, we should minimize the worst case loss $\max_{\mathbf{q}\in\mathbb{R}^n_+} \lambda_{max}\nabla^2(C(\mathbf{q}))$. Since in homogeneous market $\nabla^2(C(\lambda\mathbf{q})) = \lambda^{-1}\nabla^2(C(\lambda\mathbf{q}))$, it is only reasonable to minimize under given length of $\mathbf{q}$. Therefore, our final refined objective to minimize is $\max_{\mathbf{q}\in\mathbb{R}^n_+,\|\mathbf{q}\|_2=1} \lambda_{max}\nabla^2(C(\mathbf{q}))$.

First, we establish the relationship between $\lambda_{max}\nabla^2(C(\mathbf{q}))$ and curvature of $\partial^+\mathbb{Y}$.

Theorem 12. *Assume $\partial^+\mathbb{Y}$ is smooth in the sense that every point on $\partial^+\mathbb{Y}$ has a unique* *norm, and assume that the market is homogeneous, i.e.,* $C(\mathbf{q}) = \max_{\mathbf{p}\in\mathbb{Y}} \mathbf{p}\cdot\mathbf{q}$. *If* $\|\mathbf{q}\|_2 = 1$, *then* $\lambda_{max}\nabla^2(C(\mathbf{q}))$ *equals the reciprocal of the smallest principle curvature of* $\partial^+\mathbb{Y}$ *at* $\mathbf{p}(\mathbf{q})$.

Proof. It is easy to see that to maximize $\mathbf{p}\cdot\mathbf{q}$, $\mathbf{p}(\mathbf{q})$ must be the point on $\partial^+\mathbb{Y}$ with norm vector equal to $\mathbf{q}$ (we already assume $\|\mathbf{q}\|_2 = 1$). Therefore the price function maps a unit vector $\mathbf{q}$ to the position vector on the surface $\partial^+\mathbb{Y}$ that has norm $\mathbf{q}$. This map is exactly the inverse of Gauss Map, which is often used to characterize a surface or hypersurface. The strict non-linearity of $\partial^+\mathbb{Y}$ insures that the price map is one-one, thus invertible. The Gauss Map, as well as the following derivation, requires some differential geometry knowledge (see for example Do Carmo [1976]).

To increase clarity, denote the price map as $F : \mathbf{q} \to \mathbf{p}$ instead of the using same symbol $\mathbf{p}$. By homogeneity, $F(\mathbf{q}) = F(\frac{\mathbf{q}}{\|\mathbf{q}\|_2})$. We can write F as $i \circ G^{-1} \circ r$, where i projects a point in $\mathbb{R}^n$ to unit ball S^{n-1}, G^{-1} is the inverse of Gauss Map, r is the map from the $n-1$ dimensional surface $\partial^+\mathbb{Y}$ to position vector in $\mathbb{R}^n$.

Use $\mathrm{T}f$ to denote the tangent map of f, then we have

$$\nabla^2(C(\mathbf{q})) = \mathrm{T}F = \mathrm{T}r \circ \mathrm{T}G^{-1} \circ \mathrm{T}i = \mathrm{T}r \circ (\mathrm{T}G)^{-1} \circ \mathrm{T}i,$$

where $\mathrm{T}r$, $\mathrm{T}G$, $\mathrm{T}i$ are $n \times n-1$, $n-1 \times n-1$, $n-1 \times n$ matrices respectively. In fact, it can be shown from differential geometry that $\mathrm{T}r$ is the transpose of $\mathrm{T}i$, and the $n-1$ columns of $\mathrm{T}r$ are orthogonal. Find the vector $\mathbf{x}$ such that $\begin{bmatrix}\mathrm{T}r & \mathbf{x}^T\end{bmatrix}$ is orthogonal matrix Q, then

$$\nabla^2(C(\mathbf{q})) = \mathrm{T}r \circ (\mathrm{T}G)^{-1} \circ \mathrm{T}i = \begin{bmatrix}\mathrm{T}r & \mathbf{x}\end{bmatrix} \begin{bmatrix}(\mathrm{T}G)^{-1} & \mathbf{0} \\ \mathbf{0}^T & 0\end{bmatrix} \begin{bmatrix}\mathrm{T}i \\ \mathbf{x}^T\end{bmatrix} = Q \begin{bmatrix}(\mathrm{T}G)^{-1} & \mathbf{0} \\ \mathbf{0}^T & 0\end{bmatrix} Q^{-1}$$

The above equation means that $\nabla^2(C(\mathbf{q}))$ is similar to $\begin{bmatrix}(\mathrm{T}G)^{-1} & \mathbf{0} \\ \mathbf{0}^T & 0\end{bmatrix}$, thus the eigenvalues of $\nabla^2(C(\mathbf{q}))$ consists a zero, and the rest are the reciprocal of eigenvalues of $\mathrm{T}G$. Since the eigenvalues of $\mathrm{T}G$ are by definition principle curvatures of $\partial^+\mathbb{Y}$, we proved the theorem. $\square$

As the final piece of our theory, the next theorem shows that given requirements of hard information loss and asymptotic profit rate, the best market in the sense of minimizing $\max_{\mathbf{q}\in\mathbb{R}^n_+,\|\mathbf{q}\|_2=1} \lambda_{max}\nabla^2(C(\mathbf{q}))$ has $\mathbb{Y}$ as a Euclidean ball. In the light of previous theorem, minimizing $\max_{\mathbf{q}\in\mathbb{R}^n_+,\|\mathbf{q}\|_2=1} \lambda_{max}\nabla^2(C(\mathbf{q}))$ is equivalent to maximize the minimum of smallest curvature of all points on $\partial^+\mathbb{Y}$.

Theorem 13. *Given $0 < b < a < \sqrt{n} \cdot b$, we want to find a convex surface that is sandwiched between the two simplex $\{\mathbf{x}|x_i \geq 0, \sum x_i = a\}$ and $\{\mathbf{x}|x_i \geq 0, \sum x_i = b\}$ that maximize the minimum of smallest curvature of all points on it, then the optimal surface is the surface of Euclidean ball that is tightly sandwiched between the above two simplices.*

Proof. First of all, the optimal surface must be tightly sandwiched between the two simplex, otherwise it can be made curvier. The condition $a < \sqrt{n} \cdot b$ simply make sure that the Euclidean ball surface forms a complete price range.

To see why Euclidean ball surface is optimal, note that the curvature of all directions at all points of a Euclidean ball surface has the same value, say η. Consider first the 2-dimensional case, a non-Euclidean ball curve is either partly under the Euclidean ball curve, or totally above the Euclidean ball curve. This is shown in Figure 8.2, where the dashed curve is non-Euclidean ball curve and the solid is Euclidean ball curve. In the first case, the dashed curve between points A and B must have curvature smaller than η at least at some points. In the second case, the curvature of the dashed curve at point C must be smaller than η. Therefore, the non-Euclidean curve can not be optimal.

In general n-dimensional case, the curvature at a point is defined by the curvature of 2-dimensional curve by intersecting the surface with a hyperplane. There are also two cases. If the non-Euclidean ball surface is totally above the Euclidean ball surface, then at the point where they tangent with the simplex $\{\mathbf{x}|x_i \geq 0, \sum x_i = b\}$, the curvature of the non-Euclidean ball must be less than η. If the non-Euclidean ball surface are partly under the Euclidean ball surface, then we can find a hyperplane passing the center of the Euclidean ball on the cross section of which it looks exactly like the case on the left of Figure 8.2. And we know that there must be some point on the non-Euclidean ball surface where the smallest curvature is less than η.

Therefore, the optimal surface must be a tightly sandwiched Euclidean ball surface. □

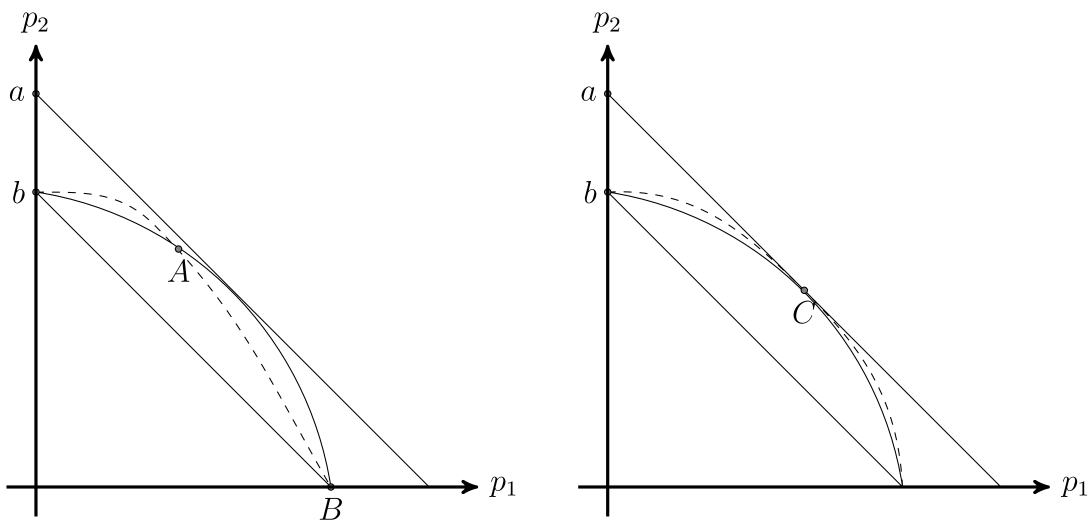


Figure 8.2: The 2-dimensional case

We call the class of market makers defined by Euclidean ball surface as Euclidean ball market maker. The center and radius of the Euclidean ball can be calculated directly from the hard information loss requirement L and profit requirement ρ.

As our last result, we give a corollary that follows from the arguments above. This corollary explicates the trade-off between information loss and profit.

Corollary 1. *Suppose the hard information loss and asymptotic profit rate of an asymptotic homogeneous market are L and ρ respectively, then $\rho < 1 - \frac{2}{2+L}$.*

Proof. As argued above, to satisfy the requirement L and ρ, the positive boundary of $extend(\mathbb{Y})$ must be sandwiched between two simplices $\{\mathbf{x}|x_i \geq 0, \sum x_i = a\}$ and $\{\mathbf{x}|x_i \geq 0, \sum x_i = b\}$, where $a = 1 + L/2$ and $b = 1/(1-\rho)$. Note that we must have $a > b$ in order that the positive boundary of $extend(\mathbb{Y})$ exists and is strictly non-linear. The corollary then follows from simple calculation. □

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