Title
Efficient Ethical Rules for Volunteer's Dilemmas

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Abstract

This paper extends the classic Volunteers Dilemma game to environments in which individuals have differing costs and private information about their own costs. It explores the nature of symmetric ethical optimum strategies for Volunteer's Dilemma games with and without differing costs. Where costs differ, ethical optima are constructed by symmetrizing the game with a Rawlsian Veil of Ignorance.

Efficient Ethical Rules for Volunteer's Dilemmas

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As you stroll along a well-traveled path, you observe water rushing from a broken water main. If you believe that nobody else will do so, you will certainly take the trouble to find a telephone and call the water department. But since the path is busy, many others will see the problem. If someone else calls, your effort will be wasted. But if everybody believes that someone else will call, the problem will go unreported.

1 The Volunteer's Dilemma

Andreas Diekmann [10] modeled situations like this with a symmetric n-player, simultaneous-move game that he called the *Volunteer's Dilemma*. In the Volunteer's Dilemma game, each player can choose to take action or not. If at least one player acts, then all n players will receive a benefit b. Those who act must pay a cost c, where $0 < c < b$ and hence receive a net benefit of $b - c$. If no player acts, then all players receive a net benefit of 0.

1.1 Symmetric Nash Equilibrium

In the Volunteer's Dilemma with two or more players, there cannot be a symmetric Nash equilibrium in which all take action, since if everybody else acts, one's own best response is not to act. Nor can there be a symmetric Nash equilibrium in which none take action, since if nobody else acts, one's own best response is to act. In the only symmetric Nash equilibrium for this game, each player uses a mixed strategy; taking action with a positive probability less than 1. These results are stated formally in Proposition 1, a proof of which is found in the Appendix.

Proposition 1. *(Diekmann) The n-player Volunteer's Dilemma has a unique symmetric Nash equilibrium. With n players, the Nash equilibrium probability that an individual player takes action is*

$$p_N(n) = 1 - \left(\frac{c}{b}\right)^{\frac{1}{n-1}}, \tag{1}$$

which decreases as n increases and asymptotically approaches 0. The probability that at least one player takes action is

$$P_N(n) = 1 - \left(\frac{c}{b}\right)^{\frac{n}{n-1}}, \tag{2}$$

which also decreases with n and asymptotically approaches $1 - \frac{c}{b}$. The expected utility of each player is constant with respect to n and equal to $b - c$.

Proposition 1 leaves us with a vexing conundrum. The technology of the Volunteer's Dilemma game offers the potential for significant benefits from the formation of larger groups; an action taken by a single person is sufficient to benefit the entire group, no matter how large the group. Yet, in the symmetric Nash equilibrium for this game, as the number of players increases, the probability that nobody takes action increases in such a way that none of these potential gains are realized. As group size grows, the expected payoff to each player remains constant at $b - c$.

1.2 Optimal Symmetric Mixed Strategies

In the Volunteer's Dilemma game, inefficiency of symmetric Nash equilibrium arises from two sources. One is the standard problem of neglected externalities. Individuals do not account for the fact that an increase in their own probability of taking action benefits all other players. The second source of inefficiency is a coordination problem. Players do not know the actions that have been taken by others. Thus, in equilibrium, there is a positive probability that more than one player takes costly action, although the action of only one is needed to produce benefits for all.

Sometimes it is possible to coordinate the actions of players so that if there is more than one volunteer, only a single volunteer will be selected to perform the task. For example, potential donors of stem cells from bone marrow or blood aphoresis join a registry of persons who have declared their willingness to donate if their contributions are needed. When a patient is in need of a transplant, if one or more potential donors of this patient's immunity type have volunteered, the registry selects exactly one of these volunteers to make the donation. [4]. Jeroen Weesie [13] and Ted Bergstrom and Greg Leo [5] analyze the comparative statics of Nash equilibrium for versions of Volunteer's Dilemma in which at most one of the volunteers is required to pay.

Sometimes duplication of effort can be avoided because potential volunteers can see immediately whether someone else has "beat them to it." Bergstrom [3] studies the case of passers-by on a more or less crowded highway, who are presented sequentially with the opportunity to help a distressed traveler. Christopher Bliss and Barry Nalebuff [9], Marc Bilodeau and Al Slivinski [8] and Weesie [12] analyze a war-of-attrition game in which the first person to take action is observed by all and where benefits diminish as time passes. In deciding when to act, players face a trade-off between the costs of postponement and the possibility that if one waits a little longer, action will be unnecessary because someone else will have done it.

This paper studies situations where such coordination is technically infeasible. In the example at the beginning of this paper, the cost of informing authorities would be minimized if only one passer-by took action. But how can this be accomplished? It would not be cost-effective for everyone who has seen the problem to assemble and choose one of their number to contact the authorities.

1.3 An Appeal to Ethics

If players could be persuaded to abide by a self-enforced ethical rule that accounts for the well-being of others, they would all be better off than in symmetric Nash equilibrium. We will show that in the absence of coordination, there is an optimal symmetric ethical rule that mandates each player takes action with a probability that exceeds the Nash equilibrium probability but is less than 1. Notice that an ethic that demanded that all players take action would not be efficient. If all followed this rule, each would have an expected payoff of $b - c$, which is no better than the Nash equilibrium payoff.

An optimal symmetric ethical rule for the symmetric Volunteer's dilemma game is a strategy that satisfies the *Kantian* principle: "Use the strategy that you would wish that everyone would use". This rule is characterized by Proposition 2, which is proved in the Appendix.

Proposition 2. *In an n-player Volunteer's Dilemma, there is an optimal symmetric rule that requires each player to use a mixed strategy in which the the probability of taking action is*

$$p_O(n) = 1 - \left(\frac{c}{bn}\right)^{\frac{1}{n-1}}, \tag{3}$$

and the probability that at least one player takes action is

$$P_O(n) = 1 - \left(\frac{c}{bn}\right)^{\frac{n}{n-1}}. \tag{4}$$

Tables 1 and 2 illustrate the results of Proposition 2 by showing the probabilities of taking action in Nash equilibrium and in the ethical optimal solution. This is done for two special cases, with cost benefit ratios, $c/b = .5$ and $c/b = .9$.

Table 1: Symmetric Nash equilibrium and Ethical Optimum with $c/b = .5$

	Nash equilibrium		Ethical Optimum	
n	$p_N(n)$	$P_N(n)$	$p_O(n)$	$P_O(n)$
2	0.50	0.75	0.75	0.94
3	0.29	0.65	0.59	0.93
4	0.21	0.60	0.50	0.94
5	0.16	0.58	0.44	0.94
25	0.03	0.51	0.15	0.98
100	0.01	0.50	0.05	1.00
∞	0.00	0.50	0.00	1.00

The tables show that when the number of players is small, the ethical optimum strategy requires players to take action with much higher

Table 2: Symmetric Nash equilibrium and Ethical Optimum with $c/b = .9$

	Nash equilibrium		Ethical Optimum	
n	$p_N(n)$	$P_N(n)$	$p_O(n)$	$P_O(n)$
2	0.10	0.19	0.55	0.80
3	0.05	0.15	0.45	0.84
4	0.04	0.13	0.39	0.86
5	0.03	0.12	0.35	0.88
25	0.00	0.10	0.14	0.85
100	0.00	0.10	0.05	0.95
∞	0.00	0.10	0.00	1.00

probability than in Nash equilibrium. They also show, as predicted by Proposition 1 that in Nash equilibrium, as the number of players increases, the probability that any individual acts declines asymptotically toward zero, while the probability that at least one player takes action declines asymptotically toward $1 - c/b$.

In these tables, it appears that if all players use the ethical optimal strategy, the probability that any individual takes action declines asymptotically toward zero and the probability that at least one takes action approaches 1. This turns out to be true in general. We state this formally in Proposition 3, which is proved in the Appendix.

Proposition 3. *If all players use the optimal ethical rule, then in the limit as the number of players approaches infinity, the probability that any single individual takes action approaches zero, but the probability that at least one player takes action approaches one.*

If all players use the optimal symmetric strategy, they will all be better off than in Nash equilibrium, but in the absence of a coordinating device, there will still be some probability of duplicated effort. If a coordinating device were available to select a single randomly-chosen player to take action, then the expected payoff of each player would be $b - c/n$, which is higher than expected payoffs in the uncoordinated ethical outcome.

In Table 3, we show the expected utility, $\hat{u}(n)$, of each player in Nash equilibrium, expected utility, $\bar{u}(n)$, when all players use the optimal ethical strategy, and expected utility, $u_c(n)$, of each player in a coordinated equilibrium where one randomly selected player is assigned to take action. We show this for two special cases, where $c = .5$ and $b = 1$ and where $c = .9$ and $b = 1$,

Table 3: Utility Comparison: Nash Equilibrium $\hat{u}(n)$; Ethical Optimum $\bar{u}(n)$; and Coordinated Solution $u_C(n)$

	$c = 0.5,\ b = 1$			$c = 0.9,\ b = 1$		
n	$\hat{u}(n)$	$\bar{u}(n)$	$\bar{u}_c(n)$	$\hat{u}(n)$	$\bar{u}(n)$	$u_c(n)$
2	0.50	0.56	0.75	0.10	0.30	0.55
3	0.50	0.64	0.83	0.10	0.43	0.70
4	0.50	0.69	0.88	0.10	0.95	0.99
5	0.50	0.73	0.90	0.10	0.57	0.82
25	0.50	0.91	0.98	0.10	0.85	0.96
100	0.50	0.97	0.995	0.10	0.95	0.99

2 Nash Equilibrium when Costs Differ

It is common practice to "simplify" game theoretic models like the Volunteer's Dilemma by assuming that all players have identical benefits and costs. While this simplification makes it easy to calculate a symmetric Nash equilibrium, the resulting mixed-strategy Nash equilibrium has an air of implausibility. In the symmetric mixed strategy equilibrium of the Volunteer's Dilemma game, all players are indifferent between the equilibrium mixed strategy and any other probability mix of the strategies "act" and "don't act." Given that this is the case, why should any player take the trouble to determine the equilibrium mixed strategy proportions and act accordingly?[1] If we allow the realistic possibility that different players have different costs of taking action, we avoid this conundrum and we can construct a manageable model in which players use pure strategies in a symmetric Nash equilibrium. In this case, we find that the optimal symmetric ethical rule recommends to each player a pure strategy that is determined by that player's realized cost of taking action.

Let us assume that the costs, c, of taking action differ among players and that if at least one player takes action, all players receive the same benefit, b. Players cannot communicate before deciding whether to act. Individuals know their own costs, but do not know the costs of the other players in the game.[2] Players costs are chosen by independent draws from a distribution that is common knowledge. The distribution from which players' costs are drawn is assumed to satisfy the following:

Assumption 1. *Players' costs are drawn randomly from a population with a cumulative distribution of costs, $F(\cdot)$ that is continuously differentiable on the interval $[\ell, h]$ where $0 \leq \ell < b \leq h$. We assume that $F(\ell) = 0$, $F(h) = 1$, and $F'(c) > 0$ for all $c \in [\ell, h]$.*

If $F(b) < 1$, then with positive probability, a player's costs will

[1] Herbert Gintis [11] describes this quandry as "the mixing problem".

[2] The assumption of incomplete information seems appropriate for games in which players are thrown together by chance for a single interaction. Situations where the same players are engaged in repeated encounters and know each other well might better be treated as games of complete information. Weesie [12] characterizes asymmetric equilibria for Volunteer's Dilemma games with differing payoffs, but complete information in which players know each other's payoffs.

exceed individual benefits. In Nash equilibrium, such a player would not act even if nobody else takes action.

The game that begins before individuals learn their costs can be modelled as a symmetric game. For each player, a strategy is a function that maps costs, once revealed, to actions. This game has a symmetric Nash equilibrium in which every player uses a *threshold strategy* of the form: "Act if and only if your costs, c, are no larger than the threshold level $\hat{c}$." The threshold strategy with threshold $\hat{c}$ will be a Nash equilibrium if and only if, when all other players follow this rule, a player with realized cost $c < \hat{c}$ will have a higher expected payoff from acting than not and a player with realized cost $c > \hat{c}$ will have a higher payoff from not acting.

Let us define $G(c) = 1 - F(c)$. If the $n - 1$ other players all use the threshold strategy with threshold $\hat{c}(n)$, then a player whose cost is $\hat{c}(n)$ must be indifferent between acting and not acting. For this player, the expected payoff from not acting is $b\left(1 - G\left(\hat{c}(n)\right)^{n-1}\right)$ and the expected payoff from acting is $b - \hat{c}(n)$. This implies that

$$b\left(1 - G(\hat{c}(n))^{n-1}\right) = b - \hat{c}(n), \tag{5}$$

or equivalently,

$$bG(\hat{c}(n))^{n-1} = \hat{c}(n). \tag{6}$$

We have the following result, which is proved in the Appendix.

Proposition 4. *In an n-player Volunteer's Dilemma game, where the distribution of costs is common knowledge and satisfies Assumption 1, there is a unique Nash equilibrium threshold strategy, with threshold $\hat{c}(n) \in (\ell, b)$ such that $\hat{c}(n)$ decreases as n increases and* $\lim_{n\to\infty} \hat{c}(n) =$

ℓ.

The equilibrium probability that nobody takes action when there are n players is $G\left(\hat{c}(n)\right)^n$. From Equation 6, it follows that

$$G\left(\hat{c}(n)\right)^n = G\left(\hat{c}(n)\right)^{n-1} G\left(\hat{c}(n)\right) = \frac{1}{b}\hat{c}(n)G\left(\hat{c}(n)\right) \tag{7}$$

The function $G(\cdot)$ is assumed to be continuous and $G(\ell) = 1$. According to Proposition 4, $\lim_{n\to\infty} \hat{c}(n) = \ell$. It follows from Equation 7 that

$$\begin{aligned} \lim_{n\to\infty} G\left(\hat{c}(n)\right)^n &= \lim_{n\to\infty} \frac{1}{b}\hat{c}(n) \lim_{n\to\infty} G\left(\hat{c}(n)\right) \\ &= \frac{\ell}{b}G(\ell) \\ &= \frac{\ell}{b} \end{aligned} \tag{8}$$

Let us define $F^*(c,n)$ to be the the probability that at least one player takes action when there are n players, each of whom uses a threshold strategy with threshold c. Then

$$F^*(c,n) = 1 - G\left(c\right)^n. \tag{9}$$

Therefore Equation 8 implies the following:

Proposition 5. *In symmetric Nash equilibrium for an $n-$player Volunteer's Dilemma where the distribution of costs satisfies Assumption 1, the limiting probability that at least one player takes action is*

$$\lim_{n\to\infty} F^*\left(\hat{c}(n), n\right) = 1 - \frac{\ell}{b}. \tag{10}$$

Proposition 5 has the following corollary.

Corollary 1. *If $\ell = 0$, then in symmetric Nash equilibrium, the probability that someone takes action approaches one as n gets large.*

Where $\ell > 0$, the limiting value of the probability that someone takes action is less than one. It is of some interest to explore whether this probability increases or decreases with the number of players.

Proposition 4 informs us that $\hat{c}(n)$ is a decreasing function of n. Therefore the probability that in equilibrium, nobody takes action increases (decreases) as n increases if $cG(c)$ is a decreasing (increasing) function of c over the interval $(0, 1)$. Differentiating, we find that

$$\frac{d}{dc}\left(cG(c)\right) = G(c) + cG'(c) = G(c)\left(1 + \frac{cG'(c)}{G(c)}\right). \tag{11}$$

The ratio $cG'(c)/G(c)$ in Equation 11 could be written as $\frac{c}{G}\frac{dG}{dc}$, which is recognizable as the elasticity of the function G with respect to c. It is useful to give this expression a name of its own.

Definition 1. *The cost elasticity of refusals is the ratio $\eta_r(c) = \frac{cG'(c)}{G(c)}$. Refusals are cost-elastic at c if $\eta_r(c) < -1$ and cost-inelastic if $\eta_r(c) > -1$.*

An immediate consequence of Equation 11 and Definition 1 is the following:

Lemma 1. *The function $cG(c)$ is increasing in c over the interval $(0, 1)$ if refusals are cost-inelastic and decreasing over this interval if refusals are cost-elastic.*

According to Proposition 4, $\hat{c}(n)$ is a decreasing function of n. With n players, the Nash equilibrium probability that nobody takes action is $\hat{c}(n)G\left(\hat{c}(n)\right)$. It follows from Lemma 1 that $\hat{c}(n)G\left(\hat{c}(n)\right)$ increases with n if refusals are cost-elastic, and decreases with n if refusals are cost-inelastic. Since the probability that someone takes action is one minus the probability that nobody takes action, we have the following:

Proposition 6. *If Assumption 1 is satisfied, then in Nash equilibrium, the probability $F^*(\hat{c}(n), n)$ that at least one player takes action decreases with n if refusals are cost-elastic and increases with n if refusals are cost-inelastic. In either case, as n approaches infinity, this probability approaches $1 - \frac{\ell}{b}$.*

3 Optimal Ethical Strategies when Costs Differ

In order to formulate an optimal ethical rule, we impose a "veil of ignorance" on differences in costs and benefits in such a way that before the veil is removed, all players seek to maximize the same objective function. We imagine an initial position in which players do not yet know their own costs, but expect them to be drawn at random from a probability distribution that is common knowledge. In the initial position, players have identical prospects. We assume that it is common knowledge that the distribution $F(\cdot)$ from which individual costs are drawn satisfies Assumption 1 of the previous section. We consider symmetric strategies that take the form of a threshold cost level $\bar{c}$ and a mandate that any player should take action if and only if this player has costs $c \leq \bar{c}$.

For every $n > 1$, there is an ethical optimum threshold strategy, with threshold $\bar{c}(n)$. Viewed from the initial position, this strategy, if followed by all players yields a higher expected utility for each than would any other strategy used by all players.

Proposition 7. *If the distribution of costs satisfies Assumption 1, and if $nb > \ell$, then there is a unique ethical optimum strategy $\bar{c}(n)$. The threshold $\bar{c}(n)$ satisfies the equation*

$$bG\left(\bar{c}(n)\right)^{n-1} = \frac{\bar{c}(n)}{n}. \tag{12}$$

Not surprisingly, the ethical optimal threshold $\bar{c}(n)$ exceeds the Nash equilibrium threshold $\hat{c}(n)$. We show this as follows:

Corollary 2. *For all $n > 1$, $\bar{c}(n) > \hat{c}(n)$ where $\hat{c}(n)$ is the symmetric Nash equilibrium threshold.*

Proof. Where $\hat{c}(n)$ is the symmetric Nash equilibrium threshold, for n players, it follows from Equations 6 and 12 that for all $n > 1$

$$b\frac{G\left(\bar{c}(n)\right)^{n-1}}{\bar{c}(n)} = \frac{1}{n} < 1 = b\frac{G\left(\hat{c}(n)\right)^{n-1}}{\hat{c}(n)}. \tag{13}$$

Since $G(c) > 0$ and $G'(c) < 0$ it must be that that $\frac{G(c)^{n-1}}{c}$ is a strictly decreasing function of c for $c > 0$. Therefore the inequality in Expression 13 implies that $\bar{c}(n) > \hat{c}(n)$. $\square$

Proposition 8 informs us that in the limit, for large n, the ethical optimal threshold, $\bar{c}(n)$, like the Nash equilibrium threshold, $\hat{c}(n)$, approaches ℓ, the lower bound of the support of the distribution of costs. However, $\bar{c}(n)$ approaches ℓ more slowly than does $\hat{c}(n)$, so that the limiting probability that at least one player takes action is 1 if all players use the ethical optimal threshold and $1 - \frac{\ell}{b}$ in symmetric Nash equilibriium.

This result is stated formally in Proposition 8, which is proved in the Appendix.

Proposition 8. *If the distribution of costs satisfies Assumption 1, then* $\lim_{n\to\infty} \bar{c}(n) = \ell$ *and* $\lim_{n\to\infty} F^* \left(\bar{c}(n), n\right) = 1$.

4 Examples

We illustrate our general results with examples from two special distribution families, the Pareto distributions and the uniform distributions.

4.1 Pareto Distribution

The Pareto distribution with parameters $\ell > 0$ and $\lambda > 0$ has a cumulative distribution function of the form

$$F(c) = 1 - \left(\frac{\ell}{c}\right)^{\lambda} \tag{14}$$

with support (ℓ, ∞). For the Pareto distribution,

$$G(c) = \left(\frac{\ell}{c}\right)^{\lambda} \tag{15}$$

and thus the cost elasticity of refusal is $\eta_r(c) = -\lambda$.

Nash Equilibrium

From Proposition 6, it is immediate that:

Remark 1. *If the distribution of costs is a Pareto distribution with parameters* ℓ *and* λ*, then the symmetric Nash equilibrium probability,* $F^* \left(\hat{c}(n), n\right)$*, that at least one player takes action decreases with group size if* $\lambda > 1$ *and increases with group size if* $\lambda < 1$.

If costs are Pareto distributed with parameters ℓ and λ, we can solve directly for $\hat{c}(n)$ and $\bar{c}(n)$. According to Equation 6, it must be that $bG\left(\hat{c}(n)\right)^{n-1} = \hat{c}(n)$. For the Pareto distribution, this implies that

$$b\left(\frac{\ell}{\hat{c}(n)}\right)^{\lambda(n-1)} = \hat{c}(n) \tag{16}$$

Rearranging terms of 16, we have

$$\hat{c}(n) = b^{\frac{1}{1+\lambda(n-1)}}\,\ell^{\frac{\lambda(n-1)}{1+\lambda(n-1)}} \tag{17}$$

and hence the probability that any individual takes action is

$$1 - G\left(\hat{c}(n)\right) = 1 - \left(\frac{\ell}{\hat{c}(n)}\right)^{\lambda} = 1 - \left(\frac{\ell}{b}\right)^{\frac{\lambda}{1+\lambda(n-1)}}. \tag{18}$$

The probability that at least one player takes action is then

$$F^{*}\left(\hat{c}(n), n\right) = 1 - G\left(\hat{c}(n)\right)^{n} = 1 - \left(\frac{\ell}{b}\right)^{\frac{\lambda n}{1+\lambda(n-1)}} \tag{19}$$

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From Equation 12 of Proposition 7 it follows that

$$b\left(\frac{\ell}{\bar{c}(n)}\right)^{\lambda(n-1)} = \frac{\bar{c}(n)}{n} \tag{20}$$

From Equations 16 and 20, it follows that

$$\frac{\bar{c}(n)}{\hat{c}(n)} = n^{\frac{1}{1+\lambda(n-1)}}. \tag{21}$$

Equations 15 and 21 imply that

$$\frac{G\left(\bar{c}(n)\right)}{G\left(\hat{c}(n)\right)} = \left(\frac{\hat{c}(n)}{\bar{c}(n)}\right)^{\lambda(n-1)} = n^{\frac{-\lambda(n-1)}{1+\lambda(n-1)}}. \tag{22}$$

Numerical Examples

Tables 4 and 5 show outcome probabilities for special cases where $b = 2$, $\ell = 1$ and with $\lambda = .5$ and $\lambda = 2$, respectively. Table 4 shows that if $\lambda = .5$, the Nash equilibrium probability, $F^*\left(\hat{c}(n), n\right)$, that at least one player takes action diminishes as n increases. Table 5 shows that if $\lambda = 2$, this probability increases with n. In both cases, $F^*\left(\hat{c}(n), n\right)$ approaches $\ell/b = .5$. The tables also show that for small n, the ethical optimum recommends a much higher probability of acting than the Nash equilibrium probability.

Table 4: Pareto Distribution with
$\lambda = .5$, $\ell = 1$, and $b = 2$

	Nash equilibrium			Ethical Optimum		
n	$F(\hat{c}(n))$	$F^*(\hat{c}(n), n)$	$\hat{u}(n)$	$F(\bar{c}(n))$	$F^*(\bar{c}(n))$	$\bar{u}(n)$
2	.21	.37	.48	.37	.60	0.62
3	.16	.41	.62	.36	.74	0.91
4	.13	.43	.70	.34	.81	1.11
5	.11	.44	.76	.32	.85	1.24
25	.03	.49	.95	.14	.98	1.79
100	.01	.50	.99	.05	.99	1.94
∞	.00	.50	1.00	0	1.00	2.00

Table 5: Pareto Distribution with
$\lambda = 2$, $\ell = 1$ and $b = 2$

	Nash equilibrium			Ethical Optimum		
n	$F(\hat{c}(n))$	$F^*(\hat{c}(n), n)$	$\hat{u}(n)$	$F(\bar{c}(n))$	$F^*(\bar{c}(n), n)$	$\bar{u}(n)$
2	.37	.60	.79	.60	.84	0.95
3	.24	.56	.87	.51	.88	1.17
4	.18	.55	.91	.45	.91	1.30
5	.14	.54	.93	.40	..92	1.39
25	.03	.51	.99	.15	.98	1.81
100	.01	.50	1.00	.05	1.00	1.94
∞	.00	.50	1.00	.00	1.00	2.00

4.1.1 Uniform Distribution

Consider the uniform distributed on an interval $[\ell, h]$ where $0 \leq \ell < b \leq h$. Then $F(c) = \frac{c-\ell}{h-\ell}$ and $G(c) = 1 - F(c) = \frac{h-c}{h-\ell}$.

With the uniform distribution, the cost elasticity of refusal is

$$\eta_r(c) = \frac{cG'(c)}{G(c)} = \frac{-c}{h-c}. \tag{23}$$

In this case, the cost elasticity of refusal is not constant but depends on c. From Equation 23, it follows that if $c < h/2$, refusals are cost-inelastic, with $\eta_r(c) > -1$ and if $c > h/2$, refusals are cost-elastic, with $\eta_r(c) < -1$.

Although the cost-elasticity of refusals at the threshold $\hat{c}(n)$ depends on the value of $c(n)$, some simple conditions on the parameters of the distribution can confine $\hat{c}(n)$ to either the cost-inelastic or the cost-elastic region. According to Proposition 4, it must be that $\ell < \hat{c}(n) < b$ for all $n \geq 2$. Therefore if $b < h/2$, it must be that $\hat{c}(n) < h/2$ and hence refusals are cost-inelastic at $\hat{c}(n)$. If, on the other hand, $h/2 < \ell$, then since $\hat{c}(n) > \ell$, it must be that if $\hat{c}(n) > h/2$ and therefore refusals are cost-elastic at $\hat{c}(n)$. From these facts and Proposition 6 we conclude that

Remark 2. *If the distribution of costs is uniform on the interval* $[\ell, h]$ *and if* $b < h/2$*, the Nash equilibrium probability that at least one player takes action decreases with the number of players. If* $h/2 < \ell$*, this probability increases with the number of players.*

Tables 6 and 7 illustrate the general result of Remark 2. In the example of Table 6, $b < h/2$ and hence refusals are cost-inelastic at $\hat{c}(n)$ for $n \geq 2$.. In the example of Table 7, $h/2 < \ell$ and refusals are cost-elastic at $\hat{c}(n)$ for $n \geq 2$. In the former case, the Nash equilibrium probability that at least one person acts increases and in the latter case this probability decreases with n.

Table 6: Uniform Distribution on the interval $[1, 5]$ with $b = 2$

	Nash equilibrium				Ethical Optimum		
n	$\hat{c}(n)$	$\eta_r(\hat{c}(n))$	$F(\hat{c}(n))$	$F^*(\hat{c}(n), n)$	$\bar{c}(n)$	$F(\bar{c}(n))$	$F^*(\bar{c}(n), n)$
2	1.67	-0.50	0.17	0.31	2.50	0.38	0.61
3	1.52	-0.44	0.13	0.34	2.45	0.36	0.74
4	1.43	-0.40	0.11	0.36	2.34	0.34	0.81
5	1.36	-0.37	0.09	0.38	2.08	0.27	0.79
10	1.20	-0.31	0.05	0.43	1.92	0.23	0.93
25	1.10	-0.28	0.03	0.46	1.52	0.13	0.97
100	1.00	-0.25	0.00	0.50	1.20	0.05	0.99
∞	1.00	-0.25	0.00	0.50	1.00	0.00	1.00

Table 7: Uniform Distribution on the interval $[3, 5]$ with $b = 4$

	Nash equilibrium				Ethical Optimum		
n	$\hat{c}(n)$	$\eta_r(\hat{c}(n))$	$F(\hat{c}(n))$	$F^*(\hat{c}(n), n)$	$\bar{c}(n)$	$F(\bar{c}(n))$	$F^*(\bar{c}(n), n)$
2	3.33	-2.00	0.17	0.31	4.00	0.50	0.75
3	3.21	-1.79	0.10	0.28	3.86	0.43	0.82
4	3.15	-1.71	0.08	0.27	3.77	0.38	0.85
5	3.12	-1.66	0.06	0.27	3.69	0.34	0.90
10	3.00	-1.58	0.03	0.26	3.48	0.24	0.93
25	3.02	-1.53	0.01	0.25	3.25	0.13	0.96
100	3.01	-1.51	0.00	0.25	3.10	0.05	0.99
∞	3.00	-1.50	0.00	0.25	3.00	0.00	1.00

Remark 2 does not apply when $\ell < h/2 < b$. In this case, examples can be found in which refusals are cost-elastic for small n and cost-inelastic for large n. Hence the probability that at least one person acts decreases with n for small n and increases with n for large n. Table 8 shows an example in which $\ell < h/2 < b$. In this example, the Nash equilibrium probability that at least one player decreases with n for $n = 2$ and $n = 3$ and increases with n for $n > 3$.

Table 8: Uniform Distribution on the interval $[2,5]$ with $b=4$

	Nash equilibrium				Ethical Optimum		
n	$\hat{c}(n)$	$\eta_r(\hat{c}(n))$	$F(\hat{c}(n))$	$F^*(\hat{c}(n),n)$	$\bar{c}(n)$	$F(\bar{c}(n))$	$F^*(\bar{c}(n),n)$
2	2.86	-1.33	0.29	0.4898	3.63	0.54	0.79
3	2,59	-1.07	0.20	0.4798	3.48	0.49	0.87
4	2.45	-0.96	0.15	0.4796	3.24	0.41	0.88
5	2.39	-0.90	0.12	0.4802	3.12	0.37	0.90
10	2.19	-0.78	0.07	0.4875	2.77	0.26	0.95
25	2.08	-0.71	0.03	0.4938	2.43	0.14	0.98
100	2.02	-0.68	0.01	0.4979	2.15	0.05	0.99
∞	2.00	-0.67	0.00	0.5000	2.00	0.00	1.00

Tables 6-8 illustrate the general result that in Nash equilibrium and also in the ethical optimum, the probability that any single individual takes action approaches zero as n becomes large. However, as n becomes large, the Nash equilibrium probability that at least one takes action approaches ℓ/b, while if all players use the ethical optimum strategy, the probability that at least one takes action approaches unity.

5 Conclusion

Diekmann's model of a Volunteer's Dilemma with identical players yields a surprising and somewhat distressing conclusion. This game displays strong technical returns to scale for large groups—if at least one player takes a costly action, all will benefit. But the potential gains from group size are entirely dissipated by the "free-rider problem." As group size increases, the Nash equilibrium probability that at least one player takes action is reduced, and the equilibrium expected utility of each player remains constant.

The dissipation of returns to scale is, in general, less severe in a Volunteer's Dilemma if benefits and/or costs of action differ among players. We consider a model in which if at least one player takes action, all group members receive the same benefit, but costs of acting differ between individuals. Nash equilibrium strategies take the form of threshold strategies, where players will take costly action if and only if their costs are below the threshold. In this model, although the equilibrium threshold and hence the probability that any individual takes action decreases, the probability that at least one player takes action may either increase or decrease, depending on the elasticity of the distribution of costs.

We show examples in which the probability that someone takes action increases with group size and decreases with group size. In the limit, as group size approaches infinity, the Nash equilibrium probability that any individual acts approaches zero, while the probability that at least one player takes action approaches $1 - \ell/b$ where b is the benefit received by all if someone takes action and ℓ is the lower bound of the support of the distribution of costs.

Achieving efficient outcomes in the Volunteer's Dilemma game requires a balance between two competing forces, the externality that arises when the action of one player can benefit all, and the wasted resources that arise when more than one player takes costly action. If full coordination were possible, in an efficient outcome, only the player with lowest cost would take action. But in many situations it would be time-consuming and expensive to coordinate the actions of all players. Furthermore, a player's costs of acting are often private information, known only by the potential actor.

In this environment, we explore the nature of an "optimal ethical strategy." Suppose that players have common priors about the distribution of costs. Before their own costs are revealed to them, they consider alternative cost thresholds, where players are mandated to take action if and only if their costs are below this threshold. Under our assumptions, there is some threshold $\bar{c}$ that gives all players the highest expected payoff before they learn their type. The threshold strategy with threshold $\bar{c}$ is the optimal ethical strategy for this group of players.

For any group of players the optimal ethical strategy prescribes a higher threshold than the Nash-equilibrium threshold value and hence leads to a higher probability of acting. If all players use the ethical optimal threshold strategy, then as n gets large, the probability that any individual acts approaches 0, while the probability that at least one acts approaches 1.

This discussion of optimal ethical strategies takes a fully Kantian approach in which the chosen threshold strategy is the one that would be best for all if all were to use the same strategy. An alternative approach, taken by Bergstrom [6, 7] and by Alger and Weibull [1, 2] is to consider "partially" Kantian rules that ask players to use the strategy that would be best for them if they believed that each of the other group members would, with some probability between 0 and 1, use the same strategy that they use. It would be interesting to explore the implications of such rules for Volunteer's Dilemma games in which costs of helping differ between individuals.

In the Volunteer's Dilemma model, a costly effort by a single player results in a benefit b for every group member, regardless of the size of the group. In many realistic scenarios of mutual aid, as the group gets larger, there are likely to be more group members in need, so the cost of helping the needy increases with group size. In such situations it is often the case that the efforts of more than one donor can be pooled. There remains interesting work to be done in studying the effects of group size on Nash equilibrium and on ethical rules.

Appendix-Proofs of Propositions

5.1 Proof of Proposition 1

Proof. In a mixed strategy equilibrium, each player is indifferent between taking action and not doing so. Anyone who takes action is certain to have a net payoff of $b - c$. In equilibrium, all players must be indifferent between taking action and not taking action. Therefore, regardless of the number of players, the expected utility of each player in a symmetric Nash equilibrium must be $b - c$.

In a mixed strategy equilibrium for n players who each take action with independent probability p, a player who chooses the strategy "do not act" will not pay any cost and will enjoy the benefit b if at least one other player takes action. Let $q = 1 - p$. If all other players take action with probability p then the probability that at least one of the others takes action is $1 - q^{n-1}$. Therefore the expected payoff from the strategy "do not act" is $b\left(1 - q^{n-1}\right)$.

Let us define $p_N(n) = 1 - q_N(n)$ and $q_N(n)$ to be the probabilities respectively that a player acts and does not act in a symmetric mixed-strategy Nash equilibrium. In this mixed strategy equilibrium, it must be that the expected payoff is the same from taking action and not taking action. Therefore it must be that

$$b\left(1 - q_N(n)^{n-1}\right) = b - c. \tag{24}$$

Rearranging terms in Equation 24, we see that in symmetric equilibrium each player takes action with probability $p_N(n) = 1 - q_N(n)$ where

$$q_N(n) = \left(\frac{c}{b}\right)^{\frac{1}{n-1}}. \tag{25}$$

Let us define $Q_N(n) = q_N(n)$ to be the symmetric Nash equilibrium probability that no player takes action and $P_N(n) = 1 - Q_N(n)$ the probability that at least one player takes action. Equation 25 implies that

$$P_N(n) = 1 - Q_N(n) = 1 - q_N(n)^n = 1 - \left(\frac{c}{b}\right)^{\frac{n}{n-1}} \tag{26}$$

A simple calculation shows that the equilibrium probability $P_N(n)$ that someone takes action is a decreasing function of n, which asymptotically approaches $1 - c/b$. □

5.2 Proof of Proposition 2

Proof. Where the mandated strategy is of the form: take action with probability $1 - x$, the probability that at least one player takes action is $1-x^n$, and the expected cost to each player of following this strategy is $c(1 - x)$. The expected utility of every player is

$$b(1 - x^n) - c(1 - x). \tag{27}$$

Taking the derivative of expression 27, and arranging terms, we see that expected utility is maximized at $x = x_n$, when

$$x_n = n^{\frac{-1}{n-1}} \left(\frac{c}{b}\right)^{\frac{1}{n-1}}. \tag{28}$$

From Equation 28, it follows that $x_n^{n-1} = \frac{1}{n}\frac{c}{b}$, which is a decreasing function of n. This implies that for $n \geq 2$, x_n is also a decreasing function of n,

From Equations 28 and Equation 25, it follows that

$$x_n = n^{\frac{-1}{n-1}} q_n. \tag{29}$$

Since $n^{\frac{-1}{n-1}} < 1$ for all $n > 1$, it must be that $0 < x_n < q_n$ and hence the $1 > 1 - x_n > 1 - q_n$, which means that the probability that an individual takes action under the optimal symmetric rule is less than one, but greater than the probability of taking action in Nash equilibrium. $\square$

5.3 Proof of Proposition 3

Proof. Equation 28 implies that

$$\begin{aligned}
\lim_{n\to\infty} \ln x_n &= \lim_{n\to\infty} \left(\frac{-1}{n-1}\right) \ln n + \lim_{n\to\infty} \left(\frac{1}{n-1}\right) \frac{c}{b} \\
&= \lim_{n\to\infty} \left(\frac{-\ln n}{n-1}\right) \\
&= 0. \qquad (30)
\end{aligned}$$

It also follows form Equation 28 that

$$\begin{aligned}
\lim_{n\to\infty} \ln x_n^n &= \lim_{n\to\infty} \left(\frac{-n}{n-1}\right) \ln n + \lim_{n\to\infty} \left(\frac{n}{n-1}\right) \frac{c}{b} \\
&= \lim_{n\to\infty} \left(\frac{-n \ln n}{n-1}\right) + \frac{c}{b} \\
&= -\infty, \qquad (31)
\end{aligned}$$

where the final equalities in Equations 30 and 31 are direct consequences of application of L'Hospital's rule. Since $\lim_{n\to\infty} \ln x_n = 0$, it must be that $\lim_{n\to\infty} x_n = 1$, and since $\lim_{n\to\infty} \ln x_n^n = -\infty$, it must be that $\lim_{n\to\infty} x_n^n = 0$. Therefore as $n \to \infty$, the limiting probability that any single individual acts is $1 - \lim_{n\to\infty} x_n = 0$ and the probability that at least one individual acts is $1 - \lim_{n\to\infty} x_n^n = 1$.

□

5.4 Proof of Proposition 4

Proof. Assumption 1 implies that $G(\cdot) = 1 - F(\cdot)$ is a decreasing function and that $G(\ell) = 1$ and $G(b) < 1$. Let $H(c, n) = bG(c)^{n-1} - c$. Then H is a continuous, strictly decreasing function with $H(\ell, n) = b - \ell > 0$ and $H(b, n) = bG(b)^{n-1} - b < 0$. Therefore for any $n > 1$, there is exactly one solution $\hat{c}(n) \in (\ell, b)$ such that $H(\hat{c}(n), n) = 0$.

To show that $\hat{c}(n)$ decreases with n, take logs of both sides of Equation 6 and differentiate with respect to n. This yields the equation

$$\ln G(\hat{c}(n)) + (n-1)\frac{\hat{c}(n)G'(\hat{c}(n))}{G(\hat{c}(n))}\left(\frac{\hat{c}'(n)}{\hat{c}(n)}\right) = \frac{\hat{c}'(n)}{\hat{c}(n)} \tag{32}$$

and hence

$$\frac{\hat{c}'(n)}{\hat{c}(n)} = \frac{\ln G(\hat{c}(n))}{1 - (n-1)\frac{G'(\hat{c}(n))}{G(\hat{c}(n))}} \tag{33}$$

Since $0 \leq G(\hat{c}(n)) \leq 1$ and $G'(\hat{c}(n)) \leq 0$, the numerator of Equation 33 must be negative and the denominator must be positive. It follows that $\hat{c}'(n) < 0$ and hence $\hat{c}(\cdot)$ is a decreasing function of n.

The sequence $\hat{c}(n)$ is a bounded monotone sequence. Hence, by the monotone convergence theorem, this sequence converges to a limit $\hat{c} \geq \ell$. Suppose that $\hat{c} > \ell \geq 0$. Then $G(\hat{c}) < 1$ and therefore for N sufficiently large, $G(\hat{c})^{N-1} < \hat{c}$. Since $\hat{c}$ is a lower bound for the sequence of $c(n)$'s, it must be that $\hat{c}(N) > \hat{c}$. Since $G(\cdot)$ is a decreasing function, it must be that $G(\hat{c}(N))^{N-1} < G(\hat{c})^{N-1} < \hat{c} < \hat{c}(N)$. But this contradicts the requirement that $G(\hat{c}(N))^{N-1} = \hat{C}(N)$. Therefore it cannot be that $\hat{c} > \ell$. It follows that $\hat{c} = \ell$. $\square$

5.5 Proof of Proposition 7

Proof. With threshold set at c, the probability that any single player will not take action is $G(c)$ and the probability that at least one player will take action is $1 - G(c)^n$. Individuals will take costly action if and only if their costs lie below the threshold level c, Before individuals learn their own costs, the expected value of the costs that each will have to pay is

$$\int_0^c xF'(x)dx = -\int_0^c xG'(x)dx.$$

Thus, if there are n players and if the threshold is set at c, then, before individual costs are revealed, the expected utility of every player must be

$$b\left(1 - G(c)^n\right) + \int_0^c xG'(x)dx. \tag{34}$$

The first-order necessary condition for $c = \bar{c}(n)$ to maximize Expression 34 is

$$bnG\left(\bar{c}(n)\right)^{n-1} G'\left(\bar{c}(n)\right) - \bar{c}(n)G'\left(\bar{c}(n)\right) = 0. \tag{35}$$

Let us define

$$H(c,n) = bnG\left(\bar{c}(n)\right)^{n-1} - \bar{c}(n) = 0 \tag{36}$$

Since $G'(c) > 0$ for all $c \in (\ell, h)$, Equation 35 is equivalent to

$$H(c,n) = 0 \tag{37}$$

Since, by assumption, $nb > \ell$ and $G(\ell) = 1$ it must be that $H(\ell, n) = nb - \ell > 0$. Since $G(h) = 0$, it must be that $H(h, n) = -h < 0$. Since $H(c, n)$ is a decreasing function of c, it follows that there is exactly one solution $\bar{c}(n)$ to the equation $H(c, n) = 0$. This solution $\bar{c}(n)$ is the unique symmetric optimal threshold. $\square$

5.6 Proof of Proposition 8

Proof. Notice that $\lim_{n\to\infty} \bar{c}(n) = \ell$ if and only if for every $\epsilon > 0$, there exists $N(\epsilon)$ such that if $n > N(\epsilon)$, then $|\bar{c}(n) - \ell| < \epsilon$ for all $n > N(\epsilon)$. To show that $\lim_{n\to\infty} \bar{c}(n) = \ell$, let $\epsilon > 0$ and $c \in (\ell, h)$ with $|c - \ell| \geq \epsilon$. Then $c \geq \ell + \epsilon$ and since G is strictly decreasing in c, it must be that $G(c) \leq G(\ell + \epsilon) < 1$. Since $G(c) < 1$, it must be that $\lim_{n\to\infty} bnG(c)^{n-1} = 0$. It follows that there exists $N(\epsilon)$ such that $bnG(c)^{n-1} < \epsilon \leq \ell + \epsilon$ for all $n > N(\epsilon)$. Therefore if $|c - \ell| > \epsilon$, $bnG(c)^{n-1} > c$. This implies $|c(n) - \ell| < \epsilon$ for all $n > N(\epsilon)$. Therefore $\lim_{\to\infty} \bar{c}(n) = \ell$.

If players use the optimal symmetric threshold strategy, then with n players, the probability that no player takes action is $G\left(\bar{c}(n)\right)^n$ From Equation 12, it follows that

$$G\left(\bar{c}(n)\right)^n = \frac{\bar{c}(n)}{n} G\left(\bar{c}(n)\right) \tag{38}$$

and hence

$$\begin{aligned} \lim_{n\to\infty} G\left(\bar{c}(n)\right)^n &= \lim_{n\to\infty} \frac{\bar{c}(n)}{n} \lim_{n\to\infty} G\left(\bar{c}(n)\right) \\ &= \lim_{n\to\infty} \frac{\ell G(\ell)}{n} \\ &= 0 \end{aligned} \tag{39}$$

The limiting probability that at least one player takes action is then

$$1 - \lim_{n\to\infty} G\left(\bar{c}(n)\right)^n = 1 \tag{40}$$

□

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