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(Non-)Abelian discrete anomalies

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Abstract

We derive anomaly constraints for Abelian and non-Abelian discrete symmetries using the path integral approach. We survey anomalies of discrete symmetries in heterotic orbifolds and find a new relation between such anomalies and the so-called ‘anomalous’ $U(1)$.

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1. Introduction

Symmetries play a key role in the understanding of fundamental laws of physics. Apart from continuous, in particular gauge, symmetries, discrete symmetries provide a useful tool in field-theoretic model building and arise often in top–down models.

Very much like continuous symmetries, discrete symmetries can be broken by quantum effects, i.e. have an anomaly [1]. If this is the case, one expects that the corresponding conservation laws be violated through non-perturbative effects. The criteria for discrete symmetries to be non-anomalous, and thus to be exact, have been extensively studied in the Abelian (\mathbb{Z}_N) case [2,3]. Anomaly criteria for non-Abelian discrete symmetries have been discussed first in specific examples [4]. Here, we use the path integral approach [5,6] to derive anomaly constraints on non-Abelian discrete symmetries. We follow the discussion of [7], and extend it such as to include gravitational anomaly constraints. We further re-derive the conditions for Abelian discrete

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symmetries to be anomaly-free, using the path integral method. This derivation allows for an alternative, perhaps more intuitive understanding of the criteria, which does not rely on contributions from heavy states.

We explore the issue of discrete anomalies in string compactifications, focusing on heterotic orbifolds. The question we seek to clarify is whether discrete anomalous symmetries can appear in string-derived models [3,8]. The discrete symmetries on orbifolds reflect certain geometrical symmetries of internal space. Since the geometrical operations, i.e. space group transformations, are embedded into the gauge group, one might suspect that the discrete anomalies are related to gauge anomalies. We find that this is indeed the case, specifically we find that the so-called ‘anomalous’ $U(1)$, which occurs frequently in heterotic orbifolds, determines the anomalies of discrete symmetries.

The paper is organized as follows. In Section 2 we first re-derive anomaly constraints for Abelian discrete symmetries and then derive the constraints for non-Abelian discrete symmetries, using the path integral approach. In Section 3 we consider heterotic orbifolds and identify a geometric operation on the orbifold, which we would like to refer to as ‘anomalous space group element’, as the source of all discrete anomalies. Section 4 contains our conclusions. We also include four appendices where we present the calculation of anomalies of the dihedral group D_4 (A), D_4 anomalies in a concrete model from the literature (B) and the anomaly coefficients in two concrete string models (C & D).

2. Anomaly-free discrete symmetries

2.1. A few words on symmetries

Consider a theory described by a Lagrangean \mathcal{L} with a set of fermions $\Psi = [\psi^{(1)}, \dots, \psi^{(M)}]$, where $\psi^{(m)}$ denotes a field transforming in the irreducible representation (irrep) $\mathbf{R}^{(m)}$ of all internal symmetries. A general transformation $\Psi \rightarrow U\Psi$ or, more explicitly,

$$\begin{bmatrix} \psi^{(1)} \\ \vdots \\ \psi^{(M)} \end{bmatrix} \rightarrow \begin{pmatrix} U^{(1)} & & 0 \\ & \ddots & \\ 0 & & U^{(M)} \end{pmatrix} \begin{bmatrix} \psi^{(1)} \\ \vdots \\ \psi^{(M)} \end{bmatrix}, \quad (1)$$

which leaves \mathcal{L} invariant (up to a total derivative) denotes a classical symmetry. By Noether’s theorem, *continuous* symmetries imply, at the classical level, conserved currents, $D_\mu j^\mu = 0$. For instance, in the case of an Abelian continuous symmetry one can define the charge $Q = \int d^3x j^0$ which satisfies the conservation law $\frac{d}{dt} Q = 0$.

In the case of a discrete symmetry, the situation is similar. Consider, for simplicity, an Abelian discrete symmetry, i.e. \mathbb{Z}_N . Under this symmetry, the fermions of the theory transform as

$$\psi^{(m)} \rightarrow e^{2\pi i q^{(m)}/N} \psi^{(m)}, \quad (2)$$

where (by convention) the discrete charges $q^{(m)}$ are integer and only defined modulo N . If (2) is a symmetry of \mathcal{L} , the corresponding charge $q^{(m)}$ is conserved modulo N .¹

¹ A familiar example for such a conservation law is due to R -parity, which implies that superpartners can only be produced in pairs.

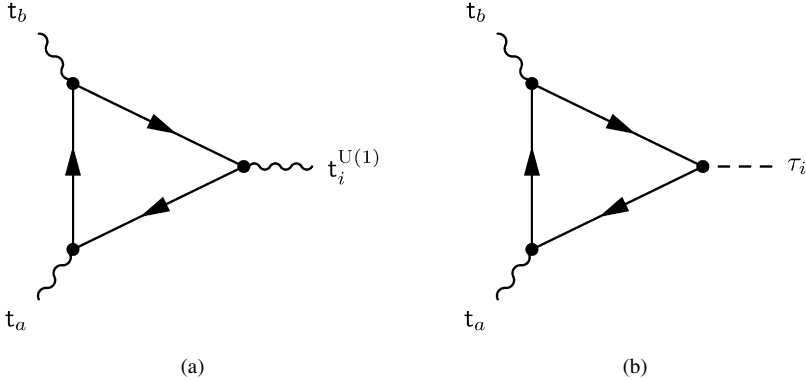


Fig. 1. Triangle diagrams. (a) U(1)–G–G. (b) D–G–G.

2.2. Basics of anomalies

Classical chiral symmetries can be broken by quantum effects, i.e. have an anomaly. Specifically, consider a chiral transformation

$$\Psi(x) \rightarrow \Psi'(x) = \exp(i\alpha P_L)\Psi(x), \tag{3}$$

where $\alpha = \alpha^A T_A$ with T_A denoting the generators of the transformation, and P_L is the left-chiral projector. It is well known that at the quantum level the classically conserved current $j^\mu(x)$ is not necessarily conserved any more, that is (cf. e.g. [9])

$$\langle D_\mu j^\mu(x) \rangle = \mathcal{A}(x; \alpha) \neq 0. \tag{4}$$

The anomaly $\mathcal{A}(x; \alpha)$ can be derived using Fujikawa’s method, i.e. by calculating the transformation of the path integral measure [5,6], which in our case reads

$$\mathcal{D}\Psi \mathcal{D}\bar{\Psi} \rightarrow J(\alpha) \mathcal{D}\Psi \mathcal{D}\bar{\Psi}, \tag{5}$$

where the Jacobian of the transformation is given by

$$J(\alpha) = \exp\left\{i \int d^4x \mathcal{A}(x; \alpha)\right\}. \tag{6}$$

The anomaly function \mathcal{A} decomposes into a gauge and a gravitational part [10–12],

$$\mathcal{A} = \mathcal{A}_{\text{gauge}} + \mathcal{A}_{\text{grav}}. \tag{7}$$

The gauge part $\mathcal{A}_{\text{gauge}}$ corresponds to the triangle diagram α –gauge–gauge (Fig. 1). This anomaly is given by²

$$\mathcal{A}_{\text{gauge}}(x; \alpha) = \frac{1}{32\pi^2} \text{Tr}\left[\alpha \mathcal{F}^{\mu\nu}(x) \tilde{\mathcal{F}}_{\mu\nu}(x)\right]. \tag{8}$$

Here $\mathcal{F}_{\mu\nu} = [D_\mu, D_\nu]$ is the field strength of the gauge symmetry, such that $\mathcal{F}_{\mu\nu} = g(\partial_\mu A_\nu - \partial_\nu A_\mu)$ for a U(1) symmetry, and $\tilde{\mathcal{F}}^{\mu\nu} = \varepsilon^{\mu\nu\rho\sigma} \mathcal{F}_{\rho\sigma}$ denotes its dual. The trace ‘Tr’ runs over all internal indices.

² Note that there is a factor 1/2 discrepancy to Fujikawa’s result [5,6] because we are considering only fermions of one chirality (cf. e.g. [13] and [9, p. 271]).

Analogously, the gravitational part $\mathcal{A}_{\text{grav}}$ is the mixed α –gravity–gravity anomaly. It is known that it takes the form [10–12]

$$\mathcal{A}_{\text{grav}} = -\mathcal{A}_{\text{grav}}^{\text{Weyl fermion}} \sum_m \text{tr}[\alpha(\mathbf{R}^{(m)})], \tag{9}$$

where the summation runs over the (spin-1/2) fermions in the representations $\mathbf{R}^{(m)}$. The subscript ‘ m ’ indicates that each representation $\mathbf{R}^{(m)}$ appears only once in the sum. $\alpha(\mathbf{R}^{(m)})$ denotes $\alpha_A T^A(\mathbf{R}^{(m)})$ in the representation $\mathbf{R}^{(m)}$, and might therefore be thought of as a $\dim \mathbf{R}^{(m)} \times \dim \mathbf{R}^{(m)}$ matrix such that ‘tr’ is the standard (matrix) trace. The contribution of a single Weyl fermion to the gravitational anomaly is given by [10–12]

$$\mathcal{A}_{\text{grav}}^{\text{Weyl fermion}} = \frac{1}{384\pi^2} \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} R_{\mu\nu}{}^{\lambda\gamma} R_{\rho\sigma\lambda\gamma}. \tag{10}$$

To evaluate the anomaly (7), we split the set of all generators T_A into generators of continuous symmetries t_a and those of discrete symmetries τ_i . Therefore, we shall discuss separately the two cases:

- (i) anomalies of continuous symmetries with $\alpha = \alpha^a t_a$;
- (ii) anomalies of discrete symmetries with $\alpha = \alpha^i \tau_i$.

Note that we (implicitly) assume that all symmetries are gauged.

For the evaluation of the anomalies, it is useful to recall the powerful index theorems [10,11], which imply

$$\int d^4x \frac{1}{32\pi^2} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^b \text{tr}[t_a t_b] \in \mathbb{Z}, \tag{11a}$$

$$\frac{1}{2} \int d^4x \frac{1}{384\pi^2} \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} R_{\mu\nu}{}^{\lambda\gamma} R_{\rho\sigma\lambda\gamma} \in \mathbb{Z}, \tag{11b}$$

where t_a are in the fundamental representation of a particular gauge factor G . Note that in our conventions $\text{tr}[t_a t_b] = \frac{1}{2} \delta_{ab}$. The factor $\frac{1}{2}$ in Eq. (11b) follows from Rohlin’s theorem [14], as discussed in [15].

2.3. Anomaly constraints for continuous symmetries

We start by reviewing the anomaly constraints for the continuous symmetries. They arise from demanding that $\mathcal{A}(x; \alpha^a t_a)$ vanish for arbitrary α^a in order for the Jacobian $J(\alpha)$ to be trivial. Consider first the mixed U(1)– G – G anomaly, where G is a non-Abelian gauge factor with generators t_a . Representations under G are denoted by $\mathbf{r}^{(f)}$. This anomaly can be related to Fig. 1(a). From Eq. (8) and the index theorem (11a), one finds that it only vanishes if

$$A_{\text{U(1)-}G\text{-}G} \equiv \sum_{\mathbf{r}^{(f)}} q^{(f)} \ell(\mathbf{r}^{(f)}) = 0. \tag{12}$$

In analogy to Eq. (9), ‘ $\sum_{\mathbf{r}^{(f)}}$ ’ means that each representation $\mathbf{r}^{(f)}$ is only summed once.³ $q^{(f)}$ denote the respective U(1) charges. The Dynkin indices $\ell(\mathbf{r}^{(f)})$ are defined by

$$\ell(\mathbf{r}^{(f)}) \delta_{ab} = \text{tr}[t_a(\mathbf{r}^{(f)}) t_b(\mathbf{r}^{(f)})]. \tag{13}$$

³ Of course, the dimensions of representations w.r.t. further symmetry factors have to be taken into account.

Our conventions are such that $\ell(\mathbf{M}) = 1/2$ for $SU(M)$ and $\ell(\mathbf{M}) = 1$ for $SO(M)$. Consider next the $U(1)$ –grav–grav anomaly, Eq. (9). From the index theorem (11b), it vanishes if

$$A_{U(1)\text{-grav-grav}} \equiv \sum_f q^{(f)} = \sum_m q^{(m)} \dim(\mathbf{R}^{(m)}) = 0. \quad (14)$$

The sum ‘ \sum_f ’ indicates a plain summation over all fermions.

In summary, we see that the continuous symmetries are non-anomalous if and only if the Jacobian (6) is trivial for arbitrary α .

2.4. (Re-)derivation of anomaly constraints for \mathbb{Z}_N symmetries

Now consider a discrete symmetry, i.e. $\alpha = \alpha^i \tau_i$ where, by convention, α^i takes only the discrete values $2\pi/N^i$ and the eigenvalues of τ_i are integer. Like before, we demand that $J(\alpha)$ be trivial. It is now important to note that the Jacobian can also be trivial for non-zero arguments of the exponential. Let us specify the conditions for this to happen. Consider first the Abelian case, i.e. a \mathbb{Z}_N symmetry with $\alpha = 2\pi\tau/N$. From the gauge and gravitational parts of the anomaly function, equations (8) and (9), and the index theorems (11), we see that the Jacobian is trivial if

$$A_{\mathbb{Z}_N\text{-G-G}} = \frac{1}{N} \sum_{\mathbf{r}^{(f)}} q^{(f)} (2\ell(\mathbf{r}^{(f)})) \in \mathbb{Z}, \quad (15a)$$

$$A_{\mathbb{Z}_N\text{-grav-grav}} = \frac{2}{N} \sum_m q^{(m)} \dim \mathbf{R}^{(m)} \in \mathbb{Z}. \quad (15b)$$

The factor 2 in front of the Dynkin index in (15a) is due to our conventions ($\text{tr}[\mathbf{t}_a \mathbf{t}_b] = \frac{1}{2} \delta_{ab}$). This means that the constraints for a \mathbb{Z}_N symmetry to be anomaly-free are

$$\mathbb{Z}_N\text{-G-G}: \quad \sum_{\mathbf{r}^{(f)}} q^{(f)} \ell(\mathbf{r}^{(f)}) = 0 \pmod{N/2}, \quad (16a)$$

$$\mathbb{Z}_N\text{-grav-grav}: \quad \sum_m q^{(m)} \dim \mathbf{R}^{(m)} = 0 \pmod{N/2}. \quad (16b)$$

If N is odd, we can always make the \mathbb{Z}_N charges even by shifting them by integer multiples of N . This explains why the sums in (16a) and (16b) can always be made integer. Hence in the case of an odd N one can replace $N/2$ by N after a suitable shift of the charges. The constraints (16a) and (16b) coincide with the ones of the literature [2,3,15–19]. We would like to emphasize that, in our derivation, we did not invoke the contributions from heavy Majorana fermions.⁴ Rather, the anomaly constraints (including the condition mod $N/2$) are a consequence of the index theorems and follow from demanding that the Jacobian be trivial. We also note that in our approach one immediately sees that there are no cubic anomaly constraints for discrete symmetries, which is in agreement with [3].

⁴ The mod $N/2$ condition for even N has been justified as follows [2]: one can always introduce Majorana fermions ψ with \mathbb{Z}_N charges $N/2$. Their contribution to the sum is $N/2$, on the other hand the Majorana mass term $m\psi\psi$ is allowed by the discrete symmetry. Since m can be arbitrarily large, ψ can be ‘removed from the theory’. While the argument leads to the correct result, one might nevertheless wonder if the anomaly conditions change if one considers more constrained settings (such as string-derived theories) where extra degrees of freedom cannot be introduced at will. Our derivation shows that the anomaly conditions remain unchanged.

2.5. Anomalies of non-Abelian discrete symmetries

We now turn to non-Abelian discrete symmetries D . Consider a specific transformation U . Since we are considering a discrete symmetry, there is a positive integer N such that $U^N = \mathbb{1}$, i.e. U is generating a \mathbb{Z}_N symmetry; we take N to be the smallest such integer. Denote the (discrete) representations of D by $\mathbf{d}^{(f)}$. Moreover, an element $U \in D$ in a representation $\mathbf{d}^{(f)}$ is given by $U(\mathbf{d}^{(f)}) = e^{i\alpha(\mathbf{d}^{(f)})}$ with $\alpha(\mathbf{d}^{(f)}) = 2\pi\tau(\mathbf{d}^{(f)})/N$ and $\tau(\mathbf{d}^{(f)})$ having integer eigenvalues. In the evaluation of the anomaly functions, Eqs. (8) and (9), we note that $\text{tr}[\tau(\mathbf{d}^{(f)})]$ takes the role of the \mathbb{Z}_N charge. This charge, denoted by $\delta^{(f)}$, can be expressed in terms of the group elements $U(\mathbf{d}^{(f)})$ as (cf. [7])

$$\delta^{(f)} \equiv \text{tr}[\tau(\mathbf{d}^{(f)})] = N \frac{\ln \det U(\mathbf{d}^{(f)})}{2\pi i}. \tag{17}$$

As usual, the \mathbb{Z}_N charges $\delta^{(f)}$ are defined modulo N only (such that they can consistently be expressed through the multi-valued logarithm).

From the index theorems (11), we find that demanding that the Jacobian be trivial amounts to requiring

$$\sum_{(\mathbf{r}^{(f)}, \mathbf{d}^{(f)})} \delta^{(f)} \cdot \ell(\mathbf{r}^{(f)}) \stackrel{!}{=} 0 \pmod{\frac{N}{2}}, \tag{18a}$$

$$\sum_{\mathbf{d}^{(f)}} \delta^{(f)} \stackrel{!}{=} 0 \pmod{\frac{N}{2}}, \tag{18b}$$

where the sum ‘ $\sum_{(\mathbf{r}^{(f)}, \mathbf{d}^{(f)})}$ ’ indicates that only over representations is summed which are non-trivial w.r.t. both G and D ; the symbol ‘ $\sum_{\mathbf{d}^{(f)}}$ ’ in (18b) means that the sum extends over all non-trivial representations $\mathbf{d}^{(f)}$.

These constraints have to be fulfilled for each discrete transformation U separately. However, elements with $\det U = 1$ do not lead to anomalies, cf. Eq. (17).

Non-Abelian discrete groups D have more than one element. Assume that we have verified that the constraints (18) are fulfilled for $U, U' \in D$. It is then obvious that this implies that for both elements $U'' = U \cdot U'$ and $U''' = U' \cdot U$ Eqs. (18) hold as well. This means that in practice one only has to check anomaly constraints for the generators of D . In Appendix A, we discuss D_4 anomalies as an example for non-Abelian discrete anomalies. In Appendix B we present a sample calculation.

2.6. Summary of anomaly constraints

The anomaly constraints for discrete symmetries can be summarized as follows:

(i) Anomalies of \mathbb{Z}_N symmetries

$$\mathbb{Z}_N\text{-G-G}: \sum_{\mathbf{r}^{(f)}} q^{(f)} \cdot \ell(\mathbf{r}^{(f)}) \stackrel{!}{=} 0 \pmod{\frac{N}{2}}, \tag{19a}$$

$$\mathbb{Z}_N\text{-grav-grav}: \sum_m q^{(m)} \cdot \dim \mathbf{R}^{(m)} \stackrel{!}{=} 0 \pmod{\frac{N}{2}}. \tag{19b}$$

(ii) Anomalies of non-Abelian discrete symmetries D : one has to verify that for all generators of D the following two equations hold

$$D\text{-}G\text{-}G: \sum_{(\mathbf{r}^{(f)}, \mathbf{d}^{(f)})} \delta^{(f)} \cdot \ell(\mathbf{r}^{(f)}) \stackrel{!}{=} 0 \pmod{\frac{N}{2}}, \quad (20a)$$

$$D\text{-}grav\text{-}grav: \sum_{\mathbf{d}^{(f)}} \delta^{(f)} \stackrel{!}{=} 0 \pmod{\frac{N}{2}}. \quad (20b)$$

Here, the sum $\sum_{\mathbf{d}^{(f)}}$ extends over all non-trivial representations of D , $\delta^{(f)}$ is defined in Eq. (17), and N denotes the order of the generator.

2.7. Consequences of discrete anomalies

Now we turn to study the implications of an anomalous discrete symmetry. One might envisage several scenarios in which such a symmetry appears. In what follows, we focus on a particular one: we start with a so-called ‘anomalous’ $U(1)$ and break it to a discrete subgroup. Later, in Section 3, where we investigate string-derived models, we will attempt to realize different situations.

In a fundamental theory, anomalies of a continuous symmetry are not acceptable. However, there is the well-understood situation in which a $U(1)$ factor appears ‘anomalous’, i.e. the usual anomaly conditions seem not to be satisfied. This is the case when the anomaly is canceled by the Green–Schwarz (GS) mechanism [20]. To discuss this scenario, consider a supersymmetric gauge theory. Under the (‘anomalous’) $U(1)_{\text{anom}}$ transformation, the chiral superfields $\Phi^{(f)}$ containing the chiral fermions $\psi^{(f)}$ and the vector superfield V transform as

$$\Phi^{(f)} \rightarrow e^{-iq^{(f)}\Lambda} \Phi^{(f)}, \quad V \rightarrow V + i(\Lambda - \bar{\Lambda}). \quad (21)$$

The anomaly is canceled by the transformation of the dilaton S (or possibly a different chiral field), which gets shifted under the $U(1)_{\text{anom}}$ transformation as

$$S \rightarrow S + \frac{i}{2} \delta_{\text{GS}} \Lambda, \quad (22)$$

where δ_{GS} is proportional to the trace of the generator of $U(1)_{\text{anom}}$, $\text{tr} t_{\text{anom}}$, (see below). The tree-level Kähler potential for the dilaton is

$$K_{\text{dilaton}}(S + \bar{S}) = -\ln(S + \bar{S}). \quad (23)$$

As usual, the kinetic terms for the scalar components of S arise from the corresponding D -term, $[K_{\text{dilaton}}(S + \bar{S})]_D$, i.e.

$$\frac{1}{4s^2} (\partial^\mu s \partial_\mu s + \partial^\mu a \partial_\mu a), \quad (24)$$

where $s = \text{Re } S$ and $a = \text{Im } S$. Consider now the axionic shift (22),

$$a \rightarrow a + \theta/2. \quad (25)$$

The kinetic term (24) is invariant under this shift when θ is constant. However, as the parameter θ depends on x for $U(1)_{\text{anom}}$ transformations, the kinetic term (24) is not invariant under $U(1)_{\text{anom}}$. To make it invariant, we have to introduce the terms, $A^\mu \partial_\mu a$ and $A^\mu A_\mu$, in the Stückelberg form.

That implies the $U(1)_{\text{anom}}$ -invariant Kähler potential for the dilaton is

$$K_{\text{dilaton}} \left(S + \bar{S} - \frac{\delta_{\text{GS}}}{2} V \right), \tag{26}$$

which also includes the s -dependent Fayet–Iliopoulos (FI) D-term.

It is convenient to define a normalized $U(1)_{\text{anom}}$ generator \hat{t}_{anom} , whose charges \hat{q}_{anom} fulfill the consistency conditions (cf. [21])

$$\frac{1}{3} \sum_f (\hat{q}_{\text{anom}}^{(f)})^3 = \frac{1}{24} \sum_f \hat{q}_{\text{anom}}^{(f)} =: 8\pi^2 \delta_{\text{GS}}. \tag{27}$$

Our conventions are such that δ_{GS} is positive. (As before, \sum_f means plain summation.) The mixed $U(1)_{\text{anom}}-G-G$ anomaly coefficients, as defined in (12), have to satisfy the consistency conditions

$$\frac{1}{k} A_{U(1)_{\text{anom}}-G-G} = 8\pi^2 \delta_{\text{GS}}, \tag{28}$$

where k denotes the Kač–Moody level of G . For the Green–Schwarz mechanism to work, this relation has to hold for all gauge group factors.

The first question is whether $U(1)_{\text{anom}}$ can be used to forbid couplings. To answer this question, consider a product of fields, $\Phi^{(1)} \dots \Phi^{(n)}$, with $\sum_i q_i < 0$. In the case of a usual $U(1)$ symmetry, $\Phi^{(1)} \dots \Phi^{(n)}$ cannot denote an allowed coupling. However, in the case of $U(1)_{\text{anom}}$, this conclusion does not apply; instead one finds that the non-perturbative coupling

$$e^{-pS/\delta_{\text{GS}}} \Phi^{(1)} \dots \Phi^{(n)} \tag{29}$$

with an appropriate p can be induced (cf. [3,22,23]). In other words, the field $\Sigma = e^{-pS/\delta_{\text{GS}}}$ transforms under the $U(1)$ with a charge that is opposite to $tr q_{\text{anom}}$. That means that $U(1)_{\text{anom}}$ does not forbid products of fields with $\sum_i q_i < 0$.

What can one say about products of fields $\Phi^{(1)} \dots \Phi^{(n)}$ with $\sum_i q_i > 0$? Here the answer is that an anomalous $U(1)$ implies the existence of a FI D-term (cf. Eq. (26)). To obtain a supersymmetric vacuum, the FI term has to be canceled. That is, certain fields with net negative anomalous charge have to attain a VEV in the vacuum. Multiplication of $\Phi^{(1)} \dots \Phi^{(n)}$ by such fields can lead to allowed couplings, hence in supersymmetric vacua couplings of the type $\Phi^{(1)} \dots \Phi^{(n)}$ will generically be allowed.

Given these considerations, it is also clear what happens if one breaks $U(1)_{\text{anom}}$ to a discrete, anomalous subgroup. Since $U(1)_{\text{anom}}$ is violated by terms of the form (29), also the discrete subgroup is expected not to be exact.⁵

An anomaly of an Abelian discrete symmetry does not necessarily signal an inconsistency of the model. Symmetries might just be accidental or approximate, and, therefore, need not to be gauged. Further, if the anomalies are universal, they can be canceled by a Green–Schwarz mechanism. In practice, this means that they are broken by the VEVs of certain fields; in addition there are non-perturbatively induced terms with hierarchically small coefficients, as in (29).

⁵ A special situation arises if $U(1)_{\text{anom}}$ gets broken to a \mathbb{Z}_N subgroup which, however, is non-anomalous by the criteria (16). Here, either the terms (29) appear nevertheless, or there is a subclass of terms, which are forbidden by the non-anomalous \mathbb{Z}_N , and where the coefficient happens to be zero. That is, if the second possibility is true, non-perturbative effects break $U(1)_{\text{anom}}$ to a non-anomalous \mathbb{Z}_N subgroup. To find out which situation is realized would be, by itself, an interesting question, which is, however, beyond the scope of this study.

These small corrections might turn out to be a virtue rather than a problem in concrete models.

2.8. A comment on the ‘SUSY zero mechanism’

We conclude this section by commenting on supersymmetric texture zeros [24,25], which go sometimes also under the term ‘SUSY zero mechanism’. It is stated that, due to holomorphicity of the superpotential, an anomalous U(1) symmetry can enforce absence of certain couplings even though the symmetry is broken in supersymmetric vacua, where the FI D -term is canceled. Let us briefly review the argument: cancellation of the FI term requires certain field with certain, say negative, sign of ‘anomalous’ charge to attain a vacuum expectation value (VEV). Now one might envisage a situation in which only fields with non-positive charges get a VEV. Consider then a combination of some other fields, $\Phi^{(1)} \dots \Phi^{(n)}$, which has total negative anomalous charge. To be neutral w.r.t. the $U(1)_{\text{anom}}$ symmetry, this combination needs to be multiplied by fields with positive $U(1)_{\text{anom}}$ charge. However, so the argument goes, those fields do not attain VEVs, and hence $\Phi^{(1)} \dots \Phi^{(n)}$ cannot denote an allowed coupling. That is, couplings of the type $\Phi^{(1)} \dots \Phi^{(n)}$ appear to be absent. On the other hand, in many applications of the ‘SUSY zero mechanism’ it is not possible to specify a symmetry that forbids those couplings.

With what we have discussed above, we are able to resolve the puzzle: $\Sigma = e^{-pS/\delta_{\text{GS}}}$ carries positive charge and hence couplings of the form $\Sigma \Phi^{(1)} \dots \Phi^{(n)}$ can arise. The induced effective coupling is suppressed (so that, as far as textures are concerned, a ‘zero’ can be a good approximation), however, in contrast to what is often assumed, in general it is not related to the scale of supersymmetry breakdown.

3. Anomalies in heterotic orbifold models

An interesting question is whether discrete anomalies occur in top–down constructions, in particular in string compactifications [3,8]. Since string theory is believed to be UV complete, one would expect that there are no (uncanceled) anomalies in this framework. While this has been extensively checked for continuous gauge symmetries, the case of discrete symmetries is somewhat more subtle. Construction of string models exhibiting discrete anomalies would lead to a playground in which the ‘quantum gravity effects’, which are commonly believed to spoil the discrete conservation laws, can be specified in somewhat more detail than usual.

Specifically, we study anomalies of discrete symmetries in heterotic orbifold models. In our presentation, we mainly focus on the \mathbb{Z}_6 -II orbifold, yet in our computations we also considered different orbifolds, so that our results are more generally valid. We start with a very brief review on orbifolds, summarize the essentials of (discrete) string selection rules, continue by relating the so-called ‘anomalous U(1)’ to a discrete transformation in compact space, which we refer to as the ‘anomalous space group element g^{anom} ’, and conclude by relating anomalies in discrete symmetries to the anomaly in the discrete transformation g^{anom} .

3.1. Orbifold basics

A heterotic orbifold emerges by dividing a six-dimensional torus \mathbb{T}^6 by one of its symmetries θ [26,27] (see [28] for a recent review). \mathbb{T}^6 can be parametrized by three complex coordinates z_i ($i = 1, 2, 3$). Then we denote $\theta = \text{diag}(e^{2\pi i v_1}, e^{2\pi i v_2}, e^{2\pi i v_3})$. For example, in \mathbb{Z}_6 -II orbifolds one has $v_i = (1/6, 1/3, -1/2)$. A model is defined by the compactification lattice, the twist vector

v_i , the shift V and the Wilson lines W_α . Given these data, the massless spectrum (at the orbifold point) is completely determined (for recent explicit examples see e.g. [29,30]). A rather common feature of these constructions is the occurrence of a so-called ‘anomalous $U(1)$ ’, $U(1)_{\text{anom}}$. [21, 31] (cf. Section 2.7), which implies that, at one-loop, a FI D -term is induced [32]. As we shall see, the ‘anomalous’ $U(1)$ plays a prominent role in the discussion of discrete anomalies.

3.2. Stringy discrete symmetries

Couplings on heterotic orbifolds are governed by certain selection rules [33,34] (see also [29,30,35,36]), some of which can be interpreted as discrete symmetries of the effective field theory emerging as ‘low-energy’ limit in these constructions. These symmetries fall into two classes, depending on whether they reflect space group rules or R -charge (or H -momentum) conservation.

3.2.1. Space group rules

The space group selection rules are stated by

$$\prod_r (\theta^{k^{(r)}} , n_\alpha^{(r)} e_\alpha) \simeq (\mathbb{1}, 0), \tag{30}$$

where we label the states entering the coupling by r . $(\theta^{k^{(r)}} , n_\alpha^{(r)} e_\alpha)$ is the space group element representing the string boundary condition with $n_\alpha^{(r)} = \text{integer}$, e_α are lattice vectors defining \mathbb{T}^6 , and ‘ \simeq ’ means that the product on the l.h.s. lies in the same equivalence class as the identity element. The rotational part of (30) gives rise to the point group selection rule, and here we refer to it as the k -rule, which in \mathbb{Z}_6 -II orbifolds reads

$$\sum_r k^{(r)} = 0 \pmod 6. \tag{31}$$

The translational part can be rewritten as

$$\text{SO}(4) \text{ plane: } \sum_{r=1}^n k^{(r)} n_2^{(r)} = 0 \pmod 2, \tag{32a}$$

$$\sum_{r=1}^n k^{(r)} n_2^{(r)'} = 0 \pmod 2, \tag{32b}$$

$$\text{SU}(3) \text{ plane: } \sum_{r=1}^n k^{(r)} n_3^{(r)} = 0 \pmod 3. \tag{32c}$$

The quantum numbers $n_3^{(r)}$, $n_2^{(r)}$ and $n_2^{(r)'}$ specify the localization of the states on the orbifold; we follow the conventions of [30].

The space group rules (32) can be interpreted as $\mathbb{Z}_2 \times \mathbb{Z}'_2 \times \mathbb{Z}_3$ flavor symmetries, denoted $\mathbb{Z}_2^{\text{flavor}} \times \mathbb{Z}_2^{\text{flavor}'}$ $\times \mathbb{Z}_3^{\text{flavor}}$ in what follows. Under this symmetry, each state comes with two \mathbb{Z}_2 charges and one \mathbb{Z}_3 charge,

$$\mathbb{Z}_2^{\text{flavor}}: \quad q_2 = kn_2 \pmod 2, \tag{33a}$$

$$\mathbb{Z}_2^{\text{flavor}'}: \quad q'_2 = kn'_2 \pmod 2, \tag{33b}$$

$$\mathbb{Z}_3^{\text{flavor}}: \quad q_3 = kn_3 \pmod 3. \tag{33c}$$

In models where certain Wilson lines are absent, these symmetries combine with permutation symmetries of equivalent fixed points to non-Abelian discrete flavor symmetries [29,37]. As we are interested in anomalies, we focus on the Abelian subgroups of these discrete symmetries (cf. Section 2.5).

3.2.2. *Discrete R-symmetries*

The discrete R -symmetries in \mathbb{Z}_6 -II orbifolds based on the Lie lattice $G_2 \times SU(3) \times SO(4)$ are expressed by [29,30]

$$\sum_{r=1}^n R_1^{(r)} = -1 \pmod 6, \tag{34a}$$

$$\sum_{r=1}^n R_2^{(r)} = -1 \pmod 3, \tag{34b}$$

$$\sum_{r=1}^n R_3^{(r)} = -1 \pmod 2. \tag{34c}$$

Hereby, $R_i^{(r)}$ denotes the i th component of the H -momentum of the bosonic components of chiral superfields,

$$R_i = q_{\text{sh},i} - \Delta N_i, \tag{35}$$

where $q_{\text{sh},i}$ denote the $SO(6)$ shifted momenta of bosonic states and $\Delta N_i = \tilde{N}_i - \tilde{N}_i^*$ is the difference of oscillator numbers $\tilde{N}_i, \tilde{N}_i^*$. For twisted sectors, it can be shown that $q_{\text{sh},i} = kv_i - \text{int}(kv_i)$, with $\text{int}(kv_i)$ being the smallest integer, such that $\text{int}(kv_i) \geq kv_i$.

3.2.3. *Modular symmetries*

In orbifold constructions, T -duality transformations act as discrete reparametrizations of the moduli space. In general, there are three T -moduli T_i ($i = 1, 2, 3$), each of which corresponds to the i th complex plane z_i . For example, the modulus, T_1, T_2 and T_3 , in \mathbb{Z}_6 -II orbifolds correspond to the overall sizes of $G_2, SU(3)$ and $SO(4)$ tori, respectively. Modular symmetry is in a sense different from other symmetries, where moduli T_i are singlets. Under the modular symmetry, the moduli T_i transform as

$$T_i \rightarrow \frac{a_i T_i - ib_i}{ic_i T_i + d_i}, \tag{36}$$

where $a_i, b_i, c_i, d_i \in \mathbb{Z}$ and $ad - bc = 1$. The Kähler potential K_{matter} of matter fields $\Phi^{(f)}$ depends in general on the T_i moduli as

$$K_{\text{matter}} = \prod_i (T_i + \bar{T}_i)^{m_i} |\Phi^{(f)}|^2, \tag{37}$$

where the so-called modular weights m_i are given by [39–41]

$$m_i = \begin{cases} 1, & \text{if } q_{\text{sh},i} = -1, \\ 0, & \text{if } q_{\text{sh},i} = 0, \\ q_{\text{sh},i} + 1 - \Delta N_i, & \text{if } q_{\text{sh},i} \neq 0, -1. \end{cases} \tag{38}$$

We require that the Kähler potential K_{matter} be invariant under (36). This implies that the matter fields with the modular weight m_i transform under (36) as the following chiral rotation⁶:

$$\Phi^{(f)} \rightarrow \Phi^{(f)} \prod_i (ic_i T_i + d_i)^{m_i}. \tag{39}$$

Once the T_i attain vacuum expectation values, these symmetries are (in general) completely broken. That is, the T -duality symmetries are not expected to contribute to discrete symmetries which survive to low energies.

3.3. Discrete anomalies on orbifolds

According to the various discrete symmetries described in the previous subsections, we now define the corresponding anomaly coefficients. We further conduct a scan over many models, based on several orbifold geometries, and elicit whether there the symmetries of Section 3.2 are anomalous or not.

3.3.1. $\mathbb{Z}_n^{\text{flavor}}$ anomalies

Let us start by studying anomalies in the $\mathbb{Z}_n^{\text{flavor}}$ symmetries. A special class of $\mathbb{Z}_n^{\text{flavor}}$ anomalies is given by

$$A_{\mathbb{Z}_n^{\text{flavor}}-G-G} = \frac{1}{n} \sum_{\mathbf{r}^{(f)}} q_n^{(f)} 2\ell(\mathbf{r}^{(f)}), \tag{40}$$

where the sum extends over all non-trivial representations $\mathbf{r}^{(f)}$ of a non-Abelian gauge factor G and the $q_n^{(f)}$ are defined in (33). $A_{\mathbb{Z}_n^{\text{flavor}}-G-G}$ is only defined up to twice the smallest non-vanishing Dynkin index $\ell_{\min} = \min\{\ell(\mathbf{r}^{(f)})\}$ that appears, i.e. up to 1 if fundamental representations of $SU(N)$ groups are present.

We have investigated various heterotic orbifolds, and find that, in general, they exhibit flavor anomalies (see Appendices C and D for specific examples).

3.3.2. Discrete R anomalies

The R anomalies are given by [38]

$$A_G^{R_i} = -c_2(G) + \sum_{\mathbf{r}^{(f)}} \left(R_i^{(f)} + \frac{1}{2} \right) 2\ell(\mathbf{r}^{(f)}), \tag{41}$$

with c_2 denoting the quadratic Casimir. The sum extends over all irreps $\mathbf{r}^{(f)}$ denoting the representation of the field f w.r.t. the gauge factor G . The discrete R charges in this orbifold are only defined modulo $(6, 3, 2)$. Therefore, the anomalies can only be specified up to $(6, 3, 2)$ times twice the smallest non-vanishing Dynkin index ℓ_{\min} appearing in the sum in (41).

We find empirically that the R anomalies are not universal (for specific examples see Appendices C and D).

⁶ The Kähler potential of moduli fields, $K_{\text{moduli}} = -\sum_i \ln(T_i + \bar{T}_i)$ is not invariant under (36). T -duality invariance requires that the holomorphic superpotential W transform as $W \rightarrow W \prod_i (T_i + \bar{T}_i)^{-1}$, such that the combination $G = K_{\text{moduli}} + K_{\text{matter}} + \ln|W|^2$, which appears in the supergravity Lagrangean, is invariant.

3.3.3. T -duality anomalies

By considering the one-loop effective supersymmetric Lagrangean, one finds that the gauge coupling constant is not invariant under the discrete modular group of T -duality transformations. The coefficients of this T -duality anomaly are given by [38,40–42]

$$A_G^{T_i} = 2c_2(G) + \sum_{r^{(f)}} (2m_i^{(f)} - 1) 2\ell(\mathbf{r}^{(f)}), \quad (42)$$

where $m_i^{(f)}$ denotes the modular weight of the state $\Phi^{(f)}$ w.r.t. the plane i (cf. (38)).

As is well known, T -duality anomalies can be canceled in two different ways. One part of it is removed by the Green–Schwarz mechanism whereas a second part only disappears after considering one-loop threshold corrections to the gauge coupling constants. Only universal anomalies, i.e. those $A_G^{T_i}$ in (42) with fixed i whose values do not depend on G , can be canceled by the Green–Schwarz mechanism. In contrast, cancellation of non-universal T -duality anomalies requires additionally threshold corrections. According to [41], in orbifold models the anomaly associated to the modulus T_i is non-universal only if any of the orbifold twists acts trivially on the corresponding i th complex plane of the underlying six-torus. This means in particular, that for \mathbb{Z}_6 -II orbifolds the anomalies of T_2 and T_3 are non-universal and therefore the associated moduli appear in the threshold corrections. Further, since the orbifold twist acts non-trivially on the first complex plane, the T_1 -anomaly must be universal to be completely canceled by the Green–Schwarz mechanism.

We have conducted a scan over T -anomaly in \mathbb{Z}_6 -II orbifold models, and confirm that only the T_1 -anomalies are universal (for our conventions for labeling the two-tori see [29,30,36]). However, this does not imply that there are uncanceled T -anomalies in the other tori. Rather, as we shall see in the next section, some T -anomalies are inherited from what we will call the ‘ k -anomaly’, which can be canceled by the Green–Schwarz mechanism.

Discrete anomalies can also be canceled by the Green–Schwarz mechanism, just like in the $U(1)_{\text{anom}}$ case [3,22]. Under discrete transformation, the dilaton S (more precisely the axion) gets shifted according to (22), (25). Note that for the discrete transformation, the shift Λ and θ are constant (cf. [7]), while for the anomalous $U(1)_{\text{anom}}$ the shift $\Lambda(x)$ and $\theta(x)$ are x -dependent. Hence, both forms of the Kähler potential (23) and (26) are invariant under the anomalous discrete transformation. This implies that the term $\Sigma = e^{-aS}$ has a definite charge under the discrete transformation. Then, stringy non-perturbative effects induce terms of the form $\Phi_1 \cdots \Phi_n \cdot e^{-aS}$, where the Φ_i transform under (anomalous) discrete symmetries. These terms transform trivially (although they appear to be forbidden by the discrete symmetry) because the transformation of the fields gets compensated by the dilaton [3,22]. Note that a superpotential term $\Phi_1 \cdots \Phi_n \cdot e^{-aS}$ has to transform trivially both for the anomalous $U(1)_{\text{anom}}$ and anomalous discrete symmetries. Furthermore, anomaly cancellation by the Green–Schwarz mechanism requires that discrete anomalies be universal for different gauge group up to modulo the structure (27). We will examine the universality conditions for discrete anomalies in Section 3.4.4.

3.4. Relations between discrete anomalies

In orbifolds there are certain quantum numbers like k (denoting the twisted sector), p_{sh} (shifted $E_8 \times E_8$ momentum), q_{sh} (shifted $SO(8)$ momentum) and oscillator numbers. From these, one can derive other useful quantum numbers such as the discrete R -charges and modular weights, as defined in Eqs. (35) and (38). It is hence clear that the derived quantum numbers

are related. On the other hand, the discrete R -charges and modular weights represent discrete charges relevant for the string selection rules. Clearly, since the \mathbb{Z}_n charges derive from the same set of quantum numbers, the different \mathbb{Z}_n symmetries entailing different string selection rules cannot be completely independent.

To see what this means, consider the discrete R -charges and the corresponding selection rule. At first sight, one might think that the R_1 , R_2 and R_3 rules in \mathbb{Z}_6 -II orbifolds entail \mathbb{Z}_{36} , \mathbb{Z}_9 and \mathbb{Z}_4 symmetries, respectively. However, it is obvious that, once the k -rule (31) is satisfied, the discrete R symmetries boil down to $\mathbb{Z}_6 \times \mathbb{Z}_3 \times \mathbb{Z}_2$ discrete symmetries. That is, one can factorize this subset of discrete symmetries as

$$\mathbb{Z}_6^k \times [\mathbb{Z}_6 \times \mathbb{Z}_3 \times \mathbb{Z}_2]_R. \tag{43}$$

3.4.1. A k -anomaly

This raises the question whether the \mathbb{Z}_6^k symmetry (which is implied by the selection rule (31)) has an anomaly. To clarify this, define the k -anomalies as

$$A_{\mathbb{Z}_6^k-G-G} = \frac{1}{6} \sum_{\mathbf{r}^{(f)}} k^{(f)} 2\ell(\mathbf{r}^{(f)}), \tag{44}$$

where the sum extends over all non-trivial representations of G . Similarly as for the flavor anomalies, the k -anomaly is only defined modulo twice the smallest non-vanishing Dynkin index ℓ_{\min} appearing in the sum in (44). Condition (15a) implies that, if $A_{\mathbb{Z}_6^k-G-G}$ is not integer, one has a \mathbb{Z}_6^k anomaly.

3.4.2. R - vs. k -anomalies

Now let us evaluate the R_1 anomaly, using the prescription of [38]. One has

$$\begin{aligned} A_G^{R_1} &= -c_2(G) + \sum_{\mathbf{r}^{(f)}} \left(R_1^{(f)} + \frac{1}{2}\right) 2\ell(\mathbf{r}^{(f)}) \\ &= -c_2(G) + \sum_{\mathbf{r}^{(t)}} \left(k^{(t)} v_1 - \underbrace{\text{int}(k^{(t)} v_1)}_{=1} - \Delta N_1^{(t)} + \frac{1}{2}\right) 2\ell(\mathbf{r}^{(t)}) \\ &\quad + \sum_{\mathbf{r}^{(u)}} \left(R_1^{(u)} + \frac{1}{2}\right) 2\ell(\mathbf{r}^{(u)}) \\ &= A_{\mathbb{Z}_6^k-G-G} - c_2(G) - \sum_{\mathbf{r}^{(t)}} (\Delta N_1^{(t)} + \frac{1}{2}) 2\ell(\mathbf{r}^{(t)}) + \sum_{\mathbf{r}^{(u)}} \left(R_1^{(u)} + \frac{1}{2}\right) 2\ell(\mathbf{r}^{(u)}), \end{aligned} \tag{45a}$$

where we have used that $v_1 = 1/6$. The summations $\sum_{\mathbf{r}^{(u)}}$ and $\sum_{\mathbf{r}^{(t)}}$ extend, respectively, over untwisted and twisted representations of the gauge factor G . This calculation shows that $A_G^{R_1}$ and $A_{\mathbb{Z}_6^k-G-G}$ are related. Repeating the calculation for R_2 and R_3 yields

$$\begin{aligned} A_G^{R_2} &= 2A_{\mathbb{Z}_6^k-G-G} - c_2(G) + \sum_{\mathbf{r}^{(t)}} \left(-\Delta N_2^{(t)} + \frac{1}{2} - \text{int}(k^{(t)} v_2)\right) 2\ell(\mathbf{r}^{(t)}) \\ &\quad + \sum_{\mathbf{r}^{(u)}} \left(R_2^{(u)} + \frac{1}{2}\right) 2\ell(\mathbf{r}^{(u)}), \end{aligned} \tag{45b}$$

$$\begin{aligned}
 A_G^{R_3} &= 3A_{\mathbb{Z}_6^k-G-G} - c_2(G) + \sum_{\mathbf{r}^{(t)}} \left(-\Delta N_3^{(t)} + \frac{1}{2} - \text{int}(k^{(t)} v_3) \right) 2\ell(\mathbf{r}^{(t)}) \\
 &\quad + \sum_{\mathbf{r}^{(u)}} \left(R_3^{(u)} + \frac{1}{2} \right) 2\ell(\mathbf{r}^{(u)}).
 \end{aligned}
 \tag{45c}$$

That is, whenever $A_{\mathbb{Z}_6^k-G-G}$ is non-zero, the R_i anomalies can be fractional.

3.4.3. An ‘anomalous space group element’

In this subsection, we put the k - and $\mathbb{Z}_3^{\text{flavor}}$ anomalies into a greater perspective. It turns out that they can be related to the so-called ‘anomalous U(1)’ direction. Denote the corresponding generator by $\mathfrak{t}_{\text{anom}}$.⁷ Obviously, $\mathfrak{t}_{\text{anom}}$ is a function of the input, i.e. shift and Wilson lines,

$$\mathfrak{t}_{\text{anom}} = \mathfrak{t}_{\text{anom}}(V, \{W_\alpha\}).
 \tag{46}$$

This direction is fixed up to rescaling, our conventions are to normalize $\mathfrak{t}_{\text{anom}}$ such that (for $\mathfrak{t}_{\text{anom}} \neq 0$)

$$\sum_i \frac{\mathfrak{t}_{\text{anom}} \cdot p_{\text{sh}}^{(i)}}{\mathfrak{t}_{\text{anom}} \cdot \mathfrak{t}_{\text{anom}}} = 12,
 \tag{47}$$

where the sum extends over all states.⁸ Together with the other properties $U(1)_{\text{anom}}$, this implies

$$\mathfrak{t}_{\text{anom}} = \frac{1}{12} \sum_i p_{\text{sh}}^{(i)}.
 \tag{48}$$

Now perform a Weyl rotation of the input,

$$(V, \{W_\alpha\}) \rightarrow (\Omega V, \{\Omega W_\alpha\})
 \tag{49}$$

with $\Omega \in \mathcal{W}$ and \mathcal{W} denoting the Weyl group. This is nothing but a change of the basis, hence

$$\mathfrak{t}_{\text{anom}} \rightarrow \Omega \mathfrak{t}_{\text{anom}}
 \tag{50}$$

under (49). This fixes $\mathfrak{t}_{\text{anom}}$ to be a linear superposition of V and the W_α with coefficients that are invariant under Weyl transformations. Because we are working on the lattice $\Lambda_{E_8 \times E_8}$, this relation holds only up to lattice vectors, i.e.

$$\mathfrak{t}_{\text{anom}} = k^{\text{anom}} V + \sum_\alpha n_\alpha^{\text{anom}} W_\alpha + \lambda,
 \tag{51}$$

where $\lambda \in \Lambda_{E_8 \times E_8}$ is a lattice vector. This relation between $\mathfrak{t}_{\text{anom}}$ and the orbifold parameters indicates that the presence of an anomalous U(1) can be attributed to a geometrical operation in the six dimensional compactified space. This transformation is then encoded in the space group element $g^{\text{anom}} = (\theta^{k^{\text{anom}}}, n_\alpha^{\text{anom}} e_\alpha)$.

⁷ In heterotic orbifolds, the normalization of $\mathfrak{t}_{\text{anom}}$ is determined, so that the first equality sign in (27) represents a non-trivial condition which can be used to check the consistency of the model.

⁸ This normalization differs from the one used in Section 2.7 above Eq. (27). In heterotic orbifolds, one can use the scalar product of the $E_8 \times E_8$ lattice, which also appears in (47). With this scalar product, $\hat{\mathfrak{t}}_{\text{anom}}$ fulfills $\hat{\mathfrak{t}}_{\text{anom}} \cdot \hat{\mathfrak{t}}_{\text{anom}} = 1/2$.

We would like to comment that one cannot trade k^{anom} for n_α^{anom} (and vice versa) as long as $0 \leq k^{\text{anom}} < N$ and $0 \leq n_\alpha^{\text{anom}} < N_\alpha$ with N_α denoting the order of the Wilson line. That is, the coefficients k^{anom} and n_α^{anom} are fixed mod N and N_α , respectively. Further, if $t_{\text{anom}} \in \Lambda_{E_8 \times E_8}$, one has $k^{\text{anom}} = n_\alpha^{\text{anom}} = 0$, i.e. if k^{anom} or n_α^{anom} are non-zero, one can infer that $t_{\text{anom}} \neq 0$, but the converse is in general not true.

As we shall see in the next section, it turns out that the coefficients k^{anom} and n_α^{anom} are related to the k - and flavor anomalies. We have verified that the decomposition (51) is possible, i.e. that there exist k^{anom} and n_α^{anom} such that $[t_{\text{anom}} - (k^{\text{anom}}V + \sum_\alpha n_\alpha^{\text{anom}}W_\alpha)] \in \Lambda_{E_8 \times E_8}$, for several \mathbb{Z}_N and $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds with and without Wilson lines.

3.4.4. Survey of anomaly relations

As we have seen, not all discrete anomalies are independent in orbifold constructions. Specifically, we found that the k - and R -anomalies are related by (45). Given the decomposition (51), one is tempted to suspect that discrete anomalies are related to and determined by the coefficients k^{anom} and n_α^{anom} . To figure out whether this is so, we have conducted a scan over several thousands of models with various geometries and have calculated the k -, R - and T -duality anomalies. We obtain the following (empirical) relations:

- Relation between the k -anomaly and k^{anom} :

$$A_{\mathbb{Z}_6^k-G-G} = \frac{k^{\text{anom}}}{6} \pmod{1}. \tag{52}$$

In particular, the $A_{\mathbb{Z}_6^k-G-G}$ anomalies are universal. Furthermore, the mixed \mathbb{Z}_6^k -grav-grav anomaly

$$A_{\mathbb{Z}_6^k\text{-grav-grav}} = \sum_m k^{(m)} \cdot \dim \mathbf{R}^{(m)} \tag{53}$$

turns out to be always 0 mod 3, thus consistent with the anomaly constraints (19).

- Relation between $A_{\mathbb{Z}_\alpha^{\text{flavor}}-G-G}$ and n_α^{anom} :

$$A_{\mathbb{Z}_3^{\text{flavor}}-G-G} = \frac{n_3^{\text{anom}}}{3} \pmod{1}, \tag{54}$$

$$A_{\mathbb{Z}_2^{\text{flavor}}-G-G} = \frac{n_2^{\text{anom}}}{2} \pmod{1}. \tag{55}$$

These anomalies turn out to be universal for different gauge groups in the models under consideration.

- Relation between the k - and R_i -anomalies: Only if there is a k -anomaly, the R -anomalies can be fractional. We find that the R_i -anomalies are ‘inherited’ from the k -anomaly, specifically

$$A_G^{R_1} = A_{\mathbb{Z}_6^k-G-G} \pmod{1}, \tag{56a}$$

$$A_G^{R_2} = 2A_{\mathbb{Z}_6^k-G-G} \pmod{1}, \tag{56b}$$

$$A_G^{R_3} = 3A_{\mathbb{Z}_6^k-G-G} \pmod{1}. \tag{56c}$$

- Relation between the k - and T -duality anomalies. Similarly to (56), we have found that the T -duality anomaly is related to the k -anomaly by

$$A_G^{T_1} = 2A_{\mathbb{Z}_6^k-G-G} \pmod{1}, \tag{57a}$$

$$A_G^{T_2} = 4A_{\mathbb{Z}_6^k-G-G} \pmod 1, \quad (57b)$$

$$A_G^{T_3} = 6A_{\mathbb{Z}_6^k-G-G} \pmod 1. \quad (57c)$$

These statements apply also to the models presented in [Appendices C and D](#).

• *Relation between the k -, T -duality and R_i -anomalies.* The previous relations (56) and (57) imply

$$A_G^{T_1} - A_G^{R_1} = A_{\mathbb{Z}_6^k-G-G} \pmod 1, \quad (58a)$$

$$A_G^{T_2} - A_G^{R_2} = 2A_{\mathbb{Z}_6^k-G-G} \pmod 1, \quad (58b)$$

$$A_G^{T_3} - A_G^{R_3} = 3A_{\mathbb{Z}_6^k-G-G} \pmod 1. \quad (58c)$$

To summarize, we have conducted a search for discrete anomalies in heterotic orbifolds. As in previous searches [3,8],⁹ we find that all basic discrete anomalies are universal in the models we studied, and all anomalies can be canceled by the discrete Green–Schwarz mechanism. We identify previously unknown relations between the occurrence of discrete anomalies and the so-called ‘anomalous U(1)’. The anomalous U(1) is in one-to-one correspondence to the ‘anomalous space group element’ g^{anom} , whose gauge embedding is the generator of the ‘anomalous’ U(1). T -duality anomalies can be canceled by two ways: the Green–Schwarz mechanism and T -dependent threshold corrections as said in Section 3.3.3. It is widely believed [41] that T -dependent threshold corrections would be non-universal and there would be no certain relation among T -duality anomalies for T_i , which appear in threshold corrections, e.g. T_2 and T_3 in \mathbb{Z}_6 -II orbifolds. On the other hand, our (empirical) results (57), which have been checked in several thousands of models with different geometries, show that there exist certain relations among T -duality anomalies. That is, T -duality anomalies are related to some basic anomalies that are cancelled only by the Green–Schwarz mechanism. This issue will be studied in more detail elsewhere.

3.5. Breaking of anomalous U(1) and discrete symmetries

As already mentioned, an ‘anomalous’ U(1) implies the existence of a FI term, which needs to be canceled in supersymmetric vacua (as well as in settings with low-energy supersymmetry). That means that certain fields which have negative $U(1)_{\text{anom}}$ charges need to attain vacuum expectation values; hence $U(1)_{\text{anom}}$ is broken in (almost) supersymmetric vacua. In other words, there are no ‘anomalous-looking’ unbroken U(1) factors. The requirement of keeping the D -terms of the other symmetries zero leads typically to a situation in which more than one field attains a VEV and in which the various VEVs are related. Achieving D -flatness translates in the construction of gauge invariant monomials which carry net negative anomalous charge [43,44] (see [30,36,45] for more details).

One may wonder if one could break $U(1)_{\text{anom}}$ by canceling the FI term as usual while leaving the anomalous flavor symmetries intact. We have tried to do this in a large set of models with ‘anomalous’ U(1) (including the models presented in [46]), i.e. we searched for gauge invariant monomials with net negative charge under $U(1)_{\text{anom}}$ whose constituents transform trivially under the anomalous discrete symmetries. In most models it is hard, if not impossible, to find

⁹ Our findings are not completely consistent with the relations presented in [38].

such a monomial. In other words, according to what we find, the requirement of keeping supersymmetry unbroken forces one not only to break the ‘anomalous’ $U(1)$, as is well known, but generically also implies that ‘anomalous’ discrete symmetries get broken (which is somewhat surprising because they do of course not have a D -term). However, in a couple of models we did find a monomial whose constituents transform trivially under some of the anomalous discrete symmetries. In these models, an anomalous \mathbb{Z}_2 subgroup of the original \mathbb{Z}_6^k remains unbroken. We posted the details of the model at a web site [47]. Implications will be studied elsewhere.

4. Conclusions

We have studied various aspects of discrete anomalies. We started by reproducing the well-known anomaly constraints for \mathbb{Z}_N symmetries, taking a different route than usual, namely using the path integral approach. Unlike in the conventional approach, our derivation does not rely on contributions from heavy Majorana fermions; only massless fermions enter the computation. We have used the path integral approach to derive anomaly constraints for non-Abelian discrete symmetries; the constraints are given in Eq. (20).

In the second part of the study, we have explored discrete anomalies in string-derived models, focusing on heterotic orbifolds. We find that discrete anomalies can only occur if there is an ‘anomalous’ $U(1)$. One can then rotate the anomalous symmetries into two basic symmetries, corresponding to the rotational and translational part of the space group selection rules, i.e. the k rule and n_α rules. All other anomalies, such as R_1 -anomalies and T -duality anomalies, derive from these basic anomalies. The coefficients of the basic anomalies are connected to an ‘anomalous space group element’, whose gauge embedding arises from the generator of the ‘anomalous $U(1)$ ’. We find that the basic anomalies are always universal, such that they might be canceled by the same Green–Schwarz mechanism that cancels the $U(1)$ anomaly.

We have also searched for models where the ‘anomalous’ $U(1)$ symmetry can be broken (i.e. the FI term can be canceled) without breaking the ‘anomalous’ discrete symmetries. While it is hard to find a model with these properties, we could find a few examples in which an anomalous \mathbb{Z}_2 symmetry survives. The implication of these anomalous \mathbb{Z}_2 symmetries will be discussed elsewhere.

Of course, discrete and continuous symmetries that are broken by a suppressed VEV, as is the case in the ‘anomalous’ $U(1)$, are known to be a useful tool in model building. Indeed, our results indicate that in string models discrete cousins of ‘anomalous’ $U(1)$ symmetries are frequently present, whereby, according to what we find, cancellation of the FI term triggers symmetry breakdown. Since the FI term is loop suppressed, the vacuum expectation value of the field that breaks the symmetry can be small. The emerging approximate symmetries can play an important role in understanding the observed pattern of fermion masses and mixings.

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Appendix A. Anomalies of discrete non-Abelian D_4 symmetry

In this appendix, we discuss anomalies of the discrete symmetry D_4 . The D_4 symmetry is one of the simplest non-Abelian discrete symmetries.¹⁰

The non-Abelian finite group D_4 has eight elements, which can be written as products of the two generators g and h , i.e.

$$\mathcal{G}_{D_4} = \{\mathbb{1}, g, h, gh, hg, hgh, ghg, ghgh\}. \quad (\text{A.1})$$

D_4 has five irreps: $\mathbf{2}$, $\mathbf{1}_{++}$, $\mathbf{1}_{+-}$, $\mathbf{1}_{-+}$ and $\mathbf{1}_{--}$. The action of g and h on these irreps is

$$\begin{aligned} \mathbf{2}: \quad g &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & h &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \mathbf{1}_{++}: \quad g &= 1, & h &= 1, \\ \mathbf{1}_{+-}: \quad g &= 1, & h &= -1, \\ \mathbf{1}_{-+}: \quad g &= -1, & h &= 1, \\ \mathbf{1}_{--}: \quad g &= -1, & h &= -1. \end{aligned} \quad (\text{A.2})$$

According to our discussion in Section 2.5, all we need to do for D_4 anomalies is to study the anomalies for the group elements g and h (or another combination).

The D_4 flavor symmetry can appear from \mathbb{Z}_6 -II orbifold models [29,37] (and other orbifold models whose compact spaces include the 1D \mathbb{Z}_2 sub-orbifold). In \mathbb{Z}_6 -II orbifold models, the group element g corresponds to $\mathbb{Z}_2^{\text{flavor}}$ or $\mathbb{Z}_2^{\text{flavor}'}$. There are two fixed points on the 1D \mathbb{Z}_2 sub-orbifold. Massless spectra on these two fixed points are degenerate, when there is no Wilson line on the 1D \mathbb{Z}_2 sub-orbifold. Then, these modes correspond to $\mathbf{2}$ and the group element h corresponds to the permutation of these modes. In \mathbb{Z}_6 -II orbifold models, only the doublet $\mathbf{2}$ and the trivial singlet $\mathbf{1}_{++}$ can appear as fundamental modes. In this case, anomalies are constrained. We denote

$$h' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (\text{A.3})$$

Now note that $h = h'g$ for the doublet $\mathbf{2}$ and $\det h' = 1$. Thus, all eight elements of the D_4 group can be written as products of g and h' , and the generator h' does not lead to anomalies. That implies that all of D_4 anomalies originate from $\mathbb{Z}_2^{\text{flavor}}$ anomalies, that is, D_4 anomalies, e.g. anomalies for the permutation h , appear in \mathbb{Z}_6 -II orbifold models only if there are $\mathbb{Z}_2^{\text{flavor}}$ anomalies, i.e. anomalies for the group element g . The situation is the same for D_4 anomalies in heterotic orbifold models with the 1D \mathbb{Z}_2 sub-orbifold such as $\mathbb{Z}_2 \times \mathbb{Z}_M$.

The situation would change if we had heterotic orbifold models including non-trivial singlets of the D_4 flavor symmetry, in $\mathbf{1}_{+-}$ and $\mathbf{1}_{-+}$, because in these representation the determinants of

¹⁰ The D_4 flavor symmetry happens to occur in certain, potentially realistic string models [29,30,36,46], which have been constructed recently within the framework of heterotic orbifolds.

Table 1
Transformation properties of the lepton and Higgs fields in [48]

	D_e	(D_μ, D_τ)	e_R, ν_{eR}	(μ_R, τ_R)	$(\nu_{\mu R}, \nu_{\tau R})$	ϕ_1, ϕ_2	ϕ_3	(χ_1, χ_2)
D_4	$\mathbf{1}_{++}$	$\mathbf{2}$	$\mathbf{1}_{++}$	$\mathbf{2}$		$\mathbf{1}_{++}$	$\mathbf{1}_{+-}$	$\mathbf{2}$
$SU(2)_L$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$		$\mathbf{2}$	$\mathbf{2}$	$\mathbf{1}$

g and h differ. Indeed, in heterotic orbifold models including the 2D \mathbb{Z}_4 sub-orbifold, non-trivial singlets can appear as fundamental modes [37]. However, massless states corresponding to $\mathbf{1}_{+-}$ and $\mathbf{1}_{-+}$ are always degenerate. This can only be changed by introducing a Wilson line, which, however, breaks the D_4 flavor symmetry. Thus, in these models, non-trivial singlets $\mathbf{1}_{+-}$ and $\mathbf{1}_{-+}$ do not contribute to anomalies. Therefore, the situation is the same as \mathbb{Z}_6 -II orbifold models, that is, all of D_4 anomalies originate from $\mathbb{Z}_2^{\text{flavor}}$ anomalies.

Appendix B. Sample calculation of discrete anomalies

In this appendix we present a sample calculation in order to demonstrate how the anomaly constraints can be applied. We will base the calculations on the Grimus–Lavoura model [48], which is not supersymmetric. The lepton and Higgs fields are assigned the transformation properties displayed in Table 1. $\phi_{2,3}$ are extra $SU(2)_L$ doublet Higgs fields and $\chi_{1,2}$ are extra gauge singlet Higgs fields. All quark fields are assumed to be trivial D_4 singlets, i.e. to transform as $\mathbf{1}_{++}$.

Let us now calculate the anomaly coefficients of the mixed anomaly D_4 – $SU(2)_L$ – $SU(2)_L$. According to our discussion in Section 2.5, all we need to do is to study the generators, i.e. the group elements g and h , in order to check whether this model is anomalous or not. As $g^2 = h^2 = \mathbb{1}$, this then amounts to checking the conditions for \mathbb{Z}_2 anomalies. For g and h , only (D_μ, D_τ) contributes to the calculation of the anomaly. Hence we find

$$\mathbb{Z}_2^g\text{-SU(2)}_L\text{-SU(2)}_L: \sum_{(\mathbf{r}^{(f)}, \mathbf{d}^{(f)})} \frac{2 \ln \det g(\mathbf{d}^{(f)})}{2\pi i} \ell(\mathbf{r}^{(f)}) = \frac{1}{2} \pmod{1}, \tag{B.1}$$

$$\mathbb{Z}_2^h\text{-SU(2)}_L\text{-SU(2)}_L: \sum_{(\mathbf{r}^{(f)}, \mathbf{d}^{(f)})} \frac{2 \ln \det h(\mathbf{d}^{(f)})}{2\pi i} \ell(\mathbf{r}^{(f)}) = \frac{1}{2} \pmod{1}. \tag{B.2}$$

Therefore, the symmetry generated by g and, hence, the D_4 symmetry of this model is anomalous.

Repeating the calculation for $U(1)_Y$ yields

$$\mathbb{Z}_2^g\text{-U(1)}_Y\text{-U(1)}_Y: \sum_{(\mathbf{r}^{(f)}, \mathbf{d}^{(f)})} \frac{2 \ln \det g(\mathbf{d}^{(f)})}{2\pi i} \left(\frac{q_Y^{(f)}}{2}\right)^2 = \frac{1}{2} \pmod{1}, \tag{B.3}$$

$$\mathbb{Z}_2^h\text{-U(1)}_Y\text{-U(1)}_Y: \sum_{(\mathbf{r}^{(f)}, \mathbf{d}^{(f)})} \frac{2 \ln \det h(\mathbf{d}^{(f)})}{2\pi i} \left(\frac{q_Y^{(f)}}{2}\right)^2 = \frac{1}{2} \pmod{1}, \tag{B.4}$$

where the summation runs over all non-trivial D_4 representations with non-zero hypercharge. We close by stating that the anomalies do not necessarily invalidate the model. As discussed in

Table 2
Summary of R anomalies in the KRZ model

G	R_1	R_2	R_3
SU(4)	$\frac{13}{3} \bmod 6$	$\frac{5}{3} \bmod 3$	1 mod 2
SU(2) _L	$\frac{13}{3} \bmod 6$	$\frac{8}{3} \bmod 3$	1 mod 2
SU(2) _R	$\frac{13}{3} \bmod 6$	$\frac{5}{3} \bmod 3$	1 mod 2
SO(10)	$\frac{7}{3} \bmod 12$	$\frac{17}{3} \bmod 6$	3 mod 4
SU(2)′	$\frac{7}{3} \bmod 6$	$\frac{2}{3} \bmod 3$	1 mod 2

the conclusions, it just means that the symmetry gets broken by certain fields attaining VEVs, which can be suppressed.

Appendix C. Anomalies in the KRZ model

This appendix summarizes the discrete anomalies in the KRZ model A1 [29].

C.1. R anomalies

We obtain for the R anomalies

$$A_{\text{SU}(4)}^{\vec{R}} = (13/3, 5/3, 1), \quad (\text{C.1a})$$

$$A_{\text{SU}(2)_L}^{\vec{R}} = (13/3, 8/3, 1), \quad (\text{C.1b})$$

$$A_{\text{SU}(2)_R}^{\vec{R}} = (13/3, 5/3, 1). \quad (\text{C.1c})$$

The anomalies are only fixed up to $(6, 3, 2)$. Here, the R_3 anomalies match while the others do not. They satisfy only

$$A_{\text{SU}(4)}^{R_2} = A_{\text{SU}(2)_R}^{R_2} \neq A_{\text{SU}(2)_L}^{R_2} \bmod 3. \quad (\text{C.2})$$

One can repeat the analysis for the non-Abelian subgroups of the second E_8 . This leads again to the result that anomalies are not universal. The R anomalies for the KRZ model are summarized in Table 2.

C.2. Flavor anomalies in the KRZ model

Let us calculate the flavor anomalies in the KRZ model. The \mathbb{Z}_3 symmetry is anomalous, but the G – G – \mathbb{Z}_3 anomalies are universal (see Table 3). Note, however, that there is no gravitational \mathbb{Z}_3 anomaly if one considers the charged fields only. This means that there is an uncharged (modulus) field that contributes to the gravitational anomaly.

C.3. T -duality anomalies

The T -duality anomalies are calculated according to Eq. (38) of [38]; the result is listed in Table 4.

Table 3
Summary of \mathbb{Z}_n anomalies in the KRZ model

G	\mathbb{Z}_2	\mathbb{Z}'_2	\mathbb{Z}_3
SU(4)	0 mod 1	0 mod 1	$\frac{1}{3}$ mod 1
SU(2) _L	0 mod 1	0 mod 1	$\frac{1}{3}$ mod 1
SU(2) _R	0 mod 1	0 mod 1	$\frac{1}{3}$ mod 1
SO(10)	0 mod 2	0 mod 2	$\frac{4}{3}$ mod 2
SU(2)'	0 mod 1	0 mod 1	$\frac{1}{3}$ mod 1

Table 4
Summary of T -duality anomalies in the KRZ model

SU(4)	SU(2) _L	SU(2) _R	SO(10)	SU(2)'
$(\frac{62}{3}, -\frac{14}{3}, -2)$	$(\frac{62}{3}, -\frac{14}{3}, -2)$	$(\frac{62}{3}, -\frac{14}{3}, -2)$	$(\frac{62}{3}, \frac{34}{3}, -6)$	$(\frac{62}{3}, -\frac{14}{3}, -18)$

C.4. Anomalous U(1)

The coefficients of the anomalous U(1) (cf. Eq. (51)) are $(k^{\text{anom}}, n_2^{\text{anom}}, n_3^{\text{anom}}) = (2, 0, 1)$.

Appendix D. Calculation of anomalies in the BHLR model

D.1. R anomalies

Let us now consider the model described in [30]. Let us focus on the non-Abelian subgroups of the first E_8 factor, i.e. SU(3) and SU(2). Start with SU(3). We have 10 **3**-plets and 10 $\bar{\mathbf{3}}$ -plets under SU(3) (quark doublets give rise to two **3**-plets each). By performing the sum (41), one obtains (cf. Table 5)

$$A_{\text{SU}(3)}^{R_i} = (0, 1, 1) \text{ mod}(6, 3, 2). \tag{D.1}$$

Continue with SU(2). We have 30 **2**-plets. By performing the sum (41), one obtains

$$A_{\text{SU}(2)}^{R_i} = (0, 0, 1) \text{ mod}(6, 3, 2). \tag{D.2}$$

While $A_{\text{SU}(3)}^{R_1} = A_{\text{SU}(2)}^{R_1} \text{ mod } 6$ and $A_{\text{SU}(3)}^{R_3} = A_{\text{SU}(2)}^{R_3} \text{ mod } 2$, one finds

$$A_{\text{SU}(3)}^{R_2} \neq A_{\text{SU}(2)}^{R_2} \text{ mod } 3. \tag{D.3}$$

D.2. Anomalies of discrete flavor symmetries

The flavor anomalies (cf. Eq. (40)) in this model are

$$A_{\text{SU}(3)}^{(\mathbb{Z}_2, \mathbb{Z}'_2, \mathbb{Z}_3)} = \left\{ 0, 0, \frac{2}{3} \right\}, \tag{D.4a}$$

$$A_{\text{SU}(2)}^{(\mathbb{Z}_2, \mathbb{Z}'_2, \mathbb{Z}_3)} = \left\{ 0, 0, \frac{2}{3} \right\}. \tag{D.4b}$$

Table 5
Summary of R anomalies in the BHLR model

G	R_1	R_2	R_3
SU(3)	0 mod 6	1 mod 3	1 mod 2
SU(2)	0 mod 6	0 mod 3	1 mod 2
SU(4)	3 mod 6	0 mod 3	1 mod 2
SU(2)′	0 mod 6	1 mod 3	1 mod 2

Table 6
Summary of \mathbb{Z}_n anomalies in the BHLR model

G	\mathbb{Z}_2	\mathbb{Z}_2'	\mathbb{Z}_3
SU(3)	0 mod 1	0 mod 1	$\frac{2}{3}$ mod 1
SU(2)	0 mod 1	0 mod 1	$\frac{2}{3}$ mod 1
SU(4)	0 mod 1	0 mod 1	$\frac{2}{3}$ mod 1
SU(2)′	0 mod 1	0 mod 1	$\frac{2}{3}$ mod 1

Table 7
Summary of T -duality anomalies in the BHLR model.

SU(3)	SU(2)	SU(4)	SU(2)′
(10, 10, −6)	(10, 10, −6)	(10, 10, −6)	(10, 2, −2)

That is, the \mathbb{Z}_3 symmetry has anomalies, but they appear to be universal. This applies also to \mathbb{Z}_3 – G – G anomalies where G denotes a subgroup of the second E_8 (see Table 6). Notice, on the other hand, that the gravitational \mathbb{Z}_3 anomalies seem to vanish.

D.3. T -duality anomalies

The T -duality anomalies are calculated according to Eq. (38) of [38]; the result is listed in Table 7.

D.4. Anomalous $U(1)$

The coefficients of the anomalous $U(1)$ (cf. Eq. (51)) are $(k^{\text{anom}}, n_2^{\text{anom}}, n_3^{\text{anom}}) = (0, 0, 2)$.

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