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ABSTRACT: When a charged particle travels through a material medium Cerenkov radiation is emitted if the velocity of the particle is greater than that of light in the medium. An accompanying effect is radiation emitted when the particle enters or leaves the medium. This is called transition radiation. In this paper the theory of transition radiation is presented in considerable detail. The work constitutes a review of earlier studies but some new results are presented. In particular, the case in which the particle enters the medium obliquely is treated and equations are developed for the situation where the boundaries of the medium are not sharp. These equations are then used to show that the sharp boundary approximation is always valid for relativistic particles since the distance over which the radiation develops is always large compared to the thickness of the boundary.

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1. Introduction

The Cerenkov radiation which can be emitted when a charged particle moves with constant velocity through a material medium occurs because of the acceleration of the bound charges in the medium by the forces exerted by the particle as it passes by. Transition radiation, which was predicted by Ginzburg and Frank[1)] in 1946, arises for similar reasons when a particle crosses a boundary between different media. Much of the later work on transition radiation has been carried out by Garibian, although many other authors have contributed to its theoretical development. Garibian[2)] has written an excellent review of the subject which contains an exhaustive list of references to other work on transition radiation. This paper is strongly recommended to the reader.

The transition radiation is of great interest to high energy physicists since it depends on the energy (γ) of the charged particle rather than its velocity as in the Cerenkov radiation. Furthermore, over an interesting range of energies (γ's in the tens of thousands range) the radiation is approximately proportional to γ. It thus offers a useful method for particle detection which is being widely studied[3)].

In view of the widespread interest and the potential of methods using this phenomenon for experimental purposes, the theory of transition radiation is presented in considerable detail in the following sections so that it may be as widely accessible as possible. The case of arbitrary particle incidence angles on the interface between two media is presented and this is then used to study the radiation from a slab imbedded in another medium. It is shown that the radiation vanishes as the slab thickness goes to zero, due to cancellation of the fields radiated from the two surfaces. The significance of the so-called "formation zone" for transition radiation is also discussed.

Since there has been considerable interest in the question of modification of the theory when the boundaries are not sharp, equations which cover this case are developed in Section 8 and it is shown that due to the size of the formation zone the sharp boundary theory remains applicable.

2. Equations and boundary conditions

We begin by considering the radiation emitted at the interface of two different media (1 and 2) by a charged particle traveling from the left in medium 1 across the boundary into medium 2. The charge and current densities of the particle are

$$\rho(\vec{r},t) = e\,\delta(\vec{r}-\vec{v}t)$$
$$\vec{j}(\vec{r},t) = e\,\vec{v}\,\delta(\vec{r}-\vec{v}t), \tag{1}$$

where δ is the Dirac delta function, $\vec{v}$ is the particle velocity and e is its charge. These densities have been chosen so that the particle crosses the interface point, taken as the origin of the spatial coordinate system, when t is zero.

The Maxwell equations, which govern the process, are:

$$\vec{\nabla}\times\vec{H} = \frac{1}{c}\frac{\partial\vec{D}}{\partial t} + \frac{4\pi}{c}\vec{j} \qquad \vec{\nabla}\cdot\vec{B} = 0$$
$$\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \qquad \nabla\cdot\vec{D} = 4\pi\rho\,. \tag{2}$$

The Gaussian system of units is employed.

The constitutive relations between $\vec{E}$ and $\vec{D}$, and $\vec{B}$ and $\vec{H}$ are expressed in terms of spectral components, so we first make this decomposition:

$$\vec{E}(\vec{r},t) = \int \vec{E}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, d^3k\, d\omega$$
$$\vec{D}(\vec{r},t) = \int \vec{D}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, d^3k\, d\omega$$
$$\vec{B}(\vec{r},t) = \int \vec{B}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, d^3k\, d\omega$$
$$\vec{H}(\vec{r},t) = \int \vec{H}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, d^3k\, d\omega. \tag{3}$$

For convenience we have used the same label for a function and its Fourier transform. All integrals are over the infinite domain.

The constitutive relations are

$$\vec{D}(\vec{k},\omega) = \epsilon(\vec{k},\omega)\,\vec{E}(\vec{k},\omega)$$
$$\vec{B}(k,\omega) = \mu(\vec{k},\omega)\,\vec{H}(\vec{k},\omega), \tag{4}$$

where ϵ and μ represent the dielectric constant and permeability of the medium in question [either (ϵ_1, μ_1) or (ϵ_2, μ_2)].

The spectral decomposition of the charge and current gives

$$\rho(\vec{k},\omega) = \frac{1}{(2\pi)^4}\int \rho(\vec{r},t)e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\, d^3r\, dt = \frac{e}{(2\pi)^3}\,\delta(\omega-\vec{k}\cdot\vec{v})$$
$$\vec{j}(\vec{k},\omega) = \frac{1}{(2\pi)^4}\int \vec{j}(\vec{r},t)\, e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\, d^3r\, dt = \frac{e\,\vec{v}}{(2\pi)^3}\,\delta(\omega-\vec{k}\cdot\vec{v}) \tag{5}$$

so that the Maxwell equations become:

$$i\,\vec{k}\times\vec{H} = -i\frac{\omega}{c}\epsilon\,\vec{E} + \frac{4\pi}{c}\vec{v}\,\rho \qquad i\,\vec{k}\cdot(\mu\,\vec{H}) = 0$$

$$i\,\vec{k}\times\vec{E} = i\frac{\omega}{c}\,\mu\,\vec{H} \qquad i\,\vec{k}\cdot(\epsilon\,\vec{E}) = 4\pi\rho\ . \tag{6}$$

The equation for $\vec{E}$ obtained by combining these equations in the usual fashion is

$$\left(k^2 - \mu\,\epsilon\,\frac{\omega^2}{c^2}\right)\vec{E} = \frac{4\pi i\rho}{\epsilon}\left(-\vec{k} + \epsilon\,\mu\,\frac{\vec{v}\omega}{c^2}\right). \tag{7}$$

Solutions of this equation correspond to a linear combination of the field due to the charge and a field of plane waves which satisfy the homogeneous equations. These must be suitably chosen to satisfy the boundary conditions at the interface. The field due to the charge is

$$\vec{E}_0 = \frac{4\pi i\rho}{\epsilon}\left(\vec{k} - \chi\,\frac{\vec{v}\omega}{c^2}\right)\Delta^{-1}\ , \tag{8}$$

where we have set

$$\chi = \epsilon\,\mu$$

and

$$\Delta = \chi(\omega/c)^2 - k^2\ . \tag{9}$$

The plane waves which make up the solution of the homogeneous equation have a propagation vector which satisfies

$$k^2 = \chi(\omega/c)^2\ . \tag{10}$$

The equation for the magnetic field $\vec{H}$ obtained from equation 6 is

$$\left(\frac{\omega^2}{c^2}\chi - k^2\right)\vec{H} = -\frac{4\pi i\rho}{c}(\vec{k}\times\vec{v})\ . \tag{11}$$

This has a particular solution

$$\vec{H}_0 = -\frac{4\pi i\rho}{c}(\vec{k}\times\vec{v})\ \Delta^{-1} \tag{12}$$

due to the charge as well as plane wave solutions satisfying equation 10.

Let us now consider the boundary conditions to be imposed. We first note that there must be no radiation coming from ∞ in any direction. In addition, at a plane boundary between the two media passing through the origin and perpendicular to the z axis we have the usual electromagnetic continuity conditions across the boundary:

$$\begin{aligned}(D_\perp)_1 &= (D_\perp)_2\\ (E_\parallel)_1 &= (E_\parallel)_2\\ (B_\perp)_1 &= (B_\perp)_2\\ (H_\parallel)_1 &= (H_\parallel)_2\ .\end{aligned} \tag{13}$$

The subscripts $\perp$ and $\parallel$ refer to the components perpendicular and parallel to the boundary. These imply that no surface charges or currents are present.

3. Determination of the electric and magnetic fields

Let us now consider the electric field in medium 1 as inferred from equations 3 and 8. First, the field due to the charge is

$$\vec{E}_0^{(1)}(\vec{r},t) = \frac{ie}{2\pi^2\epsilon_1}\int(\vec{k} - \chi_1\,\vec{v}\,\frac{\omega}{c^2})\,\Delta_1^{-1}\,\delta(\vec{k}\cdot\vec{v} - \omega) \times e^{i(\vec{k}\cdot\vec{r}-\omega t)}\ d^3k\,d\omega\ . \tag{14}$$

We divide $\vec{k}$ into a component $\vec{\kappa}$ lying parallel to the plane of the boundary and a component k_z lying perpendicular to it. Thus

$$d^3k = d^2\kappa \, dk_z . \tag{15}$$

The integration over k_z may be carried out to give

$$\vec{E}_0^{(1)}(\vec{r},t) = \frac{4\pi i\rho}{\epsilon_1} \int (\vec{k} - \chi_1 \vec{v} \frac{\omega}{c^2}) \Delta_1^{-1} e^{i(\vec{\kappa}\cdot\vec{r}_\perp + k_z z - \omega t)} d^2\kappa \frac{d\omega}{v_z} . \tag{16}$$

In this expression k_z has the value

$$k_z = \frac{\omega - \vec{\kappa}\cdot\vec{v}}{v_z} . \tag{17}$$

We thus write

$$\vec{E}_0^{(1)}(\vec{r},t) = \int \vec{\mathcal{E}}_0^{(1)}(\kappa,\omega) \, e^{i(\vec{\kappa}\cdot\vec{r}_\perp + k_z z - \omega t)} d^2\kappa \frac{d\omega}{v_z} \tag{18}$$

where $\vec{\mathcal{E}}_0^{(1)}(\vec{\kappa},\omega)$ is easily obtained from equation 16. For the radiation field, in which

$$\kappa^2 + k_{1z}^2 - \chi_1(\omega/c)^2 = 0 \tag{19}$$

we have a corresponding expression:

$$\vec{E}_r^{(1)}(\vec{r},t) = \int \vec{\mathcal{E}}_r^{(1)}(\vec{\kappa},\omega) \, e^{i(\vec{\kappa}\cdot\vec{r}_\perp + k_{1z} - \omega t)} d^2\kappa \frac{d\omega}{v_z} . \tag{20}$$

This expression defines $\vec{\mathcal{E}}_r^{(1)}(\vec{\kappa},\omega)$, which must yet be determined. Similar equations hold in medium 2 for $\vec{E}_0^{(2)}$ and $\vec{E}_r^{(2)}$, with k_{1z} replaced by k_{2z}. The total field in medium 1 is thus

$$\vec{E}^{(1)}(\vec{r},t) = \int \Big(\vec{\mathcal{E}}_0^{(1)}(\vec{\kappa},\omega) e^{i(\vec{\kappa}\cdot\vec{r}_\perp + k_z z - \omega t)} + \vec{\mathcal{E}}_r^{(1)}(\vec{\kappa},\omega) e^{i(\vec{\kappa}\cdot\vec{r}_\perp + k_{1z} z - \omega t)} \Big) d^2\kappa \frac{d\omega}{v_z} \tag{21}$$

and a similar expression involving $\vec{\mathcal{E}}_0^{(2)}$, $\mathcal{E}_r^{(2)}$ and k_{2z} holds in the second region for $\vec{E}^{(2)}(r,t)$.

The boundary conditions which $\vec{E}$ and $\vec{D}$ must satisfy at $z = 0$, equation 13, yield

$$\begin{aligned} \epsilon_1 \mathcal{E}_{rz}^{(1)} - \epsilon_2 \mathcal{E}_{rz}^{(2)} &= -\epsilon_1 \mathcal{E}_{0z}^{(1)} + \epsilon_2 \mathcal{E}_{0z}^{(2)} \\ \vec{\kappa} \cdot (\vec{\mathcal{E}}_r^{(1)} - \vec{\mathcal{E}}_r^{(2)}) &= -\vec{\kappa} \cdot (\vec{\mathcal{E}}_0^{(1)} - \mathcal{E}_0^{(2)}) . \end{aligned} \tag{22}$$

The second of this pair may be combined with the radiation conditions

$$\begin{aligned} \vec{\kappa} \cdot \mathcal{E}_r^{(1)} + k_{1z} \mathcal{E}_{rz}^{(1)} &= 0 \\ \vec{\kappa} \cdot \mathcal{E}_r^{(2)} + k_{2z} \mathcal{E}_{rz}^{(2)} &= 0 \end{aligned} \tag{23}$$

to give

$$-k_{1z} \mathcal{E}_{rz}^{(1)} + k_{2z} \mathcal{E}_{rz}^{(2)} = -\vec{\kappa} \cdot (\vec{\mathcal{E}}_0^{(1)} - \vec{\mathcal{E}}_0^{(2)}). \tag{24}$$

Equations 22 and 24 may now be solved for the radiation fields, $\mathcal{E}_{rz}^{(1)}$ and $\mathcal{E}_{rz}^{(2)}$. One finds:

$$\mathcal{E}_{rz}^{(1)} = \frac{-ie}{2\pi^2(\epsilon_1 k_{2z} - \epsilon_2 k_{1z})}$$

$$\times \left\{ \frac{(k_{2z}/\omega)\left[k_z(\vec{\kappa}\cdot\vec{v}) - \kappa^2 v_z\right] + (\epsilon_2/\epsilon_1)\left[\kappa^2 - \chi_1\omega(\vec{\kappa}\cdot\vec{v})/c^2\right]}{\chi_1 \ \omega^2/c^2 - k^2} - \frac{(k_{2z}/\omega)\left[k_z(\vec{\kappa}\cdot\vec{v}) - \kappa^2 v_z\right] + \left[\kappa^2 - \chi_1 \omega(\vec{\kappa}\cdot\vec{v})/c^2\right]}{\chi_2 \ \omega^2/c^2 - k^2} \right\} \quad (25)$$

and

$$\mathcal{E}_{rz}^{(2)} = \frac{ie}{2\pi^2(\epsilon_1 k_{2z} - \epsilon_2 k_{1z})}$$

$$\times \left\{ \frac{(k_{1z}/\omega)\left[k_z(\vec{\kappa}\cdot\vec{v}) - \kappa^2 v_z\right] + (\epsilon_1/\epsilon_2)\left[\kappa^2 - \chi_2 \omega(\vec{\kappa}\cdot\vec{v})/c^2\right]}{\chi_2 \ \omega^2/c^2 - k^2} - \frac{(k_{1z}/\omega)\left[k_z(\vec{\kappa}\cdot\vec{v}) - \kappa^2 v_z\right] + \left[\kappa^2 - \chi_2 \omega(\vec{\kappa}\cdot\vec{v})/c^2\right]}{\chi_1 \ \omega^2/c^2 - k^2} \right\}. \quad (26)$$

The reader may note that $\mathcal{E}_{rz}^{(2)}$ may be obtained from $\mathcal{E}_{rz}^{(1)}$ simply by exchanging the labels 1 and 2.

At this point, we may also note that for normal incidence of the particle, E_{rz} fully determines the Poynting vector. This is easily seen by observing that the magnetic analogs of equations 22 and 24 (equations 30 and 32) have zero on the right-hand side because

then $\vec{H}_0 \propto [\vec{k} \times \vec{v}] = [\vec{\kappa} \times \vec{v}]$. Therefore $H_{rz} = 0$, and from the orthogonality of $\vec{k}$, $\vec{E}$, and $\vec{H}$ in the radiation field, $\vec{E}$ is in the plane of $\vec{\kappa}$ and z, and

$$|\vec{E}_r^{(1)}| = \frac{|\vec{\kappa}|}{k_{1z}} E_{rz}^{(1)},$$

with a similar result for medium 2. Thus, since $\epsilon E^2 = \mu H^2$ for plane waves, the magnitude of the Poynting vector, S, in region i is

$$S = (\epsilon_i/\mu_i)^{1/2} (c/4\pi) (\kappa^2/k_{iz}^2) E_{rz}^{(i)2}.$$

Thus, if the reader is not interested in oblique incidence the balance of this section may be ignored.

We now turn to the determination of the magnetic fields $\vec{B}$ and $\vec{H}$. We first write, in analogy with equation 16

$$\vec{H}_0^{(1)}(\vec{r},t) = -\frac{ie}{2\pi^2 c}\int(\vec{k} \times \vec{v}) \, \Delta_1^{-1} \, e^{i(\vec{\kappa}\cdot\vec{r}_\perp + k_z z - \omega t)} \, d^2\kappa \, \frac{d\omega}{v_z}. \quad (27)$$

In this expression k_z is given by equation 17. We then define $\vec{\mathcal{H}}_0^{(1)}$ by

$$\vec{H}_0^{(1)}(\vec{r},t) = \int \vec{\mathcal{H}}_0^{(1)}(\vec{\kappa},\omega) \, e^{i(\vec{\kappa}\cdot\vec{r}_\perp + k_z z - \omega t)} \, d^2\kappa \, \frac{d\omega}{v_z}. \quad (28)$$

The radiation field, which satisfies equation 19, is defined in analogy with equation 20 by

$$\vec{H}_r^{(1)}(r,t) = \int \vec{\mathcal{H}}_r^{(1)}(\vec{\kappa},\omega) \, e^{i(\vec{\kappa}\cdot\vec{r}_\perp + k_{1z} z - \omega t)} \, d^2\kappa \, \frac{d\omega}{v_z}. \quad (29)$$

Similar expressions for $\vec{\mathcal{H}}_0^{(2)}$ and $\vec{\mathcal{H}}_r^{(2)}$ hold in region 2.

The boundary conditions which $\vec{B}$ and $\vec{H}$ must satisfy at $z = 0$, equation 13, lead to

$$\mu_1 \mathcal{H}^{(1)}_{rz} - \mu_2 \mathcal{H}^{(2)}_{rz} = -\mu_1 \mathcal{H}^{(1)}_{0z} + \mu_2 \mathcal{H}^{(2)}_{0z}$$
$$\vec{\kappa}\cdot(\vec{\mathcal{H}}^{(1)}_r - \vec{\mathcal{H}}^{(2)}_r) = -\vec{\kappa}\cdot(\vec{\mathcal{H}}^{(1)}_0 - \vec{\mathcal{H}}^{(2)}_0) \ . \quad (30)$$

Again the radiation conditions analogous to equation 23,

$$\vec{\kappa}\cdot\vec{\mathcal{H}}^{(1)}_r + k_{1z}\mathcal{H}^{(1)}_{rz} = 0$$
$$\vec{\kappa}\cdot\vec{\mathcal{H}}^{(2)}_r + k_{2z}\mathcal{H}^{(2)}_{rz} = 0 \ ,$$

may be combined with the second member of equation 30 to give

$$-k_{1z}\mathcal{H}^{(1)}_{rz} + k_{2z}\mathcal{H}^{(2)}_{rz} = -\vec{\kappa}\cdot(\vec{\mathcal{H}}^{(1)}_0 - \vec{\mathcal{H}}^{(2)}_0) \ . \quad (32)$$

Equations 30 and 32 may be solved for the radiation fields $\mathcal{H}^{(1)}_{rz}$ and $\mathcal{H}^{(2)}_{rz}$. One finds

$$\mathcal{H}^{(1)}_{rz} = \frac{ie(\vec{k}\times\vec{v})_z}{2\pi^2 c(\mu_1 k_{2z} - \mu_2 k_{1z})}\left\{\frac{k_{2z}\mu_1 - k_z\mu_2}{\chi_1(\omega^2/c^2) - k^2} - \frac{(k_{2z} - k_z)\mu_2}{\chi_2(\omega^2/c^2) - k^2}\right\} \quad (33)$$

and

$$\mathcal{H}^{(2)}_{rz} = \frac{-ie(\vec{k}\times\vec{v})_z}{2\pi^2 c(\mu_1 k_{2z} - \mu_2 k_{1z})}\left\{\frac{k_{1z}\mu^2 - k_z\mu_1}{\chi_2(\omega^2/c^2) - k^2} - \frac{(k_{1z} - k_z)\mu_1}{\chi_1(\omega^2/c^2) - k^2}\right\} . \quad (34)$$

As in the case of the electric fields, $\mathcal{H}^{(2)}_{rz}$ may be obtained from $\mathcal{H}^{(1)}_{rz}$ by interchanging the labels 1 and 2.

We must now find the other components of $\vec{\mathcal{E}}$ and $\vec{\mathcal{H}}$. It is convenient for this purpose to introduce the set of unit vectors $\hat{z}$, $\hat{\kappa}$, $\hat{z}\times\hat{\kappa}$ along the $\vec{z}$, $\vec{\kappa}$ directions and the direction mutually perpendicular to them. We thus have for the radiation field in region 1

$$\vec{k} = \vec{\kappa} + k_{1z}\hat{z} \ . \quad (35)$$

The Maxwell equation 6 may thus be written

$$\vec{k}\times\vec{\mathcal{E}}^{(1)}_r = \vec{\kappa}\times\vec{\mathcal{E}}^{(1)}_r + k_{1z}\hat{z}\times\vec{\mathcal{E}}^{(1)}_r = \frac{\omega\mu}{c}\vec{\mathcal{H}}^{(1)}_r . \quad (36)$$

Upon taking the scalar product of this equation with $\hat{z}$ one finds

$$\hat{z}\cdot(\vec{\kappa}\times\vec{\mathcal{E}}^{(1)}_r) = \frac{\omega\mu}{c}\hat{z}\cdot\vec{\mathcal{H}}^{(1)}_r = \frac{\omega\mu}{c}\mathcal{H}^{(1)}_{rz} . \quad (37)$$

Thus the component of $\vec{\mathcal{E}}^{(1)}_r$ along the unit vector $\hat{z}\times\hat{\kappa}$ is

$$(\hat{z}\times\hat{\kappa})\cdot\vec{\mathcal{E}}^{(1)}_r = \frac{\omega\mu_1}{c\kappa}\mathcal{H}^{(1)}_{rz} , \quad (38)$$

where κ is the magnitude of $\vec{\kappa}$. The component parallel to $\hat{\kappa}$ is given by equation 23 and is

$$\hat{\kappa}\cdot\vec{\mathcal{E}}^{(1)}_r = -\frac{k_{1z}}{\kappa}\mathcal{E}^{(1)}_{rz} . \quad (39)$$

One thus finds

$$\vec{\mathcal{E}}^{(1)}_r = \mathcal{E}^{(1)}_{rz}\hat{z} - \frac{k_{1z}}{\kappa}\mathcal{E}^{(1)}_{rz}\hat{\kappa} + \frac{\omega\mu_1}{c\kappa}\mathcal{H}^{(1)}_{rz}(\hat{z}\times\hat{\kappa}). \quad (40)$$

The remaining fields may be obtained in an identical manner. One finds

$$\vec{\mathcal{E}}_r^{(2)} = \mathcal{E}_{rz}^{(2)}\hat{z} - \frac{k_{2z}}{\kappa}\mathcal{E}_{rz}^{(2)}\hat{\kappa} + \frac{\omega\mu_2}{c\kappa}\mathcal{H}_{rz}^{(2)}(\hat{z}\times\hat{\kappa}) . \qquad (41)$$

The magnetic fields may be obtained similarly by using equation 6 for medium 1, i.e.,

$$\vec{\mathcal{H}}^{(1)} = \frac{c}{\omega\mu}\vec{k}\times\vec{\mathcal{E}}^{(1)} .$$

One thus finds for the magnetic fields,

$$\vec{\mathcal{H}}_r^{(1)} = \mathcal{H}_{rz}^{(1)}\hat{z} - \frac{k_{1z}}{\kappa}\mathcal{H}_{rz}^{(1)}\hat{\kappa} - \frac{\omega\epsilon_1}{c\kappa}\mathcal{E}_{rz}^{(1)}(\hat{z}\times\hat{\kappa})$$

$$\vec{\mathcal{H}}_r^{(2)} = \mathcal{H}_{rz}^{(2)}\hat{z} - \frac{k_{2z}}{\kappa}\mathcal{H}_{rz}^{(2)}\hat{\kappa} - \frac{\omega\epsilon_2}{c\kappa}\mathcal{E}_{rz}^{(2)}(\hat{z}\times\hat{\kappa}). \qquad (42)$$

This completes the determination of the electric and magnetic fields.

4. Wave zone fields for a particle normal to media interface

In order to obtain the energy radiated in the transition radiation process, it is necessary to integrate the results of the previous section using equations 18 and 28 when $r, t \to \infty$. The general, oblique-incidence case will be dealt with in Appendix A, while in this section we review in detail the case of normal incidence similar to that treated by Garibian[4] except that we choose to evaluate the radiation in medium 2. Thus it is necessary to evaluate the fields

$$\vec{E}_r^{(2)}(\vec{r},t) = \int \vec{\mathcal{E}}_r^{(2)}(\vec{\kappa},\omega)\, e^{i(\vec{\kappa}\cdot\vec{r}+k_{2z}z-\omega t)}\, d^2\kappa\, \frac{d\omega}{v_z} \qquad (43)$$

and

$$\vec{H}_r^{(2)}(\vec{r},t) = \int \vec{\mathcal{H}}_r^{(2)}(\vec{\kappa},\omega)\, e^{i(\vec{\kappa}\cdot\vec{r}+k_{2z}z-\omega t)}\, d^2\kappa\, \frac{d\omega}{v_z} \qquad (44)$$

for large values of $|\vec{r}|$. This analysis will be carried out in the conventional way using the method of steepest descent.

It is convenient to introduce a rectangular coordinate system specified by unit vectors $\hat{x}, \hat{y}, \hat{z}$, with its origin at the interface of the two media. The vectors $\vec{k}$, $\vec{r}$ and their projections on the $\hat{x}, \hat{y}$ plane are illustrated in fig. 1. In this case one has cylindrical symmetry so there is no loss of generality in choosing the observation vector $\vec{r}$ in the $\hat{x}, \hat{z}$ plane with $\vec{\rho}$ its projection along the $\hat{x}$ axis. Following Garibian we write

$$\vec{\rho} = r\sin\theta\,\hat{x}$$
$$\vec{z} = r\cos\theta\,\hat{z} \qquad (45)$$

and we take the angle between $\vec{\rho}$ and $\vec{\kappa}$ to be Φ.

Consider now the components of the electric field vector $\vec{\mathcal{E}}_r^{(2)}$. These are

$$\mathcal{E}_{rx}^{(2)} = -\frac{k_{2z}}{\kappa}\mathcal{E}_{rz}^{(2)}\cos\Phi$$

$$\mathcal{E}_{ry}^{(2)} = -\frac{k_{2z}}{\kappa}\mathcal{E}_{rz}^{(2)}\sin\Phi \qquad (46)$$

$$\mathcal{E}_{rz}^{(2)} = \frac{ie\,\kappa^2}{2\pi^2(\epsilon_1 k_{2z} - \epsilon_2 k_{1z})}\left[\frac{(\epsilon_1/\epsilon_2) - k_{1z}v_z/\omega}{\chi_2(\omega_2/c^2) - k^2} - \frac{1 - k_{1z}v_z/\omega}{\chi_1(\omega^2/c^2) - k^2}\right] .$$

These are to be substituted into equation 43. The integral over κ may be written with surface element: $\kappa\, d\kappa\, d\phi$, so since the exponential is an even function of ϕ only the terms involving $\cos\phi$ make any contribution. Thus $E^{(2)}_{ry}$ vanishes and

$$E^{(2)}_{rx} = -\int \frac{k_{2z}}{v_z} \mathcal{E}^{(2)}_{rz} \cos\phi \; e^{i(\kappa\rho\cos\phi + k_{2z}z - \omega t)} \, d\kappa\, d\phi\, d\omega \qquad (47)$$

$$E^{(2)}_{rz} = \int \frac{\mathcal{E}^{(2)}_{rz}}{v_z} e^{i(\kappa\rho \operatorname{cox}\phi + k_{2z}z - \omega t)} \, \kappa\, d\kappa\, d\phi\, d\omega \,. \qquad (48)$$

The electric field thus lies in the plane including the vector to the point of observation and the velocity. As we shall soon see it is, of course, perpendicular to the direction of observation.

The integrals over the azimuthal angle ϕ may now be carried out using the integral representation for the Bessel function,[5] J_n,

$$J_n(z) = \frac{i^{-n}}{\pi} \int_0^{\pi} e^{iz\cos\phi} \cos n\phi \, d\phi \,. \qquad (49)$$

We thus find

$$E^{(2)}_{rx} = -2\pi i \int \frac{k_{2z}}{v_z} \mathcal{E}^{(2)}_{rz} \, J_1(\kappa\rho)\, e^{i(k_{2z}z-\omega t)} \, d\kappa\, d\omega \,, \qquad (50)$$

$$E^{(2)}_{rz} = 2\pi \int \frac{\mathcal{E}^{(2)}_{rz}}{v_z} J_0(\kappa\rho)\, e^{i(k_{2z}z-\omega t)} \, \kappa\, d\kappa\, d\omega \,. \qquad (51)$$

In order to find the wave zone fields we must evaluate these equations for very large values of r. This may be done by using the asymptotic formulae for the Bessel functions[6]:

$$J_n(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cos\left(z - n\pi/2 - \pi/4\right) . \qquad (52)$$

We thus have

$$E^{(2)}_{rx} = -\left(\frac{8\pi}{\rho}\right)^{\frac{1}{2}} i \int_0^{\infty} \frac{k_{2z}}{v_z} \mathcal{E}^{(2)}_{rz} \, \kappa^{-\frac{1}{2}} \cos(\kappa\rho - 3\pi/4) e^{i(k_{2z}z-\omega t)} \, d\kappa\, d\omega \qquad (53)$$

$$E^{(2)}_{rz} = \left(\frac{8\pi}{\rho}\right)^{\frac{1}{2}} \int_0^{\infty} \frac{\mathcal{E}^{(2)}_{rz}}{v_z} \kappa^{\frac{1}{2}} \cos(\kappa\rho - \pi/4) e^{i(k_{2z}z-\omega t)} \, d\kappa\, d\omega \,. \qquad (54)$$

Following Garibian, we now express the cosines in the integrand in terms of exponentials so that for $E^{(2)}_{rx}$ we have

$$E^{(2)}_{rx} = -\left(\frac{2\pi}{\rho}\right)^{\frac{1}{2}} i \int_0^{\infty} \frac{k_{2z}}{v_z} \mathcal{E}^{(2)}_{rz} \kappa^{-\frac{1}{2}} \times \left[e^{i(\kappa\rho - 3\pi/4)} + e^{-i(k\rho - 3\pi/4)} \right] e^{i(k_{2z}z-\omega t)} \, d\kappa\, d\omega \,. \qquad (55)$$

As one sees from equations 19, 26, 53, and 54, our integrand contains both branch cuts and poles so we now turn to a discussion of how these are to be treated. The branch points are determined by

$$\kappa^2 = \chi_1 \, \omega^2/c^2 \,, \qquad \kappa^2 = \chi_2 \, \omega^2/c^2 , \qquad (56)$$

and

$$\kappa = 0,$$

while the poles are given by

$$\kappa^2 = \frac{\omega^2}{v^2}\left(\chi_2 \frac{v^2}{c^2} - 1\right),$$

or (57)

$$\kappa^2 = \frac{\omega^2}{v^2}\left(\frac{v^2}{c_0^2} - 1\right),$$

where c_0 is the speed of light in the medium,

$$c_0^2 = c^2/\chi_2 \ . \tag{58}$$

Consider now the pole contributions only and imagine that we are to carry out the integration over ω first. We are, of course, interested in the retarded fields; i.e., for $t \to -\infty$ the field must vanish. The pole contributions to the integrand given by equation 57 are at either real or imaginary values of ω according to whether the speed of the particle exceeds or is less than the speed of light in the medium. In the latter case, one easily finds that the field is exponentially small as $|t| \to \infty$. In the former case, however, as is well known, if $t < 0$ the field will vanish if the contour is taken above the poles and this, then, is the desired integration path for the retarded solution. In the latter case we have

$$\omega = \pm i \kappa v_z (1 - v_z^2/c_0^2)^{-\frac{1}{2}}$$

or (59)

$$\kappa = \mp i \frac{\omega}{v_z} (1 - v_z^2/c_0^2)^{\frac{1}{2}},$$

while in the former case

$$\kappa = \pm \frac{\omega}{v_z} (v_z^2/c_0^2 - 1)^{\frac{1}{2}} . \tag{60}$$

With this requirement established, let us now return to the evaluation of the integral over κ in equation 55. As we have just seen, an off-the-real-axis pole does not contribute to the radiation. On the other hand, a pole on the positive real κ axis does contribute and for positive κ one readily finds that a contour in ω which goes above the pole translates into a contour in κ which goes below the pole. This term gives the Cerenkov radiation.

We now turn to the question of the branch points encountered in the integrand of equation 55. First, if the integral over κ is to converge, the value of k_{2z} on the real axis when κ is large must be given by

$$k_{2z} = i(\kappa^2 - \chi_2 \omega^2/c^2)^{\frac{1}{2}} . \tag{61}$$

Cuts associated with the branch points of this function at $\kappa = \pm \chi_2^{\frac{1}{2}} \omega/c$ may be taken along the real axis to $\pm \infty$.

Branch points and cuts associated with k_{1z} may be dealt with in a similar manner but the reader must be careful to note that since k_{1z} represents a reflected wave its value between the branch points, $\pm \chi_1^{\frac{1}{2}} \omega/c$, is negative, in contrast to k_{2z} which is positive between the branch points. Unfortunately, the arguments used for the pole and the branch point in k_{2z} do not apply to the branch point in k_{1z}. We have already deduced that the path of integration in κ around the pole and the branch point from k_{2z} must go below the singularities.

To deduce the requirement for the k_{1z} branch point, we introduce the physical argument that, because of radiation damping effects in the medium, χ will in fact be complex rather than real, so that the singularities are slightly displaced from the real axis. When it is noted that $\mathrm{Im}(\omega\chi) > 0$, the previous results are obtained, and in addition one finds that the contour desired goes below the k_{1z} branch point as well. Except for the effect of defining the path, the imaginary part of χ is negligibly small in the region of interest, so we will assume χ is real from this point on. Finally a cut may be placed along the negative real κ axis to deal with the branch point associated with $\kappa^{\frac{1}{2}}$.

An integral of the form encountered in equation 55 may now be evaluated by deforming our integration path to the path of steepest descent. In what follows the poles corresponding to the Cerenkov radiation will be ignored and only the transition radiation will be treated.

The functions in the exponential of the two terms in the integrand of equation 55 are

$$f(\kappa) = ir(\kappa \sin\theta + k_{2z}\cos\theta)$$

$$g(\kappa) = ir(-\kappa \sin\theta + k_{2z}\cos\theta) \ . \tag{63}$$

Consider now the integration of each term separately. Saddle points for these terms are determined by

$$\frac{df}{d\kappa} = ir(\sin\theta - \frac{\kappa}{k_{2z}}\cos\theta)$$

and (64)

$$\frac{dg}{d\kappa} = ir(-\sin\theta - \frac{\kappa}{k_{2z}}\cos\theta) \ .$$

In the first case

$$\kappa = k_{2z}\tan\theta \tag{65}$$

so that κ and k_{2z} have the same sign, while in the second

$$\kappa = -k_{2z}\tan\theta \ . \tag{66}$$

In either case it is easy to see that

$$k_{2z} = \chi_2^{\frac{1}{2}}\ \frac{\omega}{c}\cos\theta \ . \tag{67}$$

When equation 65 holds, the saddle point is on the positive κ axis at

$$\kappa = +\chi_2^{\frac{1}{2}}\ \frac{\omega}{c}\ \sin\theta \ , \tag{69}$$

while if equation 66 holds it lies on the negative κ axis at

$$\kappa = -\chi_2^{\frac{1}{2}}\ \frac{\omega}{c}\ \sin\theta \ . \tag{70}$$

The values of the functions f and g at their respective saddle points are equal and are

$$f = g = ir\ \chi_2^{\frac{1}{2}}\ \omega/c \ . \tag{71}$$

This implies that in the ω integration only the retarded time, τ ,

$$\tau = t - \chi_2^{\frac{1}{2}}\ r/c \ , \tag{72}$$

occurs. Note that the values of the second derivatives at either saddle point are

$$f'' = g'' = -irc/(X_2^{\frac{1}{2}} \omega \cos^3 \theta) . \tag{73}$$

The paths of steepest descent through the saddle points thus make an angle of $-\pi/4$ with the real κ axis.

Let us now concentrate on the integral involving f. Note that the value of f at the origin is pure imaginary

$$f(0) = + ir X_2^{\frac{1}{2}} (\omega/c) \cos \theta , \tag{74}$$

and that in the neighborhood of the origin

$$f(\kappa) = ir \left[X_2^{\frac{1}{2}}(\omega/c)\cos \theta + \kappa \sin \theta - \kappa^2 \cos \theta/(2X_2^{\frac{1}{2}} \omega/c)\right] . \tag{75}$$

A path of constant phase (steepest descent) is thus given by

$$\mathrm{Re}\left[\kappa \sin \theta - (\kappa^2 c \cos \theta/2 X_2^{\frac{1}{2}} \omega)\right] = 0$$

or

$$(\kappa_R - \tan \theta X_2 \omega/c)^2 - \kappa_I^2 = \tan^2 \theta X_2 \omega^2/c^2 \tag{76}$$

where κ_R and κ_I are the real and imaginary parts of κ.

The branch of this hyperbola which passes through the origin lies on a path of steepest descent which goes to ∞ along the upper side of the left-branch cut. See fig. 2. To obtain a path which finally ends at $+\infty$ under the right-hand cut a possible path of integration is along OABS. The contribution $E_{rx}^{(2)}$ from $OA(E_{OA})$ (since only small values of κ contribute, we use only the small κ approximation for the integrand) is proportional to the integral

$$E_{OA} \sim \frac{1}{(r \sin \theta)^{\frac{1}{2}}} \int_0^{i\infty} \kappa^{3/2} e^{i\kappa r \sin \theta} d\kappa . \tag{77}$$

It thus falls off as r^{-3} and makes no contribution to the wave zone fields. Next, the contribution of the path AB is also small since the real part of the integrand is small and slowly varying while the phase is rapidly varying. Let us now examine the main contribution from the path; i.e., the part through the saddle point (E_{BS}). Near the saddle point we have approximately

$$E_{BS} \sim i \left(\frac{2\pi}{\rho}\right)^{\frac{1}{2}} \int \frac{k_{2z} \mathcal{E}_{rz}^{(2)}}{v_z} \kappa^{-\frac{1}{2}} e^{-at^2} dt\, e^{-i\omega(t-X_2^{\frac{1}{2}} r/c)} dt\, d\omega, \tag{78}$$

where

$$a = \frac{r}{2 X_2^{\frac{1}{2}}(\omega \cos^2 \theta)/c} , \tag{79}$$

and all slowly varying quantities can be evaluated at κ_0. We have also written

$$\kappa = t e^{-i \pi/4} . \tag{80}$$

Upon integrating over the saddle point from $-\infty$ to ∞ we find

$$E_{rx}^{(2)} = - \frac{e \sin \theta \cos^2 \theta}{\pi r v_z} \int \xi(\omega) e^{-i\omega(t-X_2^{\frac{1}{2}} r/c)} d\omega \tag{81}$$

where

$$\xi(\omega) = \frac{(\omega/c)^3 X_2^{3/2}}{(\epsilon_1 k_{2z} - \epsilon_2 k_{1z})} \left[\frac{(\epsilon_1/\epsilon_2) - k_{1z} v_z/\omega}{X_2(\omega^2/c^2) - \kappa^2 - \omega^2/v_z^2} - \frac{1 - k_{1z} v_z/\omega}{X_1(\omega^2/c^2) - \kappa^2 - \omega^2/v_z^2} \right] \tag{82}$$

and

$$k_{2z} = + \chi_2^{\frac{1}{2}} (\omega/c) \cos \theta$$

$$k_{1z} = -(\omega/c)(\chi_1 - \chi_2 \sin^2 \theta)^{\frac{1}{2}} . \qquad (83)$$

This result is in complete agreement* with the case considered in Garibian's paper[4] (equation 27) in which the particle passes from

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* The reader will note that we have corrected a misprint since $\cos \theta$ should be squared in Garibian's equation 27.

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medium to vacuum and when

$$\epsilon_1 = \epsilon , \qquad \epsilon_2 = \mu_2 = \mu_1 = 1 . \qquad (84)$$

The result is (not including the Cerenkov term),

$$E_{rx}^{(2)} = \frac{e\,\beta^2 \sin \theta \cos^2 \theta}{\pi r\, v_z} \int e^{-i\omega(t-r/c)} \left[\frac{1}{\epsilon \cos \theta + (\epsilon - \sin^2 \theta)^{\frac{1}{2}}} \right]$$

$$\times \left[\frac{\epsilon + \beta(\epsilon - \sin^2 \theta)^{\frac{1}{2}}}{1 - \beta^2 \cos^2 \theta} - \frac{1}{1 - \beta(\epsilon - \sin^2 \theta)^{\frac{1}{2}}} \right] \qquad (85)$$

where $\beta \equiv v_z/c$.

We now turn to the evaluation of the integral in equation 55, which involves the function $g(\kappa)$. In this case, as we have seen, the saddle point is on the negative κ axis. Reasoning similar to that in the foregoing analysis leads to the picture of possible contours of integration given in fig. 3.

In this case the path of steepest descent connecting the origin to ∞ follows along the branch of the hyperbola, through the origin, which lies along the bottom of the cut in the right-half plane. It should not be deformed to pass through the saddle point. Arguments similar to those given in the discussion of the contribution of the path OA to the former integration involving $f(\kappa)$ show that, as in equation 77, such contributions fall off as r^{-3} and do not contribute to the wave zone field. The entire radiation field is thus given by equation 81 since the g term does not contribute.

The expression for $E_{rz}^{(2)}$, equation 54, may be treated in a similar manner but the calculation is unnecessary since we only need to know the magnitude of the radiation field $E_r^{(2)}$. This is related to $E_{rx}^{(2)}$ by

$$E_{rx}^{(2)} = E_r^{(2)} \cos \theta . \qquad (86)$$

Thus

$$E_r^{(2)} = \frac{e \sin \theta \cos \theta}{\pi\, r\, v_z} \int \xi(\omega)\, e^{-i\omega(t-r/c)}\, d\omega . \qquad (87)$$

The magnitude of the magnetic field may also be evaluated directly but it is easily obtained from the fact that for a radiation field $\epsilon E^2 = \mu H^2$. Electric and magnetic fields in the wave zone are, of course, perpendicular to each other and to $\vec{k}$.

In the case when the particle passes from vacuum into the medium, equation 82 yields

$$\xi(\omega) = \frac{-1}{\cos\theta + \epsilon(1 - \epsilon\sin^2\theta)^{\frac{1}{2}}} \left[\frac{1 + \beta\epsilon(1 - \epsilon\sin^2\theta)^{\frac{1}{2}}}{1 - \beta^2\epsilon\cos^2\theta} - \frac{\epsilon + \beta\epsilon(1 - \epsilon\sin^2\theta)^{\frac{1}{2}}}{1 - \beta^2(1 - \epsilon\sin^2\theta)} \right]. \qquad (88)$$

This may be used to obtain the radiation field from the first surface.

5. The radiated energy

The total radiated energy is given by the time integral of the Poynting vector, $(c/4\pi)(\vec{E}^{(2)} \times \vec{H}^{(2)})$. We thus have for the energy radiated into a given solid angle

$$\frac{dW}{d\Omega} = \frac{c\, e^2 \sin^2\theta \cos^2\theta}{\pi^2 v_z^2} (\epsilon^2/\mu_2)^{\frac{1}{2}} \int_0^\infty |\xi(\omega)|^2\, d\omega . \qquad (89)$$

For the case in which the particle passes from vacuum into the medium, for which $\mu_2 = 1$, we have

$$\epsilon_1 = \mu_1 = \mu_2 = 1, \qquad \epsilon_2 = \epsilon \qquad (90)$$

we find

$$|\xi(\omega)|^2 = \beta^4 \left| \frac{(\epsilon - 1)\left[1 - \beta^2 - \beta^3\epsilon(1 - \epsilon\sin^2\theta)^{\frac{1}{2}} - \beta^2\epsilon\cos^2\theta\right]}{\left\{\cos\theta + \left[\epsilon(1 - \epsilon\sin^2\theta)\right]^{\frac{1}{2}}\right\}(1 - \beta^2\epsilon\cos^2\theta)(1 - \beta^2 + \epsilon\beta^2\sin^2\theta)} \right|^2 . \qquad (91)$$

The reader will note that in equation 89 the contribution from negative ω has been combined with that from positive ω so that the integration goes from zero to infinity.

When the particle passes from medium to vacuum we find the result obtained by Garibian[4]:

$$|\xi(\omega)|^2 = \frac{\beta^4}{(1 - \beta^2\cos^2\theta)} \left| \frac{(\epsilon - 1)\left[1 - \beta^2 - \beta(\epsilon - \sin^2\theta)^{\frac{1}{2}}\right]}{\left[\epsilon\cos\theta + (\epsilon - \sin^2\theta)^{\frac{1}{2}}\right]\left[1 - \beta(\epsilon - \sin^2\theta)^{\frac{1}{2}}\right]} \right|^2 . \qquad (92)$$

6. The formation zone for transition radiation

In this section we consider the fields produced when a particle passes through a strip of dielectric of thickness d , from vacuum to dielectric and back to vacuum. It will be assumed that the dielectric constant differs but little from 1. We write

$$\epsilon - 1 = \delta . \qquad (93)$$

Only the leading terms in δ in the expressions for ξ in terms of δ will be retained. We find

$$\xi = \frac{\pm\beta^2\delta}{2\cos\theta} \left[\frac{1 - \beta\cos\theta - \beta^2}{(1 - \beta^2\cos^2\theta)(1 - \beta\cos\theta)} \right]. \qquad (94)$$

where the upper sign is to be chosen for the forward field produced in the dielectric at the vacuum-dielectric interface and the lower for the forward field produced in the vacuum at the second, dielectric-vacuum interface.

The field produced in the dielectric will be subject to many internal reflections and transmissions all of which are determined by the Fresnel formulae[7]. A backward field will also be produced at the second interface and this will be subject to the same phenomena. We know, however, that when the dielectric thickness tends to zero the total field must be zero. Equation 94 is the precise expression of this fact in the limit in which the secondary fields are neglected. The opposite signs of ξ given by equation 94 give the required cancellation.

Let us now turn to the case of finite thickness, d, in the limit where δ is very small so that we may neglect the secondary fields described above. Equation 47 describes the radiation field and we shall base our discussion on it. As has been shown the principal contribution to the integrand comes from the saddle point, where $\mathcal{E}_{rz}^{(2)}$ is proportional to ξ. We may thus write for the value of the field produced at the first surface,

$$E_{rx}^{(2)} = -\int \left(\frac{k_{2z}}{v_z} \mathcal{E}_{r2}^{(2)} \right) \cos \phi \; e^{i(\vec{x}\cdot\vec{\rho}+k_{2z}z-\omega t)} \, d\kappa \, d\phi \, d\omega \tag{95}$$

where the subscript denotes evaluation at the saddle point.

The field produced at the second boundary is similar to this except that the sign of $\mathcal{E}_{rz}^{(2)}$ is changed and the phase of the exponential must be corrected, if our formulas are to be applicable, since the formulas have been developed assuming that the surface of discontinuity is at $z = 0$, and the particle passes through it at $t = 0$, so that

$$t \rightarrow t - \frac{d}{v_z} \tag{96}$$

$$z \rightarrow z - d \, .$$

We thus find that the total field, $E_{rx}^{(2)}$, evaluated at the second interface $(z = d)$ is proportional to

$$E_{rx}^{(2)} \sim \left[\xi \, e^{i(\vec{\kappa}\cdot\vec{\rho}+k_{2z}d-\omega t)} - \xi \, e^{i(\vec{\kappa}\cdot\vec{\rho}+\frac{\omega d}{v}-\omega t)} \right] . \tag{97}$$

The condition that the two fields will reinforce each other is that

$$\left| \sin \frac{k\, d_F}{2} \left[\frac{k_{2z}}{k} - \frac{1}{\beta} \right] \right| = 1 \tag{98}$$

where $k = \omega/c$. We thus find that

$$d_F = \left[\frac{\lambda\beta/2}{1 - \beta\sqrt{\epsilon}\ \cos\theta} \right] . \tag{99}$$

When θ is small and β is close to one this may be written as

$$\gamma \equiv (1 - \beta^2)^{-\frac{1}{2}}$$

$$d_F \cong \lambda(\gamma^{-2} - \delta + \theta^2)^{-1} . \tag{100}$$

The "formation zone" for transition radiation is defined by d_F. As will be seen in Section 9, there is a cutoff frequency in the radiation spectrum where $\delta \cong -1/\gamma^2$. Evidently for high energies d_F will be very large compared to the wavelength under consideration. For this reason, as we shall see in the last section, sharp dielectric boundaries require no special treatment when the wavelength of the radiation being considered is comparable to the region over which the dielectric constant varies rapidly.

7. Transition radiation from multiple layered media

We now treat the case of transition radiation from N dielectric layers of dielectric constant ϵ and thickness d separated by vacuum regions of thickness D. Our aim is to find the modification of $|\xi|^2$ which is to be substituted in equation 89. This new factor will be denoted by $|\Xi|^2$. The same approximations which have been used in the previous section will be used. As an example, we consider the electric field evaluated at a distance 3d + 3D. The six interfaces will contribute fields proportional to the exponential factor in equation 95, where the phases, ϕ, are (see fig. 4)

$$\phi_1 = 3k_{2z}d + 3k_{1z}D ,$$

$$\phi_2 = 2k_{2z}d + 3k_{1z}D + k_z d + \pi ,$$

$$\phi_3 = 2k_{2z}t + 2k_{1z}D + k_z(d + D) ,$$

$$\phi_4 = k_{2z}d + 2k_{1z}D + k_z(2d + D) + \pi ,$$

$$\phi_5 = k_{2z}d + k_{1z}D + k_z(2d + 2D) ,$$

$$\phi_6 = k_{1z}D + k_z(3d + 2D) + \pi . \qquad (101)$$

If we introduce the notation

$$A = i(k_z - k_{2z})d \qquad (102)$$

and

$$B = i(k_z - k_{1z})D \qquad (103)$$

we find

$$|\Xi|^2 = |\xi|^2 \left| \left[1 - e^A + e^{A+B} - e^{2A+B} + e^{2A+2B} - e^{3A+2B} \right] \right|^2 .$$

This may be written as

$$|\Xi|^2 = |\xi|^2 \, | 1 - e^A |^2 \left| 1 + e^{A+B} + e^{2(A+B)} \right|^2 .$$

If N dielectric strips are considered and the field is evaluated at N(d + D) we evidently have

$$|\Xi|^2 = |\xi|^2 \, | 1 - e^A |^2 \left| 1 + e^{A+B} + \cdots + e^{(N-1)(A+B)} \right|^2 . \qquad (104)$$

The first factor $(1 - e^A)$ is just that encountered in the previous section dealing with the formation zone. The second factor may be summed. We thus find

$$|\Xi|^2 = |\xi|^2 \, | 1 - e^A |^2 \left| \frac{1 - e^{N(A+B)}}{1 - e^{A+B}} \right|^2 .$$

If our basic variables are inserted in this expression it becomes

$$|\Xi|^2 = 4|\xi|^2 \sin^2 \frac{(k_z - k_{2z})d}{2} \; \frac{\sin^2 \frac{N}{2}\left[(k_z - k_{2z})d + (k_z - k_{1z})D\right]}{\sin^2 \frac{1}{2}\left[(k_z - k_{2z})d + (k_z - k_{1z})D\right]} . \qquad (105)$$

In the foregoing no account has been taken of absorption of radiation. We may treat this case by writing

$$A + B = i\,a - b , \qquad (106)$$

where

$$a = i\left[(k_z - k_{2z})d + (k_z - k_{1z})D\right] \qquad (107)$$

and

$$b = \alpha\, d. \tag{108}$$

The absorption coefficient α is expected to be a strong function of frequency. If we neglect absorption in the first region and assume the number of dielectric regions, N, to be very large we find that $|\xi|^2$ must be replaced by

$$|\Xi|^2 = 4\,|\xi|^2 \left\{ \sin^2\left[(k_z - k_{2z})\,\frac{d}{2}\right]\right\} e^{-Nb} \left[\frac{\cosh Nb - \cos Na}{\cosh b - \cos a}\right]. \tag{109}$$

This reduces exactly to the value given by equation 105 when absorption is negligible $(b = 0)$.

8. Continuously varying dielectric constant

In the foregoing sections we have employed the idealization of a sharp dielectric boundary. This does not exist in nature so the reader may enquire whether our formulae must be significantly altered when the wavelength of the radiation becomes comparable to the distance over which the dielectric constant varies rapidly. (For high energy applications, the wavelength of important radiation becomes of the order of atomic sizes.) In what follows we will treat the case of normal particle incidence at the dielectric boundary and, as a model, the dielectric constant will be taken to be a continuous function of z. One might question whether the medium can be treated as continuous, or whether one must deal with a collection of electron-atom interactions. If the formation zone is large, as is true even when the important wavelengths become small, it seems reasonable that such a collection of atoms may be described in terms of space-time average properties such as the dielectric constant and the permeability. This belief is

reinforced by the result from our model that the formation zone provides the appropriate scale of length for evaluating the effects of diffuseness of a medium boundary. In our model we assume that the change in the dielectric constant takes place in a distance t which is small compared to the length of the formation zone.

In our model, the permeability μ will be taken to be unity. Thus we write

$$D(\vec{r},\omega) = \epsilon(z,\omega)\, E(\vec{r},\omega)\,. \tag{110}$$

The Maxwell equations, specialized to the case of frequency ω, may then be combined to give differential equations for $\vec{E}$ and $\vec{B}$. For $\vec{E}$ we find:

$$\nabla^2 \vec{E} + \frac{\omega^2}{c^2}\,\epsilon\vec{E} + \vec{\nabla}\left(\frac{\vec{E}\cdot\vec{\nabla}\epsilon}{\epsilon}\right) = \vec{\nabla}\left(\frac{4\pi\rho}{\epsilon}\right) - 4\pi\,\frac{i\omega\rho}{c^2}\,\vec{v}\,. \tag{111}$$

In this equation

$$\rho(\vec{r},\omega) = \frac{e}{2\pi}\,\frac{\delta(x)\delta(y)}{v}\,e^{ik_z z}\,. \tag{112}$$

If we now write

$$E(\vec{r},\omega) = \int E(\vec{\kappa},\omega,z)\,e^{i\vec{\kappa}\cdot\vec{r}_\perp}\,d^2\kappa\,, \tag{113}$$

we obtain the following equations for $\vec{E}_\perp(\vec{\kappa},\omega,z)$ and $E_z(\vec{\kappa},\omega,z)$:

$$\frac{d^2\vec{E}_\perp}{dz^2} + \left(\epsilon\,\frac{\omega^2}{c^2} - \kappa^2\right)\vec{E}_\perp + i\vec{\kappa}\left(\frac{\epsilon'\,E_z}{\epsilon}\right) = 4\pi i\rho_0\,\frac{\vec{\kappa}}{\epsilon}\,e^{ik_z z}\,, \tag{114}$$

and

$$\frac{d^2E_z}{dz^2} + \left(\epsilon\,\frac{\omega^2}{c^2} - \kappa^2\right)E_z + \frac{d}{dz}\left(\frac{\epsilon'\,E_z}{\epsilon}\right) = 4\pi\rho_0\left[\frac{d}{dz}\left(\frac{e^{ik_z z}}{\epsilon}\right) - ik_z\beta^2 e^{ik_z z}\right]. \tag{115}$$

In a similar manner, one may obtain the equation for $\vec{B}$. If the direction of observation is taken to lie in the x,z plane, the only nonvanishing component of the magnetic field is then B_y:

$$\frac{d^2B_y}{dz^2} - \frac{\epsilon'}{\epsilon}\frac{dB_y}{dz} + (\epsilon\frac{\omega^2}{c^2} - \kappa^2)B_y = 4\pi\rho_0\, i\,\beta\,\kappa_x\, e^{ik_z z} . \qquad (116)$$

In the above equations,

$$\epsilon' = \frac{d\epsilon}{dz}, \qquad \rho_0 = \frac{e}{(2\pi)^3\, v}, \qquad \text{and} \qquad \beta = \frac{v}{c} . \qquad (117)$$

In principle, a discussion of the emitted radiation may be based either on equation 115 or on equation 116. However, in what follows we wish to demonstrate the adequacy of the sharp boundary approximation when ϵ is small, as is true in cases of interest, and when the formation zone is large compared to t. In making this approximation we must be sure that the radiation field is divergenceless, i.e.

$$\vec{\kappa}\cdot\vec{E}^{(2)}_{r\perp} + \frac{dE^{(2)}_{rz}}{dz} = 0 . \qquad (118)$$

The reader will note that equation 26 for $E^{(2)}_{rz}$ is proportional to κ^2 so that when $\kappa = 0$, $E^{(2)}_{rz}$ vanishes. Our approximation must also lead to this result since it follows generally that there is no radiation as $\theta \to 0$. We see that if we work with equation 115, care must be taken to satisfy equation 118. On the other hand, if we choose to work with equation 116 for B_y, the divergence condition for the radiation field

$$\vec{\kappa}\cdot\vec{B}_{r\perp} = 0 ,$$

is automatically satisfied. When $\kappa = 0$, the right-hand side of

equation 116 vanishes. For this methodological reason we choose to work with the magnetic field.

We now turn to a discussion of the solution of equation 117. The direction of observation will be chosen in the x, z plane so we will write $\kappa = \kappa_x$ in the following. It will be assumed that ϵ is a continuous function of ϵ which changes from unity in region 1 to a value ϵ in region 2. If we define λ by

$$\lambda^2 = \epsilon\frac{\omega^2}{c^2} - \kappa^2 , \qquad (119)$$

then in region 1, $\lambda = -k_{1z}$, and in region 2, $\lambda = k_{2z}$. Since k_{1z} is in fact $-[(\omega^2/c^2) - \kappa^2]^{\frac{1}{2}}$, λ is positive in both regions and has a small change between the two regions. We then have

$$\frac{d^2B_y}{dz^2} - \frac{\epsilon'}{\epsilon}\frac{dB_y}{dz} + \lambda^2 B_y = 4\pi\rho_0\, i\,\kappa\,\beta\, e^{ik_z z} . \qquad (120)$$

To solve this inhomogeneous equation we first construct a Green's function, $\mathcal{G}$, for which

$$\frac{d^2\mathcal{G}}{dz^2} - \frac{\epsilon'}{\epsilon}\frac{d\mathcal{G}}{dz} + \lambda^2\mathcal{G} = \delta(z - z'). \qquad (121)$$

In this equation δ represents the Dirac delta function, and $\mathcal{G}$ may be constructed in terms of the two solutions χ_1 and χ_2 of the homogeneous part of equation 121. These satisfy

$$\frac{d^2\chi}{dz^2} - \frac{\epsilon'}{\epsilon}\frac{d\chi}{dz} + \lambda^2\chi = 0 . \qquad (122)$$

Because of the asymptotic behavior of χ in regions 1 and 2 we write:

$$X = a(z) \exp\left(i \int_0^z \lambda(z')dz'\right) + b(z)\exp\left(-i \int_0^z \lambda(z')dz'\right). \qquad (123)$$

Furthermore a and b are expected to be approximately constant in the case of interest. In calculating dX/dz $(\equiv X')$ we write

$$X' = i\lambda\left[a \exp\left(i \int_0^z \lambda(z')dz'\right) - b \exp\left(-i\int_0^z \lambda(z')dz'\right)\right], \qquad (124)$$

and thus impose the condition that

$$a' \exp\left(i\int_0^z \lambda(z')dz'\right) + b' \exp\left(-i\int_0^z \lambda(z')dz'\right) = 0 . \qquad (125)$$

We then find the equations for a and b:

$$a' + \frac{1}{2}\left(\frac{\lambda'}{\lambda} - \frac{\epsilon'}{\epsilon}\right)\left[a - b \exp\left(-2i\int_0^z \lambda(z')dz'\right)\right] = 0$$

$$-b' + \frac{1}{2}\left(\frac{\lambda'}{\lambda} - \frac{\epsilon'}{\epsilon}\right)\left[a \exp\left(2i\int_0^z \lambda(z')dz'\right) - b\right] = 0 \qquad (126)$$

where primes denote differentiation with respect to z. The substitutions

$$a = A\left(\frac{\epsilon}{\lambda}\right)^{\frac{1}{2}}, \qquad b = B\left(\frac{\epsilon}{\lambda}\right)^{\frac{1}{2}} \qquad (127)$$

then lead to*

* The well-known W.K.B. approximation follows from these transformations by noting that if $\lambda'/\lambda \ll 1$, a rapidly oscillating exponential makes A, B approximately constant.

$$A' = \frac{1}{2}\left(\frac{\lambda'}{\lambda} - \frac{\epsilon'}{\epsilon}\right) B \exp\left(-2i\int_0^z \lambda(z')dz'\right) \qquad (128)$$

and

$$B' = \frac{1}{2}\left(\frac{\lambda'}{\lambda} - \frac{\epsilon'}{\epsilon}\right) A \exp\left(2i\int_0^z \lambda(z')dz'\right). \qquad (129)$$

Our object is to construct two linearly independent solutions of equation 122 from which we can obtain the Green's function. These may be defined by choosing solutions X_1 and X_2 of the form given by equation 123 with

$$A_1(-\infty) = 1 ; \qquad B_1(-\infty) = 0 , \qquad (130)$$

and

$$A_2(-\infty) = 0 ; \qquad B_2(-\infty) = 1 .$$

We note here that the Wronskian, W, of these two solutions

$$W = (X_1X_2' - X_2X_1') = -2i\epsilon(A_1B_2 - A_2B_1) \qquad (131)$$

has the value

$$W = -2i\epsilon(z) . \qquad (132)$$

We choose to construct a Green's function which then gives a particular solution of equation 120 for B_y in the form

$$B_y(z) = \int_0^z \mathcal{G}(z,z')\, R(z')dz' \qquad (133)$$

where $R(z')$ denotes the right-hand side of equation 120. The Green's function may be written as[8)]

$$\mathcal{G}(z,z') = F(z')\left[X_1(z)X_2(z') - X_2(z)X_1(z')\right], \text{ for } z > z'$$

$$\mathcal{G}(z,z') = 0, \quad \text{for } z \leqslant z'. \tag{134}$$

The function F is determined by the condition:

$$\left.\frac{d\mathcal{G}(z,z')}{dz}\right|_{z=z'} = 1. \tag{135}$$

One finds that for our choice of X's, $F = -W^{-1}$, where W is the Wronskian. Thus $\mathcal{G}$ is given by:

$$\mathcal{G}(z,z') = \frac{1}{2i\,\epsilon(z)}\left[X_1(z)\,X_2(z') - X_2(z)\,X_1(z')\right]. \tag{136}$$

The general solution of equation 120 may now be written as

$$B_y(z) = \left[C_1\,X_1(z) + C_2\,X_2(z)\right] + \int_0^z \mathcal{G}(z,z')\,R(z')dz'. \tag{137}$$

In this expression C_1 and C_2 are constants which must be chosen to fit the boundary conditions.

In what follows it will be convenient to introduce the functions, X_{1A} and X_{2A} which represent the asymptotic behavior of X_1 and X_2 for large positive values of z:

$$X_{1A} \equiv \left(\frac{\epsilon}{k_{2z}}\right)^{\frac{1}{2}}\left[A_1(\infty)\,e^{i(\sigma+k_{2z}z)} + B_1(\infty)\,e^{-i(\sigma+k_{2z}z)}\right]$$

$$X_{2A} \equiv \left(\frac{\epsilon}{k_{2z}}\right)^{\frac{1}{2}}\left[A_2(\infty)\,e^{i(\sigma+k_{2z}z)} + B_2(\infty)\,e^{-i(\sigma+k_{2z}z)}\right] \tag{138}$$

where

$$\sigma \equiv \int_0^t \left[\lambda(z') - k_{2z}\right] dz'. \tag{139}$$

It is also useful to introduce the function $\mathcal{G}_A(z,z')$, defined by

$$\mathcal{G}_A(z,z') = \frac{1}{2i\epsilon}\left[X_1(z)\,X_{2A}(z') - X_2(z)\,X_{1A}(z')\right]. \tag{140}$$

When z becomes very large $(z \rightarrow \infty)$, $\mathcal{G}_A$ has the value

$$\mathcal{G}_A(z,z') = \frac{1}{2ik_{2z}}\left[e^{ik_{2z}(z-z')} - e^{-ik_{2z}(z-z')}\right], \quad z > z'. \tag{141}$$

We now turn to the determination of the constant C_1 in equation 137. Consider the case when $z \rightarrow -\infty$. Since ϵ is constant for $z < 0$, and then $\lambda = -k_{1z}$, one easily finds that

$$B_y(z) = \left[\frac{C_1}{(k_{1z})^{\frac{1}{2}}}\,e^{-ik_{1z}z} + \frac{C_2}{(k_{1z})^{\frac{1}{2}}}\,e^{+ik_{1z}z}\right]$$

$$- \frac{1}{2ik_{1z}}\int_0^z \left[e^{-ik_{1z}(z-z')} - e^{+ik_{1z}(z-z')}\right] R(z')dz'. \tag{142}$$

We now define

$$R(z') \equiv r\;e^{ik_z z'} \tag{143}$$

where

$$r = 4\pi\,\rho_0\,i\,\kappa\,\beta. \tag{144}$$

Equation 142 may be integrated to yield

$$B_y(z) = \left[\frac{C_1}{(-k_{1z})^{\frac{1}{2}}} - \frac{r}{2k_{1z}}\left(\frac{1}{k_z + k_{1z}}\right)\right] e^{-ik_{1z}z}$$

$$+ \left[\frac{C_2}{(-k_{1z})^{\frac{1}{2}}} + \frac{r}{2k_{1z}}\left(\frac{1}{k_z - k_{1z}}\right)\right] e^{+ik_{1z}z} - \frac{r\, e^{ik_z z}}{(k_z^2 - k_{1z}^2)} \quad . \tag{145}$$

The terms in $B_y(z)$, in the order in which they appear, represent incoming radiation from $z = -\infty$, outgoing radiation and the field of the particle. Since there is no incoming radiation

$$C_1 + \frac{r}{2(-k_{1z})^{\frac{1}{2}}} \cdot \frac{1}{k_z + k_{1z}} = 0 \ . \tag{146}$$

This determines the constant C_1 .

To determine the constant C_2 it is necessary to consider the field as $z \to \infty$. Since we are interested here in the asymptotic fields it is convenient to write

$$B_y(z) = \left[C_1\, X_1(z) + C_2\, X_2(z)\right] + \int_0^z \mathcal{G}_A(z,a')r \ e^{ik_z z'}\, dz'$$

$$+ \int_0^z \left[\mathcal{G}(z,z') - \mathcal{G}_A(z,z')\right] r\, e^{ik_z z'}\, dz' \ . \tag{147}$$

Between the two integrals, the major term in the asymptotic field is given by the first integral involving $\mathcal{G}_A$, since the difference $\left[\mathcal{G}(z,z') - \mathcal{G}_A(z,z')\right]$ vanishes when z and z' are in the region

$z,z' > t$. This will be shown in detail in what follows, but first let us consider the contributions from $\mathcal{G}_A$ and the free-field terms to $B_y(z)$:

$$B_y^{\,A}(z) = \left[C_1 X_1(z) + C_2 X_2(z)\right] + \int_0^z \mathcal{G}_A(z,z')r \ e^{ik_z z'}\, dz' \ . \tag{148}$$

When z is large we have

$$B_y^{\,A}(z) = \left(\frac{\epsilon}{k_{2z}}\right)^{\frac{1}{2}} \left\{ C_1\left[A_1(\infty)\, e^{i(\sigma + k_{2z}z)} + B_1(\infty)\, e^{-i(\sigma+k_{2z}z)}\right]\right.$$

$$\left. + C_2\left[A_2(\infty)\, e^{i(\sigma+k_{2z}z)} + B_2(\infty)\, e^{-i(\sigma+k_{2z}z)}\right]\right\}$$

$$+ \frac{1}{2ik_{2z}} \int_0^z \left[e^{ik_{2z}(z-z')} - e^{-ik_{2z}(z-z')}\right] r\, e^{ik_z z'}\, dz' \ . \tag{149}$$

In this equation we have written ϵ for the value of the dielectric constant in the second medium. The integral for σ (equation 139) is of the order t/d_F , where d_F is the "formation zone" length and so by assumption is exceedingly small and may be neglected. We thus have

$$B_y^{\,A}(z) \cong \left(\frac{\epsilon}{k_{2z}}\right)^{\frac{1}{2}} \left\{ C_1\left[A_1(\infty)\, e^{ik_{2z}z} + B_1(\infty)\, e^{-ik_{2z}z}\right]\right.$$

$$\left. + C_2\left[A_2(\infty)\, e^{ik_{2z}z} + B_2(\infty)\, e^{-ik_{2z}z}\right]\right\} - \frac{r\, e^{ik_z z}}{(k_z^2 - k_{2z}^2)}$$

$$+ \frac{r\, e^{ik_{2z}z}}{2k_{2z}(k_z - k_{2z})} - \frac{r\, e^{-ik_{2z}z}}{2k_{2z}(k_z + k_{2z})} \ , \tag{150}$$

a result which may be compared with equation 145. As there, the various terms represent incoming and outgoing radiation and the field of the charge.

The condition that there be no incoming radiation from $+\infty$ is

$$\left(\frac{\epsilon}{k_{2z}}\right)^{\frac{1}{2}}\left[C_1 B_1(\infty) + C_2 B_2(\infty)\right] - \frac{r}{2k_{2z}(k_z + k_{2z})} = 0 . \qquad (151)$$

The field of the charge is again given by $B_{oy}(z)$:

$$B_{oy}(z) = -\frac{r\ e^{ik_z z}}{(k_z^{\ 2} - k_{2z}^{\ 2})}$$

and the radiation field, $B_{ry}^{(2)}(z)$, is

$$B_{ry}^{(2)}(z) \cong \left(\frac{\epsilon}{k_{2z}}\right)^{\frac{1}{2}}\left[C_1\ A_1(\infty)\ e^{ik_{2z}z} + C_2\ A_2(\infty)\ e^{ik_{2z}z}\right] + \frac{r\ e^{ik_{2z}z}}{2k_{2z}(k_z - k_{2z})} . \qquad (152)$$

Since a detailed comparison of these results would be both tedious and unrewarding we now note that two simplifications may be made. First, for the case of interest the change in dielectric constant between the two media is extremely small and second, $(k_z + k_{2z})$ is very large compared to $k_z - k_{2z}$. Accordingly the comparison which we shall make between these two results will be to first order in $\delta\epsilon$, the change in dielectric constants, and terms involving $(k_z + k_{2z})^{-1}$ will be neglected.

Let us now consider the approximate solution of equations 128 and 129 for A_1 and B_1. The region over which λ' is different

from zero is t. It is easily seen that the quantity $(\lambda'/\lambda - \epsilon'/\epsilon)$ is a monotonic function of z if $\epsilon(z)$ is monotonic. Therefore, since

$$\left|\frac{dA}{dz}\right| = \frac{1}{2}\left|\frac{\lambda'}{\lambda} - \frac{\epsilon'}{\epsilon}\right| \cdot |B| ,$$

we see that

$$\frac{dA}{dz} \leqslant -\frac{1}{2}\left(\frac{\lambda'}{\lambda} - \frac{\epsilon'}{\epsilon}\right) \cdot B ,$$

assuming that $B > 0$, and that λ/ϵ is monotonic decreasing. The latter is true if $\theta < 45^{\circ}$, and for $\theta > 45^{\circ}$ the sign of the inequality must be reversed. A similar relation holds for B. Both A and B will be bounded if the inequalities are replaced by equalities, in which case the equations can be integrated to obtain:

$$A(z) \leqslant \left[\left(\frac{\lambda_0\ \epsilon}{\lambda}\right)^{\frac{1}{2}} + \left(\frac{\lambda}{\lambda_0\ \epsilon}\right)^{\frac{1}{2}}\right]\frac{A_0}{2} ,$$

and

$$B(z) \leqslant \left[\left(\frac{\lambda_0\ \epsilon}{\lambda}\right)^{\frac{1}{2}} - \left(\frac{\lambda}{\lambda_0\ \epsilon}\right)^{\frac{1}{2}}\right]\frac{A_0}{2} . \qquad (153)$$

In this case we have assumed that $B_0 = 0$. A similar result holds if $A_0 = 0$. Since B reaches its final value $B(\infty)$ at t we have

$$B_1(\infty) \lessapprox \frac{1}{2}\left[\left(-\frac{\epsilon\ k_{1z}}{k_{2z}}\right)^{\frac{1}{2}} - \left(-\frac{k_{2z}}{\epsilon\ k_{1z}}\right)^{\frac{1}{2}}\right] . \qquad (154)$$

The quantity $[k_{2z}/(\epsilon k_{1z})]^2$ may be written as

$$\left(\frac{k_{2z}}{\epsilon\ k_{1z}}\right)^2 = 1 - \frac{(\epsilon - 1)}{\epsilon^2\ k_{1z}^{\ 2}}\left[\epsilon\frac{\omega^2}{c^2} - (\epsilon + 1)\ \kappa^2\right] \qquad (155)$$

so that B_1 is proportional to $(\epsilon - 1) = \delta\epsilon$, and

$$B_1(\infty) \gtrsim \frac{|(\epsilon - 1)|\,\omega^2}{4\epsilon^2 k_{1z}{}^2 c^2} . \qquad (156)$$

Since B_1 is proportional to $\delta\epsilon$ it follows that the change in A_1, δA_1, is proportional to $(\delta\epsilon)^2$ and so may be neglected. A similar result holds for A_2 and B_2 with the roles of A and B interchanged. Thus, to first order in $\delta\epsilon$ we may take $A_1(\infty) = B_2(\infty) = 1$ and replace $B_1(\infty) = A_2(\infty)$ by the value given in equation 156.

With the approximations discussed above we find from equation 151:

$$C_2 = C_1 B_1(\infty). \qquad (157)$$

Thus C_2 is of order $\delta\epsilon$. If we now turn to the expression for $B_y(z)$, the term involving C_2A_2 is proportional to $(\delta\epsilon)^2$ and may be neglected. Thus

$$B^{(2)}_{ry}(z) \cong (\epsilon/k_{2z})^{\frac{1}{2}}\, C_1\, e^{ik_{2z}z} + \frac{r}{2k_{2z}}\, \frac{e^{ik_{2z}z}}{(k_z - k_{2z})} . \qquad (158)$$

Upon combining this with equation 146 one finds

$$B^{(2)}_{ry}(z) \cong \left[\frac{1}{k_{2z}(k_z - k_{2z})} - \frac{\epsilon^{\frac{1}{2}}}{(-k_{1z}\, k_{2z})^{\frac{1}{2}}\,(k_z + k_{1z})}\right] \frac{r}{2}\, e^{ik_{2z}z}. \qquad (159)$$

It is interesting but not surprising to note that the "backward" radiation given in equation 145 is very small compared to the "forward" radiation since it contains no small denominator like those in equation 159. This completes the determination of $B^{(2)}_{ry}$.

We may now find $E^{(2)}_{rz}(z)$ using the Maxwell equation for radiation as used to obtain equation 42, to obtain

$$E^{(2)}_{rz} = -\frac{\kappa\, c}{\epsilon\,\omega}\, B^{(2)}_{rz}.$$

The final result is

$$E^{(2)}_{rz}(z) = \frac{i\, e\, \kappa^2}{4\pi^2\, k_z\, v_z}\left[\frac{-1/\epsilon}{k_{2z}(k_z - k_{2z})} + \frac{1}{(-\epsilon k_{1z}k_{2z})^{\frac{1}{2}}\,(k_z + k_{1z})}\right]. \qquad (160)$$

This is now to be compared with the result, for this case, which follows from equations 20 and 26:

$$E^{(2)}_{rz}(z) = \frac{i\, e\, \kappa^2}{2\pi^2 v_z k_z}\left[\frac{1}{(k_{2z} - \epsilon k_{1z})}\right]\left[\frac{(-1/\epsilon)(k_z - \epsilon k_{1z})}{(k_z{}^2 - k_{2z}{}^2)} + \frac{1}{k_z + k_{1z}}\right]. \qquad (161)$$

To make the comparison, we note that k_z, $-k_{1z}$ and k_{2z} are approximately equal. Furthermore, $\delta\epsilon \approx -\omega_p{}^2/\omega^2$ where ω_p is the plasma frequency for the atomic electrons, and, as will be shown in the next section (see equations 174 and 175), the important frequency region for the radiated power lies below $\gamma\omega_p$, so that in this region $|\delta\epsilon| > \gamma^{-2}$, and as $|\delta\epsilon| \to \gamma^{-2}$ the radiation falls to an insignificant value.

We now consider the terms $(k_z + k_{1z})$ and $(k_z - k_{2z})$ in either of the above expressions. We have, since $\kappa_0 = (\epsilon^{\frac{1}{2}}\,\omega/c)\sin\theta$,

$$(k_z + k_{1z})^{-1} \cong \frac{2\gamma^2 c}{\omega(1 + \gamma^2\theta^2)}$$

and

$$(k_z - k_{2z})^{-1} \cong \frac{2c}{\omega(\gamma^{-2} + \delta\epsilon + \theta^2)} \ .$$

Thus the terms involving $(k_z + k_{1z})^{-1}$ are much larger than those involving $(k_z - k_{2z})^{-1}$ and we conclude that equations 160 and 161 agree to a high degree of precision. This is the major result of this section.

Our remaining task is to justify the neglect of the terms in equation 146 which involve $\left[\mathcal{J}(z,z') - \mathcal{J}_A(z,z')\right]$. These are given by

$$B_y^{\Delta}(z') = \int_0^z \left[\mathcal{J}(z,z') - \mathcal{J}_A(z,z')\right] r\, e^{ikz'}\, dz' \ . \tag{162}$$

Consider a typical term in this expression:

$$I = -\frac{\chi_2(z)}{2i} \int_0^z \left[\frac{\chi_1(z')}{\epsilon(z')} - \frac{\chi_{1A}(z')}{\epsilon}\right] r\, e^{ik_z z'}\, dz' \ . \tag{163}$$

If we now use equations 123 and 138, I may be written as

$$I = \frac{\chi_2(z)}{2i} \int_0^t \Bigg\{ \left[\epsilon(z')\lambda(z')\right]^{-\frac{1}{2}} \left[A_1(z') \exp\left(i \int_0^{z'} \lambda(z'')dz''\right) + B_1(z') \exp\left(-i \int_0^{z'} \lambda(z'')dz''\right)\right]$$
$$- (\epsilon k_{2z})^{-\frac{1}{2}} \left[A_1(\infty)\, e^{i(\sigma + k_{2z}z')} + B_1(\infty)\, e^{-i(\sigma + k_{2z}z')}\right] \Bigg\} r\, e^{ik_z z'} \ . \tag{164}$$

It is to be noted that the integral extends only to $z = t$ since the integrand vanishes beyond that point. Consider first the term I_1

arising from the A's in equation 164:

$$I_1 = \frac{\chi_2(z)}{2i} \int_0^t \Bigg\{ \left[\epsilon(z')\lambda(z')\right]^{-\frac{1}{2}} A_1(z') \exp\left[i\left(\int_0^{z'} \lambda(z'')dz'' - \sigma - k_{2z}z'\right)\right]$$
$$- (\epsilon\, k_{2z})^{-\frac{1}{2}} A_1(\infty) \Bigg\} r \exp\left[i(k_z z' + k_{2z} z' + \sigma)\right] dz' \ . \tag{165}$$

We see that the terms in the exponential multiplying $A_1(z')$ are negligible if the formation zone is large so that

$$I_1 \cong \frac{\chi_2(z)}{2i} \int_0^t \left\{ \left[\epsilon(z')\lambda(z')\right]^{-\frac{1}{2}} A_1(z') - \left(\epsilon\, k_{2z}\right)^{-\frac{1}{2}} A_1(\infty) \right\}$$
$$\times\ r\, e^{i(k_z + k_{2z})z'}\, dz' \ . \tag{166}$$

As k_z becomes small compared to t, this term will become very small because of the cancelling oscillations associated with the exponential. Even if the cancellations are ignored, however, in this expression one can demonstrate that $\epsilon(z)\,\lambda(z)$ is monotonic in ϵ, and thus the integral is less than $\approx \left\{(-k_{1z})^{-\frac{1}{2}} - (\epsilon\, k_{2z})^{-\frac{1}{2}}\right\} rt$. Compared to the corresponding terms in B_y^{A}, this term is of order $t\delta\epsilon$ whereas the other is of order $d_F\, \delta\epsilon$. Hence the integral is small compared to those terms which have been discussed above and so may be neglected. Next consider the terms, I_2, arising from the B's in equation 164.

$$I_2 = \frac{\chi_2(z)}{2i}\int_0^t \left\{\left[\epsilon(z')\lambda(z')\right]^{-\frac{1}{2}} B_1(z') \exp\left[-i\left(\int_0^{z'}\lambda(z'')dz'' + k_{2z}z' - \sigma\right)\right]\right.$$

$$\left. - (\epsilon k_{2z})^{-\frac{1}{2}} B_1(\infty)\right\} r \exp\left[i(k_z z' - k_{2z}z' - \sigma)\right] dz' . \qquad (167)$$

Using the same approximations which led to equation 165 we have

$$I_2 \cong \frac{\chi_2(z)}{2i}\int_0^t \left\{\left[\epsilon(z')\lambda(z')\right]^{-1} B_1(z')\right.$$

$$\left. - (\epsilon k_{2z})^{\frac{1}{2}} B_1(\infty)\right\} r \; e^{i(k_z - k_{2z})z'} dz' . \qquad (168)$$

The exponential may be set equal to one since it depends on the formation zone d_F. Further, since $\delta\epsilon \ll 1$, and B_1 is of order $\delta\epsilon$, the remaining integral is of order $t\delta\epsilon$ and hence can be neglected as well. Corresponding arguments can be applied to the other parts of $B_y^{\triangle}$ to show that they too are negligible in our approximation. We thus conclude that when the formation zone is large compared to the region over which the dielectric constant varies, the simple result, obtained by assuming a sharp dielectric boundary, is valid.

9. The dependence of transition radiation on particle energy*

To determine the total energy radiated into a given frequency range, $d\omega$, when the particle passes from vacuum into a medium we must

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* All the results of Section 9 are contained in ref. 2. We include them here for completeness.

integrate equation 90 using the expression for $|\xi(\omega)|^2$ in equation 91. For an extremely relativistic particle we may take $v_z \doteq c$, and we expect that radiation will be mainly confined to small angles because of the terms, $(1 - \beta^2 \epsilon \cos^2 \theta)$ and $(1 - \beta^2 + \epsilon \beta^2 \sin^2 \theta)$ in the denominator of ξ, which become very small in the forward direction. In all but these rapidly varying factors we take θ to be zero. Also since ξ is proportional to $\delta\epsilon$ we set $\epsilon = 1$, except in these rapidly varying factors. We thus have

$$1 - \beta^2 - \beta^3\epsilon(1 - \epsilon \sin^2 \theta)^{\frac{1}{2}} - \beta^2\epsilon \cos^2 \theta \cong -2 ,$$

and

$$\cos\theta + \left\{\epsilon(1 - \epsilon \sin^2 \theta)\right\}^{\frac{1}{2}} \cong 2 ,$$

so that

$$W \cong \frac{2e^2}{\pi c}\int_0^\infty d\omega \int_0^\pi d\theta \sin^3\theta \cos^2\theta$$

$$\times \left|\frac{(\epsilon - 1)}{(1 - \beta^2\epsilon \cos^2 \theta)(1 - \beta^2 + \epsilon\beta^2 \sin^2 \theta)}\right|^2 . \qquad (169)$$

To further simplify this integral one may write

$$\sin\theta \cong \theta ,$$

$$(1 - \beta^2\epsilon \cos^2 \theta) \cong (1 - \beta^2\epsilon + \theta^2) ,$$

$$(1 - \beta^2 + \epsilon\beta^2 \sin \theta) \cong (1 - \beta^2 + \theta^2) .$$

With these approximations W is given by

$$W \cong \frac{2e^2}{\pi c} \int_0^\infty d\omega \int_0^\infty \theta^3 \, d\theta \left| \frac{(\epsilon - 1)}{(1 - \beta^2 \epsilon + \theta^2)(1 - \beta^2 + \theta^2)} \right|^2 . \tag{170}$$

In the X-ray frequency range ϵ may be expressed as

$$\epsilon = 1 - \left(\frac{\omega_p}{\omega}\right)^2 \tag{171}$$

where ω_p is the plasma frequency

$$\omega_p = \frac{4\pi N e^2}{m} .$$

In this expression, N is the number of electrons per unit volume and e and m are the electron charge and mass, respectively. Since the imaginary part of ϵ is negligible the absolute value signs in equation 170 may be ignored and the integration over θ may be carried out to yield

$$W = \frac{2e^2}{\pi c} \int_0^\infty d\omega \left\{ \left[\frac{1}{2} + \frac{\omega^2(1 - \beta^2)}{\beta^2 \, \omega_p^2}\right] \ell n \left[1 + \frac{\beta^2 \omega_p^2}{\omega^2(1 - \beta^2)}\right] - 1 \right\} . \tag{172}$$

Before carrying out the integration over ω we note that equation 172 holds only when $\omega >> \omega_p$. In the X-ray region $\omega \sim 10^3 \, \omega_p$. There will also be a selection or cut-off effect at the lower limit because of the X-ray detector efficiency. Beyond this we note that there is a high energy cut-off frequency, ω_c, in equation 172

$$\omega_c = \frac{\omega_p}{(1 - \beta^2)^{\frac{1}{2}}} . \tag{173}$$

When $\omega << \omega_c$ the integrand behaves as

$$\frac{dW}{d\omega} \cong \frac{2e^2}{\pi c} \, \ell n \left(\frac{\omega_c}{\omega}\right) , \tag{174}$$

while when $\omega >> \omega_c$ it becomes

$$\frac{dW}{d\omega} = \frac{e^2}{6\pi c} \left(\frac{\omega_c}{\omega}\right)^4 . \tag{175}$$

Since the low frequency part of the integral contributes little and because of the natural high frequency cut-off, we conclude that equation 174 may be integrated to give the radiated energy,

$$W = \frac{e^2}{3c} \, \omega_c . \tag{176}$$

Thus W is proportional to $(1 - \beta^2)^{-\frac{1}{2}}$ or the ratio E/m for the incident particle. This is the basis for studies of the use of transition counters[9)] in high energy physics. It should be noted that this result assumes that the X-ray detector efficiency is unity. If the detector becomes inefficient for an $\omega_x < \omega_c$, the total transition radiated energy detected will only be proportional to $\log \gamma$, rather than γ , as given in equation 176.

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APPENDIX

In this appendix we will complete our treatment of the case when the particle is not incident normally to the dielectric boundary.

Consider the expression for $E_{rx}^{(2)}$, equation 43. The integrand is a function of $\cos \phi$ and κ so we may write (fig. 1a)

$$E_{rx}^{(2)} = \int_{-\infty}^{\infty} d\omega \int_{0}^{\infty} d\kappa \int_{0}^{2\pi} d\phi \; G(\kappa, \cos \phi)$$

$$\times \exp\left[i(\kappa r \sin \theta \cos \phi + k_{2z} r \cos \theta - \omega t) \right] . \quad (A1)$$

where G is given by (we choose $\mu_1 = \mu_2 = 1$)

$$G(\kappa, \cos \phi) = - \frac{k_{2z}}{v_z} \mathcal{E}_{rz}^{(2)} \cos \phi - \frac{\omega}{c v_z} \mathcal{H}_{rz}^{(2)} \sin \phi . \quad (A2)$$

We have suppressed the dependence of G on ω, α, β for convenience. The integrations over both ϕ and κ will now be carried out using the saddle-point method described in the main part of this paper. Consider first the ϕ integration denoted by I :

$$I = \int_{0}^{2\pi} d\phi \; G(\kappa, \cos \phi) \, e^{iz \cos \phi} . \quad (A3)$$

Here z is given by

$$z = \kappa r \sin \theta . \quad (A4)$$

If I is divided into two parts, I_1 and I_2,

$$I_1 = \int_{0}^{\pi} G(\kappa, \cos \phi) \, e^{iz \cos \phi} \, d\phi , \quad (A5)$$

and

$$I_2 = \int_{\pi}^{2\pi} G(\kappa, \cos \phi) \, e^{iz \cos \phi} \, d\phi , \quad (A6)$$

we see that upon changing ϕ to $\phi + \pi$, I_2 is given by

$$I_2 = \int_{0}^{\pi} G(\kappa, -\cos \phi) \, e^{-iz \cos \phi} \, d\phi . \quad (A7)$$

We thus see that

$$I_2^{*} = \int_{0}^{\pi} G(\kappa, -\cos \phi) \, e^{iz \cos \phi} \, d\phi . \quad (A8)$$

Let us now apply the saddle-point method to evaluate I_1. If we write

$$\psi(\phi) = iz \cos \phi , \quad (A9)$$

then

$$\frac{d\psi}{d\phi} = - iz \sin \phi . \quad (A10)$$

The saddle points are located at

$$\phi = 0, \pi, 2\pi, \cdots . \quad (A11)$$

Our region of integration thus begins and ends at a saddle point. The second derivative of ψ is

$$\frac{d^2\psi}{d\phi^2} = - iz \cos \phi \quad (A12)$$

so we may write for the contribution from the saddle points

$$I_1 \cong e^{iz} G(\kappa,1) \int e^{-\frac{1}{2}iz\phi^2} d\phi + e^{-iz} G(\kappa,-1) \int e^{\frac{1}{2}iz\phi^2} d\phi . \tag{A13}$$

Upon writing

$$\phi = e^{-i\pi/4} t ,$$

in the first integral of this equation and

$$\phi = e^{i\pi/4} t ,$$

in the second we have

$$I_1 \cong e^{i(z-\pi/4)} G(\kappa,1) \int_0^\infty e^{-\frac{1}{2}zt^2} dt + e^{-i(z-\pi/4)} G(\kappa,-1) \int_{-\infty}^0 e^{-\frac{1}{2}zt^2} dt . \tag{A14}$$

Thus:

$$I_1 \cong \frac{1}{2} (2\pi/z)^{\frac{1}{2}} \left[e^{i(z-\pi/4)} G(\kappa,1) + e^{-i(z-\pi/4)} G(\kappa,-1) \right] . \tag{A15}$$

When the same method is applied to equation A8 we find

$$I_2^* \cong \frac{1}{2} (2\pi/z)^{\frac{1}{2}} \left[e^{i(z-\pi/4)} G(\kappa,-1) + e^{-i(z-\pi/4)} G(\kappa,1) \right] .$$

Thus

$$I_2 = I_1 \tag{A16}$$

and

$$I = (2\pi/z)^{\frac{1}{2}} \left[e^{i(z-\pi/4)} G(\kappa,1) + e^{-i(z-\pi/4)} G(\kappa,-1) \right] . \tag{A17}$$

The expression for $E_{rx}^{(2)}$ becomes

$$E_{rx}^{(2)} = \left(2\pi/r \sin\theta\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} d\omega \int_0^{\infty} d\kappa \ \kappa^{-\frac{1}{2}}$$

$$\times \left[e^{i(\kappa r \sin\theta - \pi/4)} G(\kappa,1) + e^{-i(\kappa r \sin\theta - \pi/4)} G(\kappa,-1) \right]$$

$$\times \ e^{i(k_{2z} r \cos\theta - \omega t)} . \tag{A18}$$

This result may be compared with equation 55 of the main part of this paper. The more detailed arguments given there show that only the first term in the integrand of equation A18 will contribute to the asymptotic behavior of $E_{rx}^{(2)}$.

In what follows we must take care to note that k_z is given now by

$$k_z = \frac{\omega - \vec{\kappa}\cdot\vec{v}}{v_z} ,$$

or

$$k_z = \frac{\omega - \kappa v_x}{v_z} , \tag{A19}$$

when ϕ is set equal to zero.

Since only the terms involving $\phi = 0, \pi$ contribute to $E_{rx}^{(2)}$ we see that the term in equation A2 involving $H_{rz}^{(2)}$ does not contribute to the radiation field, $E_{rx}^{(2)}$.

The saddle-point method used to evaluate the κ integration in equation 55 may be also used to evaluate equation A18. We thus find the analogue of equation 81

$$E_{rx}^{(2)} = -\frac{e \sin\theta \cos^2\theta}{\pi\, r\, v_z} \int \xi(\omega) \exp\left[-i\omega(t - \chi_2^{\frac{1}{2}}\, r/c)\right] d\omega \tag{A20}$$

where ξ is now given by

$$\xi = \frac{(\omega/c)^3\, \chi_2^{3/2}}{(\epsilon_1 k_{2z} - \epsilon_2 k_{1z})} \left\{ \frac{\frac{\epsilon_1}{\epsilon_2} - \frac{k_{1z}^0 v_z}{\omega} + \frac{v_x}{\kappa}\left(\frac{k_{1z}k_z}{\omega} - \frac{\epsilon_1}{\epsilon_2}\chi_2 \frac{\omega}{c^2}\right)}{\chi_2(\omega^2/c^2) - k^2} \right.$$

$$\left. - \frac{1 - \frac{k_{1z}v_z}{\omega} + \frac{v_x}{\kappa}\left(\frac{k_{1z}k_z}{\omega} - \chi_2 \frac{\omega^2}{c^2}\right)}{\chi_1(\omega^2/c^2) - k^2} \right\}. \tag{A21}$$

This reduces to equation 82 when v_x is zero.

We now turn to a new facet of our problem. Since it no longer has axial symmetry about the direction of the particle velocity, $E_{ry}^{(2)}$ does not vanish. For $E_{ry}^{(2)}$ we find an expression identical in form to equation A1 but with G replaced by

$$G(\kappa, \cos\not{\phi}) = -\frac{k_{2z}}{v_z}\, \mathcal{E}_{rz}^{(2)} \sin\not{\phi} + \frac{\omega\mu_2}{c v_z}\, \mathcal{H}_{rz}^{(2)} \cos\not{\phi}\,. \tag{A22}$$

The expression for $\mathcal{H}_{rz}^{(2)}$, equation 34, is proportional to

$$[\vec{k} \times \vec{v}] \cdot \hat{z} = \kappa[v_y \cos\not{\phi} - v_x \sin\not{\phi}]\,. \tag{A23}$$

Since terms in $\sin\not{\phi}$ do not contribute, $E_{ry}^{(2)}$ is proportional to

κv_y. For the case when $\mu_1 = \mu_2 = 1$, G may therefore be replaced by

$$G(\kappa, \cos\not{\phi}) \rightarrow \frac{i e \kappa v_y \omega^3 (k_z - k_{1z})}{2\pi^2 v_z c^3 (k_{1z} - k_{2z})} \cdot \frac{(\epsilon_2 - \epsilon_1)\cos^2\not{\phi}}{[\epsilon_1(\omega^2/c^2) - k^2][\epsilon_2(\omega^2/c^2) - k^2]}\,. \tag{A24}$$

Evaluation of the expression for $E_{ry}^{(2)}$ leads to the analog of equation A18

$$E_{ry}^{(2)} = (2\pi/r \sin\theta)^{\frac{1}{2}} \int_{-\infty}^{\infty} d\omega \int_0^{\infty} \kappa^{-\frac{1}{2}}\, d\kappa$$

$$\times \left[e^{i(\kappa r \sin\theta - \pi/4)}\, G(\kappa, 1) + e^{-i(\kappa r \sin\theta - \pi/4)}\, G(\kappa, -1)\right]$$

$$\times\, e^{i(k_{2z} r \cos\theta - \omega t)}\,. \tag{A25}$$

As before we apply the saddle-point method to carry out the integration over κ and find

$$E_{ry}^{(2)} = \frac{e}{\pi r} \frac{v_y}{v_z} \cos\theta \int \eta(\omega)\, e^{-i\omega(t - \epsilon^{\frac{1}{2}} r/c)}\, d\omega \tag{A26}$$

where η is given by

$$\eta = \frac{(\omega/c)^4\, \epsilon^{\frac{1}{2}} (\epsilon - 1)(k_z - k_{1z})}{(k_{1z} - k_{2z})[(\omega^2/c^2) - k^2]\,[\epsilon(\omega^2/c^2) - k^2]}\,. \tag{A27}$$

The integral for $E_{rz}^{(2)}$ may be similarly evaluated. We find

$$E_{rz}^{(2)} = \frac{e}{\pi r v_z} \frac{\cos\theta}{\sin\theta} \int_{-\infty}^{\infty} \zeta(\omega)\, e^{-i\omega(t - \epsilon^{\frac{1}{2}} r/c)}\, d\omega\,, \tag{A28}$$

where ζ is given by

$$\zeta(\omega) = \frac{1}{(k_{2z} - \epsilon k_{1z})}\left\{ \frac{(k_{1z}/\omega)\left[k_z(\vec{\kappa}\cdot\vec{v}) - \kappa^2 v_z\right] + \frac{1}{\epsilon}\left[\kappa^2 - \epsilon\omega(\vec{\kappa}\cdot\vec{v})/c^2\right]}{\epsilon(\omega^2/c^2) - k^2} - \frac{(k_{1z}/\omega)\left[k_z(\vec{\kappa}\cdot\vec{v}) - \kappa^2 v_z\right] + \left[\kappa^2 - \epsilon\,\omega(\vec{\kappa}\cdot\vec{v})/c^2\right]}{\left[(\omega^2/c^2) - k^2\right]} \right\} . \quad \text{(A29)}$$

The foregoing equations allow us to calculate E_r^2. As before H_r^2 may be calculated by using the relation, valid for radiation fields,

$$\epsilon E_r^2 = \mu H_r^2 .$$

The magnitude of the Poynting vector is therefore

$$|\vec{S}| = \frac{c\,\epsilon^{\frac{1}{2}}}{4\pi} E_r^2 .$$

In view of the complicated equations for the components of $\vec{E}_r$ we will not present an analytic expression for $|\vec{S}|$. The evaluation of the radiated energy using our equations is best handled directly by computer.

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FIGURE CAPTIONS

Fig. 1. The wave number $\vec{k}$, observation vector $\vec{r}$, and their projections $\vec{\kappa}$ and $\vec{\rho}$ on the $\hat{x}$, $\hat{y}$ plane. $\vec{v}$ is normal to the $\hat{x}$, $\hat{y}$ media interface.

Fig. 2. Shows saddle point, paths of steepest descent (arrows show direction of the increasing function), branch points and values of k_{2z}, along the real axis, all involved in integration of f term.

Fig. 3. Shows saddle point, paths of steepest descent (opposite to arrows), branch points and values of k_{2z}, along the real axis, all involved in integration of g term.

Fig. 4. Shows three dielectric media. Field is evaluated at a distance D from the last dielectric slab, the point P.

Fig. 1A. Shows the various angles involved when the particle is not normally incident on the dielectric boundary. The velocity is described by polar angle α and aximuth β. The other angles are the same as for the case of normal incidence.

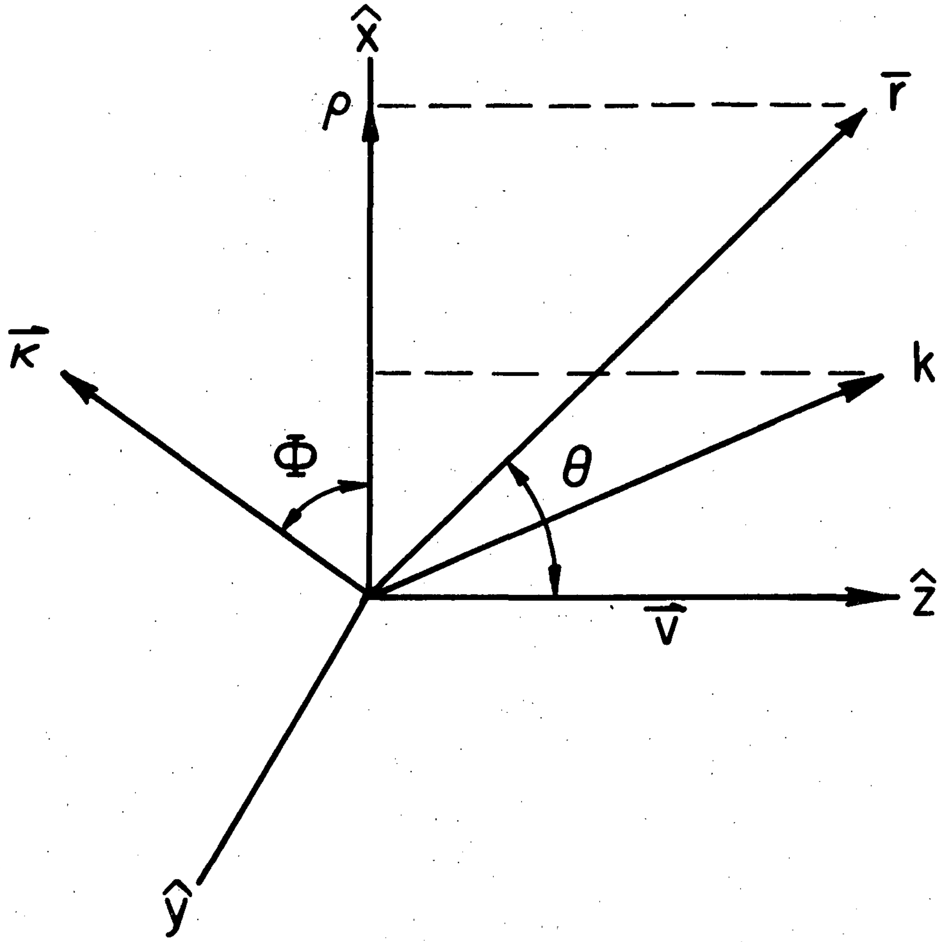

Fig. 1

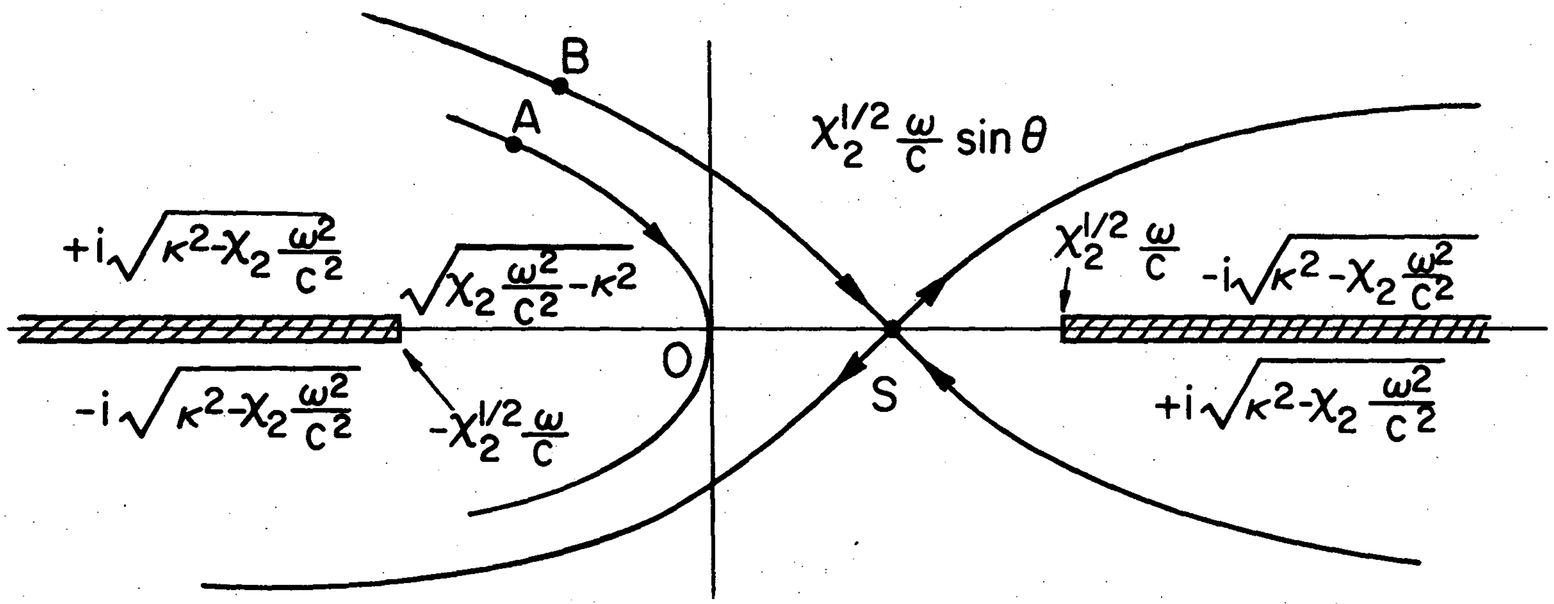

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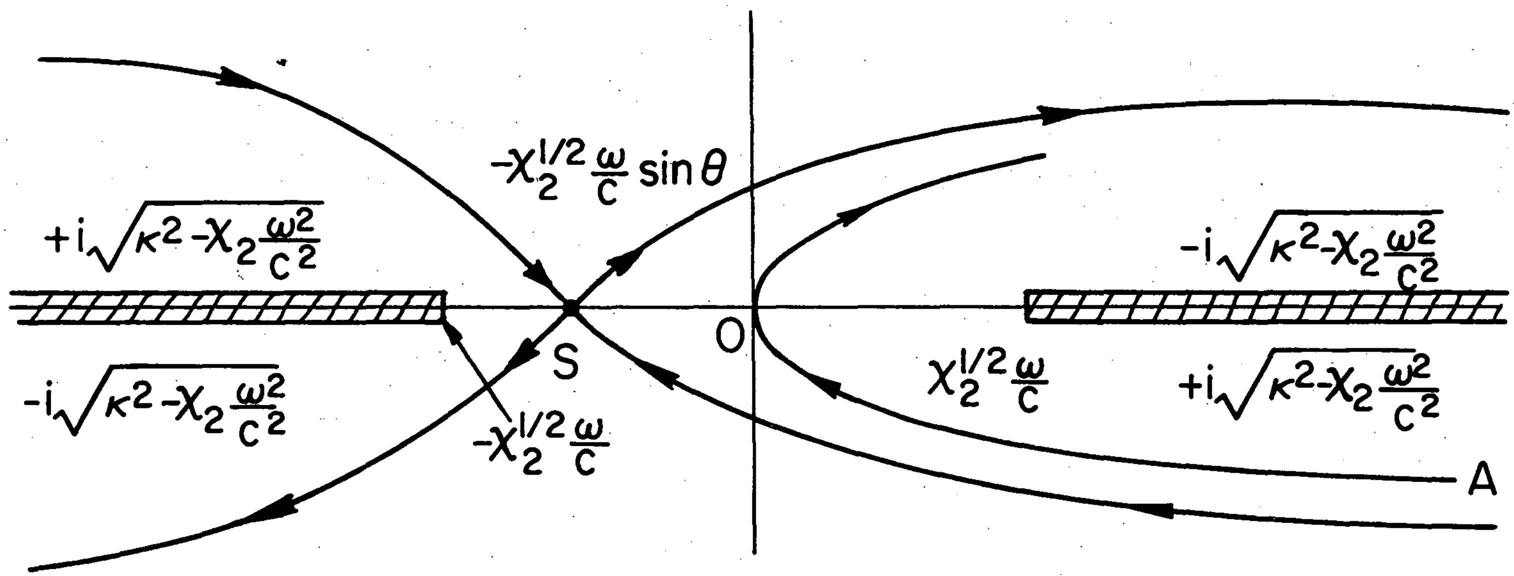
Fig. 2



$-\chi_2^{1/2}\frac{\omega}{c}\sin\theta$
$+i\sqrt{\kappa^2-\chi_2\frac{\omega^2}{c^2}}$
$-i\sqrt{\kappa^2-\chi_2\frac{\omega^2}{c^2}}$
S
O
$-i\sqrt{\kappa^2-\chi_2\frac{\omega^2}{c^2}}$
$\chi_2^{1/2}\frac{\omega}{c}$
$+i\sqrt{\kappa^2-\chi_2\frac{\omega^2}{c^2}}$
$-\chi_2^{1/2}\frac{\omega}{c}$
A
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Fig. 3

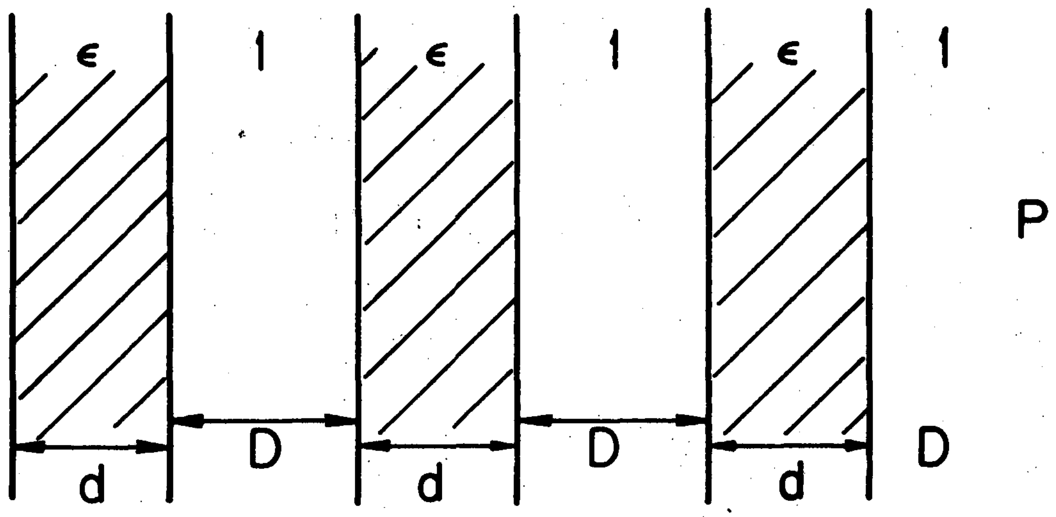

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Fig. 4

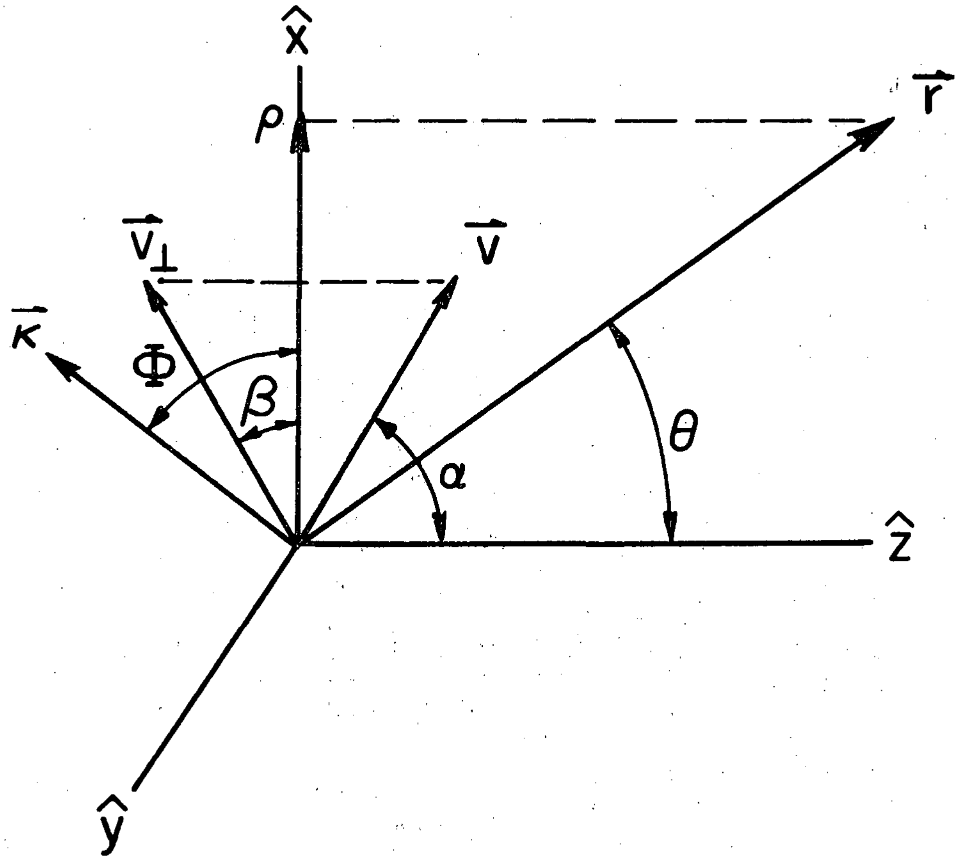

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Fig. 1a