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Abstract

Equivalence of the core and the set of Walrasian allocations has long been taken as one of the basic tests of perfect competition. The present paper examines this basic test of perfect competition in economies with an infinite dimensional space of commodities and a large *finite* number of agents. In this context we cannot expect *equality* of the core and the set of Walrasian allocations; rather, as in the finite dimensional context, we look for theorems establishing *core convergence* (that is, approximate decentralization of core allocations in economies with a large finite number of agents).

Previous work in this area has established that core convergence for replica economies and core equivalence for economies with a continuum of agents continue to be valid under assumptions much the same as those usual in the finite dimensional context. For general large finite economies, however, we present here a sequence of examples of the *failure* of core convergence. These examples point to a serious disconnection between replica economies and continuum economies on the one hand an general large finite economies on the other hand. We identify the source of this disconnection as the measurability requirements that are implicit in the continuum model, and which correspond to compactness requirements that have especially serious economic content in the infinite dimensional context.

We also obtain positive results. When the commodity space is a Riesz space, we show that familiar assumptions lead to a kind of "local" core convergence; strong assumptions lead to "global" core convergence. In the differentiated commodities context, we obtain core convergence results that are quite parallel to known equivalence results for continuum economies. Our positive results depend on infinite dimensional versions of the Shapley-Folkman theorem.

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Contents

1 Introduction

Since Edgeworth (1881), equivalence of the core and the set of Walrasian allocations has been taken as one of the basic tests of perfect competition. If the core is much larger than the set of competitive allocations (or the set of approximately competitive allocations), the price-taking assumptions underlying the whole competitive story are seriously in question.[1] The purpose of the present paper is to examine this basic test of perfect competition in economies with an infinite dimensional commodity space and a large *finite* number of agents. In this context, of course, we cannot expect *equality* of the core and the set of Walrasian allocations; rather, as in the finite dimensional case, we look for theorems establishing that the core converges, i.e. that core allocations of economies with a sufficiently large number of agents are approximately competitive in an appropriate sense.

Our motivation for studying the core in economies with an infinite dimensional commodity space and a large but finite number of agents is two-fold. First, an extensive literature on the existence of Walrasian equilibrium in economies with a finite number of agents and an infinite dimensional commodity space has developed over the last twenty-five years. This literature is motivated by a variety of economic issues, including choice under uncertainty (especially finance), choice over an infinite time horizon, and choice among subtly differentiated commodities. It is important to know whether the price-taking assumption implicit in the definition of Walrasian equilibrium can be justified in these models.

Second, our study of core convergence in economies with an infinite dimensional commodity space and a large finite number of agents provides us with a way to study properties of economies with a large *finite* number of commodities and a large finite number of agents. Consider, for example, a model in which every agent is endowed with his/her labor. Each person's

[1] We do not suggest that equivalence of the core and the set of competitive allocations is *sufficient* for perfect competition, only that it is *necessary*. We would make a similar argument for other tests of perfect competition that have been offered, such as Ostroy's no surplus condition, or the coincidence of Walrasian equilibria with the Nash equilibria of the corresponding Shapley–Shubik market game. See Gretsky, Ostroy and Zame (1994) for a comparative analysis of various tests of perfect competition in the context of the assignment model.

labor is subtly different from that of every other agent, and very different from that of some other agents. If we consider an Arrow-Debreu model with each agent's labor treated as a wholly separate commodity, we have no convenient way to capture the fact that the labor of certain individuals are close substitutes. In a sequence of economies in which the number of agents is growing and the labor endowments are treated as distinct components in the Arrow-Debreu commodity space, the number of distinct goods is growing as rapidly as the number of agents. As a consequence, none of the core convergence theorems in the literature suffices to show that core allocations become approximately competitive as the number of agents grows. However, by treating the various kinds of labor as elements of an appropriate infinite dimensional commodity space, we can capture the fact that the labor of certain individuals will be cl ose substitutes, and core convergence can be established.

For economies with a finite dimensional commodity space, the "classical" results connecting the core and the set of competitive allocations are the Debreu–Scarf theorem (which identifies the competitive allocations of an economy with a finite number of traders with those in the core of every replication), Aumann's theorem (which identifies the competitive allocations of an economy with a continuum of traders with the core of the economy), and the convergence theorems of Anderson (1978, 1981, 1987), Bewley (1973a), Brown and Robinson (1974), Cheng (1981, 1982, 1983a, 1983b), Debreu (1975), E. Dierker (1975), H. Dierker (1975), Keiding (1974), Geller (1987), Grodal (1975), Grodal and Hildenbrand (1974), Kannai (1970), Khan (1974, 1976), Khan and Rashid (1976), Trockel (1976), Vind (1965) and others (which assert that core allocations of "large" finite economies can be "approximately decentralized by prices"). For our purposes in this paper, we say (following E. Dierker (1975) and Anderson (1978)) that an allocation is approximately decentralized by a price p if

- the average amount by which the allocation fails to be budget feasible with respect to p is small, and
- the average saving that can be effected at prices p while still improving on the given allocation is small.

We say an allocation can be approximately decentralized by prices if there is a price vector p which approximately decentralizes it.[2]

For economies with an infinite dimensional commodity space, analogs of the Debreu–Scarf theorem and of Aumann's theorem have been known for some time (Aliprantis, Brown and Burkinshaw (1989), Bewley (1973b), Gabszewicz (1968), Mas-Colell (1975), Ostroy and Zame (1993), Rustichini and Yannelis (1991), Zame (1986)). However, no analogs of the non-replica convergence theorems have been established.[3]

In this paper we argue that, for economies with an infinite dimensional commodity space, there may be a substantial disconnection between replica economies and continuum economies on the one hand and general large finite economies on the other hand. One way to understand the source of this disconnection is to review the assumptions that underlie Aumann's theorem and its extensions. Aumann's theorem assumes that the preference mapping, the endowment mapping, the core allocation mapping, and all comparison bundle mappings are measurable. In the finite dimensional context, these requirements are usually interpreted as slightly unpleasant technical conditions, but without real economic import.[4] In the infinite dimensional context, however, measurability has a great deal more bite. The reason is that, in either the finite dimensional or the infinite dimensional situation, a measurable mapping has "almost compact" range; this is a much more serious restriction in the infinite dimensional setting because compact subsets of infinite

[2]Some of the finite dimensional results establish stronger conclusions, for example, that individual's core consumptions are close to their demand sets at the price p, under additional hypotheses, such as equiconvexity of preferences. Many of these conclusions can be derived, under the additional hypotheses, from approximate decentralization by prices in the sense used here.

[3]Nomura (1993) asserts, for the commodity space L^2, the existence of a price which approximately decentralizes core allocations up to a constant which depends on a measure of nonconvexity. However, it appears to us that his measure of nonconvexity will often be infinite, even for a single well-behaved convex preference on L^2. If his measure of nonconvexity is indeed infinite, his conclusion will be satisfied by any allocation and any price.

[4]A little care should be exercised even in the finite dimensional context. Integrability of the endowment mapping implies that the endowments of most agents lie in a compact set. Measurability of the preference mapping together with monotonicity of individual preferences implies a kind of equimonotonicity. Clearly, these conclusions have economic import.

dimensional spaces are typically very thin subsets of the underlying spaces.[5] Thus, the infinite dimensional versions of Aumann's theorem point at infinite dimensional versions of the convergence theorems that are significantly different from the finite dimensional versions.

For replica economies, of course, compactness restrictions on preferences, on endowments and on core consumptions have no additional bite. By assumption, preferences, endowments, and core consumptions all lie in a finite, hence compact set. When preferences are strictly convex, the equal treatment property implies that the core is homeomorphic to a closed subset of the Pareto set of the *unreplicated* economy, which in turn can be shown to be homeomorphic to a simplex of dimension one less than the number of types; thus, the core is compact. It is less obvious, but true, that a compactness restriction on comparison bundles also carries no bite. The proof of the Debreu-Scarf theorem, in either the finite or infinite dimensional context, proceeds one allocation at a time. Once a non-Walrasian allocation is specified, one immediately finds a profile of profile of comparison bundles, a convex combination of which lies in the negative orthant. One then need only choose n sufficiently large that the convex coefficients can be well approximated by rational numbers with denominator less than or equal to n; once aga in, the convexifying effect of large numbers takes place in a simplex of dimension one less than the number of types. Thus, one establishes that every allocation which is in the core of every replica is Walrasian. Moreover, since the replica cores are nested and compact, it follows that given any neighborhood U, there is a sufficiently large n_0 such that every core allocation of the n-fold replica for $n \geq n_0$ is within U of a Walrasian equilibrium. Thus, the replica context is actually very similar to the continuum context.

For general large finite economies, these compactness restrictions turn out to have a great deal of bite. Indeed — and this is a central point of this paper — these compactness restrictions have real *economic* content.

We emphasize this point through four examples of sequences of finite economies and core allocations that *cannot* be approximately decentralized by prices. The examples are:

[5]It is not clear how to interpret the measurability restriction on preferences, since no convincing topology on spaces of preferences has been offered in the infinite dimensional context.

- a sequence of economies in which consumers retain *monopsony* power; the source of monopsony power is that *core consumptions* do not lie in a compact set

- a sequence of economies in which consumers retain *monopoly* power; the source of monopoly power is that *endowments* do not lie in a compact set

- a sequence of economies in which consumers retain neither monopoly power nor monopsony power, but the (unique) core allocation cannot be approximately decentralized by prices; the source of the failure of approximate decentralization is that the *relevant comparison bundles* do not lie in a compact set

- a sequence of economies in which there is no Walrasian equilibrium, and core allocations cannot be approximately decentralized by prices; in the continuum analogue of the sequence, core equivalence holds because both the core and the set of Walrasian equilibria are empty; the source of the failure of approximate decentralization is that the preferences exhibit *increasing returns* to specialization in consumption. This example also establishes a failure of upper hemicontinuity of the core, demonstrating that it will be extremely difficult to study core convergence by topologizing economies in the manner of Hildenbrand (1974).

The examples can be imbedded in most of the infinite dimensional commodity spaces that have been used in economic analysis.

It may be useful to comment on the connection between compactness on the one hand and monopoly or monopsony on the other. Suppose all agents' endowments lie in a compact set. Given $\varepsilon > 0$, this compact set can be covered by a finite number of balls of radius ε. Suppose the number of agents in the economy is large. Then the proportion of agents with isolated endowments (in the sense that there are few other agents with endowments in the same ε ball) must be small. The agents whose endowments are not isolated cannot have monopoly power.[6] The agents whose endowments are

[6]The argument is actually a little more complex than we have indicated here. If preferences of the agents were equicontinuous over the whole commodity space, then agents whose endowments are not isolated cannot have monopoly power. However, equicontinuity

isolated may have monopoly power, but they are few in number, and their monopoly power does not prevent approximate decentralization of core allocations. On the other hand, if endowments do not come from a compact set, it is possible to have all the endowments a uniform distance away from every other endowment. In this case, i t is possible that every agent possesses significant monopoly power, resulting in a big core. Dually, if individual agents' core consumptions all lie in a compact set, then few agents will have isolated consumptions. As in the case of monopoly, this implies that few agents will have monopsony power, and these few agents will not upset approximate decentralization. On the other hand, if core consumptions do not lie in a compact set, it is possible that all consumptions lie a uniform distance from every other consumption, because each agent has a special preference for a personal good. In this case, every agent will possess significant monopsony power, and once again the core will be large.

Of course, monopoly power and monopsony power are the antithesis of perfect competition; what the first two examples demonstrate is that monopoly power and monopsony power are particularly easy to come by — and particularly hard to avoid — in infinite dimensional commodity spaces.[7] It has long been known that the form of core convergence may change if certain standard assumptions — notably convexity — are relaxed. The striking work of Manelli (1991) shows that core convergence and approximate decentralization may fail outright, even in the finite dimensional context, if the "standard" assumption of monotonicity is weakened slightly. We emphasize, therefore, that the examples described above satisfy natural infinite dimensional analogs of all the "standard" assumptions; in particular, preferences are strictly monotone and marginal rates of substitution are bounded.

These examples suggest that, in the infinite dimensional context, the

over the whole commodity space is a strong assumption. Accordingly, in our positive theorems, we impose an equicontinuity assumption only on a subset of the commodity space, and then either assume, or prove under additional hypotheses, that the core allocations and relevant consumption bundles are contained the set on which equicontinuity holds.

[7]For the first two of these examples, we also show that the Walrasian equilibria of these sequences of economies fail Ostroy's approximate no-surplus test of perfect competition; we conjecture that the Nash equilibria of the corresponding Shapley–Shubik market games do not approximate Walrasian equilibria. The third example has a different character, in that only *infeasible* allocations demonstrate the impossibility of approximately decentralizing core allocations.

failure of core convergence and approximate decentralization is far more pervasive than in the finite dimensional context. One way to understand why the equivalence theorems for continuum economies in the infinite dimensional context are compatible with our counterexamples for large finite economies is to take literally the compactness restrictions implicit in the measurability requirement, and formulate an approximate decentralization result for large finite economies which incorporates these compactness restrictions. We say an allocation can be approximately decentralized with respect to X if there is a price for which

- the average amount by which the allocation fails to be budget feasible is small, and
- the average saving that can be effected *by consumption in* X while still improving on the given allocation is small.

Following this stratagem leads to a peculiar-looking result:

Most Peculiar Theorem 7.1 *Fix the commodity space $L = C(\Omega)$, where Ω is a compact Hausdorff space, and a compact subset $K \subset L_+$. Consider a finite economy for which endowments lie in K. If the number of consumers is sufficiently large, then every core allocation with consumptions lying in K can be approximately decentralized with respect to K.*

As we have argued, it seems to us that this peculiar result is actually the finite agent analog of Aumann's equivalence theorem for continuum economies because the measurability assumptions in Aumann's theorem correspond to compactness assumptions in the large finite economies.

For Dedekind complete Riesz spaces, we can do substantially better than Most Peculiar Theorem 7.1 by proceeding in a different direction. For most commodity spaces of interest, order bounded sets are, roughly speaking, much bigger than compact sets.[8] In particular, Edgeworth boxes are always order-bounded, but almost never compact. In essence, Slightly Silly Theorem 8.2

[8]In L^∞ and ℓ_∞, compact sets are necessarily order bounded, but this is not true generally, for example in L^1. Thus, Slightly Silly Theorem 8.2 is not a *generalization* of Most Peculiar Theorem 7.1.

is obtained by substituting order bounded sets for compact sets in the statement of Most Peculiar Theorem 7.1.[9] Requiring endowments to come from an order bounded set is a weaker requirement than requiring endowments to come from a compact set. Restricting attention to core allocations from an order bounded set is more problematic because it is an endogenous assumption; however, such a restriction is not nearly as objectionable as restricting attention to core allocations from a compact set. Similarly, while decentralization with respect to an order bounded set is a disappointing conclusion, it is far less troublesome than decentralization with respect to a compact set.

Our convergence theorems for Dedekind complete Riesz spaces depend on an assumption on preferences called *equimonotonicity*, a kind of uniform strict monotonicity in a given direction. This assumption is weaker than uniform properness. Indeed, in finite dimensions, the assumption is weaker than monotonicity. The preferences in Manelli's (1991) nonconvergence examples are equimonotone in the sense used here.

Slightly Silly Theorem 8.2 *Fix the commodity space L, a Dedekind complete Riesz space with an order continuous topology.*[10] *Fix an order bounded set B, and a family $\mathcal{P}$ of preferences which is equimonotone on order bounded sets. Consider a finite economy for which endowments lie in B and preferences lie in $\mathcal{P}$. If the number of consumers is sufficiently large, then every core allocation with consumptions lying in B can be approximately decentralized with respect to B.*

With one additional assumption (which is strong but natural in some contexts) — that marginal utilities decrease to zero as consumption increases to infinity — we can obtain "global" approximate decentralization for endowments that lie in an order bounded set:

Theorem 8.8 *If the commodity space is a Dedekind complete Riesz space with an order continuous topology, endowments lie in an order bounded set, and preferences belong to a family that has uniformly vanishing marginal utility at infinity and is equimonotone on order bounded sets, then core allocations*

[9] In fact, we can weaken order boundedness to a kind of "uniform integrability" condition with respect to the order.

[10] The spaces $L^p(\mu)$ $(1 \leq p < \infty)$ and $L^\infty(\mu)$ with the Mackey topology all satisfy this hypothesis.

of sufficiently large finite economies can be approximately decentralized by prices.[11]

As we have argued, compactness is a strong restriction in infinite dimensional spaces because compact sets are typically very thin. There is one important context however in which compact sets may be quite big: if the commodity space is a dual space endowed with the weak star topology, then closed norm balls are compact. One such space is the space of measures (on some compact Hausdorff space of characteristics), which is a natural setting for the study of commodity differentiation; see Mas-Colell (1975), and Jones (1984). Here we obtain two results for large finite economies that are quite reminiscent of the continuum results of Ostroy and Zame (1993). In each of these results, an equimonotonicity assumption rules out monopsony power, and bounds on endowments rule out monopoly power. Together, these two assumptions amount to a kind of "economic thickness." (The first result assumes uniform bounds on marginal rates of substitution; the second result dispenses with these bounds, at the expense of requiring stronger bounds on endowments.)

Theorem 9.4 *If the commodity space is $M(\Omega)$, endowments are restricted to lie in a norm bounded set, and preferences belong to a family that exhibits uniformly bounded marginal rates of substitution and is weak star equimonotone on norm bounded sets, then core allocations of sufficiently large finite economies can be approximately decentralized by prices.*

Theorem 9.5 *If the commodity space is $M(\Omega)$, endowments are restricted to lie in an order bounded set, and preferences belong to a family that is weak star equimonotone on norm bounded sets, then core allocations of sufficiently large finite economies can be approximately decentralized by prices.*

We emphasize that, in these results, we make no assumption about the core allocation and we obtain *global* approximate decentralization by prices

As in the finite dimensional case, our positive results depend on the convexifying effect of large numbers, through two infinite dimensional versions of the Shapley–Folkman theorem, which may be of interest in themselves:

[11] Again, we can relax order boundedness to a kind of "uniform integrability" with respect to the order.

Theorem 5.1 *In a locally convex topological vector space, the mean of a large (finite) number of sets, each lying in a given* compact *set, differs from the convex hull of the mean of the sets by an arbitrarily small amount.*

Theorem 5.2 *In a Dedekind complete Riesz space with an order continuous topology, the mean of a large (finite) number of sets, each lying in a given* order bounded *set, differs from the convex hull of the mean of the sets by an arbitrarily small amount.*

As we have said, we have used "approximate decentralization" to mean the existence of a price for which (a) the average amount by which the allocation fails to be budget feasible is small, and (b) the average saving that can be effected while still improving on the given allocation is small. As in the finite dimensional setting, approximate decentralization in this sense does *not* generally imply the stronger conclusion that core consumptions are close to demands at this price or at any other price (see Anderson and Mas-Colell(1988)). We are able to show that this stronger conclusion holds when the commodity space is $M(\Omega)$, and marginal rates of substitution are bounded, and the preferences satisfy an equiconvexity assumption:

Theorem 10.2 *If the commodity space is $M(\Omega)$, endowments are restricted to lie in a norm bounded set, and preferences belong to a family that exhibits uniformly bounded marginal rates of substitution and is weak star equimonotone and weak star equiconvex on norm bounded sets, then core allocations of sufficiently large finite economies are close to demands.*

We conjecture that this stronger conclusion will hold in other contexts (with appropriate assumptions), but the arguments may not be entirely straightforward since a characteristic of many infinite dimensional commodity spaces is that demand sets (for many non-pathological preferences) may well be empty at many prices. For a detailed analysis and discussion of the connection between approximate decentralization and core consumptions close to demands in the finite dimensional setting, see Anderson (1981).

In none of our positive results do we establish a "rate of convergence." That is, we do not estimate the number of agents required to guarantee decentralization to within a fixed amount ε. Straightforward adaptation of our arguments would probably yield such rates of convergence, but the rates that could be obtained in this way might be very slow. Indeed, in Section

11, we provide an example in the space of measures in which convergence is *arbitrarily slow*.[12] It remains to be seen whether there are natural assumptions which will guarantee a reasonably fast rate of convergence. Because a very slow rate of convergence raises questions about the interpretation of approximate decentralization, we believe that the rate of convergence is an important topic that deserves further exploration.

We believe that the results of this paper raise some questions as to when and whether Walrasian equilibrium, with its implicit assumption of price-taking behavior, is appropriate as a positive equilibrium notion in economies with an infinite dimensional commodity space, even if the number of agents is large. Of course, in addition to its positive significance, Walrasian equilibrium also has normative significance as a benchmark to which other economic outcomes can be compared. Our results also suggest that much work remains to be done to understand the relationship of Walrasian equilibrium to other economic outcomes in the infinite dimensional setting.

The remainder of the paper is organized in the following way. Section 2 discusses lower bounds on the social endowment, the normalization of prices, and the notion of decentralization. Section 3 contains our central nonconvergence examples. Section 4 gives some basic background about Riesz spaces, and states a representation theorem for Riesz spaces due to Abramovich, Aliprantis and Zame (1995). Section 5 contains our two versions of the Shapley–Folkman Theorem. An algebraic result connecting a separation property to decentralization is presented in Section 6. Our approximate decentralization result in the presence of compactness restrictions is in Section 7. Our approximate decentralization results for Riesz spaces are in Section 8; the corresponding results for the space of measures are in Section 9. In Section 10, we use an approximate decentralization result in the

[12]In the finite dimensional context, Anderson (1978) and E. Dierker (1975) show that the number of agents required to guarantee approximate decentralization to within ε is bounded by a constant times $1/\varepsilon$; no smoothness or regularity is required. If one measures the distance of the core to the nearest Walrasian equilibrium in *utility* terms, the situation is more complex. Aumann (1979) gives examples showing that the convergence of the core to the set of Walrasian allocations in the sense of utilities can be arbitrarily slow. If preferences are smooth, Debreu (1975) and H. Dierker (1975) show that the rate of convergence of the core to the set of Walrasian allocations in the sense of utilities is $1/\varepsilon$ *for a generic set of economies* (namely, the regular economies).

space of measures to establish a stronger conclusion — core consumptions are close to demand sets — under strict convexity assumptions. Finally, in Section 11, we provide an example in the space of measures in which convergence (measured in terms of approximate decentralization) is arbitrarily slow.

2 The Notion of Decentralization, the Social Endowment and the Price Normalization

E. Dierker (1975) and Anderson (1978) show that core allocations in finite economies can be approximately decentralized — whether or not the social endowment is strictly positive. Specifically, the following definition establishes a measure of how well a given price decentralizes an allocation.

Definition 2.1 Let $\chi : A \to \mathcal{P} \times L_+$ be an exchange economy, where A is a finite set, L_+ is the nonnegative cone of an ordered vector space, and $\mathcal{P}$ is a set of preferences on L_+. We define the endowment $e(a)$ and preference $\succ_a$ of an agent $a \in A$ by $(\succ_a, e(a)) = \chi(a)$. Suppose $X \subset L_+$. If $f : A \to L_+$ and p is in the vector space dual of L, we define the following measures of the budget, support, and approximate decentralization gaps:

- budget gap

$$\begin{aligned} \rho_B(f,a,p) &= |p \cdot (f(a) - e(a))| \\ \rho_B(f,p) &= \frac{1}{|A|}\sum_{a\in A} \rho_B(f,a,p) \end{aligned}$$

- support gap

$$\begin{aligned} \rho_S(f,a,p,X) &= \max\{0, \sup\{p \cdot (f(a) - x) : x \in X,\ x \succ_a f(a)\}\} \\ \rho_S(f,a,p) &= \rho_S(f,a,p,L_+) \\ \rho_S(f,p,X) &= \frac{1}{|A|}\sum_{a\in A} \rho_S(f,a,p,X) \\ \rho_S(f,a,p) &= \rho_S(f,a,p,L_+) \end{aligned}$$

- approximate decentralization gap

$$\begin{aligned} \rho(f,a,p,X) &= \rho_B(f,a,p) + \rho_S(f,a,p,X) \\ \rho(f,a,p) &= \rho(f,a,p,L_+) \\ \rho(f,p,X) &= \frac{1}{|A|}\sum_{a\in A} \rho(f,a,p,X) \qquad (1) \\ \rho(f,p) &= \rho(f,p,L_+) \end{aligned}$$

ρ_B measures the budget deviation of f at the price p; if we know that $\rho_B(f)$ is small, then we know that the net trades of most agents are almost in the hyperplane perpendicular to p. ρ_S measures how well p serves as a support price to agents' preferences at f; $\rho_S(f, p, X)$ measures how well p serves as a support to agents' preferences *when comparison bundles are restricted to lie in* X. In the definition of $\rho_S(f, a, p, X)$, we take the maximum of the supremum and 0 for two reasons. First, there may not be any $x \in X$ which is preferred to $f(a)$, in which case the supremum would be $-\infty$. Second, if every $x \in X$ with $x \succ_a f(a)$ satisfied, say, $p \cdot x \geq p \cdot f(a) + 1$ (so that p supports the preferred set too well), the supremum would be negative; since we do not want to allow negative values for some agents to offset support failures for other agents, we want $\rho_S(f, a, p, X)$ to be 0 in this case.

E. Dierker (1975) and Anderson (1978) establish, under minimal assumptions on preferences, that when $L_+ = \mathbf{R}^k_+$, any core allocation f satisfies

$$\rho(f, p) \leq \frac{6k \max_{a \in A} \|e(a)\|_\infty}{|A|}$$

for some price p satisfying $\|p\|_1 = 1$.

The theorem just described normalizes prices in a way which, while common in finite-dimensional theory, is inappropriate in infinite-dimensional theory. Specifically, prices are normalized to lie in the standard price simplex $\{p \in \mathbf{R}^k_+ : \|p\|_1 = 1\}$; if the social endowment is not strictly positive, the value of the social endowment could well be 0. Indeed, if the social endowment of good i is 0, the conclusion of the approximate decentralization theorem is trivially satisfied by taking $p = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 occurs in the i-th component. In infinite-dimensional commodity spaces, the corresponding normalization would be to require that the L^1 norm of the price be 1; this makes sense only if the commodity space is L^∞ and the price space is L^1. For this reason, a new normalization is needed. To avoid triviality, it is natural to choose p so that the mean social endowment has value 1, i.e.

$$p \cdot \frac{1}{|A|} \sum_{a \in A} e(a) = 1. \tag{2}$$

Definition 2.2 Let $\chi : A \to \mathcal{P} \times L_+$ be an exchange economy, and $f : A \to L_+$. We say that p ε-decentralizes f with respect to X if p satisfies Equation

(2) and

$$\rho(f,p,X) < \varepsilon \tag{3}$$

We say that p ε-decentralizes f if p ε-decentralizes f with respect to $X = L_+$. We say that f is ε-decentralizable with respect to X if there is a price p which ε-decentralizes f with respect to X. We say that f is ε-decentralizable if it is ε-decentralizable with respect to $X = L_+$.

When we normalize prices by the social endowment, however, approximate decentralization may fail even in finite dimensional spaces.

Example 2.3 We construct a replica sequence χ_n of finite exchange economies (with two goods and one type of agent) and core allocations that cannot be approximately decentralized by prices assigning value 1 to the mean social endowment.

In the nth economy, the set of agents is $A_n = \{1,\ldots,n\}$. For each agent $a \in A_n$, we have

$$e(a) = (1,0), \quad u_a(x) = x^1 x^2.$$

Let $f_n(a) = (1,0)$ for each $a \in A_n$. Clearly, f_n belongs to the core of χ_n.

Suppose p_n ε-decentralizes f_n for $\varepsilon = \frac{1}{2}$. Since the per capita social endowment is $(1,0)$, we must have $p_n = (1,\alpha_n)$ for some $\alpha_n \in \mathbf{R}$. If $\alpha_n < 0$, then $\inf\{p_n \cdot (x - f_n(a)) : x \succ_a f_n(a)\} = -\infty$. Thus, we may assume that $\alpha_n \geq 0$. But then $\rho_S(f_n, a, p_n) = |\inf\{p_n \cdot (x - f_n(a)) : x \succ_a f_n(a)\}| = 1$ for all $a \in A$, so $\rho_S(f_n, p_n) = 1$, which is a contradiction.

The difficulty posed by this example could be ruled out by requiring that the mean social endowment be bounded away from 0.[13] It is not entirely clear how to translate such a requirement on the mean social endowment from the finite dimensional context to the infinite dimensional context. The path we shall follow is to require that the mean social endowment be bounded below by a fixed (strictly) positive vector.

[13]The example is easily modified so that the mean social endowment in each economy is strictly positive, but cannot be modified so that the mean social endowments are bounded away from 0.

3 Examples

We have argued in the Introduction that the finite agent analog of Aumann's theorem (approximate decentralizability by prices) requires that endowments, core allocations, and comparison bundles all lie in small subsets of the commodity space. In this Section we show that these requirements have economic content, and are not simply artifacts of interpretation. In the absence of any one of these requirements, core allocations may fail to be approximately decentralizable by prices. For each of these requirements, we describe a sequence of economies and core allocations for which the requirement fails and show that the core allocations *cannot* be approximately decentralized by prices. We conclude with a fourth example of a sequence of economies with no Walrasian equilibria and core allocations which cannot be approximately decentralized by prices. Every agent in every economy has the *same* preference and endowment; thus, there is no sense in which the example is driven by monopsony or monopoly. Because the preferences and endowments are constant along the sequence, the sequence has a well-defined limit economy with a continuum of agents. In this limit economy, *core equivalence holds because the core and the set of Walrasian equilibria are both empty.*

It is convenient to set all four examples in essentially the same environment. (As we shall see later, they can each be cast in a wide variety of environments.) Let λ denote Lebesgue measure on the unit interval $[0,1]$, and let $L^1 = L^1(\lambda)$ be the space of (equivalence classes of) real-valued integrable functions on $[0,1]$ (as usual, we identify functions which are equal almost everywhere.) Similarly, L^2 denotes the space of square-integrable functions. The first three examples are set in L^1, while the fourth is in L^2. Write $\mathbf{1}$ for the function which is 1 everywhere. For $E \subset [0,1]$ a measurable set, write $\mathbf{1}_E$ for the characteristic function of E; that is, the function which is 1 on E and 0 elsewhere. The first example, adapted from Anderson (1990), shows how monopsony power may lead to the failure of perfect competition. Here, the expression of monopsony power is that core consumptions are not order bounded (although they are norm bounded).

Example 3.1 We describe a sequence of economies. Fix an even integer N; write $A = \{1, 2, \ldots, N\}$ for the set of agents in the N^{th} economy. The endowment of each agent $n \in A$ is $e(n) = \mathbf{1}$. To describe preferences, first define intervals $F_n = [\frac{n-1}{N}, \frac{n}{N})$. (Note that these intervals form a partition of $[0, 1)$ into disjoint subintervals of equal measure $1/N$. Let $\phi_n : [0, 1] \to \mathbf{R}$ be the function which is 2 on F_n and 1 elsewhere; agent n's utility function is

$$u_n(x) = \int_{[0,1]} x(t)\phi_n(t)\, d\lambda(t)$$

Thus, agent n's utility function is linear, with constant marginal utility equal to 2 for commodities in his/her "preferred interval" F_n, and with constant marginal utility equal to 1 for commodities outside this preferred interval. Note that these utility functions are quite well-behaved: norm continuous, strictly monotone, and having marginal rates of substitution bounded above and bounded away from zero (hence uniformly proper).

This economy has a unique Walrasian equilibrium; the equilibrium price is $p = \mathbf{1}$ and the equilibrium allocation gives consumer n the consumption bundle:

$$x(n) = N\mathbf{1}_{F_n}$$

The core of this economy, however, is quite large. In particular, we assert that the allocation f defined by

$$f(n) = \begin{cases} N\mathbf{1}_{F_n} + \frac{1}{8}N\mathbf{1}_{F_{n+1}} & \text{if } n \text{ is odd} \\ \\ \frac{7}{8}N\mathbf{1}_{F_n} & \text{if } n \text{ is even} \end{cases}$$

is in the core.

To see that f is in the core, it is useful to begin by calculating the total utility achievable by a group $B \subset A$ of size $M \leq N$, using its own resources; as we shall see, this total utility depends on the size of B but not on its composition. The total endowment of such a group is $M\mathbf{1}$. The utilitarian allocation of these total resources (that is, the allocation which maximizes the sum of individual utilities) has consumptions:

$$y_M(m) = M\mathbf{1}_{F_m} + \sum_{n \in A \setminus B} \mathbf{1}_{F_n}$$

This yields (identical) individual utility levels of

$$u_M = \frac{2M}{N} + \frac{N-M}{N} = 1 + \frac{M}{N}$$

and total utility to the coalition B of

$$v(B) = M + \frac{M^2}{N}$$

Thus utility achievable by a coalition exhibits strictly increasing returns to scale; as is well-known, this provides a recipe for a big core. (In game-theoretic language: the transferable utility game (A, v) we have just constructed is *strictly convex*, and strictly convex games have big cores.)

Now consider a blocking coalition $B \subset A$; say that B contains k odd consumers and l even consumers, and suppose that $g : B \to L^1$ is a feasible allocation for the coalition B which improves on f. We first analyze total utilities, in order to obtain an estimate for the size and population distribution of B. The allocation f yields utility of $17/8$ for each even consumer and $7/4$ for each even consumer; in order that g be an improvement on f, it must provide each odd member utility exceeding $17/8$, and each even member utility exceeding $7/4$. In particular, the total utility provided by g must exceed the total utility provided to B by f. The calculation above shows that the total utility available to members of B, using only the resources of B, is $v(B) = (k+l)^2/N + (k+l)$. Since B can block, we must have

$$\frac{(k+l)^2}{N} + (k+l) \ > \ \frac{17}{8}k + \frac{7}{4}l \ \geq \ \frac{7}{4}(k+l)$$

Simplifying the extremes of this inequality implies that $k + l > 3N/4$. As we have already noted, the total utility available to a coalition depends on its size but not on its composition; since f gives odd consumers greater utility than even consumers, a new blocking coalition could be constructed from B by replacing odd consumers with even consumers (to the limit of the number of even consumers). Thus there is no loss in assuming that $l = N/2$ and $k > N/4$ (so that all the even consumers and more than half the odd consumers belong to B). Write B_o for the set of odd consumers in B, and B_e for the set of even consumers. We now analyze actual consumptions. For each $b \in B$, write

$$c(b) = \int g(b)(t)\, d\lambda(t)$$

(the mean consumption of agent b). Since g is an improvement on f, it must provide each even agent in B utility exceeding $7/4$, and hence must provide each even agent mean consumption exceeding 7/8. Since B_e contains all $N/2$ even agents, it follows that

$$\sum_{b \in B_e} c(b) > \frac{7N}{16}$$

On the other hand, the total endowment of members of B is just $(k + N/2)\mathbf{1}$, so the sum (over all agents in B) of the mean endowments is $\mathbf{1}$. Because g is feasible for B, it follows that

$$\sum_{b \in B_o} c(b) < k + \frac{N}{2} - \frac{7N}{16} = k + \frac{N}{16}$$

Hence there is at least one consumer $a \in B_o$ for whom

$$c(a) < \frac{k + \frac{N}{16}}{k} = 1 + \frac{N}{16k}$$

Recall that the total endowment of the coalition B is just $(k + N/2)\mathbf{1}$; thus the utility of consumer a is at most

$$\frac{2(k + \frac{N}{2})}{N} + 1 + \frac{N}{16k} - \frac{k + \frac{N}{2}}{N} = \frac{k}{N} + \frac{N}{16k} + \frac{3}{2}$$

Since $f(a)$ yields consumer a the utility 17/8, and $g(a)$ yields higher utility, it follows that

$$\frac{k}{N} + \frac{N}{16k} + \frac{3}{2} > \frac{17}{8}$$

or equivalently that

$$\frac{k}{N} + \frac{N}{16k} > \frac{5}{8} \tag{4}$$

If we view k as a real variable, and differentiate, we find that

$$\frac{\mathrm{d}}{\mathrm{dk}}\left[\frac{k}{N} + \frac{N}{16k}\right] = \frac{1}{N} - \frac{N}{16k^2}$$

The right hand side is non-negative for k in the interval $[N/4, N/2]$, so the maximum value of $k/N + N/16k$ is obtained for $k = N/2$, where $k/N + N/16k = 5/8$. It follows that equation(4) has no solution for $N/4 \leq k \leq N/2$, which is a contradiction. Hence there are no blocking coalitions, and f is in the core, as asserted.

As we have already noted, f yields utility 17/8 for odd consumers and 7/4 for even consumers, independently of the size of the economy. Thus the core *does not shrink* to the unique Walrasian allocation as $N \to \infty$. More to the point (for our purposes), we claim that there is no price p for which the mean social endowment has value 1 and which 1/100-decentralizes the core allocation f.

To see this, note first that all individual endowments, and hence the mean social endowment, is $\mathbf{1}$. Let p be a linear functional on L^1 for which $p \cdot \mathbf{1} = 1$, and suppose that p 1/100-decentralizes f. Since all consumers have the same endowment, they all have wealth 1.

We claim first of all that p is a positive linear functional. For if not, there would be a positive element $x \in L^1_+$ such that $p \cdot x \leq 0$. Since marginal utilities for all consumers lie between 1 and 2, the consumption bundle $3x/\|x\|$ would be preferred to the core consumption f by every consumer, and would have a negative or zero cost, a contradiction since p 1/100 decentralizes and wealth is 1. We conclude that p is a positive linear functional, and hence continuous (because every positive linear functional on a Banach lattice is continuous (Aliprantis and Burkinshaw (1985))). Since the dual space of L^1 is L^∞, we may identify p with a positive function on $[0, 1]$.

Since the price p 1/100-decentralizes f, and the number of even and odd consumers is each precisely 1/2 the total number of consumers (recall that N is even), it follows that there is at least one odd integer k such that the odd consumer k and the even consumer $k+1$ are each within 1/25 of optimization in their budget sets. That is

- $p \cdot f(k) < 1 + \frac{1}{25}$ and
- $\not\exists y$ such that $u_k(y) > u_k(f(k))$ and $p \cdot y < 1 - \frac{1}{25}$

and similarly

- $p \cdot f(k+1) < 1 + \frac{1}{25}$ and
- $\not\exists y$ such that $u_{k+1}(y) > u_{k+1}(f(k+1))$ and $p \cdot y < 1 - \frac{1}{25}$

By assumption, $p \cdot f(k) < 1 + 1/25$. If $p \cdot f(k) < 1 - 1/25$, strict monotonicity would provide a consumption bundle preferred to $f(k)$ and costing less than $1 - 1/25$, a contradiction. Hence

$$1 - \frac{1}{25} \leq p \cdot f(k) < 1 + \frac{1}{25}$$

Similarly,

$$1 - \frac{1}{25} \leq p \cdot f(k+1) < 1 + \frac{1}{25}$$

Recall that $f(k) = N\mathbf{1}_{F_k} + (1/8)N\mathbf{1}_{F_{k+1}}$ and that $f(k+1) = (7/8)N\mathbf{1}_{F_{k+1}}$; substituting into the previous equations and combining yields

$$1 - \frac{1}{25} - \frac{1 + \frac{1}{25}}{7} \leq p \cdot N\mathbf{1}_{F_k} \leq 1 + \frac{1}{25} - \frac{1 - \frac{1}{25}}{7}$$

On the other hand, the commodity bundle $(17/16)N\mathbf{1}_{F_k}$ yields consumer k the same utility as $f(k)$ (because the marginal utility of consumer k is identically 2 on F_k and identically 1 elsewhere); hence strict monotonicity of preferences implies

$$p \cdot \frac{17}{16} N\mathbf{1}_{F_k} \geq 1 - \frac{1}{25}$$

Combining the last two equations yields

$$\frac{17}{16}\left(1 + \frac{1}{25} - \frac{1 + \frac{1}{25}}{7}\right) \geq 1 - \frac{1}{25}$$

Simplifying yields

$$\frac{10}{231} < \frac{1}{25}$$

which is absurd. We conclude that f cannot be 1/100-decentralized by any price, as asserted.

As a final note, it is useful to view the economy as if it had transferable utility, and calculate marginal contributions and individual surplus in the sense of Ostroy (1980). At the Walrasian allocation, each consumer obtains utility 2. However, each consumer contributes to society the difference between the total utility society could obtain with his presence and the total utility society could obtain in his absence. According to our previous calculations, this means that each individual's contribution to society is

$$2N - \left[\frac{(N-1)^2}{N} + (N-1)\right] = 3 - \frac{1}{N}$$

Hence, each consumer's marginal contribution to society exceeds his utility at the Walrasian allocation by $1 - 1/N$, and this difference remains bounded away from 0 as $N \to \infty$. Therefore individuals do not (approximately) extract their marginal contributions to society, and the economy fails what Ostroy calls the *asymptotic no-surplus test* of perfect competition.

Our second example, a variation on a theme of Ostroy and Zame (1993), shows how monopoly power may again lead to the failure of perfect competition. Here, the expression of monopoly power is that endowments are not order bounded (although they are norm bounded). As we will see, these two examples are essentially dual to one another, reflecting the essential duality of monopoly and monopsony.

Example 3.2 We again construct a sequence of economies. Fix an even integer N; write $A = \{1, 2, \ldots, N\}$ for the set of agents in the N^{th} economy. The endowment of consumer $n \in A_N$ is

$$e(n) = N\mathbf{1}_{F_n}$$

where $F_n = [(n-1)/N, n/N)$, as before. To define preferences, define a felicity function $U : \mathbf{R}^+ \to \mathbf{R}^+$by[14]

$$U(t) = \begin{cases} 2t & \text{if } 0 \leq t \leq 1 \\ t+1 & \text{if } 1 \leq t < \infty \end{cases}$$

Now define the common utility function u of all consumers by

$$u(x) = \int u(x(t))\, d\lambda(t)$$

[14]The particular form of this felicity function is chosen simply for computational convenience; the non-differentiability at 1 can easily be smoothed without affecting the qualitative behavior of the example. Alternatively, any strictly concave felicity function would serve.

Thus, consumer n has a monopoly on commodities in the interval F_n, and all consumers prefer to spread out consumption on the interval $[0,1)$. Again, this economy has a unique Walrasian equilibrium; the equilibrium price is $p = \mathbf{1}$ and the equilibrium allocation gives each consumer n the identical consumption bundle $x = \mathbf{1}$. And again, the core of this economy is quite large. In particular, we assert that the allocation f defined by:

$$f(n) = \begin{cases} \frac{9}{8}\mathbf{1} & \text{if } n \text{ is odd} \\ \\ \frac{7}{8}\mathbf{1} & \text{if } n \text{ is even} \end{cases}$$

is in the core.

To see this, we begin once again by computing the total utility that can be obtained by a coalition $B \subset A$ having M members. The total endowment of such a coalition is

$$\sum_{b \in B} N\mathbf{1}_{F_b}$$

The utilitarian allocation gives each member of B the identical consumption

$$\frac{N}{M}\sum_{b \in B} \mathbf{1}_{F_b}$$

and the identical utility

$$2\frac{M}{N} + \left(\frac{N}{M} - 1\right)\left(\frac{M}{N}\right) = \frac{M}{N} + 1$$

Thus the total utility that can be obtained by B is

$$v(B) = \frac{M^2}{N} + M$$

which is precisely the total utility we computed for such a coalition in Example 3.1.

Now consider a blocking coalition $B \subset A$; say that B contains k odd consumers and l even consumers, and suppose that $g : B \to L^1$ is a feasible allocation for the coalition B which improves on f. Since the utilities at the core allocation f and the total utility obtainable by B are the same as in Example 3.1, our previous calculation applies to show that $l = N/2$ and $k > N/4$. To complete the argument that f is in the core, we analyze actual consumptions; the argument is only very slightly different than in Example 3.1. Write B_o for the set of odd members of B and B_e for the set of even members. For each $b \in B$, write

$$c(b) = \int g(b)(t)\, d\lambda(t)$$

(the mean consumption of agent b). Since g is an improvement on f, it must provide each even agent in B utility exceeding $7/4$, and hence must provide each even agent mean consumption exceeding $7/8$. Since B_e contains all $N/2$ even agents, it follows that

$$\sum_{b \in B_e} c(b) > \frac{7N}{16}$$

On the other hand, the total endowment of members of B is just $(k + N/2)\mathbf{1}$, so the sum (over all agents in B) of the mean endowments is 1. Because g is feasible for B, it follows that

$$\sum_{b \in B_o} c(b) < k + \frac{N}{2} - \frac{7N}{16} = k + \frac{N}{16}$$

Hence there is at least one consumer $a \in B_o$ for whom

$$c(a) < \frac{k + \frac{N}{16}}{k} = 1 + \frac{N}{16k}$$

Recall that the total endowment of the coalition B is just $\sum_{b \in B} N\mathbf{1}_{F_b}$; thus the utility of consumer a is at most

$$2\left(k + \frac{N}{2}\right) + 1 + \frac{N}{16k} - \frac{k + \frac{N}{2}}{N} \;=\; \frac{k}{N} + \frac{N}{16k} + \frac{3}{2}$$

Since $f(a)$ yields consumer a the utility $17/8$, and $g(a)$ yields higher utility, it follows that

$$\frac{k}{N} + \frac{N}{16k} + \frac{3}{2} > \frac{17}{8}$$

or equivalently that

$$\frac{k}{N} + \frac{N}{16k} > \frac{5}{8}$$

As we know from Example 3.1, this equation has no solution for $N/4 \leq k \leq N/2$. Hence there are no blocking coalitions, and f is in the core, as asserted.

As we have already noted, f yields utility 17/8 for odd consumers and 7/4 for even consumers, independent of the size of the economy. Thus the core *does not shrink* to the unique Walrasian allocation as $N \to \infty$. We claim that the allocation f cannot be 1/504-decentralized by any price. To see this, let p be a linear functional on L^1 for which $p \cdot \mathbf{1} = 1$, and suppose that p 1/504-decentralizes f. Just as in Example 3.1, the price functional p is necessarily positive, hence continuous,and may be identified with a positive function on $[0, 1]$. Since the price p 1/504-decentralizes f, and the number of even and odd consumers is each precisely 1/2 the total number of consumers (recall that N is even), it follows that more than 1/2 of all consumers are within 1/126 of optimization in their budget sets; in particular, there is some odd consumer k and some even consumer l for which

- $\frac{9}{8} = p \cdot f(k) < p \cdot e(k) + \frac{1}{126}$ and
- $\nexists y$ such that $u(y) > u(f(k))$ and $p \cdot y < p \cdot e(k) - \frac{1}{126}$

and

- $\frac{7}{8} = p \cdot f(l) < p \cdot e(l) + \frac{1}{126}$ and
- $\nexists y$ such that $u(y) > u(f(l))$ and $p \cdot y < p \cdot e(l) - \frac{1}{126}$

The bundle $\mathbf{1} + (N/8)\mathbf{1}_{F_l}$ yields consumer k the same utility as $f(k) = (9/8)\mathbf{1}$, so approximate optimization in the budget set and strict monotonicity for consumer k guarantee that

$$p \cdot \left[\mathbf{1} + \frac{N}{8}\mathbf{1}_{F_l}\right] > p \cdot e(k) - \frac{1}{126} > \frac{9}{8} - \frac{2}{126}$$

Hence

$$p \cdot e(l) = p \cdot N\mathbf{1}_{F_l} > 1 - \frac{16}{126}$$

On the other hand, approximate optimization in the budget set for consumer l guarantees that

$$p \cdot e(l) < p \cdot f(l) + \frac{1}{126} = \frac{7}{8} + \frac{1}{126}$$

Combining the last two inequalities and simplifying yields

$$\frac{1}{8} > \frac{17}{126} = \frac{1}{8}$$

which is absurd. We conclude that f cannot be 1/504-decentralized by any price, as asserted.

Finally, note that the calculation of marginal contributions and individual surplus gives the same results as in Example 3.1: At the Walrasian allocation, each consumer obtains utility 2, but each individual's contribution to society is $3 - 1/N$. Hence, each consumer's marginal contribution to society exceeds his utility at the Walrasian allocation by $1 - 1/N$, so once again individuals do not(approximately) extract their marginal contributions to society, and the economy fails the asymptotic no-surplus test of perfect competition.

In each of the preceding examples, we have constructed a core allocation that cannot be approximately decentralized by prices. In each case, the distance in consumption from the core allocation to the Walrasian allocation is quite large(the agents who do worse lose 1/8 of their consumption) but the failure of approximate decentralization by prices is much smaller (the relative error being only 1/100 in the first example, 1/504 in the second). However, this apparent disparity reflects only the difference in the metric; the same phenomenon occurs in the finite dimensional context. Consider, for example, an N-consumer economy in $\mathbf{R}^2$, and assume that preferences are strictly convex and that individual endowments are bounded by the vector $(1, 1)$. Anderson (1987) shows that any core allocation can be $1/N^2$-decentralized, but the mean distance (in consumption) from a core allocation to a Walrasian allocation is generically $1/N$ (Debreu(1975)).

The third example has a more technical — and perhaps less economic — interpretation. As we have shown, failure of approximate decentralizability by prices may occur when core consumptions fail to be order bounded (monopsony) or when endowments fail to be order bounded (monopoly). As the following example shows, failure of approximate decentralizability by prices may occur even when endowments are order bounded and core consumptions are order bounded, but the relevant comparison bundles are not order bounded.

Example 3.3 Again, we describe a sequence of economies. The N^{th} economy is composed of N identical agents, with endowments $e = \mathbf{1}$. To define utility functions, let $U : \mathbf{R}_+ \to \mathbf{R}_+$ be any smooth function such that U' is continuous, nonnegative and satisfies

(a) $1 \leq U'(x) \leq 2$ for $0 \leq x \leq 2N$

(b) $U'(x) \geq 5$ for $3N \leq x \leq 6N$

Now let utility functions be defined by

$$u(x) = \int_{[0,1]} U(x(t))\, d\lambda(t)$$

The endowment is the unique core allocation (indeed, the unique Pareto optimal allocation); the crucial point is that the *feasible* consumptions are bounded by $N\mathbf{1}$. However, this allocation cannot be approximately decentralized by prices.

To see this, let p be any price with $p \cdot \mathbf{1} = 1$. We can find an interval $E \subset [0,1]$ of length $1/8N$ for which $\int_E p\, d\lambda(t) \leq 1/8N$. Then $p \cdot (6N\mathbf{1}_E) \leq 3/4$ and $u(6N\mathbf{1}_E) \geq 2\frac{1}{8} > 2 \geq u(e)$; that is, $6N\mathbf{1}_E$ is preferred (by every consumer) to the core allocation, and costs much less, a contradiction.

We have cast each of these examples in $L^1(\lambda)$, but they may easily be recast in other commodity spaces. Indeed, since all the allocations in question are bounded (although not uniformly bounded), each of these examples may be interpreted in $L^p(\lambda)$ for any p with $1 \leq p \leq \infty$. In each case, it is the failure of the relevant consumption bundles to be order bounded that leads to the failure of perfect competition.

Alternatively, since we may view $L^1(\lambda)$ as a subspace of the space of measures $M[0,1]$, we may also interpret these examples in that commodity space.[15] Looking ahead to Section 9, in $M[0,1]$ we note that the failure of perfect competition in these examples may be traced, not to the unbounded-

[15] A little care must be taken in the interpretation of the preferences in Example 3.1; see Ostroy and Zame (1993).

ness of endowments, core consumptions, or comparison bundles, but rather to the failure of preferences to be weak star equimonotone on bounded sets.[16]

The final example constructs a sequence of economies with core allocations that cannot be approximately decentralized by prices, and with no Walrasian equilibria; these features result from increasing returns to specialization in consumption. The economies in the sequence also have the property that the preferences and endowments are the same for all agents, and are independent of which economy in the sequence is chosen. As a consequence, the sequence has a well-defined continuum limit economy. Core equivalence holds in this continuum limit economy — but it holds vacuously: both the core and the set of Walrasian equilibria are empty. The disconnection between the continuum limit and the large finite economies arises because the core allocations of the finite economies are becoming very spiky; as a consequence, there is no corresponding limit allocation in the continuum economy. This example also demonstrates that it will be extremely hard to topologize the space of economies in such a way that core convergence results in large finite economies can be deduced from properties of economies with a continuum of agents, as in Hildenbrand (1974). Every agent in the finite economies has the *same* endowment and preference as every agent in the continuum limit economy; if *any* sequence of economies converges, surely this one must. However, we see that the core is *not* upper hemicontinuous along this sequence. In the finite economies, the cores are nonempty; in the continuum economy, the core is empty.[17]

Example 3.4 Again, we construct a sequence of economies. The commodity space is $L^2(\lambda)$. Fix an integer N and write $A = \{1, \ldots, N\}$ for the set of agents in the N^{th} economy. Each agent $n \in A$ has endowment $\mathbf{1}$ and utility function

$$u(X) = \int_{[0,1]} X(t)^2 d\lambda$$

[16] In fact the preferences we have given are not weak star continuous, but this can be remedied; the failure of weak star equimonotonicity is irremediable.

[17] Another example showing the difficulty of following the topological approach is given in Anderson (1990). Two sequences of economies χ_N and χ'_N are considered. χ_N is the same sequence of economies which appears in Example 3.1. χ'_N is the same as χ_N, except that it contains N^2 agents, with N clones of each of the N types of agents present in χ_N. The core of χ'_N can be approximately decentralized by prices, while the core of χ_N cannot. Note that χ_N and χ'_N generate exactly the same distribution of agents' characteristics.

Let

$$f(n) = N\mathbf{1}_{F_n}$$

where as before $F_N = \left[\frac{n-1}{N}, \frac{n}{N}\right)$. We claim that f is in the core of the N^{th} economy. Note first that $u(f(n)) = N^2/N = N$. Given a coalition S with M agents, the maximum aggregate utility it can achieve for its members is M^2, while to block it must achieve aggregate utility strictly exceeding $MN \geq M^2$, which is impossible; thus, f is in the core. Now suppose p approximately decentralizes f. Since $p \cdot \bar{e} = 1$, we can find a measurable set E with $\lambda(E) = \frac{1}{4N}$ such that $\int_E p d\lambda \leq \frac{1}{4N}$. Let $X = 3N\mathbf{1}_E$. Then $p \cdot x \leq \frac{3}{4}$, while $u(X) = \frac{9N}{4}$; in other words, X yields more than twice as much utility as $f(n)$ and costs at most three-quarters as much as the endowment.

Since every agent has the same preference and endowment, which is independent of N, the sequence has a well-defined continuum limit economy with $A = [0,1]$, $e(a) = \mathbf{1}$ and $u(X) = \int X(t)^2 d\lambda$ for all a. We claim that the core of this economy is empty. For if $f : A \to L^2$ is an allocation, then $u(f(a))$ is finite for λ-almost all a. Find α such that if $A^* = \{a : u(f(a)) \leq \alpha\}$, then $\lambda(A^*) = \beta > 0$. Let $T : A^* \to [0,1]$ be defined by $T(a) = \frac{\lambda(A^* \cap [0,a])}{\beta}$. Let $\delta = \frac{1}{2\alpha}$ and define $g : A^* \to L^2$ by

$$g(a) = \frac{1}{\delta}\mathbf{1}_{[T(a), T(a)+\delta)}$$

where the interval is interpreted modulo 1. Then $\int_{A^*} g(a) d\lambda = \beta\mathbf{1} = \int_{A^*} e(a)$, so g is feasible for A^*. $u(g(a)) = \frac{\delta}{\delta^2} = \frac{1}{\delta} = 2\alpha > u(f(a))$, which shows that A^* blocks f via g. Since the core is empty, the economy also has no Walrasian equilibria, so core equivalence is vacuously satisfied.

4 Riesz Spaces

We collect here some necessary facts about Riesz spaces, including a recent representation theorem of Abramowicz, Aliprantis and Zame (1995). The books of Aliprantis and Burkinshaw (1978, 1985) provide excellent general references.

A *Riesz space* (or *vector lattice*) is an ordered vector space L which is a lattice in its ordering; that is, every pair $x, y \in L$ of elements has a supremum (least upper bound) $x \vee y$ and an infimum (greatest lower bound) $x \wedge y$. We write $x^+ = x \vee 0$ for the *positive part* of x, $x^- = x \wedge 0$ for the *negative part* of x, and $|x| = x^+ + x^-$ for the *absolute value* of x. We denote by $[x, y]$ the *order interval* $\{z : x \leq z \leq y\}$. A set is *order bounded* if it is contained in some order interval. We write L_+ for the positive cone of L (the set of non-negative elements). The Riesz space L is *Dedekind complete* if every subset $A \subset L$ which has an upper bound has a least upper bound.

We write $x_\alpha \uparrow x$ (respectively, $x_\alpha \downarrow x$) to mean that $\{x_\alpha\}$ is an increasing (decreasing) net in L with supremum (infimum) x.

An *order ideal* is a vector subspace $K \subset L$ with the property that $y \in K$ whenever $y \in L$ and there is an $x \in K$ with $|y| \leq |x|$. If $x \in L_+$ is a positive element, then the *principal order ideal* L_x generated by x is the smallest order ideal containing x:

$$L_x = \{y \in L : \exists M \ |y| \leq Mx\}$$

An order ideal K is a *band* if $x \in L, \{x_\alpha\} \subset K$ and $x_\alpha \uparrow x$ imply $x \in K$. A positive element $x \in L$ is a *weak order unit* if the smallest band containing x is L itself.

A linear functional $f : L \to \mathbf{R}$ is *positive* if $f(x) \geq 0$ whenever $x \geq 0$ and *strictly positive* if $f(x) > 0$ whenever $x > 0$ (i.e., $x \geq 0, x \neq 0$). A positive linear functional f is *order continuous* if $f(x_\alpha) \to 0$ whenever $x_\alpha \downarrow 0$.

A subset $E \subset L$ is *solid* if $|x| \in E$ whenever $x \in E$. A *locally-convex solid topology* τ on L is a linear topology which has a neighborhood base at 0 consisting of convex, solid sets. The locally-convex solid topology τ is *order continuous* if $x_\alpha \to 0$ in the topology τ whenever $x_\alpha \downarrow x$. If τ is a locally-convex solid topology on L then the dual space $(L, \tau)'$ (the space of

τ-continuous linear functionals) is itself a Riesz space in the natural ordering: $f \geq g$ if $f(x) \geq g(x)$ for all $x \in L_+$. As usual, we frequently suppress explicit mention of the topology and write L' rather than $(L, \tau)'$.

If τ is a locally convex solid topology on L, a vector $x \in L$ is *strictly positive* if $f(x) > 0$ whenever $f \in (L, \tau)'$, f is positive, and $f \neq 0$ (i.e., v acts as a strictly positive functional on $(L, \tau)'$). Strictly positive vectors are always weak order units, but a weak order unit need not be strictly positive.

Our Riesz space results depend on a representation theorem of Abramowicz, Aliprantis and Zame (1995), which has several parts; the following collects the assumptions required to obtain the parts we desire.

Assumption 4.1 L is a Riesz space and τ is a Hausdorff locally convex solid, order continuous topology on L such that

- L is Dedekind complete
- L has a weak order unit $x \in L$
- the dual space $(L, \tau)'$ contains a strictly positive order continuous linear functional

Assumption 4.1 holds for every separable Banach lattice with order continuous norm, including (for any probability measure μ) each of the Banach lattices $L = L^p(\mu)$, $(1 \leq p < \infty)$ as well as $L^\infty(\mu)$, equipped with the Mackey topology. If $L = L^p(\mu)$ for $1 \leq p \leq \infty$, $\mathbf{1}$ denotes the element of L which is 1 almost surely.

The representation theorem we require is:

Theorem 4.2 (Abramovich, Aliprantis and Zame) *Let L be a Riesz space satisfying Assumption 4.1, let $v \in L$ be a weak order unit, and let $\varphi \in L'$ be a strictly positive order continuous linear functional. Then there is a countably additive probability space $(\Omega, \mathcal{F}, \mu)$ and a continuous mapping $R : L \to L^1(\mu)$ such that*

(i) R is a vector lattice isomorphism onto its range $R(L)$

(ii) $R(v) = \mathbf{1}$ *and* $R(L_v) = L^\infty(\mu)$

(iii) the range $R(L)$ *is a dense order ideal in* $L^1(\mu)$

(iv) for each positive real number $M > 0$, *the restriction of* R *to the order interval* $[-Mv, Mv]$ *is a homeomorphism onto* $R([-Mv, Mv]) = [-M\mathbf{1}, M\mathbf{1}]$

5 The Convexifying Effect of Large Numbers

In this Section we establish two analogs of the Shapley–Folkman Theorem for infinite dimensional spaces. In both cases we show that the mean of a sufficiently large finite number of sets differs from its convex hull by an arbitrarily small amount, provided that the sets in question are subsets of a given "sufficiently small" set. Our first result applies to the setting of locally convex topological vector spaces; in this setting, "sufficiently small" means "compact."

Theorem 5.1 *Let L be a locally convex topological vector space, and X a compact subset of L. For every neighborhood U of 0 in L, there exists an integer n_0 such that if $A_1, \ldots, A_n \subset X$ and $n \geq n_0$, then*

$$\text{con}\left(\frac{1}{n}\sum_{i=1}^{n} A_i\right) \subset \left(\frac{1}{n}\sum_{i=1}^{n} A_i\right) + U. \tag{5}$$

Proof: Let U be a neighborhood of 0. Since L is locally convex, we can find a convex open set V with $0 \in 4V \subset U$. The collection $\{x + V : x \in X\}$ is an open cover of X. Since X is compact, there is a finite subcover $\{x_i + V : 1 \leq i \leq M\}$. Fix a function $\phi : X \to \{x_1, \ldots, x_M\}$ such that $\phi(x) = x_i \Rightarrow x \in x_i + V$. Let span$X$ denote the linear span of $\{x_1, \ldots, x_M\}$, and let k be the dimension of span$X$. Since scalar multiplication is continuous, the collection $\{\frac{n}{k}V : n \in \mathbf{N}\}$ is an open cover of X, so there exists n_0 such that $X \subset \frac{n_0}{k}V$. If $s_1, \ldots, s_m \in X$ and $m \leq k$, and $n \geq n_0$,

$$\sum_{i=1}^{m} s_i \in m\left(\frac{n_0}{k}V\right) \subset nV, \tag{6}$$

since V is convex.

Suppose $x \in \text{con}\,(A_1 + \cdots A_n)$, with $n \geq n_0$. Then $x = x_1 + \cdots + x_n$, where $x_i \in \text{con}\,A_i$. Therefore,

$$x_i = \sum_{j=1}^{m_i} \lambda_{ij} x_{ij}, \tag{7}$$

where $x_{ij} \in A_i$, $\lambda_{ij} > 0$, and $\sum_j \lambda_{ij} = 1$ for each i. Let

$$y_{ij} = \phi(x_{ij}),\ y_i = \sum_{j=1}^{m_i} \lambda_{ij} y_{ij},\ y = \sum_{i=1}^{n} y_i,\ B_i = \{y_{ij} : j = 1, \ldots, m_i\}. \tag{8}$$

$$x - y = \underbrace{\sum_{i=1}^{n} \underbrace{\sum_{j=1}^{m_i} \lambda_{ij} \underbrace{(x_{ij} - y_{ij})}_{\in V}}_{\in V}}_{\in nV}, \tag{9}$$

since V is convex. By the usual finite-dimensional Shapley-Folkman Theorem, we can find $\hat{y}_i \in \text{con } B_i$ such that $y = \sum_i \hat{y}_i$, $\hat{y}_i \in B_i$ for $i \in S$ and $|S| \geq n - k$. Let $\hat{y} = \sum_{i \in S} \hat{y}_i$.

$$y - \hat{y} = \sum_{i \notin S} \hat{y}_i \in nV, \tag{10}$$

by Equation (6).

For $i \in S$, $\hat{y}_i \in B_i$, so $\hat{y}_i = y_{ij}$ for some j; let $\hat{x}_i = x_{ij}$. For $i \notin S$, choose $\hat{x}_i \in A_i$ arbitrarily. Let $\hat{x} = \sum_{i=1}^{n} \hat{x}_i$. Observe that $\hat{x} \in A_1 + \cdots A_n$.

$$\hat{y} - \hat{x} = \underbrace{\underbrace{\sum_{i \in S} \underbrace{\hat{y}_i - \hat{x}_i}_{\in V}}_{\in nV} - \underbrace{\sum_{i \notin S} \hat{x}_i}_{\in nV \text{ by Equation (6)}}}_{\in 2nV} \tag{11}$$

Therefore,

$$x - \hat{x} = (x - y) + (y - \hat{y}) + (\hat{y} - \hat{x}) \in 4nV \subset nU, \tag{12}$$

so

$$\frac{x}{n} \in \left(\frac{1}{n} \sum_{i=1}^{n} A_i \right) + U. \tag{13}$$

■

Our second result shows that, in a wide variety of interesting commodity spaces, we may weaken the compactness assumption of Theorem 5.1 to order boundedness. It makes use of the representation Theorem 4.2, and hence requires Assumption 4.1.

Theorem 5.2 *Let L be a Riesz space and τ a Hausdorff locally-convex solid topology on L, satisfying Assumption 4.1. Let $X \subset L$ be an order bounded set. For every neighborhood U of 0 in L, there exists an integer n_0 such that if $A_1, \ldots, A_n \subset X$ and $n \geq n_0$, then*

$$\text{con}\left(\frac{1}{n}\sum_{i=1}^{n} A_i\right) \subset \left(\frac{1}{n}\sum_{i=1}^{n} A_i\right) + U. \tag{14}$$

Proof: Since X is order bounded, there is a positive vector $x \in L$ such that $X \subset \left[-\frac{x}{2}, \frac{x}{2}\right]$. By translation, we can assume without loss that $X \subset [0, x]$. Let $y \in L$ be a weak order unit, and set $v = x + y$; since x is positive, the band generated by v contains y, and so contains the band generated by y, so v is also a weak order unit. Choose a probability space (Ω, μ) and a representation $R : L \to L^1(\mu)$ having the properties given in Theorem 4.2; in particular, $R(v) = \mathbf{1}$, so $R(X) \subset [0, \mathbf{1}]$.

Since μ is a probability measure, every bounded measurable function on Ω is integrable and square integrable; in particular, the order interval $[0, \mathbf{1}]$ is contained in $L^2(\mu)$. Moreover, if $h \in [0, \mathbf{1}]$ then $\|h\|_2 \leq 1$. In particular, the diameter (computed in $L^2(\mu)$) of any subset of the order interval $[0, \mathbf{1}]$ is at most 1. Cassels (1975) asserts that there is a constant K such that, if $B_1, \ldots, B_n \subset [0, \mathbf{1}]$ then

$$\text{dist}_2\left(\sum_{i=1}^{n} B_i, \text{con}\left(\sum_{i=1}^{n} B_i\right)\right) \leq K^{1/2} n^{1/2}$$

where dist_2 is the Hausdorff distance, computed with respect to the L^2 norm. Dividing by n yields

$$\text{dist}_2\left(\frac{1}{n}\sum_{i=1}^{n} B_i, \text{con}\left(\frac{1}{n}\sum_{i=1}^{n} B_i\right)\right) \leq K^{1/2} n^{-1/2}$$

Equivalently,

$$\text{con}\left(\frac{1}{n}\sum_{i=1}^{n} B_i\right) \quad \subset \quad \left(\frac{1}{n}\sum_{i=1}^{n} B_i \; + \; B^2(0, K^{1/2} n^{-1/2})\right) \tag{15}$$

where $B^2(0, r)$ is the L^2 ball of center 0, radius r.

We now prove the desired assertion. Let U be a neighborhood of 0 in L. Since the restriction of R to each order interval is a homeomorphism, we can find a neighborhood V^1 of 0 in L^1 such that $R(U \cap [-v, +v]) = V^1 \cap [-\mathbf{1}, +\mathbf{1}]$. As a simple computation shows, the fact that μ is a finite measure implies that the L^1 norm and the L^2 norm define equivalent topologies on the order interval $[-\mathbf{1}, +\mathbf{1}]$. Hence there is an L^2 neighborhood V^2 of 0 such that

$$V^2 \cap [-\mathbf{1}, +\mathbf{1}] \subset V^1 \cap [-\mathbf{1}, +\mathbf{1}]$$

There is no loss of generality in supposing that V^2 is a ball, say $V^2 = B^2(0, r)$. Set $n_0 = K/r^2$. Suppose that $n \geq n_0$ and that $A_1, \ldots, A_n \subset X$. For each i, write $B_i = R(A_i)$. Keeping in mind that $X \subset [0, v]$, and hence that $B_i \subset [0, \mathbf{1}]$ for each i, we may apply equation (15) and obtain

$$\begin{aligned} \operatorname{con}\left(\frac{1}{n}\sum_{i=1}^{n} B_i\right) &\subset \frac{1}{n}\sum_{i=1}^{n} B_i + \left(B^2(0, K^{1/2}n^{-1/2}) \cap [-\mathbf{1}, \mathbf{1}]\right) \\ &\subset \frac{1}{n}\sum_{i=1}^{n} B_i + \left(V^1 \cap [-\mathbf{1}, \mathbf{1}]\right) \end{aligned}$$

Applying the inverse of R gives

$$\begin{aligned} \operatorname{con}\left(\frac{1}{n}\sum_{i=1}^{n} A_i\right) &\subset \frac{1}{n}\sum_{i=1}^{n} A_i + (U \cap [-v, +v]) \\ &\subset \frac{1}{n}\sum_{i=1}^{n} A_i + U \end{aligned}$$

which is the desired result. ■

6 Separation and Decentralization

In this section, we isolate the basic algebraic result relating a separation property to approximate decentralization. This result will be used in establishing all of our convergence theorems. The essential idea of the argument is contained in the proof of a finite-dimensional core decentralization result in Anderson (1978); since our context is more general, we shall give a proof.

Proposition 6.1 *Let $\chi : A \to \mathcal{P} \times L_+$ be an exchange economy, where A is a finite set of agents, L is an ordered vector space with nonnegative cone L_+ and $\mathcal{P}$ is a set of preferences on L_+. Let f be an allocation and $X \subset L_+$. Suppose that the preferences are locally nonsatiated with respect to X at f for $a \in S$ i.e.*

$$\forall a \in S\ \exists v \in L_+\ \forall \lambda \in (0,1)\ \begin{cases} f(a) + \lambda v & \succ_a & f(a) \\ f(a) + \lambda v & \in & X \end{cases}$$

Define

$$\begin{aligned} \gamma'(a) &= \{x - e(a) :\ x \in X,\ x \succ_a f(a)\} \cup \{0\} \\ \Gamma' &= \sum_{a \in A} \gamma'(a) \end{aligned}$$

Then, for every nonnegative linear functional $p : L \to \mathbf{R}$,

$$\rho_B(f,p) \leq \frac{-2 \inf p \cdot \Gamma' + 2p \cdot \sum_{a \in A \setminus S} e(a)}{|A|} \tag{16}$$

and

$$\rho_S(f,p,X) \leq \frac{-4 \inf p \cdot \Gamma' + 3p \cdot \sum_{a \in A \setminus S} e(a)}{|A|} \tag{17}$$

where $\inf p \cdot \Gamma' = \inf\{p \cdot G : G \in \Gamma'\}$

Proof: For $a \in S$, choose $v(a) \in X$ such that

$$f(a) + \delta v(a) \succ_a f(a) \quad \text{and} \quad f(a) + \delta v(a) \in X$$

and hence

$$f(a) + \delta v(a) - e(a) \in \gamma'(a)$$

for all $\delta \in (0,1)$.

To obtain Equation (16), decompose $A = A_1 \cup A_2$ into two disjoint sets:

$$\begin{aligned} A_1 &= \{a \in A : p \cdot (f(a) - e(a)) < 0\} \\ A_2 &= \{a \in A : p \cdot (f(a) - e(a)) \geq 0\} \end{aligned}$$

We want to find an upper bound for

$$\sum_{a \in A} |p \cdot (f(a) - e(a))|$$

To accomplish this, consider first the sum over the set A_1; we will later use the feasibility of f to show that the sum over A_2 equals the sum over A_1. Since $0 \in \gamma'(a)$ for all a, $\inf p \cdot \gamma'(a) \leq 0$. For $a \in A_1 \cap S$, $p \cdot (f(a) + \delta v(a) - e(a)) < 0$ for δ sufficiently small, since p is linear. Since δ is arbitrary, p is nonnegative and $p \cdot (f(a) - e(a)) < 0$ for $a \in A_1$,

$$\begin{aligned} \sum_{a \in A_1} |p \cdot (f(a) - e(a))| &\leq \sum_{a \in (A_1 \cap S)} (-p \cdot (f(a) - e(a))) + \sum_{a \in A \setminus S} p \cdot e(a) \\ &\leq -\inf p \cdot \Gamma' + p \cdot \sum_{a \in A \setminus S} e(a) \end{aligned}$$

Since $\sum_{a \in A}(f(a) - e(a)) = 0$,

$$\begin{aligned} \sum_{a \in A} |p \cdot (f(a) - e(a))| &= 2 \sum_{a \in A_1} |p \cdot (f(a) - e(a))| \\ &\leq -2 \inf p \cdot \Gamma' + 2p \cdot \sum_{a \in A \setminus S} e(a) \end{aligned}$$

It remains to establish Equation (17). It is enough to show that

$$\begin{aligned} &\sum_{a \in A} |\min\{0, \inf\{p \cdot (x - e(a)) : x \in X,\ x \succ_a f(a)\}\}| \\ &\qquad \leq -2 \inf p \cdot \Gamma' + p \cdot \sum_{a \in A \setminus S} e(a) \qquad (18) \end{aligned}$$

for then

$$\sum_{a \in A} \rho_S(f, a, p, X)$$

$$
\begin{aligned}
&\leq \sum_{a\in A} |\min\{0, \inf\{p\cdot(x-e(a)) :\ x\in X,\ x\succ_a f(a)\}\}| \\
&\qquad + \sum_{a\in A} |p\cdot(f(a)-e(a))| \\
&\leq -2\inf p\cdot\Gamma' + p\cdot \sum_{a\in A\setminus S} e(a) - 2\inf p\cdot\Gamma' + 2p\cdot \sum_{a\in A\setminus S} e(a) \\
&= -4\inf p\cdot\Gamma' + 3p\cdot \sum_{a\in A\setminus S} e(a)
\end{aligned}
$$

We now establish Equation (18). Since $f(a)+\delta v(a)-e(a)\in\gamma'(a)$ for all $\delta\in(0,1)$ and all $a\in S$, and p is linear and nonnegative, we can draw three conclusions depending on the location of a: if $a\in A_1\cap S$, then

$$|\inf\{p\cdot(x-e(a)) :\ x\in X,\ x\succ_a f(a)\}| = -\inf p\cdot\gamma'(a)$$

if $a\in A_2\cap S$, then

$$
\begin{aligned}
&|\inf\{p\cdot(x-e(a)) :\ x\in X,\ x\succ_a f(a)\}| \\
&\qquad \leq \max\{-\inf p\cdot\gamma'(a), p\cdot(f(a)-e(a))\}
\end{aligned}
$$

if $a\in A\setminus S$, then

$$|\min\{0,\inf\{p\cdot(x-e(a)) :\ x\in X,\ x\succ_a f(a)\}\}| \leq p\cdot e(a)$$

Therefore,

$$
\begin{aligned}
&\sum_{a\in A} |\min\{0, \inf\{p\cdot(x-e(a)) :\ x\in X,\ x\succ_a f(a)\}\}| \\
&\quad \leq -\sum_{a\in A} p\cdot\gamma'(a) + \sum_{a\in A_2} |p\cdot(f(a)-e(a))| + \sum_{a\in A\setminus S} p\cdot e(a) \\
&\quad \leq -\inf p\cdot\Gamma' - \inf p\cdot\Gamma' + p\cdot\sum_{a\in A\setminus S} e(a) \\
&\quad = -2\inf p\cdot\Gamma' + p\cdot \sum_{a\in A\setminus S} e(a)
\end{aligned}
$$

which establishes Equation (18) and completes the proof. ■

7 A Most Peculiar Theorem

At first sight, the most puzzling aspect of the results reported in this paper is the stark contrast between the nonconvergence examples of Section 3 and the known versions of Aumann's Theorem and the Debreu-Scarf Theorem in infinite-dimensional commodity spaces. In this section, we discuss how these nonconvergence examples can diverge so dramatically from the equivalence results.

The central reason for this divergence is that measurable mappings have "almost compact" range. Consider, for example, a complete metrizable topological vector space L and a measurable mapping $f : [0,1] \to L$. For every $\varepsilon > 0$ there is a compact set $K \subset L$ such that, if μ is the distribution of f, then $\mu(K) > 1 - \varepsilon$. This conclusion is valid whether L is a finite dimensional space (i.e., $\mathbf{R}^k$) or an infinite dimensional space, but the implications of the conclusion are dramatically different in the two contexts. Compact subsets of $\mathbf{R}^k$ may be relatively large: balls are compact, and the whole space is a union of countably many compact sets. By contrast, compact subsets of infinite dimensional spaces are relatively thin: balls are never compact, and the whole space will typically not be the union of countably many compact sets. Indeed, if L is an infinite dimensional Banach lattice, the Edgeworth box of feasible allocations will typically not be compact.

To make the same point in another way, suppose the commodity space L is any metrizable topological vector space. Suppose $X \subset L$ is compact and $\varepsilon > 0$. Then X is covered by a finite collection of ε balls centered at points $x_1, \ldots, x_{m_\varepsilon} \in X$. Write X_ε for the linear span of $\{x_1, \ldots, x_{m_\varepsilon}\}$. Obviously, X_ε is a finite dimensional subspace of L, and every point in X is within ε of this finite dimensional subspace; X is thus "almost finite dimensional."

Because measurability has such strong implications in infinite dimensional spaces, the "technical" measurability assumptions needed to formulate Aumann's Theorem correspond, in economies with a finite number of agents, to *compactness* requirements which have quite unpleasant *economic* interpretations.

- *Endowments come from a compact set.* This is a very stringent assumption, but at least it is exogenous.

- *Core consumptions come from a compact set.* This is a very stringent assumption, and it is *endogenous*. Even if endowments come from a compact set, there is no reason why core consumptions should also come from a compact set (when the commodity space is infinite dimensional).[18]

- *The comparison bundles used in testing approximate decentralization come from a compact set.* When approximate decentralization is weakened in this way, it does not rule out the possibility that there are consumptions *outside the compact set* which are preferred to the core allocation, and which are much cheaper than the endowment.

- *The blocking allocations considered in the definition of the core come from a compact set.* Note that this restriction on blocking makes the core bigger than it would otherwise be.

To make this point, we state and prove Most Peculiar Theorem 7.1, which incorporates explicitly the first three of these points, producing a very peculiar result. The counterexamples of Section 3 are consistent with Most Peculiar Theorem 7.1 precisely because this theorem *assumes the problems away*.

For convenience, we take $L = C(\Omega)$, the space of continuous functions on a compact Hausdorff space, endowed with the sup norm topology. We choose to work in $C(\Omega)$ because the nonnegative cone has nonempty interior. this simplifies the statement and proof of the theorem, because:

- The fact that an allocation is in the core guarantees that a certain set Γ' does not intersect the negative orthant. In order to achieve separation, we need to show that the *convex hull* of Γ' does not intersect an appropriate convex cone. The infinite dimensional Shapley-Folkman Theorem guarantees that con Γ' is contained in a neighborhood of Γ'. The fact that the negative orthant has nonempty interior then allows us to conclude that con Γ' does not intersect a translate of the negative orthant.

[18] When the commodity space is finite dimensional, Bewley (1973a) shows that compactness of the space of preferences implies that core consumptions also lie in a compact set.

- The fact that the negative orthant has nonempty interior then guarantees that there is a price separating it from con Γ'.

The proof turns out to be quite easy — it is essentially the same as in the finite dimensional case (Anderson (1978)). (Just as the proof of Aumann's theorem in this context (Gabszewicz (1968)) is very much the same as in the finite dimensional case.)

We write $\mathcal{P}$ for the space of preference relations on L_+ that are irreflexive, transitive, monotone and algebraically continuous.

Most Peculiar Theorem 7.1 *Let $L = C(\Omega)$, where Ω is a compact Hausdorff space. For every compact convex set X, every $\delta > 0$, and every $\varepsilon > 0$, there is an integer N_0 such that:*

If $\chi : A \to \mathcal{P} \times L_+$ is an exchange economy for which

(a) $e(a) \in X$ for all $a \in A$

(b) $\delta \mathbf{1} \leq \bar{e} = \frac{1}{N} \sum e(a)$

(c) $|A| = N \geq N_0$

and f is a core allocation of χ with $f(a) \in X$ for all $a \in A$, then there is a price $p \in (L')_+$ that ε-decentralizes f with respect to X.[19]

Proof: Fix X compact and $\varepsilon > 0$. Set

$$X' = X + [0, \delta]\mathbf{1}$$

and note that X' is compact.

Let

$$\alpha = \frac{\varepsilon\delta}{7}$$

[19]Recall that decentralization with respect to X is defined in Section 2.

By Theorem 5.1, we may choose an integer N_0 such that, if $N \geq N_0$ and $X_1, \ldots, X_N \subset X' - X = \{x' - x : x' \in X', x \in X\}$ then

$$\text{con}\left(\frac{1}{N}\sum_{n=1}^{N} X_n\right) \subset \frac{1}{N}\sum_{n=1}^{N} X_n + B\left(0, \alpha\right)$$

Now let $\chi : A \to \mathcal{P} \times L_+$ be an exchange economy satisfying assumptions (a) – (d), and fix a core allocation f with $f(a) \in X$ for all $a \in A$. We construct a price p that approximately decentralizes f with respect to comparison bundles in X.

For each $a \in A$, set

$$\gamma'(a) = \{y - e(a) : \ y \in X', \ y \succ_a f(a)\} \cup \{0\}$$

and write

$$\Gamma' = \sum_{a \in A} \gamma'(a)$$

Note that $\gamma'(a)$ is a truncated version of the net preferred set of agent a (together with $\{0\}$), and that Γ' is a truncated version of the aggregate net preferred set.

We claim that

$$\Gamma' \cap (-L_+) = \{0\}$$

To see this, suppose not. Then we can find a coalition $A^* \subset A$ and vectors $y(a) - e(a) \in \gamma'(a), 0 \neq c \in L_+$ with

$$\sum_{a \in A^*} [y(a) - e(a)] = -c$$

Since $c \neq 0$, $A^* \neq \emptyset$.

For $a \in A^*$, let $z(a) = y(a) + \frac{c}{|A^*|}$. Then $z(a) \geq y(a) \succ_a f(a)$, so $z(a) \succ_a f(a)$.

$$\begin{aligned}\sum_{a \in A^*} z(a) &= \left(\sum_{a \in A^*} y(a)\right) + c \\ &= -c + \left(\sum_{a \in A^*} e(a)\right) + c \\ &= \left(\sum_{a \in A^*} e(a)\right)\end{aligned}$$

so z is feasible for A^*. Therefore, A^* can block f via z, so f is not in the core, a contradiction. Thus, we have shown that $\Gamma' \cap (-L_+) = \{0\}$, so $\frac{1}{N}\Gamma' \cap (-L_+) = \{0\}$.

Since $(1/N)\text{con }\Gamma' \subset (1/N)\Gamma' + B(0, \alpha)$, and the norm is the sup norm,

$$\frac{1}{N}\text{con }\Gamma' \bigcap (-\alpha\mathbf{1} - L_+) = \emptyset$$

Since $(1/N)\text{con }\Gamma'$ is convex and $(-\alpha\mathbf{1} - L_+)$ is convex and has nonempty interior, we can find a nonzero continuous linear functional p on L (that is, a continuous function $p \in L'$) such that

$$\sup p \cdot (-\alpha\mathbf{1} - L_+) \leq \inf p \cdot \frac{1}{N}\text{con }\Gamma' \tag{19}$$

Note that we must have $p > 0$; if not, there exists $x \in -L_+$ such that $p \cdot x > 0$, so $\sup p \cdot (-\alpha\mathbf{1} - L_+) = +\infty$, while $0 \in \Gamma'$, which implies that $\inf p \cdot \frac{1}{N}\text{con }\Gamma' \leq 0$, contradicting Equation 19. Since $\mathbf{1}$ is strictly positive, $p \cdot e \geq p \cdot (\delta\mathbf{1}) > 0$.

By renormalizing, we may assume that $p \cdot \bar{e} = 1$. Thus,

$$\inf p \cdot \frac{\Gamma'}{N} \geq -\left(+\frac{\varepsilon\delta}{7}\right) p \cdot \bar{\mathbf{1}} \geq -\frac{\varepsilon}{7} p \cdot \bar{e} = -\frac{\varepsilon}{7} \tag{20}$$

We are now in a position to establish the approximate decentralization conclusion. Monotonicity, the requirement that $f(a) \in X$ for all $a \in A$, and the definition of X' implies that for all $a \in A$,

$$\begin{aligned} f(a) + \xi v &\succ_a f(a) \text{ for all } \xi \in (0,1) \\ f(a) + \xi v &\in X' \end{aligned}$$

so $\succ_a$ is locally nonsatiatied with respect to X' for all $a \in A$. Proposition 6.1 implies that

$$\begin{aligned} \rho(f, p, X) &\leq -7 \inf p \cdot \frac{\Gamma'}{N} \\ &\leq \left(\frac{6\varepsilon}{7}\right) \\ &< \varepsilon \end{aligned}$$

which is the desired result. ■

8 Decentralization in Riesz Spaces

In this Section we obtain (approximate) decentralization in certain Riesz spaces (those satisfying Assumption 4.1). We show that if endowments are bounded, then bounded core allocations can be decentralized with respect to bounded comparisons. Roughly speaking, "local" decentralization is always possible; the examples of Section 3 show clearly that "global" decentralization generally will not be possible. In one interesting case, however — when marginal utility tends to 0 at infinity — we can show that local decentralization leads to global decentralization.

Throughout this section, we assume that the Riesz space L satisfies Assumption 4.1, so that the representation theorem of Abramovich, Aliprantis and Zame (1995) (cited as Theorem 4.2 above) holds. As before, we write $\mathcal{P}$ for the space of preference relations on L that are irreflexive, transitive, monotone and algebraically continuous. For $v \in L_+$, a preference relation $\succ \in \mathcal{P}$ is *strictly monotone in the direction* v if $x + tv \succ x$ for every $x \in L_+, t \in \mathbf{R}_+$. Write $\mathcal{P}_v$ for the set of all preferences in $\mathcal{P}$ that are strictly monotone in the direction v.

Definition 8.1 Let $v \in L_+$. A set of preferences $\mathcal{P}_0$ is said to be *equimonotone in the direction* v if, for every order bounded set $X \subset L_+$ and every $\alpha > 0$, there is an open set $W \subset L$ such that for every preference $\succ \in \mathcal{P}_0$,

$$z \in y + \alpha v + W; y, z \in X \Longrightarrow z \succ y$$

Equimonotonicity of a single preference relation is an assertion that strict monotonicity in the direction v holds uniformly over the order bounded set X; equimonotonicity of a family of preferences is an assertion that strict monotonicity holds uniformly over the order bounded set X *and* over the family of preferences. Note that equimonotonicity of a single preference relation is a consequence of either

- uniform properness in the direction v, *or*
- strict monotonicity in the direction v, continuity, and compactness of order intervals

In the following result, we require that all preferences be strictly monotone in the direction v and that there be an equimonotone family of preferences such that all but a small fraction of agents have preferences in the family. The reader may wonder why we have not assumed simply that *all* agents have preferences in the equimonotone family. We believe there is good reason for not doing so. There can be little doubt that there are some individuals whose preferences are highly unconventional. For a number of years, Polka Dot Man was a fixture on the Berkeley campus. Polka Dot Man maximized his utility by spending the entire day posing in Sproul Plaza wearing blue clothing with large white polka dots. After several years of this behavior, he received an important revelation. He then took to spending the entire day posing in Sproul Plaza wearing white clothing with medium size black crosses. The presence of unconventional individuals such as Polka Dot Man makes us reluctant to impose compactness conditions on *all* agents. It is important to know that the presence of a few unconventional individuals will not upset the decentralization conclusions for the vast majority of individuals whose tastes are more conventional. One might at first think that Polka Dot Man and other unconventional people can simply be excluded from consideration in our exchange economy, permitting us to assume that *all* agents' preferences come from the equimonotone family. The problem with that approach is that Polka Dot Man *trades* with other agents; it is conceivable that the presence of Polka Dot Man could upset approximate decentralization for *other* agents (in particular, the core consumptions of the *other* agents may be infeasible if Polka Dot Man is excluded from consideration). Intuition may suggest that the presence of a small number of people with very unconventional tastes should not matter very much; our formulations and proofs validate this intuition.

The first main result of this Section is the following.

Slightly Silly Theorem 8.2 *Suppose L satisfies Assumption 4.1. For every strictly positive $v \in L_+$, every $K, \varepsilon > 0$, and every uniformly integrable set $\mathcal{F}$ of probability distributions, there exists $\delta > 0$ such that for every set of preferences $\mathcal{P}_0 \subset \mathcal{P}_v$ which is equimonotone in the direction v, there is an integer N_0 such that:*

If $\chi : A \to \mathcal{P}_v \times L_+$ is an exchange economy for which *(a)* $\frac{1}{N}|\{a \in A : \succ_a \in \mathcal{P}_0\}| > 1 - \delta$

(b) $v \leq \bar{e} = \frac{1}{N}\sum e(a) \leq Kv$

(c) if

$$E(t) = \frac{|\{a \in A : e(a) \leq t\bar{e}\}|}{N}$$

then $E \in \mathcal{F}$ [20]

(d) $|A| = N \geq N_0$

and f *is a core allocation of* χ *for which*

(e) if

$$F(t) = \frac{|\{a \in A : f(a) \leq t\bar{e}\}|}{N}$$

then $F \in \mathcal{F}$

then there is a price $p \in (L')_+$ *that* ε*-decentralizes* f *with respect to* $[0, K^2 v]$. [21]

Proof: Fix $v \in L_+$, $K, \varepsilon > 0$, and a uniformly integrable set $\mathcal{F}$ of probability distributions. Because L satisfies Assumption 4.1, we can apply Theorem 4.2, to obtain a countably additive probability space (Ω, μ) and a vector lattice isomorphism $R : L \to L^1(\mu)$ onto a dense order ideal $R(L) \subset L^1(\mu)$, satisfying the various conclusions of 4.2. In particular, $R(v) = \mathbf{1}$ and, for all M, the restriction of R to the order interval $[0, Mv] \subset L$ (endowed with the topology of L) is a homeomorphism onto the order interval $[0, M\mathbf{1}] \subset L^1(\mu)$ (endowed with the norm topology of $L^1(\mu)$).

Because R is an vector lattice isomorphism, we may and shall identify L and $R(L)$ as sets and as vector lattices (and we shall henceforth refer to $L \subset L^1(\mu)$); we identify the vector $v \in L$ with $\mathbf{1} \in L^1(\mu)$. Because R is not a homeomorphism, however, the given topology τ on L may differ from

[20] Note that E is the cumulative distribution function of $\|e\|_{\bar{e}}$, where $\|x\|_{\bar{e}} = \inf\{t : |x| \leq t\bar{e}\}$.

[21] One can actually show that f is ε-decentralized with respect to uniformly integrable comparison bundles. Specifically, there exists a price p such that (i) $p \cdot \bar{e} = 1$; (ii) $\frac{1}{N}\sum_{a \in A} |p \cdot (f(a) - e(a))| < \varepsilon/2$; and (iii) if $g : A \to L_+$ satisfies (f) $g(a) \succ_a f(a)$ for all $a \in A$ (g) if $G(t) = \frac{|\{a \in A : g(a) \leq t\bar{e}\}|}{N}$ then $G \in \mathcal{F}$; then $\frac{1}{N}\sum_{a \in A}(p \cdot (g(a) - f(a)))^- < \varepsilon/2$.

the restriction to L of the norm topology of $L^1(\mu)$; we shall write τ^1 for the latter topology. Of course, continuity of R means that the topology τ is stronger than the topology τ^1; and the fact that the restriction of R to order intervals is a homeomorphism means that the topologies τ, τ^1 agree on order intervals $[0, Mv] = [0, M\mathbf{1}]$. Note that the definition of equimonotonicity involves a particular topology, and that we have assumed equimonotonicity only with respect to the topology τ, but that all our other assumptions about preferences are purely algebraic.

Fix $\mathcal{P}_0$, a set of preferences which is equimonotone in the direction $v = \mathbf{1}$. Fix $\varepsilon > 0$. Find T such that

$$\int_T^\infty t \, dG(t) < \varepsilon/72$$

for all $G \in \mathcal{F}$. There is no loss of generality in assuming $\varepsilon < 1 < T \leq K$. Set

$$\delta = \frac{\varepsilon}{72T}$$

It is convenient at this point to introduce constants C, D; later in the proof we shall take $C = D = 1$. The point of introducing these constants is to allow us, in the proof of Theorem 8.8, to quote verbatim a large section of the present proof.

We work in the order interval $[0, 2CDK^2\mathbf{1}] = [0, 2CDK^2v]$. Equimonotonicity with respect to the topology τ means that there is a τ-neighborhood Q of 0 such that $z \succ y$ whenever $y, z \in [0, 2CDK^2\mathbf{1}]$, $z \in (y + \frac{\varepsilon}{36}v + Q)$ and $\succ \in \mathcal{P}_0$. Because the topologies τ, τ^1 agree on order bounded sets, we can find a τ^1-neighborhood W_1 of 0 such that

$$\begin{aligned} Q \cap \left[-\left(4CDK^2 + \frac{\varepsilon}{36}\right)\mathbf{1}, \left(4CDK^2 + \frac{\varepsilon}{36}\right)\mathbf{1}\right] \\ = W_1 \cap \left[-\left(4CDK^2 + \frac{\varepsilon}{36}\right)\mathbf{1}, \left(4CDK^2 + \frac{\varepsilon}{36}\right)\mathbf{1}\right] \end{aligned}$$

Thus, $z \succ y$ whenever $y, z \in [0, 2CDK^2\mathbf{1}]$, $z \in (y + \frac{\varepsilon}{36}v + W_1)$ and $\succ \in \mathcal{P}_0$; in other words, the preference is equimonotone with respect to τ_1. Without loss of generality, we may assume that

$$W_1 = \{w \in L : \|w\|_1 < \varepsilon_1\}$$

for some $\varepsilon_1 \in (0, 1]$.

Using the same argument, we can also find a τ^1-neighborhood W_2 of 0 such that $z \succ x$ whenever $y, z \in [0, 2CDK^2\mathbf{1}]_+$, $z \in (y + CDK^2 v + W_2)$, $\succ \in \mathcal{P}_0$ and $y \succ x$. Without loss of generality, we may assume that

$$W_2 = \{w \in L : \|w\|_1 < \varepsilon_2\}$$

for some $\varepsilon_2 > 0$.

Let

$$U = \left\{u \in L : \|u\|_1 < \frac{\varepsilon}{72CDK^2} \min\{\varepsilon_1, \varepsilon_2\}\right\}$$

By Theorem 5.2, we may choose an integer N_0 such that, if $N \geq N_0$ and $X_1, \ldots, X_N \subset [-K\mathbf{1}, CDK^2\mathbf{1}]$ then

$$\text{con}\left(\frac{1}{N}\sum_{n=1}^{N} X_n\right) \subset \frac{1}{N}\sum_{n=1}^{N} X_n + U$$

Now let $\chi : A \rightarrow \mathcal{P}_v \times M(\Omega)_+$ be an exchange economy satisfying assumptions (a) – (d), and fix a core allocation f satisfying assumption (e). We construct a price p that approximately decentralizes f with respect to uniformly integrable comparison bundles.

Let

$$\begin{aligned} E(t) &= \frac{|\{a \in A : e(a) \leq t\bar{e}\}|}{|A|} \\ F(t) &= \frac{|\{a \in A : f(a) \leq t\bar{e}\}|}{|A|} \end{aligned}$$

By assumption, $E, F \in \mathcal{F}$. Let

$$\begin{aligned} A_1 &= \{a \in A : \succ_a \notin \mathcal{P}_0\} \\ A_2 &= \{a \in A : e(a) \not\leq K\bar{e}\} \\ A_3 &= \{a \in A : \ f(a) \not\leq K\bar{e}\} \end{aligned}$$

A_1, A_2, and A_3 are sets of agents who are, from the point of view of our decentralization argument, badly behaved. However, the members of $A_1 \cup A_2 \cup A_3$

will not matter in the overall decentralization result because $\frac{1}{N}\sum_{a\in A_1\cup A_2\cup A_3} e(a)$ is small:

$$\begin{aligned}
\frac{1}{N}\sum_{a\in A_1} e(a) &= \frac{1}{N}\sum_{\succ_a\notin \mathcal{P}_0} e(a) \\
&\leq \left(T\delta\bar{e} + \frac{1}{N}\sum_{e(a)\nleq T\bar{e}} e(a)\right) \\
&= \left(\frac{\varepsilon}{72} + \int_T^\infty t\, dE(t)\right)\bar{e} \\
&< \left(\frac{\varepsilon}{72} + \frac{\varepsilon}{72}\right)\bar{e} = \frac{\varepsilon}{36}\bar{e} \qquad (21)\\
\frac{1}{N}\sum_{a\in A_2} e(a) &= \frac{1}{N}\sum_{e(a)\nleq K\bar{e}} e(a) \\
&\leq \frac{1}{N}\sum_{e(a)\nleq T\bar{e}} e(a) \\
&\leq \frac{1}{N}\sum_{e(a)\nleq T\bar{e}} e(a) \\
&\leq \int_T^\infty t\, dE(t) \\
&\leq \frac{\varepsilon}{72T}T\bar{e} \qquad (22)\\
\frac{1}{N}\sum_{a\in A_3} e(a) &= \frac{1}{N}\sum_{f(a)\nleq K\bar{e}} e(a) \\
&\leq \frac{1}{N}\sum_{f(a)\nleq T\bar{e}} e(a) \\
&\leq \frac{|\{a\in A : f(a)\nleq T\bar{e}\}|}{N}T\bar{e} + \frac{1}{N}\sum_{e(a)\nleq T\bar{e}} e(a) \\
&\leq \frac{\int_T^\infty t\, dF(t)}{T}T\bar{e} + \int_T^\infty t\, dE(t) \\
&\leq \frac{\varepsilon}{72T}T\bar{e} + \frac{\varepsilon}{72}\bar{e} \\
&= \frac{\varepsilon}{36}\bar{e} \qquad (23)
\end{aligned}$$

For each $a \in A$, set

$$\gamma'(a) = \{y - e(a) \ : y \in [0, (CDK^2+1)\mathbf{1}],\ y \succ_a f(a)\} \cup \{0\}$$

and write

$$\Gamma' = \sum_{a \in A} \gamma'(a)$$

Note that $\gamma'(a)$ is a truncated version of the net preferred set of agent a (together with $\{0\}$), and that Γ' is a truncated version of the aggregate net preferred set.

We claim that

$$\frac{1}{N}\Gamma' \cap \left(-\frac{\varepsilon}{9}\bar{e} - L_+ + 2U\right) = \emptyset$$

To see this, suppose not. Then we can find a coalition $A^{**} \subset A$ and vectors $y(a) - e(a) \in \gamma'(a), d \in L_+, u \in 2U$ with

$$\frac{1}{N}\sum_{a \in A^{**}} [y(a) - e(a)] = -\frac{\varepsilon}{9}\bar{e} - d + u \leq -\frac{\varepsilon}{9}\bar{e} + u$$

Let $A^* = A^{**} \setminus (A_1 \cup A_3)$. Then

$$\begin{aligned}
\frac{1}{N}\sum_{a \in A^*} [y(a) - e(a)] &\leq \frac{1}{N}\sum_{a \in A^{**}} [y(a) - e(a)] + \frac{1}{N}\sum_{a \in (A_1 \cup A_3)} e(a) \\
&\leq -\frac{\varepsilon}{9}\bar{e} + u + \frac{\varepsilon}{36}\bar{e} + \frac{\varepsilon}{36}\bar{e} \\
&= -\frac{\varepsilon}{18}v + u
\end{aligned}$$

by Equation (21). Furthermore,

$$\left\|\frac{\varepsilon}{18}v\right\|_1 = \frac{\varepsilon}{18} \not\leq 2\left(\frac{\varepsilon\varepsilon_1}{72CDK^2}\right)$$

so $\frac{\varepsilon}{18}v \notin 2U$, which shows that $A^* \neq \emptyset$.

$$\begin{aligned}
\sum_{a \in A^*}\left[y(a) + \frac{\varepsilon}{36}v\right] &\leq \sum_{a \in A^*} e(a) - \frac{N\varepsilon}{18}v + Nu + \frac{|A^*|\varepsilon}{36}v \\
&\leq \sum_{a \in A^*} e(a) - \frac{N\varepsilon}{36}v + Nu - \left(\frac{N\varepsilon}{36}v - \frac{|A^*|\varepsilon}{36}v\right) \\
&\leq \sum_{a \in A^*} e(a) - \frac{N\varepsilon}{36}v + Nu
\end{aligned}$$

Let

$$\begin{aligned}
x(a) &= y(a) + \frac{\varepsilon}{36}v \\
x^* &= \sum_{a \in A^*} x(a) \\
e^* &= \sum_{a \in A^*} e(a) \\
e^{**} &= \left[e^* - N\left(\frac{\varepsilon}{36}v - u\right)\right] \wedge e^* \\
&= e^* - N\left(\frac{\varepsilon}{36}v - u\right)^+ \\
z^* &= x^* \wedge e^{**}
\end{aligned}$$

Note that

$$\begin{aligned}
x^* - z^* &= (x^* - e^{**})^+ \\
&\leq e^* - N\left(\frac{\varepsilon}{36}v - u\right) - e^* + N\left(\frac{\varepsilon}{36}v - u\right)^+ \\
&= N\left[-\left(\frac{\varepsilon}{36}v - u\right)^+ + \left(\frac{\varepsilon}{36}v - u\right)^- + \left(\frac{\varepsilon}{36}v - u\right)^+\right] \\
&= N\left(\frac{\varepsilon}{36}v - u\right)^- \\
&\leq Nu^+
\end{aligned}$$

By the Riesz Decomposition Property, there exists $z(a)$ with

$$\begin{aligned}
z(a) &\in [0, x(a)] \\
\sum_{a \in A^*} z(a) &= z^*
\end{aligned}$$

We will show that $z(a) \succ_a f(a)$ for all but a few agents $a \in A^*$, and will then alter the allocation for those few agents. Let

$$A^{***} = \{a \in A^* : \ z(a) \notin x(a) + W_1\}$$

If $a \in A^* \setminus A^{***}$, then

$$z(a) \quad \in \quad y(a) + \frac{\varepsilon}{36}v + W_1$$

$$\begin{aligned}
z(a) &\leq x(a) = y(a) + \frac{\varepsilon}{36}v \leq 2CDK^2v \\
y(a) &\leq 2CDK^2v \\
y(a) &\succ_a f(a) \\
\succ_a &\in \mathcal{P}_0
\end{aligned}$$

so $z(a) \succ_a f(a)$ by the definition of W_1.

Since

$$\left\| \sum_{A^*} x(a) - z(a) \right\|_1 = \|x^* - z^*\|_1 \leq N\|u^+\|_1 < \frac{N\varepsilon\varepsilon_1}{72CDK^2}$$

we have

$$|A^{***}| = |\{a \in A^* : \ \|x(a) - z(a)\|_1 \geq \varepsilon_1\}| < \frac{N\varepsilon}{72CDK^2}$$

Define

$$\hat{z}(a) = \begin{cases} z(a) & \text{if } \ a \in A^* \setminus A^{***} \\ \left(2CDK^2v - \frac{72CDK^2}{\varepsilon}u^+\right)^+ & \text{if } \ a \in A^{***} \end{cases} \tag{24}$$

If $a \in A^{***}$, write $y' = f(a)$. (As with the constants C, D, this extra bit of notation is introduced here so that we can cite a large section of this proof in the proof of Theorem 8.8.) Note that $\succ_a \in \mathcal{P}_0$ and that

$$\begin{aligned}
y' &\leq CDK^2v \\
\hat{z}(a) &= \left(2CDK^2v - \frac{72CDK^2}{\varepsilon}u^+\right)^+ \leq 2CDK^2v \\
\hat{z}(a) &= \left(2CDK^2v - \frac{72CDK^2}{\varepsilon}u^+\right)^+ \\
&\geq \left(y' + CDK^2v - \frac{72CDK^2}{\varepsilon}u^+\right)^+ \\
\left\| \frac{72CDK^2}{\varepsilon}u \right\|_1 &= \frac{72CDK^2}{\varepsilon}\|u\|_1 \\
&< \frac{72CDK^2}{\varepsilon}\left(\frac{\varepsilon_2\varepsilon}{72CDK^2}\right) = \varepsilon_2
\end{aligned}$$

so that

$$\left(y' + CDK^2 v - \frac{72CDK^2}{\varepsilon} u^+\right)^+ \succ_a f(a)$$

by the definition of W_2. Hence

$$\hat{z}(a) \succ_a f(a) \tag{25}$$

by monotonicity and transitivity.

We now show that the allocation $\hat{z}$ is feasible for the coalition A^*:

$$\begin{aligned}
\sum_{a \in A^*} \hat{z}(a) &\leq z^* + |A^{***}| \left(2CDK^2 v - \frac{72CDK^2}{\varepsilon} u^+\right)^+ \\
&\leq e^* - N\left(\frac{\varepsilon}{36} v - u\right)^+ + \frac{N\varepsilon}{72CDK^2}\left(2CDK^2 v - \frac{72CDK^2}{\varepsilon} u^+\right)^+ \\
&= e^* - N\left(\frac{\varepsilon}{36} v - u\right)^+ + N\left(\frac{\varepsilon}{36} v - u^+\right)^+ \\
&\leq e^*
\end{aligned}$$

We have shown that $\sum_{a \in A^*} \hat{z}(a) \leq e^*$ and $\hat{z}(a) \succ_a f(a)$ for all $a \in A$. We may find $z' : A^* \to L_+$ with $z'(a) \geq z(a)$ and $\sum_{a \in A^*} z'(a) = e^*$. By monotonicity and transitivity, $z'(a) \succ_a f(a)$ for all $a \in A^*$, which contradicts the assumption that f is a core allocation. We conclude that

$$\frac{1}{N}\Gamma' \cap \left(-\frac{\varepsilon}{9}\bar{e} - L_+ + 2U\right) = \emptyset$$

as claimed.

Let

$$\Gamma'' = \sum_{a \in A \setminus A_2} \gamma'(a) \subset \Gamma'$$

For $A \in A \setminus A_2$, $\gamma'(a) \subset [-K\mathbf{1}, (CD^2K + 1)\mathbf{1}]$, so

$$\frac{1}{N}\text{con}\,\Gamma'' \subset \frac{1}{N}\Gamma'' + U \subset \frac{1}{N}\Gamma' + U$$

Suppose that

$$\frac{1}{N}\text{con}\,\Gamma'' \cap \left(-\frac{\varepsilon}{9}\bar{e} - L_+ + U\right) \neq \emptyset$$

Then we can find $u \in U$ and $x \in L_+$ such that

$$\left(-\frac{\varepsilon}{9}\bar{e} - x + u\right) \in \frac{1}{N}\text{con } \Gamma'' \subset \frac{1}{N}\Gamma' + U$$

so there exists $G \in \frac{1}{N}\Gamma'$ and $u' \in U$ such that

$$\left(-\frac{\varepsilon}{9}\bar{e} - x + u\right) = G + u'$$

so

$$G = \left(-\frac{\varepsilon}{9}\bar{e} - x + (u - u')\right)$$

Since U is symmetric, $-u' \in U$; since U is convex,

$$u - u' = 2\left(\frac{u - u'}{2}\right) \in 2U$$

which shows

$$\frac{1}{N}\Gamma' \cap \left(-\frac{\varepsilon}{9}\bar{e} - L_+ + 2U\right) \neq \emptyset$$

a contradiction which establishes that

$$\frac{1}{N}\text{con } \Gamma'' \cap \left(-\frac{\varepsilon}{9}\bar{e} - L_+ + U\right) = \emptyset$$

Since $(1/N)\text{con } \Gamma''$ is convex and $(-(\varepsilon/9)\bar{e} - L_+ + U)$ is convex and open, we can find a τ^1-continuous linear functional p on L that separates $\frac{1}{N}\text{con } \Gamma''$ from $\left(-\frac{\varepsilon\bar{e}}{9} - L_+ + U\right)$; note that p is necessarily nonnegative.[22] Since v is strictly positive and $\bar{e} \geq v$, we conclude that $p \cdot \bar{e} \neq 0$; renormalizing if necessary, we may assume that $p \cdot \bar{e} = 1$, and thus

$$\begin{aligned}
\inf p \cdot \left(\frac{1}{N}\right)\Gamma' &\geq \inf p \cdot \left(\frac{1}{N}\Gamma''\right) - p \cdot \left(\frac{1}{N}\sum_{a \in A_2} e(a)\right) \\
&\geq -\frac{\varepsilon}{9} - p \cdot \left(\frac{\varepsilon}{72}\bar{e}\right) \\
&> -\frac{\varepsilon}{9} - \frac{\varepsilon}{72} \\
&> -\frac{5\varepsilon}{36}
\end{aligned} \tag{26}$$

[22]Since L is dense in $L^1(\mu)$, p defines a continuous linear functional on $L^1(\mu)$, and hence may be identified with an element of $L^\infty(\mu)$.

We are now in a position to establish the approximate decentralization conclusions. We have already normalized so that $p \cdot \bar{e} = 1$. Taking X to be the order bounded set $[0, (CDK^2 + 1)v]$, strict monotonicity of $\{\succ_a\}$ in the direction v implies that for all $a \in A \setminus A_3$, and all $\xi \in (0, 1)$

$$\begin{aligned} f(a) + \xi v &\succ_a f(a) \\ f(a) + \xi v &\in [0, (CDK^2 + 1)v] \end{aligned}$$

so $\succ_a$ is locally nonsatiated at $f(a)$ with respect to $[0, (CDK^2 + 1)v]$.

Now take $C = D = 1$. Proposition 6.1 implies that

$$\begin{aligned} \rho(f, p, [0, K^2 v]) &\leq \rho(f, p, [0, (K^2 + 1)v]) \\ &\leq -9 \inf p \cdot \frac{\Gamma'}{N} + 5p \cdot \sum_{a \in A_3} e(a) \\ &< \frac{30\varepsilon}{36} + \frac{5\varepsilon}{36} < \varepsilon \end{aligned}$$

which shows that p ε-decentralizes f with respect to $[0, K^2 v]$.

By construction, the price functional p is continuous in the topology τ^1; since the topology τ is stronger than the topology τ^1, the price functional p is *a fortioi* continuous in the topology τ, so the proof is complete. ■

We now turn to the case in which marginal utility tends to zero at infinity. The most natural examples lie in the commodity space $L^1([0, 1])$, representing contingent claims over states of the world. If agents are expected utility maximizers, their preferences are represented by utility functions of the form

$$u(x) = \int_{[0,1]} v(x(t)) \, d\mu$$

where v is a felicity function. It is very natural to suppose that, in states where consumption is very large, the marginal utility of additional consumption is vanishingly small compared to the marginal utility of additional consumption in states where consumption is moderate. The following definition embodies this idea for commodity spaces that are vector sublattices of $L^1(\mu)$.

Definition 8.3 Let L be a vector sublattice of $L^1(\mu)$. A set of preferences $\mathcal{P}_0$ on L is said to exhibit *uniformly vanishing marginal utility at infinity* if, for every $\varepsilon > 0$ and every $\beta > 0$, there exists $\sigma > 0$ such that, for every preference $\succ \in \mathcal{P}_0$, if

$$\begin{aligned} \sigma < x(t) &\Rightarrow y(t) = \sigma \\ \beta < x(t) \leq \sigma &\Rightarrow x(t) = y(t) \\ x(t) \leq \beta &\Rightarrow x(t) \leq y(t) \leq \beta \\ \|(y-x)^+\|_1 &> \varepsilon \|(y-x)^-\|_1 \end{aligned}$$

then $y \succ x$.

Remark 8.4 Some comments may help in understanding this definition. One should think of β as being of moderate size, while σ is extremely large. The function y is formed from x in two stages. First, we truncate x to a consumption level of σ (i.e. we take $x \wedge \sigma \mathbf{1}$). Second, we compensate for the consumption lost in the truncation by adding back at least ε times the amount of consumption lost, in such a way that all of the consumption added back is consumed at density levels at most β. Our assumption requires that for every β and ε, there is a σ such that whenever y is constructed from x in this way, then y is preferred to x.

A simple example may help to illustrate.

Example 8.5 Let λ denote Lebesgue measure on the interval $[0, 1]$. Let $\mathcal{V}$ be a family of C^1 felicity functions $v : \mathbf{R}_+ \to \mathbf{R}_+$ satisfying

$$\begin{aligned} \sup_{v \in \mathcal{V}} v'(t) &\to 0 \text{ as } t \to \infty \\ \inf_{v \in V,\ t \in [0,T]} v'(t) &> 0 \text{ for every } T \in \mathbf{R} \end{aligned}$$

Given $v \in \mathcal{V}$, define the utility function u_v on $L^\infty(\lambda) \subset L^1(\lambda)$ by

$$u_v(x) = \int_{[0,1]} v(x(t))\, d\lambda$$

Then $\{u_v : v \in \mathcal{V}\}$ exhibits uniformly vanishing marginal utility at infinity.

For general Riesz spaces, we extend the definition by using the representation theorem.

Definition 8.6 Let L be a Riesz space satisfying Assumption 4.1. A set of preferences $\mathcal{P}_0$ on L is said to exhibit *uniformly vanishing marginal utility at infinity* if there exists a representation (in the sense of Theorem 4.2) $R : L \to L^1(\mu)$ such that the set of induced preferences on $R(L)$ exhibits uniformly vanishing marginal utility at infinity.

It might be desirable to define uniformly vanishing marginal utility at infinity in terms of the intrinsic properties of a general Riesz space, without reference to the representation provided by Theorem 4.2. Unfortunately, this does not seem possible because — as the next example demonstrates — a preference relation may exhibit uniformly vanishing marginal utility at infinity in one representation, but not in another.

Example 8.7 Let λ denote Lebesgue measure on $[0,1]$, and μ the measure whose Radon-Nikodym derivative, with respect to λ, is $f(t) = \frac{1}{2\sqrt{t}}$. Then μ and λ are mutually absolutely continuous, so $L^\infty(\mu) = L^\infty(\lambda)$. However, μ is not boundedly absolutely continuous with respect to λ, which allows us to find a preference which exhibits uniformly vanishing marginal utility at infinity as a preference on $L^\infty(\mu) \subset L^1(\mu)$ but not as a preference on $L^\infty(\lambda) \subset L^1(\lambda)$.

For $x \in L^\infty(\mu) = L^\infty(\lambda)$, define

$$u(x) = \int_{[0,1]} \sqrt{x(t)}\, d\mu$$

u obviously exhibits uniformly vanishing marginal utility at infinity when we view $L^\infty(\mu) = L^\infty(\lambda)$ as a sublattice of $L^1(\mu)$. However, when we view we view $L^\infty(\mu) = L^\infty(\lambda)$ as a sublattice of $L^1(\lambda)$ we compute:

$$\begin{aligned} u(x) &= \int_{[0,1]} \sqrt{x(t)}\, d\mu \\ &= \frac{1}{2}\int_{[0,1]} \sqrt{\frac{x(t)}{t}}\, d\lambda \end{aligned}$$

Let $\beta = 1$, $\varepsilon = 1/2$, and define

$$x_n(t) = \begin{cases} n & \text{if } \; t \in \left[0, \frac{1}{n}\right] \\ 0 & \text{if } \; t \in \left(\frac{1}{n}, 1\right] \end{cases}$$

If $\{u\}$ exhibits uniformly vanishing marginal utility at infinity with respect to λ, there exists σ satisfying Definition 8.6. Let y_n be formed by truncating x_n to σ and spreading half of the removed mass evenly over the interval $\left(\frac{1}{n}, 1\right]$, i.e.

$$y_n(t) = \begin{cases} \sigma & \text{if } t \in \left[0, \frac{1}{n}\right] \\ \frac{n-\sigma}{2(n-1)} & \text{if } t \in \left(\frac{1}{n}, 1\right] \end{cases}$$

Then we should have $u(x_n) \leq u(y_n)$ for all n; however, an easy calculation shows that $u(x_n) = 1$, while $u(y_n) \to \frac{1}{\sqrt{2}}$, a contradiction which shows that $\{u\}$ does not exhibit uniformly vanishing marginal utility at infinity when we view $L^\infty(\mu) = L^\infty(\lambda)$ as a sublattice of $L^1(\lambda)$.

The following result shows that the assumption of uniformly vanishing marginal utility leads to global approximate decentralization.

Theorem 8.8 *Suppose L satisfies Assumption 4.1. For every strictly positive $v \in L_+$, every $K, \varepsilon^* > 0$, and every uniformly integrable set $\mathcal{F}$ of probability distributions, there exists $\delta > 0$ such that for every set of preferences $\mathcal{P}_0 \subset \mathcal{P}_v$ which exhibits uniformly vanishing marginal utility at infinity and is equimonotone in the direction v, there is an integer N_0 such that:*

If $\chi : A \to \mathcal{P}_v \times L_+$ is an exchange economy for which

(a) $\frac{1}{N}|\{a \in A : \succ_a \in \mathcal{P}_0\}| > 1 - \delta$

(b) $v \leq \bar{e} = \frac{1}{N}\sum e(a) \leq Kv$

(c) if

$$E(t) = \frac{|\{a \in A : e(a) \leq t\bar{e}\}|}{N}$$

then $E \in \mathcal{F}$

(d) $|A| = N \geq N_0$

and f is a core allocation of χ then there is a price $p \in (L')_+$ which ε^*-decentralizes f.

Proof: As in the proof of Slightly Silly Theorem 7.2, we may find a probability space (Ω, μ) and identify L with a dense order ideal in $L^1(\mu)$, in such a way that the vector v is identified with the function $\mathbf{1}$. As before, write τ for the given topology on L and τ^1 for the restriction to L of the norm topology of $L^1(\mu)$.

Fix $\varepsilon^* > 0$ and let $\varepsilon = \frac{12}{27}\varepsilon^*$. Find T such that

$$\int_T^\infty t\, dG(t) < \varepsilon/72$$

for all $G \in \mathcal{F}$. There is no loss of generality in assuming $\varepsilon < 1 < T \leq \frac{K}{8}$. Set

$$\delta = \frac{\varepsilon}{72T}$$

Fix $\mathcal{P}_0$, a set of preferences which exhibits uniformly vanishing marginal utility at infinity and is equimonotone in the direction $v = \mathbf{1}$. Let

$$\begin{aligned} \varepsilon_3 &= \frac{\varepsilon}{18K} \\ D &= \frac{18}{\varepsilon} \end{aligned}$$

Choose C such that if

$$\begin{aligned} x(t) \leq DK^2 &\Rightarrow x(t) \leq y(t) \leq DK^2 \\ DK^2 < x(t) \leq CDK^2 &\Rightarrow x(t) = y(t) \\ CDK^2 < x(t) &\Rightarrow y(t) = CDK^2 \\ \|(y-x)^+\|_1 &> \frac{\varepsilon_3}{2}\|(y-x)^-\|_1 \\ \succ &\in \mathcal{P}_0 \end{aligned}$$

then $y \succ x$.

We work first on the order interval $[0, CDK^2\mathbf{1}] = [0, CDK^2v]$. Equimonotonicity means that we can find a τ-neighborhood W_1 of 0 such that $z \succ y$ whenever $y, z \in [-K\mathbf{1}, CDK^2\mathbf{1}]$, $z \in (y + \frac{\varepsilon}{36}v + W_1)$, $\succ \in \mathcal{P}_0$. As before, the fact that the topologies τ, τ' agree on order bounded sets means that we may assume

$$W_1 = \{w \in L : \|w\|_1 < \varepsilon_1\}$$

for some $\varepsilon_1 \in (0,1]$. In addition, equimonotonicity means that we can find a neighborhood W_2 of 0 such that $z \succ y$ whenever $y, z \in [0, 2CDK^2\mathbf{1}]$, $z \in (y + CDK^2v + W_2)$ and $\succ \in \mathcal{P}_0$. Without loss of generality, we may assume that

$$W_2 = \{w \in L : \|w\|_1 < \varepsilon_2\}$$

for some $\varepsilon_2 > 0$. Let

$$U = \left\{u \in L : \|u\|_1 < \frac{\varepsilon}{72CDK^2} \min\{\varepsilon_1, \varepsilon_2\}\right\}$$

By Theorem 5.2, we may choose an integer N_0 such that, if $N \geq N_0$ and $X_1, \ldots, X_N \subset [-K\mathbf{1}, CDK^2\mathbf{1}]$ then

$$\text{con}\left(\frac{1}{N}\sum_{n=1}^{N} X_n\right) \subset \frac{1}{N}\sum_{n=1}^{N} X_n + U$$

Now let $\chi : A \to \mathcal{P}_v \times M(\Omega)_+$ be an exchange economy satisfying assumptions (a) – (d). We will first find a decentralizing price with respect to comparison bundles in $[0, CDK^2\mathbf{1}]$; then, we shall modify the price so that for every comparison bundle y outside $[0, CDK^2\mathbf{1}]$ which is preferred to $f(a)$, we can find a comparison bundle y' inside $[0, CDK^2\mathbf{1}]$ which is also preferred to $f(a)$ and such that $p \cdot y' \leq p \cdot y$.

Let

$$E(t) = \frac{|\{a \in A : e(a) \leq t\bar{e}\}|}{|A|}$$

By assumption, $E \in \mathcal{F}$. Let

$$\begin{aligned} A_1 &= \{a \in A : \succ_a \notin \mathcal{P}_0\} \\ A_2 &= \{a \in A : e(a) \not\leq K\bar{e}\} \\ A_3 &= \{a \in A : \|f(a)\|_1 > DK^2\} \end{aligned}$$

A_1, A_2, and A_3 are sets of agents who are, from the point of view of our decentralization argument, badly behaved. However, the members of $A_1 \cup A_2 \cup A_3$ do not upset the overall decentralization result because $\frac{1}{N}\sum_{a \in A_1 \cup A_2 \cup A_3} e(a)$ is small.

$$\frac{1}{N}\sum_{a \in A_1} e(a) = \frac{1}{N}\sum_{\succ_a \notin \mathcal{P}_0} e(a)$$

$$
\begin{aligned}
&\le \left(T\delta\bar{e} + \frac{1}{N}\sum_{e(a)\not\le T\bar{e}} e(a)\right) \\
&= \left(\frac{\varepsilon}{72} + \int_T^\infty t\, dE(t)\right)\bar{e} \\
&< \left(\frac{\varepsilon}{72} + \frac{\varepsilon}{72}\right)\bar{e} = \frac{\varepsilon}{36}\bar{e} \qquad (27)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{N}\sum_{a\in A_2} e(a) &= \frac{1}{N}\sum_{e(a)\not\le K\bar{e}} e(a) \\
&\le \frac{1}{N}\sum_{e(a)\not\le T\bar{e}} e(a) \\
&= \left(\int_T^\infty t\, dE(t)\right)\bar{e} \\
&< \frac{\varepsilon}{72}\bar{e} \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{N}\sum_{a\in A_3} e(a) &= \frac{1}{N}\sum_{\|f(a)\|_1 > DK^2} e(a) \\
&\le \frac{|\{a\in A : \|f(a)\|_1 > DK^2\}|}{N} T\bar{e} + \frac{1}{N}\sum_{e(a)\not\le T\bar{e}} e(a) \\
&\le \frac{\|\bar{e}\|_1}{DK^2} T\bar{e} + \int_T^\infty t\, dE(t) \\
&\le \frac{T}{DK}\bar{e} + \frac{\varepsilon}{72}\bar{e} \\
&\le \frac{\varepsilon}{72}\bar{e} + \frac{\varepsilon}{72}\bar{e} \\
&= \frac{\varepsilon}{36}\bar{e} \qquad (29)
\end{aligned}
$$

Note that Equations (27), (28) and (29) are identical to Equations (21), (22) and (23) in the proof of Slightly Silly Theorem 8.2. Follow the proof of Slightly Silly Theorem 8.2 *verbatim* from the end of Equation (23) through the end of Equation (24), then continue as follows:

Suppose $a \in A^{***}$, so in particular $a \notin A_1 \cup A_3$. We will construct a consumption vector y' by truncating $f(a)$ to CDK^2 and adding an equal mass so that all of the added mass is consumed at a density of at most DK^2.

Let

$$y' = f(a) \wedge CDK^2\mathbf{1} + \frac{\|(f(a) - CDK^2\mathbf{1})^+\|_1}{\|(DK^2\mathbf{1} - f(a))^+\|_1}(DK^2\mathbf{1} - f(a))^+$$

Since $a \notin A_3$, $\|f(a)\|_1 \leq DK^2$, so

$$\begin{aligned} \|(f(a) - CDK^2\mathbf{1})^+\|_1 &\leq \|(f(a) - DK^2\mathbf{1})^+\|_1 \\ &\leq \|(f(a) - DK^2\mathbf{1})^-\|_1 \\ &= \|(DK^2\mathbf{1} - f(a))^+\|_1 \end{aligned}$$

and hence

$$\begin{aligned} f(a)(t) \leq DK^2 &\Rightarrow f(a)(t) \leq y'(t) \leq DK^2 \\ DK^2 < f(a)(t) \leq CDK^2 &\Rightarrow f(a)(t) = y'(t) \\ CDK^2 < f(a)(t) &\Rightarrow y'(t) = CDK^2 \\ \|(y' - f(a))^+\|_1 &= \|(y' - f(a))^-\|_1 \\ &\geq \frac{\varepsilon_3}{2}\|(y' - f(a))^-\|_1 \end{aligned}$$

with strict inequality in the last line unless $y' = f(a)$. Thus, either $y' = f(a)$ or $y' \succ_a f(a)$, by our definition of C; in either case, $y' \in [0, CDK^2\mathbf{1}]$.

Now return to the proof of Slightly Silly Theorem 8.2 at the sentence beginning "Note that" shortly after Equation (24). Follow that proof *verbatim* through the end of Equation (26). We conclude that we can find $q \in (L')_+$ (which is called p in Equation (26)) such that $q \cdot \bar{e} = 1$ and

$$\inf q \cdot \frac{\Gamma'}{N} \geq -\frac{5\varepsilon}{36} \tag{30}$$

Since L is dense in $L^1(\mu)$, the functional q may be identified with an element of $L^\infty(\mu)$.

Proposition 6.1 implies that the price q approximately decentralizes the core allocation *with respect to comparison bundles in* $[0, CDK^2\mathbf{1}]$. Since we want to decentralize with respect to comparison bundles chosen anywhere in F_+, we modify the price q, finding a price p so that if x is a comparison bundle preferred to $f(a)$, there is a comparison bundle $y \in [0, CDK^2\mathbf{1}]$ such that $y \succ_a f(a)$ and $p \cdot y \leq p \cdot x$. In particular, we will produce a price p which is bounded away from 0, eliminating the difficulties which arise because we cannot say anything about the ratio $\max q / \min q$.

Define $P, p \in (L')_+$ by $P = q + \frac{2\varepsilon}{9K}\mathbf{1}$ and $p = \frac{P}{P \cdot \bar{e}}$.

$$\begin{aligned}
\min_\omega p(\omega) &= \frac{\frac{2\varepsilon}{9K}}{\left(q + \frac{2\varepsilon}{9K}\right) \cdot \bar{e}} \\
&\geq \frac{\frac{2\varepsilon}{9K}}{1 + \frac{2\varepsilon}{9}} \\
&> \frac{\varepsilon}{9K}
\end{aligned}$$

On the other hand, let

$$\Omega_0 = \{\omega \in \Omega : \; p(\omega) \leq 2\}$$

Then

$$\begin{aligned}
\mu(\Omega_0) &= 1 - \mu(\{\omega : p(\omega) > 2\}) \\
&\geq 1 - \frac{p \cdot v}{2} \\
&\geq 1 - \frac{p \cdot \bar{e}}{2} \\
&= \frac{1}{2}
\end{aligned}$$

Observe that if $y = x - e(a) \in \gamma'(a)$, $a \in A \setminus A_2$ and $p \cdot y < 0$, then

$$\begin{aligned}
p \cdot y &> P \cdot (x - e(a)) \\
&= \left(q + \frac{2\varepsilon}{9K}\mathbf{1}\right) \cdot (x - e(a)) \\
&\geq q \cdot (x - e(a)) - \frac{2\varepsilon}{9K}\|e(a)\|_1
\end{aligned} \tag{31}$$

Let

$$\begin{aligned}
\gamma(a) &= \{y - e(a) : y \succ_a f(a)\} \cup \{0\} \\
\Gamma &= \sum_{a \in A} \gamma(a)
\end{aligned}$$

We will show that

$$\inf p \cdot \frac{\Gamma}{N} \geq -\frac{\varepsilon}{4} \tag{32}$$

Since $\gamma(a) \supset \gamma'(a)$, $\inf p \cdot \gamma(a) \leq \inf p \cdot \gamma'(a) \leq 0$. On the other hand, if $x \in \gamma(a) \setminus \gamma'(a)$, then $x = y - e(a)$ where $y \succ_a f(a)$ and $\|y\|_\infty > CDK^2$.

Now suppose $a \in A \setminus A_2$ and $y - e(a) \in \gamma'(a) \setminus \gamma(a)$. We consider two cases:

- *Case 1:* $\|y\|_1 > \frac{DK^2}{2}$. Then

$$\begin{aligned} p \cdot y &\geq (\min p)\frac{DK^2}{2} \\ &\geq \frac{\varepsilon}{9K}\frac{18K^2}{2\varepsilon} \\ &= K \end{aligned}$$

  Since $a \in A \setminus A_2$, $p \cdot e(a) \leq Kp \cdot \bar{e} = K$, so $p \cdot (y - e(a)) \geq 0 \geq \inf p \cdot \gamma'(a)$.

- *Case 2:* $\|y\|_1 \leq \frac{DK^2}{2}$. We will construct a consumption vector y' such that $y' - e(a) \in \gamma'(a)$ and $p \cdot (y' - e(a)) \leq p \cdot (y - e(a))$; we do this by truncating y to CDK^2 and adding back a small fraction of the mass removed, so that all of the added mass is consumed at a density of at most DK^2 and at point where the price is at most 2. Let

$$y' = y \wedge CDK^2\mathbf{1} + \varepsilon_3 \frac{\|(y - CDK^2\mathbf{1})^+\|_1}{\|(DK^2\chi_{\Omega_0} - y)^+\|_1}\left(DK^2\chi_{\Omega_0} - y\right)^+$$

  Since $\|y\|_1 \leq \frac{DK^2}{2}$,

$$\begin{aligned} \|(y - CDK^2\mathbf{1})^+\|_1 &\leq \|(y - DK^2\chi_{\Omega_0})^+\|_1 \\ &\leq \|(y - DK^2\chi_{\Omega_0})^-\|_1 \\ &= \|(DK^2\chi_{\Omega_0} - y)^+\|_1 \end{aligned}$$

  and hence

$$\begin{aligned} y(t) \leq DK^2 &\Rightarrow y(t) \leq y'(t) \leq DK^2 \\ DK^2 < y(t) \leq CDK^2 &\Rightarrow y(t) = y'(t) \\ CDK^2 < y(t) &\Rightarrow y'(t) = CDK^2 \\ \|(y' - y)^+\|_1 &= \varepsilon_3\|(y' - y)^-\|_1 \\ &> \frac{\varepsilon_3}{2}\|(y' - y)^-\|_1 \end{aligned}$$

$y' \succ_a f(a)$, by our definition of C; $y' \in [0, CDK^2\mathbf{1}]$ so $y' - e(a) \in \gamma'(a)$.

$$\begin{aligned}
p \cdot (y' - e(a)) &= p \cdot (y - e(a)) + p \cdot (y' - y) \\
&= p \cdot (y - e(a)) + p \cdot (y' - y)^+ - p \cdot (y' - y)^- \\
&\leq p \cdot (y - e(a)) + 2\|(y' - y)^+\|_1 - \frac{\varepsilon}{9K}\|(y' - y)^-\|_1 \\
&= p \cdot (y - e(a)) + 2\varepsilon_3\|(y' - y)^-\|_1 - 2\varepsilon_3\|(y' - y)^-\|_1 \\
&= p \cdot (y - e(a))
\end{aligned}$$

so

$$\inf p \cdot \gamma'(a) \leq \inf p \cdot \gamma(a)$$

and hence

$$\inf p \cdot \gamma'(a) = \inf p \cdot \gamma(a)$$

Therefore, by Equations (30) and (31),

$$\begin{aligned}
\inf p \cdot \frac{1}{N} \sum_{a \in A \setminus A_2} \gamma(a) &= \inf p \cdot \frac{1}{N} \sum_{a \in A \setminus A_2} \gamma'(a) \\
&\geq \inf q \cdot \frac{1}{N} \sum_{a \in A \setminus A_2} \gamma'(a) - \frac{2\varepsilon}{9NK} \sum_{a \in A \setminus A_2} \|e(a)\|_1 \\
&\geq \inf q \cdot \frac{\Gamma'}{N} - \frac{2\varepsilon}{9} \\
&\geq -\frac{5\varepsilon}{36} - \frac{2\varepsilon}{9} = -\frac{13\varepsilon}{36}
\end{aligned}$$

By Equation (28),

$$\begin{aligned}
\inf p \cdot \frac{1}{N}\Gamma &= \frac{1}{N} \sum_{a \in A \setminus A_2} \inf p \cdot \gamma(a) + \frac{1}{N} \sum_{a \in A_2} \inf p \cdot \gamma(a) \\
&\geq -\frac{13\varepsilon}{36} - \frac{1}{N} \sum_{a \in A_2} p \cdot e(a) \\
&> -\frac{13\varepsilon}{36} - \frac{\varepsilon}{72} = -\frac{27\varepsilon}{72} = -\frac{\varepsilon^*}{6} \qquad (33)
\end{aligned}$$

We are now in a position to show that the price p approximately decentralizes the core allocation f. We have already normalized so that $p \cdot \bar{e} = 1$.

Taking X_a to be the compact set $f(a) + [0,1]v$, equimonotonicity of $\{\succ_a\}$ in the direction v implies that $f(a) + \xi v \succ_a f(a)$ for all $\xi \in (0,1)$ and all $a \in A$, so $\succ_a$ is locally nonsatiated with respect to L_+ at $f(a)$. The decentalizing conclusion then follows from Proposition 6.1. Finally, continuity of the price functional p in the topology τ follows, as before, because τ is stronger than τ^1. ■

9 Commodity Differentiation

In this Section we address decentralization in the space of measures, a setting which has been used by Mas-Colell (1975) and Jones (1984) to model commodity differentiation, and by Ostroy and Zame (1994) and Anderson (1990) in their work on perfect competition.

Throughout, we let Ω be a compact Hausdorff space. Write $C(\Omega)$ for the Banach lattice of continuous real-valued functions on Ω, equipped with the supremum norm. Write $M(\Omega)$ for the space of regular Borel measures on Ω, which is the dual space of $C(\Omega)$. The pairing between a continuous function $\varphi \in C(\Omega)$ and a measure $\mu \in M(\Omega)$ is

$$\varphi \cdot \mu = \int_\Omega \varphi \, d\mu$$

We consider two topologies on $M(\Omega)$: the norm topology induced by the total variation norm, and the *weak star* topology, which is the weakest topology for which the map

$$\mu \mapsto \varphi \cdot \mu : M(\Omega) \to \mathbf{R}$$

is continuous for each $\varphi \in C(\Omega)$. Note that $M(\Omega)$ is a Banach lattice in its norm topology, but that the weak star topology is not a lattice topology (the operations of sup and inf are not continuous).

If $\mu \in M(\Omega)$ and $\varphi \in C(\Omega)$ then $\varphi\mu$ is the measure on Ω defined by

$$\varphi\mu(E) = \int_E \varphi \, d\mu$$

Note that

$$\|\varphi\mu\| = \|\varphi\| \cdot \|\mu\| = \int_\Omega |\varphi| \, d|\mu|$$

As before, we take consumption sets for all consumers to be the nonnegative cone $M(\Omega)_+$. As before, we write $\mathcal{P}$ for the space of preference relations on L that are irreflexive, transitive, monotone and algebraically continuous. For $v \in L_+$, a preference relation $\succ \in \mathcal{P}$ is *strictly monotone in the direction* v if $x + tv \succ x$ for every $x \in L_+, t \in \mathbf{R}_+$. Write $\mathcal{P}_v$ for the set of all preferences in $\mathcal{P}$ that are strictly monotone in the direction v. An *exchange economy* is a map

$$\chi : A \to \mathcal{P} \times M(\Omega)_+$$

where A is a finite set of agents and $\mathcal{P}$ is a set of binary relations on $M(\Omega)_+$. We continue to write $\succ_a$ for the preference relation and $e(a)$ for the endowment of agent a, and

$$\bar{e} = \frac{1}{|A|}\sum_{a\in A} e(a)$$

for the mean endowment.

Definition 9.1 A set of probability distributions $\mathcal{F}$ on $\mathbf{R}_+$ is said to be *uniformly integrable* if, for every $\varepsilon > 0$, there exists $T \in \mathbf{R}$ such that for all $F \in \mathcal{F}$

$$\int_T^\infty t dF(t) < \varepsilon$$

Definition 9.2 Let $v \in M(\Omega)_+$. A set of preferences $\mathcal{P}_0$ is said to be *weak star equimonotone in the direction* v if, for every weak star compact set $X \subset M(\Omega)_+$ and every $\alpha > 0$, there is a weak star open set W such that for every preference $\succ \in \mathcal{P}_0$,

$$z \in y + \alpha v + W,\ y, z \in X \Longrightarrow z \succ y$$

Note that equimonotonicity of a single preference relation is a consequence of strong monotonicity in the direction v together with weak star continuity and transitivity; it is also a consequence of uniform weak star properness in the direction v and transitivity. Equimonotonicity is a compactness condition on the family $\mathcal{P}_0$.

Definition 9.3 The family $\mathcal{P}_0 \subset \mathcal{P}$ has *bounded marginal rates of substitution* if there is a constant $R > 0$ such that $y \succ x$ whenever $x, y \in M(\Omega)_+$ and $\|(y-x)^+\| > R\|(y-x)^-\|$. (Note that this condition implies strong monotonicity.)

The following result shows that the combination of equimonotonicity and bounded marginal rates of substitution is enough to imply approximate decentralization in the sense that the average deviation from the budget set is small and the average support error is also small; if preferences are equiconvex, the average distance from core consumptions to agents' budget sets is small. This result is very much in the spirit of Theorem 5 of Ostroy and Zame (1994), which establishes core equivalence in the presence of an economic thickness assumption; an early version of this result appeared in Anderson (1989). In the statement, δ is needed only to allow a small proportion of the agents to have preferences outside the equimonotone set $\mathcal{P}_0$.

Theorem 9.4 *For every nonzero $v \in M(\Omega)_+$, every $R, \varepsilon > 0$, and every uniformly integrable set $\mathcal{F}$ of probability distributions, there exists $\delta > 0$ such that for every set of preferences $\mathcal{P}_0 \subset \mathcal{P}_v$ which is weak star equimonotone in the direction v and has bounded marginal rates of substitution with constant R, there is an integer N_0 such that:*

If $\chi : A \to \mathcal{P}_v \times M(\Omega)_+$ is an exchange economy for which

(a) $\frac{1}{N}|\{a \in A : \succ_a \in \mathcal{P}_0\}| > 1 - \delta$

(b) $\bar{e} = \frac{1}{N}\sum e(a) \geq v$

(c) if

$$E(t) = \frac{|\{a \in A : \|e(a)\| \leq t\}|}{N}$$

then $E \in \mathcal{F}$

(d) $|A| = N \geq N_0$

and f is a core allocation of χ, then there is a price $p \in C(\Omega)_+$ that ε-decentralizes f and which satisfies

$$\begin{aligned} \frac{\max_{\omega \in \Omega} p(\omega)}{\min_{\omega \in \Omega} p(\omega)} &\leq R \\ \frac{1}{R\|\bar{e}\|} \leq \min_{\omega \in \Omega} p(\omega) &\leq \frac{1}{\|v\|} \\ \frac{1}{\|\bar{e}\|} \leq \max_{\omega \in \Omega} p(\omega) &\leq \frac{R}{\|v\|} \end{aligned}$$

Proof: Fix $\varepsilon > 0$. Find T such that

$$\int_T^\infty t \, dE(t) < \frac{\varepsilon\|v\|}{20R}$$

for all $F \in \mathcal{F}$. There is no loss of generality in assuming $\varepsilon < 1 < T$ and $R > 1$. Set

$$\begin{aligned} K &= T + \frac{\varepsilon\|v\|}{20R} \\ \delta &= \frac{\varepsilon\|v\|}{20TR} \\ C &= R \end{aligned}$$

Fix $\mathcal{P}_0$, a set of transitive preferences which is weak star equimonotone in the direction v. Let $B(0, 4CK)_+$ be the nonnegative part of the closed ball of radius $4CK$. Equimonotonicity means that we can find a weak star neighborhood W of 0 such that $z \succ x$ whenever $y, z \in B(0, 4CK)_+$, $z \in (y + \frac{\varepsilon}{20}v + W)$, $\succ \in \mathcal{P}_0$ and $y \succ x$. Without loss of generality, we may suppose that W is of the form

$$W = \{\lambda \in M(\Omega) \ : \ |p_i \cdot \lambda| < 1 \ , \ i = 1, \ldots, I\}$$

where $p_1, \ldots, p_I \in C(\Omega)$. Set

$$\varepsilon_1 = \frac{1}{2CK}$$

Continuity of the functions p_i and compactness of Ω enable us to find a finite covering $\{\Delta_j : j = 1, \ldots, J\}$ of Ω by open sets with the property that

$$\omega_1, \omega_2 \in \Delta_j \Rightarrow |p_i(\omega_1) - p_i(\omega_2)| < \varepsilon_1$$

for each i. Choose a partition of unity $\{\varphi_j : j = 1, \ldots J\}$ subordinate to the cover $\{\Delta_j\}$; thus

$$0 \le \varphi_j \le 1 \quad \text{for each } j$$

$$\sum_{j=1}^{J} \varphi_j = 1$$

$$\text{supp } \varphi_j \subset \Delta_j \quad \text{for each } j$$

Set

$$\varepsilon_2 = \frac{\varepsilon\|v\|}{40RJ}$$

and define

$$U^* = \{\lambda \in M(\Omega) \ : \ |\varphi_j \cdot \lambda| < \varepsilon_2 \quad \text{for each } j\}$$

Define the cone

$$D = \{d \in M(\Omega) : \|d^+\| \leq \frac{\|d^-\|}{R}\}$$

and

$$U^{**} = U^* \bigcap \left[\frac{1}{2}\left(M(\Omega) \setminus \left(-\frac{\varepsilon}{10}v + D\right)\right)\right]$$

Notice that

$$0 \notin \left(-\frac{\varepsilon}{10}v + D + 2U^{**}\right) \quad \text{and} \quad 0 \in U^{**} \tag{34}$$

We claim that U^{**} is weak star open. It is enough to show that D is weak star closed. Suppose $\{d_\lambda\}_{\lambda\in\Lambda}$ is a net, $d_\lambda \in D$, and $d_\lambda \to d$ in the weak star topology. It is not true generally that the positive and negative parts of a weak star convergent sequence or net of measures need be norm bounded. However, we will show *in this situation* that $(d_\lambda)^+$ and $(d_\lambda)^-$ *are* norm bounded. Observe that

$$\|(d_\lambda)^-\| - \|(d_\lambda)^+\| = -\mathbf{1} \cdot d_\lambda \to -\mathbf{1} \cdot d = \|d^-\| - \|d^+\|$$

Moreover,

$$\|(d_\lambda)^-\| - \|(d_\lambda)^+\| \geq \left(1 - \frac{1}{R}\right) \|(d_\lambda)^-\|$$

which implies, since $R > 1$, that $\|(d_\lambda)^-\|$ is bounded, so $\|(d_\lambda)^+\|$ is also bounded. Since norm bounded sets are weak star compact by Alaoglu's Theorem, we may, by choosing a subnet and relabelling, assume without loss of generality that

$$\begin{aligned} (d_\lambda)^- &\to x \\ (d_\lambda)^+ &\to y \\ y - x &= d \end{aligned}$$

Since $(d_\lambda)^-$, $(d_\lambda)^+$ are nonnegative,

$$\|y\| = \mathbf{1}\cdot y = \lim_\lambda \mathbf{1}\cdot(d_\lambda)^+ = \lim_\lambda \|(d_\lambda)^+\| \leq \lim_\lambda \frac{\|(d_\lambda)^-\|}{R} = \lim_\lambda \frac{\mathbf{1}\cdot(d_\lambda)^-}{R} = \frac{\mathbf{1}\cdot x}{R} = \frac{\|x\|}{R}$$

x and y need not be disjoint, but

$$\|x\| - \|d^-\| = \|y\| - \|d^+\| \geq 0$$

so

$$\begin{aligned} \|d^+\| &= \|y\| + \left(\|d^+\| - \|y\|\right) \\ &= \|y\| + \left(\|d^-\| - \|x\|\right) \\ &\leq \frac{\|x\|}{R} + \frac{\|d^-\| - \|x\|}{R} \\ &= \frac{\|d^-\|}{R} \end{aligned}$$

so $d \in D$; thus, D is weak star closed, so U^{**} is weak star open. Let U be a symmetric (i.e. $x \in U \to -x \in U$) convex open set with $0 \in U \subset U^{**}$.

Alaoglu's Theorem guarantees that $B(0, 2CK)$ is weak star compact, so we may use Theorem 4.1 to choose an integer N_0 such that, if $N \geq N_0$ and $X_1, \ldots, X_N \subset B(0, 2CK)$ then

$$\text{con}\left(\frac{1}{N}\sum_{n=1}^{N} X_n\right) \subset \frac{1}{N}\sum_{n=1}^{N} X_n + U$$

Now let $\chi : A \to \mathcal{P}_v \times M(\Omega)_+$ be an exchange economy satisfying assumptions (a)–(e), and fix a core allocation f. We construct a price p that approximately decentralizes f.

Let

$$E(t) = \frac{|\{a \in A : \|e(a)\| \leq t\}|}{|A|}$$

By assumption, $E \in \mathcal{F}$. Accordingly,

$$\|\bar{e}\| \leq T + \frac{\varepsilon\|v\|}{20R} = K$$

Let

$$\begin{aligned} A_1 &= \{a \in A : \succ_a \notin \mathcal{P}_0\} \\ A_2 &= \{a \in A : \|e(a)\| > K\} \end{aligned}$$

A_1 and A_2 are sets of agents who are, from the point of view of our decentralization argument, badly behaved. However, the members of $A_1 \cup A_2$ will not matter in the overall decentralization result because $\left\|\frac{1}{N}\sum_{a\in A_1\cup A_2} e(a)\right\|$ is small:

$$\begin{aligned}
\left\|\frac{1}{N}\sum_{a\in A_1} e(a)\right\| &= \left\|\frac{1}{N}\sum_{\succ_a\notin \mathcal{P}_0} e(a)\right\| \\
&\leq \left(T\delta + \frac{1}{N}\sum_{\|e(a)\|\not\leq T} \|e(a)\|\right) \\
&= \left(\frac{\varepsilon\|v\|}{20R} + \int_T^\infty t\, dE(t)\right) \\
&< \left(\frac{\varepsilon\|v\|}{20R} + \frac{\varepsilon\|v\|}{20R}\right) = \frac{\varepsilon\|v\|}{10R} \qquad (35) \\
\left\|\frac{1}{N}\sum_{a\in A_2} e(a)\right\| &= \left\|\frac{1}{N}\sum_{\|e(a)\|\not\leq K} e(a)\right\| \\
&\leq \frac{1}{N}\sum_{\|e(a)\|\not\leq T} \|e(a)\| \\
&= \left(\int_T^\infty t\, dE(t)\right) \\
&< \frac{\varepsilon\|v\|}{20R} \qquad (36) \\
& \qquad (37)
\end{aligned}$$

For each $a \in A$ define

$$\gamma'(a) = \{y - e(a) \ :\ \|y\| \leq CK, y \succ_a f(a)\} \cup \{0\}$$

and write

$$\Gamma' = \sum_{a\in A} \gamma'(a)$$

Note that $\gamma'(a)$ is a truncated version of the net preferred set of agent a (together with $\{0\}$), and that Γ' is a truncated version of the aggregate net preferred set. We will first find a decentralizing price with respect to comparison bundles in $B(0, CK)_+$; then, we shall show that comparison bundles outside $B(0, CK)_+$ are too expensive to upset the decentralization.

We claim that

$$\frac{1}{N}\Gamma' \cap \left(-\frac{\varepsilon}{5}\bar{e} + D + 2U\right) = \emptyset$$

To see this, suppose not. Then we can find a coalition $A^{**} \subset A$ and vectors $y(a) - e(a) \in \gamma'(a), y(a) - e(a) \neq 0,\ d^* \in D, u \in 2U$ with

$$\frac{1}{N}\sum_{a\in A^{**}}[y(a) - e(a)] = -\frac{\varepsilon}{5}\bar{e} + d^* + u$$

Let $A^* = A^{**} \setminus A_1$. Then

$$\begin{aligned}\frac{1}{N}\sum_{a\in A^*}[y(a) - e(a)] &\leq \frac{1}{N}\sum_{a\in A^{**}}[y(a) - e(a)] + \frac{1}{N}\sum_{a\in A_1} e(a)\\ &\leq -\frac{\varepsilon}{10}\bar{e} + d^* + u + \left(-\frac{\varepsilon}{10}\bar{e} + \frac{1}{N}\sum_{a\in A_1} e(a)\right)\end{aligned}$$

From Equation (35),

$$\begin{aligned}\left\|\left(-\frac{\varepsilon}{10}\bar{e} + \frac{1}{N}\sum_{a\in A_1} e(a)\right)^+\right\| &= \left\|\frac{1}{N}\sum_{a\in A_1} e(a)\right\| < \frac{\varepsilon\|v\|}{10R}\\ \left\|\left(-\frac{\varepsilon}{10}\bar{e} + \frac{1}{N}\sum_{a\in A_1} e(a)\right)^-\right\| &= \left\|-\frac{\varepsilon}{10}\bar{e}\right\| \geq \frac{\varepsilon\|v\|}{10}\end{aligned}$$

Setting

$$d = d^* + \left(-\frac{\varepsilon}{10}\bar{e} + \sum_{a\in A_1} e(a)\right) \in D$$

we have

$$\frac{1}{N}\sum_{a\in A^*}[y(a) - e(a)] \leq -\frac{\varepsilon}{10}\bar{e} + d + u \leq -\frac{\varepsilon}{10}v + d + u$$

Since $0 \notin -\frac{\varepsilon}{10}v + D + 2U$ by Equation (34), we must have $A^* \neq \emptyset$.

$$\sum_{a\in A^*}\left[y(a) + \frac{\varepsilon}{20}v\right] \leq \sum_{a\in A^*} e(a) - \frac{N\varepsilon}{10}v + Nd + Nu + \frac{|A^*|\varepsilon}{20}v$$

$$\leq \sum_{a\in A^*} e(a) - \frac{N\varepsilon}{20}v + Nd + Nu - \left(\frac{N\varepsilon}{20}v - \frac{|A^*|\varepsilon}{20}v\right)$$

$$\leq \sum_{a\in A^*} e(a) - \frac{N\varepsilon}{20}v + Nd + Nu$$

We shall use the left hand side as a template to redistribute $\sum_{a\in A^*} e(a)$, obtaining a contradiction to the fact that f is a core allocation. To this end, write $x(a) = y(a) + (\varepsilon/20)v$ for each $a \in A^*$ and define

$$\begin{aligned}
x^* &= \sum_{a\in A^*} x(a) \\
e^* &= \sum_{a\in A^*} e(a) \\
a_j &= \min\{\varphi_j \cdot e^*, \varphi_j \cdot x^*\} \\
b_j &= \varphi_j \cdot x^* - a_j \\
b &= \sum_{j=1}^{J} b_j \\
\beta_j &= [1 - \frac{a_j}{\varphi_j \cdot e^*}]\varphi_j e^* \\
\beta &= \sum_{j=1}^{J} \beta_j
\end{aligned}$$

(In the definition of β_j, we use the convention that $0/0 = 0$.) For each $a \in A^*$, define

$$z(a) = \sum_{j=1}^{J} \frac{\varphi_j \cdot x(a)}{(\varphi_j \cdot x^*)(\varphi_j \cdot e^*)}(a_j \varphi_j e^*) + \left[\sum_{j=1}^{J} \frac{(\varphi_j \cdot x(a))b_j}{(\varphi_j \cdot x^*)b}\right]\beta$$

At this point, we pause to provide some motivation for the construction just given. It is useful in this discussion to pretend (contrary to fact) that φ_j is the characteristic function of a set Ω_j, where $\{\Omega_j : j = 1, \cdots, J\}$ is a partition of Ω. Then $\varphi_j e^*$ is just the endowment possessed by the coalition A^*, restricted to the set Ω_j, while $\varphi_j x^*$ is the consumption on the set Ω_j prescribed by x. The first term in the definition of $z(a)$ distributes all or part of $\varphi_j e^*$ among the members of A^* in proportion to $\varphi_j \cdot x(a) = \|\varphi_j x(a)\|$. In the event that $\|\varphi_j x^*\| > \|\varphi_j \cdot e^*\|$, all of $\varphi_j e^*$ is distributed; note that this will leave members of the coalition less well off than at x, so they will need to be compensated by getting additional consumption on other sets in the partition. In the event that $\|\varphi_j x^*\| \leq \|\varphi_j \cdot e^*\|$, the total mass distributed is $\|\varphi_j x^*\|$, as this is enough to give the same utility level as at x, modulo an element of W; the remaining mass is retained in the second term. The second term then compensates individuals for losses on the partition sets on which $\|\varphi_j x^*\| > \|\varphi_j e^*\|$ by handing out the retained mass in proportion to the losses on those partition sets. The amount of mass retained is large enough so that the norm of the compensating handout is at least R times the norm of the losses. We then use the assumption that marginal rates of substitution are bounded by R, and the equimonotonicity assumption (which involves W) to show that $z(a) \succ_a f(a)$.

We now return to the formal proof that A^* can block f via the allocation z. The first step is to show that it is feasible for the coalition A^* to achieve the consumption specified by z:

$$\begin{aligned}
&\sum_{a\in A^*} z(a) \\
&= \sum_{a\in A^*}\left[\sum_{j=1}^{J}\frac{\varphi_j\cdot x(a)}{(\varphi_j\cdot x^*)(\varphi_j\cdot e^*)}(a_j\varphi_j e^*)+\left[\sum_{j=1}^{J}\frac{(\varphi_j\cdot x(a))b_j}{(\varphi_j\cdot x^*)b}\right]\beta\right] \\
&= \sum_{j=1}^{J}\frac{\varphi_j\cdot\sum_{a\in A^*}x(a)}{(\varphi_j\cdot x^*)(\varphi_j\cdot e^*)}(a_j\varphi_j e^*)+\left[\sum_{j=1}^{J}\frac{(\varphi_j\cdot\sum_{a\in A^*}x(a))b_j}{(\varphi_j\cdot x^*)b}\right]\beta \\
&= \sum_{j=1}^{J}\frac{\varphi_j\cdot x^*}{(\varphi_j\cdot x^*)(\varphi_j\cdot e^*)}(a_j\varphi_j e^*)+\left[\sum_{j=1}^{J}\frac{(\varphi_j\cdot x^*)b_j}{(\varphi_j\cdot x^*)b}\right]\beta \\
&= \sum_{j=1}^{J}\frac{1}{(\varphi_j\cdot e^*)}(a_j\varphi_j e^*)+\left[\sum_{j=1}^{J}\frac{b_j}{b}\right]\beta \\
&= \sum_{j=1}^{J}(\varphi_j e^*-\beta_j)+\beta \\
&= e^*-\beta+\beta=e^*
\end{aligned}$$

The next step is to show that $z(a) \succ_a y(a)$ — and hence that $z(a) \succ_a f(a)$ — for each $a \in A^*$. We first show that the norm of the consumption handed out in the second term of the definition of $z(a)$ is at least R times the norm of consumption lost in going from x to the first term in the definition of $z(a)$. This requires that we given an estimate on β. Write $\mathcal{J}$ for the set of indices j for which $\varphi_j \cdot e^* \leq \varphi_j \cdot x^*$ and $\mathcal{K}$ for the complementary set of indices for which $\varphi_j \cdot e^* > \varphi_j \cdot x^*$. We want to establish the following estimate:

$$\|\beta\| > R \sum_{j \in \mathcal{J}} [\varphi_j \cdot x^* - \varphi_j \cdot e^*]$$

To this end, we first use the fact that the norm of the sum of positive measures is the sum of the norms to write

$$\begin{aligned}
\|\beta\| &= \|\sum_j \beta_j\| \\
&= \sum_j \|\beta_j\| \\
&= \sum_j \left\| \frac{\varphi_j \cdot e^* - a_j}{\varphi_j \cdot e^*} \varphi_j e^* \right\| \\
&= \sum_j |\varphi_j \cdot e^* - a_j| \\
&= \sum_j [\varphi_j \cdot e^* - \min\{\varphi_j \cdot e^*, \varphi_j \cdot x^*\}]
\end{aligned}$$

Thus

$$\|\beta\| = \sum_{j \in \mathcal{K}} [\varphi_j \cdot e^* - \varphi_j \cdot x^*]$$

Recall that

$$x^* \leq e^* - \frac{N\varepsilon}{20} v + Nd + Nu$$

Substituting and collecting terms yields

$$\begin{aligned}
&\|\beta\| - R \sum_{j \in \mathcal{J}} [\varphi_j \cdot x^* - \varphi_j \cdot e^*] \\
&\geq \sum_{j \in \mathcal{K}} \varphi_j \cdot (e^* - x^*) - R \sum_{j \in \mathcal{J}} \varphi_j \cdot (x^* - e^*) \\
&\geq \sum_{j \in \mathcal{K}} \varphi_j \cdot \left(\frac{N\varepsilon}{20} v - Nd - Nu \right) - R \sum_{j \in \mathcal{J}} \varphi_j \cdot \left(-\frac{N\varepsilon}{20} v + Nd + Nu \right) \\
&= N \left\{ \frac{\varepsilon}{20} \sum_{j=1}^{J} \varphi_j \cdot v + (R-1) \sum_{j \in \mathcal{J}} \varphi_j \cdot v \right\}
\end{aligned}$$

$$
\begin{aligned}
& +N\left\{-\sum_{j=1}^{J}\varphi_j\cdot d-(R-1)\sum_{j\in\mathcal{J}}\varphi_j\cdot d\right\} \\
& +N\left\{-\sum_{j\in\mathcal{K}}\varphi_j\cdot u-R\sum_{j\in\mathcal{J}}\varphi_j\cdot u\right\} \\
\geq\; & N\left\{\frac{\varepsilon}{20}\mathbf{1}\cdot v+0\right\} \\
& +N\left\{-\mathbf{1}\cdot d-(R-1)\|d^+\|\right\} \\
& +N\left\{-RJ\frac{\varepsilon\|v\|}{20RJ}\right\} \\
\geq\; & N\left[\frac{\varepsilon}{20}\|v\|-\frac{\varepsilon}{20}\|v\|-\left(\|d^-\|-\|d^+\|-(R-1)\|d^+\|\right)\right] \\
=\; & N\left[\|d^-\|-R\|d^+\|\right]\geq 0
\end{aligned}
$$

since $d \in D$. Therefore,

$$\|\beta\| \geq R\sum_{j\in\mathcal{J}}[\varphi_j\cdot x^*-\varphi_j\cdot e^*]$$

as desired.

We now are in a position to complete the demonstration that $z(a) \succ_a y(a)$ for each $a \in A^*$. The idea is to connect $z(a)$ to $y(a)$ in two steps, the first of which can be handled using the assumption that marginal rates of substitution are bounded by R, and the second of which can be handled using equimonotonicity. Fix $a \in A^*$. Write

$$\alpha_j = a_j\frac{\varphi_j\cdot x(a)}{(\varphi_j\cdot x^*)(\varphi_j\cdot e^*)}(\varphi_j e^*)$$

and

$$\mu = \left[\sum_{j=1}^{J} b_j\frac{\varphi_j\cdot x(a)}{(\varphi_j\cdot x^*)b}\right]\beta$$

so that

$$z(a) \;=\; \sum_j \varphi_j\alpha_j+\mu$$

$$= \quad (1 - \sum_j \varphi_j)(y(a) + \frac{\varepsilon}{20}v) + \sum_j \varphi_j \alpha_j + \mu$$
$$= \quad y(a) + \frac{\varepsilon}{20}v + \sum_j \varphi_j \alpha_j - \sum_j \varphi_j x(a) + \mu$$

We have

$$\sum_{j=1}^{J} \varphi_j \alpha_j - \sum_{j=1}^{J} \varphi_j x(a)$$
$$= \quad \sum_{\mathcal{K}} [\varphi_j \alpha_j - \varphi_j x(a)] + \sum_{\mathcal{J}} [\varphi_j \alpha_j - \frac{\varphi_j \cdot e^*}{\varphi_j \cdot x^*} \varphi_j x(a)] - \sum_{\mathcal{J}} (1 - \frac{\varphi_j \cdot e^*}{\varphi_j \cdot x^*}) \varphi_j x(a)$$

Write

$$w_1 = \sum_{\mathcal{K}} [\varphi_j \alpha_j - \varphi_j x(a)] + \sum_{\mathcal{J}} [\varphi_j \alpha_j - \frac{\varphi_j \cdot e^*}{\varphi_j \cdot x^*} \varphi_j x(a)]$$

and

$$w_2 = \sum_{\mathcal{J}} \left(1 - \frac{\varphi_j \cdot e^*}{\varphi_j \cdot x^*}\right) \varphi_j x(a)$$

so that

$$z(a) = y(a) + \frac{\varepsilon}{20} v + w_1 - w_2 + \mu$$

We claim that $w_1 \in W$ and $w_2 - \mu \in D$. By construction,

$$\|\varphi_j \alpha_j\| = a_j \frac{\varphi_j \cdot x(a)}{\varphi_j \cdot x^*}$$

Note that $a_j = \varphi_j \cdot e^*$ when $j \in \mathcal{J}$, and $a_j = \varphi_j \cdot x^*$ when $j \in \mathcal{K}$ so

$$\|\varphi_j \alpha_j\| = \begin{cases} \frac{\varphi_j \cdot e^*}{\varphi_j \cdot x^*} \|\varphi_j x(a)\| & \text{for } j \in \mathcal{J} \\ \|\varphi_j x(a)\| & \text{for } j \in \mathcal{K} \end{cases}$$

$$\begin{aligned} |p_i \cdot w_1| \quad &\leq \quad \sum_{j \in \mathcal{K}} |p_i \cdot (\varphi_j \alpha_j - \varphi_j x(a))| + \sum_{j \in \mathcal{J}} \left| p_i \cdot \left(\varphi_j \alpha_j - \frac{\varphi_j \cdot e^*}{\varphi_j \cdot x^*} \varphi_j (x(a)) \right) \right| \\ &\leq \quad \sum_{j \in \mathcal{K}} \frac{1}{2CK} \|\varphi_j x(a)\| + \sum_{j \in \mathcal{J}} \frac{1}{2CK} \frac{\varphi_j \cdot e^*}{\varphi_j \cdot x^*} \|\varphi_j x(a)\| \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{j=1}^{J} \frac{\|\varphi_j x(a)\|}{2CK} \\
&= \frac{1}{2CK}\left\|\sum_{j=1}^{J} \varphi_j x(a)\right\| \\
&= \frac{\|x(a)\|}{2CK} \\
&= \frac{\left\|y(a) + \frac{\varepsilon}{20}v\right\|}{2CK} \\
&\leq \frac{\|y(a)\| + \frac{\varepsilon}{20}\|v\|}{2CK} \\
&\leq \frac{CK + R(K-T)}{2CK} \\
&< \frac{2CK}{2CK} = 1
\end{aligned}
$$

so $w_1 \in W$. Our estimate for $\|\beta\|$ guarantees that $\|\mu\| > R\|w_2\|$, so our assumption that marginal rates of substitution are bounded by R guarantees that

$$z(a) = y(a) + \frac{\varepsilon}{20}v + w_1 - w_2 + \mu \succ_a y(a) + \frac{\varepsilon}{20}v + w_1$$

Since

$$
\begin{aligned}
\|y + \frac{\varepsilon}{20}v\| &= \|y(a)\| + \frac{\varepsilon}{20}\|v\| \\
&\leq CK + R(K-T) \\
&\leq 2CK \\
\|y + \frac{\varepsilon}{20}v + w_1\| &\leq 2CK + \|w_1\| \\
&\leq 2CK + 2\|x(a)\| \\
&= 2CK + \|y(a) + \frac{\varepsilon}{20}v\| \\
&\leq 4CK
\end{aligned}
$$

equimonotonicity guarantees that

$$z(a) = y(a) + \frac{\varepsilon}{20}v + w_1 \succ_a y(a)$$

Finally, $y(a) \succ_a f(a)$ by construction. Since $A^* \subset A \setminus A_1$, $\succ_a \in \mathcal{P}_0$, and thus $\succ_a$ is transitive, so $z(a) \succ_a f(a)$, as asserted. However, since z is feasible for the coalition A^* and f is a core allocation, this is a contradiction. We conclude that

$$\frac{1}{N}\Gamma' \cap \left(-\frac{\varepsilon}{5}\bar{e} + D + 2U\right) = \emptyset$$

as claimed.

From our choice of N_0 it follows that

$$\frac{1}{N}\text{con } \Gamma' \subset \frac{1}{N}\Gamma' + U$$

Suppose that

$$\frac{1}{N}\text{con } \Gamma' \cap \left(-\frac{\varepsilon}{5}\bar{e} + D + U\right) \neq \emptyset$$

Then we can find $d \in D$, $u \in U$ such that

$$\left(-\frac{\varepsilon}{5}\bar{e} + d + u\right) \in \frac{1}{N}\text{con } \Gamma' \subset \frac{1}{N}\Gamma' + U$$

so there exists $G \in \frac{1}{N}\Gamma'$ and $u' \in U$ such that

$$\left(-\frac{\varepsilon}{5}\bar{e} + d + u\right) = G + u'$$

so

$$G = \left(-\frac{\varepsilon}{5}\bar{e} + d + (u - u')\right)$$

Since U is symmetric, $-u' \in U$; since U is convex,

$$u - u' = 2\left(\frac{u - u'}{2}\right) \in 2U$$

which shows

$$\frac{1}{N}\Gamma' \cap \left(-\frac{\varepsilon}{5}\bar{e} + D + 2U\right) \neq \emptyset$$

a contradiction which establishes that

$$\frac{1}{N}\text{con } \Gamma' \cap \left(-\frac{\varepsilon}{5}\bar{e} + D + U\right) = \emptyset$$

Since $\frac{1}{N}\text{con } \Gamma'$ is convex and $\{-(\varepsilon/5)\bar{e} + D + U\}$ is convex and weak star open, we can find a weak star continuous linear functional p on $M(\Omega)$ (that is, a continuous function $p \in C(\Omega)$) that separates $\frac{1}{N}\mathrm{con}\,\Gamma'$ from $\{-(\varepsilon/5)\bar{e} + D + U\}$. Proposition 6.1 then implies that the price p approximately decentralizes the core allocation *with respect to comparison bundles in* $B(0, CK)_+$. Since we want to show we have decentralization with respect to comparison bundles anywhere in $M(\Omega)_+$, we need to show that bundles outside $B(0, CK)_+$ are too expensive to upset the decentralization. The definition of D guarantees that p satisfies

$$\max_{\omega\in\Omega} p(\omega) / \min_{\omega\in\Omega} p(\omega) \le R$$

Since this implies $\min_{\omega\in\Omega} p(\omega) > 0$, $p \cdot \bar{e} > 0$; renormalizing, we may assume that $p \cdot \bar{e} = 1$, so

$$\begin{aligned} \inf p \cdot \left(\frac{1}{N}\Gamma'\right) &\ge -\frac{\varepsilon}{5} p \cdot \bar{e} = -\frac{\varepsilon}{5} \qquad (38) \\ \frac{\max_{\omega\in\Omega} p(\omega)}{\min_{\omega\in\Omega} p(\omega)} &\le R \\ \frac{1}{R\|\bar{e}\|} \le \min_{\omega\in\Omega} p(\omega) &\le \frac{1}{\|v\|} \\ \frac{1}{\|\bar{e}\|} \le \max_{\omega\in\Omega} p(\omega) &\le \frac{R}{\|v\|} \end{aligned}$$

Let

$$\begin{aligned} \gamma(a) &= \{y - e(a) : y \succ_a f(a)\} \cup \{0\} \\ \Gamma &= \sum_{a\in A} \gamma(a) \end{aligned}$$

We will show that

$$\inf p \cdot \frac{\Gamma}{N} \ge -\frac{\varepsilon}{4} \qquad (39)$$

Since $\gamma(a) \supset \gamma'(a)$, $\inf p \cdot \gamma(a) \le \inf p \cdot \gamma'(a) \le 0$. On the other hand, if $x \in \gamma(a) \setminus \gamma'(a)$, then $x = y - e(a)$ where $y \succ_a f(a)$ and $\|y\| > CK$.

Therefore,

$$\begin{aligned} p \cdot y &> (\min_{\omega\in\Omega} p(\omega))\|y\| \\ &\ge \frac{CK}{\|v\|} = \frac{RK}{\|v\|} \end{aligned}$$

Now suppose $a \in A \setminus A_2$. Then

$$\begin{aligned} p \cdot e(a) &\leq \max_{\omega \in \Omega} p(\omega) \|e(a)\| \\ &\leq \frac{R}{\|v\|} K = \frac{RK}{\|v\|} \end{aligned}$$

so $p \cdot (y - e(a)) > 0 \geq \inf p \cdot \gamma'(a)$, so $\inf p \cdot \gamma(a) = \inf p \cdot \gamma'(a)$. Therefore, by Equations (36) and (38),

$$\begin{aligned} \inf p \cdot \frac{1}{N}\Gamma &= \frac{1}{N} \sum_{a \in A \setminus A_2} \inf p \cdot \gamma(a) + \frac{1}{N} \sum_{a \in A_2} \inf p \cdot \gamma(a) \\ &\geq \frac{1}{N} \sum_{a \in A \setminus A_2} \inf p \cdot \gamma'(a) - \frac{1}{N} \sum_{a \in A_2} p \cdot e(a) \\ &\geq -\frac{\varepsilon}{5} - \max_{\omega \in \Omega} p(\omega) \left\| \frac{1}{N} \sum_{a \in A_2} e(a) \right\| \\ &> -\frac{\varepsilon}{5} - \frac{R}{\|v\|} \times \frac{\varepsilon \|v\|}{20R} \\ &= -\frac{\varepsilon}{5} - \frac{\varepsilon}{20} = -\frac{\varepsilon}{4} \end{aligned} \tag{40}$$

We are now in a position to show that the price p approximately decentralizes the core allocation f. We have already normalized so that $p \cdot \bar{e} = 1$, which is (i). Taking X_a to be the compact set $f(a) + [0,1]v$, equimonotonicity of $\{\succ_a\}$ in the direction v implies that $f(a) + \xi v \succ_a f(a)$ for all $\xi \in (0,1)$ and all $a \in A$, so $\succ_a$ is locally nonsatiated with respect to $M(\Omega)_+$ at $f(a)$. Proposition 6.1 (modulo a slight rejuggling of the epsilonics) then implies that p ε-decentralizes f. The conclusions about $\min_{\omega \in \Omega} p(\omega)$ and $\max_{\omega \in \Omega} p(\omega)$ are contained in Equation (38). ■

The argument of Theorem 9.4 uses the assumption of bounded marginal rates of substitution to provide upper and lower bounds on the separating price, which in turn guarantee that anything outside the ball of radius CK is too expensive for anyone to afford. In the absence of a bound on marginal rates of substitution, however, stronger restrictions on the endowments will still yield approximate decentralization. The following result is in the spirit of Theorem 4 of Ostroy and Zame (1994), which yields core equivalence in the presence of physical thickness assumptions.

Theorem 9.5 *For every $v \in M(\Omega)_+$ with supp $v = \Omega$, every $K, \varepsilon > 0$, and every uniformly integrable set $\mathcal{F}$ of probability distributions, there exists $\delta > 0$ such that for every set of preferences $\mathcal{P}_0 \subset \mathcal{P}_v$ which is weak star equimonotone in the direction v, there is an integer N_0 such that:*

If $\chi : A \to \mathcal{P}_v \times M(\Omega)_+$ is an exchange economy for which

(a) $\frac{1}{N}|\{a \in A : \succ_a \in \mathcal{P}_0\}| > 1 - \delta$

(b) $\bar{e} = \frac{1}{N} \sum e(a) \geq v$

(c) $\|\bar{e}\| \leq K$

(d) if

$$E(t) = \frac{|\{a \in A : e(a) \leq t\bar{e}\}|}{N}$$

then $E \in \mathcal{F}$[23]

(e) $|A| = N \geq N_0$

and f is a core allocation of χ, then there is a price $p \in C(\Omega)_+$ that ε-decentralizes f.

Proof: The argument has much in common with the proof of Theorem 9.4, but there are some significant differences. Fix $\varepsilon > 0$. Find T such that

$$\int_T^{\infty} t \, dE(t) < \varepsilon/36$$

for all $F \in \mathcal{F}$. There is no loss of generality in assuming $\varepsilon < 1 < T \leq K$. Set

$$\delta = \frac{\varepsilon}{36T}$$

Set

$$C = \frac{36K^2}{\varepsilon}$$

Fix $\mathcal{P}_0$, a set of preferences which is weak star equimonotone in the direction v. Let $B(0, 2CK)_+$ be the nonnegative part of the closed ball of radius

[23] Note in particular we do not require $e(a)$ to lie in the order ideal generated by v.

$2CK$. Equimonotonicity means that we can find a weak star neighborhood W of 0 such that $z \succ x$ whenever $y, z \in B(0, 2CK)_+$, $z \in (y + \frac{\varepsilon}{36}v + W)$, $\succ \in \mathcal{P}_0$ and $y \succ x$. Without loss of generality, we may suppose that W is of the form

$$W = \{\lambda \in M(\Omega) \ : \ |p_i \cdot \lambda| < 1 \ , \ i = 1, \ldots, I\}$$

where $p_1, \ldots, p_I \in C(\Omega)$.

Set

$$\varepsilon_1 = \frac{1}{2CK}$$

Continuity of the functions p_i and compactness of Ω enable us to find a finite covering $\{\Delta_j : j = 1, \ldots, J\}$ of Ω by open sets with the property that

$$\omega_1, \omega_2 \in \Delta_j \Rightarrow |p_i(\omega_1) - p_i(\omega_2)| < \varepsilon_1$$

for each i. Choose a partition of unity $\{\varphi_j : j = 1, \ldots J\}$ subordinate to the cover $\{\Delta_j\}$; thus

$$0 \leq \varphi_j \leq 1 \quad \text{for each } j$$

$$\sum_{j=1}^{J} \varphi_j = 1$$

$$\text{supp } \varphi_j \subset \Delta_j \quad \text{for each } j$$

Without loss of generality, we may assume that $\{\Delta_j : j = 1, \ldots, J\}$ contains no proper subcover, and hence that no φ_j is identically zero. Our assumption that supp $v = \Omega$ guarantees that $\varphi_j \cdot v > 0$ for each j; set

$$\varepsilon_2 = \min\left\{\frac{\varepsilon\varphi_1 \cdot v}{72}, \ldots, \frac{\varepsilon\varphi_J \cdot v}{72}\right\}$$

and define

$$U = \{\lambda \in M(\Omega) \ : \ |\varphi_j \cdot \lambda| < \varepsilon_2 \quad \text{for each } j\}$$

Alaoglu's Theorem guarantees that $B(0, CK)_+$ is weak star compact, so we may use Theorem 4.1 to choose an integer N_0 such that, if $N \geq N_0$ and $X_1, \ldots, X_N \subset B(0, CK)_+$ then

$$\text{con}\left(\frac{1}{N}\sum_{n=1}^{N} X_n\right) \subset \frac{1}{N}\sum_{n=1}^{N} X_n + U$$

Now let $\chi : A \to \mathcal{P}_v \times M(\Omega)_+$ be an exchange economy satisfying assumptions (a) – (e), and fix a core allocation f. We construct a price p that approximately decentralizes f.

Let

$$E(t) = \frac{|\{a \in A : e(a) \leq t\bar{e}\}|}{|A|}$$

By assumption, $E \in \mathcal{F}$. Let

$$\begin{aligned} A_1 &= \{a \in A : \succ_a \notin \mathcal{P}_0\} \\ A_2 &= \{a \in A : \ e(a) \not\leq K\bar{e}\} \end{aligned}$$

A_1 and A_2 are sets of agents who are, from the point of view of our decentralization argument, badly behaved. However, the members of $A_1 \cup A_2$ will not matter in the overall decentralization result because $\frac{1}{N}\sum_{a \in A_1 \cup A_2} e(a)$ is small:

$$\begin{aligned} \frac{1}{N}\sum_{a \in A_1} e(a) &= \frac{1}{N}\sum_{\succ_a \notin \mathcal{P}_0} e(a) \\ &\leq \left(T\delta\bar{e} + \frac{1}{N}\sum_{e(a)\not\leq T\bar{e}} e(a)\right) \\ &= \left(\frac{\varepsilon}{36} + \int_T^\infty t\ dE(t)\right)\bar{e} \\ &< \left(\frac{\varepsilon}{36} + \frac{\varepsilon}{36}\right)\bar{e} = \frac{\varepsilon}{18}\bar{e} \qquad (41) \\ \frac{1}{N}\sum_{a \in A_2} e(a) &= \frac{1}{N}\sum_{e(a)\not\leq K\bar{e}} e(a) \\ &\leq \frac{1}{N}\sum_{e(a)\not\leq T\bar{e}} e(a) \\ &= \left(\int_T^\infty t\ dE(t)\right)\bar{e} \\ &< \frac{\varepsilon}{36}\bar{e} \qquad (42) \end{aligned}$$

For each $a \in A$, set

$$\gamma'(a) = \{y - e(a)\ :\ \|y\| \leq CK,\ y \succ_a f(a)\} \cup \{0\}$$

and write

$$\Gamma' = \sum_{a \in A} \gamma'(a)$$

Note that $\gamma'(a)$ is a truncated version of the net preferred set of agent a (together with $\{0\}$), and that Γ' is a truncated version of the aggregate net preferred set. We will first find a decentralizing price with respect to comparison bundles in $B(0, CK)_+$; then, we shall modify the price so that comparison bundles outside $B(0, CK)_+$ are too expensive to upset the decentralization.

We claim that

$$\frac{1}{N}\Gamma' \cap \left(-\frac{\varepsilon}{9}\bar{e} - M(\Omega)_+ + 2U\right) = \emptyset$$

To see this, suppose not. Then we can find a coalition $A^{**} \subset A$ and vectors $y(a) - e(a) \in \gamma'(a), d \in M(\Omega)_+, u \in 2U$ with

$$\frac{1}{N}\sum_{a \in A^{**}} [y(a) - e(a)] = -\frac{\varepsilon}{9}\bar{e} - d + u \leq -\frac{\varepsilon}{9}\bar{e} + u$$

Let $A^* = A^{**} \setminus A_1$. Then

$$\begin{aligned}
\frac{1}{N}\sum_{a \in A^*} [y(a) - e(a)] &\leq \frac{1}{N}\sum_{a \in A^{**}} [y(a) - e(a)] + \frac{1}{N}\sum_{a \in A_1} e(a) \\
&\leq -\frac{\varepsilon}{9}\bar{e} + u + \frac{\varepsilon}{18}\bar{e} \leq -\frac{\varepsilon}{18}v + u
\end{aligned}$$

by Equation (41). Since $\frac{\varepsilon}{18}v \notin 2U$, we must have $A^* \neq \emptyset$. Therefore,

$$\begin{aligned}
\sum_{a \in A^*}\left[y(a) + \frac{\varepsilon}{36}v\right] &\leq \sum_{a \in A^*} e(a) - \frac{N\varepsilon}{18}v + Nu + \frac{|A^*|\varepsilon}{36}v \\
&\leq \sum_{a \in A^*} e(a) - \frac{N\varepsilon}{36}v + Nu - \left(\frac{N\varepsilon}{36}v - \frac{|A^*|\varepsilon}{36}v\right) \\
&\leq \sum_{a \in A^*} e(a) - \frac{N\varepsilon}{36}v + Nu
\end{aligned}$$

As in the proof of Theorem 9.4, we use the left hand side as a template to redistribute $\sum_{a \in A^*} e(a)$, obtaining a contradiction to the fact that f is a core allocation. To this end, write $x(a) = y(a) + \frac{\varepsilon}{36}v$ for each a, and $e^* = \sum_{a\in A^*} e(a)$; for each $a \in A^*$, define

$$z(a) = \sum_j \frac{\varphi_j \cdot x(a)}{\varphi_j \cdot e^*}\varphi_j e^*$$

We claim that z is a feasible allocation for the coalition A^*, and is preferred to $f(a)$ by each member of A^*. To see that z is feasible, we simply expand $\sum_{a\in A^*} z(a) - e(a)$, using the facts at our disposal:

$$\begin{aligned}
\sum_{a\in A^*}(z(a) - e(a)) &= \sum_{a\in A^*}\sum_j \left(\frac{\varphi_j \cdot x(a)}{\varphi_j \cdot e^*}\varphi_j e^* - \varphi_j e(a)\right) \\
&= \sum_j \left(\sum_{a\in A^*}\left[\frac{\varphi_j \cdot x(a)}{\varphi_j \cdot e^*}\varphi_j e^*\right] - \varphi_j e^*\right) \\
&= \sum_j \left(\varphi_j \cdot \left[\sum_{a\in A^*} x(a)\right]\frac{\varphi_j e^*}{\varphi_j \cdot e^*} - \varphi_j e^*\right) \\
&\leq \sum_j \left(\varphi_j \cdot \left[e^* - N\frac{\varepsilon}{36}v + Nu\right]\frac{\varphi_j e^*}{\varphi_j \cdot e^*} - \varphi_j e^*\right) \\
&= N\sum_j \left(\left[-\varphi_j \cdot \left(\frac{\varepsilon}{36}v\right) + \varphi_j \cdot u\right]\frac{\varphi_j e^*}{\varphi_j \cdot e^*}\right)
\end{aligned}$$

Our choice of ε_2 guarantees that $\varphi_j \cdot u < 2\varphi_j \cdot (\varepsilon v/72) = \varphi_j(\varepsilon v/36)$ for each j. We conclude that

$$\sum_{a\in A^*} z(a) - e(a) \leq 0$$

That is, z is feasible for the coalition A^*.

On the other hand, we claim that z is preferred to the core allocation f by every member of A^*. To see this, fix $a \in A^*$ and note that

$$\frac{\varphi_j \cdot x(a)}{\varphi_j \cdot e^*}\varphi_j e^* \quad \text{and} \quad \varphi_j x(a)$$

are both nonnegative, have support contained in Δ_j, and have the same norm:

$$\left\|\frac{\varphi_j \cdot x(a)}{\varphi_j \cdot e^*}\varphi_j e^*\right\| = \frac{|\varphi_j \cdot x(a)|}{|\varphi_j \cdot e^*|}\|\varphi_j e^*\| = \frac{|\varphi_j \cdot x(a)|}{|\varphi_j \cdot e^*|}|\varphi_j \cdot e^*| = \|\varphi_j x(a)\|$$

Therefore,

$$\begin{aligned}|p_i \cdot (z(a) - x(a))| &= \left| p_i \cdot \left(\sum_j \frac{\varphi_j \cdot x(a)}{\varphi_j \cdot e^*} \varphi_j e^* - \sum_j \varphi_j x(a) \right) \right| \\ &\leq \sum_j \left| p_i \cdot \left(\frac{\varphi_j \cdot x(a)}{\varphi_j \cdot e^*} \varphi_j e^* - \varphi_j x(a) \right) \right| \\ &\leq \sum_j \varepsilon_1 \|\varphi_j x(a)\| \\ &\leq \varepsilon_1 \|x(a)\| \leq \frac{CK}{2CK} < 1\end{aligned}$$

for each i, so $z(a) - x(a) \in W$. Moreover, $\|z(a)\| = \|x(a)\| \leq \|y(a)\| + \frac{\varepsilon}{36}\|v\| \leq CK+K \leq 2CK$, $\|y\| \leq CK$ and $\succ_a \in \mathcal{P}_0$. Accordingly, $z(a) \succ_a f(a)$ by the definition of weak star equimonotonicity.

We have shown that $\sum_{a\in A^*} z(a) \leq e^*$ and $z(a) \succ_a f(a)$ for all $a \in A$. We may find $z' : A^* \to M(\Omega)_+$ with $z'(a) \geq z(a)$ and $\sum_{a\in A^*} z'(a) = e^*$. By free disposal, $z'(a) \succ_a f(a)$ for all $a \in A^*$, which contradicts the assumption that f is a core allocation. We conclude that

$$\frac{1}{N}\Gamma' \cap \left(-\frac{\varepsilon}{9}\bar{e} - M(\Omega)_+ + 2U\right) = \emptyset$$

as claimed.

From our choice of N_0 it follows that

$$\frac{1}{N}\text{con } \Gamma' \subset \frac{1}{N}\Gamma' + U$$

Suppose that

$$\frac{1}{N}\text{con } \Gamma' \cap \left(-\frac{\varepsilon}{9}\bar{e} - M(\Omega)_+ + U\right) \neq \emptyset$$

Then we can find $u \in U$ and $\mu \in M(\Omega)_+$ such that

$$\left(-\frac{\varepsilon}{9}\bar{e} - \mu + u\right) \in \frac{1}{N}\text{con } \Gamma' \subset \frac{1}{N}\Gamma' + U$$

so there exists $G \in \frac{1}{N}\Gamma'$ and $u' \in U$ such that

$$\left(-\frac{\varepsilon}{9}\bar{e} - \mu + u\right) = G + u'$$

so

$$G = \left(-\frac{\varepsilon}{9}\bar{e} - \mu + (u - u')\right)$$

Since U is symmetric, $-u' \in U$; since U is convex,

$$u - u' = 2\left(\frac{u - u'}{2}\right) \in 2U$$

which shows

$$\frac{1}{N}\Gamma' \cap \left(-\frac{\varepsilon}{9}\bar{e} - M(\Omega)_+ + 2U\right) \neq \emptyset$$

a contradiction which establishes that

$$\frac{1}{N}\text{con } \Gamma' \cap \left(-\frac{\varepsilon}{9}\bar{e} - M(\Omega)_+ + U\right) = \emptyset$$

Since $(1/N)\text{con } \Gamma'$ is convex and $(-(\varepsilon/9)\bar{e} - M(\Omega)_+ + U)$ is convex and weak star open, we can find a weak star continuous linear functional q on $M(\Omega)$ (that is, a continuous function $q \in C(\Omega)$) that separates $\left(\frac{1}{N}\text{con } \Gamma'\right)$ from $(\frac{\varepsilon\bar{e}}{9} - M(\Omega)_+ + U)$; note that q is necessarily nonnegative. Since supp $v = \Omega$ and $\bar{e} \geq v$, we conclude that $q \cdot \bar{e} \neq 0$; renormalizing if necessary, we may assume that $q \cdot \bar{e} = 1$, and thus

$$\inf q \cdot \frac{\Gamma'}{N} \geq -\frac{\varepsilon}{9} \tag{43}$$

Proposition 6.1 implies that the price q approximately decentralizes the core allocation *with respect to comparison bundles in* $B(0, CK)_+$. Since we want to decentralize with respect to comparison bundles chosen anywhere in $M(\Omega)_+$, we modify the price q so that bundles outside $B(0, CK)_+$ are too expensive to upset the decentralization. In particular, we will produce a price p which is bounded away from 0, eliminating the difficulties which arise because we cannot say anything about the ratio $\max q / \min q$.

Define $P, p \in C(\Omega)$ by $P(\omega) = q(\omega) + \frac{2(1+\|\bar{e}\|)}{C}\mathbf{1}$ and $p = \frac{P}{P \cdot \bar{e}}$. Note that

$$\begin{aligned}
\min_{\omega \in \Omega} p(\omega) &\geq \frac{\frac{2(1+\|\bar{e}\|)}{C}}{\left(q + \frac{2(1+\|\bar{e}\|)}{C}\mathbf{1}\right) \cdot \bar{e}} \\
&\geq \frac{\frac{2}{C}}{1 + \frac{2(1+\|\bar{e}\|)\|\bar{e}\|}{C}} \\
&\geq \frac{\frac{2}{C}}{2} = \frac{1}{C}
\end{aligned}$$

Observe that if $y = x - e(a) \in \gamma'(a)$, $a \in A \setminus A_2$ and $p \cdot y < 0$, then

$$\begin{aligned} p \cdot y &> P \cdot (x - e(a)) \\ &= \left(q + \frac{2(1 + \|\bar{e}\|)}{C}\mathbf{1}\right) \cdot (x - e(a)) \\ &\geq q \cdot (x - e(a)) - \frac{2(1 + \|\bar{e}\|)}{C}\|e(a)\| \\ &\geq q \cdot (x - e(a)) - \frac{2(1 + K)}{\frac{36K^2}{\varepsilon}}\|e(a)\| \\ &\geq q \cdot (x - e(a)) - \frac{\varepsilon}{9K}\|e(a)\| \end{aligned} \tag{44}$$

Let

$$\begin{aligned} \gamma(a) &= \{y - e(a) : y \succ_a f(a)\} \cup \{0\} \\ \Gamma &= \sum_{a \in A} \gamma(a) \end{aligned}$$

We will show that

$$\inf p \cdot \frac{\Gamma}{N} \geq -\frac{\varepsilon}{4} \tag{45}$$

Since $\gamma(a) \supset \gamma'(a)$, $\inf p \cdot \gamma(a) \leq \inf p \cdot \gamma'(a) \leq 0$. On the other hand, if $x \in \gamma(a) \setminus \gamma'(a)$, then $x = y - e(a)$ where $y \succ_a f(a)$ and $\|y\| > CK$. Therefore,

$$\begin{aligned} p \cdot y &\geq \min_{\omega \in \Omega} p(\omega)\|y\| \\ &\geq \frac{1}{C}CK = K \end{aligned}$$

Now suppose $a \in A \setminus A_2$. Then $p \cdot e(a) \leq Kp \cdot \bar{e} = K$, so $p \cdot (y - e(a)) \geq 0 \geq \inf p \cdot \gamma'(a)$, so $\inf p \cdot \gamma(a) = \inf p \cdot \gamma'(a)$. Therefore, by Equations (43) and (44),

$$\begin{aligned} \inf p \cdot \frac{1}{N} \sum_{a \in A \setminus A_2} \gamma(a) &= \inf p \cdot \frac{1}{N} \sum_{a \in A \setminus A_2} \gamma'(a) \\ &\geq \inf q \cdot \frac{1}{N} \sum_{a \in A \setminus A_2} \gamma'(a) - \frac{\varepsilon}{9KN} \sum_{a \in A \setminus A_2} \|e(a)\| \end{aligned}$$

$$\begin{aligned}
&\geq \inf q \cdot \frac{\Gamma'}{N} - \frac{\varepsilon}{9K}\|\bar{e}\| \\
&\geq -\frac{\varepsilon}{9} - \frac{\varepsilon}{9} = -\frac{2\varepsilon}{9}
\end{aligned}$$

By Equation (42),

$$\begin{aligned}
\inf p \cdot \frac{1}{N}\Gamma &= \frac{1}{N}\sum_{a\in A\setminus A_2} \inf p \cdot \gamma(a) + \frac{1}{N}\sum_{a\in A_2} \inf p \cdot \gamma(a) \\
&\geq -\frac{2\varepsilon}{9} - \frac{1}{N}\sum_{a\in A_2} p \cdot e(a) \\
&> -\frac{2\varepsilon}{9} - \frac{\varepsilon}{36} = -\frac{\varepsilon}{4} \qquad (46)
\end{aligned}$$

We are now in a position to show that the price p approximately decentralizes the core allocation f. We have already normalized so that $p \cdot \bar{e} = 1$, which is conclusion (i). Taking X_a to be the compact set $f(a) + [0,1]v$, equimonotonicity of $\{\succ_a\}$ in the direction v implies that $f(a) + \xi v \succ_a f(a)$ for all $\xi \in (0,1)$ and all $a \in A$, so $\succ_a$ is locally nonsatiated with respect to $M(\Omega)_+$ at $f(a)$. Proposition 6.1 then implies, modulo a slight rejuggling of the epsilonics, that p ε-decentralizes f. ■

10 A Stronger Form of Decentralization

The form of decentralization established in Theorems 8.2, 8.8, 9.4 and 9.5 involves a measure of how well the constructed price p decentralizes the given core allocation f. In the finite dimensional literature, this conclusion has been shown to imply, in the presence of equiconvexity conditions, a stronger form of convergence: $f(a)$ is, on average, close to agent a's demand set, $D(p,a)$; see in particular Anderson (1981). This stronger conclusion is desirable, since it shows that $\frac{1}{N}\sum_{a\in A} D(p,a)$ is close to $\bar{e}$, or in other words, that p is an approximate Walrasian price. For a discussion of this interpretation, see Anderson (1993).

In this section, we take a first step toward extending our infinite dimensional results in the same way. In Theorem 10.2, we show, under the assumptions of Theorem 9.4 and the additional assumption of equiconvexity of preferences, that core consumptions are on average close to agents' demand sets. The essential idea of the proof is the same as part of the proof in Anderson (1981): if prices are bounded away from zero, and preferences are equiconvex, the fact that p nearly supports f implies that $f(a)$ is close to $D(p,a)$. We obtain the lower bound on the prices from the bound on marginal rates of substitution assumed in Theorem 9.4.

It would be highly desirable to obtain this stronger conclusion also in the context of Theorems 8.8 and 9.5. The principal obstacle is that the lower bound established on price in those results depends on ε, while the argument used in Theorem 10.2 requires that the lower bound on price be independent of ε. It might be possible to show directly that equiconvexity implies that approximately decentralizing prices are bounded away from zero in a manner independent of ε; indeed, this is done in the finite dimensional case in Anderson (1981). It may also be possible to establish the conclusion by a different argument altogether.

Definition 10.1 A set of preferences $\mathcal{P}_0$ is said to be *weak star equiconvex* if, for every weak star compact set $X \subset M(\Omega)_+$ and every weak star open set V, there exists a norm open set U such that for every preference $\succ \in \mathcal{P}_0$,

$$x, y \in X, \ x \notin y + V$$

$$\Longrightarrow \quad \left(\frac{x+y}{2}+U\right)\cap X \succ x \text{ or } \left(\frac{x+y}{2}+U\right)\cap X \succ y$$

Note that equiconvexity of a single preference relation is a consequence of strong convexity together with weak star continuity; indeed, we could have required that the set U be weak star open. Equiconvexity is a compactness condition on the family $\mathcal{P}_0$.

Theorem 10.2 *Suppose that the assumptions of Theorem 9.4 are satisfied. If in addition $\mathcal{P}_0$ is equiconvex and V is weak star open, there is an N_0 such that if χ is an exchange economy satisfying assumptions (a)–(e), then there is a family $\{t_a : a \in A\}$ such that*

$$\begin{aligned} f(a) - D(p,a) &\subset t_a V \\ \frac{1}{N}\sum_{a\in A} t_a &< 1 \end{aligned}$$

If in addition Ω is a metric space, there is an N_0 such that

$$\frac{1}{N}\sum_{a\in A}\zeta(f(a), D(p,a)) < \varepsilon$$

where ζ is the Prohorov metric on $\mathcal{M}(\Omega)$.

Proof: Fix a weak star open set V. Find $\hat{M}$ such that

$$B\left(0, \frac{1}{\hat{M}}\right) \subset V$$

Find $\hat{T}$ such that

$$\int_{\hat{T}}^{\infty} t dE(t) \leq \frac{1}{22R\hat{M}}$$

Without loss of generality, we may assume that $\hat{T} \geq 1$.

Let

$$\begin{aligned} \hat{K} &= \hat{T} + \frac{1}{22R\hat{M}} \\ \hat{\delta} &= \frac{1}{11R\hat{M}\hat{T}} \\ \hat{X} &= \{\mu \in M(\Omega)_+ : \|\mu\| \leq R\hat{K} + 1\} \end{aligned}$$

Note that

$$\|\bar{e}\| \leq \hat{T} + \frac{1}{22R\hat{M}} = \hat{K}$$

Fix $\mathcal{P}_0$, an equiconvex equimonotone set of preferences. Since $\mathcal{P}_0$ is equiconvex, there exists $\rho > 0$ such that

$$\begin{aligned} &\mu,\, \nu \in X, \quad \mu - \nu \notin \frac{V}{2} \\ &\qquad \Longrightarrow B\left(\frac{\mu+\nu}{2}, \rho\right) \succ \mu \text{ or } B\left(\frac{\mu+\nu}{2}, \rho\right) \succ \nu \end{aligned}$$

Without loss of generality, we may assume that $\rho \leq 1$. Let

$$\hat{\varepsilon} = \frac{\rho}{44R^2\hat{K}\hat{T}\hat{M}}$$

From Theorem 9.4, there exists $\hat{N}_0$ such that if χ satisfies assumptions (a)–(f) (with $\hat{\delta}$ and $\hat{N}_0$ substituted for δ and N_0) and f is in the core of χ there exists $p \in C(\Omega)_+$ such that

$$p \cdot \bar{e} = 1$$

$$\tfrac{1}{N} \textstyle\sum_{a \in A} |p \cdot (f(a) - e(a))| < \hat{\varepsilon}$$

$$\tfrac{1}{N} \textstyle\sum_{a \in A} |\inf\{p \cdot (x - f(a)) : x \succ_a f(a)\}| < \hat{\varepsilon}$$

$$\tfrac{1}{R\hat{K}} \leq \tfrac{1}{R\|\bar{e}\|} \leq \min_{\omega \in \Omega} p(\omega) \leq \tfrac{1}{\|v\|}$$

$$\tfrac{1}{\hat{K}} \leq \tfrac{1}{\|\bar{e}\|} \leq \max_{\omega \in \Omega} p(\omega) \leq \tfrac{R}{\|v\|}$$

$$\tfrac{\max_{\omega \in \Omega} p(\omega)}{\min_{\omega \in \Omega} p(\omega)} \leq R$$

Since $\min_{\omega \in \Omega} p(\omega) > 0$, each agent's budget set at the price p is norm bounded, and thus weak star compact. Since $\succ_a$ is transitive for all $a \in A$, the demand set $D(p, a)$ is nonempty.[24]

[24]It is easy to see that $D(p, a)$ has only one element if $\{\succ_a\}$ is equiconvex. For those agents whose demand set has more than one element, our proof will show that $f(a) - x \in t_a V$ for some $x \in D(p, a)$.

Let

$$\begin{aligned}
\hat{A}_1 &= \{a \in A : \succ_a \notin \mathcal{P}_0\} \\
\hat{A}_2 &= \{a \in A : \|e(a)\| > \hat{K}\} \\
\hat{A}_4 &= \{a \in A : \ |p \cdot (f(a) - e(a))| \geq 22R\hat{M}\hat{T}\hat{\varepsilon} \\
&\quad \text{or } |\inf\{p \cdot (x - f(a)) : x \succ_a f(a)\}| \geq 22R\hat{M}\hat{T}\hat{\varepsilon}\}
\end{aligned}$$

$$\begin{aligned}
\frac{|\hat{A}_1 \cup \hat{A}_4|}{N} &\leq \hat{\delta} + \frac{2\hat{\varepsilon}}{22R\hat{M}\hat{T}\hat{\varepsilon}} \\
&= \hat{\delta} + \hat{\delta} = 2\hat{\delta}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left\| \frac{1}{N} \sum_{a \in \hat{A}_1 \cup \hat{A}_2 \cup \hat{A}_4} e(a) \right\| &\leq \left(\hat{T} 2\hat{\delta} + \frac{1}{N} \sum_{\|e(a)\| \not\leq \hat{T}} \|e(a)\| \right) \\
&= \left(\frac{2\hat{T}}{11R\hat{M}\hat{T}} + \int_{\hat{T}}^{\infty} t \; dE(t) \right) \bar{e} \\
&< \left(\frac{2}{11R\hat{M}} + \frac{1}{22R\hat{M}} \right) \bar{e} = \frac{5\bar{e}}{22R\hat{M}}
\end{aligned}$$

so

$$\begin{aligned}
&\frac{1}{N} \sum_{a \in \hat{A}_1 \cup \hat{A}_2 \cup \hat{A}_4} \|f(a) - D(p,a)\| \\
&\leq \frac{1}{N} \sum_{a \in \hat{A}_1 \cup \hat{A}_2 \cup \hat{A}_4} \|f(a)\| + \|D(p,a)\| \\
&\leq \frac{1}{N} \sum_{a \in \hat{A}_1 \cup \hat{A}_2 \cup \hat{A}_4} \frac{p \cdot f(a)}{\min_{\omega \in \Omega} p(\omega)} + \frac{p \cdot D(p,a)}{\min_{\omega \in \Omega} p(\omega)} \\
&\leq \frac{1}{N} \sum_{a \in \hat{A}_1 \cup \hat{A}_2 \cup \hat{A}_4} \frac{p \cdot e(a) + |p \cdot (f(a) - e(a))|}{\min_{\omega \in \Omega} p(\omega)} + \frac{p \cdot e(a)}{\min_{\omega \in \Omega} p(\omega)} \\
&\leq \frac{1}{N} \sum_{a \in \hat{A}_1 \cup \hat{A}_2 \cup \hat{A}_4} 2\frac{\max_{\omega \in \Omega} p(\omega) \|e(a)\|}{\min_{\omega \in \Omega} p(\omega)} + \frac{1}{N} \sum_{a \in \hat{A}} \frac{|p \cdot (f(a) - e(a))|}{\min_{\omega \in \Omega} p(\omega)} \\
&\leq \frac{1}{N} \sum_{a \in \hat{A}_1 \cup \hat{A}_2 \cup \hat{A}_4} 2R\|e(a)\| + R\hat{K}\hat{\varepsilon}
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{10}{22\hat{M}} + \frac{\rho}{22R\hat{T}\hat{M}} \\
&\le \frac{10}{22\hat{M}} + \frac{1}{22\hat{M}} = \frac{1}{2\hat{M}}
\end{aligned}
$$

Now suppose $a \in A \setminus (\hat{A}_1 \cup \hat{A}_2 \cup \hat{A}_4)$. We claim that $f(a) - D(p,a) \in \frac{V}{2}$. Note that

$$
\begin{aligned}
\|D(p,a)\| &\le \frac{p \cdot D(p,a)}{\min_{\omega\in\Omega} p(\omega)} \\
&\le \frac{p \cdot e(a)}{\min_{\omega\in\Omega} p(\omega)} \\
&\le \frac{\max_{\omega\in\Omega} p(\omega)}{\min_{\omega\in\Omega} p(\omega)}\|e(a)\| \\
&\le R\hat{K} \\
\|f(a)\| &\le \frac{p \cdot f(a)}{\min_{\omega\in\Omega} p(\omega)} \\
&\le \frac{p \cdot e(a)}{\min_{\omega\in\Omega} p(\omega)} + \frac{|p \cdot (f(a) - e(a))|}{\min_{\omega\in\Omega} p(\omega)} \\
&\le \frac{\max_{\omega\in\Omega} p(\omega)}{\min_{\omega\in\Omega} p(\omega)}\|e(a)\| + \frac{22R\hat{M}\hat{T}\hat{\varepsilon}}{\min_{\omega\in\Omega} p(\omega)} \\
&\le R\hat{K} + 22R\hat{M}\hat{T}(R\hat{K})\frac{\rho}{44R^2\hat{K}\hat{T}\hat{M}} \\
&< R\hat{K} + \rho \le R\hat{K} + 1
\end{aligned}
$$

so $f(a), D(p,a) \in \hat{X}$. Let

$$
\begin{aligned}
x &= \frac{f(a) + D(p,a)}{2} \\
y &= \frac{\left(\|x\| - \frac{3\rho}{4}\right)_+}{\|x\|}x
\end{aligned}
$$

Note that

$$
\begin{aligned}
\|y - x\| &\le \left(1 - \frac{\|x\| - \frac{3\rho}{4}}{\|x\|}\right)\|x\| \\
&\le \frac{3\rho}{4} < \rho
\end{aligned}
$$

Since $a \notin \hat{A}_1$, either $y \succ_a f(a)$ or $y \succ_a D(p,a)$. If $\|x\| \leq \frac{3\rho}{4}$, we have $y = 0$, which shows that we cannot have either $y \succ_a f(a)$ or $y \succ_a D(p,a)$. If $\|x\| > \frac{3\rho}{4}$, we have

$$
\begin{aligned}
p \cdot y &\leq p \cdot x - \min_{\omega \in \Omega} p(\omega) \frac{3\rho}{4} \\
&\leq p \cdot \left(\frac{f(a) + D(p,a)}{2} \right) - \frac{3\rho}{4R\hat{K}} \\
&\leq p \cdot f(a) + \frac{p \cdot (D(p,a) - f(a))}{2} - 33R\hat{T}\hat{M}\hat{\varepsilon} \\
&\leq p \cdot f(a) + \left| \frac{p \cdot (e(a) - f(a))}{2} \right| - 33R\hat{T}\hat{M}\hat{\varepsilon} \\
&\leq p \cdot f(a) + 11R\hat{T}\hat{M}\hat{\varepsilon} - 33R\hat{T}\hat{M}\hat{\varepsilon} \\
&\leq p \cdot f(a) - 22R\hat{T}\hat{M}\hat{\varepsilon}
\end{aligned}
$$

Since $a \notin \hat{A}_4$, this shows that $y \not\succ_a f(a)$. In addition, note that

$$
\begin{aligned}
p \cdot y &\leq p \cdot x - \min_{\omega \in \Omega} p(\omega) \frac{3\rho}{4} \\
&\leq p \cdot \left(\frac{f(a) + D(p,a)}{2} \right) - \frac{3\rho}{4R\hat{K}} \\
&\leq p \cdot e(a) + \left| \frac{p \cdot (f(a) - e(a))}{2} \right| - 33R\hat{T}\hat{M}\hat{\varepsilon} \\
&< p \cdot e(a) + 11R\hat{T}\hat{M}\hat{\varepsilon} - 33R\hat{T}\hat{M}\hat{\varepsilon} \\
&< p \cdot e(a)
\end{aligned}
$$

which contradicts $y \succ_a D(p,a)$. This contradiction establishes the claim, i.e. $a \in A \setminus (\hat{A}_1 \cup \hat{A}_2 \cup \hat{A}_4)$ implies $f(a) - D(p,a) \in \frac{V}{2}$.

For each $a \in A$, let

$$s_a = \inf\{s : f(a) - D(p,a) \subset sV\}$$

Then

$$\frac{1}{N} \sum_{a \in A} s_a \leq \frac{1}{N} \sum_{a \in \hat{A}_1 \cup \hat{A}_2 \cup \hat{A}_4} s_a + \sum_{a \in A \setminus (\hat{A}_1 \cup \hat{A}_2 \cup \hat{A}_4)} s_a$$

$$
\begin{aligned}
&< \frac{1}{N}\sum_{a\in\hat{A}_1\cup\hat{A}_2\cup\hat{A}_4}\|f(a)-D(p,a)\|\hat{M}+\frac{1}{N}\sum_{a\in\hat{A}\setminus(\hat{A}_1\cup\hat{A}_2\cup\hat{A}_4)}\frac{1}{2}\\
&\le \frac{\hat{M}}{2\hat{M}}+\frac{1}{2}=1
\end{aligned}
$$

so there exits t_a such that $f(a)-D(p,a)\subset t_aV$ for all $a\in A$ and such that $\sum_{a\in A}t_a<1$.

Now suppose that Ω is metric. Since Ω is also compact, it is separable, so $\mathcal{M}(\Omega)$ is metrizable by the Prohorov metric (Billingsley (1968)). Choose $V=B\left(0,\frac{\varepsilon}{2}\right)$. It is easy to see that

$$
t_aV\subset\begin{cases} V & \text{if } t_a\le 1\\ B\left(0,\frac{t_a\varepsilon}{2}\right) & \text{if } t_a>1\end{cases}
$$

Then

$$
\begin{aligned}
\frac{1}{N}\sum_{a\in A}\sigma(f(a),D(p,a)) &\le \frac{\varepsilon}{2N}\sum_{a\in A}\max\{1,t_a\}\\
&\le \frac{\varepsilon}{2N}\sum_{a\in A}(1+t_a)<\varepsilon
\end{aligned}
$$

■

11 Slow Convergence

In Sections 8 and 9, we gave conditions sufficient to rule out the kind of monopsony and monopoly power and the other difficulties demonstrated in the Examples, and showed that those conditions lead to approximate decentralization. It is natural to ask about the rate at which approximate decentralization takes place — in other language, about the "rate of convergence." Some results in this direction could presumably be extracted from our positive results, but it does not seem easy to obtain satisfactory results in this way. Part of the difficulty is that our positive results depend on various forms of equimonotonicity in a way that seems hard to quantify. The following example illustrates this point by showing that the positive results of Section 9 are consistent with *arbitrarily slow rates of convergence.*

Our example is a modification of Example 3.1. However, it is convenient to work, not on the unit interval $[0, 1]$, but rather on a Cantor subset $\Omega \subset [0, 1]$ having positive measure. (The example could be recast in the unit interval, but the construction would be much messier and the verifications would be very unpleasant.) Let λ denote Lebesgue measure on the unit interval $[0, 1]$. We construct a Cantor set $\Omega \subset [0, 1]$ in the following way. Let Q_1 be the open middle $1/4$ of $[0, 1]$, and write $X_1 = [0, 1] \setminus Q_1$. Note that $\lambda(X_1) = 3/4$ and that X_1 is the union of two closed intervals, each of length $3/8$. Let Q_2 be the union of the open middle $1/6$ of each of these two intervals, and let $X_2 = X_1 \setminus Q_2$. Note that $\lambda(X_2) = 5/8$ and that X_2 is the union of four closed intervals, each of length $5/32$. Continuing in this way we obtain a descending sequence $X_1, X_2, X_3, \ldots$ of closed subsets of $[0, 1]$; $\lambda(X_k) = (2^k + 1)/2^{k+1}$, and X^k is the union of 2^k intervals X_k^n, each of length $(2^k + 1)/2^{2k+1}$. Define $\Omega = \bigcap X_k$. Our construction guarantees that $\lambda(\Omega) = 1/2$; since the sets $\Omega \cap X_k^n$ are all translates of each other, it follows that $\lambda(\Omega \cap X_k^n) = \frac{\frac{1}{2}}{2^k} = 2^{-k-1}$ for each k, n.

Write $\mu = 2\lambda|_\Omega$ (so that μ is the restriction of λ to Ω, re-normalized so as to have total mass 1). For $E \subset \Omega$ a measurable set, write $\mathbf{1}_E$ for the characteristic function of E; that is, the function which is 1 on E and 0 elsewhere.

Example 11.1 We describe a sequence of economies in the commodity space

$M(\Omega)$ for which the results of Section 9 guarantee approximate decentralization, but for which the rate at which this approximate decentralization takes place is arbitrarily slow. To this end, we begin by fixing a strictly decreasing sequence $\{\alpha_N\}$ of real numbers converging to 0. In the N^{th} economy, there are 2^N agents: $A_N = \{1, 2, \ldots, 2^N\}$. The endowment of each agent $n \in A_N$ is $e(n) = \mu$. To describe preferences, define $F_{N,n} = X_N^n \cap \Omega$; our construction guarantees that (for fixed N) the sets $F_{N,1}, F_{N,2}, \ldots F_{N,2^N}$ form a partition of Ω into disjoint closed subsets of equal μ-measure 2^{-N}. Let $f_{N,n} : \Omega \to \mathbf{R}$ be the (continuous) function which is $1 + \alpha_N$ on $F_{N,n}$ and 1 elsewhere. Agent n's utility function is

$$u_{N,n}(\gamma) = \int_\Omega f_n(t)\, d\gamma(t)$$

Thus, agent n's utility function is linear, with constant marginal utility equal to $1+\alpha_N$ for commodities in his/her "preferred set" $F_{N,n}$, and with constant marginal utility equal to 1 forother commodities. These utility functions are well-behaved: strictly monotone, weak star continuous (because Ω is totally disconnected), with marginal rates of substitution bounded between 1 and $1 + \alpha_1$. Moreover, it is easily verified (using the fact that $\alpha_n \to 0$) that the family $\{u_{N,n}\}$ is weak star equimonotone.

Note that the endowments— hence all feasible consumption bundles — are absolutely continuous with respect to μ. Hence, all feasible consumption bundles lie in the subspace $L^1(\mu) \subset M(\Omega)$. Analysis of this example is almost identical to that of Example 3.1. In particular, it is easily seen that this economy has a unique Walrasian equilibrium; the equilibrium price is $p = 1 \in C(\Omega)$ and the equilibrium allocation gives consumer n the consumption bundle:

$$x(n) = 2^N \mathbf{1}_{F_{N,n}}\mu$$

However, the core of this economy is large. In particular, the allocation f defined by

$$f(n) = \begin{cases} 2^N \mathbf{1}_{F_{N,n}}\mu + \frac{\alpha_N}{8} 2^N \mathbf{1}_{F_{N,n+1}}\mu & \text{if } n \text{ is odd} \\ [1 - \frac{\alpha_N}{8}] 2^N \mathbf{1}_{F_{N,n}}\mu & \text{if } n \text{ is even} \end{cases}$$

is in the core. Moreover, there is no price for which the mean social endowment has value 1 and which $(\alpha_N)/100$-decentralizes the core allocation f.

(Again, the analysis is essentially the same as for Example 3.1; we leave the details to the reader.)

Since we are free to choose the sequence $\{\alpha_N\}$ so that convergence to 0 is as slow as we like, we can arrange that the rate of approximate decentralization is arbitrarily slow.

References

[1] Abramovich, Y. A., C. D. Aliprantis, and W. R. Zame (1995), "A Representation Theorem for Riesz Spaces and its Applications to Economics," *Economic Theory* forthcoming.

[2] Aliprantis, Charalambos D., Donald J. Brown and Owen Burkinshaw (1989), *Existence and Optimality of Competitive Equilibrium*. Berlin: Springer-Verlag.

[3] Aliprantis, Charalambos D. and Owen Burkinshaw (1985), *Positive Operators.* New York: Academic Press.

[4] Aliprantis, Charalambos D., and Owen Burkinshaw (1991), "When is the Core Equivalence Theorem Valid?" *Economic Theory* 1, 169-182.

[5] Anderson, Robert M. (1978), "An Elementary Core Equivalence Theorem," *Econometrica*, 46:1483-1487.

[6] Anderson, Robert M. (1981), "Core Theory with Strongly Convex Preferences," *Econometrica*, 49:1457-1468.

[7] Anderson, Robert M. (1987), "Gap-Minimizing Prices and Quadratic Core Convergence," *Journal of Mathematical Economics*, 16:1-15. Correction, *Journal of Mathematical Economics*, 20:599-601.

[8] Anderson, Robert M. (1990), "Large Square Economies: An Asymptotic Interpretation," preprint, Department of Economics, University of California at Berkeley, April.

[9] Anderson, Robert M. (1993), "The Core in Perfectly Competitive Economies," in Robert J. Aumann and Sergiu Hart (eds.), *Handbook of Game Theory with Economic Applications*, Volume I. Amsterdam: North-Holland.

[10] Anderson, Robert M. and Andreu Mas-Colell (1988), "An Example of Pareto Optima and Core Allocations Far from Agents' Demand Correspondences," *Econometrica*, 56:361-382.

[11] Arrow, Kenneth J. and F. H. Hahn (1971), *General Competitive Analysis.* San Francisco: Holden-Day, Inc.

[12] Aumann, Robert J. (1964), "Markets with a Continuum of Traders," *Econometrica*, 32:39-50.

[13] Aumann, Robert J. (1979), "On the Rate of Convergence of the Core," *International Economic Review*, 20:349-357.

[14] Bewley, Truman F. (1973a), "Edgeworth's Conjecture," *Econometrica*, 41:425-454.

[15] Bewley, Truman F. (1973b), "The Equality of the Core and the Set of Equilibria in Economies with Infinitely Many Commodities and a Continuum of Agents," *International Economic Review*, 14:383-394.

[16] Billingsley, Patrick (1968), *Convergence of Probability Measures.* New York: John Wiley and Sons.

[17] Brown, Donald J. and M. Ali Khan (1980), "An Extension of the Brown-Robinson Equivalence Theorem," *Applied Mathematics and Computation*, 6:167-175.

[18] Brown, Donald J. and Abraham Robinson (1974), "The Cores of Large Standard Exchange Economies," *Journal of Economic Theory*, 9:245-254.

[19] Cheng, Hsueh-Cheng (1981), "What is the Normal Rate of Convergence of the Core (Part I)," *Econometrica*, 49:73-83.

[20] Cheng, Hsueh-Cheng (1982), "Generic Examples on the Rate of Convergence of the Core," *International Economic Review*, 23:309-321.

[21] Cheng, Hsueh-Cheng (1983a), "The Best Rate of Convergence of the Core," *International Economic Review*, 24:629-636.

[22] Cheng, Hsueh-Cheng (1983b), "A Uniform Core Convergence Result for Non-convex Economies," *Journal of Economic Theory* 31:269-282.

[23] Cheng, Harrison H.-C. (1987), "The Principle of Equivalence," Working Paper, University of Southern California.

[24] Debreu, Gerard (1975), "The Rate of Convergence of the Core of an Economy," *Journal of Mathematical Economics*, 2:1-7.

[25] Debreu, Gerard and Herbert Scarf (1963), "A Limit Theorem on the Core of an Economy," *International Economic Review*, 4:236-246.

[26] Dierker, Egbert (1975), "Gains and Losses at Core Allocations," *Journal of Mathematical Economics*, 2:119-128.

[27] Dierker, Hildegard (1975), "Equilibria and Core of Large Economies," *Journal of Mathematical Economics*, 2:155-169.

[28] Edgeworth, Francis Y. (1881), *Mathematical Psychics*. London: Kegan Paul.

[29] Gabszewicz, J. (1968), "Coeurs et Allocations Concurrentielles dans des Economies d'Echange avec un Continu de Biens," unpublished dissertation, Librairie Universitaire, Université Catholique de Louvain, published in M. Ali Khan and Nicholas Yannelis (eds.), *Equilibrium Theory in Infinite Dimensional Spaces, Studies in Economic Theory* 1. New York: Springer-Verlag, 1991.

[30] Geller, William (1987), "Almost Quartic Core Convergence," presentation to Econometric Society North American Summer Meeting, Berkeley, June 1987.

[31] Gretsky, Neil and Joseph Ostroy (1985), "Thick and Thin Market Non-Atomic Exchange Economies," in Charalambos D. Aliprantis, Owen Burkinshaw and N.J. Rothman, eds., *Advances in Equilibrium Theory*, Springer-Verlag Lecture Notes in Economics and Mathematical Systems, 244.

[32] Grodal, Birgit (1975), "The Rate of Convergence of the Core for a Purely Competitive Sequence of Economies," *Journal of Mathematical Economics*, 2:171-186.

[33] Grodal, Birgit and Werner Hildenbrand (1974), "Limit Theorems for Approximate Cores," Working Paper IP-208, Center for Research in Management, University of California, Berkeley.

[34] Hildenbrand, Werner (1974), *Core and Equilibria of a Large Economy*. Princeton: Princeton University Press.

[35] Jones, Larry E. (1984), "A Competitive Model of Commodity Differentiation," *Econometrica*, 52, 507-530.

[36] Kannai, Yakar (1970), "Continuity Properties of the Core of a Market," *Econometrica*, 38:791-815.

[37] Keiding, Hans (1974), "A Limit Theorem on the Cores of Large but Finite Economies," preprint, University of Copenhagen.

[38] Khan, M. Ali (1974), "Some Equivalence Theorems," *Review of Economic Studies*, 41:549-565.

[39] Khan, M. Ali (1976), "Oligopoly in Markets with a Continuum of Traders: An Asymptotic Interpretation," *Journal of Economic Theory*, 12:273-297.

[40] Khan, M. Ali and Salim Rashid (1976), "Limit Theorems on Cores with Costs of Coalition Formation," preprint, Johns Hopkins University.

[41] Khan, M. Ali and Nicholas Yannelis, "Equilibria in Markets with a Continuum of Agents and Commodities," in M. Ali Khan and Nicholas Yannelis (eds.), *Equilibrium Theory in Infinite Dimensional Spaces*, *Studies in Economic Theory* 1. New York: Springer-Verlag, 1991.

[42] Manelli, Alejandro (1990), "Core Convergence without Monotone Preferences or Free Disposal," preprint, Department of Managerial Economics and Decision Sciences, Kellogg Graduate School of Management, Northwestern University, May.

[43] Manelli, Alejandro (1991), "Monotonic Preferences and Core Equivalence," *Econometrica*, 59: 123-138.

[44] Mas-Colell, Andreu (1975), "A Model of Equilibrium with Differentiated Commodities," *Journal of Mathematical Economics*, 2, 263-295.

[45] Mas-Colell, Andreu and William R. Zame (1991), "Equilibrium Theory in Infinite Dimensional Spaces," Chapter 34 in Werner Hildenbrand

and Hugo Sonnenschein (eds.), *Handbook of Mathematical Economics*, Volume IV. Amsterdam: North-Holland.

[46] Mertens, Jean-Francois (1970), "An Equivalence Theorem for the Core of an Economy with Commodity Space $L_\infty - \tau(L_\infty, L_1)$," Discussion Paper 7028, Center for Operations Research and Econometrics, Université Catholique de Louvain, published in in M. Ali Khan and Nicholas Yannelis (eds.), *Equilibrium Theory in Infinite Dimensional Spaces, Studies in Economic Theory* 1. New York: Springer-Verlag, 1991.

[47] Nomura (1992), "Elementary Core Equivalence Theorems with Infinitely Many Commodities," Working Paper #92-E009, Economics Association for Chiba University, Chiba, Japan.

[48] Ostroy, Joseph M. and William R. Zame (1994), "Nonatomic Economies and the Boundaries of Perfect Competition," *Econometrica* 62, 593-633.

[49] Rustichini, Aldo and Nicholas Yannelis (1991), "Edgeworth's Conjecture in Economies with a Continuum of Agents and Commodities," *Journal of Mathematical Economics*, 20: 307-326.

[50] Shapley, Lloyd S. (1975), "An Example of a Slow-Converging Core," *International Economic Review*, 16:345-351.

[51] Shubik, Martin (1959), "Edgeworth Market Games," *Contributions to the Theory of Games IV, Annals of Mathematics Studies*, 40:267-278.

[52] Trockel, Walter (1976), "A Limit Theorem on the Core," *Journal of Mathematical Economics*, 3:247-264.

[53] Vind, Karl (1965), "A Theorem on the Core of an Economy," *Review of Economic Studies*, 32:47-48.

[54] Zame, William R. (1986), "Markets with a Continuum of Traders and Infinitely Many Commodities," Working Paper, Department of Mathematics, State University of New York at Buffalo, December.

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