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# Polarization of massive fermions in a vortical fluid

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Fermions become polarized in a vortical fluid due to spin-vorticity coupling. Such a polarization can be calculated from the Wigner function in a quantum kinetic approach. By extending previous results for chiral fermions, we derive the Wigner function for massive fermions up to next-to-leading order in spatial gradient expansion. The polarization density of fermions can be calculated from the axial vector component of the Wigner function and is found to be proportional to the local vorticity $\boldsymbol{\omega}$. The polarizations per particle for fermions and antifermions decrease with the chemical potential and increase with energy (mass). Both quantities approach the asymptotic value $\hbar\boldsymbol{\omega}/4$ in the large energy (mass) limit. The polarization per particle for fermions is always smaller than that for antifermions, whose ratio of fermions to antifermions also decreases with the chemical potential. The polarization per particle on the Cooper-Frye freeze-out hypersurface can also be formulated and is consistent with the previous result of Becattini *et al.* [11,27].

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## I. INTRODUCTION

In noncentral high-energy heavy-ion collisions, the large orbital angular momentum present in the colliding system can lead to nonvanishing local vorticity in the hot and dense fluid [1–6]. The vorticity induced by global orbital angular momentum in the fluid can be considered as local rotational motion of particles [3,4,7,8]. It is closely related to the rapidity dependence of the $v_1$ flow and shear of the longitudinal flow velocity inside the reaction plane [5,9,10].

As a result of spin-orbital coupling, quarks and antiquarks can become polarized along the normal direction of the reaction plane [1,2,5]. Through hadronization of polarized quarks and antiquarks, hyperons can also be polarized in the same direction in the final state [1,2,11]. Measurements of such global hyperon polarization are feasible through the parity-violating decay of hyperons [12,13]. Such measurements will shed light on properties of the vortical structures of the strongly coupled quark-gluon plasma in high-energy heavy-ion collisions.

Quark and antiquark polarizations in a vortical fluid are also closely related to the chiral magnetic and vortical effects [14–19]. From the solutions of Wigner functions for chiral or massless fermions in a quantum kinetic approach, one can derive the axial current $j_5^\mu = \rho_5 u^\mu + \xi_5 \omega^\mu + \xi_5^B B^\mu$, where $\rho_5$ is the axial charge density, $u^\mu$ is the fluid velocity, $\omega^\mu \equiv \frac{1}{2}\epsilon^{\mu\sigma\alpha\beta} u_\sigma \partial_\alpha u_\beta$ is the vorticity four-vector, and $B^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma} u_\nu F_{\lambda\sigma}$ is the four-vector of the magnetic field with $F_{\lambda\rho}$ being the strength tensor of the electromagnetic field. The coefficients $\xi_5$ and $\xi_5^B$ are all functions of temperatures and chemical potentials $\mu$ and $\mu_5$ [19]. In a three-flavor quark matter with $u, d$, and $s$ quarks and their antiquarks, $\xi_5^B = 0$. In other words, the axial current in a three-flavor quark matter is blind to the magnetic field and solely induced by the vorticity. Such an axial current leads to the local polarization effect [19] which is also connected to the spin-vorticity coupling for chiral or massless fermions [20].

In this paper, we will extend our Wigner function method for massless fermions to massive ones and formulate the polarization of massive fermions induced by vorticity. In Sec. II, we will give a brief introduction to the Wigner function method and derive the equations for the Wigner function components for massive fermions based on Refs. [21,22]. The Wigner function components can be determined perturbatively by gradient expansion. In Sec. III, we will derive the Wigner function at leading order by definition. Using the projection method we can extract each component of the Wigner function at leading order. We will propose the first-order solution for the axial vector component in Sec. IV by extending the solution for massless fermions. In Sec. V, we will show that the axial vector component can be regarded as the spin density in phase space. We can obtain the polarization density after completion of momentum integration of the axial vector component in Sec. VI. We will also formulate the fermion polarization on the freeze-out hypersurface by extending the Cooper-Frye formula. We will give a summary of the results in the final section.

We adopt the same sign conventions for fermion charge $Q$ as in Refs. [19,20,22,23] and the same sign convention for the axial vector $A^\mu \sim \langle \bar{\psi}\gamma^\mu\gamma^5\psi \rangle$ as in Refs. [19,20,23] but different sign convention from Ref. [22].

## II. WIGNER FUNCTION FOR MASSIVE FERMIONS

In this section we will give a brief introduction to the Wigner function and its kinetic equation for massive fermions based on Refs. [21,22]. There are also other earlier works in the literature along this line [24,25]. In a background electromagnetic field, the quantum mechanical analog of a classical phase-space distribution for fermions is the gauge invariant Wigner function $W_{\alpha\beta}(x,p)$ defined by

$$W_{\alpha\beta}(x,p) = \int \frac{d^4y}{(2\pi)^4} e^{-ipy} \left\langle \bar{\psi}_\beta\left(x+\frac{1}{2}y\right) \times PU\left(G,x+\frac{1}{2}y,x-\frac{1}{2}y\right)\psi_\alpha\left(x-\frac{1}{2}y\right)\right\rangle, \quad (1)$$

where $\psi_\alpha$ and $\bar{\psi}_\beta$ are fermionic quantum fields, $\langle\hat{O}\rangle$ denotes the grand canonical ensemble averaging and normal ordering, $x=(x_0,\mathbf{x})$ and $p=(p_0,\mathbf{p})$ are time-space and energy-momentum four-vectors, respectively, and the gauge link $PU(G,x_1,x_2)$ is to ensure the gauge invariance of the Wigner function and given by

$$PU\left(G,x+\frac{1}{2}y,x-\frac{1}{2}y\right) = P\,\exp\left[-iQy^\mu \int_0^1 ds\, G_\mu\left(x-\frac{1}{2}y+sy\right)\right], \quad (2)$$

where $G^\mu$ is the gauge potential of the classical electromagnetic field.

The Wigner function in (1) satisfies the following equation of motion:

$$(\gamma_\mu K^\mu - m)W(x,p) = 0, \quad (3)$$

where the operator $K^\mu$ is given by

$$K^\mu = p_{\rm W}^\mu + i\hbar\tfrac{1}{2}\nabla^\mu, \quad (4)$$

with

$$\begin{aligned} p_{\rm W}^\mu &= p^\mu - \hbar\tfrac{1}{2}Qj_1(\Delta)F^{\mu\nu}\partial_{p\,\nu},\\ \nabla^\mu &= \partial_x^\mu - Qj_0(\Delta)F^{\mu\nu}\partial_{p\,\nu}, \end{aligned} \quad (5)$$

where we have used $\Delta \equiv \frac{1}{2}\hbar\partial_p\cdot\partial_x$ with the operator $\partial_x$ in $\Delta$ acting only on the strength tensor $F^{\mu\nu}$ and $j_0(x)=\sin(x)/x$ and $j_1(x)=[\sin(x)-x\cos(x)]/x^2$ are spherical Bessel functions. If $F^{\mu\nu}$ is a constant we have simpler forms of these operators,

$$p_{\rm W}^\mu = p^\mu, \quad \nabla^\mu = \partial_x^\mu - QF^{\mu\nu}\partial_{p\nu}. \quad (6)$$

The Wigner function is a $4\times 4$ matrix in Dirac indices and can be decomposed into 16 independent generators of Clifford algebra,

$$W = \tfrac{1}{4}\left[F + i\gamma^5 P + \gamma^\mu V_\mu + \gamma^5\gamma^\mu A_\mu + \tfrac{1}{2}\sigma^{\mu\nu}S_{\mu\nu}\right], \quad (7)$$

where the generators of Clifford algebra are

$$\Gamma_i = 1,\ \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3,\ \gamma^\mu,\ \gamma^5\gamma^\mu,\ \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu], \quad (8)$$

corresponding to the scalar, pseudoscalar, vector, axial vector, and tensor components, respectively. The coefficients in the decomposition (7) can be obtained by projection of corresponding Dirac matrices on the Wigner function and taking traces,

$$\begin{aligned} &F = {\rm Tr}[W], \quad P = -i\,{\rm Tr}[\gamma^5 W], \quad V^\mu = {\rm Tr}[\gamma^\mu W],\\ &A^\mu = {\rm Tr}[\gamma^\mu\gamma^5 W], \quad S^{\mu\nu} = {\rm Tr}[\sigma^{\mu\nu}W]. \end{aligned} \quad (9)$$

Substituting Eq. (7) into Eq. (3) with (6) and comparing common terms in the basis of Clifford algebra, we obtain the following system of equations:

$$\begin{aligned} K^\mu V_\mu - mF &= 0,\\ K^\mu A_\mu + imP &= 0,\\ K_\mu F + iK^\nu S_{\nu\mu} - mV_\mu &= 0,\\ iK_\mu P + \tfrac{1}{2}\epsilon_{\mu\beta\nu\sigma}K^\beta S^{\nu\sigma} + mA_\mu &= 0,\\ -i(K_\mu V_\nu - K_\nu V_\mu) - \epsilon_{\mu\nu\alpha\beta}K^\alpha A^\beta - mS_{\mu\nu} &= 0. \end{aligned} \quad (10)$$

The real parts of the above equations are

$$\begin{aligned} p^\mu V_\mu - mF &= 0,\\ \tfrac{1}{2}\hbar\nabla^\mu A_\mu + mP &= 0,\\ p_\mu F - \tfrac{1}{2}\hbar\nabla^\nu S_{\nu\mu} - mV_\mu &= 0,\\ -\tfrac{1}{2}\hbar\nabla_\mu P + \tfrac{1}{2}\epsilon_{\mu\beta\nu\sigma}p^\beta S^{\nu\sigma} + mA_\mu &= 0,\\ \tfrac{1}{2}\hbar(\nabla_\mu V_\nu - \nabla_\nu V_\mu) - \epsilon_{\mu\nu\alpha\beta}p^\alpha A^\beta - mS_{\mu\nu} &= 0. \end{aligned} \quad (11)$$

The imaginary parts are

$$\begin{aligned} \hbar\nabla^\mu V_\mu &= 0,\\ p^\mu A_\mu &= 0,\\ \tfrac{1}{2}\hbar\nabla_\mu F + p^\nu S_{\nu\mu} &= 0,\\ p_\mu P + \tfrac{1}{4}\hbar\epsilon_{\mu\beta\nu\sigma}\nabla^\beta S^{\nu\sigma} &= 0,\\ (p_\mu V_\nu - p_\nu V_\mu) + \tfrac{1}{2}\hbar\epsilon_{\mu\nu\alpha\beta}\nabla^\alpha A^\beta &= 0. \end{aligned} \quad (12)$$

From the third and the fifth lines of the imaginary part equations (13) we obtain,

$$p\cdot\nabla F = 0, \quad (13)$$

and

$$\hbar(\nabla^\lambda A^\rho - \nabla^\rho A^\lambda) - 2\epsilon^{\mu\nu\lambda\rho}p_\mu V_\nu = 0, \quad (14)$$

respectively, where we have multiplied $\epsilon^{\mu\nu\lambda\rho}$ to the equation and used $\epsilon^{\mu\nu\lambda\rho}\epsilon_{\mu\nu\alpha\beta} = -2(\delta_\alpha^\lambda\delta_\beta^\rho - \delta_\beta^\lambda\delta_\alpha^\rho)$. Taking contraction of the above equation with $p^\lambda$, we obtain

$$p\cdot\nabla A^\rho = p_\lambda\nabla^\rho A^\lambda = QF^{\rho\xi}A_\xi, \quad (15)$$

where we have used $p^\mu A_\mu = 0$ from the second line of Eqs. (13).

From the first and third lines of real part equations (12), we obtain

$$(p^2 - m^2)F = \tfrac{1}{2}\hbar p^\mu\nabla^\nu S_{\nu\mu} \approx \tfrac{1}{2}\hbar QF^{\mu\nu}S_{\mu\nu}, \quad (16)$$

where we have neglected the second-order term $\hbar\nabla^\nu(p^\mu S_{\nu\mu}) \sim \hbar^2$. Inserting the fifth line into the fourth line in Eqs. (12) and neglecting the second-order term $\hbar\nabla_\mu P \sim \hbar^2$, we obtain

$$(p^2 - m^2)A_\mu = \tfrac{1}{2}\hbar\epsilon_{\mu\beta\nu\sigma}p^\beta\nabla^\nu V^\sigma = -\tfrac{1}{2}\hbar Q\epsilon_{\mu\beta\nu\sigma}F^{\beta\nu}V^\sigma = -\hbar Q\tilde{F}_{\mu\sigma}V^\sigma, \quad (17)$$

where we have neglected the second-order term $\hbar\epsilon_{\mu\beta\nu\sigma}\nabla^\nu(p^\beta V^\sigma) \sim \hbar^2$ following the last line of Eqs. (13). Here we have used $\tilde{F}^{\rho\lambda} = \frac{1}{2}\epsilon^{\rho\lambda\mu\nu}F_{\mu\nu}$.

From the second, third, and fifth lines of Eqs. (12), the pseudoscalar, vector, and tensor components are

$$P = -\frac{1}{2m}\hbar\nabla^\mu A_\mu, \quad V_\mu = \frac{1}{m}p_\mu F - \frac{1}{2m}\hbar\nabla^\nu S_{\nu\mu},$$
$$S^{\nu\sigma} = \frac{1}{2m}\hbar(\nabla^\nu V^\sigma - \nabla^\sigma V^\nu) - \frac{1}{m}\epsilon^{\nu\sigma\alpha\beta}p_\alpha A_\beta. \quad (18)$$

Substituting the above into Eqs. (16) and (17), we obtain a closed system of on-shell equations for $F$ and $A^\mu$ up to $O(\hbar)$. We now collect all equations for $F$ and $A^\mu$,

$$p^\mu A_\mu = 0, \quad p\cdot\nabla A^\rho = QF^{\rho\xi}A_\xi, \quad p\cdot\nabla F = 0,$$
$$(p^2 - m^2)F = -\frac{1}{2m}\hbar QF_{\mu\nu}\epsilon^{\mu\nu\alpha\beta}p_\alpha A_\beta, \quad (19)$$
$$(p^2 - m^2)A_\mu = -\frac{1}{m}\hbar Q\tilde{F}_{\mu\sigma}p^\sigma F,$$

which make a closed system of equations for $F$ and $A^\mu$ and can be solved perturbatively in powers of $\hbar$. The last two equations relate the solutions of the lower order to the higher order. Having $F$ and $A^\mu$, we can determine $P$, $V^\mu$, and $S^{\mu\nu}$ through Eq. (19).

## III. WIGNER FUNCTION COMPONENTS AT LEADING ORDER

At leading order of an electromagnetic interaction, the gauge link in the Wigner function in Eq. (1) can be set to 1, then we have following simple form:

$$W_{\alpha\beta}(x,p) = \int\frac{d^4y}{(2\pi)^4}e^{-ipy}\left\langle\bar{\psi}_\beta\left(x+\frac{y}{2}\right)\psi_\alpha\left(x-\frac{y}{2}\right)\right\rangle. \quad (20)$$

We can expand fermionic fields in momentum space using creation and destruction operators as

$$\psi(x) = \frac{1}{\sqrt{\Omega}}\sum_{\mathbf{k},s}\frac{1}{\sqrt{2E_k}}[a(\mathbf{k},s)u(\mathbf{k},s)e^{-ikx} + b^\dagger(\mathbf{k},s)v(\mathbf{k},s)e^{ikx}],$$
$$\bar{\psi}(x) = \frac{1}{\sqrt{\Omega}}\sum_{\mathbf{k},s}\frac{1}{\sqrt{2E_k}}[a^\dagger(\mathbf{k},s)\bar{u}(\mathbf{k},s)e^{ikx} + b(\mathbf{k},s)\bar{v}(\mathbf{k},s)e^{-ikx}], \quad (21)$$

where $\Omega$ is the volume and $s = \pm$ denotes the spin state parallel or antiparallel to the spin quantization direction $\mathbf{n}$ in the rest frame of the particle. By inserting the above into Eq. (20), we obtain

$$W_{\alpha\beta}(x,p) = \frac{1}{(2\pi)^3}\delta(p^2-m^2)\left\{\theta(p^0)\sum_s f_{\rm FD}(E_p-\mu_s)u_\alpha(\mathbf{p},s)\bar{u}_\beta(\mathbf{p},s) - \theta(-p^0)\sum_s f_{\rm FD}(E_p+\mu_s)v_\alpha(-\mathbf{p},s)\bar{v}_\beta(-\mathbf{p},s)\right\}, \quad (22)$$

where we have used $\langle a^\dagger(\mathbf{p},s)a(\mathbf{p},s)\rangle = f_{\rm FD}(E_p-\mu_s)$ and $\langle b^\dagger(-\mathbf{p},s)b(-\mathbf{p},s)\rangle = f_{\rm FD}(E_p+\mu_s)$ with the Fermi-Dirac distribution defined by $f_{\rm FD} = 1/(e^{\beta x}+1)$ ($\beta \equiv 1/T$, $T$ is temperature) and $\mu_s$ is the chemical potential for the fermions with spin state $s$.

From Eq. (22) we can extract the scalar, vector, and axial vector components by applying Eq. (10). We extract the scalar component as

$$F_{(0)} = \mathrm{Tr}[W] = m\delta(p^2-m^2)V, \quad (23)$$

where we have used $\bar{u}(\mathbf{p},s)u(\mathbf{p},s) = 2m$ and $\bar{v}(-\mathbf{p},s)v(-\mathbf{p},s) = -2m$, and

$$V \equiv \frac{2}{(2\pi)^3}\sum_s[\theta(p^0)f_{\rm FD}(p_0-\mu_s) + \theta(-p^0)f_{\rm FD}(-p_0+\mu_s)]. \quad (24)$$

For the vector component, we have

$$V^\mu_{(0)} = \mathrm{Tr}[\gamma^\mu W] = p^\mu\delta(p^2-m^2)V, \quad (25)$$

where we have used $\bar{u}(\mathbf{p},s)\gamma^\mu u(\mathbf{p},s) = 2(E_p,\mathbf{p})$ and $\bar{v}(-\mathbf{p},s)\gamma^\mu v(-\mathbf{p},s) = 2(E_p,-\mathbf{p})$. For the axial vector component, we obtain

$$A^\mu_{(0)} = \mathrm{Tr}[\gamma^\mu\gamma^5 W] = m[\theta(p_0)n^\mu(\mathbf{p},\mathbf{n}) - \theta(-p_0)n^\mu(-\mathbf{p},-\mathbf{n})] \times\delta(p^2-m^2)A, \quad (26)$$

where we have defined

$$A \equiv \frac{2}{(2\pi)^3}\sum_s s[\theta(p^0)f_{\rm FD}(p_0-\mu_s) + \theta(-p^0)f_{\rm FD}(-p_0+\mu_s)], \quad (27)$$

and used $\bar{u}(\mathbf{p},s)\gamma^\mu\gamma^5 u(\mathbf{p},s) = 2msn^\mu(\mathbf{p},\mathbf{n})$ and $\bar{v}(-\mathbf{p},s)\gamma^\mu\gamma^5 v(-\mathbf{p},s) = 2msn^\mu(-\mathbf{p},-\mathbf{n})$ with $n^\mu(\mathbf{p},\mathbf{n})$ given by

$$n^\mu(\mathbf{p},\mathbf{n}) = \Lambda^\mu_\nu(\mathbf{v})n^\nu(\mathbf{0},\mathbf{n}) = \left(\frac{\mathbf{n}\cdot\mathbf{p}}{m}, \mathbf{n} + \frac{(\mathbf{n}\cdot\mathbf{p})\mathbf{p}}{m(m+E_p)}\right). \quad (28)$$

Here $\Lambda^\mu_\nu(\mathbf{v})$ is the Lorentz transformation for $\mathbf{v} = \mathbf{p}/E_p$ and $n^\nu(\mathbf{0},\mathbf{n}) = (0,\mathbf{n})$ is the four-vector of the spin quantization direction in the rest frame of the fermion. One can check that $n^\mu(\mathbf{p},\mathbf{n})$ satisfies $n^2 = -1$ and $n\cdot p = 0$, so it behaves like a spin four-vector up to a factor of 1/2. For Pauli spinors $\chi_s$ and $\chi_{s'}$ in $u(\mathbf{p},s)$ and $v(-\mathbf{p},s')$, respectively, we have

$\chi_s^\dagger \boldsymbol{\sigma} \chi_s = s\mathbf{n}$ and $\chi_{s'}^\dagger \boldsymbol{\sigma} \chi_{s'} = -s'\mathbf{n}$. We can take the massless limit by setting $\mathbf{n} = \hat{\mathbf{p}}$, then we have $mn^\mu(\mathbf{p},\mathbf{n}) \to (|\mathbf{p}|,\mathbf{p})$ and $mn^\mu(-\mathbf{p},-\mathbf{n}) \to (|\mathbf{p}|,-\mathbf{p})$. This way we can recover the previous result of the axial vector component for massless fermions [19,23],

$$A_{(0)}^\mu \to \delta(p^2)\frac{2}{(2\pi)^3} p^\mu \sum_s s\{\theta(p^0) f_{\rm FD}(p_0 - \mu_s) + \theta(-p^0) f_{\rm FD}(-p_0 + \mu_s)\}, \tag{29}$$

where $s = \pm$ now denote the right-handed and left-handed fermions.

## IV. AXIAL VECTOR COMPONENT AT NEXT-TO-LEADING ORDER

We start with the solution to the Wigner function for chiral or massless fermions [19,20,23]. It is well known that in this case the vector and axial vector components decouple from the rest of other components. Their solutions can be recombined into the chiral components of right-handed and left-handed fermions,

$$\begin{aligned} \mathscr{J}_{(0)s}^\rho(x,p) &= p^\rho f_s \delta(p^2), \\ \mathscr{J}_{(1)s}^\rho(x,p) &= -\frac{s}{2}\hbar\tilde{\Omega}^{\rho\sigma} p_\sigma \frac{df_s}{d(\beta p_0)}\delta(p^2) \\ &\quad - sQ\hbar\tilde{F}^{\rho\lambda} p_\lambda f_s \frac{\delta(p^2)}{p^2}, \end{aligned} \tag{30}$$

where $s = \pm$ denote right-hand and left-hand helicities, $p_0 \equiv u\cdot p$, $\tilde{\Omega}^{\rho\sigma} = \frac{1}{2}\epsilon^{\rho\sigma\mu\nu}\partial_\mu(\beta u_\nu)$, $\tilde{F}^{\rho\lambda} = \frac{1}{2}\epsilon^{\rho\lambda\mu\nu}F_{\mu\nu}$, and $f_s$ are distribution functions of chiral fermions defined by

$$f_s(x,p) = \frac{2}{(2\pi)^3}[\theta(p_0) f_{\rm FD}(p_0 - \mu_s) + \theta(-p_0) f_{\rm FD}(-p_0 + \mu_s)]. \tag{31}$$

and

$$\frac{df_s}{d(\beta p_0)} = \frac{2}{(2\pi)^3}\left[\theta(p_0)\frac{d}{d(\beta p_0)} f_{\rm FD}(p_0 - \mu_s) - \theta(-p_0)\frac{d}{d(-\beta p_0)} f_{\rm FD}(-p_0 + \mu_s)\right]. \tag{32}$$

Note that in the definition of the dual vorticity tensor $\tilde{\Omega}^{\rho\beta}$ in Eq. (30) we have included the factor $\beta = 1/T$ inside $\partial_\mu$, which is different from the convention (without such a factor) in Refs. [19,20,23]. The chiral components in Eq. (30) are related to the vector and axial vector components by

$$\begin{aligned} V^\rho(x,p) &= \mathscr{J}_+^\rho(x,p) + \mathscr{J}_-^\rho(x,p), \\ A^\rho(x,p) &= \mathscr{J}_+^\rho(x,p) - \mathscr{J}_-^\rho(x,p). \end{aligned} \tag{33}$$

Now we try to extend Eq. (30) to massive fermions. We recall that the vector and axial vector components at leading or zeroth order are given by Eqs. (25) and (26),

$$\begin{aligned} V_{(0)}^\mu &= p^\mu \delta(p^2 - m^2) V, \\ A_{(0)}^\mu &= m[\theta(p_0) n_\sigma(\bar{p},n_0) - \theta(-p_0) n_\sigma(-\bar{p}, -n_0)] \\ &\quad \times \delta(p^2 - m^2) A, \end{aligned} \tag{34}$$

where $V = f_+ + f_-$ and $A = f_+ - f_-$ are given by Eqs. (24) and (27). Note that we have written relevant quantities in covariant forms with fluid velocity: $p_0 \to u\cdot p, (0,\mathbf{p}) \to \bar{p}^\alpha = p^\alpha - (u\cdot p)u^\alpha$, $E_p = \sqrt{m^2 - \bar{p}^2} = |u\cdot p|$. In particular, we have rewritten $n^\mu(\mathbf{p},\mathbf{n})$ and $n^\mu(-\mathbf{p},-\mathbf{n})$ from Eq. (26) as

$$\begin{aligned} n^\mu(\mathbf{p},\mathbf{n}) &\to n^\mu(\bar{p},n_0) \\ &= -\frac{n_0\cdot\bar{p}}{m}u^\mu + n_0^\mu - \frac{n_{0\xi}\bar{p}^\xi \bar{p}^\mu}{m(m+E_p)}, \\ n^\mu(-\mathbf{p},-\mathbf{n}) &\to n^\mu(-\bar{p},-n_0) \\ &= -\frac{n_0\cdot\bar{p}}{m}u^\mu - n_0^\mu + \frac{n_{0\xi}\bar{p}^\xi \bar{p}^\mu}{m(m+E_p)}, \end{aligned} \tag{35}$$

where $n_0^\alpha = (0,\mathbf{n})$ is the four-vector in the comoving frame of the fluid cell and satisfies $n_0\cdot u = 0$. We now propose the following form for the axial component at first order for massive fermions based on the solution in Eq. (30),

$$\begin{aligned} A_{(1)}^\alpha(x,p) &= -\frac{1}{2}\hbar\tilde{\Omega}^{\alpha\sigma} p_\sigma \frac{dV}{d(\beta p_0)}\delta(p^2 - m^2) \\ &\quad - Q\hbar\tilde{F}^{\alpha\lambda} p_\lambda V \frac{\delta(p^2 - m^2)}{p^2 - m^2}, \end{aligned} \tag{36}$$

where the first term is induced by the vorticity. We can check that the above $A_{(1)}^\alpha(x,p)$ satisfies the first and last equations of (20). The kinetic equation, the second equation of Eq. (20), can be imposed for $A_{(1)}^\alpha(x,p)$. We will show in the next section that the axial vector can give the spin four-vector, so we can calculate the polarization density from the vorticity term of $A_{(1)}^\alpha(x,p)$ in Eq. (36).

## V. ENERGY-MOMENTUM AND SPIN TENSOR OR VECTOR DENSITY FROM THE WIGNER FUNCTION

The symmetrized Lagrange density for a free Dirac particle is

$$L = \bar{\psi}\left(\tfrac{1}{2}i\gamma^\mu \overleftrightarrow{\partial}_\mu - m\right)\psi, \tag{37}$$

where $\overleftrightarrow{\partial} = \overrightarrow{\partial} - \overleftarrow{\partial}$. The energy-momentum tensor can be obtained

$$\begin{aligned} T^{\mu\nu} &= \frac{\partial L}{\partial(\partial_\mu\psi)}\partial^\nu\psi + \partial^\nu\psi^\dagger\frac{\partial L}{\partial(\partial_\mu\psi^\dagger)} - g^{\mu\nu}L \\ &= \frac{1}{2}i\bar{\psi}\gamma^\mu \overleftrightarrow{\partial}^\nu \psi - g^{\mu\nu}\bar{\psi}\left(\frac{1}{2}i\gamma^\mu \overleftrightarrow{\partial}_\mu - m\right)\psi. \end{aligned} \tag{38}$$

When taking the ensemble average of $T^{\mu\nu}$, we will use the Dirac equation and assume all fields are on shell.

So we have

$$\begin{aligned}\langle T^{\mu\nu}(x)\rangle &= \frac{1}{2}i\langle\bar{\psi}(x)\gamma^{\mu}\overleftrightarrow{\partial}_{x}^{\nu}\psi(x)\rangle - g^{\mu\nu}\left\langle\bar{\psi}\left(\frac{1}{2}i\gamma^{\alpha}\overleftrightarrow{\partial}_{\alpha}-m\right)\psi\right\rangle \\ &= \int d^4p\, p^{\nu}\mathrm{Tr}(\gamma^{\mu}W) - g^{\mu\nu}\int d^4p[p_{\mu}\mathrm{Tr}(\gamma^{\mu}W) - m\,\mathrm{Tr}(W)] \\ &= \int d^4p\, p^{\nu}V^{\mu},\end{aligned} \tag{39}$$

where we have used $p^{\mu}V_{\mu}=mF$, the first line of Eqs. (12), and

$$\begin{aligned} W_{\alpha\beta}(x,p) &= \int\frac{d^4y}{(2\pi)^4}e^{-ipy}\left\langle\bar{\psi}_{\beta}\left(x+\frac{y}{2}\right)\psi_{\alpha}\left(x-\frac{y}{2}\right)\right\rangle \\ \lim_{y\to0}\partial_y^{\mu}\left\langle\bar{\psi}_{\beta}\left(x+\frac{y}{2}\right)\psi_{\alpha}\left(x-\frac{y}{2}\right)\right\rangle &= \frac{1}{2}\langle[\partial_x^{\mu}\bar{\psi}_{\beta}(x)]\psi_{\alpha}(x) - \bar{\psi}_{\beta}(x)\partial_x^{\mu}\psi_{\alpha}(x)\rangle \\ &= i\int d^4p\, p^{\mu}W_{\alpha\beta}(x,p).\end{aligned} \tag{40}$$

The spin tensor density is defined by

$$M^{\alpha\beta}(x) = \psi^{\dagger}(x)\tfrac{1}{2}\sigma^{\alpha\beta}\psi(x) = \tfrac{1}{2}\mathrm{Tr}[\gamma_0\sigma^{\alpha\beta}\psi(x)\bar{\psi}(x)]. \tag{41}$$

Taking the ensemble average of the spin tensor, we can also express it in terms of the Wigner function,

$$\begin{aligned}\langle M^{\alpha\beta}(x)\rangle &= \frac{1}{2}\lim_{y\to0}\mathrm{Tr}\left[\gamma_0\sigma^{\alpha\beta}\psi\left(x-\frac{y}{2}\right)\bar{\psi}\left(x+\frac{y}{2}\right)\right] \\ &= \frac{1}{2}\int d^4p\,\mathrm{Tr}[\gamma_0\sigma^{\alpha\beta}W(x,p)].\end{aligned} \tag{42}$$

Then we can define the spin tensor component in the Wigner function as

$$\begin{aligned}M^{\alpha\beta}(x,p) &\equiv \tfrac{1}{2}\mathrm{Tr}[\gamma_0\sigma^{\alpha\beta}W(x,p)] \\ &= \tfrac{1}{2}[-\epsilon^{0\alpha\beta\rho}A_{\rho} + ig^{\alpha0}\mathrm{Tr}(\gamma^{\beta}W) - ig^{\beta0}\mathrm{Tr}(\gamma^{\alpha}W)],\end{aligned} \tag{43}$$

where we have used $\gamma^{\mu}\sigma^{\nu\alpha} = i(g^{\mu\nu}\gamma^{\alpha} - g^{\mu\alpha}\gamma^{\nu}) + \epsilon^{\mu\nu\alpha\lambda}\gamma^5\gamma_{\lambda}$. If we take $\alpha\beta = ij$ (spatial indices), we have a simple relation,

$$M^{ij}(x,p) = -\tfrac{1}{2}\epsilon^{ijk}A_k(x,p) = \tfrac{1}{2}\epsilon^{ijk}A^k(x,p), \tag{44}$$

where $\epsilon_{ijk}$ is three-dimensional antisymmetric tensor. The above property can also be seen by the spatial components of $A^{\mu}(x)$,

$$\begin{aligned}A^i(x) &= \bar{\psi}(x)\gamma^i\gamma^5\psi(x) \\ &= \psi^{\dagger}(x)\gamma^0\gamma^i\gamma^5\psi(x) = \psi^{\dagger}(x)\Sigma_i\psi(x),\end{aligned} \tag{45}$$

where $\Sigma_i = \mathrm{diag}(\sigma_i,\sigma_i)$ with $\sigma_i$ being the Pauli matrices. Thus we recognize that $A^i(x,p)/2$ corresponds to the spin vector component of the Wigner function from which we can calculate the polarization density.

## VI. POLARIZATION FROM THE AXIAL VECTOR COMPONENT

We can now calculate the polarization of massive fermions from the axial vector component obtained in Sec. V. At leading order, we can obtain the polarization density by integrating $A^{\alpha}_{(0)}$ in Eq. (26) or (34) over the four-momentum,

$$\begin{aligned}\Pi^{\alpha}_{(0)}(x) &= \frac{1}{2}\int d^4p\, A^{\alpha}_{(0)}(x,p) \\ &= \frac{1}{2}m\int\frac{d^3p}{(2\pi)^3}\frac{1}{E_p}\sum_s s\left[n^{\alpha}(\bar{p},n_0)\frac{1}{e^{\beta(E_p-\mu_s)}+1} - n^{\alpha}(-\bar{p},-n_0)\frac{1}{e^{\beta(E_p+\mu_s)}+1}\right] \\ &= -\frac{1}{2}u^{\alpha}\int\frac{d^3p}{(2\pi)^3}\frac{n_0\cdot\bar{p}}{E_p}\sum_s s\left[\frac{1}{e^{\beta(E_p-\mu_s)}+1} - \frac{1}{e^{\beta(E_p+\mu_s)}+1}\right] \\ &\quad + \int\frac{d^3p}{(2\pi)^3}\frac{m}{2E_p}\left[n_0^{\alpha} - \frac{(n_0\cdot\bar{p})\bar{p}^{\alpha}}{m(m+E_p)}\right]\sum_s s\left[\frac{1}{e^{\beta(E_p-\mu_s)}+1} + \frac{1}{e^{\beta(E_p+\mu_s)}+1}\right].\end{aligned} \tag{46}$$

If $\mu_s = \mu$ does not depend on $s$, we see immediately that $\Pi^\alpha = 0$. In this case the nonvanishing polarization can only come from the first-order contribution from the vorticity term of $A^\alpha_{(1)}(x,p)$ in Eq. (36),

$$\begin{aligned}\Pi^\alpha(x) = \Pi^\alpha_{(1)}(x) &= -\frac{1}{4}\int d^4p\, \hbar\tilde{\Omega}^{\alpha\sigma} p_\sigma \frac{dV}{d(\beta p_0)}\delta(p^2-m^2)\\ &= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\hbar\tilde{\Omega}^{\alpha\sigma}\frac{1}{E_p}\left\{p_\sigma|_{p_0=E_p}\frac{e^{\beta(E_p-\mu)}}{[e^{\beta(E_p-\mu)}+1]^2} - p_\sigma|_{p_0=-E_p}\frac{e^{\beta(E_p+\mu)}}{[e^{\beta(E_p+\mu)}+1]^2}\right\}\\ &= \frac{1}{2}\hbar\omega^\alpha \int \frac{d^3p}{(2\pi)^3}\left\{\frac{e^{\beta(E_p-\mu)}}{[e^{\beta(E_p-\mu)}+1]^2} + \frac{e^{\beta(E_p+\mu)}}{[e^{\beta(E_p+\mu)}+1]^2}\right\},\end{aligned} \tag{47}$$

where we have removed the spin dependence in the chemical potential $\mu_s = \mu$ and we have used the fact that the spatial part of $p_\sigma$ gives a vanishing momentum integral. We see that the polarization density is proportional to the vorticity vector $\omega^\alpha = \tilde{\Omega}^{\alpha\sigma}u_\sigma$ and is the sum over contributions from fermions and antifermions.

We can also obtain the polarization density from the second (electromagnetic-field) term of $A^\alpha_{(1)}(x,p)$ in Eq. (36),

$$\begin{aligned}\Pi^\alpha_B(x) &= \frac{1}{2}\hbar Q\int d^4p\, \tilde{F}^{\alpha\lambda}p_\lambda V\frac{d}{dp_0^2}\delta(p^2-m^2)\\ &= -\frac{1}{4}\hbar Q\int d^4p\, \tilde{F}^{\alpha\lambda}u_\lambda\frac{dV}{dp_0}\delta(p^2-m^2)\\ &= \frac{1}{2}\hbar Q\beta B^\alpha\int\frac{d^3p}{(2\pi)^3}\frac{1}{E_p}\left\{\frac{e^{\beta(E_p-\mu)}}{[e^{\beta(E_p-\mu)}+1]^2} - \frac{e^{\beta(E_p+\mu)}}{[e^{\beta(E_p+\mu)}+1]^2}\right\},\end{aligned} \tag{48}$$

where we have used $\delta'(x) = -\delta(x)/x$ and that the spatial part of $p_\sigma$ gives vanishing momentum integral. Also we have dropped the complete derivative term which is vanishing at the boundary in momentum space.

We see from Eqs. (47) and (48) that there is a correspondence between $\Pi^\alpha(x)$ from the vorticity and $\Pi^\alpha_B(x)$ from the magnetic field: $E_p\omega^\alpha \leftrightarrow Q\beta B^\alpha$. Note that there is a factor $\beta$ in the definition of $\omega^\alpha$, $\omega^\alpha \equiv (1/2)\epsilon^{\alpha\rho\mu\nu}u_\rho\partial_\mu(\beta u_\nu)$. At zero temperature, the antifermion parts in Eqs. (47) and (48) are vanishing, and the momentum integrals can be carried out analytically from the Fermi sphere distribution. The correspondence at zero temperature now becomes $\mu\omega^\alpha \leftrightarrow Q\beta B^\alpha$, where the $\beta$ factor cancels the one in the definition of $\omega^\alpha$ so the correspondence does not have temperature dependence. From such a correspondence, we see that $\Pi^\alpha_B(x)$ always comes with the charge $Q$ whereas $\Pi^\alpha(x)$ does not, therefore the contributions from fermions and antifermions in $\Pi^\alpha(x)$ have the same sign whereas they have opposite signs in $\Pi^\alpha_B(x)$ since fermions and antifermions carry opposite charges.

In this paper we consider only the polarization induced by the vorticity since it lasts longer and is stronger than the magnetic effect in the later stage of hydrodynamical evolution for massive hadrons.

To estimate the magnitude of $\Pi^\mu(x)$ for fermions from Eq. (47), we can carry out the momentum integral in the comoving frame. After completing the integral over the momentum direction, we obtain the spin-polarization density,

$$\mathbf{\Pi}(x) = \hbar\boldsymbol{\omega}\frac{1}{4\pi^2}\int_0^\infty d|\mathbf{p}||\mathbf{p}|^2\frac{e^{\beta(E_p\mp\mu)}}{[e^{\beta(E_p\mp\mu)}+1]^2} \tag{49}$$

for fermions (−) and antifermions (+). The particle number density for fermions and antifermions is given by

$$\begin{aligned}\rho(x) &= 2\int\frac{d^3p}{(2\pi)^3}\frac{1}{e^{\beta(E_p\mp\mu)}+1}\\ &= \frac{1}{\pi^2}\int_0^\infty d|\mathbf{p}|\frac{|\mathbf{p}|^2}{e^{\beta(E_p\mp\mu)}+1}.\end{aligned} \tag{50}$$

The integrated polarization per particle $\mathbf{\Pi}(x)/\rho(x)$ for fermions or antifermions can be obtained by completing the momentum integrals in Eqs. (49) and (50). We can also define the unintegrated ones with momentum dependence, which is given by the following formula in the comoving frame:

$$\frac{\mathbf{\Pi}(x,\mathbf{p})}{\rho(x,\mathbf{p})} = \hbar\frac{\boldsymbol{\omega}}{4}\frac{e^{\beta(E_p\mp\mu)}}{e^{\beta(E_p\mp\mu)}+1}, \tag{51}$$

where we have defined $\mathbf{\Pi}(x,\mathbf{p}) \equiv d\mathbf{\Pi}(x)/d|\mathbf{p}|$ and $\rho(x,\mathbf{p}) \equiv d\rho(x)/d|\mathbf{p}|$.

At zero temperature, the spin-polarization density in (49) and the particle number density in (50) for the antifermions are vanishing, and the fermion parts can be worked out following the Fermi sphere distribution,

$$\begin{aligned}\mathbf{\Pi}_{T=0}(x) &= \frac{1}{4\pi^2}\hbar\beta^{-1}\boldsymbol{\omega}\mu\sqrt{\mu^2-m^2}\,\theta(\mu-m),\\ \rho_{T=0}(x) &= \frac{1}{3\pi^2}(\mu^2-m^2)^{3/2}\theta(\mu-m).\end{aligned} \tag{52}$$

We can also obtain from Eq. (48) the polarization density from electromagnetic fields at zero temperature,

$$\mathbf{\Pi}_{B,T=0}(x) = \frac{1}{4\pi^2}\hbar Q\mathbf{B}\sqrt{\mu^2-m^2}\,\theta(\mu-m). \tag{53}$$

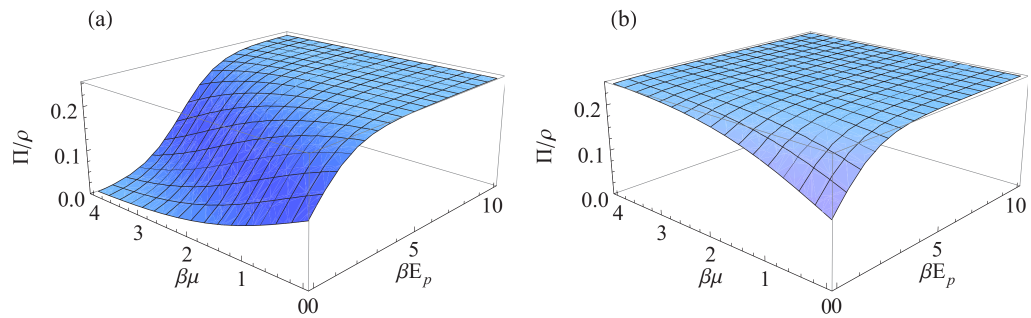


FIG. 1. The unintegrated polarization per particle defined in Eq. (51) for (a) fermions and (b) antifermions at momentum $\mathbf{p}$ in the unit of the local vorticity $\hbar\boldsymbol{\omega}$ as functions of $\beta E_p$ and $\beta\mu$.

We can see the correspondence between $\mathbf{\Pi}_{T=0}(x)$ and $\mathbf{\Pi}_{B,T=0}(x)$ is $\mu\boldsymbol{\omega} \leftrightarrow Q\beta\mathbf{B}$. The integrated polarization per particle $\mathbf{\Pi}(x)/\rho(x)$ for fermions at zero temperature has a simple form

$$\frac{\mathbf{\Pi}_{T=0}(x)}{\rho_{T=0}(x)} = \frac{3}{4}\hbar\beta^{-1}\boldsymbol{\omega}\frac{\mu}{\mu^2 - m^2}\theta(\mu - m), \tag{54}$$

which is a decreasing function of $\mu$. Note that the factor $\beta^{-1}$ in Eqs. (52) and (54) is to cancel the factor $\beta$ in the definition of $\boldsymbol{\omega}$ so that there is no temperature dependence in the results.

The numerical results for the unintegrated polarization per particle in Eq. (51) in the unit of the local vorticity $\hbar\boldsymbol{\omega}$ are shown in Fig. 1 in the ranges of $\beta E_p = [0,10]$ and $\beta\mu = [0,4]$. At fixed values of energy $\beta E_p$, we see that $\mathbf{\Pi}(x,\mathbf{p})/\rho(x,\mathbf{p})$ is a decreasing (increasing) function of $\beta\mu$ for fermions (antifermions), but it always increases with $\beta E_p$ at fixed $\beta\mu$ for both fermions and antifermions. The numerical results for the ratio of $\mathbf{\Pi}(x,\mathbf{p})/\rho(x,\mathbf{p})$ for fermions to antifermions,

$$R = \frac{[\mathbf{\Pi}(x,\mathbf{p})/\rho(x,\mathbf{p})]_{\text{fermion}}}{[\mathbf{\Pi}(x,\mathbf{p})/\rho(x,\mathbf{p})]_{\text{antifermion}}} \tag{55}$$

are shown in Fig. 2. We see that $\mathbf{\Pi}(x,\mathbf{p})/\rho(x,\mathbf{p})$ for fermions is always less than that for antifermions, i.e., $R < 1$ and $R$ decreases with $\beta\mu$ and increases with $\beta E_p$. When $\beta E_p$ is very large, the Fermi-Dirac distributions become Boltzmann ones, and $\mathbf{\Pi}(x,\mathbf{p})/\rho(x,\mathbf{p})$ reaches its asymptotic value 1/4 (in the unit of $\hbar\boldsymbol{\omega}$) for both fermions and antifermions, which leads to $R \to 1$.

The numerical results for the integrated polarization per particle $\mathbf{\Pi}(x)/\rho(x)$ for fermions (left panel) and antifermions

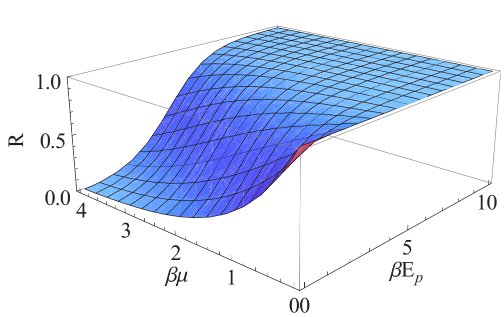


FIG. 2. The ratio $R$ of polarization per particle in Eq. (55) for fermions to antifermions as a function of $\beta E_p$ and $\beta\mu$.

(right panel) are shown in Fig. 3 as functions of $\beta m$ and $\beta\mu$. The numerical results for the ratio of $\mathbf{\Pi}(x)/\rho(x)$,

$$R = \frac{[\mathbf{\Pi}(x)/\rho(x)]_{\text{fermion}}}{[\mathbf{\Pi}(x)/\rho(x)]_{\text{antifermion}}} \tag{56}$$

are shown in Fig. 4. In the left panel we show $R$ as a function of $\beta m$ and $\beta\mu$, whereas in the right panel we show $R$ at three values of $\beta\mu$ as functions of $\beta m$. The dependences of $\mathbf{\Pi}(x)/\rho(x)$ on $\beta m$ and $\beta\mu$ are similar to $\mathbf{\Pi}(x,\mathbf{p})/\rho(x,\mathbf{p})$ on $\beta E_p$ and $\beta\mu$, but the variation in the values of $\mathbf{\Pi}(x)/\rho(x)$ on $\beta m$ is much smaller than $\mathbf{\Pi}(x,\mathbf{p})/\rho(x,\mathbf{p})$ as shown in Figs. 1 and 2.

We see that $R < 1$, i.e., the polarization per particle for fermions is always less than that for the antifermions. This behavior is consistent with the observation in the STAR Collaboration experiment [26]. Also $R$ decreases with $\mu$ at fixed $m$. Such behaviors are based on the following facts: (a) $\mathbf{\Pi}(x)$ is actually proportional to the susceptibility $\partial\rho/\partial\mu$ and increases or decreases for fermions and antifermions with $\beta\mu$ just as $\rho(x)$; (b) $\mathbf{\Pi}_{\text{fermion}}/\mathbf{\Pi}_{\text{antifermion}}$ and $\rho_{\text{fermion}}/\rho_{\text{antifermion}}$ are all increasing functions of $\beta\mu$; (c) $\mathbf{\Pi}_{\text{fermion}}/\mathbf{\Pi}_{\text{antifermion}}$ is less than $\rho_{\text{fermion}}/\rho_{\text{antifermion}}$ and increases slower with $\beta\mu$ than $\rho_{\text{fermion}}/\rho_{\text{antifermion}}$.

In the massless case, the momentum integrals in Eqs. (49) and (50) can be worked out, so we obtain the quantities for fermions (+) and antifermions (−),

$$\begin{aligned}
\mathbf{\Pi}_{m=0}(x) &= -\hbar\boldsymbol{\omega}\frac{1}{2\pi^2}\text{Li}_2(-e^{\pm\beta\mu}),\\
\rho_{m=0}(x) &= -\frac{2}{\pi^2}\text{Li}_3(-e^{\pm\beta\mu}),\\
\left[\frac{\mathbf{\Pi}(x)}{\rho(x)}\right]_{m=0} &= \hbar\boldsymbol{\omega}\frac{1}{4}\frac{\text{Li}_2(-e^{\pm\beta\mu})}{\text{Li}_3(-e^{\pm\beta\mu})},
\end{aligned} \tag{57}$$

where the polylogarithm function is defined by the power series $\text{Li}_s(z) = \sum_{k=1}^{\infty} z^k/k^s$. Figure 5 shows the numerical results for $[\mathbf{\Pi}(x)/\rho(x)]_{m=0}$ for fermions and antifermions and their ratio $R$ defined by Eq. (56) as functions of $\beta\mu$.

If we consider the Cooper-Frye description of hadron freeze-out in hydrodynamic evolution, we can rewrite the polarization density in Eq. (47) by replacing the momentum integral with the one on the freeze-out hypersurface. For fermions, we pick up the first term in the second line of Eq. (47)

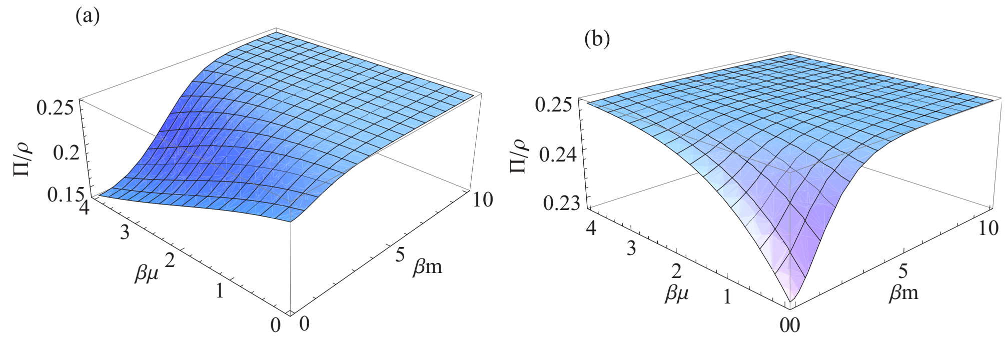


FIG. 3. The integrated polarization per particle $\mathbf{\Pi}(x)/\rho(x)$ for (a) fermions and (b) antifermions in the unit of the local vorticity $\hbar\boldsymbol{\omega}$ as functions of $\beta m$ and $\beta\mu$.

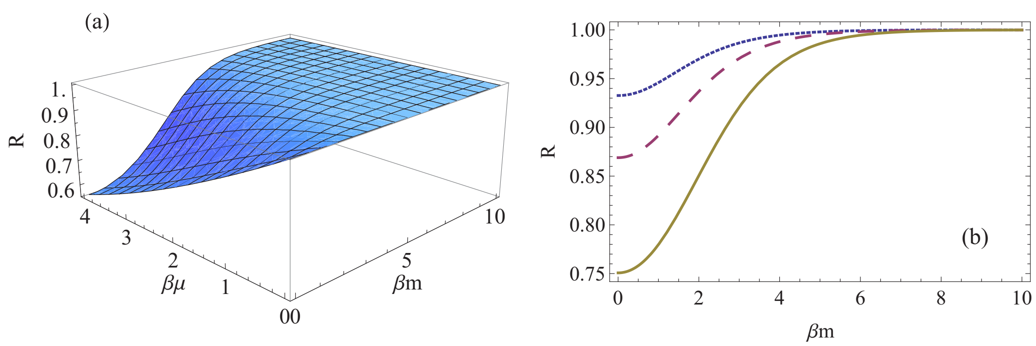


FIG. 4. The ratio $R$ of the integrated polarization per particle in Eq. (56) for fermions to antifermions. (a) $R$ as a function of $\beta m$ and $\beta\mu$. (b) $R$ as functions of $\beta m$ at three values $\beta\mu = 0.5, 1, 2$ corresponding to short-dashed, long-dashed, and solid lines, respectively.

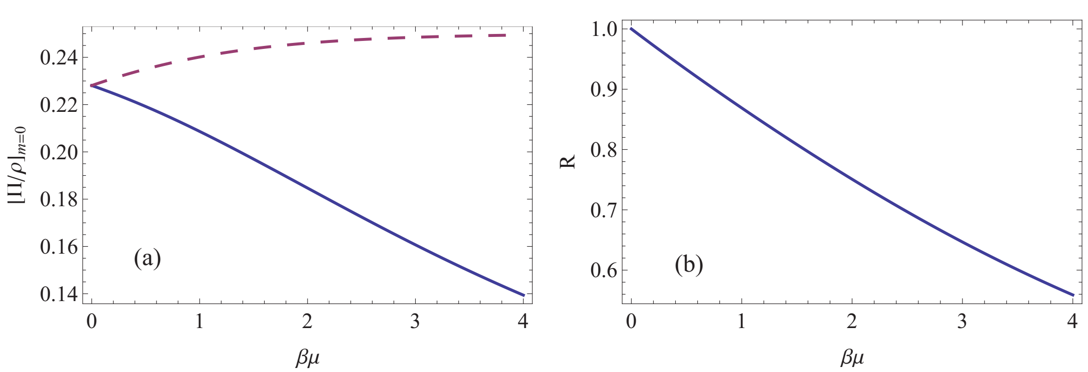


FIG. 5. (a) The integrated polarization per particle $\mathbf{\Pi}(x)/\rho$ for massless fermions (solid line) and antifermions (long-dashed line) in the unit of $\hbar\boldsymbol{\omega}$ as functions of $\beta\mu$. (b) The ratio $R$ of the integrated polarization per particle in Eq. (56) for fermions to antifermions as a function of $\beta\mu$.

and define the polarization spectra in momentum space as

$$\frac{d\Pi^{\alpha}(p)}{d^3p} \approx \frac{\hbar}{2mE_p}\int d\Sigma_{\lambda}p^{\lambda}\tilde{\Omega}^{\alpha\sigma}p_{\sigma}f_{\rm FD}(x,p)[1-f_{\rm FD}(x,p)], \tag{58}$$

where $p^{\mu}$ denotes the on-shell four-momentum and we have $p^{\mu}=(E_p,\mathbf{p})$ in the comoving frame. The particle number distribution for fermions is given by

$$f_{\rm FD}(x,p)=\frac{1}{e^{\beta(x)[u(x)\cdot p-\mu]}+1}. \tag{59}$$

In Eq. (58), we note that $\Pi^{\alpha}(p)$ is the polarization of fermions with the momentum $p$ and has the unit $\hbar$. We can verify that the Lorentz transformation rule for both sides of Eq. (58) are the same. The particle number spectra for fermions in momentum space emitting on the freeze-out hypersurface can be defined as

$$\frac{d\rho(p)}{d^3p}=\frac{2}{E_p}\int d\Sigma_{\lambda}p^{\lambda}f_{\rm FD}(x,p), \tag{60}$$

where the factor 2 is from two spin orientations. Then we obtain the polarization per particle for fermions with the momentum $p$,

$$\mathscr{P}^{\alpha}(p)\equiv\frac{d\Pi^{\alpha}(p)/d^3p}{d\rho(p)/d^3p} = \frac{\hbar}{4m}\frac{\int d\Sigma_{\lambda}p^{\lambda}\tilde{\Omega}^{\alpha\sigma}p_{\sigma}f_{\rm FD}(x,p)[1-f_{\rm FD}(x,p)]}{\int d\Sigma_{\lambda}p^{\lambda}f_{\rm FD}(x,p)}. \tag{61}$$

Equation (61) is a covariant expression for the polarization vector per particle which is the same as the result by Becattini *et al.* [27]. For antifermions, we can flip the sign of the chemical potential $\mu\to-\mu$ in the above formula. We see from Eq. (47) that the total polarization is the sum of fermion and antifermion contributions.

## VII. SUMMARY AND CONCLUSION

We have extended our previous works on the Wigner function for chiral or massless fermions to that for massive fermions. The Wigner function at leading order is derived from its definition by setting the gauge link to 1 and by expanding the free form of the fermionic quantum fields in momentum space. Then all components of the Wigner function can be extracted by projecting the corresponding Dirac matrices and taking traces. The axial vector component at next-to-leading order for massive fermions can be obtained by extending that for massless fermions and satisfies the required equations. We have shown that the axial vector component behaves like a spin four-vector in phase space up to a factor of 1/2. The polarization density can be computed by integration of the axial vector component over momentum. Our numerical results show that the polarization per particle decreases or increases with the (temperature-normalized) chemical potential for fermions or antifermions at fixed (temperature-normalized) energy (mass), whereas it always increases with the (temperature-normalized) energy (mass) at the fixed (temperature-normalized) chemical potential. We have found that the polarization per particle for fermions is always less than that for antifermions. At the large energy (mass) limit the polarization per particle approaches the asymptotic value $\hbar\boldsymbol{\omega}/4$ for both fermions and antifermions following the Boltzmann distribution. We have also formulated the polarization per particle for fermions with the specific momentum on the Cooper-Frye freeze-out hypersurface in a hydrodynamic description, which is consistent with the previous result of Becattini *et al.* [11,27].

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[1] Z.-T. Liang and X.-N. Wang, Phys. Rev. Lett. **94**, 102301 (2005); Erratum, Phys. Rev. Lett. **96**, 039901 (2006).
[2] Z.-T. Liang and X.-N. Wang, Phys. Lett. B **629**, 20 (2005).
[3] F. Becattini, F. Piccinini, and J. Rizzo, Phys. Rev. C **77**, 024906 (2008).
[4] B. Betz, M. Gyulassy, and G. Torrieri, Phys. Rev. C **76**, 044901 (2007).
[5] J.-H. Gao, S.-W. Chen, W.-t. Deng, Z.-T. Liang, Q. Wang, and X.-N. Wang, Phys. Rev. C **77**, 044902 (2008).
[6] X.-G. Huang, P. Huovinen, and X.-N. Wang, Phys. Rev. C **84**, 054910 (2011).
[7] Y. Jiang, Z.-W. Lin, and J. Liao, arXiv:1602.06580.
[8] W.-T. Deng and X.-G. Huang, Phys. Rev. C **93**, 064907 (2016).
[9] L. P. Csernai, V. K. Magas, H. Stocker, and D. D. Strottman, Phys. Rev. C **84**, 024914 (2011).
[10] D. J. Wang, Z. Néda, and L. P. Csernai, Phys. Rev. C **87**, 024908 (2013).
[11] F. Becattini, L. P. Csernai, and D. J. Wang, Phys. Rev. C **88**, 034905 (2013).
[12] B. I. Abelev *et al.* (STAR Collaboration), Phys. Rev. C **76**, 024915 (2007).
[13] J. Deng, Phys. Part. Nucl. **45**, 73 (2014).
[14] D. E. Kharzeev, L. D. McLerran, and H. J. Warringa, Nucl. Phys. A **803**, 227 (2008).

[15] K. Fukushima, D. E. Kharzeev, and H. J. Warringa, Phys. Rev. D **78**, 074033 (2008).

[16] D. T. Son and P. Surówka, Phys. Rev. Lett. **103**, 191601 (2009).

[17] D. E. Kharzeev and D. T. Son, Phys. Rev. Lett. **106**, 062301 (2011).

[18] S. Pu, J.-h. Gao, and Q. Wang, Phys. Rev. D **83**, 094017 (2011).

[19] J.-H. Gao, Z.-T. Liang, S. Pu, Q. Wang, and X.-N. Wang, Phys. Rev. Lett. **109**, 232301 (2012).

[20] J.-h. Gao and Q. Wang, Phys. Lett. B **749**, 542 (2015).

[21] H.-T. Elze, M. Gyulassy, and D. Vasak, Nucl. Phys. B **276**, 706 (1986).

[22] D. Vasak, M. Gyulassy, and H.-T. Elze, Ann. Phys. **173**, 462 (1987).

[23] J.-W. Chen, S. Pu, Q. Wang, and X.-N. Wang, Phys. Rev. Lett. **110**, 262301 (2013).

[24] P. Zhuang and U. W. Heinz, Ann. Phys. **245**, 311 (1996).

[25] S. Ochs and U. W. Heinz, Ann. Phys. **266**, 351 (1998).

[26] M. Lisa, Preliminary results by STAR, on the Workshop on Chirality, Vorticity and Magnetic Field in Heavy Ion Collisions, Los Angeles, February 23–26, 2016 (2016) (unpublished)

[27] F. Becattini, V. Chandra, L. Del Zanna, and E. Grossi, Ann. Phys. **338**, 32 (2013).