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Horizontal-strip LLT polynomials

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Horizontal-strip LLT polynomials

by

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Abstract

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Lascoux, Leclerc, and Thibon defined a remarkable family of symmetric functions that are q-deformations of products of skew Schur functions. These LLT polynomials $G_{\boldsymbol{\lambda}}(x;q)$ can be indexed by a tuple $\boldsymbol{\lambda}$ of skew diagrams. When each skew diagram is a row, we define a weighted graph $\Pi(\boldsymbol{\lambda})$. We show that a horizontal-strip LLT polynomial is determined by this weighted graph. When $\Pi(\boldsymbol{\lambda})$ has no triangles, we establish a combinatorial Schur expansion of $G_{\boldsymbol{\lambda}}(x;q)$. We also explore a connection to extended chromatic symmetric functions.

Contents

Contents **i**

1 Introduction **1**

2 Background **4**
2.1 Compositions, partitions, and skew diagrams 4
2.2 Tableaux, Schur functions, and symmetric functions 5
2.3 Multiskew partitions and LLT polynomials 6
2.4 Graphs and chromatic symmetric functions 8
2.5 Jeu de taquin and cocharge 11

3 Main results **13**

4 A combinatorial Schur expansion **17**

5 A chromatic connection **34**

6 Proof of Theorem 3.1.8 **44**

Bibliography **85**

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Chapter 1

Introduction

In 1997, Lascoux, Leclerc, and Thibon defined a curious family of symmetric functions called LLT polynomials [22], which have extensive connections in algebraic combinatorics and representation theory. Their motivation was to study the modular representation theory of the symmetric group via a natural action of the quantum affine algebra $U_q(\hat{\mathfrak{sl}}_n)$ on the space of symmetric functions. Horizontal-strip LLT polynomials are a generalization of Hall–Littlewood polynomials [22], which are the Frobenius series of cohomology rings of Springer fibers [19]. Springer fibers are subvarieties of the complete flag variety that are deeply connected to representations of Weyl groups [31]. Horizontal-strip LLT polynomials appear in the Shuffle Theorem, which is a combinatorial formula for $\Delta'_{e_{n-1}}e_n$, the Frobenius series of the space of diagonal harmonics [9]. The Extended Delta Theorem is a recent generalization of this formula to an infinite LLT series expansion of $\Delta_{h_\ell}\Delta'_{e_k}e_n$ [8]. LLT polynomials appear in a positive combinatorial expansion of Macdonald polynomials [17], which are the Frobenius series of the Garsia–Haiman S_n-submodules of the space of diagonal harmonics [20], and are a simultaneous generalization of many notable families of symmetric functions [25]. LLT polynomials are also connected to chromatic quasisymmetric functions [30] and representations of regular semisimple Hessenberg varieties [16].

LLT polynomials will be indexed by a sequence $\boldsymbol{\lambda}$ of skew diagrams. When each skew diagram of $\boldsymbol{\lambda}$ is a single cell, the unicellular LLT polynomial $G_{\boldsymbol{\lambda}}(\boldsymbol{x};q)$ is a generating function for arbitrary colourings of a unit interval graph $\Gamma(\boldsymbol{\lambda})$ associated to $\boldsymbol{\lambda}$. Huh, Nam, and Yoo [21] proved a combinatorial Schur expansion of $G_{\boldsymbol{\lambda}}(\boldsymbol{x};q)$ whenever this graph $\Gamma(\boldsymbol{\lambda})$ is a "melting lollipop", namely

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x};q) = \sum_{T\in\mathrm{SYT}_n} q^{\mathrm{wt}_a(T)} s_{\mathrm{shape}(T)}. \tag{1.1}$$

Carlsson and Mellit [9] proved a plethystic relationship between a unicellular LLT polynomial $G_{\boldsymbol{\lambda}}(\boldsymbol{x};q)$ and the corresponding chromatic quasisymmetric function $X_{\Gamma(\boldsymbol{\lambda})}(\boldsymbol{x};q)$ defined by Shareshian and Wachs [30], specifically

$$X_{\Gamma(\boldsymbol{\lambda})}(\boldsymbol{x};q) = \frac{G_{\boldsymbol{\lambda}}([\boldsymbol{x}(q-1)];q)}{(q-1)^n}, \tag{1.2}$$

which in particular implies that for sequences of cells $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, we have that

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x};q) = G_{\boldsymbol{\mu}}(\boldsymbol{x};q) \text{ if and only if } X_{\Gamma(\boldsymbol{\lambda})}(\boldsymbol{x};q) = X_{\Gamma(\boldsymbol{\mu})}(\boldsymbol{x};q). \tag{1.3}$$

Therefore, unicellular LLT polynomials are intimately connected to the major open problem of classifying equalities of chromatic symmetric functions, first posed by Stanley [32] and studied fervently thereafter [5, 6, 10, 26, 28].

In this thesis, we consider horizontal-strip LLT polynomials, meaning that each skew diagram of $\boldsymbol{\lambda}$ is a single row of cells. In Chapter 2, we introduce the necessary definitions. Lascoux, Leclerc, and Thibon [22] showed that if the rows of $\boldsymbol{\lambda}$ are left-justified, then the LLT polynomial $G_{\boldsymbol{\lambda}}(\boldsymbol{x};q)$ is the transformed modified Hall–Littlewood polynomial $\tilde{H}_\lambda(\boldsymbol{x};q)$, for which Lascoux and Schützenberger [24] have proved an elegant combinatorial Schur expansion in terms of a statistic called cocharge, namely

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x};q) = \tilde{H}_\lambda(\boldsymbol{x};q) = \sum_{T\in \mathrm{SSYT}(\lambda)} q^{\mathrm{cocharge}(T)} s_{\mathrm{shape}(T)}. \tag{1.4}$$

Alexandersson and Uhlin [3] generalized cocharge to prove a combinatorial Schur expansion if the rows of $\boldsymbol{\lambda}$ arise from a skew diagram ρ/τ with at most two cells in each column, which we can equivalently state as

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x};q) = \sum_{T\in \mathrm{SSYT}(\alpha)} q^{\mathrm{cocharge}_\tau(T)} s_{\mathrm{shape}(T)}. \tag{1.5}$$

In Chapter 3, we generalize the graph $\Gamma(\boldsymbol{\lambda})$ by defining a weighted interval graph $\Pi(\boldsymbol{\lambda})$ associated to $\boldsymbol{\lambda}$. We prove that a horizontal-strip LLT polynomial is determined by our weighted graph, in other words,

$$\text{if } \Pi(\boldsymbol{\lambda}) \text{ and } \Pi(\boldsymbol{\mu}) \text{ are isomorphic, then } G_{\boldsymbol{\lambda}}(\boldsymbol{x};q) = G_{\boldsymbol{\mu}}(\boldsymbol{x};q). \tag{1.6}$$

The proof is quite technical so it is postponed to Chapter 6. In Chapter 4, we prove a deletion-contraction relation of horizontal-strip LLT polynomials using our weighted graph and we use it to prove a combinatorial Schur expansion of $G_{\boldsymbol{\lambda}}(\boldsymbol{x};q)$ whenever the graph $\Pi(\boldsymbol{\lambda})$ is triangle-free, namely

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x};q) = \sum_{T\in \mathrm{SSYT}(\alpha)} q^{\mathrm{cocharge}_{\Pi(\boldsymbol{\lambda})}} s_{\mathrm{shape}(T)}. \tag{1.7}$$

This generalizes the formula (1.5), which applies in certain cases when our weighted graph $\Pi(\boldsymbol{\lambda})$ is a path.

In Chapter 5, we prove a plethystic relationship between a horizontal-strip LLT polynomial $G_{\boldsymbol{\lambda}}(\boldsymbol{x};q)$ and the corresponding extended chromatic symmetric function $X_{\Pi(\boldsymbol{\lambda})}(\boldsymbol{x})$ defined for weighted graphs by Crew and Spirkl [11], specifically

$$X_{\Pi(\boldsymbol{\lambda})}(\boldsymbol{x}) = \left(\frac{G_{\boldsymbol{\lambda}}([\boldsymbol{x}(q-1)];q)}{(q-1)^n} \right)\bigg|_{q=1}, \tag{1.8}$$

which in particular implies that for sequences of rows $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, we have that

$$\text{if } G_{\boldsymbol{\lambda}}(\boldsymbol{x};q) = G_{\boldsymbol{\mu}}(\boldsymbol{x};q), \text{ then } X_{\Pi(\boldsymbol{\lambda})}(\boldsymbol{x}) = X_{\Pi(\boldsymbol{\mu})}(\boldsymbol{x}). \tag{1.9}$$

We also extend results about equalities of extended chromatic symmetric functions [4] to analogous results about equalities of horizontal-strip LLT polynomials.

Chapter 2

Background

2.1 Compositions, partitions, and skew diagrams

A *composition* is a finite sequence of positive integers $\alpha = \alpha_1 \cdots \alpha_\ell$, which are called the *parts* of α. For convenience, we will concatenate the parts rather than write $(\alpha_1, \ldots, \alpha_\ell)$, and we set $\alpha_i = 0$ if $i > \ell$. The *length* of α, denoted $\ell(\alpha)$, is the number of parts of α and we say that α has *size* N if $\sum_{i=1}^{\ell(\alpha)} \alpha_i = N$. A *partition* ρ is a composition that is weakly decreasing, meaning that $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_{\ell(\rho)}$. For a composition α, the partition *determined* by α, denoted $\mathrm{sort}(\alpha)$, is that obtained by reordering the parts of α in weakly decreasing order. For a partition ρ, we define the integer $n(\rho) = \sum_{i=1}^{\ell(\rho)} (i-1)\rho_i$. For partitions ρ and τ such that $\rho_i \geq \tau_i$ for every i, we define the corresponding *skew diagram* $\lambda = \rho/\tau$ to be the subset of $\mathbb{N} \times \mathbb{N}$ given by

$$\lambda = \{(i,j) \in \mathbb{N} \times \mathbb{N} : i \geq 1,\ \tau_i + 1 \leq j \leq \rho_i\}. \tag{2.1}$$

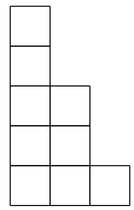
If τ is the empty partition, we will also denote the skew diagram ρ/τ by ρ. In this way, ρ denotes both a partition and a skew diagram, but which one should be clear from context. The elements of λ are called *cells* and the *content* of a cell $u = (i,j) \in \lambda$ is the integer $c(u) = j - i$. We will primarily consider *rows*, which are skew diagrams of the form

$$R = a/b = \{(1,j) : b+1 \leq j \leq a\} \tag{2.2}$$

for some $a \geq b \geq 0$. We denote by $c(R) = \{b, b+1, \ldots, a-1\}$ the set of contents of cells in R and by $l(R) = b$ and $r(R) = a-1$ the smallest and largest contents in $c(R)$ respectively. Note that $l(R)$ is the content of the leftmost cell of R, not the length of the row R, which is $|R| = r(R) - l(R) + 1 = b - a$. We also denote by $R^+ = (a+1)/(b+1)$ and $R^- = (a-1)/(b-1)$ the rows obtained by shifting R right or left by one cell respectively.

Example 2.1.1. We have that $\alpha = 21231$ is a composition of length $\ell(\alpha) = 5$ and size 9, and it determines the partition $\rho = \mathrm{sort}(\alpha) = 32211$, for which $n(\rho) = 13$. We draw the skew diagram ρ as an array of boxes as in Figure 2.1, where we draw a box of side length 1 and upper right corner at position (i,j) in the plane for each cell $(i,j) \in \lambda$. The row $R = 9/5$ has $c(R) = \{5, 6, 7, 8\}$, $l(R) = 5$, $r(R) = 8$, and $|R| = 4$.

Figure 2.1: The skew diagram $\lambda = 32211$



2.2 Tableaux, Schur functions, and symmetric functions

Let λ be a skew diagram. A *semistandard Young tableau (SSYT)* of shape λ is a function $T : \lambda \to \{1, 2, 3, \ldots\}$ that satisfies

$$T_{i,j} \leq T_{i+1,j} \text{ and } T_{i,j} < T_{i,j+1} \text{ for all } (i,j) \in \lambda, \tag{2.3}$$

where we write $T_{i,j}$ to mean $T((i,j))$. In other words, an SSYT of shape λ is a filling of the cells of λ with positive integers, called *entries*, that are weakly increasing from left to right and strictly increasing from bottom to top. The *weight* of T is the sequence $w(T) = (w_1(T), w_2(T), \ldots)$, where $w_i(T) = |T^{-1}(i)|$ is the number of times the integer i appears as an entry. We denote by SSYT_λ the set of SSYT of shape λ and by $\mathrm{SSYT}(\alpha)$ the set of SSYT of weight α, and if $T \in \mathrm{SSYT}_\lambda$, we write that $\mathrm{shape}(T) = \lambda$. We define the *skew Schur function* s_λ to be the formal power series in infinitely many commuting variables $\boldsymbol{x} = (x_1, x_2, \ldots)$ to be [33, Definition 7.10.1]

$$s_\lambda = \sum_{T \in \mathrm{SSYT}_\lambda} \boldsymbol{x}^T, \text{ where } \boldsymbol{x}^T = x_1^{w_1(T)} x_2^{w_2(T)} \cdots . \tag{2.4}$$

When λ is a partition ρ, we call s_ρ a *Schur function*. The function s_λ is a *symmetric function* [33, Theorem 7.10.2], meaning that it is invariant under any permutation of the x_i variables. More specifically, for a field F, we define the *algebra of symmetric functions* Λ_F to be the F-algebra of formal power series in $\boldsymbol{x}$ of bounded degree that are invariant under permutations of the variables. Then we have the following.

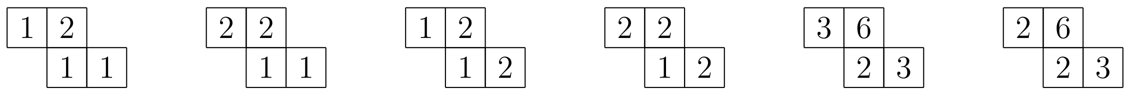
Theorem 2.2.1. [33, Corollary 7.10.6] The set $\{s_\rho : \rho \text{ a partition}\}$ forms a basis for $\Lambda_{\mathbb{Q}}$.

For an integer $n \geq 1$, the *n-th homogeneous symmetric function*, the *n-th elementary symmetric function*, and the *n-th power sum symmetric function* are

$$h_n = \sum_{1 \leq i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n}, \ e_n = \sum_{1 \leq i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}, \text{ and } p_n = \sum_{i=1}^{\infty} x_i^n. \tag{2.5}$$

For a partition $\rho = \rho_1 \cdots \rho_{\ell(\rho)}$, we define $h_\rho = h_{\rho_1} \cdots h_{\rho_{\ell(\rho)}}$, $e_\rho = e_{\rho_1} \cdots e_{\rho_{\ell(\rho)}}$, and $p_\rho = p_{\rho_1} \cdots p_{\rho_{\ell(\rho)}}$. Then we have the following.

Figure 2.2: Several semistandard Young tableaux of shape $\lambda = 32/1$



$$\begin{aligned} s_{32/1} &= x_1^3x_2 + 2x_1^2x_2^2 + x_1x_2^3 + \cdots + x_2x_3^2x_6 + x_2^2x_3x_6 + \cdots && (2.7)\\ &= s_{22} + s_{31} = h_{22} - h_4 = e_{211} - 2e_{31} + e_4 && (2.8)\\ &= \frac{5}{24}p_{1111} - \frac{1}{4}p_{211} + \frac{1}{8}p_{22} - \frac{1}{3}p_{31} - \frac{1}{4}p_4 && (2.9)\end{aligned}$$

Theorem 2.2.2. [33, Theorem 7.4.4, Corollary 7.6.2, Corollary 7.7.2] Each of the sets

$$\{h_\rho : \rho \text{ a partition}\},\ \{e_\rho : \rho \text{ a partition}\},\ \text{and } \{p_\rho : \rho \text{ a partition}\}$$

form a basis for $\Lambda_\mathbb{Q}$. Equivalently, each of the sets $\{h_n : n \geq 1\}$, $\{e_n : n \geq 1\}$, and $\{p_n : n \geq 1\}$ are algebraically independent and generate $\Lambda_\mathbb{Q}$ as a $\mathbb{Q}$-algebra, in other words

$$\Lambda_\mathbb{Q} = \mathbb{Q}[h_1, h_2, \ldots] = \mathbb{Q}[e_1, e_2, \ldots] = \mathbb{Q}[p_1, p_2, \ldots]. \qquad (2.6)$$

Example 2.2.3. Let $\lambda = 32/1$. We have drawn several SSYTs of shape λ in Figure 2.2 by writing the entry $T_{i,j}$ in the box corresponding to cell $(i, j) \in \lambda$ and we have given the corresponding monomials $\boldsymbol{x}^T$ of the skew Schur function s_λ. We have also expanded s_λ in terms of the four bases introduced above.

2.3 Multiskew partitions and LLT polynomials

A *multiskew partition* is a finite sequence of skew diagrams $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(n)})$. We define the set of sequences of SSYTs

$$\mathrm{SSYT}_{\boldsymbol{\lambda}} = \{\boldsymbol{T} = (T^{(1)}, \ldots, T^{(n)}) : T^{(i)} \in \mathrm{SSYT}_{\lambda^{(i)}}\}. \qquad (2.10)$$

The *weight* of $\boldsymbol{T}$ is $w(\boldsymbol{T}) = w(T^{(1)}) + \cdots + w(T^{(n)})$. For $1 \leq i < j \leq n$, we say that two cells $u \in \lambda^{(i)}$ and $v \in \lambda^{(j)}$ *attack* each other if $c(u) = c(v)$ or $c(u) = c(v) + 1$, we say that $\lambda^{(i)}$ and $\lambda^{(j)}$ *attack* each other if there exist $u \in \lambda^{(i)}$ and $v \in \lambda^{(j)}$ that attack each other, and we say that u and v form an *inversion* in $\boldsymbol{T} \in \mathrm{SSYT}_{\boldsymbol{\lambda}}$ if either

- $c(u) = c(v)$ and $T^{(i)}(u) > T^{(j)}(v)$, or
- $c(u) = c(v) + 1$ and $T^{(j)}(v) > T^{(i)}(u)$.

Equivalently, let us define the *adjusted content* of a cell $u \in \lambda^{(i)}$ to be $\tilde{c}(u) = c(u) + \frac{i}{n}$ and the *content reading order* of the cells $\sqcup_{i=1}^{n} \lambda^{(i)}$ to be the total ordering where $\tilde{c}(u)$ is weakly increasing and cells u and v with $\tilde{c}(u) = \tilde{c}(v)$ increase from bottom left to top right in a skew diagram. Then if $u \in \lambda^{(i)}$ precedes $v \in \lambda^{(j)}$ in the content reading order, we have that u and v attack each other if $0 < \tilde{c}(v) - \tilde{c}(u) < 1$, and that u and v form an inversion in $\boldsymbol{T} \in \mathrm{SSYT}_{\boldsymbol{\lambda}}$ if in addition $T^{(i)}(u) > T^{(j)}(v)$.

We denote by $\mathrm{inv}(\boldsymbol{T})$ the number of pairs of cells that form an inversion in $\boldsymbol{T}$. For a multiskew partition $\boldsymbol{\lambda}$, we now define the *LLT polynomial* to be the formal power series [17, Definition 3.2]

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q) = \sum_{\boldsymbol{T} \in \mathrm{SSYT}_{\boldsymbol{\lambda}}} q^{\mathrm{inv}(\boldsymbol{T})} \boldsymbol{x}^{\boldsymbol{T}}, \tag{2.11}$$

Remark 2.3.1. Lascoux, Leclerc, and Thibon originally defined LLT polynomials differently, in terms of objects called ribbon tableaux. Bylund and Haiman found this simpler formulation, which is equivalent [18, Corollary 5.2.4].

If every $\lambda^{(i)}$ consists of a single cell, we say that $\boldsymbol{\lambda}$ is *unicellular* and that $G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q)$ is a *unicellular LLT polynomial.* If every $\lambda^{(i)}$ is a row, then keeping with the terminology of Alexandersson and Sulzgruber [2], we say that $\boldsymbol{\lambda}$ is a *horizontal-strip* and that $G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q)$ is a *horizontal-strip LLT polynomial.* If $\boldsymbol{\lambda}$ is a horizontal-strip, we define $n(\boldsymbol{\lambda}) = n(\lambda)$, where λ is the partition determined by the row lengths of $\boldsymbol{\lambda}$.

Remark 2.3.2. A skew diagram $\boldsymbol{\lambda}$ that has no two cells (i, j) and (i, j') in the same column is also commonly called a horizontal-strip. We will not be using this meaning in this thesis.

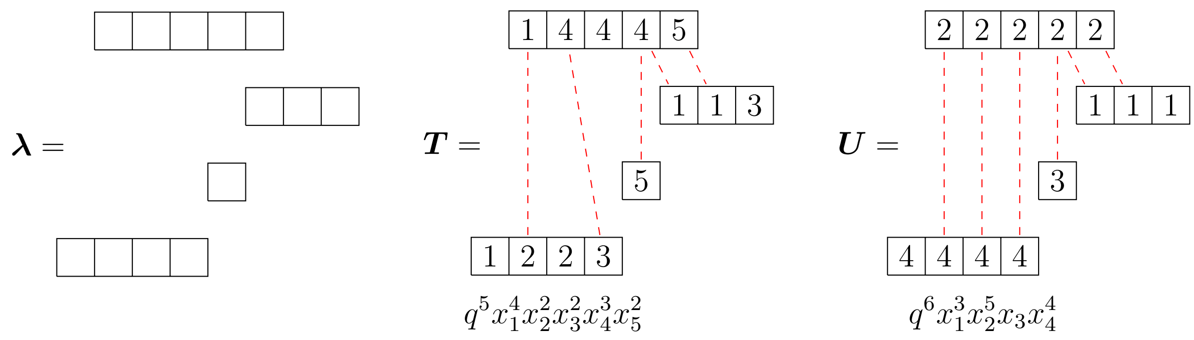
Theorem 2.3.3. [22, Theorem 6.1] The LLT polynomial $G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q)$ is a symmetric function, which in this context means that it is an element of $\Lambda_{\mathbb{Q}(q)}$.

Example 2.3.4. The horizontal-strip $\boldsymbol{\lambda} = (4/0, 5/4, 8/5, 6/1)$, two sequences of SSYTs $\boldsymbol{T}, \boldsymbol{U} \in \mathrm{SSYT}_{\boldsymbol{\lambda}}$ with their inversions marked by dotted lines, and the corresponding monomials of the LLT polynomial $G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q)$ are given in Figure 2.3. Because $\boldsymbol{\lambda}$ is a horizontal-strip, we have drawn it so that cells of the same content are aligned vertically. The horizontal-strip LLT polynomial $G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q)$ can be expanded in the Schur basis as

$$\begin{aligned} G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q) = {} & q^6 s_{5431} + q^6 s_{544} + q^6 s_{5521} + 2q^6 s_{553} + q^6 s_{6331} + 2q^6 s_{6421} + (3q^6 + q^5) s_{643} \\ & + 2q^6 s_{6511} + (4q^6 + q^5) s_{652} + (2q^6 + q^5) s_{661} + (q^6 + q^5) s_{7321} + (q^6 + 2q^5) s_{733} \\ & + (q^6 + 2q^5) s_{7411} + (2q^6 + 5q^5) s_{742} + (2q^6 + 6q^5) s_{751} + 4q^5 s_{76} + q^5 s_{8221} \\ & + (2q^5 + q^4) s_{8311} + (4q^5 + 2q^4) s_{832} + (5q^5 + 5q^4) s_{841} + (3q^5 + 4q^4) s_{85} + 2q^4 s_{9211} \\ & + 3q^4 s_{922} + (7q^4 + 2q^3) s_{931} + (5q^4 + 3q^3) s_{94} + q^3 s_{(10)111} + 6q^3 s_{(10)21} \\ & + (6q^3 + q^2) s_{(10)3} + 3q^2 s_{(11)11} + 5q^2 s_{(11)2} + 3q s_{(12)1} + s_{(13)}. \end{aligned}$$

In Example 2.3.4, we saw that $G_{(4/0,5/4,8/5,6/1)}(\boldsymbol{x}; q)$ is in fact *Schur-positive*, meaning that it is an $\mathbb{N}[q]$-linear combination of Schur functions. In fact, this property holds in general.

Figure 2.3: The horizontal-strip $\boldsymbol{\lambda} = (4/0, 5/4, 8/5, 6/1)$ and two examples of $\boldsymbol{T}, \boldsymbol{U} \in \mathrm{SSYT}_{\boldsymbol{\lambda}}$



Theorem 2.3.5. [18, Theorem 3.1.3 for horizontal-strips], [15, Corollary 6.9 in general] The LLT polynomial $G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q)$ is Schur-positive.

Theorem 2.3.5 was proven by identifying Schur coefficients as those of certain parabolic Kazhdan–Lusztig polynomials, which are known to be positive. It is a major open problem to find a combinatorial Schur expansion of LLT polynomials.

Problem 2.3.6. Find a combinatorial Schur expansion of LLT polynomials of the form

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q) = \sum_{T \in S} q^{\mathrm{stat}(T)} s_{\mathrm{partition}(T)}, \tag{2.12}$$

for some set S and some statistics $\mathrm{stat}(T)$ and $\mathrm{partition}(T)$ associated to $\boldsymbol{\lambda}$.

While there are many partial results [1, 2, 3, 15, 21], Problem 2.3.6 remains a predominant open problem in algebraic combinatorics.

In fact, although some linear relations among LLT polynomials have been found [2, 12, 27], it is unknown precisely when two LLT polynomials are equal.

Problem 2.3.7. Characterize those multiskew partitions $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ for which we have

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q) = G_{\boldsymbol{\mu}}(\boldsymbol{x}; q). \tag{2.13}$$

2.4 Graphs and chromatic symmetric functions

Let $G = (V, E)$ be a graph. We say that G is an *interval graph* if there exists a set of real intervals $[\boldsymbol{a}, \boldsymbol{b}] = \{[a_1, b_1], \ldots, [a_n, b_n]\}$ such that G is isomorphic to the graph $G_{[\boldsymbol{a},\boldsymbol{b}]}$ whose vertices are the intervals and whose edges join intersecting intervals. We say that G is a *unit interval graph* if there exists such a sequence of intervals of unit length.

Theorem 2.4.1. Let G be a graph.

1. [29, Theorem 2.1] G is an interval graph if and only if its vertices can be labelled $v_1, \ldots, v_n$ so that if $i < j < k$ and v_i is adjacent to v_k, then v_j is adjacent to v_k.

2. [14, Theorem 1] G is a unit interval graph if and only if its vertices can be labelled $v_1, \ldots, v_n$ so that if $i < j < k$ and v_i is adjacent to v_k, then v_j is adjacent to v_i and v_k.

A *colouring* of G is a function $\kappa : V \to \{1, 2, 3, \ldots\}$ and κ is *proper* if whenever two vertices $u, v \in V$ are adjacent, we have $\kappa(u) \neq \kappa(v)$. The *chromatic symmetric function* of G is the generating function of proper colourings [32, Definition 2.1]

$$X_G(\boldsymbol{x}) = \sum_{\kappa \text{ proper}} \boldsymbol{x}^\kappa, \text{ where } \boldsymbol{x}^\kappa = \prod_{v \in V} x_{\kappa(v)}. \tag{2.14}$$

Given a vertex-ordering $v_1, \ldots, v_n$ of G, we can define the *chromatic quasisymmetric function* of G to be [30, Section 4.2]

$$X_G(\boldsymbol{x}; q) = \sum_{\kappa \text{ proper}} q^{\operatorname{des}(\kappa)} \boldsymbol{x}^\kappa, \tag{2.15}$$

where $\operatorname{des}(\kappa)$ is the number of *descents* of κ, meaning pairs (v_i, v_j) with $i < j$ and $\kappa(i) < \kappa(j)$. Note that setting $q = 1$ recovers the chromatic symmetric function of G.

Alternatively, given a weight function $w : V \to \{1, 2, 3, \ldots\}$, we can define the *extended chromatic symmetric function* of (G, w) to be [11, Equation 1]

$$X_{(G,w)}(\boldsymbol{x}) = \sum_{\kappa \text{ proper}} \boldsymbol{x}^{\kappa,w}, \text{ where } \boldsymbol{x}^{\kappa,w} = \prod_{v \in V} x_{\kappa(v)}^{w(v)}. \tag{2.16}$$

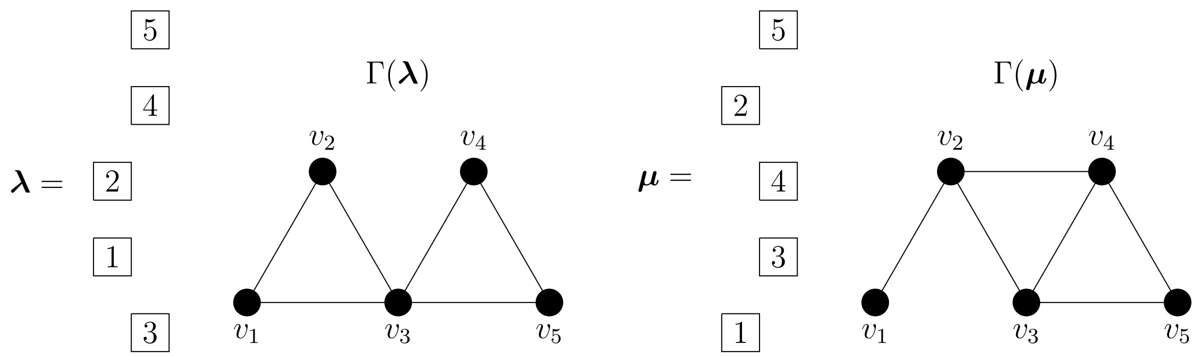
Note that if all vertex weights are 1, then we recover the chromatic symmetric function of G. Also, $X_G(\boldsymbol{x})$ and $X_{(G,w)}(\boldsymbol{x})$ are manifestly symmetric because a permutation of the variables can be right-composed with κ. Additionally, if G is a unit interval graph with a vertex-ordering as described in Theorem 2.4.1, then $X_G(\boldsymbol{x}; q) \in \Lambda_{\mathbb{Q}(q)}$ [30, Proposition 4.4].

The flexibility of this vertex weighting provides us with the following deletion-contraction relation, which exists for the chromatic polynomial but not for the chromatic symmetric function. Given an edge $e = (v_1, v_2)$ of G, the *deletion* $G \setminus e = (V, E \setminus e)$ is given by deleting the edge e and the *contraction* G/e is formed by replacing the vertices v_1 and v_2 with a new vertex v, adjacent to all neighbours of v_1 and v_2. We also set $w(v) = w(v_1) + w(v_2)$. Then we have the following relation.

Lemma 2.4.2. [11, Lemma 2] Let (G, w) be a vertex-weighted graph and let e be an edge of G. Then

$$X_{(G,w)}(\boldsymbol{x}) = X_{(G \setminus e,w)}(\boldsymbol{x}) - X_{(G/e,w)}(\boldsymbol{x}). \tag{2.17}$$

Figure 2.4: Two unicellular multiskew partitions and their corresponding graphs



We mention the two main open problems in the study of chromatic symmetric functions.

Problem 2.4.3. [32, Section 2] Prove that the chromatic symmetric function distinguishes trees, in other words, for trees T_1 and T_2, we have

$$X_{T_1}(\boldsymbol{x}) = X_{T_2}(\boldsymbol{x}) \text{ if and only if } T_1 \text{ and } T_2 \text{ are isomorphic.} \tag{2.18}$$

Problem 2.4.4. Let G be a unit interval graph. Prove that $X_G(\boldsymbol{x})$ is e-positive, meaning that it is an $\mathbb{N}$-linear combination of elementary symmetric functions [32, Conjecture 5.1]. More generally, prove that $X_G(\boldsymbol{x}; q)$ is e-positive, meaning that it is an $\mathbb{N}[q]$-linear combination of elementary symmetric functions [30, Conjecture 4.9].

When $\boldsymbol{\lambda}$ is unicellular, we can naturally associate to $\boldsymbol{\lambda}$ a graph $\Gamma(\boldsymbol{\lambda})$ whose vertices are the cells of $\boldsymbol{\lambda}$ and whose edges join attacking cells. It will often be useful to label the vertices of $\Gamma(\boldsymbol{\lambda})$ as $v_1, \ldots, v_n$ in content reading order of the corresponding cells.

Example 2.4.5. The unicellular multiskew partitions $\boldsymbol{\lambda} = (2/1, 1/0, 1/0, 2/1, 2/1)$ and $\boldsymbol{\mu} = (1/0, 2/1, 2/1, 1/0, 2/1)$ are given in Figure 2.4 with their cells labelled in content reading order, along with the associated unit interval graphs $\Gamma(\boldsymbol{\lambda})$ and $\Gamma(\boldsymbol{\mu})$.

Remark 2.4.6. If $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(n)})$ is unicellular, then the graph $\Gamma(\boldsymbol{\lambda}) \cong G_{[\boldsymbol{a},\boldsymbol{b}]}$, where a_i is the adjusted content of the cell in $\lambda^{(i)}$ and $b_i = a_i + 1$, so it is a unit interval graph. The labelling of the vertices in content reading order will satisfy the condition of Theorem 2.4.1.

Note that when $\boldsymbol{\lambda}$ is unicellular, then a sequence of tableaux $\boldsymbol{T} \in \mathrm{SSYT}_{\boldsymbol{\lambda}}$ is precisely a (not necessarily proper) colouring κ of $\boldsymbol{\lambda}$ and inversions of $\boldsymbol{T}$ are precisely descents of κ. Therefore, we can expect a unicellular LLT polynomial $G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q)$ to be very closely connected to the chromatic quasisymmetric function $X_{\Gamma(\boldsymbol{\lambda})}(\boldsymbol{x}; q)$. Indeed, Carlsson and Mellit proved the following relationship.

Theorem 2.4.7. [9, Proposition 3.5] Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(n)})$ be a unicellular multiskew partition. Then we have

$$X_{\Gamma(\boldsymbol{\lambda})}(\boldsymbol{x}; q) = \frac{G_{\boldsymbol{\lambda}}([\boldsymbol{x}(q-1)]; q)}{(q-1)^n}, \tag{2.19}$$

where the *plethystic substitution* $f(\boldsymbol{x}) \mapsto f(\boldsymbol{x}[q-1])$ is the map defined by $p_k(\boldsymbol{x}[q-1]) = (q^k - 1)p_k(\boldsymbol{x})$ and extending to a $\mathbb{Q}(q)$-algebra homomorphism $\Lambda_{\mathbb{Q}(q)} \to \Lambda_{\mathbb{Q}(q)}$. In particular, because this map is injective, we have for $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ unicellular that

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q) = G_{\boldsymbol{\mu}}(\boldsymbol{x}; q) \text{ if and only if } X_{\Gamma(\boldsymbol{\lambda})}(\boldsymbol{x}; q) = X_{\Gamma(\boldsymbol{\mu})}(\boldsymbol{x}; q). \tag{2.20}$$

Because of this connection, we can consider Problem 2.4.3 and Problem 2.4.4 in the context of LLT polynomials.

Example 2.4.8. One can check that the graphs $\Gamma(\boldsymbol{\lambda})$ and $\Gamma(\boldsymbol{\mu})$ from Example 2.4.5 have the same chromatic quasisymmetric function and therefore, by Theorem 2.4.7, the unicellular LLT polynomials $G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q)$ and $G_{\boldsymbol{\mu}}(\boldsymbol{x}; q)$ are equal as well.

2.5 Jeu de taquin and cocharge

We conclude this chapter by describing a special case in which a combinatorial Schur expansion is known for a horizontal-strip LLT polynomial.

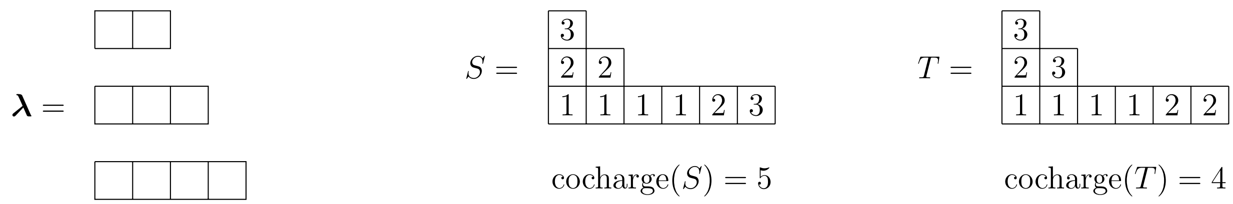
Let $\lambda = \rho/\tau$ be a skew diagram and $T \in \mathrm{SSYT}_\lambda$. An *inside corner* of λ is a cell $u \in \tau$ such that $\lambda \cup \{u\}$ is a skew diagram. The *jeu de taquin slide* of T into an inside corner u is the tableau obtained as follows. There is a cell v directly above or directly right of u. If both, let v be the one with the smaller entry, and if they are equal, let v be the cell above u. We move the entry in v to u. We then similarly consider the cells directly above and directly to the right of v and we continue until we vacate a cell on the outer boundary of λ. The *rectification* of T is the tableau obtained by successive jeu de taquin slides until the result is of partition shape. The following classical result confirms that rectification is well-defined.

Theorem 2.5.1. [33, Theorem A1.2.4] The rectification of T does not depend on the sequence of choices of inside corners into which the jeu de taquin slides are performed.

Let $T \in \mathrm{SSYT}_\lambda$. The *cocharge* of T, denoted $\mathrm{cocharge}(T)$, is the integer defined by the following properties.

- Cocharge is invariant under jeu de taquin slides.
- If λ is a partition with $\ell(\lambda) = 1$, then $\mathrm{cocharge}(T) = 0$.
- If λ is disconnected so that $T = X \cup Y$ with every entry of X is above and left of every entry of Y, no entry of X is equal to i, where i is the smallest entry of T, and S is a tableau obtained by swapping X and Y so that every entry of Y is above and left of every entry of X, then we have $\mathrm{cocharge}(T) = \mathrm{cocharge}(S) + |X|$.

Figure 2.5: The Hall–Littlewood polynomial $G_{(4/0,3/0,2/0)}(\boldsymbol{x};q) = \tilde{H}_{432}(\boldsymbol{x};q)$



$$G_{\boldsymbol{\lambda}}(\boldsymbol{x};q) = \tilde{H}_{432}(\boldsymbol{x};q) = q^7 s_{432} + \cdots + (q^5 + q^4)s_{621} + \cdots + s_9 \tag{2.22}$$

We can calculate cocharge(T) by using the following process called *catabolism*, which was introduced in [23, Problem 6.6.1]. If T consists of a single row, then cocharge(T) $= 0$, otherwise by applying jeu de taquin slides, we can slide the top row of T to the left to disconnect it, swap the pieces, and rectify to produce a new tableau S of smaller cocharge. Repeated catabolism will terminate with a single row of cocharge zero.

Example 2.5.2. Let ρ be a partition and $T \in \mathrm{SSYT}_\rho$. If every entry of T is either i or j for some $i < j$, then T has at most two rows and a single catabolism will produce a tableau with one row, so cocharge(T) $= \rho_2$ is the number of entries in the second row. We can think of cocharge(T) as measuring the extent to which there are entries above others.

Example 2.5.3. Let ρ be a partition and $T \in \mathrm{SSYT}_\rho$ such that $T_{i,j} = i$ for every $(i,j) \in \rho$. Then cocharge(T) $= n(\rho)$, the maximum possible value of cocharge of a tableau in SSYT_ρ.

Now Lascoux, Leclerc, and Thibon proved the following combinatorial Schur expansion for $G_{\boldsymbol{\lambda}}(\boldsymbol{x};q)$ if the rows of $\boldsymbol{\lambda}$ are left-justified.

Theorem 2.5.4. [22, Theorem 6.6] Let $\lambda = \lambda_1 \cdots \lambda_n$ be a partition and let $\boldsymbol{\lambda}$ be the horizontal-strip $\boldsymbol{\lambda} = (\lambda_1/0, \ldots, \lambda_n/0)$. Then the LLT polynomial $G_{\boldsymbol{\lambda}}(\boldsymbol{x};q)$ is the transformed modified Hall–Littlewood polynomial $\tilde{H}_\lambda(\boldsymbol{x};q)$, whose Schur expansion is known [24] to be

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x};q) = \tilde{H}_\lambda(\boldsymbol{x};q) = \sum_{T \in \mathrm{SSYT}(\lambda)} q^{\mathrm{cocharge}(T)} s_{\mathrm{shape}(T)}. \tag{2.21}$$

Example 2.5.5. Let $\lambda = 432$, so that $\boldsymbol{\lambda} = (4/0, 3/0, 2/0)$ is the horizontal-strip in Figure 2.5. Then $G_{\boldsymbol{\lambda}}(\boldsymbol{x};q)$ is the Hall–Littlewood polynomial $\tilde{H}_{432}(\boldsymbol{x};q)$ and the coefficient of s_{621} is calculated from the cocharge of the two tableaux of shape 621 and weight 432.

Chapter 3

Main results

In this chapter, we summarize the main results of our thesis. We begin by generalizing the construction $\Gamma(\boldsymbol{\lambda})$ to a *weighted graph* $\Pi(\boldsymbol{\lambda})$ associated to a horizontal-strip $\boldsymbol{\lambda}$. Recall our conventions for rows from (2.2).

Definition 3.1.1. [34, Definition 3.1] Let R and R' be rows. We define the integer

$$M(R, R') = \begin{cases} |R \cap R'| & \text{if } l(R) \leq l(R'), \\ |R \cap R'^{+}| & \text{if } l(R) > l(R'). \end{cases} \tag{3.1}$$

Note that we must have

$$0 \leq M(R, R') \leq \min\{|R|, |R'|\}. \tag{3.2}$$

Definition 3.1.2. [34, Definition 3.2] Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a horizontal-strip. We define a weighted graph $\Pi(\boldsymbol{\lambda})$ whose vertices are the rows of $\boldsymbol{\lambda}$. The weight of a row R_i is the number of cells $|R_i|$ and rows R_i and R_j with $i < j$ are joined by an edge of weight $M(R_i, R_j)$, where by convention we omit edges of weight zero. We also define the integer

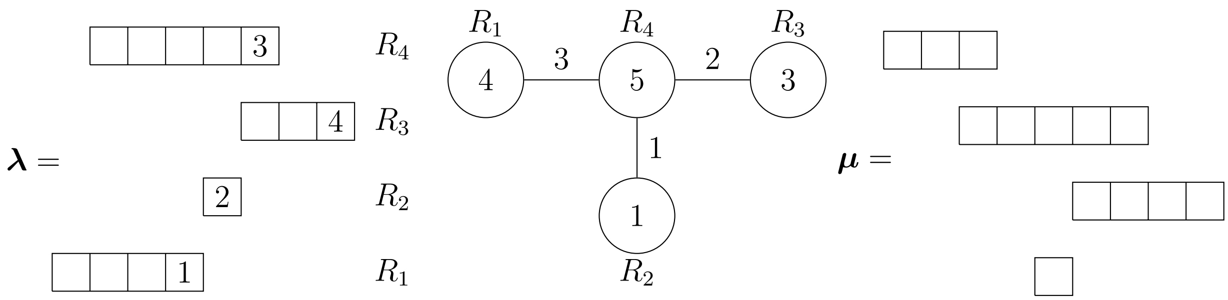
$$M(\boldsymbol{\lambda}) = \sum_{1 \leq i < j \leq n} M(R_i, R_j) \tag{3.3}$$

to be the total edge weight of $\Pi(\boldsymbol{\lambda})$. It will often be useful to label the vertices of $\Pi(\boldsymbol{\lambda})$ as $v_1, \ldots, v_n$ in content reading order of the rightmost cell of the corresponding rows.

Definition 3.1.3. A vertex-weighted and edge-weighted graph Π is *admissible* if $\Pi \cong \Pi(\boldsymbol{\lambda})$ for some horizontal-strip $\boldsymbol{\lambda}$.

Example 3.1.4. The horizontal-strip $\boldsymbol{\lambda} = (4/0, 5/4, 8/5, 6/1)$ and the weighted graph $\Pi(\boldsymbol{\lambda})$ are given in Figure 3.1. We have $M(R_1, R_4) = 3$, $M(R_2, R_4) = 1$, and $M(R_3, R_4) = 2$, so $M(\boldsymbol{\lambda}) = 6$. We have labelled the rightmost cell of each row in content reading order, so our vertex labelling is given by $(v_1, v_2, v_3, v_4) = (R_1, R_4, R_2, R_3)$. We have also drawn the horizontal-strip $\boldsymbol{\mu} = (5/4, 9/5, 7/2, 3/0)$, whose weighted graph $\Pi(\boldsymbol{\mu})$ is isomorphic to $\Pi(\boldsymbol{\lambda})$.

Figure 3.1: Two horizontal-strips $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ with $\Pi(\boldsymbol{\lambda}) \cong \Pi(\boldsymbol{\mu})$



Remark 3.1.5. If $\boldsymbol{\lambda}$ is unicellular, then the graph $\Pi(\boldsymbol{\lambda})$ is simply $\Gamma(\boldsymbol{\lambda})$ with all vertex weights and nonzero edge weights equal to 1, so this is a generalization of $\Gamma(\boldsymbol{\lambda})$. We can think of the integer $M(R_i, R_j)$ as measuring the extent to which the rows R_i and R_j attack each other. The author proved the following result, which makes this idea precise. The proof is relatively elementary and we will not need this result in this thesis, so we omit the proof.

Theorem 3.1.6. [34, Theorem 3.5] Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a horizontal-strip. For any $\boldsymbol{T} \in \mathrm{SSYT}_{\boldsymbol{\lambda}}$, the number of inversions between cells in rows R_i and R_j for $i < j$ is at most $M(R_i, R_j)$, so in particular, $\mathrm{inv}(\boldsymbol{T}) \leq M(\boldsymbol{\lambda})$. Moreover, this maximum is attained, so $M(\boldsymbol{\lambda})$ is the largest power of q in the LLT polynomial $G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q)$.

Remark 3.1.7. If $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ is a horizontal-strip, then the graph $\Pi(\boldsymbol{\lambda}) \cong G_{[\boldsymbol{a},\boldsymbol{b}]}$ as unweighted graphs, where $a_i = l(R_i) + \frac{i}{n}$ is the adjusted content of the leftmost cell in R_i and $b_i = a_i + |R_i|$, so the underlying graph of $\Pi(\boldsymbol{\lambda})$ is an interval graph. In other words, an admissible weighted graph must be an interval graph. The labelling of the vertices in content reading order of the leftmost cells will satisfy the condition of Theorem 2.4.1.

We now state the three main results of this thesis. Our first main result states that a horizontal-strip LLT polynomial is determined by our weighted graph. The proof is quite involved so it will be deferred to Chapter 6.

Theorem 3.1.8. [35, Theorem 2.7] Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be horizontal-strips.

$$\text{If } \Pi(\boldsymbol{\lambda}) \text{ and } \Pi(\boldsymbol{\mu}) \text{ are isomorphic, then } G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q) = G_{\boldsymbol{\mu}}(\boldsymbol{x}; q). \tag{3.4}$$

Theorem 3.1.8 tells us that if Π is an admissible weighted graph, then there is a well-defined horizontal-strip LLT polynomial given by setting $G_{\Pi}(\boldsymbol{x}; q) = G_{\Pi(\boldsymbol{\lambda})}(\boldsymbol{x}; q)$ for any horizontal-strip $\boldsymbol{\lambda}$ such that $\Pi \cong \Pi(\boldsymbol{\lambda})$.

Example 3.1.9. Because the horizontal-strips $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ from Example 3.1.4 have isomorphic weighted graphs, it follows from Theorem 3.1.8 that the LLT polynomials $G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q)$ and $G_{\boldsymbol{\mu}}(\boldsymbol{x}; q)$ are equal.

Our second main result gives a combinatorial Schur expansion of $G_{\boldsymbol{\lambda}}(\boldsymbol{x};q)$ whenever the weighted graph $\Pi(\boldsymbol{\lambda})$ is triangle-free. We first define a generalization of the cocharge statistic.

Definition 3.1.10. Let ρ be a partition and $T \in \mathrm{SSYT}_\rho$ with smallest entry i_0. Recall that we denote by $w_{i_0}(T)$ the number of entries in T equal to i_0. We define the integer

$$f(T) = \max\{t : 0 \leq t \leq \rho_1 - \rho_2,\ t \leq w_{i_0}(T),\ T_{2,j'} > T_{1,j'+t} \text{ for all } 1 \leq j' \leq \rho_2\}. \quad (3.5)$$

Now let $i < j$ be integers. We define $T|_{i,j}$ to be the rectification of the skew tableau obtained from T by restricting to the entries x with $i \leq x \leq j$, and we define the integer

$$\mathrm{cocharge}_{i,j}(T) = w_i(T) - f(T|_{i,j}). \quad (3.6)$$

In Chapter 4, we will prove a deletion-contraction relation for a horizontal-strip LLT polynomial, which will allow us to prove the following formula.

Theorem 3.1.11. [34, Theorem 4.6] Let $\boldsymbol{\lambda} = (R_1, \dots, R_n)$ be a horizontal-strip such that the weighted graph $\Pi(\boldsymbol{\lambda})$ is triangle-free. Let $\alpha_i = |v_i|$ be the weight of the row labelled v_i and let M_{v_i,v_j} be the weight of the edge joining vertices v_i and v_j. Then we have

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x};q) = \sum_{T \in \mathrm{SSYT}(\alpha)} q^{\mathrm{cocharge}_{\Pi(\boldsymbol{\lambda})}} s_{\mathrm{shape}(T)}, \quad (3.7)$$

where $\mathrm{cocharge}_{\Pi(\boldsymbol{\lambda})}(T) = \sum_{1 \leq i < j \leq n} \min\{\mathrm{cocharge}_{i,j}(T), M_{v_i,v_j}\}$.

Example 3.1.12. Figure 3.2 gives an example of two tableaux $S, T \in \mathrm{SSYT}_{862}$ and their restrictions $S|_{2,4}$ and $T|_{2,4}$. Informally, $f(T)$ is the maximum number of i's that we can remove from the bottom row of T so that no entry moves down when we rectify the resulting skew tableau. We have $f(S|_{2,4}) = 3$ and $\mathrm{cocharge}_{2,4}(S) = 5-3 = 2$, and we have $f(T|_{2,4}) = 3$ because we must have $t \leq w_2(T|_{2,4})$, so $\mathrm{cocharge}_{2,4}(T) = 3 - 3 = 0$.

Example 3.1.13. In the case where $j = i+1$, $\mathrm{cocharge}_{i,i+1}(T)$ is the number of entries in the second row of $T|_{i,i+1}$, which we saw in Example 2.5.2 is the same as $\mathrm{cocharge}(T|_{i,i+1})$.

Example 3.1.14. Let $\boldsymbol{\lambda} = (6/5, 9/6, 7/2, 4/0)$ be the horizontal-strip from Example 2.3.4. Because the graph $\Pi(\boldsymbol{\lambda})$ is triangle-free, Theorem 3.1.11 allows us to calculate the coefficient of s_{733} using the three tableaux of weight $\alpha = 4153$ and shape 733 in Figure 3.3. From the values of $\mathrm{cocharge}_{\Pi(\boldsymbol{\lambda})}$, we see that the coefficient is $(q^6 + 2q^5)$.

Example 3.1.15. Let $\boldsymbol{\lambda} = (R_1, R_2)$ be a horizontal-strip with exactly two rows, so that $\Pi(\boldsymbol{\lambda})$ consists of two vertices of weights a and b, joined by an edge of weight M, for some $a \geq b \geq M$. Then α is either ab or ba, but in either case, for each $0 \leq k \leq b$ there is a unique tableau T_k with content α and shape $(a+b-k)k$, which has $\mathrm{cocharge}_{\Pi(\boldsymbol{\lambda})}(T_k) = \min\{k, M\}$. Therefore, by Theorem 3.1.11, we have

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x};q) = \sum_{k=0}^{b} q^{\min\{k,M\}} s_{(a+b-k)k} = s_{(a+b)} + q s_{(a+b-1)1} + \cdots + q^M s_{(a+b-M)M} + \cdots + q^M s_{ab}. \quad (3.8)$$

Figure 3.2: Two tableaux S and T and their restrictions $S|_{2,4}$ and $T|_{2,4}$

$$S = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 3 & 4 \\ \hline 2 & 2 & 3 & 4 & 5 & 5 \\ \hline 1 & 1 & 1 & 2 & 2 & 2 & 3 & 4 \\ \hline \end{array} \qquad S|_{2,4} = \begin{array}{|c|c|c|c|c|c|c|} \hline 4 \\ \hline 3 & 3 & 4 \\ \hline 2 & 2 & 2 & 2 & 2 & 3 & 4 \\ \hline \end{array}$$

$$T = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 4 & 5 \\ \hline 2 & 2 & 4 & 4 & 5 & 5 \\ \hline 1 & 1 & 1 & 2 & 3 & 3 & 3 & 3 \\ \hline \end{array} \qquad T|_{2,4} = \begin{array}{|c|c|c|c|c|c|c|} \hline 4 & 4 & 4 \\ \hline 2 & 2 & 2 & 3 & 3 & 3 & 3 \\ \hline \end{array}$$

Figure 3.3: The three tableaux of weight α and shape 733 and the values of cocharge$_\Pi$

$$T_1 = \begin{array}{|c|c|c|c|c|c|c|} \hline 4 & 4 & 4 \\ \hline 3 & 3 & 3 \\ \hline 1 & 1 & 1 & 1 & 2 & 3 & 3 \\ \hline \end{array} \qquad T_2 = \begin{array}{|c|c|c|c|c|c|c|} \hline 4 & 4 & 4 \\ \hline 2 & 3 & 3 \\ \hline 1 & 1 & 1 & 1 & 3 & 3 & 3 \\ \hline \end{array} \qquad T_3 = \begin{array}{|c|c|c|c|c|c|c|} \hline 3 & 4 & 4 \\ \hline 2 & 3 & 3 \\ \hline 1 & 1 & 1 & 1 & 3 & 3 & 4 \\ \hline \end{array}$$

$$2+1+2=5 \qquad 3+0+2=5 \qquad 3+1+2=6$$

Our third main result relates a horizontal-strip LLT polynomial $G_{\boldsymbol{\lambda}}(\boldsymbol{x};q)$ to the extended chromatic symmetric function $X_{\Pi(\boldsymbol{\lambda})}(\boldsymbol{x})$ of the weighted graph $\Pi(\boldsymbol{\lambda})$. Note the similarity to Theorem 2.4.7.

Theorem 3.1.16. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a horizontal-strip. Then we have

$$X_{\Pi(\boldsymbol{\lambda})}(\boldsymbol{x}) = \left(\frac{G_{\boldsymbol{\lambda}}([\boldsymbol{x}(q-1)];q)}{(q-1)^n} \right) \Bigg|_{q=1}, \tag{3.9}$$

which in particular implies that for horizontal-strips $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, we have that

$$\text{if } G_{\boldsymbol{\lambda}}(\boldsymbol{x};q) = G_{\boldsymbol{\mu}}(\boldsymbol{x};q), \text{ then } X_{\Pi(\boldsymbol{\lambda})}(\boldsymbol{x}) = X_{\Pi(\boldsymbol{\mu})}(\boldsymbol{x}). \tag{3.10}$$

In Chapter 5, we will prove Theorem 3.1.16 and we will prove some other relations between horizontal-strip LLT polynomials, motivated by relations between extended chromatic symmetric functions [4, Theorem 4.12, Theorem 7.3]. In particular, we will classify equalities $G_{\boldsymbol{\lambda}}(\boldsymbol{x};q) = G_{\boldsymbol{\mu}}(\boldsymbol{x};q)$ when $\Pi(\boldsymbol{\lambda})$ and $\Pi(\boldsymbol{\mu})$ are vertex-weighted paths in Corollary 5.1.17.

Chapter 4

A combinatorial Schur expansion

In this chapter, we prove a deletion-contraction relation for horizontal-strip LLT polynomials, which will then allow us to prove Theorem 3.1.11. We will describe some operations that we can perform on a horizontal-strip $\boldsymbol{\lambda}$ while preserving both the weighted graph $\Pi(\boldsymbol{\lambda})$ and the LLT polynomial $G_{\boldsymbol{\lambda}}(\boldsymbol{x};q)$. We make the following definition.

Definition 4.1.1. Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be horizontal-strips. We say that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are *similar* if $\Pi(\boldsymbol{\lambda}) \cong \Pi(\boldsymbol{\mu})$ and $G_{\boldsymbol{\lambda}}(\boldsymbol{x};q) = G_{\boldsymbol{\mu}}(\boldsymbol{x};q)$. We denote by $\mathcal{S}(\boldsymbol{\lambda})$ the set of horizontal-strips that are similar to $\boldsymbol{\lambda}$.

Note that similarity is an equivalence relation.

Remark 4.1.2. Theorem 3.1.8 states that the first condition above implies the second, in other words that $\Pi(\boldsymbol{\lambda}) \cong \Pi(\boldsymbol{\mu})$ implies $G_{\boldsymbol{\lambda}}(\boldsymbol{x};q) = G_{\boldsymbol{\mu}}(\boldsymbol{x};q)$. However, until we prove this in Chapter 6 it will be convenient to temporarily define this notion of similarity.

Proposition 4.1.3. Let $\boldsymbol{\lambda} = (R_1, \dots, R_n)$ be a horizontal-strip.

1. Let $\boldsymbol{\lambda}^+ = (R_1^+, \dots, R_n^+)$ and $\boldsymbol{\lambda}^- = (R_1^-, \dots, R_n^-)$ denote the horizontal-strips obtained by translating all rows right by one cell or left by one cell respectively. Then we have $\boldsymbol{\lambda}^+, \boldsymbol{\lambda}^- \in \mathcal{S}(\boldsymbol{\lambda})$.

2. Define the *cycle* of $\boldsymbol{\lambda}$ to be $\kappa(\boldsymbol{\lambda}) = (R_2, \dots, R_n, R_1^-)$. Then we have $\kappa(\boldsymbol{\lambda}) \in \mathcal{S}(\boldsymbol{\lambda})$.

3. For a sufficiently large integer N, define the *rotation* of $\boldsymbol{\lambda}$ to be

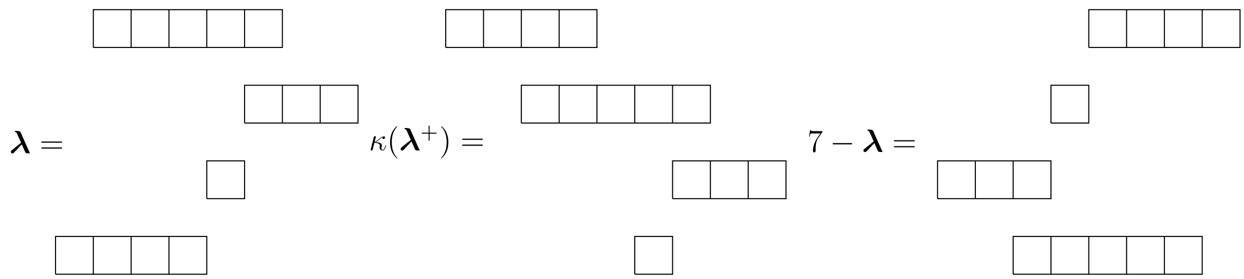
$$N - \boldsymbol{\lambda} = (N - R_n, \dots, N - R_1), \text{ where } N - R = \{(1, N-j) : (1,j) \in R\}. \tag{4.1}$$

   Then we have $N - \boldsymbol{\lambda} \in \mathcal{S}(\boldsymbol{\lambda})$.

Remark 4.1.4. In (3), we take N sufficiently large only because we assume that our cells have nonnegative content. Because of (1), the precise value of N will not matter to us.

Proof of Proposition 4.1.3.

Figure 4.1: A cycle and a rotation



1. This follows directly from the definition because $\Pi(\boldsymbol{\lambda})$ and $G_{\boldsymbol{\lambda}}(\boldsymbol{x};q)$ only depend on the relative positions of the rows of $\boldsymbol{\lambda}$.

2. This follows directly from the definition because $M(R,R') = M(R',R^-)$ and because the condition for a cell $u \in R_i$ for $i \geq 2$ to make an inversion with a cell $v = (1,j) \in R_1$ in $\boldsymbol{\lambda}$ is exactly the condition for u to make an inversion with the corresponding cell $v^- = (1, j-1) \in R_1^-$ in $\kappa(\boldsymbol{\lambda})$.

3. It follows directly from the definition that $M(N-R', N-R) = M(R,R')$ and therefore $\Pi(N-\boldsymbol{\lambda}) \cong \Pi(\boldsymbol{\lambda})$. If we restrict to a finite set of variables $(x_1,\dots,x_k)$, then by associating a sequence of tableau $\boldsymbol{T} = (T^{(1)},\dots,T^{(n)}) \in \mathrm{SSYT}_{\boldsymbol{\lambda}}$ with entries at most k to $-\boldsymbol{T} = (-T^{(1)},\dots,-T^{(n)}) \in \mathrm{SSYT}_{N-\boldsymbol{\lambda}}$ defined by $-T^{(i)}_{1,j} = k+1-T^{(n+1-i)}_{1,N-j}$, we have

$$G_{N-\boldsymbol{\lambda}}(x_1,\dots,x_k;q) = G_{\boldsymbol{\lambda}}(x_k,\dots,x_1;q), \tag{4.2}$$

and it follows that $G_{N-\boldsymbol{\lambda}}(\boldsymbol{x};q) = G_{\boldsymbol{\lambda}}(\boldsymbol{x};q)$ because LLT polynomials are symmetric.

□

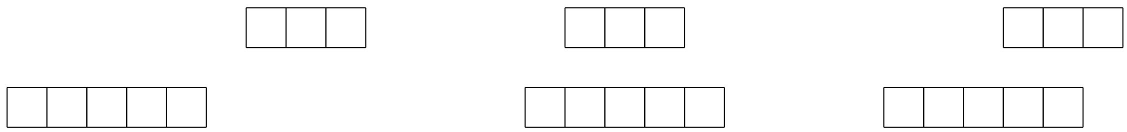
Example 4.1.5. Let $\boldsymbol{\lambda} = (4/0, 5/4, 8/5, 6/1)$. The cycle $\kappa(\boldsymbol{\lambda})$ has a negative content, so for convenience we will first translate right. Then the cycle $\kappa(\boldsymbol{\lambda}^+) = (6/5, 9/6, 7/2, 4/0)$ and the rotation $7-\boldsymbol{\lambda} = (7/2, 3/0, 4/3, 7/4)$ are drawn in Figure 4.1.

The following definition will be justified by Lemma 4.1.14, which states that we can switch commuting rows to obtain a similar horizontal-strip.

Definition 4.1.6. We say that two rows R and R' *commute*, denoted $R \leftrightarrow R'$, if $M(R,R') = M(R',R)$, and otherwise we write $R \not\leftrightarrow R'$.

Proposition 4.1.7. Let R and R' be rows and assume without loss of generality that $l(R) \leq l(R')$.

Figure 4.2: Left: $R \leftrightarrow R'$, Middle: $R \not\leftrightarrow R'$, Right: $R \leftrightarrow R'$



1. If $r(R) < l(R') - 1$, then
$$M(R, R') = M(R', R) = 0, \text{ so } R \leftrightarrow R'. \tag{4.3}$$

2. If $l(R') = l(R)$ or $r(R') \leq r(R)$, then
$$M(R, R') = M(R', R) = \min\{|R|, |R'|\}, \text{ so } R \leftrightarrow R'. \tag{4.4}$$

3. Otherwise, we have $l(R) < l(R') \leq r(R) + 1 \leq r(R')$ and
$$M(R, R') = r(R) - l(R') + 1 \text{ and } M(R', R) = r(R) - l(R') + 2, \text{ so } R \not\leftrightarrow R'. \tag{4.5}$$

Note that in particular, we see that
$$\text{if } R \leftrightarrow R', \text{ then } M(R, R') \text{ is either 0 or } \min\{|R|, |R'|\}. \tag{4.6}$$

Proof. We calculate directly from the definition. If $r(R) < l(R') - 1$, then $R \cap R' = R' \cap R^+ = \emptyset$. If $l(R) = l(R')$, then either $R \subseteq R'$ or $R' \subseteq R$, and if $l(R) < l(R')$ and $r(R') \leq r(R)$, then $R' \subseteq R$ and $R' \subseteq R^+$. Otherwise, the remaining case is when $l(R) < l(R') \leq r(R) + 1 \leq r(R')$, in which case we have
$$M(R, R') = |R \cap R'| = |\{l(R'), \ldots, r(R)\}| = r(R) - l(R') + 1, \text{ and} \tag{4.7}$$
$$M(R', R) = |R' \cap R^+| = |\{l(R'), \ldots, r(R) + 1\}| = r(R) - l(R') + 2. \tag{4.8}$$

Now all of the possibilities have been enumerated and $R \leftrightarrow R'$ only when $M(R, R') = 0$ or $\min\{|R|, |R'|\}$, which proves (4.6). □

Example 4.1.8. The three cases of Proposition 4.1.7 are illustrated in Figure 4.2. The pairs on the left and the middle commute and the pair on the right does not. As a visual description, we have that two rows commute if and only if they are either disjoint and separated by at least one cell, or if one is contained in the other.

Miller [27] defined the following notion to describe local linear relations between LLT polynomials. For our purposes, it will be enough to consider horizontal-strips. If $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ and $\boldsymbol{\mu} = (S_1, \ldots, S_m)$ are horizontal-strips, we denote the concatenation by $\boldsymbol{\lambda} \cdot \boldsymbol{\mu} = (R_1, \ldots, R_n, S_1, \ldots, S_m)$.

Definition 4.1.9. Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be horizontal-strips. We say that they are *LLT-equivalent*, denoted $\boldsymbol{\lambda} \cong \boldsymbol{\mu}$, if for every horizontal-strip $\boldsymbol{\nu}$ we have

$$G_{\boldsymbol{\lambda}\cdot\boldsymbol{\nu}}(\boldsymbol{x};q) = G_{\boldsymbol{\mu}\cdot\boldsymbol{\nu}}(\boldsymbol{x};q). \tag{4.9}$$

More generally, we have an LLT-equivalence of finite formal $\mathbb{Q}(q)$-combinations of horizontal-strips $\sum_i a_i(q)\boldsymbol{\lambda}_i \cong \sum_j b_j(q)\boldsymbol{\mu}_j$ if for every horizontal-strip $\boldsymbol{\nu}$ we have

$$\sum_i a_i(q)G_{\boldsymbol{\lambda}_i\cdot\boldsymbol{\nu}}(\boldsymbol{x};q) = \sum_j b_j(q)G_{\boldsymbol{\mu}_j\cdot\boldsymbol{\nu}}(\boldsymbol{x};q). \tag{4.10}$$

Remark 4.1.10. By cycling, we have that if $\boldsymbol{\lambda}\cdot\boldsymbol{\mu}$, then $G_{\boldsymbol{\nu}\cdot\boldsymbol{\lambda}\cdot\boldsymbol{\nu}'}(\boldsymbol{x};q) = G_{\boldsymbol{\nu}\cdot\boldsymbol{\mu}\cdot\boldsymbol{\nu}'}(\boldsymbol{x};q)$ for all horizontal-strips $\boldsymbol{\nu}$ and $\boldsymbol{\nu}'$. We can think of LLT-equivalence as a local linear relation because we can locally replace $\boldsymbol{\lambda}$ by $\boldsymbol{\mu}$ while preserving the LLT polynomial.

We can prove an LLT-equivalence combinatorially by rearranging to an equivalence of $\mathbb{N}[q]$-combinations of horizontal-strips and finding a bijection of tableaux. The condition (4.13) below will ensure that any inversions involving cells of $\boldsymbol{\nu}$ will be preserved.

Theorem 4.1.11. [27, Theorem 2.2.1] Two finite formal $\mathbb{N}[q]$-combinations of horizontal-strips $\sum_i a_i(q)\boldsymbol{\lambda}_i$ and $\sum_j b_j(q)\boldsymbol{\mu}_j$ are LLT-equivalent if there exists a bijection

$$f : \bigsqcup_i \mathrm{SSYT}_{\boldsymbol{\lambda}_i} \to \bigsqcup_j \mathrm{SSYT}_{\boldsymbol{\mu}_j} \tag{4.11}$$

such that if f maps $\boldsymbol{T} \in \mathrm{SSYT}_{\boldsymbol{\lambda}_i}$ to $\boldsymbol{U} \in \mathrm{SSYT}_{\boldsymbol{\mu}_j}$, then

$$\mathrm{inv}(\boldsymbol{T}) + a_i = \mathrm{inv}(\boldsymbol{U}) + b_j, \tag{4.12}$$

and for every $c \in \mathbb{Z}$, the multiset of entries in cells of content c is preserved, that is

$$\{\boldsymbol{T}(u) :\ c(u) = c\} = \{\boldsymbol{U}(u) :\ c(u) = c\}. \tag{4.13}$$

We now use Theorem 4.1.11 to establish three LLT-equivalence relations. These appeared in [12, Lemma 5.2] and [2, Theorem 2.1] in terms of operators on Dyck and Schröder paths.

Lemma 4.1.12. Let R and R' be rows such that $\ell(R') = r(R) + 1$. Then we have the LLT-equivalence

$$q(R, R') + (R \cup R') \cong q(R \cup R') + (R', R). \tag{4.14}$$

Proof. By Theorem 4.1.11, it suffices to find an appropriate bijection

$$f : \mathrm{SSYT}_{(R,R')} \sqcup \mathrm{SSYT}_{(R\cup R')} \to \mathrm{SSYT}_{(R\cup R')} \sqcup \mathrm{SSYT}_{(R',R)}. \tag{4.15}$$

For a sequence of tableaux $\boldsymbol{T} \in \mathrm{SSYT}_{(R,R')}$, let x and y denote the entries in the cells of content $r(R)$ and $\ell(R')$ respectively. We partition

$$\mathrm{SSYT}_{(R,R')} = \mathrm{SSYT}^{\leq}_{(R,R')} \sqcup \mathrm{SSYT}^{>}_{(R,R')} \tag{4.16}$$

depending on whether $x \le y$ or $x > y$, and we similarly partition $\mathrm{SSYT}_{(R',R)}$. We will now assemble our bijection f as the union of the unique bijections

$$f_1 : \mathrm{SSYT}^{>}_{(R,R')} \to \mathrm{SSYT}^{>}_{(R',R)}, \tag{4.17}$$
$$f_2 : \mathrm{SSYT}^{\le}_{(R,R')} \to \mathrm{SSYT}_{(R\cup R')}, \text{ and} \tag{4.18}$$
$$f_3 : \mathrm{SSYT}_{(R\cup R')} \to \mathrm{SSYT}^{\le}_{(R',R)} \tag{4.19}$$

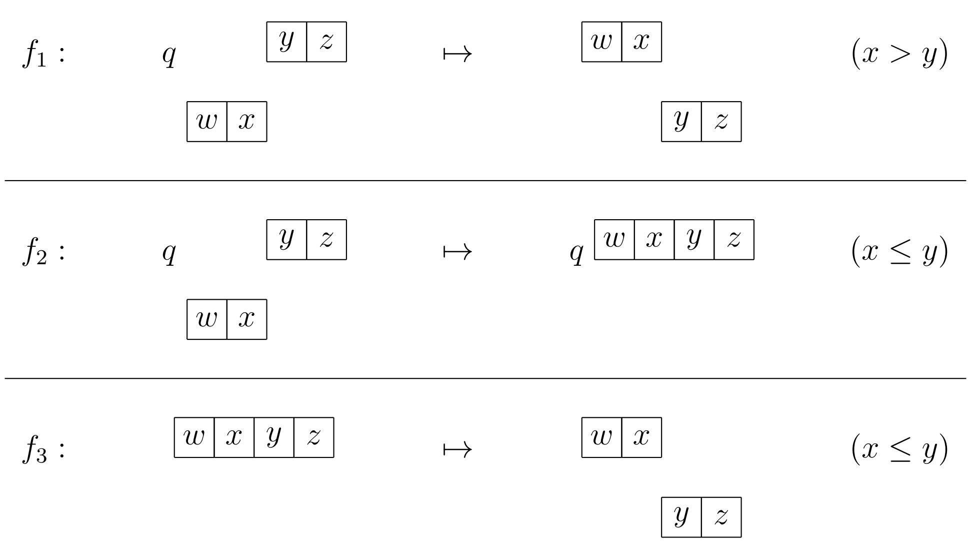
satisfying (4.13). It remains to check (4.12), which dictates how the factors of q correspond to inversions. By definition, every $\boldsymbol{T} \in \mathrm{SSYT}^{>}_{(R',R)}$ has the unique inversion (x, y), while all of the other tableaux have zero inversions, so we have

$$\mathrm{inv}(f_1(\boldsymbol{T})) = \mathrm{inv}(\boldsymbol{T}) + 1,\ \mathrm{inv}(f_2(\boldsymbol{T})) + 1 = \mathrm{inv}(\boldsymbol{T}) + 1,\ \text{and } \mathrm{inv}(f_3(\boldsymbol{T})) = \mathrm{inv}(\boldsymbol{T}),$$

as required by (4.12). This completes the proof. □

Example 4.1.13. These bijections are illustrated in Figure 4.3. We have written q's to indicate how the numbers of inversions are supposed to change.

Figure 4.3: An example of the bijections in the proof of Lemma 4.1.12



Our second LLT-equivalence relation justifies our terminology of commuting rows.

Lemma 4.1.14. Let R and R' be rows such that $R \leftrightarrow R'$. Then we have the LLT-equivalence

$$(R, R') \cong (R', R). \tag{4.20}$$

Remark 4.1.15. Suppose that $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ and $\boldsymbol{\mu} = (R_1, \ldots, R_{i+1}, R_i, \ldots, R_n)$ are horizontal-strips that differ by switching a pair of adjacent rows R_i and R_{i+1}. Lemma 4.1.14 tells us that

$$\text{if } R_i \leftrightarrow R_{i+1}, \text{ then } G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q) = G_{\boldsymbol{\mu}}(\boldsymbol{x}; q) \tag{4.21}$$

and in fact $\boldsymbol{\mu} \in \mathcal{S}(\boldsymbol{\lambda})$ because clearly $\Pi(\boldsymbol{\lambda}) \cong \Pi(\boldsymbol{\mu})$. Conversely, by Theorem 3.1.6, if $G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q) = G_{\boldsymbol{\mu}}(\boldsymbol{x}; q)$, then we must have $M(\boldsymbol{\lambda}) = M(\boldsymbol{\mu})$ and therefore $R_i \leftrightarrow R_{i+1}$ and $\Pi(\boldsymbol{\lambda}) \cong \Pi(\boldsymbol{\mu})$. Therefore, in this case, we see that equalities of LLT polynomials are precisely characterized by our weighted graph.

Proof of Lemma 4.1.14. By Theorem 4.1.11, it suffices to find an appropriate bijection

$$f : \mathrm{SSYT}_{(R,R')} \to \mathrm{SSYT}_{(R',R)}. \tag{4.22}$$

By (4.6), we have $M(R, R') = 0$ or $\min\{|R|, |R'|\}$. If $M(R, R') = 0$, then by Theorem 3.1.6 no inversions are possible, so the unique bijection f satisfying (4.13) trivially satisfies (4.12), so suppose without loss of generality that $M(R, R') = |R'|$. Denote by u_c and v_c the cells in R and R' of content c and let $\boldsymbol{T} \in \mathrm{SSYT}_{(R,R')}$. First suppose that either $\boldsymbol{T}(v_i) < \boldsymbol{T}(u_{i-1})$ for all $i \in c(R')$ or $\boldsymbol{T}(v_j) > \boldsymbol{T}(u_{j+1})$ for all $j \in c(R')$, where by convention we set $\boldsymbol{T}(u_i) = 0$ if $i < l(R)$ and $\boldsymbol{T}(u_i) = \infty$ if $i > r(R)$. Define $\boldsymbol{U} = f(\boldsymbol{T}) \in \mathrm{SSYT}_{(R',R)}$ by $\boldsymbol{U}(u_c) = \boldsymbol{T}(u_c)$ and $\boldsymbol{U}(v_c) = \boldsymbol{T}(v_c)$. Then f satisfies (4.12) because $\mathrm{inv}(\boldsymbol{U}) = \mathrm{inv}(\boldsymbol{T}) = |R'|$ and f satisfies (4.13), so we are done.

Now suppose otherwise and let $i \in c(R')$ be minimal with $\boldsymbol{T}(v_i) \geq \boldsymbol{T}(u_{i-1})$ and $j \in c(R')$ be maximal with $\boldsymbol{T}(v_j) \leq \boldsymbol{T}(u_{j+1})$. Define $\boldsymbol{U} = f(\boldsymbol{T}) \in \mathrm{SSYT}_{(R',R)}$ by $\boldsymbol{U}(u_c) = \boldsymbol{T}(u_c)$ and $\boldsymbol{U}(v_c) = \boldsymbol{T}(v_c)$ if $c < i$ or $c > j$, and $\boldsymbol{U}(u_c) = \boldsymbol{T}(v_c)$ and $\boldsymbol{U}(v_c) = \boldsymbol{T}(u_c)$ if $i \leq c$ and $c \leq j$. An example is given in Figure 4.4, where we have marked the entries of content i and j in red. Informally, the bolded entries are fixed and the unbolded entries are switched.

By construction, $\boldsymbol{U} \in \mathrm{SSYT}_{(R',R)}$ and f satisfies (4.13). By minimality of i, $\boldsymbol{T}$ has inversions $(\boldsymbol{T}(u_c), \boldsymbol{T}(v_c))$ for $l(R') \leq c < i$ and by maximality of j, $\boldsymbol{T}$ has inversions $(\boldsymbol{T}(v_c), \boldsymbol{T}(u_{c+1}))$ for $j < c \leq r(R)$. Similarly, $\boldsymbol{U}$ has inversions $(\boldsymbol{U}(u_{c-1}), \boldsymbol{U}(v_c))$ for $l(R') \leq c < i$ and $(\boldsymbol{U}(v_c), \boldsymbol{U}(u_c))$ for $j < c \leq r(R')$. The four pairs

$$(\boldsymbol{T}(v_{i-1}), \boldsymbol{T}(u_i)),\ (\boldsymbol{T}(v_j), \boldsymbol{T}(u_{j+1})),\ (\boldsymbol{U}(u_{i-1}), \boldsymbol{U}(v_i)),\ \text{and } (\boldsymbol{U}(u_j), \boldsymbol{U}(v_{j+1}))$$

are not inversions, and if $i \leq c$ and $c \leq j$ the cells of content c are unchanged so any inversions are preserved. Therefore, f also satisfies (4.12). We can recover the contents i and j from $\boldsymbol{U}$ by noting that $i \in c(R')$ is minimal with $\boldsymbol{U}(v_i) \geq \boldsymbol{U}(u_{i-1})$ and $j \in c(R')$ is maximal with $\boldsymbol{U}(v_j) \leq \boldsymbol{U}(u_{j+1})$, so f is invertible.

□

Our third LLT-equivalence relation shows that when rows do not commute, we can still switch them up to a factor of q and another correction term. We will see in Proposition 4.1.18 that we can think of this as a deletion-contraction relation.

Figure 4.4: An example of the map f in the proof of Lemma 4.1.14

$$\boldsymbol{T} = \begin{array}{l} \qquad\quad 1\;1\;5\;5\;6\;8\;9 \\ 1\;2\;3\;3\;4\;4\;6\;7\;7\;7\;8\;8\;9 \end{array} \quad \mapsto \boldsymbol{U} = \begin{array}{l} 1\;2\;3\;3\;4\;5\;5\;6\;7\;7\;8\;8\;9 \\ \qquad\quad 1\;1\;4\;6\;7\;8\;9 \end{array}$$

Lemma 4.1.16. Let R and R' be rows such that $l(R') < l(R)$ and $R \not\leftrightarrow R'$. We have the LLT-equivalence

$$(R, R') \cong q(R', R) - (q-1)(R \cup R', R \cap R'). \tag{4.23}$$

Note that by Proposition 4.1.7, the condition $R \not\leftrightarrow R'$ implies that $l(R) \leq r(R') + 1$, so the row $R \cup R'$ makes sense.

Proof. Let $R_1 = R' \setminus R$ and $R_2 = R \cap R'$, so that $R' = R_1 \cup R_2$, $R \cup R_1 = R \cup R'$, and $R \leftrightarrow R_2$. Now by Lemma 4.1.12 and Lemma 4.1.14, we have

$$\begin{aligned}
(R, R') &\cong \frac{q}{q-1}(R, R_1, R_2) - \frac{1}{q-1}(R, R_2, R_1) && (4.24)\\
&\cong \frac{q}{q-1}(q(R_1, R, R_2) - (q-1)(R_1 \cup R, R_2)) - \frac{1}{q-1}(R_2, R, R_1)\\
&\cong \frac{q^2}{q-1}(R_1, R_2, R) - q(R_1 \cup R, R_2) - \frac{1}{q-1}(q(R_2, R_1, R) - (q-1)(R_2, R_1 \cup R))\\
&\cong \frac{q^2}{q-1}(R_1, R_2, R) - q(R_1 \cup R, R_2) - \frac{q}{q-1}(R_2, R_1, R) + (R_1 \cup R, R_2)\\
&\cong \frac{q^2}{q-1}(R_1, R_2, R) - (q-1)(R_1 \cup R, R_2) - \frac{q}{q-1}(q(R_1, R_2, R) - (q-1)(R', R))\\
&= q(R', R) - (q-1)(R \cup R', R \cap R').
\end{aligned}$$

This completes the proof. □

Because Lemma 4.1.16 is an essential ingredient to our proofs of all of our main results, namely Theorem 3.1.8, Theorem 3.1.11, and Theorem 3.1.16, we take a moment to reformulate it.

Corollary 4.1.17. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a horizontal-strip with $l(R_{i+1}) < l(R_i)$ and $R_i \not\leftrightarrow R_{i+1}$. Define the horizontal-strips

$$\boldsymbol{\lambda}' = (R_1, \ldots, R_{i+1}, R_i, \ldots, R_n) \text{ and} \tag{4.25}$$
$$\boldsymbol{\lambda}'' = (R_1, \ldots, R_i \cup R_{i+1}, R_i \cap R_{i+1}, \ldots, R_n). \tag{4.26}$$

Then we have

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q) = qG_{\boldsymbol{\lambda}'}(\boldsymbol{x}; q) - (q-1)G_{\boldsymbol{\lambda}''}(\boldsymbol{x}; q). \tag{4.27}$$

We now describe the associated weighted graphs $\Pi(\boldsymbol{\lambda}')$ and $\Pi(\boldsymbol{\lambda}'')$.

Proposition 4.1.18. Let R_1, R_2, and R be rows such that $R_1 \not\leftrightarrow R_2$ and $l(R_2) < l(R_1)$, and let $M = M(R_1, R_2)$, $M_1 = M(R_1, R)$, and $M_2 = M(R_2, R)$. Then

$$M(R_2, R_1) = M - 1, \tag{4.28}$$
$$M(R_1 \cap R_2, R) = \min\{M - 1, M_1, M_2\}, \text{ and} \tag{4.29}$$
$$M(R_1 \cup R_2, R) = \min\{|R|, \max\{M_1, M_2, M_1 + M_2 - (M - 1)\}\}. \tag{4.30}$$

In particular, if $M_2 = 0$, then $M(R_1 \cap R_2, R) = 0$ and $M(R_1 \cup R_2, R) = M_1$.

Remark 4.1.19. Proposition 4.1.18 describes exactly how to obtain the weighted graphs $\Pi(\boldsymbol{\lambda}')$ and $\Pi(\boldsymbol{\lambda}'')$ from $\Pi(\boldsymbol{\lambda})$. The graph $\Pi(\boldsymbol{\lambda}')$ is obtained by reducing the weight of the edge (R_i, R_{i+1}) by one. The graph $\Pi(\boldsymbol{\lambda}'')$ is obtained by replacing the vertices R_i and R_{i+1} by new vertices $R_i \cap R_{i+1}$ and $R_i \cup R_{i+1}$ of weights $(M - 1)$ and $|R_i| + |R_{i+1}| - (M - 1)$, joined by an edge of weight $(M - 1)$, and joined to other vertices by edges of weights given in (4.29) and (4.30).

Proof. Recall that by Proposition 4.1.7, we must have $l(R_1) - 1 \leq r(R_2)$. The equation (4.28) follows from Proposition 4.1.7, Part 3. We now consider several cases. If $l(R) \geq l(R_1)$, then

$$M(R_1 \cap R_2, R) = M(R_2, R) = M_2 \leq M - 1 \leq M(R_1 \cup R_2, R) = M(R_1, R) = M_1. \tag{4.31}$$

Similarly, if $r(R) < r(R_2) - 1$, then

$$M(R_1 \cap R_2, R) = M(R_1, R) = M_1 \leq M - 1 \leq M(R_1 \cup R_2, R) = M(R_2, R) = M_2. \tag{4.32}$$

Now suppose that $l(R) \leq l(R_1) - 1 \leq r(R_2) \leq r(R)$, so that $R_1 \cap R_2 \subseteq R$ and $M(R_1 \cap R_2, R) = |R_1 \cap R_2| = M - 1 \leq M_1, M_2$. If $l(R) < l(R_2)$, then

$$M(R_1 \cup R_2, R) = |R_1 \cap R^+| + |R_2 \cap R^+| - |R_1 \cap R_2 \cap R^+| = M_1 + M_2 - (M - 1) \leq |R|. \tag{4.33}$$

Similarly, if $l(R) \geq l(R_2)$, then

$$M(R_1 \cup R_2, R) = |R_1 \cap R| + |R_2 \cap R| - |R_1 \cap R_2 \cap R| = |R_1 \cap R| + M_2 - (M - 1). \tag{4.34}$$

At this point, if $r(R) \geq r(R_1)$, then $|R_1 \cap R| = M_1$ and again

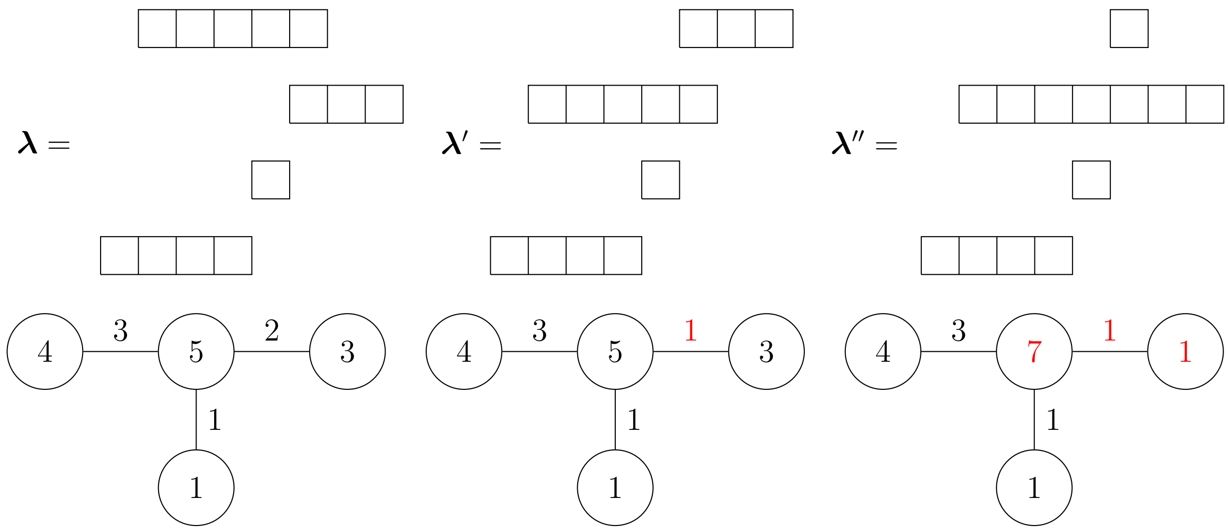
$$M(R_1 \cup R_2, R) = M_1 + M_2 - (M - 1) \leq |R|, \tag{4.35}$$

while if $r(R) < r(R_1)$, then $|R_1 \cap R| = M_1 - 1$ and $R \subseteq R_1 \cup R_2$, so

$$M(R_1 \cup R_2, R) = |R| = (M_1 - 1) + M_2 - (M - 1) < M_1 + M_2 - (M - 1). \tag{4.36}$$

This completes the proof. □

Figure 4.5: A deletion and a contraction



Example 4.1.20. Let $\boldsymbol{\lambda} = (4/0, 5/4, 8/5, 6/1)$ and note that $l(R_4) < l(R_3)$ and $R_3 \not\leftrightarrow R_4$. Therefore, letting $\boldsymbol{\lambda}' = (4/0, 5/4, 6/1, 8/5)$ and $\boldsymbol{\lambda}'' = (4/0, 5/4, 8/1, 6/5)$, we have that

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q) = qG_{\boldsymbol{\lambda}'}(\boldsymbol{x}; q) - (q-1)G_{\boldsymbol{\lambda}''}(\boldsymbol{x}; q). \tag{4.37}$$

The horizontal-strips $\boldsymbol{\lambda}$, $\boldsymbol{\lambda}'$, and $\boldsymbol{\lambda}''$, and their weighted graphs $\Pi(\boldsymbol{\lambda})$, $\Pi(\boldsymbol{\lambda}')$, and $\Pi(\boldsymbol{\lambda}'')$ are given in Figure 4.5. We can think of $\Pi(\boldsymbol{\lambda}')$ and $\Pi(\boldsymbol{\lambda}'')$ as a deletion and contraction of $\Pi(\boldsymbol{\lambda})$.

We now describe an admissible triangle-free weighted graph. We show that if $\boldsymbol{\lambda}$ is a horizontal-strip such that $\Pi(\boldsymbol{\lambda})$ is triangle-free, then $\Pi(\boldsymbol{\lambda})$ must be a union of "caterpillars", which are trees whose non-leaf vertices lie on a single path. For the remainder of this chapter, we will label the vertices of $\Pi(\boldsymbol{\lambda})$ as $v_1, \ldots, v_n$ in content reading order of the rightmost cells of the corresponding rows. We will denote by $|v|$ the weight of vertex v and by M_{v_i,v_j} the weight of the edge (v_i, v_j).

Proposition 4.1.21. Let $\boldsymbol{\lambda}$ be a horizontal-strip such that the weighted graph $\Pi(\boldsymbol{\lambda})$ is triangle-free.

1. If $i < j < k$ and v_i is adjacent to v_k, then $M_{v_j,v_k} = |v_j|$.

2. Every vertex v_i has at most one neighbour v_j for which $i < j$.

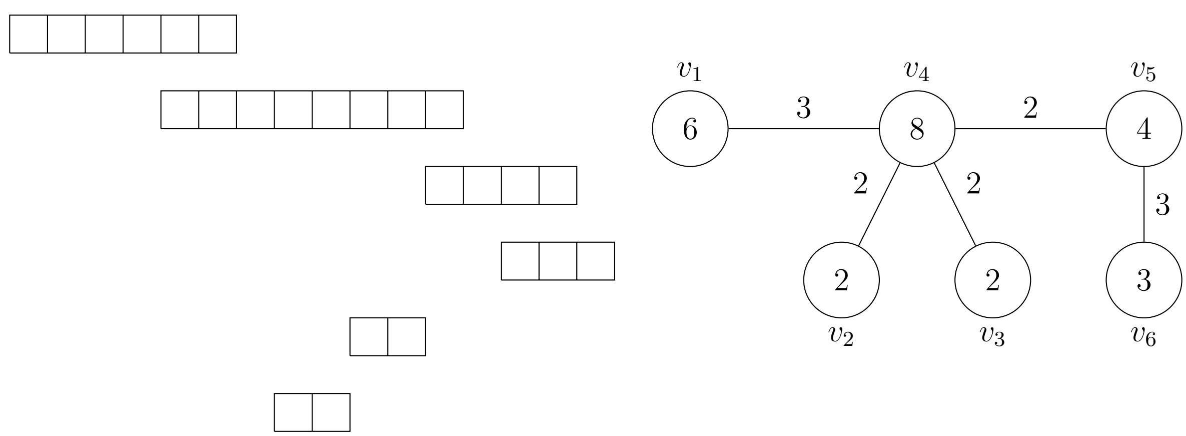
3. Let $C = (V, E)$ be a connected component of $\Pi(\boldsymbol{\lambda})$. Then C is a tree and we can partition the vertices $V = P \sqcup L$ so that the induced subgraph $C[P]$ is a path, every $v \in L$ has degree one, and if $v_j \in L$ has neighbour v_k, then $M_{v_j,v_k} = |v_j|$.

4. If the neighbours of v_i are $\{v_{j_t}\}_{t=1}^r$, then

$$|v_i| + 1 \geq \sum_{t=1}^{r} M_{v_i, v_{j_t}}. \tag{4.38}$$

Example 4.1.22. Figure 4.6 shows an example of a horizontal-strip $\boldsymbol{\lambda}$ for which the weighted graph $\Pi(\boldsymbol{\lambda})$ is triangle-free. Note that $\Pi(\boldsymbol{\lambda})$ is a caterpillar with $P = \{v_1, v_4, v_5, v_6\}$ and that $8 + 1 \geq 3 + 2 + 2 + 2$ and $4 + 1 \geq 2 + 3$.

Figure 4.6: A horizontal-strip $\boldsymbol{\lambda}$ and its corresponding caterpillar graph



Proof of Proposition 4.1.21. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$.

1. Let $R_{i'}$, $R_{j'}$, and $R_{k'}$ be the rows of $\boldsymbol{\lambda}$ corresponding to the vertices v_i, v_j, and v_k respectively. By Proposition 4.1.3, Part 2, we may assume without loss of generality that $i' = 1$. Because $i < j < k$, we have $r(R_{i'}) \leq r(R_{j'}) \leq r(R_{k'})$, and because v_i is adjacent to v_k, we must have $l(R_{k'}) \leq r(R_{i'})$, so $r(R_{j'}) \in c(R_{k'})$ and v_j is adjacent to v_k. Now using that $\Pi(\boldsymbol{\lambda})$ is triangle-free, v_i is not adjacent to v_j, so $r(R_{i'}) < \ell(R_{j'})$, $R_{j'} \subseteq R_{k'}$, and $M_{j,k} = |v_j|$.

2. If v_i is adjacent to both v_j and v_k with $i < j$ and $i < k$, then v_j and v_k are adjacent by Part 1, creating a triangle in $\Pi(\boldsymbol{\lambda})$.

3. We first note that $\Pi(\boldsymbol{\lambda})$ is acyclic because if vertices $\{v_{i_t}\}_{t=1}^r$ form a cycle with $r \geq 3$ and $i_1 < \cdots < i_r$, then the vertex v_{i_1} is adjacent to two vertices $v_{i_t}, v_{i_{t'}}$ with $i_1 < i_t, i_{t'}$, contradicting Part 2, so C must be a tree. By Part 1, we must have $V = \{v_i : i_1 \leq i \leq i_r\}$ for some i_1, i_r. Because C is connected, there must be a path $P = (v_{i_1}, v_{i_2}, \ldots, v_{i_{r-1}}, v_{i_r})$ from v_{i_1} to v_{i_r}, by Part 2, we must have $i_1 < i_2 < \cdots < i_{r-1} < i_r$,

and by Part 1, if $i_t < j < i_{t+1}$, then v_j must be adjacent to $v_{i_{t+1}}$. Because C is a tree, this accounts for all of the edges of C so indeed letting $L = V \setminus P$, each $v \in L$ has degree one, and by Part 1 again, if $v_j \in L$ is adjacent to v_k, then $M_{v_j,v_k} = |v_j|$.

4. Let $R_{i'}$ and $R_{j'_t}$ be the rows of $\boldsymbol{\lambda}$ corresponding to vertices v_i and v_{j_t} respectively, and again by cycling, we may assume that $i' = 1$. Because $\Pi(\boldsymbol{\lambda})$ is triangle-free, we have $M_{v_{j_t},v_{j_{t'}}} = 0$ for $t \neq t'$ so assuming without loss of generality that $j_1 < \cdots < j_r$, we have $l(R_{j'_{t+1}}) \geq r(R_{j'_t}) + 1$ for every t. If $l(R_{j'_t}), \ell(R_{j'_{t'}}) < \ell(R_1)$ or if $r(R_{j'_t}), r(R_{j'_{t'}}) \geq r(R_1)$, then v_t and $v_{t'}$ are adjacent, so we must have $l(R_{j'_t}) \geq l(R_1)$ for every $t \geq 2$ and $r(R_{j'_t}) \leq r(R_1) - 1$ for every $t \leq r-1$. Therefore

$$M_{v_i,v_{j_1}} \leq r(R_{j'_1}) - \ell(R_1) + 2, \tag{4.39}$$
$$M_{v_i,v_{j_2}} = |R_{j'_2}| = r(R_{j'_2}) - l(R_{j'_1}) + 1 \leq r(R_{j'_2}) - r(R_{j'_1}), \tag{4.40}$$
$$\vdots$$
$$M_{v_i,v_{j_{r-1}}} = |R_{j'_{r-1}}| = r(R_{j'_{r-1}}) - \ell(R_{j'_{r-2}}) + 1 \leq r(R_{j'_{r-1}}) - r(R_{j'_{r-2}}), \tag{4.41}$$
$$M_{v_i,v_{j_r}} \leq r(R_1) - l(R_{j'_r}) + 1 \leq r(R_1) - r(R_{j'_{r-1}}), \tag{4.42}$$

and by summing these up, we get

$$\sum_{t=1}^{r} M_{v_i,v_{j_t}} \leq r(R_1) - l(R_1) + 2 = |R_1| + 1 = |v_i| + 1. \tag{4.43}$$

This completes the proof. □

We now discuss how we can use cycling and commuting to rewrite a horizontal-strip $\boldsymbol{\lambda}$ in order to apply our deletion-contraction relation. The following idea will be applied more generally later on in Lemma 6.1.7 and Lemma 6.1.16.

Lemma 4.1.23. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a horizontal-strip such that $\Pi(\boldsymbol{\lambda})$ is triangle-free and suppose that the vertex v_1 is adjacent to some vertex v_j. Let $R_{i'}$ and $R_{j'}$ be the rows of $\boldsymbol{\lambda}$ corresponding to vertices v_1 and v_j. Then there is a horizontal-strip $\boldsymbol{\mu} = (S_1, \ldots, S_n) \in \mathcal{S}(\boldsymbol{\lambda})$ and an isomorphism of weighted graphs $\varphi : \Pi(\boldsymbol{\lambda}) \to \Pi(\boldsymbol{\mu})$ such that $\varphi(R_{i'}) = S_{k+1}$ and $\varphi(R_{j'}) = S_k$ for some k and we have $l(S_{k+1}) < l(S_k)$ and $S_k \not\leftrightarrow S_{k+1}$.

Proof. First note that by cycling, we may assume that $i' > j'$, and then $r(R_{i'}) < r(R_{j'})$ by definition of our vertex labelling. Because v_1 can only be adjacent to v_j by Proposition 4.1.21, Part 2, we have for $1 \leq t < i'$ with $t \neq j'$ that $M(R_t, R_{i'}) = 0$ and therefore $r(R_{i'}) < l(R_t) - 1$ and $R_{i'} \leftrightarrow R_t$ by Proposition 4.1.7, Part 1. Therefore, by commuting, we may now move row $R_{i'}$ down to assume that $i' = j' + 1$. Now if $R_{i'} \not\leftrightarrow R_{j'}$, we have established our claim. Otherwise, if $R_{i'} \leftrightarrow R_{j'}$, we can continue to move the row $R_{i'}$ down, then cycle to the top, decreasing $l(R_{i'})$ by one, and commute again to assume that $i' = j' + 1$. If $R_{i'}^{-} \not\leftrightarrow R_{j'}$ then again we are done, otherwise by continuing this process and decreasing $l(R_{i'})$ by one each time, we will eventually have $l(R_{i'}) < l(R_{j'})$, at which point $R_{i'} \nleftrightarrow R_{j'}$ by Proposition 4.1.7, Part 3. This completes the proof. □

Let us also recall the Littlewood–Richardson rule, which gives a combinatorial formula for the product of two Schur functions.

Theorem 4.1.24. [13, Section 5.1, Corollary 2 and Corollary 3] Let λ, μ, and ν be partitions and fix a tableau $U \in \mathrm{SSYT}_\mu$. Denote by $c^\lambda_{\mu,\nu}$ the number of tableaux in $\mathrm{SSYT}_{\lambda/\nu}$ whose rectification is U. Then the product of Schur functions s_μ and s_ν is given by

$$s_\mu s_\nu = \sum_\nu c^\lambda_{\mu,\nu} s_\nu. \tag{4.44}$$

With all of these preliminaries in place, proving Theorem 3.1.11 essentially amounts to verifying that our generalized cocharge formula (3.7) satisfies our deletion-contraction relation (4.27).

Proof of Theorem 3.1.11. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ and let $R_{i'}$ be the row corresponding to the vertex v_1. We use induction on n. If $n = 1$, then both sides of (3.7) are s_{α_1}, so we may assume that $n \geq 2$.

We first consider the case where the vertex v_1 has no neighbour. Let $\tilde{\alpha} = (0, \alpha_2, \ldots, \alpha_n)$ and note that we can associate a tableau $\tilde{T} \in \mathrm{SSYT}_{\lambda/(\alpha_1)}(\tilde{\alpha})$ with a tableau $T \in \mathrm{SSYT}_\lambda(\alpha)$ by placing α_1 1's underneath $\tilde{T}$. Because $\mathrm{cocharge}_{\Pi(\boldsymbol{\lambda})}$ is defined by restricting to the appropriate entries and rectifying, we have $\mathrm{cocharge}_{\Pi(\boldsymbol{\lambda})}(T) = \mathrm{cocharge}_{\Pi(\boldsymbol{\lambda})}(T|_{2,n})$. Now using our induction hypothesis and the Littlewood–Richardson rule, we have

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q) = G_{(R_{i'})}(\boldsymbol{x}; q) G_{(R_1, \ldots, R_{i'-1}, R_{i'+1}, \ldots, R_n)}(\boldsymbol{x}; q) \tag{4.45}$$

$$= \sum_{U \in \mathrm{SSYT}(\tilde{\alpha})} q^{\mathrm{cocharge}_{\Pi(\boldsymbol{\lambda})}(U)} s_{\mathrm{shape}(U)} s_{\alpha_1} \tag{4.46}$$

$$= \sum_{U \in \mathrm{SSYT}(\tilde{\alpha})} \sum_{\substack{T \in \mathrm{SSYT}(\alpha) \\ T|_{2,n} = U}} q^{\mathrm{cocharge}_{\Pi(\boldsymbol{\lambda})}(T)} s_{\mathrm{shape}(T)} \tag{4.47}$$

$$= \sum_{T \in \mathrm{SSYT}(\alpha)} q^{\mathrm{cocharge}_{\Pi(\boldsymbol{\lambda})}(T)} s_{\mathrm{shape(T)}}. \tag{4.48}$$

Now suppose that the vertex v_1 has a neighbour v_j corresponding to some row $R_{j'}$. Recall that by Proposition 4.1.21, Part 2, this neighbour is unique, and also the vertex v_j has at most one neighbour v_k for which $j < k$. We also use induction on $M(\boldsymbol{\lambda})$. If $M(\boldsymbol{\lambda}) = 0$, then the vertex v_1 has no neighbour, so we may assume that $M(\boldsymbol{\lambda}) \geq 1$. By Lemma 4.1.23, we may replace $\boldsymbol{\lambda}$ by a similar horizontal-strip as necessary to assume that $i' = j' + 1$, $l(R_{i'}) < l(R_{j'})$, and $R_{i'} \not\leftrightarrow R_{j'}$. Now defining

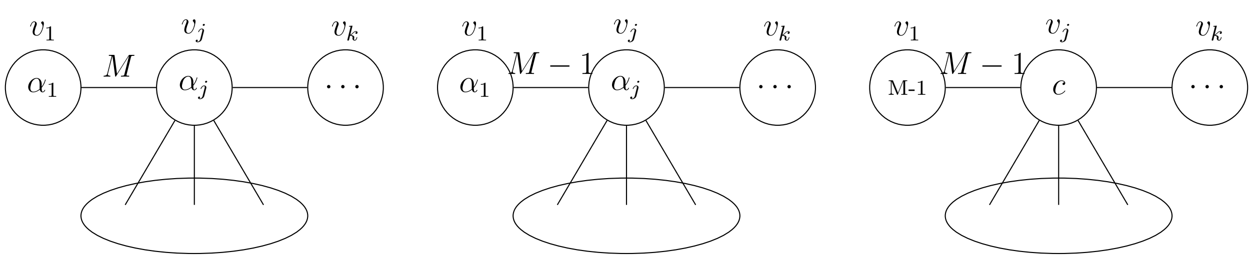
$$\boldsymbol{\lambda}' = (R_1, \ldots, R_{i'+1}, R_{i'}, \ldots, R_n) \text{ and} \tag{4.49}$$
$$\boldsymbol{\lambda}'' = (R_1, \ldots, R_{i'} \cup R_{i'+1}, R_{i'} \cap R_{i'+1}, \ldots, R_n), \tag{4.50}$$

we have by Corollary 4.1.17 that

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q) = qG_{\boldsymbol{\lambda}'}(\boldsymbol{x}; q) - (q-1)G_{\boldsymbol{\lambda}''}(\boldsymbol{x}; q). \tag{4.51}$$

By Proposition 4.1.18, the graphs $\Pi = \Pi(\boldsymbol{\lambda})$, $\Pi' = \Pi(\boldsymbol{\lambda}')$, and $\Pi'' = \Pi(\boldsymbol{\lambda}'')$ are as in Figure 4.7, where $M = M_{1,j}$ and $c = \alpha_1 + \alpha_j - (M-1)$.

Figure 4.7: The graphs Π, Π', and Π'' in the proof of Theorem 3.1.11



Let $\beta = (M-1, \alpha_2, \ldots, \alpha_{j-1}, c, \alpha_{j+1}, \ldots, \alpha_n)$ and let $t = \alpha_1 - (M-1)$. We define a map

$$\varphi : \mathrm{SSYT}(\beta) \to \{U \in \mathrm{SSYT}(\alpha) : \ \mathrm{cocharge}_{1,j}(U) \leq M-1\} \tag{4.52}$$

as follows. For $T \in \mathrm{SSYT}(\beta)$, let $\varphi(T) = U$ be the tableau of the same shape given by

$$U_{i',j'} = \begin{cases} 1 & \text{if } i' = 1 \text{ and } j' \leq t, \\ U_{1,j'-t} & \text{if } i' = 1 \text{ and } t < j' \leq j_1, \\ U_{i',j'} & \text{otherwise,} \end{cases} \tag{4.53}$$

where j_1 is the column of the rightmost j in T. Figure 4.8 gives two examples of this map. Informally, we change the two bolded 3's on the bottom row into 1's.

We first show that $\varphi(T)$ is in the codomain. Let n_j denote the number of j's in T that are not on the bottom row. Because the columns of T are strictly increasing, n_j is at most the number of entries in T less than j, so by Proposition 4.1.21, Part 4, we have

$$n_j \leq (M-1) + \alpha_2 + \cdots + \alpha_{j-1} \leq \alpha_j, \tag{4.54}$$

Figure 4.8: Two tableaux T_1 and T_2 and their images under the map φ

$$T_1 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline 3 & 4 & 5 & 5 & 5 \\ \hline 2 & 3 & 3 & 4 & 4 & 5 \\ \hline 1 & 1 & 1 & 2 & 3 & 3 & 3 & 3 & \mathbf{3} & \mathbf{3} & 5 \\ \hline \end{array} \qquad \varphi(T_1) = U_1 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline 3 & 4 & 5 & 5 & 5 \\ \hline 2 & 3 & 3 & 4 & 4 & 5 \\ \hline \mathbf{1} & \mathbf{1} & 1 & 1 & 1 & 2 & 3 & 3 & 3 & 3 & 5 \\ \hline \end{array}$$

$$T_2 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline 3 & 5 & 5 \\ \hline 2 & 3 & 3 & 4 & 4 & 5 & 5 & 5 \\ \hline 1 & 1 & 1 & 2 & 3 & 3 & 3 & 3 & \mathbf{3} & \mathbf{3} & 5 \\ \hline \end{array} \qquad \varphi(T_2) = U_2 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline 3 & 5 & 5 \\ \hline 2 & 3 & 3 & 4 & 4 & 5 & 5 & 5 \\ \hline \mathbf{1} & \mathbf{1} & 1 & 1 & 1 & 2 & 3 & 3 & 3 & 3 & 5 \\ \hline \end{array}$$

so T has at least $c - \alpha_j = t$ j's on the bottom row and indeed t j's have been replaced by t 1's and $\varphi(T) \in \mathrm{SSYT}(\alpha)$. Furthermore, by construction, $f(U) = f(T) + t \geq t$, so

$$\mathrm{cocharge}_{1,j}(U) = \mathrm{cocharge}_{1,j}(T) \leq \alpha_1 - t = M - 1. \tag{4.55}$$

Now we show that φ is a bijection. Given $U \in \mathrm{SSYT}(\beta)$ such that $\mathrm{cocharge}_{1,j}(U) \leq M - 1$, then $f(U) \geq t$ and we can define $T = \varphi^{-1}(U) \in \mathrm{SSYT}(\alpha)$ by

$$T_{i',j'} = \begin{cases} U_{1,j'+t} & \text{if } i' = 1 \text{ and } j' \leq j_1 - t, \\ j & \text{if } i' = 1 \text{ and } j_1 - t < j' \leq j_1, \\ U_{i',j'} & \text{otherwise.} \end{cases} \tag{4.56}$$

We now claim that $\mathrm{cocharge}_{\Pi''}(T) = \mathrm{cocharge}_{\Pi'}(U)$. We have already shown that $f(U) = f(T)+t$ and therefore $\mathrm{cocharge}_{1,j}(T) = \mathrm{cocharge}_{1,j}(U)$. For $2 \leq i \leq j-1$, the tableau $T|_{i,j}$ is simply $U|_{i,j}$ with t j's appended to the right of the first row, and therefore $f(T|_{i,j}) = f(U|_{i,j})$ and $\mathrm{cocharge}_{i,j}(T) = \mathrm{cocharge}_{i,j}(U)$. It remains to consider $\mathrm{cocharge}_{j,k}$. In fact, it could happen that $\mathrm{cocharge}_{j,k}(T) \neq \mathrm{cocharge}_{j,k}(U)$. However, we claim that this is only possible when both integers are at least $M_{j,k}$, so that we always have

$$\min\{M_{j,k}, \mathrm{cocharge}_{j,k}(T)\} = \min\{M_{j,k}, \mathrm{cocharge}_{j,k}(U)\}. \tag{4.57}$$

We restrict T and U to entries x with $j \leq x \leq k$ and we consider how the entries move when we rectify these tableaux. By Theorem 2.5.1, the rectification does not depend on the order of choices of inside corners, so let us begin by performing jeu de taquin slides into all inside corners that are not on the first row to produce tableaux T' and U'. Because T and U differ only in their first row, that is $T_{i',j'} = U_{i',j'}$ for all $i' \geq 2$, we also have $T'_{i',j'} = U'_{i',j'}$ for all $i' \geq 2$. Also note that the n_j j's of T' and U' that are not on the first row must now be on the second row, or in other words $T'_{2,j'} = U'_{2,j'} = j$ for $1 \leq j' \leq n_j$.

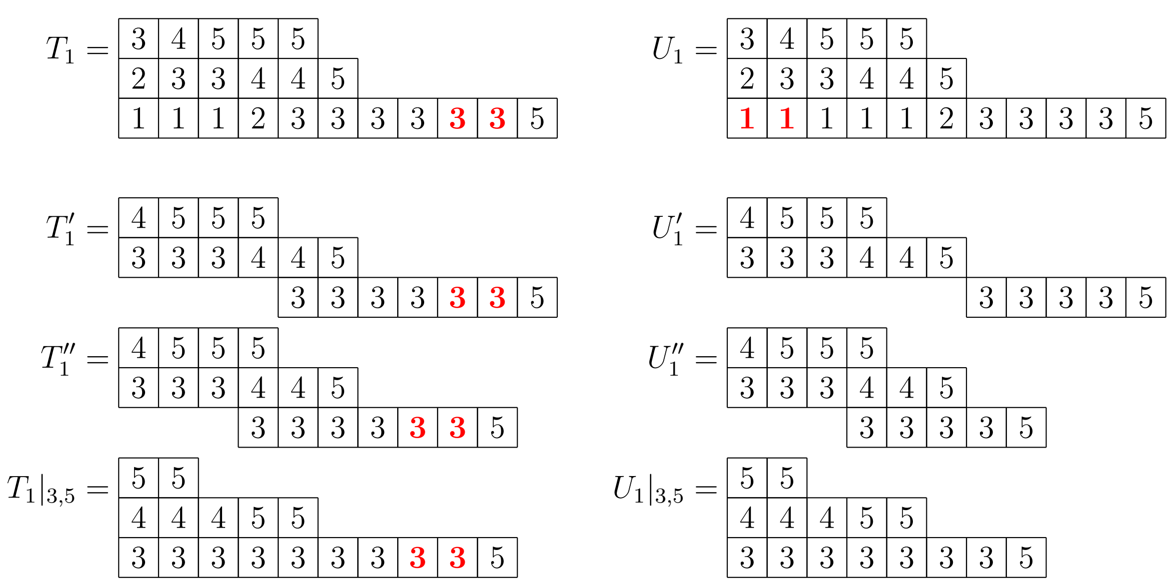
We now perform jeu de taquin slides into inside corners on the first row on T' until we obtain a skew tableau T'' of shape $\sigma/(n_j)$ for some σ and we similarly obtain U''. Let t_0 be the number of jeu de taquin slides performed on T' to produce T'', so that $t + t_0$ is the number of slides performed on U' to produce U''. We have two cases to consider.

If $U'_{2,t'} > U'_{1,t'+t+t_0}$ for all $n_j < t' \leq \sigma_2$, then no entry of U' moves down in this step, and because $U'_{2,t'} = T'_{2,t'}$ and $U'_{1,t'+t+t_0} = T'_{1,t'+t_0}$, no entry of T' moves down either. Now because $T''_{2,n_j} = U''_{2,n_j} = j$, when we perform the final n_j jeu de taquin slides, the entries of the first rows of T'' and U'' do not move, and because $T''_{i',j'} = U''_{i',j'}$ for all $i' \geq 2$, we now have

$$(T|_{j,k})_{i',j'} = \begin{cases} j & \text{if } i' = 1 \text{ and } j' \leq t, \\ (U|_{j,k})_{1,j'-t} & \text{if } i' = 1 \text{ and } j' > t, \\ (U|_{j,k})_{i',j'} & \text{otherwise.} \end{cases} \tag{4.58}$$

In particular, we have $f(T|_{j,k}) = f(U|_{j,k}) + t$ and $\text{cocharge}_{j,k}(T) = \text{cocharge}_{j,k}(U)$. Informally, the skew tableaux T' and U' look very similar at every stage in their rectifications. As an illustration, for $j = 3$ and $k = 5$, these stages in the rectification of the tableaux T'_1 and U'_1 are shown in Figure 4.9.

Figure 4.9: Tableaux arising in the rectifications of T_1 and U_1



Now suppose that $U'_{2,t'} \leq U'_{1,t'+t+t_0}$ for some t' with $n_j < t' \leq \sigma_2$. Then when we perform jeu de taquin slides to produce U'', an entry $x > j$ of the second row must move down on some slide. On subsequent slides, because the second row of U' was weakly increasing, entries of the second row will continue to move down, and we will have $j < U''_{2,t'} \leq U''_{1,t'+1}$ for some $t' > n_j$. Meanwhile, we must have $j < T''_{2,t'} \leq T''_{2,t'+t}$ for some $t' > n_j$ because if we delete t j's on the first row of T'' and then perform t jeu de taquin slides, we would obtain U'' so some entry of the second row must move down.

When we perform the final n_j jeu de taquin slides, the entries of the first rows of T'' and U'' do not move, the n_j j's in the second rows of T'' and U'' each move one cell down, and the remaining entries in the second rows move at most n_j cells to the left. Therefore, we now have $f(U|_{j,k}) \leq n_j$ and $f(T|_{j,k}) \leq n_j + t$. By Proposition 4.1.21, Part 4, we have

$$n_j \leq (M-1) + \alpha_2 + \cdots + \alpha_{j-1} \leq \alpha_j - M_{j,k}, \tag{4.59}$$

and therefore $\text{cocharge}_{j,k}(T) \geq M_{j,k}$ and $\text{cocharge}_{j,k}(U) \geq M_{j,k}$.

As an illustration, for $j = 3$ and $k = 5$, these stages in the rectification of T_2 and U_2 are shown in Figure 4.10. When we rectify U'_2, a 5 moves down from the second row so the second rows of T''_2 and U''_2 will be different. However, when we rectify T''_2 and U''_2, these 5's move at most three cells to the left, so $f(T_2|_{3,5}) \leq 5$ and $f(U_2|_{3,5}) \leq 3$, which means that $\text{cocharge}_{3,5}(T), \text{cocharge}_{3,5}(U) \geq 4$. Informally, the only way that $\text{cocharge}_{j,k}(T)$ could differ from $\text{cocharge}_{j,k}(U)$ is if a cell in the second row of U moves down prematurely, but if this happens, then $f(U|_{j,k})$ will be small enough to make $\text{cocharge}_{j,k}(U) \geq M_{j,k}$.

Figure 4.10: Tableaux arising in the rectifications of T_2 and U_2

$T_2 =$

3	5	5								
2	3	3	4	4	5	5	5			
1	1	1	2	3	3	3	3	**3**	**3**	5

$U_2 =$

3	5	5								
2	3	3	4	4	5	5	5			
1	**1**	1	1	1	2	3	3	3	3	5

$T'_2 =$

5	5									
3	3	3	4	4	5	5	5			
				3	3	3	3	3	3	5

$U'_2 =$

5	5									
3	3	3	4	4	5	5	5			
						3	3	3	3	5

$T''_2 =$

5	5								
3	3	3	4	4	5	5	**5**		
			3	3	3	3	3	3	**5**

$U''_2 =$

5	5							
3	3	3	4	4	5	**5**		
			3	3	3	3	**5**	5

$T_2|_{3,5} =$

5	5								
4	4	5	5	**5**					
3	3	3	3	3	3	3	3	3	**5**

$U_2|_{3,5} =$

5	5							
4	4	5	**5**					
3	3	3	3	3	3	3	**5**	5

In summary, the map φ is a bijection and it satisfies $\text{cocharge}_{\Pi''}(T) = \text{cocharge}_{\Pi'}(\varphi(T))$. Also note that by definition we have

$$\text{cocharge}_{\Pi}(T) = \begin{cases} \text{cocharge}_{\Pi'}(T) & \text{if } \text{cocharge}_{1,j}(T) \leq M-1, \\ \text{cocharge}_{\Pi'}(T)+1 & \text{if } \text{cocharge}_{1,j}(T) \geq M. \end{cases} \tag{4.60}$$

Therefore, using our induction hypothesis, we have

$$\begin{aligned} G_{\boldsymbol{\lambda}}(\boldsymbol{x};q) &= qG_{\boldsymbol{\lambda}'}(\boldsymbol{x};q) - (q-1)G_{\boldsymbol{\lambda}''}(\boldsymbol{x};q) \qquad (4.61)\\ &= q \sum_{T\in\text{SSYT}(\alpha)} q^{\text{cocharge}_{\Pi'}(T)} s_{\text{shape}(T)} - (q-1) \sum_{T\in\text{SSYT}(\beta)} q^{\text{cocharge}_{\Pi''}(T)} s_{\text{shape}(T)} \\ &= q \sum_{\substack{T\in\text{SSYT}(\alpha)\\ \text{cocharge}_{1,j}(T)\geq M}} q^{\text{cocharge}_{\Pi'}(T)} s_{\text{shape}(T)} + q \sum_{\substack{T\in\text{SSYT}(\alpha)\\ \text{cocharge}_{1,j}(T)\leq M-1}} q^{\text{cocharge}_{\Pi'}(T)} s_{\text{shape}(T)} \\ &\quad - (q-1) \sum_{\substack{U\in\text{SSYT}(\alpha)\\ \text{cocharge}_{1,j}(U)\leq M-1}} q^{\text{cocharge}_{\Pi'}(U)} s_{\text{shape}(U)} \\ &= \sum_{\substack{T\in\text{SSYT}(\alpha)\\ \text{cocharge}_{1,j}(T)\geq M}} q^{\text{cocharge}_{\Pi'}(T)+1} s_{\text{shape}(T)} + \sum_{\substack{T\in\text{SSYT}(\alpha)\\ \text{cocharge}_{1,j}(T)\leq M-1}} q^{\text{cocharge}_{\Pi'}(T)} s_{\text{shape}(T)} \\ &= \sum_{T\in\text{SSYT}(\alpha)} q^{\text{cocharge}_{\Pi}(T)} s_{\text{shape}(T)}. \end{aligned}$$

This completes the proof. □

We hope to be able to further extend cocharge to more general weighted graphs $\Pi(\boldsymbol{\lambda})$. If we can guess a suitable generalization of cocharge, the proof would again be a matter of verifying that our combinatorial formula satisfies our deletion-contraction relation (4.27). More specifically, if we can define an injection of tableaux

$$\varphi_t : \text{SSYT}(\beta) \to \text{SSYT}(\alpha) \tag{4.62}$$

that changes t i's to j's as in the proof of Theorem 3.1.11, then by setting

$$\text{cocharge}_{i,j}(T) = \max\{t : T \in \text{image}(\varphi_t)\}, \tag{4.63}$$

it would remain to check that $\text{cocharge}_{\Pi''}(\varphi_t(T)) = \text{cocharge}_{\Pi'}(T)$.

Some challenges can arise if the weighted graph $\Pi(\boldsymbol{\lambda})$ has triangles. Our argument in Lemma 4.1.23 requires some revision. There may be multiple vertices adjacent to v_1 and we will be able to apply the deletion-contraction relation to some edge incident to v_1, but it is not easy to identify which edge. A second complication is that the edge weights of the contracted graph $\Pi(\boldsymbol{\lambda}'')$ change according to (4.29) and (4.30), which can be difficult to get a handle on. Nevertheless, we are hopeful that the techniques we presented will be applicable toward a general solution to Problem 2.3.6.

Chapter 5

A chromatic connection

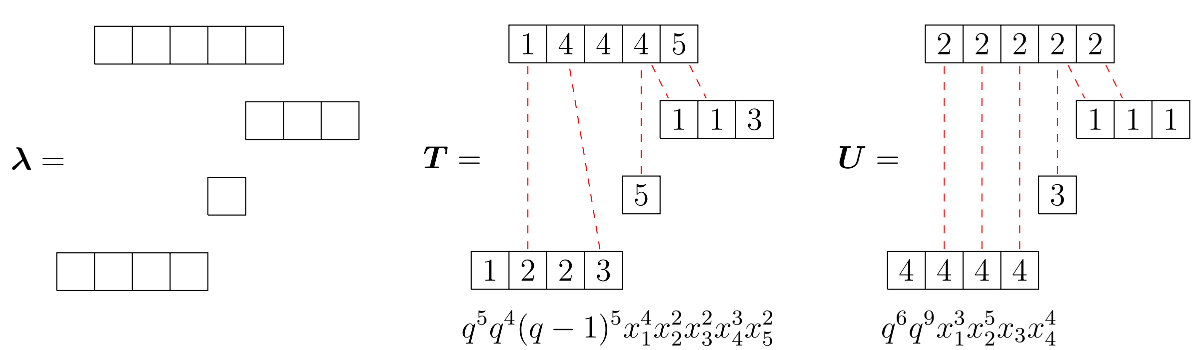
In this chapter, we show that for a horizontal-strip $\boldsymbol{\lambda}$, the LLT polynomial $G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q)$ and the extended chromatic symmetric function $X_{\Pi(\boldsymbol{\lambda})}(\boldsymbol{x})$ of the weighted graph $\Pi(\boldsymbol{\lambda})$ share many similar properties. We begin by proving the following plethystic formula, from which Theorem 3.1.16 follows.

Theorem 5.1.1. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a horizontal-strip with N cells. We will say that $\boldsymbol{T} \in \mathrm{SSYT}_{\boldsymbol{\lambda}}$ is *proper* if attacking cells have distinct entries, and we will denote by $m_i(\boldsymbol{T})$ the number of distinct entries in row R_i of $\boldsymbol{T}$. Then we have

$$\frac{G_{\boldsymbol{\lambda}}([\boldsymbol{x}(q-1)]; q)}{(q-1)^n} = \sum_{\substack{\boldsymbol{T} \in \mathrm{SSYT}_{\boldsymbol{\lambda}} \\ \boldsymbol{T} \text{ proper}}} q^{\mathrm{inv}(\boldsymbol{T})} q^{N - \sum_i m_i(\boldsymbol{T})} (q-1)^{\sum_i (m_i(\boldsymbol{T}) - 1)} \boldsymbol{x}^{\boldsymbol{T}}. \tag{5.1}$$

Example 5.1.2. The horizontal-strip $\boldsymbol{\lambda} = (4/0, 5/4, 8/5, 6/1)$, two proper sequences of SSYTs $\boldsymbol{T}, \boldsymbol{U} \in \mathrm{SSYT}_{\boldsymbol{\lambda}}$, and the corresponding terms of the sum (5.1) are given in Figure 5.1. We have $m_1(\boldsymbol{T}) = 3$, $m_2(\boldsymbol{T}) = 1$, $m_3(\boldsymbol{T}) = 2$, $m_4(\boldsymbol{T}) = 3$, and every $m_i(\boldsymbol{U}) = 1$.

Figure 5.1: The horizontal-strip $\boldsymbol{\lambda} = (4/0, 5/4, 8/5, 6/1)$ and two proper $\boldsymbol{T}, \boldsymbol{U} \in \mathrm{SSYT}_{\boldsymbol{\lambda}}$



Remark 5.1.3. We can prove Theorem 5.1.1 by using standardization to write the LLT polynomial $G_{\boldsymbol{\lambda}}(\boldsymbol{x};q)$ in terms of Gessel's fundamental quasisymmetric functions $F_\alpha(\boldsymbol{x})$, applying a result of Haglund–Haiman–Loehr–Remmel–Ulyanov [18, Corollary 2.4.3] that interprets plethysms of the form $F_\alpha[\boldsymbol{x}+\boldsymbol{y}]$ in terms of "superized tableaux", and applying two sign-reversing involutions whose fixed points are the proper $\boldsymbol{T} \in \mathrm{SSYT}_{\boldsymbol{\lambda}}$. These ideas are discussed in [17, Section 4]. In order to keep our arguments elementary, we will instead prove Theorem 5.1.1 by verifying that the sum (5.1) satisfies our deletion-contraction relation (4.27). Our argument will be very similar to our proof of Lemma 4.1.12.

Proof. We will use induction on $k = N - n = \sum_i(|R_i| - 1)$. If $k = 0$, then every $|R_i| = 1$, every $m_i(\boldsymbol{T}) = 1$, the graph $\Pi(\boldsymbol{\lambda})$ is simply $\Gamma(\boldsymbol{\lambda})$ with all vertex weights and (nonzero) edge weights equal to one, a proper $\boldsymbol{T} \in \mathrm{SSYT}_{\boldsymbol{\lambda}}$ is precisely a proper colouring κ of $\Gamma(\boldsymbol{\lambda})$, and an inversion of $\boldsymbol{T}$ is precisely a descent of κ, so the right hand side of (5.1) is the chromatic quasisymmetric function $X_{\Gamma(\boldsymbol{\lambda})}(\boldsymbol{x};q)$ and the result follows from Theorem 2.4.7.

Now suppose that $k \geq 1$, meaning that there is some row R_i with $|R_i| \geq 2$. Decompose this row as $R_i = R \sqcup R'$, where R and R' are nonempty rows with $l(R') = r(R) + 1$, and define the horizontal-strips with $(n+1)$ rows

$$\boldsymbol{\lambda}^{\nearrow} = (R_1, \ldots, R_{i-1}, R, R', R_{i+1}, \ldots, R_n) \text{ and} \tag{5.2}$$

$$\boldsymbol{\lambda}^{\searrow} = (R_1, \ldots, R_{i-1}, R', R, R_{i+1}, \ldots, R_n). \tag{5.3}$$

For a proper sequence of tableaux $\boldsymbol{T} \in \mathrm{SSYT}_{\boldsymbol{\lambda}}$, let x and y denote the entries in the cells of R_i of content $r(R)$ and $l(R')$ respectively, partition the proper sequences of tableaux

$$\mathrm{SSYT}_{\boldsymbol{\lambda},\text{proper}} = \mathrm{SSYT}^{<}_{\boldsymbol{\lambda},\text{proper}} \sqcup \mathrm{SSYT}^{=}_{\boldsymbol{\lambda},\text{proper}} \sqcup \mathrm{SSYT}^{>}_{\boldsymbol{\lambda},\text{proper}} \tag{5.4}$$

according to whether $x < y$, $x = y$, or $x > y$, and partition $\mathrm{SSYT}_{\boldsymbol{\lambda}^{\nearrow},\text{proper}}$ and $\mathrm{SSYT}_{\boldsymbol{\lambda}^{\searrow},\text{proper}}$ similarly. Note that $\mathrm{SSYT}^{>}_{\boldsymbol{\lambda},\text{proper}}$ is empty because we must have $x \leq y$ and $\mathrm{SSYT}^{=}_{\boldsymbol{\lambda}^{\searrow},\text{proper}}$ is empty because attacking cells must have distinct entries. Let us also define

$$\mathrm{wt}_q(\boldsymbol{T}) = q^{\mathrm{inv}(\boldsymbol{T})} q^{N - \sum_i m_i(\boldsymbol{T})} (q-1)^{\sum_i (m_i(\boldsymbol{T})-1)} \boldsymbol{x}^{\boldsymbol{T}}. \tag{5.5}$$

We now claim that

$$\sum_{\boldsymbol{T} \in \mathrm{SSYT}^{<}_{\boldsymbol{\lambda},\text{proper}}} \mathrm{wt}_q(\boldsymbol{T}) = q \sum_{\boldsymbol{T}' \in \mathrm{SSYT}^{<}_{\boldsymbol{\lambda}^{\nearrow},\text{proper}}} \mathrm{wt}_q(\boldsymbol{T}') - \sum_{\boldsymbol{T}'' \in \mathrm{SSYT}^{<}_{\boldsymbol{\lambda}^{\searrow},\text{proper}}} \mathrm{wt}_q(\boldsymbol{T}''), \tag{5.6}$$

$$\sum_{\boldsymbol{T} \in \mathrm{SSYT}^{=}_{\boldsymbol{\lambda},\text{proper}}} \mathrm{wt}_q(\boldsymbol{T}) = q \sum_{\boldsymbol{T}' \in \mathrm{SSYT}^{=}_{\boldsymbol{\lambda}^{\nearrow},\text{proper}}} \mathrm{wt}_q(\boldsymbol{T}') - \sum_{\boldsymbol{T}'' \in \mathrm{SSYT}^{=}_{\boldsymbol{\lambda}^{\searrow},\text{proper}}} \mathrm{wt}_q(\boldsymbol{T}''), \text{ and} \tag{5.7}$$

$$\sum_{\boldsymbol{T} \in \mathrm{SSYT}^{>}_{\boldsymbol{\lambda},\text{proper}}} \mathrm{wt}_q(\boldsymbol{T}) = q \sum_{\boldsymbol{T}' \in \mathrm{SSYT}^{>}_{\boldsymbol{\lambda}^{\nearrow},\text{proper}}} \mathrm{wt}_q(\boldsymbol{T}') - \sum_{\boldsymbol{T}'' \in \mathrm{SSYT}^{>}_{\boldsymbol{\lambda}^{\searrow},\text{proper}}} \mathrm{wt}_q(\boldsymbol{T}''). \tag{5.8}$$

To prove (5.6), we associate a proper sequence of tableaux $\boldsymbol{T} \in \mathrm{SSYT}^{<}_{\boldsymbol{\lambda},\mathrm{proper}}$ with $\boldsymbol{T}' \in \mathrm{SSYT}^{<}_{\boldsymbol{\lambda}^{\nearrow},\mathrm{proper}}$ and $\boldsymbol{T}'' \in \mathrm{SSYT}^{<}_{\boldsymbol{\lambda}^{\searrow},\mathrm{proper}}$ by fixing the entries in rows R_j for $1 \leq j \leq n$, $j \neq i$, and otherwise preserving the entries in cells of each content. We now have

$$\mathrm{wt}_q(\boldsymbol{T}) = (q-1)\mathrm{wt}_q(\boldsymbol{T}') \text{ and } \mathrm{wt}_q(\boldsymbol{T}') = \mathrm{wt}_q(\boldsymbol{T}'') \tag{5.9}$$

because there are no inversions between cells in rows R and R', any other inversions must be preserved because the entries in cells of each content are preserved, $\boldsymbol{\lambda}$ has one fewer row than $\boldsymbol{\lambda}^{\nearrow}$ and $\boldsymbol{\lambda}^{\searrow}$, and we have

$$\sum_{i=1}^{n} m_i(\boldsymbol{T}) = \sum_{i=1}^{n+1} m_i(\boldsymbol{T}') = \sum_{i=1}^{n+1} m_i(\boldsymbol{T}''). \tag{5.10}$$

This proves (5.6). Similarly, to prove (5.7), noting that $\mathrm{SSYT}^{=}_{\boldsymbol{\lambda}^{\searrow},\mathrm{proper}}$ is empty, we now associate $\boldsymbol{T} \in \mathrm{SSYT}^{=}_{\boldsymbol{\lambda},\mathrm{proper}}$ with $\boldsymbol{T}' \in \mathrm{SSYT}^{=}_{\boldsymbol{\lambda}^{\nearrow},\mathrm{proper}}$ as before and note that $\sum_i m_i(\boldsymbol{T}) = \sum_i m_i(\boldsymbol{T}') + 1$ because the entries x in rows R and R' of $\boldsymbol{\lambda}^{\nearrow}$ contribute twice to $\sum_i m_i(\boldsymbol{T})$ but only once to $\sum_i(\boldsymbol{T}')$. Finally, to prove (5.8), noting that $\mathrm{SSYT}^{>}_{\boldsymbol{\lambda},\mathrm{proper}}$ is empty, we now associate $\boldsymbol{T}' \in \mathrm{SSYT}^{>}_{\boldsymbol{\lambda}^{\nearrow},\mathrm{proper}}$ with $\boldsymbol{T}'' \in \mathrm{SSYT}^{>}_{\boldsymbol{\lambda}^{\searrow},\mathrm{proper}}$ as before and note that $\mathrm{inv}(\boldsymbol{T}'') = \mathrm{inv}(\boldsymbol{T}') + 1$ because $\boldsymbol{T}''$ has the additional inversion between the cells with entries x and y. Now using our induction hypothesis, and rearranging (4.27), we have

$$\begin{aligned}
\frac{G_{\boldsymbol{\lambda}}([\boldsymbol{x}(q-1)];q)}{(q-1)^n} &= \frac{\frac{q}{q-1}G_{\boldsymbol{\lambda}^{\nearrow}}([\boldsymbol{x}(q-1)];q) - \frac{1}{q-1}G_{\boldsymbol{\lambda}^{\searrow}}([\boldsymbol{x}(q-1)];q)}{(q-1)^n} && (5.11)\\
&= q\frac{G_{\boldsymbol{\lambda}^{\nearrow}}([\boldsymbol{x}(q-1)];q)}{(q-1)^{n+1}} - \frac{G_{\boldsymbol{\lambda}^{\searrow}}([\boldsymbol{x}(q-1)];q)}{(q-1)^{n+1}} && (5.12)\\
&= q \sum_{\boldsymbol{T}' \in \mathrm{SSYT}_{\boldsymbol{\lambda}^{\nearrow},\mathrm{proper}}} \mathrm{wt}_q(\boldsymbol{T}') - \sum_{T'' \in \mathrm{SSYT}_{\boldsymbol{\lambda}^{\searrow},\mathrm{proper}}} \mathrm{wt}_q(\boldsymbol{T}'') && (5.13)\\
&= \sum_{\boldsymbol{T} \in \mathrm{SSYT}_{\boldsymbol{\lambda},\mathrm{proper}}} \mathrm{wt}_q(\boldsymbol{T}). && (5.14)
\end{aligned}$$

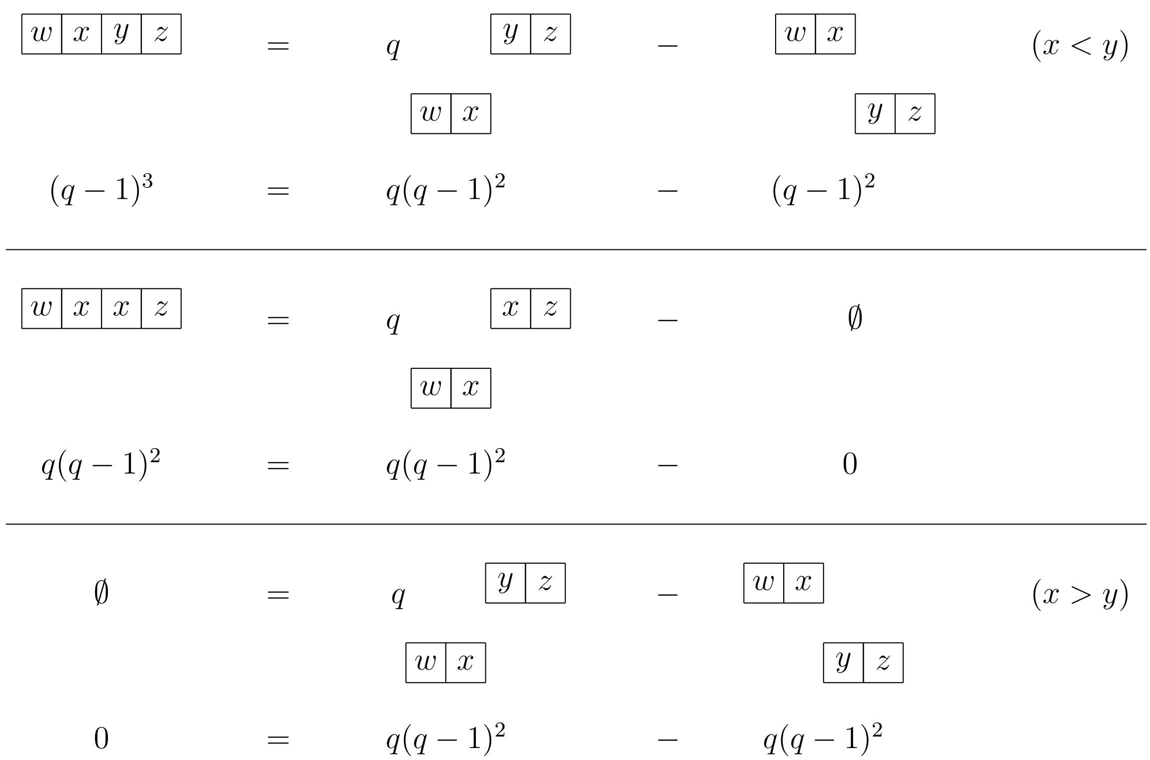
This completes the proof. □

Example 5.1.4. Figure 5.2 illustrates the idea of the proofs of (5.6), (5.7), and (5.8).

Now Theorem 3.1.16 is a straightforward consequence of Theorem 5.1.1.

Proof of Theorem 3.1.16. Setting $q = 1$ in (5.1), the only nonzero terms are where $m_i(\boldsymbol{T}) = 1$ for every i, meaning every cell of R_i is filled by the same integer a_i, which correspond to proper colourings κ of the weighted graph $\Pi(\boldsymbol{\lambda})$ by setting $\kappa(R_i) = a_i$, and we have $(\mathrm{wt}_q(\boldsymbol{T}))|_{q=1} = \boldsymbol{x}^{\boldsymbol{T}} = \boldsymbol{x}^{\kappa,w}$. As an illustration, in Example 5.1.2, the sequence of SSYT $\boldsymbol{U}$ contributes a nonzero summand to (5.1). □

Figure 5.2: An example of the calculations in the proof of Theorem 5.1.1.



We see that by Theorem 3.1.16, if two horizontal-strip LLT polynomials $G_{\boldsymbol{\lambda}}(\boldsymbol{x};q)$ and $G_{\boldsymbol{\mu}}(\boldsymbol{x};q)$ are equal, then the extended chromatic symmetric functions of the corresponding weighted graphs $X_{\Pi(\boldsymbol{\lambda})}(\boldsymbol{x})$ and $X_{\Pi(\boldsymbol{\mu})}(\boldsymbol{x})$ are also equal. We cannot in general hope for the converse to hold because the LLT polynomial $G_{\boldsymbol{\lambda}}(\boldsymbol{x};q)$ depends on the edge weights of $\Pi(\boldsymbol{\lambda})$, for example, by Theorem 3.1.6, while the extended chromatic symmetric function $X_{\Pi(\boldsymbol{\lambda})}$ does not. However, we will show that in certain cases, equalities of horizontal-strip LLT polynomials are equivalent to equalities of extended chromatic symmetric functions that were found by Aliniaeifard, Wang, and van Willigenburg [4]. We first introduce some operations on compositions.

Definition 5.1.5. Let $\alpha = \alpha_1 \cdots \alpha_n$ and $\beta = \beta_1 \cdots \beta_m$ be compositions. The *reverse* of α is $\alpha^{\mathrm{rev}} = \alpha_n \cdots \alpha_1$. The *concatenation* and *near-concatenation* of α and β are

$$\alpha \cdot \beta = \alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_m \text{ and } \alpha \odot \beta = \alpha_1 \cdots \alpha_{n-1}(\alpha_n + \beta_1)\beta_2 \cdots \beta_m. \tag{5.15}$$

The *composition* of α and β is [7, Section 3]

$$\alpha \circ \beta = \beta^{\odot \alpha_1} \cdots \beta^{\odot \alpha_n}, \tag{5.16}$$

where $\beta^{\odot k}$ denotes the k-fold near-concatenation of β. Note that all of these operations are associative. We say that β is a *coarsening* of α, denoted $\alpha \prec \beta$, if β can be obtained from α by summing some collections of contiguous parts. Alternatively, we can think of a coarsening β of α as a composition of the form

$$\beta = \alpha_1 * \cdots * \alpha_n, \text{ where each } * \text{ is } \cdot \text{ or } \odot . \tag{5.17}$$

We also define the multiset

$$\mathcal{M}(\alpha) = \{\text{sort}(\beta) : \ \alpha \prec \beta\}, \tag{5.18}$$

where sort(β) denotes the partition determined by reordering the parts of β in weakly decreasing order.

There is a bijection between the set C^N of compositions α with sum N and subsets of $\{1, \ldots, N-1\}$ given by taking the partial sums of α, other than 0 and N. By this bijection, a coarsening corresponds to a subset so the partially ordered set $(C^N, \prec)$ is (anti-)isomorphic to the boolean lattice B_{N-1}. In particular, by Möbius inversion, if A is an abelian group and $f, g : C^N \to A$, then

$$f(\alpha) = \sum_{\alpha \prec \beta} g(\beta) \text{ if and only if } g(\alpha) = \sum_{\alpha \prec \beta} (-1)^{\ell(\alpha)-\ell(\beta)} f(\beta). \tag{5.19}$$

Example 5.1.6. Consider the compositions $\alpha = 21231$, $\beta = 23121$, $\delta = 12$, and $\gamma = 21$. We have that $\delta^{\text{rev}} = \gamma$,

$$\delta \circ \gamma = \gamma^{\odot 1} \cdot \gamma^{\odot 2} = \gamma \cdot (\gamma \odot \gamma) = 21 \cdot (21 \odot 21) = 21 \cdot 231 = 21231 = \alpha, \text{ and} \tag{5.20}$$
$$\gamma \circ \gamma = \gamma^{\odot 2} \cdot \gamma^{\odot 1} = (\gamma \odot \gamma) \cdot \gamma = (21 \odot 21) \cdot 21 = 231 \cdot 21 = 23121 = \beta. \tag{5.21}$$

Some coarsenings of α include 3231, obtained by summing the first two parts, and 54, obtained by summing the first three parts and the last two parts. We have that

$$\mathcal{M}(\alpha) = \{32211, 5211, 4221, 3321, 3321, 621, 621, 531, 531, 432, 432, 81, 72, 63, 54, 9\} = \mathcal{M}(\beta). \tag{5.22}$$

The compositions α and β correspond to the subsets $\{2, 3, 5, 8\}$ and $\{2, 5, 6, 8\}$ of $\{1, \ldots, 8\}$.

Billera, Thomas, and van Willigenburg characterized when $\mathcal{M}(\alpha) = \mathcal{M}(\beta)$ in terms of compositions of compositions.

Theorem 5.1.7. [7, Theorem 4.1] Let α and β be compositions. We have $\mathcal{M}(\alpha) = \mathcal{M}(\beta)$ if and only if there are factorizations

$$\alpha = \delta^{(1)} \circ \cdots \circ \delta^{(k)} \text{ and } \beta = \gamma^{(1)} \circ \cdots \circ \gamma^{(k)} \tag{5.23}$$

so that every $\gamma^{(i)}$ is either $\delta^{(i)}$ or $\delta^{(i)\text{rev}}$.

Example 5.1.8. We saw that the compositions α and β from Example 2.1.1 can be factorized as $\alpha = \delta \circ \gamma$ and $\beta = \delta^{\text{rev}} \circ \gamma$, so it follows from Theorem 5.1.7 that $\mathcal{M}(\alpha) = \mathcal{M}(\beta)$.

Their motivation was to classify equalities of certain skew Schur functions indexed by compositions. It will be worthwhile to study their proof because we will employ a similar argument when we investigate equalities of LLT polynomials.

Definition 5.1.9. Let α be a composition. The *ribbon Schur function* $r_\alpha(\boldsymbol{x})$ is the skew Schur function indexed by the skew shape whose i-th row has α_i cells and whose adjacent rows overlap in exactly one column.

Note that when α has a single part n, then $r_n(\boldsymbol{x}) = s_n(\boldsymbol{x}) = h_n(\boldsymbol{x})$, the n-th homogeneous symmetric function.

Theorem 5.1.10. [7, Equation 2.2, Proposition 2.1, Theorem 2.6] The ribbon Schur functions satisfy the relation

$$r_\alpha(\boldsymbol{x}) r_\beta(\boldsymbol{x}) = r_{\alpha\cdot\beta}(\boldsymbol{x}) + r_{\alpha\odot\beta}(\boldsymbol{x}). \tag{5.24}$$

By iterating (5.24), we have that

$$h_{\text{sort}(\alpha)}(\boldsymbol{x}) = r_{\alpha_1}(\boldsymbol{x}) \cdots r_{\alpha_\ell}(\boldsymbol{x}) = \sum_{\alpha \prec \beta} r_\beta(\boldsymbol{x}) \tag{5.25}$$

and therefore, by (5.19), we have that

$$r_\alpha(\boldsymbol{x}) = \sum_{\alpha \prec \beta} (-1)^{\ell(\alpha)-\ell(\beta)} h_{\text{sort}(\beta)}(\boldsymbol{x}) = \sum_{\rho \in \mathcal{M}(\alpha)} (-1)^{\ell(\alpha)-\ell(\rho)} h_\rho(\boldsymbol{x}). \tag{5.26}$$

In particular, because the $h_\rho(\boldsymbol{x})$ are linearly independent, we have that $r_\alpha(\boldsymbol{x}) = r_\beta(\boldsymbol{x})$ if and only if $\mathcal{M}(\alpha) = \mathcal{M}(\beta)$.

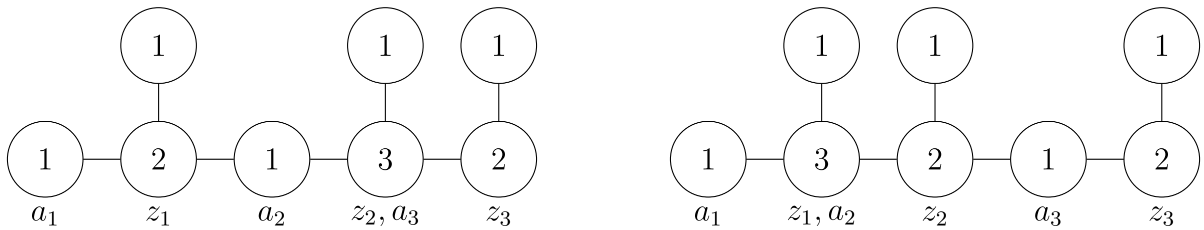
Aliniaeifard, Wang, and van Willigenburg defined similar operations for graphs in order to investigate equalities of extended chromatic symmetric functions.

Definition 5.1.11. [4, Definition 7.1] Let G and H be vertex-weighted graphs with distinguished (not necessarily distinct) vertices $a_G, z_G \in V(G)$ and $a_H, z_H \in V(H)$. The *concatenation* of G and H, denoted $G \cdot H$, is the disjoint union of G and H, with an extra edge (of weight one) joining z_G and a_H. The *near-concatenation* of G and H, denoted $G \odot H$, is the contraction of $G \cdot H$ by the edge (z_G, a_H). If $\alpha = \alpha_1 \cdots \alpha_n$ is a composition, then the *composition* of α and G is the graph

$$\alpha \circ G = G^{\odot \alpha_1} \cdots G^{\odot \alpha_n}, \tag{5.27}$$

where $G^{\odot k}$ denotes the k-fold near-concatenation of G.

Figure 5.3: The graphs $12 \circ P_{121}$ and $21 \circ P_{121}$



Note that it follows from the definition that

$$(\alpha \cdot \beta) \circ G = (\alpha \circ G) \cdot (\beta \circ G) \text{ and } (\alpha \odot \beta) \circ G = (\alpha \circ G) \odot (\beta \circ G). \tag{5.28}$$

Example 5.1.12. For a composition $\alpha = \alpha_1 \cdots \alpha_n$, denote by P_α the weighted path graph with vertices $\{v_1, \ldots, v_n\}$ of weights $w(v_i) = \alpha_i$ and with edges (v_i, v_{i+1}) of weight one for $1 \leq i \leq n-1$. Then $P_\alpha \cong \alpha \circ P_1$.

Example 5.1.13. For $G = P_{121}$ with $a = v_1$ and $z = v_2$, the graphs $12 \circ G$ and $21 \circ G$ are given in Figure 5.3.

We now state the following result of Aliniaeifard, Wang, and van Willigenburg on equalities of extended chromatic symmetric functions.

Theorem 5.1.14. [4, Theorem 7.3] Let G be a vertex-weighted graph with distinguished vertices a and z. If $\mathcal{M}(\alpha) = \mathcal{M}(\beta)$, then $X_{\alpha \circ G}(\boldsymbol{x}) = X_{\beta \circ G}(\boldsymbol{x})$. Moreover, if G is connected, then the converse holds.

We now consider an analogous statement for horizontal-strip LLT polynomials. For a horizontal-strip $\boldsymbol{\lambda}$, our distinguished vertices a and z of the weighted graph $\Pi(\boldsymbol{\lambda})$ will be those rows containing the first and last cells of $\boldsymbol{\lambda}$ in content reading order. We will say that an *admissible weighted graph with distinguished vertices* is a weighted graph of the form $\Pi \cong \Pi(\boldsymbol{\lambda})$ with distinguished vertices a and z that arise from $\boldsymbol{\lambda}$ as described above. By Theorem 3.1.8 there is a well-defined horizontal-strip LLT polynomial $G_\Pi(\boldsymbol{x}; q)$.

Lemma 5.1.15. Let $\Pi_1 = \Pi(\boldsymbol{\lambda})$ and $\Pi_2 = \Pi(\boldsymbol{\mu})$ be admissible weighted graphs with distinguished vertices. Then $\Pi_1 \cdot \Pi_2$ and $\Pi_1 \odot \Pi_2$ are admissible and we have

$$G_{\Pi_1}(\boldsymbol{x}; q) G_{\Pi_2}(\boldsymbol{x}; q) = \frac{1}{q} G_{\Pi_1 \cdot \Pi_2}(\boldsymbol{x}; q) + \frac{q-1}{q} G_{\Pi_1 \odot \Pi_2}(\boldsymbol{x}; q). \tag{5.29}$$

Proof. By cycling and translating, we may assume that $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ has a unique cell $u \in R_1$ of some maximum content $N-1$ and that $\boldsymbol{\mu} = (S_1, \ldots, S_m)$ has a unique cell $v \in S_m$

of minimum content 0. Define the horizontal-strips

$$\boldsymbol{\nu} = (N + S_1, \dots, N + S_{m-1}, N + S_m, R_1, R_2, \dots, R_n), \tag{5.30}$$

$$\boldsymbol{\nu}' = (N + S_1, \dots, N + S_{m-1}, R_1, N + S_m, R_2, \dots, R_n), \text{ and} \tag{5.31}$$

$$\boldsymbol{\nu}'' = (N + S_1, \dots, N + S_{m-1}, (N + S_m) \cup R_1, (N + S_m) \cap R_1, R_2, \dots, R_n). \tag{5.32}$$

Because $l(R_1) = N-1$ and $l(R_i) < N-1$ otherwise and $r(N+S_m) = N$ and $r(N+S_j) > N$ otherwise, we have $M(N + S_j, R_i) = 1$ if $(i, j) = (1, m)$ and 0 otherwise, and therefore $\Pi(\boldsymbol{\nu}) \cong \Pi_1 \cdot \Pi_2$. By Proposition 4.1.18, we see that $\Pi(\boldsymbol{\nu}')$ is obtained by deleting the weight of the edge $(N + S_m, R_1)$ by one, and is therefore the disjoint union $\Pi(\boldsymbol{\nu}') \cong \Pi_1 \sqcup \Pi_2$, while $\Pi(\boldsymbol{\nu}'')$ is obtained by contracting the edge $(N+S_m, R_1)$, and is therefore the weighted graph $\Pi_1 \odot \Pi_2$. Therefore, $\Pi_1 \cdot \Pi_2$ and $\Pi_1 \odot \Pi_2$ are admissible, and (5.29) follows by rearranging (4.27). □

We now state our horizontal-strip LLT polynomial analogue of Theorem 5.1.14.

Theorem 5.1.16. Let Π be an admissible weighted graph with distinguished vertices. If $\mathcal{M}(\alpha) = \mathcal{M}(\beta)$, then $G_{\alpha\circ\Pi}(\boldsymbol{x}; q) = G_{\beta\circ\Pi}(\boldsymbol{x}; q)$. Moreover, if Π is connected, then the converse holds.

Proof. By iterating (5.29), we have that

$$G_{\Pi\odot\alpha_1}(\boldsymbol{x}; q) \cdots G_{\Pi\odot\alpha_\ell}(\boldsymbol{x}; q) = \sum_{\alpha\prec\beta} \frac{(q-1)^{\ell(\alpha)-\ell(\beta)}}{q^{\ell(\alpha)-1}} G_{\beta\circ\Pi}(\boldsymbol{x}; q) \tag{5.33}$$

and rearranging, we have that

$$q^{\ell(\alpha)-1}(q-1)^{-\ell(\alpha)} \prod_{i=1}^{\ell} G_{\Pi\odot\alpha_i}(\boldsymbol{x}; q) = \sum_{\alpha\prec\beta} (q-1)^{-\ell(\beta)} G_{\beta\circ\Pi}(\boldsymbol{x}; q). \tag{5.34}$$

Therefore, by (5.19), we have that

$$(q-1)^{-\ell(\alpha)} G_{\alpha\circ\Pi}(\boldsymbol{x}; q) = \sum_{\alpha\prec\beta} q^{\ell(\beta)-1}(q-1)^{-\ell(\beta)} \prod_{i=1}^{\ell} G_{\Pi\odot\beta_i}(\boldsymbol{x}; q) \tag{5.35}$$

and rearranging again, we have that

$$G_{\alpha\circ\Pi}(\boldsymbol{x}; q) = \sum_{\alpha\prec\beta} q^{\ell(\beta)-1}(q-1)^{\ell(\alpha)-\ell(\beta)} \prod_{i=1}^{\ell} G_{\Pi\odot\beta_i}(\boldsymbol{x}; q) \tag{5.36}$$

$$= \sum_{\rho\in\mathcal{M}(\alpha)} q^{\ell(\rho)-1}(q-1)^{\ell(\alpha)-\ell(\rho)} \prod_{i=1}^{\ell} G_{\Pi\odot\rho_i}(\boldsymbol{x}; q). \tag{5.37}$$

In particular, if $\mathcal{M}(\alpha) = \mathcal{M}(\beta)$, then $G_{\alpha\circ\Pi}(\boldsymbol{x}; q) = G_{\beta\circ\Pi}(\boldsymbol{x}; q)$. Moreover, if Π is connected and $G_{\alpha\circ\Pi}(\boldsymbol{x}; q) = G_{\beta\circ\Pi}(\boldsymbol{x}; q)$, then by Theorem 3.1.16 we have $X_{\alpha\circ\Pi}(\boldsymbol{x}) = X_{\beta\circ\Pi}(\boldsymbol{x})$ and by Theorem 5.1.14 we have $\mathcal{M}(\alpha) = \mathcal{M}(\beta)$, so the converse holds. □

Corollary 5.1.17. Let α be a composition. The weighted path graph P_α is admissible and we have

$$G_{P_\alpha}(\boldsymbol{x};q) = \sum_{\rho\in\mathcal{M}(\alpha)} q^{\ell(\rho)-1}(q-1)^{\ell(\alpha)-\ell(\rho)}h_\rho(\boldsymbol{x}). \tag{5.38}$$

In particular, for compositions α and β, we have that $G_{P_\alpha}(\boldsymbol{x};q) = G_{P_\beta}(\boldsymbol{x};q)$ if and only if $\mathcal{M}(\alpha) = \mathcal{M}(\beta)$.

Proof. This follows from Theorem 5.1.16 and (5.37) because $P_\alpha = \alpha \circ P_1$, P_1 is connected, $G_{P_1^{\odot k}}(\boldsymbol{x};q) = G_{P_k}(\boldsymbol{x};q) = s_k(\boldsymbol{x}) = h_k(\boldsymbol{x})$, and the $h_\rho(\boldsymbol{x})$ are linearly independent. □

We mention another connection between extended chromatic symmetric functions and horizontal-strip LLT polynomials.

Theorem 5.1.18. [4, Theorem 5.5] For each partition ρ, let H_ρ be any weighted graph whose multiset of vertex weights are the parts of ρ. Then the set $\{X_{(H_\rho,w)}(\boldsymbol{x}) : \rho \text{ a partition}\}$ forms a basis for $\Lambda_{\mathbb{Q}}$.

Theorem 5.1.19. For each partition ρ, let Π_ρ be an admissible weighted graph whose multiset of vertex weights are the parts of ρ. Then the set $\{G_{\Pi_\rho}(\boldsymbol{x};q) : \rho \text{ a partition}\}$ forms a $\mathbb{Q}(q)$-basis for $\Lambda_{\mathbb{Q}(q)}$.

Proof. It suffices to show that for every $n \geq 1$, the set $\{G_{\Pi_\rho}(\boldsymbol{x};q) : \rho \text{ a partition of } n\}$ forms a $\mathbb{Q}(q)$-basis for $\Lambda^n_{\mathbb{Q}(q)}$, the space of symmetric functions of degree n, and because this space has dimension equal to the number of partitions of n, it suffices to show that this set is linearly independent. Suppose that $a_\rho(q) \in \mathbb{Q}(q)$ are rational functions not all 0 such that

$$\sum_{\rho \text{ a partition of } n} a_\rho(q)G_{\Pi_\rho}(\boldsymbol{x};q) = 0. \tag{5.39}$$

By clearing denominators, we may assume that the $a_\rho(q)$ are polynomials and that they do not all share a common factor. By setting $q = 1$, we now have

$$\sum_{\rho \text{ a partition of } n} a_\rho(1)h_\rho(\boldsymbol{x}) = 0, \tag{5.40}$$

and because the $h_\rho(\boldsymbol{x})$ are linearly independent, this means that every $a_\rho(1) = 0$ and therefore $a_\rho(q)$ is a multiple of $(q-1)$, contradicting that they do not all share a common factor. □

Remark 5.1.20. We can alternatively prove Theorem 5.1.19 by applying the injective map $f(\boldsymbol{x}) \mapsto f(\boldsymbol{x}[q-1])$ and applying Theorem 3.1.16 and Theorem 5.1.18.

It would be interesting to continue proving analogous results of extended chromatic symmetric functions for horizontal-strip LLT polynomials. For example, Crew and Spirkl proved results about acyclic orientations [11, Theorem 8] and connected partitions [11, Lemma 11]. It might be possible to deduce such results by finding a more direct relationship between these objects. Theorem 3.1.16 and Theorem 5.1.1 suggest that given an admissible weighted graph Π with n vertices and total vertex weight N, we might reasonably define the "extended chromatic quasisymmetric function" of Π to be

$$X_\Pi(\boldsymbol{x};q) = \frac{G_{\boldsymbol{\lambda}}([\boldsymbol{x}(q-1)];q)}{(q-1)^n} = \sum_{\substack{\boldsymbol{T}\in\mathrm{SSYT}_{\boldsymbol{\lambda}}\\ \boldsymbol{T}\text{ proper}}} q^{\mathrm{inv}(\boldsymbol{T})} q^{N-\sum_i m_i(\boldsymbol{T})}(q-1)^{\sum_i(m_i(\boldsymbol{T})-1)}\boldsymbol{x}^{\boldsymbol{T}}, \tag{5.41}$$

where $\boldsymbol{\lambda}$ is any horizontal-strip such that $\Pi \cong \Pi(\boldsymbol{\lambda})$. Note that this is well-defined by Theorem 3.1.8. In particular, this construction would generalize both the extended chromatic symmetric function and the chromatic quasisymmetric function of an unweighted graph. We can think of $\boldsymbol{T} \in \mathrm{SSYT}_{\boldsymbol{\lambda}}$ as an "extended colouring" of the weighted graph $\Pi(\boldsymbol{\lambda})$, where a vertex v is assigned a multiset of $w(v)$ colours, although it is unclear how to interpret the factor $q^{\mathrm{inv}(\boldsymbol{T})}$ in this context. We conclude this chapter by posing the following problem.

Problem 5.1.21. Let (G,w) be a vertex-weighted, vertex-ordered, and edge-weighted graph. Define an *extended chromatic quasisymmetric function* $X_{(G,w)}(\boldsymbol{x};q)$ such that the following properties hold.

1. $X_{(G,w)}(\boldsymbol{x};q)$ is easily seen to be quasisymmetric from our definition.
2. When all vertex weights and nonzero edge weights are 1, then we recover the chromatic quasisymmetric function $X_G(\boldsymbol{x};q)$.
3. When $q=1$, then we recover the extended chromatic symmetric function $X_{(G,w)}(\boldsymbol{x})$.
4. If G is the disjoint union of graphs G_1 and G_2, then
$$X_{(G,w)}(\boldsymbol{x};q) = X_{(G_1,w|_{G_1})}(\boldsymbol{x};q)X_{(G_2,w|_{G_2})}(\boldsymbol{x};q). \tag{5.42}$$
5. If G has an edge, then for some (possibly different) edge e, we have the deletion-contraction relation
$$X_{(G,w)}(\boldsymbol{x};q) = qX_{(G\backslash e,w)}(\boldsymbol{x};q) - X_{(G/e,w)}(\boldsymbol{x};q). \tag{5.43}$$
6. If $(G,w)\cong\Pi(\boldsymbol{\lambda})$ for some horizontal-strip $\boldsymbol{\lambda}$, then (5.41) holds.

Note that for graphs of the form $(G,w)\cong\Pi(\boldsymbol{\lambda})$, then the expression (5.41) satisfies all of these conditions. If we are able to solve Problem 5.1.21 for such weighted graphs by finding a definition that depends only on $\Pi(\boldsymbol{\lambda})$ and not on $\boldsymbol{\lambda}$, then this would immediately give a constructive proof of Theorem 3.1.8. Note that (5.41) could in theory depend on $\boldsymbol{\lambda}$ because of the $q^{\mathrm{inv}(\boldsymbol{T})}$ factor, although by Theorem 3.1.8, it turns out to only depend on $\Pi(\boldsymbol{\lambda})$. If we are able to solve Problem 5.1.21 in general, then the deletion-contraction relation and the flexibility of our parameter q may be key innovations toward Problem 2.4.3 and Problem 2.4.4.

Chapter 6

Proof of Theorem 3.1.8

In this chapter, we prove Theorem 3.1.8. The general idea will be to use rotating, cycling, and commuting to arrange the rows of $\boldsymbol{\lambda}$ so that we can apply our deletion-contraction relation (4.27) and use induction. One possible obstruction to our approach will be a *noncommuting path*, which is a concept we will introduce in Definition 6.1.15. However, Lemma 6.1.26 and Lemma 6.1.28 will describe the structure of a minimal noncommuting path very precisely, and in such a case we can apply a delicate induction in Lemma 6.1.53 and a series of *local rotations*, which we introduce in Lemma 6.1.50. We begin by making the following definition, which will be useful to describe relationships between rows within a fixed horizontal-strip.

Definition 6.1.1. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a horizontal-strip and $1 \leq i, j \leq n$ with $i \neq j$. We define $M_{i,j}(\boldsymbol{\lambda})$ to be the weight of the edge in $\Pi(\boldsymbol{\lambda})$ joining R_i and R_j, that is

$$M_{i,j}(\boldsymbol{\lambda}) = \begin{cases} M(R_i, R_j) & \text{if } i < j, \\ M(R_j, R_i) & \text{if } i > j. \end{cases} \tag{6.1}$$

We abbreviate $M_{i,j}(\boldsymbol{\lambda})$ as $M_{i,j}$ if the context is clear. Note that

$$0 \leq M_{i,j} \leq \min\{|R_i|, |R_j|\}. \tag{6.2}$$

Definition 6.1.2. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ and $\boldsymbol{\mu} = (S_1, \ldots, S_n)$ be horizontal-strips and let $\varphi : \Pi(\boldsymbol{\lambda}) \to \Pi(\boldsymbol{\mu})$ be an isomorphism of weighted graphs. We denote this by $\varphi : \Pi(\boldsymbol{\lambda}) \xrightarrow{\sim} \Pi(\boldsymbol{\mu})$. We will also denote by φ the underlying bijection

$$\varphi : \{1, \ldots, n\} \to \{1, \ldots, n\}, \tag{6.3}$$

which must satisfy $|R_i| = |S_{\varphi_i}|$ and $M_{i,j}(\boldsymbol{\lambda}) = M_{\varphi_i, \varphi_j}(\boldsymbol{\mu})$ for all i, j.

Example 6.1.3. In Example 3.1.4, we have an isomorphism $\varphi : \Pi(\boldsymbol{\lambda}) \xrightarrow{\sim} \Pi(\boldsymbol{\mu})$ with underlying bijection given by $\varphi_1 = 2$, $\varphi_2 = 1$, $\varphi_3 = 4$, and $\varphi_4 = 3$.

Let us also recall Proposition 4.1.7 from Chapter 4, because we will be using it extensively in this chapter. We restate it slightly, in terms of a fixed horizontal-strip.

Proposition 6.1.4. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a horizontal-strip and let $1 \leq i, j \leq n$. Without loss of generality, assume that $l(R_i) \leq l(R_j)$.

1. If $r(R_i) < l(R_j) - 1$, then $M(R_i, R_j) = M(R_j, R_i) = 0$, so $R_i \leftrightarrow R_j$.

2. If $l(R_i) = l(R_j)$ or $r(R_j) \leq r(R_i)$, then $M(R_i, R_j) = M(R_j, R_i) = \min\{|R_i|, |R_j|\}$, so $R_i \leftrightarrow R_j$.

3. Otherwise, we have $l(R_i) < l(R_j) \leq r(R_i) + 1 \leq r(R_j)$, and

$$M(R_i, R_j) = r(R_i) - l(R_j) + 1 \text{ and } M(R_j, R_i) = r(R_i) - l(R_j) + 2, \text{ so } R_i \not\leftrightarrow R_j. \tag{6.4}$$

   In particular, we have

$$M_{i,j} = r(R_i) - l(R_j) + 1 + \chi(i > j), \tag{6.5}$$

   where $\chi(i > j) = 1$ if $i > j$ and 0 otherwise.

Note that in particular, we see that

$$\text{if } R_i \leftrightarrow R_j, \text{ then } M_{i,j} \text{ is either 0 or } \min\{|R_i|, |R_j|\}. \tag{6.6}$$

Corollary 6.1.5. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a horizontal-strip.

1. If $R_i \not\leftrightarrow R_j$, then we either have $l(R_i) < l(R_j)$ and $r(R_i) < r(R_j)$, or we have $l(R_i) > l(R_j)$ and $r(R_i) > r(R_j)$. In other words, the integers $l(R_i) - l(R_j)$ and $r(R_i) - r(R_j)$ are nonzero and have the same sign.

2. Suppose that $i < j$ and $R_i \not\leftrightarrow R_j$. If $M_{i,j} = 0$, then $l(R_j) = r(R_i) + 1$, so in particular $l(R_i) < l(R_j)$. If $M_{i,j} = |R_i|$, then $r(R_j) = r(R_i) - 1$, while if $M_{i,j} = |R_j|$, then $l(R_j) = l(R_i) - 1$, so in particular we have $l(R_j) < l(R_i)$ in both cases.

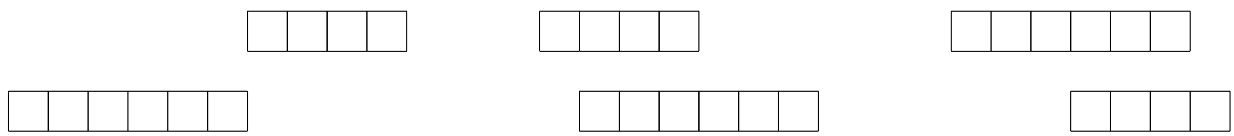
Proof.

1. Assuming without loss of generality that $l(R_i) \leq l(R_j)$, then all of the possibilities are enumerated in Proposition 6.1.4 and we have $R_i \not\leftrightarrow R_j$ only when $l(R_i) < l(R_j)$ and $l(R_j) \leq r(R_i) + 1 \leq r(R_j)$, so in particular $r(R_i) < r(R_j)$.

2. By (6.4), if $M_{i,j} = M(R_i, R_j) = 0$, then if $l(R_j) < l(R_i)$ we would have $M(R_j, R_i) = M(R_i, R_j) - 1 = -1$, contradicting (6.2), so we must have $l(R_i) < l(R_j)$ and $M_{i,j} = 0 = r(R_i) - l(R_j) + 1$. Similarly, by (6.4), if $M_{i,j} = M(R_i, R_j) = \min\{|R_i|, |R_j|\}$, then if $l(R_i) < l(R_j)$ we would have $M(R_j, R_i) = \min\{|R_i|, |R_j|\} + 1$, contradicting (6.2), so we must have $l(R_j) < l(R_i)$ and $M_{i,j} = r(R_j) - l(R_i) + 2$. The result now follows by noting that $|R_i| = r(R_i) - l(R_i) + 1$ and $|R_j| = r(R_j) - l(R_j) + 1$.

∎

Figure 6.1: Left: $M(R_i, R_j) = 0$, but $R_i \not\leftrightarrow R_j$, Middle and Right: $M(R_i, R_j) = \min\{|R_i|, |R_j|\}$, but $R_i \not\leftrightarrow R_j$



Remark 6.1.6. Note that the converse to (6.6) does not hold. It is possible to have rows R_i and R_j with $M(R_i, R_j) = 0$ or $M(R_i, R_j) = \min\{|R_i|, |R_j|\}$, but $R_i \not\leftrightarrow R_j$ as in Figure 6.1. However, Part 2 tells us that if this occurs, then R_i must be on a particular side of R_j.

We now show how we can use cycling and commuting to prove Theorem 3.1.8 in the special case of Hall–Littlewood polynomials. Recall that if $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$, then we have

$$M(\boldsymbol{\lambda}) = \sum_{1 \le i < j \le n} M(R_i, R_j) \le \sum_{1 \le i < j \le n} \min\{|R_i|, |R_j|\} = n(\boldsymbol{\lambda}). \tag{6.7}$$

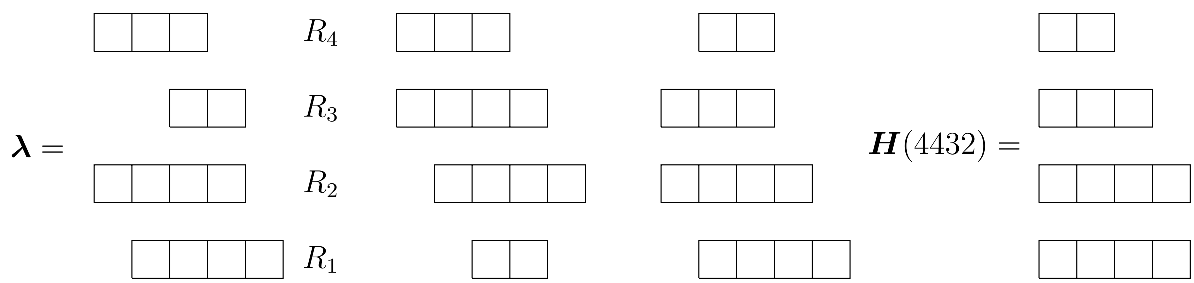
Lemma 6.1.7. Theorem 3.1.8 holds if $M(\boldsymbol{\lambda}) = n(\boldsymbol{\lambda})$.

Proof. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$. By (6.7), we have that if $M(\boldsymbol{\lambda}) = n(\boldsymbol{\lambda})$, then $M(R_i, R_j) = \min\{|R_i|, |R_j|\}$ for all $1 \le i < j \le n$. Recall that we denote by λ the partition determined by the row lengths of $\boldsymbol{\lambda}$. We now show that the horizontal-strip $\boldsymbol{H}(\lambda) = (\lambda_1/0, \ldots, \lambda_n/0)$ is similar to $\boldsymbol{\lambda}$, meaning that the LLT polynomial $G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q)$ only depends on λ and therefore only on the weighted graph $\Pi(\boldsymbol{\lambda})$. By translating, we may assume without loss of generality that $\min\{l(R_i) : 1 \le i \le n\} = 0$, and suppose that $l(R_a) = 0$. Because $M_{i,j} = \min\{|R_i|, |R_j|\}$ for every i, j, we have by Proposition 6.1.4, Part 2 an upper bound $l(R_j) \le r(R_a) + 1 = |R_a|$, so let us further assume that $\boldsymbol{\lambda}$ has $\sum_{i=1}^n l(R_i)$ minimal among all horizontal-strips similar to $\boldsymbol{\lambda}$.

We now claim that $l(R_i) = 0$ for every $1 \le i \le n$. If not, let j be such that $l(R_j) \ge 1$ is maximal. By Corollary 6.1.5, Part 3, if $i < j$ and $R_i \not\leftrightarrow R_j$, then because $M_{i,j} = \min\{|R_i|, |R_j|\}$, we must have $l(R_j) < l(R_i)$, contradicting maximality of $l(R_j)$, so we must have $R_i \leftrightarrow R_j$ for every $i < j$. By Proposition 4.1.3, we can now commute and cycle to find that $(R_1, \ldots, R_n, R_j^-) \in \mathcal{S}(\boldsymbol{\lambda})$, contradicting minimality of $\sum_{i=1}^n l(R_i)$. Therefore, we indeed have $l(R_i) = 0$ for every $1 \le i \le n$, so by Proposition 6.1.4, Part 2, we have $R_i \leftrightarrow R_j$ for every $1 \le i, j \le n$ and by commuting once again we have $\boldsymbol{H}(\lambda) \in \mathcal{S}(\boldsymbol{\lambda})$. This completes the proof. □

Example 6.1.8. Figure 6.2 illustrates the idea of the proof of Lemma 6.1.7. The row R_3, which has $l(R_3)$ maximal, commutes with all rows below, so by commuting and cycling, we can move it to the left. Continuing in this way, the horizontal-strip $\boldsymbol{\lambda}$ is shown to be similar to $\boldsymbol{H}(4432)$ on the right.

Figure 6.2: An example of using commuting and cycling to show that $\boldsymbol{H}(\lambda) \in \mathcal{S}(\boldsymbol{\lambda})$



Our general strategy will be to apply a similar sequence of transformations to replace $\boldsymbol{\lambda}$ with a similar horizontal-strip to which we can apply our deletion-contraction relation (4.27) and induction. We will take an unusual approach in that the base case of our induction will be when $M(\boldsymbol{\lambda})$ is the maximum possible value $n(\boldsymbol{\lambda})$, from which we will deduce Theorem 3.1.8 for smaller values of $M(\boldsymbol{\lambda})$. Rather than delete an edge and contract, we will add an edge and contract. This seemed to give us the right induction hypothesis. We make the following definition.

Definition 6.1.9. Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be horizontal-strips with $\Pi(\boldsymbol{\lambda}) \cong \Pi(\boldsymbol{\mu})$. A *good substitute* for $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is a pair of horizontal-strips $(\boldsymbol{\lambda}', \boldsymbol{\mu}')$, where $\boldsymbol{\lambda}' = (R_1, \ldots, R_n) \in \mathcal{S}(\boldsymbol{\lambda})$ and $\boldsymbol{\mu}' = (S_1, \ldots, S_n) \in \mathcal{S}(\boldsymbol{\mu})$ satisfy

$$l(R_1) < l(R_2),\ R_1 \nleftrightarrow R_2,\ l(S_1) < l(S_2),\ S_1 \nleftrightarrow S_2, \tag{6.8}$$

and $\varphi_1 = 1$ and $\varphi_2 = 2$, where $\varphi : \Pi(\boldsymbol{\lambda}') \xrightarrow{\sim} \Pi(\boldsymbol{\mu}')$. A single horizontal-strip $\boldsymbol{\lambda}$ is *good* if for any horizontal-strip $\boldsymbol{\mu}$ such that $\Pi(\boldsymbol{\lambda}) \cong \Pi(\boldsymbol{\mu})$, there is a good substitute for $(\boldsymbol{\lambda}, \boldsymbol{\mu})$.

We now state a key Lemma, from which Theorem 3.1.8 will follow.

Lemma 6.1.10. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a horizontal-strip with $n(\boldsymbol{\lambda}) - M(\boldsymbol{\lambda}) \geq 1$. Suppose that $\boldsymbol{\lambda}$ satisfies the condition that

$$\begin{gathered}\text{Theorem 3.1.8 holds for horizontal-strips } \boldsymbol{\lambda}' \text{ and } \boldsymbol{\mu}' \text{ with either} \\ n(\boldsymbol{\lambda}') < n(\boldsymbol{\lambda}), \text{ or with } n(\boldsymbol{\lambda}') = n(\boldsymbol{\lambda}) \text{ and } M(\boldsymbol{\lambda}') > M(\boldsymbol{\lambda}).\end{gathered} \tag{6.9}$$

Then $\boldsymbol{\lambda}$ is good.

Proof of Theorem 3.1.8 assuming Lemma 6.1.10. We use induction on $n(\boldsymbol{\lambda})$. If $n(\boldsymbol{\lambda}) = 0$, then $\boldsymbol{\lambda}$ has only one row and the result follows from Proposition 4.1.3, Part 1, so assume that $n(\boldsymbol{\lambda}) \geq 1$ and that Theorem 3.1.8 holds for horizontal-strips $\boldsymbol{\lambda}'$ and $\boldsymbol{\mu}'$ with $n(\boldsymbol{\lambda}') < n(\boldsymbol{\lambda})$. We also use induction on $n(\boldsymbol{\lambda}) - M(\boldsymbol{\lambda})$. If $n(\boldsymbol{\lambda}) - M(\boldsymbol{\lambda}) = 0$, then the result follows from

Lemma 6.1.7, so assume that $n(\boldsymbol{\lambda}) - M(\boldsymbol{\lambda}) \geq 1$ and that Theorem 3.1.8 holds for horizontal-strips $\boldsymbol{\lambda}'$ and $\boldsymbol{\mu}'$ with $n(\boldsymbol{\lambda}') = n(\boldsymbol{\lambda})$ and $n(\boldsymbol{\lambda}') - M(\boldsymbol{\lambda}') < n(\boldsymbol{\lambda}) - M(\boldsymbol{\lambda})$. This is exactly the condition (6.9), so assuming Lemma 6.1.10, we have that $\boldsymbol{\lambda}$ is good.

Now by replacing $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ and $\boldsymbol{\mu} = (S_1, \ldots, S_n)$ by a good substitute as necessary, we may assume that $l(R_1) < l(R_2)$, $R_1 \not\leftrightarrow R_2$, $l(S_1) < l(S_2)$, $S_1 \not\leftrightarrow S_2$, and $\varphi_1 = 1$ and $\varphi_2 = 2$, where $\varphi : \Pi(\boldsymbol{\lambda}) \xrightarrow{\sim} \Pi(\boldsymbol{\mu})$. Consider the horizontal-strips

$$\boldsymbol{\lambda}' = (R_2, R_1, R_3, \ldots, R_n) \text{ and } \boldsymbol{\lambda}'' = (R_1 \cup R_2, R_1 \cap R_2, R_3, \ldots, R_n) \tag{6.10}$$

and similarly define $\boldsymbol{\mu}'$ and $\boldsymbol{\mu}''$. By Proposition 4.1.18, the graphs $\Pi(\boldsymbol{\lambda}')$ and $\Pi(\boldsymbol{\mu}')$ are constructed by increasing the weight of the edge (R_1, R_2) and (S_1, S_2) by one, and therefore we have $\Pi(\boldsymbol{\lambda}') \cong \Pi(\boldsymbol{\mu}')$. Proposition 4.1.18 also describes exactly how to construct $\Pi(\boldsymbol{\lambda}'')$ from $\Pi(\boldsymbol{\lambda}')$, and therefore we have $\Pi(\boldsymbol{\lambda}'') \cong \Pi(\boldsymbol{\mu}'')$. Because

$$M(\boldsymbol{\lambda}') = M(\boldsymbol{\lambda}) + 1,\ n(\boldsymbol{\lambda}') = n(\boldsymbol{\lambda}),\ \text{and } n(\boldsymbol{\lambda}'') < n(\boldsymbol{\lambda}), \tag{6.11}$$

our induction hypothesis implies that $G_{\boldsymbol{\lambda}'}(\boldsymbol{x}; q) = G_{\boldsymbol{\mu}'}(\boldsymbol{x}; q)$ and $G_{\boldsymbol{\lambda}''}(\boldsymbol{x}; q) = G_{\boldsymbol{\mu}''}(\boldsymbol{x}; q)$. Therefore, by rearranging (4.27), we have

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q) = \frac{1}{q} G_{\boldsymbol{\lambda}'}(\boldsymbol{x}; q) + \frac{q-1}{q} G_{\boldsymbol{\lambda}''}(\boldsymbol{x}; q) = \frac{1}{q} G_{\boldsymbol{\mu}'}(\boldsymbol{x}; q) + \frac{q-1}{q} G_{\boldsymbol{\mu}''}(\boldsymbol{x}; q) = G_{\boldsymbol{\mu}}(\boldsymbol{x}; q). \tag{6.12}$$

This completes the proof. □

Remark 6.1.11. Our notation for $\boldsymbol{\lambda}'$ and $\boldsymbol{\lambda}''$ in the above proof differs from our notation in Corollary 4.1.17 in that the roles of $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ have switched because we are adding an edge, rather than deleting an edge. This is why we rearranged the deletion-contraction formula.

It now remains to prove Lemma 6.1.10. The following definition will describe a relationship between rows that will be convenient to refer to.

Definition 6.1.12. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a horizontal-strip. We write $R_i \prec R_j$ if $M_{i,j} = |R_i|$ and $R_i \not\prec R_j$ otherwise. We also write $R_i \precnsim R_j$ to mean that $R_i \prec R_j$ and $R_j \not\prec R_i$.

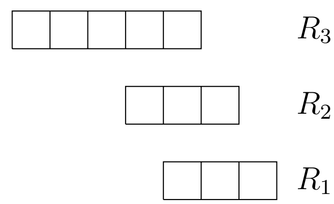
Proposition 6.1.13. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a horizontal-strip.

1. Suppose that $i < j$. Then we have $R_i \prec R_j$ if and only if either $R_i \subseteq R_j$ or $R_i \subseteq R_j^+$.
2. Suppose that $i > j$. Then we have $R_i \prec R_j$ if and only if either $R_i \subseteq R_j$ or $R_i \subseteq R_j^-$.
3. We have $R_i \prec R_j$ and $R_i \leftrightarrow R_j$ if and only if $R_i \subseteq R_j$.

Proof.

Figure 6.3: We have $R_1 \prec R_2$, $R_2 \prec R_1$, $R_2 \precnsim R_3$, and $R_1 \not\prec R_3$

$\boldsymbol{\lambda} =$



1. If $R_i \prec R_j$, then by definition we have

$$M_{i,j} = |R_i| = \begin{cases} |R_i \cap R_j| & \text{if } l(R_i) \leq l(R_j), \\ |R_i \cap R_j^+| & \text{if } l(R_i) > l(R_j), \end{cases} \tag{6.13}$$

   and therefore we must have $R_i \subseteq R_j$ or $R_i \subseteq R_j^+$. Conversely, if $R_i \subseteq R_j^+$, then $l(R_i) > l(R_j)$ and $M_{i,j} = |R_i|$ by Proposition 6.1.4, Part 2, while if $R_i \not\subseteq R_j^+$ and $R_i \subseteq R_j$, then $l(R_i) = l(R_j)$ and again $M_{i,j} = |R_i|$ by Proposition 6.1.4, Part 2.

2. This follows from the previous part by considering a rotation of $\boldsymbol{\lambda}$.

3. If $R_i \subseteq R_j$, then $R_i \prec R_j$ by the previous parts and we have $l(R_j) \leq l(R_i) \leq r(R_i) \leq r(R_j)$, so $R_i \leftrightarrow R_j$ by Proposition 6.1.4, Part 2. Conversely, if $i < j$, $R_i \prec R_j$, and $R_i \not\subseteq R_j$, then by Part 1 we have $R_i \subseteq R_j^+$, so $l(R_j) < l(R_i) \leq r(R_j) + 1 \leq r(R_i)$ and $R_i \nleftrightarrow R_j$ by Proposition 6.1.4, Part 3. The case where $i > j$ follows by rotating.

□

Example 6.1.14. Let $\boldsymbol{\lambda} = (R_1, R_2, R_3) = (7/4, 6/3, 5/0)$ as in Figure 6.3. We have $M(R_1, R_2) = 3 = |R_1| = |R_2|$, so $R_1 \prec R_2$ and $R_2 \prec R_1$. We have $M(R_1, R_3) = 2 < |R_1|$, so $R_1 \not\prec R_3$, and we have $M(R_2, R_3) = 3 = |R_2| < |R_3|$, so $R_2 \precnsim R_3$. Informally, we can think of the relation $R_i \prec R_j$ as being very similar to the relation $R_i \subseteq R_j$, except that we may need to shift a row by one cell. Because of this possible shift, the relation $\prec$ is not transitive. In this example, we have $R_1 \prec R_2$ and $R_2 \prec R_3$, but $R_1 \not\prec R_3$.

The following concept will be important to define the potential obstruction to our technique from the proof of Lemma 6.1.7 of using commuting to rearrange rows.

Definition 6.1.15. A sequence of $n \geq 3$ rows $(R_1, \ldots, R_n)$ is a *noncommuting path* from R_1 to R_n if $R_i \nleftrightarrow R_{i+1}$ for every $1 \leq i \leq n-1$. A noncommuting path is *minimal* if there is no subsequence of rows $(R_1 = R_{i_1}, R_{i_2}, \ldots, R_{i_k} = R_n)$ with $i_1 < i_2 < \cdots < i_k$ and $3 \leq k < n$ that forms a noncommuting path from R_1 to R_n.

In particular, if $(R_1, \ldots, R_n)$ is a minimal noncommuting path, then we have $R_i \leftrightarrow R_j$ for every i, j with $1 < |j - i| < n - 1$. Because we require a noncommuting path to have length at least 3, we cannot conclude that $R_1 \leftrightarrow R_n$ in a minimal noncommuting path.

Our next Lemma shows that, given two rows R_i and R_j of $\boldsymbol{\lambda}$ with $i < j$, either there is a minimal noncommuting path in $\boldsymbol{\lambda}$ from R_i to R_j, or we may assume that $j = i + 1$.

Lemma 6.1.16. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a horizontal-strip and let $1 \leq i < j \leq n$. Then one of the following holds.

1. There is a minimal noncommuting path $(R_i = R_{i_1}, \ldots, R_{i_k} = R_j)$ in $\boldsymbol{\lambda}$ from R_i to R_j.
2. There is a horizontal-strip $\boldsymbol{\mu} = (S_1, \ldots, S_n) \in \mathcal{S}(\boldsymbol{\lambda})$ and an isomorphism of weighted graphs $\varphi : \Pi(\boldsymbol{\lambda}) \xrightarrow{\sim} \Pi(\boldsymbol{\mu})$ such that $\varphi_j = \varphi_i + 1$ and $l(S_{\varphi_t}) = l(R_t)$ for all $1 \leq t \leq n$.

Proof. We use induction on $j - i$. If $j - i = 1$, then the second possibility holds by simply taking $\boldsymbol{\lambda} = \boldsymbol{\mu}$, so assume that $j - i \geq 2$. If $R_j \leftrightarrow R_{j-1}$, then by commuting we have $(R_1, \ldots, R_j, R_{j-1}, \ldots, R_n) \in \mathcal{S}(\boldsymbol{\lambda})$ and we are done by our induction hypothesis on $j - i$, so we may assume that $R_j \not\leftrightarrow R_{j-1}$. Similarly, if $R_{j-1} \leftrightarrow R_t$ for every $i \leq t \leq j - 2$, then by commuting we would have $(R_1, \ldots, R_{j-1}, R_i, \ldots, R_j, \ldots, R_n) \in \mathcal{S}(\boldsymbol{\lambda})$ and we are again done by induction. So we may assume that $R_{j-1} \not\leftrightarrow R_t$ for some $i \leq t \leq j - 2$, and continuing in this way there must be a noncommuting path in $\boldsymbol{\lambda}$ from R_i to R_j. Finally, if this noncommuting path is not minimal, then it contains a minimal one. □

Examples of minimal noncommuting paths are given in Figure 6.4. Informally, our next goal is to show that all minimal noncommuting paths look like these examples. Specifically, our goal is to prove Lemma 6.1.26, which describes what a minimal noncommuting path may look like, and Lemma 6.1.28, which describes the extent to which our weighted graph determines the structure of a minimal noncommuting path. This will allow us to prove Corollary 6.1.30, which proves Lemma 6.1.10 in many cases. We will first prove some elementary Propositions.

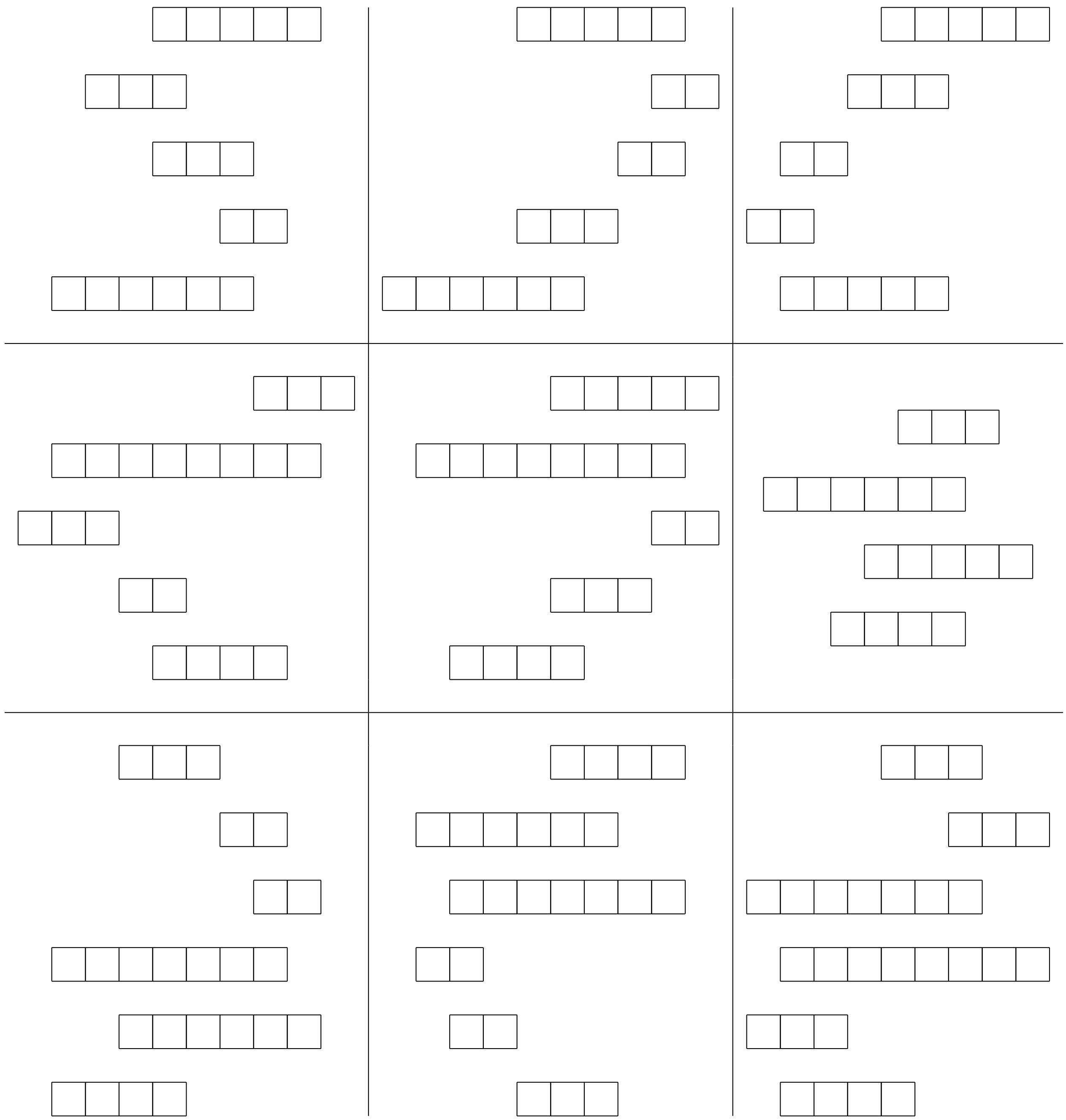
Proposition 6.1.17. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a horizontal-strip.

1. Suppose that $R_i \not\leftrightarrow R_j$, $R_j \not\leftrightarrow R_k$, $R_i \leftrightarrow R_k$, and that the integers $l(R_j) - l(R_i)$ and $l(R_k) - l(R_j)$ have the same sign. Then $M_{i,k} = 0$.
2. Suppose that $R_i \not\leftrightarrow R_j$, $R_j \not\leftrightarrow R_k$, $R_i \leftrightarrow R_k$, and that the integers $l(R_j) - l(R_i)$ and $l(R_k) - l(R_j)$ have opposite signs. Then $R_i \prec R_k$ or $R_k \prec R_i$.
3. Suppose that $R_i \leftrightarrow R_j$, $R_i \leftrightarrow R_k$, and $R_j \not\leftrightarrow R_k$. Then $R_j \prec R_i$ if and only if $R_k \prec R_i$.

Note that in (1) and (2), because $R_i \not\leftrightarrow R_j$ and $R_j \not\leftrightarrow R_k$, the integers $l(R_j) - l(R_i)$ and $l(R_k) - l(R_j)$ are nonzero by Corollary 6.1.5, Part 1.

Proof of Proposition 6.1.17.

Figure 6.4: Some examples of minimal noncommuting paths



1. Without loss of generality, we may assume that $l(R_i) < l(R_j) < l(R_k)$, and then because $R_i \not\leftrightarrow R_j$ and $R_j \not\leftrightarrow R_k$, by Corollary 6.1.5, Part 1, we must have $r(R_i) < r(R_j) < r(R_k)$ as well. But now we cannot have $R_i \subseteq R_k$ or $R_k \subseteq R_i$, so by Proposition 6.1.13, Part 3, we cannot have $R_i \prec R_k$ or $R_k \prec R_i$. Therefore, because $R_i \leftrightarrow R_k$, by (6.6), we must have $M_{i,k} = 0$.

2. By rotating, we may assume without loss of generality that $l(R_i) \leq l(R_j) - 1$ and $l(R_k) \leq l(R_j) - 1$. Because $R_i \not\leftrightarrow R_j$ and $R_k \not\leftrightarrow R_j$, by Proposition 6.1.4, Part 1, we have $r(R_i) \geq l(R_j) - 1$ and $r(R_k) \geq l(R_j) - 1$. But now $M_{i,k} > 0$ and because $R_i \leftrightarrow R_k$, we have by (6.6) that $R_i \prec R_k$ or $R_k \prec R_i$.

3. By symmetry, it suffices to prove that $R_j \prec R_i$ implies $R_k \prec R_i$. Suppose that $R_j \prec R_i$, and then because $R_i \leftrightarrow R_j$, by Proposition 6.1.13, Part 1, we have $R_j \subseteq R_i$, that is $l(R_i) \leq l(R_j) \leq r(R_j) \leq r(R_i)$. Suppose that $l(R_k) < l(R_i)$. By Proposition 6.1.4 Parts 1 and 2, if $r(R_k) < l(R_i) - 1 \leq l(R_j) - 1$, then $R_j \leftrightarrow R_k$, if $l(R_i) - 1 \leq r(R_k) \leq r(R_i) - 1$, then $R_i \not\leftrightarrow R_k$, and if $r(R_k) \geq r(R_i) \geq r(R_j)$, then again $R_j \leftrightarrow R_k$, a contradiction in all cases, so we must have $l(R_k) \geq l(R_i)$. Similarly, by rotating, we must have $r(R_k) \leq r(R_i)$, so we have $R_k \subseteq R_i$ and $R_k \prec R_i$.

□

Proposition 6.1.18. Let $\boldsymbol{\lambda} = (R_1, \dots, R_n)$ be a minimal noncommuting path.

1. If $R_i \prec R_j$ and $R_j \prec R_i$ for some $j \geq i + 2$, then we must have $n = 3$ or $n = 4$ and $j = i + 2$. In particular, if $n \geq 5$, then $R_i \prec R_j$ implies that in fact $R_i \precnsim R_j$.

2. If $R_j \prec R_i$ for some $j \geq i + 2$, then $R_k \prec R_i$ for every $k \geq i + 2$, with the possible exception of $k = n$ if $i = 1$. Similarly, if $R_j \prec R_i$ for some $j \leq i - 2$, then $R_k \prec R_i$ for every $k \leq i - 2$, with the possible exception of $k = 1$ if $i = n$.

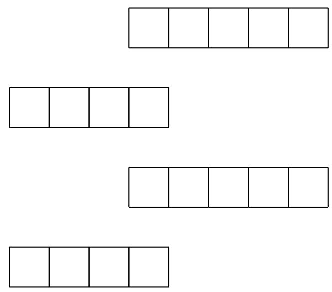
3. Suppose that $R_j \precnsim R_i$ for some $i \geq j + 2$, that i is minimal with these two properties, and that $i \neq n$ if $j = 1$. Then either $l(R_{i-1}) > \cdots > l(R_j)$ and $l(R_i) < l(R_{i-1})$, or $l(R_{i-1}) < \cdots < l(R_j)$ and $l(R_i) > l(R_{i-1})$.

Remark 6.1.19. Figure 6.5 shows that if $n \leq 4$, then it is possible to have a minimal noncommuting path with $R_1 \prec R_3$ and $R_3 \prec R_1$. If $n \geq 5$, then this will not happen because there will be some $R_t \not\leftrightarrow R_{t'}$ with $t' \geq t + 2$ and $(t, t') \neq (1, n)$, contradicting minimality. Informally, (2) states that if we are contained in R_i, then we must remain stuck in R_i and (3) states that if R_j is contained in some minimal R_i, then we must move in the same direction until R_i.

Proof of Proposition 6.1.18.

1. Because $j \geq i+2$, we have $R_i \leftrightarrow R_j$ by minimality and therefore $R_i = R_j$ by Proposition 6.1.13. Now $R_{i+1} \not\leftrightarrow R_j$, so by minimality of the noncommuting path we must have

Figure 6.5: A minimal noncommuting path with $R_1 \prec R_3$ and $R_3 \prec R_1$



$j = i+2$. Similarly, if $i \geq 2$, then $R_{i-1} \not\leftrightarrow R_j$ and so we must have $i = 2$ and $j = n = 4$, if $j \leq n-1$, then $R_i \not\leftrightarrow R_{j+1}$ and so we must have $i = 1$ and $j = 3 = n-1$ so again $n = 4$, and otherwise we have $i = 1$ and $j = 3 = n$.

2. If $i+2 \leq k \leq n-1$, $R_k \prec R_i$, and $k+1 \neq n$ in the case of $i = 1$, then because $R_i \leftrightarrow R_k$, $R_i \leftrightarrow R_{k+1}$, and $R_k \not\leftrightarrow R_{k+1}$, by Proposition 6.1.17, Part 3, we have $R_k \prec R_i$ if and only if $R_{k+1} \prec R_i$, so the first statement follows by induction on k and the second statement follows by rotating.

3. Recall that because $R_j \prec R_i$ and $R_j \leftrightarrow R_i$, by Proposition 6.1.13 we must have $l(R_i) \leq l(R_j)$ and $r(R_i) \geq r(R_j)$. Suppose that either $l(R_{j+1}) > l(R_j)$ and $l(R_{t+1}) < l(R_t)$ for some minimal $j+1 \leq t \leq i-2$, or $l(R_{j+1}) < l(R_j)$ and $l(R_{t+1}) > l(R_t)$ for some minimal $j+1 \leq t \leq i-2$, so in particular, $n \geq 5$. Then by Proposition 6.1.17, Part 2, we have either $R_{t-1} \prec R_{t+1}$ or $R_{t+1} \prec R_{t-1}$. If $R_{t-1} \prec R_{t+1}$, then by the previous two parts we have $R_j \not\precsim R_{t+1}$, contradicting minimality of i. If $R_{t+1} \prec R_{t-1}$, then by the previous two parts we have $R_i \not\precsim R_{t-1}$, but now by Proposition 6.1.13, Part 3, we have $R_j \subsetneq R_i \subsetneq R_{t-1}$, so $R_j \leftrightarrow R_{t-1}$, $t-1 \geq j+2$, and $R_j \not\precsim R_{t-1}$, again contradicting minimality of i. Therefore, we must have either $l(R_{i-1}) > \cdots > l(R_j)$ and $l(R_i) \leq l(R_j) < l(R_{i-1})$, or $l(R_{i-1}) < \cdots < l(R_j)$ and $r(R_i) \geq r(R_j) > r(R_{i-1})$, so $l(R_i) > l(R_{i-1})$ by Corollary 6.1.5, Part 2.

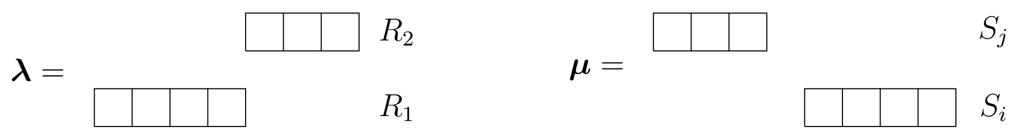
□

Proposition 6.1.20. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a horizontal-strip with $R_1 \not\leftrightarrow R_2$. Let $\boldsymbol{\mu} = (S_1, \ldots, S_n)$ be a horizontal-strip with $\varphi : \Pi(\boldsymbol{\lambda}) \xrightarrow{\sim} \Pi(\boldsymbol{\mu})$ and let $i = \varphi_1$ and $j = \varphi_2$. If $l(R_2) > l(R_1)$, also assume that $l(S_j) > l(S_i)$. Then

$$l(S_j) - l(S_i) \geq l(R_2) - l(R_1) \text{ with equality only if } i < j. \tag{6.14}$$

Remark 6.1.21. Informally, Proposition 6.1.20 states that the leftmost possible position of S_j, given the weighted graph data, occurs when S_j is above S_i and $S_i \not\leftrightarrow S_j$. Figure 6.6 demonstrates the necessity of the hypothesis that if $l(R_2) > l(R_1)$, then $l(S_j) > l(S_i)$.

Figure 6.6: An example where $R_1 \not\leftrightarrow R_2$, but $l(S_j) - l(S_i) < 0 < l(R_2) - l(R_1)$



Proof of Proposition 6.1.20. We calculate directly, using (6.5) and noting that $|R| = r(R) - l(R) + 1$. If $l(R_2) < l(R_1)$, then the statement holds unless $l(S_j) < l(S_i)$, and because $R_1 \not\leftrightarrow R_2$, we have by Corollary 6.1.5, Part 2, that $M_{1,2}(\boldsymbol{\lambda}) = M_{i,j}(\boldsymbol{\mu}) > 0$. By Proposition 6.1.4, Parts 2 and 3, we have $M_{1,2}(\boldsymbol{\lambda}) = r(R_2) - l(R_1) + 2$ and we either have $M_{i,j}(\boldsymbol{\mu}) = r(S_j) - l(S_i) + 1 + \chi(i < j)$, or $M_{i,j}(\boldsymbol{\mu}) = \min\{|S_i|, |S_j|\} \leq r(S_j) - l(S_i) + 1 + \chi(i < j)$. Therefore we have

$$\begin{aligned} l(S_j) - l(S_i) &= r(S_j) - |S_j| + 1 - l(S_i) \geq M_{i,j}(\boldsymbol{\mu}) - |S_j| - \chi(i < j) \\ &\geq M_{1,2}(\boldsymbol{\lambda}) - |R_2| - 1 = r(R_2) - |R_2| + 1 - l(R_1) = l(R_2) - l(R_1), \end{aligned} \tag{6.15}$$

with equality only if $i < j$.

Similarly, if $l(R_2) > l(R_1)$, then by hypothesis we have $l(S_j) > l(S_i)$, and because $R_1 \not\leftrightarrow R_2$, we have by Corollary 6.1.5, Part 2, that $M_{i,j}(\boldsymbol{\mu}) < \min\{|S_i|, |S_j|\}$. By Proposition 6.1.4, Parts 1 and 3, we have $M_{1,2}(\boldsymbol{\lambda}) = r(R_1) - l(R_2) + 1$ and we either have $M_{i,j}(\boldsymbol{\mu}) = r(S_i) - l(S_j) + 1 + \chi(i < j)$ or $M_{i,j}(\boldsymbol{\mu}) = 0 \geq r(S_i) - l(S_j) + 1 + \chi(i < j)$. Therefore we have

$$\begin{aligned} l(S_j) - l(S_i) &= l(S_j) - r(S_i) - 1 + |S_i| \geq |S_i| - M_{i,j}(\boldsymbol{\mu}) + \chi(i > j) \\ &\geq |R_1| - M_{1,2}(\boldsymbol{\lambda}) = l(R_2) - r(R_1) - 1 + |R_1| = l(R_2) - l(R_1), \end{aligned} \tag{6.16}$$

with equality only if $i < j$. □

Proposition 6.1.22. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a horizontal-strip with $l(R_1) < l(R_3)$ and $R_1 \not\leftrightarrow R_3$. Let $\boldsymbol{\mu} = (S_1, \ldots, S_n)$ be a horizontal-strip and $\varphi : \Pi(\boldsymbol{\lambda}) \xrightarrow{\sim} \Pi(\boldsymbol{\mu})$ such that $\varphi_1 < \varphi_3$, $l(S_{\varphi_1}) < l(S_{\varphi_3})$, and $S_{\varphi_1} \not\leftrightarrow S_{\varphi_3}$. Then if $l(R_2) > l(R_1)$ and $R_1 \not\leftrightarrow R_2$, then $l(S_{\varphi_2}) > l(S_{\varphi_1})$. Similarly, if $l(R_3) > l(R_2)$ and $R_2 \not\leftrightarrow R_3$, then $l(S_{\varphi_3}) > l(S_{\varphi_2})$.

Remark 6.1.23. Informally, Proposition 6.1.22 describes the extent to which rows R_1 and R_3 with $l(R_1) < l(R_3)$ and $R_1 \not\leftrightarrow R_3$, given the weighted graph data, determine the relative horizontal position of another row R_2.

Proof of Proposition 6.1.22. We first suppose for a contradiction that $l(R_2) > l(R_1)$ and $R_1 \not\leftrightarrow R_2$, but $l(S_{\varphi_2}) \leq l(S_{\varphi_1})$. Because $l(R_2) > l(R_1)$ and $R_1 \not\leftrightarrow R_2$, we have by Corollary 6.1.5, Part 1, that $r(R_2) > r(R_1)$, and by Part 2, that $M_{1,2}(\boldsymbol{\lambda}) < \min\{|R_1|, |R_2|\}$. Therefore, we must have $M_{\varphi_1,\varphi_2}(\boldsymbol{\mu}) < \min\{|S_{\varphi_1}|, |S_{\varphi_2}|\}$ and by Proposition 6.1.4, Part 2, we have that $l(S_{\varphi_2}) < l(S_{\varphi_1}) < l(S_{\varphi_3})$ and $r(S_{\varphi_2}) < r(S_{\varphi_1}) < r(S_{\varphi_3})$. In particular, we have

$M_{\varphi_2,\varphi_3}(\boldsymbol{\mu}) < \min\{|S_{\varphi_2}|, |S_{\varphi_3}|\}$, so $M_{2,3}(\boldsymbol{\lambda}) < \min\{|R_2|, |R_3|\}$ and $l(R_2) \neq l(R_3)$.

Now if $l(R_2) < l(R_3)$, then by (6.4) we have

$$M_{2,3}(\boldsymbol{\lambda}) = r(R_2) - l(R_3) + 1 > r(R_1) - l(R_3) + 1 = M_{1,3}(\boldsymbol{\lambda}), \tag{6.17}$$

but if $r(S_{\varphi_2}) < l(S_{\varphi_3}) - 1$, then $M_{\varphi_2,\varphi_3}(\boldsymbol{\mu}) = 0 \leq M_{\varphi_1,\varphi_3}(\boldsymbol{\mu})$, and if $r(S_{\varphi_2}) \geq l(S_{\varphi_3}) - 1$, then

$$M_{\varphi_2,\varphi_3}(\boldsymbol{\mu}) = r(S_{\varphi_2}) - l(S_{\varphi_3}) + 1 + \chi(\varphi_2 > \varphi_3) \leq r(S_{\varphi_1}) - l(S_{\varphi_3}) + 1 = M_{\varphi_1,\varphi_3}(\boldsymbol{\mu}), \tag{6.18}$$

a contradiction in either case. Similarly, if $l(R_2) > l(R_3)$, then by (6.4) we have

$$M_{2,3}(\boldsymbol{\lambda}) = r(R_3) - l(R_2) + 2 > r(R_1) - l(R_2) + 1 = M_{1,2}(\boldsymbol{\lambda}), \tag{6.19}$$

but if $r(S_{\varphi_2}) < l(S_{\varphi_3}) - 1$, then $M_{\varphi_2,\varphi_3}(\boldsymbol{\mu}) = 0 \leq M_{\varphi_1,\varphi_2}(\boldsymbol{\mu})$, and if $r(S_{\varphi_2}) \geq l(S_{\varphi_3}) - 1$, then

$$M_{\varphi_2,\varphi_3}(\boldsymbol{\mu}) = r(S_{\varphi_2}) - l(S_{\varphi_3}) + 1 + \chi(\varphi_2 > \varphi_3) \leq r(S_{\varphi_2}) - l(S_{\varphi_3}) + 1 = M_{\varphi_1,\varphi_2}(\boldsymbol{\mu}), \tag{6.20}$$

a contradiction in either case. This proves the first claim and the second follows by rotating. □

Proposition 6.1.24. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a minimal noncommuting path with $l(R_n) > \cdots > l(R_1)$. Let $\boldsymbol{\mu} = (S_1, \ldots, S_n)$ be a horizontal-strip, $\varphi : \Pi(\boldsymbol{\lambda}) \xrightarrow{\sim} \Pi(\boldsymbol{\mu})$, and let i be such that $l(S_i)$ is minimal and j be such that $l(S_j)$ is maximal. Then

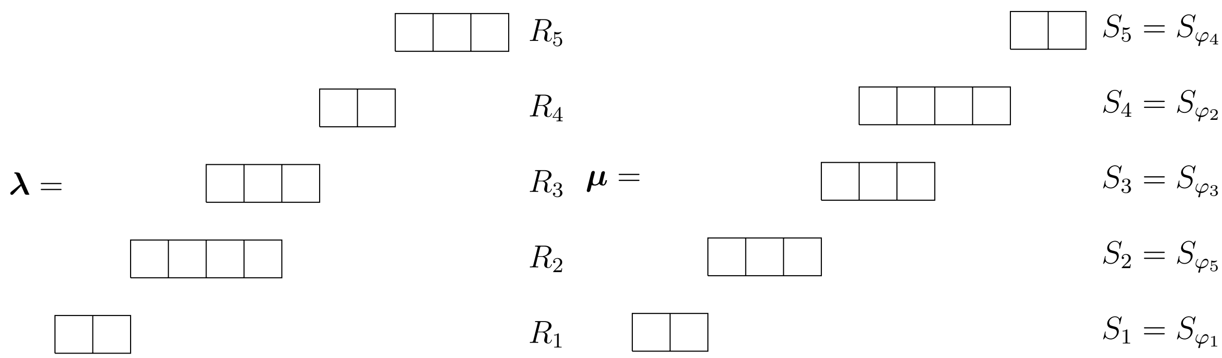
$$r(S_j) - l(S_i) \geq r(R_n) - l(R_1) \text{ with equality only if } i = 1 \text{ and } j = n. \tag{6.21}$$

Example 6.1.25. Proposition 6.1.24 describes a situation like the one in Figure 6.7. Informally, when the rows of $\boldsymbol{\lambda}$ move to the right, we could have $M_{t,t+1}(\boldsymbol{\lambda}) = 0$ and the corresponding rows could be permuted in $\boldsymbol{\mu}$. This is a crucial example to keep in mind. We might not have $\ell(S_{\varphi_{t+1}}) > \ell(S_{\varphi_t})$ for every t, but we can still deduce the leftmost possible position of the rightmost row S_j.

Proof of Proposition 6.1.24. Note that we cannot directly apply Proposition 6.1.20 because we do not know that $l(S_{\varphi_{t+1}}) > l(S_{\varphi_t})$ for every t. Instead, we will reorder the rows of $\boldsymbol{\mu}$ and compute directly. By Corollary 6.1.5, Part 2, we have $M_{t,t+1}(\boldsymbol{\lambda}) < \min\{|R_t|, |R_{t+1}|\}$, so $M_{\varphi_t,\varphi_{t+1}}(\boldsymbol{\mu}) < \min\{|S_{\varphi_t}|, |S_{\varphi_{t+1}}|\}$ and in particular the $l(S_{\varphi_t})$ are distinct. Let σ be the permutation that sorts the rows of $\boldsymbol{\mu}$ so that

$$l(S_i) = l(S_{\sigma_1}) < l(S_{\sigma_2}) < \cdots < l(S_{\sigma_{n-1}}) < l(S_{\sigma_n}) = l(S_j). \tag{6.22}$$

Figure 6.7: A minimal noncommuting path $\boldsymbol{\lambda} = (R_1, \dots, R_5)$ with $l(R_5) > \cdots > l(R_1)$ and a horizontal-strip $\boldsymbol{\mu} = (S_1, \dots, S_5)$ with $\Pi(\boldsymbol{\lambda}) \cong \Pi(\boldsymbol{\mu})$



By Proposition 6.1.17, Part 1, we have $M_{t,t'}(\boldsymbol{\lambda}) = 0$ if $|t' - t| \geq 2$. Now by (6.5), we have

$$\begin{aligned} r(S_j) - l(S_i) &= l(S_{\sigma_n}) - l(S_{\sigma_1}) + |S_{\sigma_n}| - 1 \qquad (6.23) \\ &= l(S_{\sigma_{n-1}}) - l(S_{\sigma_1}) + |S_{\sigma_n}| + |S_{\sigma_{n-1}}| - M_{\sigma_{n-1},\sigma_n}(\boldsymbol{\mu}) - 1 + \chi(\sigma_{n-1} > \sigma_n) \\ \cdots &= \sum_{t=1}^{n} |S_{\sigma_t}| - M(\boldsymbol{\mu}) - 1 + \sum_{t=1}^{n-1} \chi(\sigma_t > \sigma_{t+1}) \\ &\geq \sum_{t=1}^{n} |R_t| - M(\boldsymbol{\lambda}) - 1 \\ \cdots &= l(R_{n-1}) - l(R_1) + |R_n| + |R_{n-1}| - M_{n-1,n}(\boldsymbol{\lambda}) - 1 \\ &= r(S_n) - l(S_1) + |R_n| - 1 = r(R_n) - l(R_1), \end{aligned}$$

with equality only if $i = \sigma_1 < \cdots < \sigma_n = j$, so $i = 1$ and $j = n$. □

We now describe the structure of a minimal noncommuting path $(R_1, \dots, R_n)$, where $l(R_1) < l(R_n)$ and $R_1 \not\leftrightarrow R_n$. Informally, such a minimal noncommuting path must look loosely like one of the examples in Figure 6.4.

Lemma 6.1.26. Let $\boldsymbol{\lambda} = (R_1, \dots, R_n)$ be a minimal noncommuting path with $n \geq 4$, $l(R_1) < l(R_n)$, and $R_1 \not\leftrightarrow R_n$.

1. Suppose that there is no $i \geq 3$ for which $R_1 \precsim\!\!\!\!/\; R_i$ and that there is no $j \leq n-2$ for which $R_n \precsim\!\!\!\!/\; R_j$. Then one of the following holds.
    - We have $l(R_2) > l(R_1)$, $l(R_{n-1}) < \cdots < l(R_2)$, and $l(R_n) > l(R_{n-1})$.
    - We have $l(R_{n-1}) > \cdots > l(R_1)$, $R_t \prec R_n$ for $2 \leq t \leq n-2$, and $l(R_n) < l(R_{n-1})$.

- We have $l(R_2) < l(R_1)$, $l(R_n) > \cdots > l(R_2)$, and $R_t \prec R_1$ for $3 \leq t \leq n-1$.

2. Now suppose that $R_1 \precnsim R_i$ for some minimal $i \geq 3$, and that in fact $i = n-1$. Then $l(R_n) > l(R_{n-1})$ and one of the following holds.
   - We have $l(R_{i-1}) < \cdots < l(R_1)$ and $l(R_i) > l(R_{i-1})$.
   - We have $l(R_{i-1}) > \cdots > l(R_1)$, $R_t \prec R_n$ for $2 \leq t \leq i-1$, and $l(R_i) < l(R_{i-1})$.
   - We have $n = 4$, $l(R_2) > l(R_1)$, $l(R_3) < l(R_2)$, and $R_4 \precnsim R_2$.
3. Now suppose that $R_1 \precnsim R_i$ for some minimal $i \geq 3$, and that $i \leq n-2$. Then $R_n \precnsim R_i$, so $R_n \precnsim R_j$ for some maximal $i \leq j \leq n-2$. Additionally, one of the following holds.
   - We have $l(R_{i-1}) < \cdots < l(R_1)$ and $l(R_i) > l(R_{i-1})$.
   - We have $i = j = 3$, $l(R_2) > l(R_1)$, $R_n \precnsim R_2$, and $l(R_3) < l(R_2)$.

   Similarly, one of the following holds.
   - We have $l(R_n) < \cdots < l(R_{j+1})$ and $l(R_{j+1}) > l(R_j)$.
   - We have $i = j = n-2$, $l(R_n) > l(R_{n-1})$, $R_1 \precnsim R_{n-1}$, and $l(R_{n-1}) < l(R_{n-2})$.

   Finally, we must have either $j = i$, or $j = i+1$ and $l(R_j) < l(R_i)$.

Remark 6.1.27. If the hypothesis of (1) does not hold, then $R_1 \precnsim R_i$ for some $i \geq 3$ or $R_n \precnsim R_j$ for some $j \leq n-2$. By rotating, we may assume that $R_1 \precnsim R_i$ for some $i \geq 3$, and by Corollary 6.1.5, Part 2, we cannot have $R_1 \prec R_n$, so $i \leq n-1$. Therefore (2) and (3) cover all of the cases we will need.

Proof of Lemma 6.1.26. Note that because $n \geq 4$, we must have $R_2 \leftrightarrow R_n$ by minimality.

Suppose that there is no $i \geq 3$ for which $R_1 \precnsim R_i$ and that there is no $j \leq n-2$ for which $R_n \precnsim R_j$. Additionally, suppose that $l(R_2) > l(R_1)$. Because $l(R_1) < l(R_n)$, by Proposition 6.1.17, Part 2, we must have $R_2 \prec R_n$ or $R_n \prec R_2$, so by our hypothesis we must have $R_2 \prec R_n$ and by Proposition 6.1.18, Part 2, that $R_t \prec R_n$ for $2 \leq t \leq n-2$. Note that if $R_2 \prec R_k$ for any $4 \leq k \leq n-1$, then we would have $R_1 \prec R_k$ by Proposition 6.1.18, Part 2, and because $n \geq 5$, $R_1 \precnsim R_k$ by Proposition 6.1.18 Part 1, contradicting our hypothesis. Therefore, by Proposition 6.1.18, Part 3, we have either $l(R_{n-1}) < \cdots < l(R_2)$ and $l(R_n) > l(R_{n-1})$ and the first possibility holds, or we have $l(R_{n-1}) > \cdots > l(R_2) > l(R_1)$ and $l(R_n) < l(R_{n-1})$ and the second possibility holds. Now suppose that $l(R_2) < l(R_1)$. If $l(R_n) > l(R_{n-1})$, then by rotating we can reduce to the case where $l(R_2) > l(R_1)$ and it follows that the third possibility holds, so it remains to consider the case where $l(R_n) < l(R_{n-1})$. Because $l(R_{n-1}) > l(R_n) > r(R_2) \geq l(R_2)$, we have $R_2 \leftrightarrow R_{n-1}$ and $n \geq 5$, so Proposition 6.1.18, Part 1 applies. Also, we must have $l(R_{t+1}) > l(R_t)$ for some minimal $2 \leq t \leq n-2$. By Proposition 6.1.17, Part 2, we must have $R_{t-1} \prec R_{t+1}$ or $R_{t+1} \prec R_{t-1}$. However, by Proposition 6.1.18, Part 2, if $R_{t-1} \prec R_{t+1}$, then $R_1 \precnsim R_{t+1}$, and if $R_{t+1} \prec R_{t-1}$, then

$R_n \precsim R_{t-1}$, contradicting our hypothesis in both cases.

Now suppose that $R_1 \precsim R_i$ for some minimal $i \geq 3$, and that in fact $i = n-1$. Then $l(R_{n-1}) \leq l(R_1) < l(R_n)$ by Proposition 6.1.13. By Proposition 6.1.18, Part 2, we must have either $l(R_{i-1}) < \cdots < l(R_1)$ and $l(R_i) > l(R_{i-1})$ and the first possibility holds, or we have $l(R_{i-1}) > \cdots > l(R_1) \geq l(R_i)$ and $l(R_i) < l(R_{i-1})$. Then because $l(R_n) > l(R_1)$, by Proposition 6.1.17, Part 2, we must have either $R_2 \prec R_n$, in which case we have $R_t \prec R_n$ for $2 \leq t \leq n-2 = i-1$ by Proposition 6.1.18, Part 2, and the second possibility holds, or we have $R_n \precsim R_2$. In this case, we have $R_i \prec R_2$ and $R_2 \prec R_i$ by Proposition 6.1.18, Part 2, so by Proposition 6.1.18, Part 1, we must have $n = 4$ and the third possibility holds.

Now suppose that $R_1 \precsim R_i$ for some minimal $i \geq 3$, and that $i \leq n-2$, so in particular, $n \geq 5$ and Proposition 6.1.18, Part 1 applies. Then because $R_1 \not\leftrightarrow R_n$, by Proposition 6.1.17, Part 3, we have $R_n \precsim R_i$, so $R_n \precsim R_j$ for some maximal $i \leq j \leq n-2$. By Proposition 6.1.18, Part 3 we must have either $l(R_{i-1}) < \cdots < l(R_1)$ and $l(R_i) > l(R_{i-1})$, in which case the first possibility holds, or we have $l(R_{i-1}) > \cdots > l(R_1)$ and $l(R_i) < l(R_{i-1})$. But in this case, because $l(R_n) > l(R_1)$, by Proposition 6.1.17, Part 2, we have $R_2 \prec R_n$ or $R_n \prec R_2$. If $R_2 \prec R_n$, then by Proposition 6.1.18, Part 2, we would have $R_j \prec R_n$, which is impossible by definition of j, so we must have $R_n \precsim R_2$. If $j \geq 4$, then we would have $R_j \prec R_2$ by Proposition 6.1.18, Part 2, but because $R_n \prec R_j$ and $R_1 \not\leftrightarrow R_n$, by Proposition 6.1.17, Part 3, we have $R_1 \prec R_j$ and $R_2 \prec R_j$ by Proposition 6.1.18, Part 2 again, which is impossible by Proposition 6.1.18, Part 1. Therefore, we must have $i = j = 3$ and the second possibility holds. This proves the first claim and the second follows by rotating. Finally, if $j \geq i+2$, then by Proposition 6.1.18, Part 2, we would have $R_i \prec R_j$ and $R_j \prec R_i$, which is impossible by Proposition 6.1.18, Part 1, so we must have either $j = i$ or $j = i+1$. If $j = i+1$, then by Proposition 6.1.18, Part 2, we have $R_{i-1} \prec R_j$ and now $l(R_j) \leq l(R_{i-1}) < l(R_i)$ by Proposition 6.1.13. This completes the proof. □

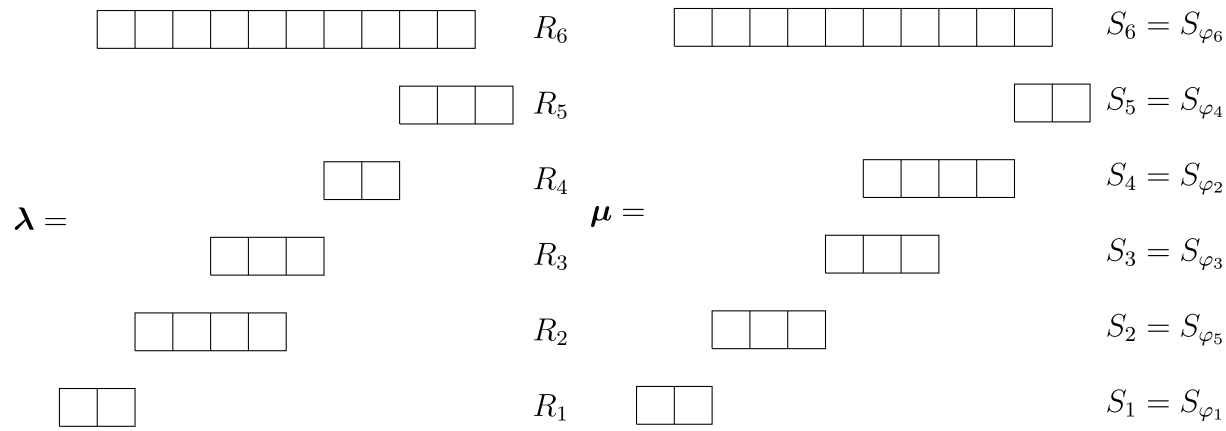
We now describe the image of a minimal noncommuting path under an isomorphism of weighted graphs.

Lemma 6.1.28. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a minimal noncommuting path with $l(R_1) < l(R_n)$ and $R_1 \not\leftrightarrow R_n$. Let $\boldsymbol{\mu} = (S_1, \ldots, S_n)$ and $\varphi : \Pi(\boldsymbol{\lambda}) \xrightarrow{\sim} \Pi(\boldsymbol{\mu})$ satisfy $\varphi_1 = 1$, $l(S_1) < l(S_{\varphi_n})$, and $S_1 \not\leftrightarrow S_{\varphi_n}$. Then $\varphi_n = n$.

Example 6.1.29. Informally, Lemma 6.1.28 describes a situation like the one in Figure 6.8. The conditions $l(R_1) < l(R_6)$, $R_1 \not\leftrightarrow R_6$, $l(S_1) < l(S_{\varphi_6})$, and $S_1 \not\leftrightarrow S_{\varphi_6}$ fix the horizontal positions of R_1, R_6, S_1, and S_{φ_6}. Then if we have a minimal noncommuting path in $\boldsymbol{\lambda}$ from R_1 to R_6, the row S_{φ_6} must be above the other rows in $\boldsymbol{\mu}$. Note that the intermediate rows can be permuted as in this example.

Proof of Lemma 6.1.28. By translating all rows, we may assume without loss of generality that $l(R_1) = l(S_1)$, and then by (6.5) we have $l(R_n) = l(S_{\varphi_n})$ as well. The idea is to repeatedly apply Proposition 6.1.20 and Proposition 6.1.24 to write the inequality $l(S_{\varphi_n}) \geq l(R_n)$,

Figure 6.8: The image of a minimal noncommuting path



for which equality holds only if $\varphi_n = n$. Because equality does indeed hold, we will conclude that $\varphi_n = n$.

Case 0: We have $n = 3$.
By Proposition 6.1.22, if $l(R_2) > l(R_1)$, then $l(S_{\varphi_2}) > l(S_1)$, so by Proposition 6.1.20 we have $l(S_{\varphi_2}) \geq l(R_2)$. By Proposition 6.1.22 again, if $l(R_3) > l(R_2)$, then $l(S_{\varphi_3}) > l(S_{\varphi_2})$, so by Proposition 6.1.20 we have $l(S_{\varphi_3}) \geq l(R_3)$ with equality only if $\varphi_3 > \varphi_2$. Because equality does indeed hold, we must have $\varphi_3 > \varphi_2$ and therefore $\varphi_3 = 3$.

We may now assume that $n \geq 4$, so it remains to consider the several possibilities outlined in Lemma 6.1.26. Cases 1a and 1b illustrate the main ideas.

Case 1: There is no $i \geq 3$ for which $R_1 \precnsim R_i$ and there is no $j \leq n-2$ for which $R_n \precnsim R_j$.

Case 1a: We have $l(R_2) > l(R_1)$, $l(R_{n-1}) < \cdots < l(R_2)$, and $l(R_n) > l(R_{n-1})$.
By Proposition 6.1.22, we have $l(S_{\varphi_2}) > l(S_1)$, so by Proposition 6.1.20 we have $l(S_{\varphi_2}) \geq l(R_2)$. By Proposition 6.1.20 again, we have $l(S_{\varphi_{n-1}}) \geq l(R_{n-1})$ with equality only if $\varphi_{n-1} > \cdots > \varphi_2$. By Proposition 6.1.22, we have $l(S_{\varphi_n}) > l(S_{\varphi_{n-1}})$, so by Proposition 6.1.20 we have $l(S_{\varphi_n}) \geq l(R_n)$ with equality only if we also have $\varphi_n > \varphi_{n-1}$. Because equality does indeed hold, we must have $\varphi_n > \varphi_{n-1} > \cdots > \varphi_2$, so $\varphi_n = n$.

Case 1b: We have $l(R_{n-1}) > \cdots > l(R_1)$, $R_t \prec R_n$ for $2 \leq t \leq n-2$, and $l(R_n) < l(R_{n-1})$.
As in Case 1a, by Proposition 6.1.22, we have $l(S_{\varphi_2}) > l(S_1)$, so by Proposition 6.1.20

we have $l(S_{\varphi_2}) \geq l(R_2)$. However, we must now be careful because we need not have $l(S_{\varphi_{n-1}}) > \cdots > l(S_{\varphi_2})$, so we take a slightly different approach. Let $k \in \{\varphi_t : 2 \leq t \leq n-1\}$ be such that $l(S_k)$ is maximal. By Proposition 6.1.24, we have $r(S_k) \geq r(R_{n-1})$ with equality only if $k = n-1$. If $R_{n-1} \not\prec R_n$, then we must have $k = \varphi_{n-1}$ because $S_{\varphi_t} \prec S_{\varphi_n}$ for every $2 \leq t \leq n-1$ except for S_k. Now by Proposition 6.1.20, we have $l(S_{\varphi_n}) \geq l(R_n)$ with equality only if $\varphi_n > \varphi_{n-1} = k = n-1$. Because equality does indeed hold, we must have $\varphi_n = n$. On the other hand, if $R_{n-1} \prec R_n$, then $S_k \prec S_{\varphi_n}$ and by Corollary 6.1.5, Part 2, we have $r(S_{\varphi_n}) \geq r(S_k) - 1 \geq r(R_{n-1}) - 1 = r(R_n)$ with equality only if $\varphi_n > k = n-1$. Because equality does indeed hold, we must have $\varphi_n = n$.

Case 1c: We have $l(R_2) < l(R_1)$, $l(R_n) > \cdots > l(R_2)$, and $R_t \prec R_1$ for $3 \leq t \leq n-1$.

By rotating, the conclusion follows from Case 1b.

We now assume that $R_1 \precnsim R_i$ for some minimal $i \geq 3$ or $R_n \precnsim R_j$ for some maximal $j \leq n-2$. By rotating, we may assume that $R_1 \precnsim R_i$ for some minimal $i \geq 3$. Note that by Corollary 6.1.5, Part 2, we cannot have $R_1 \prec R_n$, so we have $i \leq n-1$. It remains to consider the cases where $i = n-1$ and where $i \leq n-2$.

Case 2: We have $R_1 \precnsim R_i$ for some minimal $i \geq 3$, and in fact $i = n-1$.

Case 2a: We have $l(R_{i-1}) < \cdots < l(R_1)$ and $l(R_i) > l(R_{i-1})$.

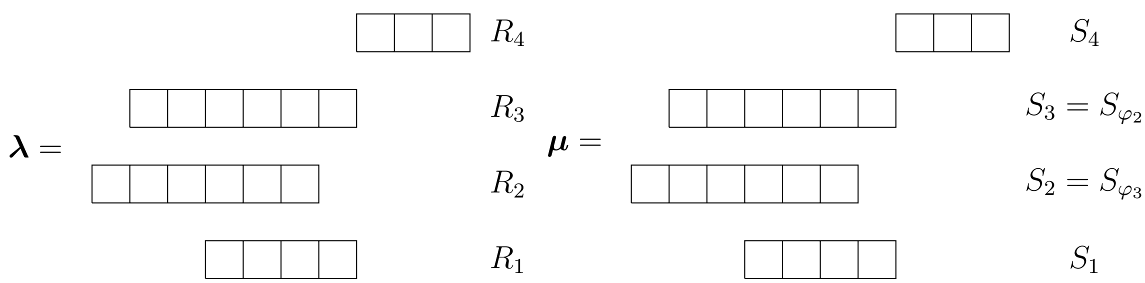
By Proposition 6.1.20, we have $l(S_{\varphi_{i-1}}) \geq l(R_{i-1})$ with equality only if $\varphi_{i-1} > \cdots > \varphi_2$. If $l(S_{\varphi_i}) > l(S_{\varphi_{i-1}})$, then by Proposition 6.1.20 again, we have $l(S_{\varphi_n}) \geq l(R_n)$ with equality only if $\varphi_n > \varphi_i > \varphi_{i-1} > \cdots > \varphi_2$. Because equality does indeed hold, we must have $\varphi_n = n$.

Now suppose that $l(S_{\varphi_i}) \leq l(S_{\varphi_{i-1}})$. We will show that this is narrowly possible, but $\boldsymbol{\mu}$ will be so specifically determined that we will be able to reduce to the previous case. By Corollary 6.1.5, Part 2, we have $M_{i-1,i}(\boldsymbol{\lambda}) < \min\{|R_{i-1}|, |R_i|\}$, so we must have $r(S_{\varphi_i}) < r(S_{\varphi_{i-1}})$. Again, by Corollary 6.1.5, Part 2, we have $M_{t,t+1}(\boldsymbol{\lambda}) > 0$ for $1 \leq t \leq i-2$, and therefore $M_{\varphi_t,\varphi_{t+1}}(\boldsymbol{\mu}) > 0$ and the rows $S_1, S_{\varphi_2}, \ldots, S_{\varphi_{i-1}}$ must overlap; to be precise, $S_1 \cup \cdots \cup S_{\varphi_{i-1}}$ must be a row. Now because $r(S_1) < r(S_{\varphi_n})$ and $M_{\varphi_t,\varphi_n}(\boldsymbol{\mu}) = 0$ for $2 \leq t \leq i-1$, we must have $r(S_{\varphi_{i-1}}) < l(S_{\varphi_n})$. Because $S_1 \prec S_{\varphi_i}$, we now have

$$r(S_1) \leq r(S_{\varphi_i}) + 1 \leq r(S_{\varphi_{i-1}}) \leq l(S_{\varphi_n}) - 1 \leq r(S_1), \tag{6.24}$$

so we have equality everywhere and in particular, $M_{1,i-1}(\boldsymbol{\lambda}) = \min\{|R_1|, |R_{i-1}|\} > 0$. Because $M_{1,t}(\boldsymbol{\lambda}) = 0$ for $3 \leq t \leq i-1$ by Proposition 6.1.17, Part 1, this means that in fact $i = 3$. Now by (6.24), we have $M_{1,4}(\boldsymbol{\lambda}) = M_{3,4}(\boldsymbol{\lambda}) = 0$, so $r(R_1) = r(R_3) = l(R_4) - 1$ by Corollary 6.1.5, Part 2, and $l(R_3) < l(R_1)$ because $R_1 \precnsim R_3$. Because $M_{1,2}(\boldsymbol{\lambda}) = \min\{|R_1|, |R_2|\}$, we either have $R_1 \prec R_2$ or $R_2 \prec R_1$, but because $l(R_2) < l(R_3) < l(R_1)$, we cannot have $R_2 \prec R_1$ by Proposition 6.1.13, Part 2, so $R_1 \prec R_2$ and $r(R_2) = r(R_1) - 1$ by Corollary 6.1.5, Part 2. But now $|R_3| - 1 = M_{2,3}(\boldsymbol{\lambda}) = M_{\varphi_2,\varphi_3}(\boldsymbol{\mu}) = |S_{\varphi_2}| - 1 = |R_2| - 1$, so $|R_2| = |R_3|$.

Figure 6.9: An example of Case 2a where $l(S_{\varphi_i}) \leq l(S_{\varphi_{i-1}})$



Moreover, we have $R_1 \prec R_2$, $R_1 \prec R_3$, and $M_{2,4}(\boldsymbol{\lambda}) = M_{3,4}(\boldsymbol{\lambda}) = 0$ so in fact these two vertices are equivalent in $\Pi(\boldsymbol{\lambda})$, and we can swap the roles of these two vertices to reduce to the case where $l(S_{\varphi_i}) > l(S_{\varphi_{i-1}})$. This possibility is illustrated in Figure 6.9. It is barely possible to have $l(S_{\varphi_3}) \leq l(S_{\varphi_2})$, but this requires S_{φ_2} and S_{φ_3} to play equivalent roles in $\Pi(\boldsymbol{\mu})$, so by instead considering the identity isomorphism $\tilde{\varphi} : \Pi(\boldsymbol{\lambda}) \xrightarrow{\sim} \Pi(\boldsymbol{\mu})$, we can reduce to the previous case.

Case 2b: We have $l(R_{i-1}) > \cdots > l(R_1)$, $R_t \prec R_n$ for $2 \leq t \leq i-1$, and $l(R_i) < l(R_{i-1})$.
Let $k \in \{\varphi_t : 2 \leq t \leq i-1\}$ be such that $l(S_k)$ is maximal. By Proposition 6.1.24, we have $r(S_k) \geq r(R_{i-1})$ with equality only if $k = i-1$. If $R_{i-1} \not\prec R_i$, then we must have $k = \varphi_{i-1}$ because $S_{\varphi_t} \prec S_{\varphi_i}$ for every $2 \leq t \leq i-1$ except for S_k. Now by Proposition 6.1.20, we have $l(S_{\varphi_n}) \geq l(R_n)$ with equality only if $\varphi_n > \varphi_i > \varphi_{i-1} = k = i-1$. Because equality does indeed hold, we must have $\varphi_n = n$. On the other hand, if $R_{i-1} \prec R_i$, then $S_k \prec S_{\varphi_i}$, and by Corollary 6.1.5, Part 2, we have $r(S_{\varphi_i}) \geq r(S_k) - 1 \geq r(R_{i-1}) - 1 = r(R_i)$ with equality only if $\varphi_i > k = i-1$. Then by Proposition 6.1.20 again, we have $l(S_{\varphi_n}) \geq l(R_n)$ with equality only if $\varphi_n > \varphi_i$. Because equality does indeed hold, we must have $\varphi_n = n$.

Case 2c: We have $n = 4$, $l(R_2) > l(R_1)$, $l(R_3) < l(R_2)$, and $R_4 \precnsim R_2$.
By Proposition 6.1.22, we have $l(S_{\varphi_2}) > l(S_1)$ and $l(S_{\varphi_4}) > l(S_{\varphi_3})$, so by Proposition 6.1.20, we have $l(S_{\varphi_4}) \geq l(R_4)$ with equality only if $\varphi_4 > \varphi_3 > \varphi_2$. Because equality does indeed hold, we must have $\varphi_4 = 4$.

Case 3: We have $R_1 \precnsim R_i$ for some minimal $3 \leq i \leq n-2$.
We first show that $l(S_{\varphi_i}) \geq l(R_i)$ with equality only if $\varphi_i > \varphi_t$ for all $t < i$. If $i = j = 3$, $l(R_2) > l(R_1)$, $R_n \precnsim R_2$, and $l(R_3) < l(R_2)$, then we have $l(S_{\varphi_2}) > l(S_1)$ by Proposition 6.1.22 and the result follows as before from Proposition 6.1.20. Now suppose that $l(R_{i-1}) < \cdots < l(R_1)$ and $l(R_i) > l(R_{i-1})$. We show that $l(S_{\varphi_i}) > l(S_{\varphi_{i-1}})$. Suppose that $l(S_{\varphi_i}) \leq l(S_{\varphi_{i-1}})$. By Corollary 6.1.5, Part 2, we have $M_{i-1,i}(\boldsymbol{\lambda}) < \min\{|R_{i-1}|, |R_i|\}$, so we must have $r(S_{\varphi_n}) \leq r(S_{\varphi_i}) + 1 \leq r(S_{\varphi_{i-1}})$. Because $M_{t,n}(\boldsymbol{\lambda}) = 0$ for $2 \leq t \leq i-1$ and $M_{t,t+1}(\boldsymbol{\lambda}) > 0$

for $1 \leq t \leq i-2$ by Corollary 6.1.5, Part 2, we must have $r(S_{\varphi_{i-1}}) < l(S_{\varphi_n}) \leq r(S_{\varphi_n})$, a contradiction, so indeed $l(S_{\varphi_i}) > l(S_{\varphi_{i-1}})$. Now by Proposition 6.1.20, we have $l(S_{\varphi_i}) \geq l(R_i)$ with equality only if $\varphi_i > \varphi_t$ for all $t < i$. Because either $j = i$, or $j = i+1$ and $l(R_j) < l(R_i)$, we in fact have $l(S_{\varphi_j}) \geq l(R_j)$ with equality only if $\varphi_j > \varphi_t$ for all $t < j$. Finally, by rotating and repeating the previous argument, we have $l(S_{\varphi_n}) \geq l(R_n)$ with equality only if $\varphi_n > \varphi_t$ for all $t < n$. Because equality does indeed hold, we must have $\varphi_n = n$. This completes the proof. □

The payoff of all our work so far is the following Corollary.

Corollary 6.1.30. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a horizontal-strip with $l(R_i) < l(R_{i+1})$ and $R_i \not\leftrightarrow R_{i+1}$. Let $\boldsymbol{\mu} = (S_1, \ldots, S_n)$ and $\varphi : \Pi(\boldsymbol{\lambda}) \xrightarrow{\sim} \Pi(\boldsymbol{\mu})$ be such that $S_{\varphi_i} \not\leftrightarrow S_{\varphi_{i+1}}$. Then there exists a good substitute for $(\boldsymbol{\lambda}, \boldsymbol{\mu})$.

Proof. By cycling and rotating, we may assume without loss of generality that $i = 1$, $l(S_{\varphi_1}) < l(S_{\varphi_2})$, and $\varphi_1 = 1$. Let $j = \varphi_2$. By applying Lemma 6.1.16 to $\boldsymbol{\mu}$, either we may replace $\boldsymbol{\mu}$ by a similar horizontal-strip to assume that $j = 2$, in which case we have our good substitute and we are done, or there is a minimal noncommuting path $(S_1 = S_{j_1}, \ldots, S_{j_k} = S_j)$ in $\boldsymbol{\mu}$ from S_1 to S_j. However, in this case, by considering the corresponding rows in $\boldsymbol{\lambda}$, we would have $2 = \varphi_j^{-1} = \varphi_{j_k}^{-1} \geq k \geq 3$ by Lemma 6.1.28, a contradiction. Therefore, there is indeed a good substitute for $(\boldsymbol{\lambda}, \boldsymbol{\mu})$. □

Remark 6.1.31. The hypothesis that $S_{\varphi_i} \not\leftrightarrow S_{\varphi_{i+1}}$ is essential to the proof of Corollary 6.1.30. Note that this hypothesis is not always satisfied, as Figure 6.7 shows an example where $R_1 \not\leftrightarrow R_2$, but $S_{\varphi_1} \leftrightarrow S_{\varphi_2}$. In other words, it is possible for rows not to commute in $\boldsymbol{\lambda}$ but for the corresponding rows to commute in $\boldsymbol{\mu}$.

Our next goal is to extend Corollary 6.1.30 by describing properties of the weighted graph $\Pi(\boldsymbol{\lambda})$ that will force certain rows not to commute.

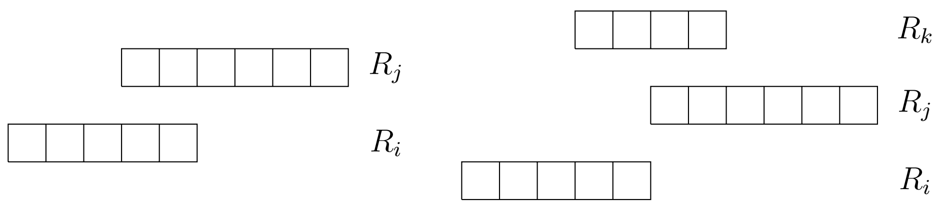
Definition 6.1.32. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a horizontal-strip. A pair of rows (R_i, R_j) of $\boldsymbol{\lambda}$ with $i < j$ and $l(R_i) < l(R_j)$ is *strict* if either

1. $0 < M_{i,j} < \min\{|R_i|, |R_j|\}$, or
2. $M_{i,j} = 0$ and $M_{i,k} + M_{j,k} \geq |R_k| + 1$ for some k.

Example 6.1.33. The two possibilities for strictness are given in Figure 6.10. Note that on the right, we have $M_{i,k} + M_{j,k} = 2 + 3 = |R_k| + 1$. Informally, in the second possibility where $M_{i,j} = 0$, the weighted graph normally would not know about the relationship between R_i and R_j. However, the presence of this row R_k glues the rows R_i and R_j together and means that the weighted graph data forces rows R_i and R_j not to commute.

Remark 6.1.34. Because we define strictness using the weighted graph data, it is preserved under isomorphisms. To be specific, if $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ and $\boldsymbol{\mu} = (S_1, \ldots, S_n)$ are horizontal-strips with $\varphi : \Pi(\boldsymbol{\lambda}) \xrightarrow{\sim} \Pi(\boldsymbol{\mu})$ and the pair (R_i, R_{i+1}) is strict, then by rotating and cycling

Figure 6.10: Examples of strict pairs of rows



to assume that $l(S_{\varphi_i}) < l(S_{\varphi_{i+1}})$ and $\varphi_i < \varphi_{i+1}$, we have that the pair $(S_{\varphi_i}, S_{\varphi_{i+1}})$ is strict as well.

We now present the key property that motivates our definition of a strict pair.

Proposition 6.1.35. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a horizontal-strip and suppose that the pair of rows (R_i, R_j) is strict. Then $R_i \not\leftrightarrow R_j$.

Proof. If $0 < M_{i,j} < \min\{|R_i|, |R_j|\}$, then $R_i \not\leftrightarrow R_j$ by (6.6), so suppose that $M_{i,j} = 0$ and $M_{i,k} + M_{j,k} \geq |R_k| + 1$ for some k. Because $l(R_i) < l(R_j)$ and $M_{i,j} = 0$, we have that $l(R_i) \geq r(R_j) + 1$ and because $M_{i,k}, M_{j,k} \leq |R_k|$, we must have $M_{i,k}, M_{j,k} > 0$ so $l(R_k) \leq r(R_i) + 1 \leq l(R_j) \leq r(R_k) + 1$ by Proposition 6.1.4, Part 1. Now we either have $M_{i,k} = r(R_i) - l(R_k) + 1 + \chi(i > k)$ or $M_{i,k} = \min\{|R_i|, |R_k|\} \leq r(R_i) - l(R_k) + 1 + \chi(i > k)$ and similarly we have $M_{j,k} \leq r(R_k) - l(R_j) + 1 + \chi(k > j)$. Because $i < j$, we have $\chi(i > k) + \chi(k > j) \leq 1$, so

$$|R_k| + 1 \leq M_{i,k} + M_{j,k} \leq |R_k| + r(R_i) - l(R_j) + 1 + \chi(i > k) + \chi(k > j) \leq |R_k| + 1, \quad (6.25)$$

so we must have equality everywhere and in particular, $r(R_j) - l(R_i) + 1 = 0$, so $R_i \not\leftrightarrow R_j$. □

Remark 6.1.36. This proof shows that if $l(R_i) < l(R_j)$ and $M_{i,j} = 0$, then in fact $M_{i,k} + M_{j,k} \leq |R_k| + 1$ for all k, so we could replace the condition $M_{i,k} + M_{j,k} \geq |R_k| + 1$ with the equivalent condition $M_{i,k} + M_{j,k} = |R_k| + 1$.

Corollary 6.1.37. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a horizontal-strip with a pair of adjacent strict rows (R_i, R_{i+1}). Then $\boldsymbol{\lambda}$ is good.

Proof. Let $\boldsymbol{\mu} = (S_1, \ldots, S_n)$ and $\varphi : \Pi(\boldsymbol{\lambda}) \xrightarrow{\sim} \Pi(\boldsymbol{\mu})$. By definition, we have $M_{i,i+1}(\boldsymbol{\lambda}) < \min\{|R_i|, |R_{i+1}|\}$, so $M_{\varphi_i, \varphi_{i+1}}(\boldsymbol{\mu}) < \min\{|S_{\varphi_i}|, |S_{\varphi_{i+1}}|\}$ and $l(S_{\varphi_i}) \neq l(S_{\varphi_{i+1}})$. Therefore, by cycling and rotating, we may assume without loss of generality that $l(S_{\varphi_i}) < l(S_{\varphi_{i+1}})$ and $\varphi_i < \varphi_{i+1}$, so the pair $(S_{\varphi_i}, S_{\varphi_{i+1}})$ is strict. By Proposition 6.1.35, we have $S_{\varphi_i} \not\leftrightarrow S_{\varphi_{i+1}}$, so by Corollary 6.1.30, there exists a good substitute for $(\boldsymbol{\lambda}, \boldsymbol{\mu})$. □

We now investigate some useful properties of strict pairs.

Proposition 6.1.38. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a horizontal-strip with $i < j$, $l(R_i) < l(R_j)$, and $R_i \nleftrightarrow R_j$. Suppose that $R_i \nleftrightarrow R_k$ and $R_j \nleftrightarrow R_k$ for some k with $k < i$ or $k > j$. Then the pair (R_i, R_j) is strict.

Proof. By Corollary 6.1.5, Part 2, we cannot have $M_{i,j} = \min\{|R_i|, |R_j|\}$, and if $0 < M_{i,j} < \min\{|R_i|, |R_j|\}$, then we are done, so suppose that $M_{i,j} = 0$ and therefore $l(R_j) = r(R_i) + 1$ by Corollary 6.1.5, Part 2. Because $R_i \nleftrightarrow R_k$, by Proposition 6.1.4, Part 1 we must have $l(R_k) \le r(R_i) + 1 = l(R_j)$, and because $R_j \nleftrightarrow R_k$, we must have $r(R_k) \ge l(R_j) - 1 = r(R_i)$, so by Corollary 6.1.5, Part 1, we in fact have $l(R_i) < l(R_k) < l(R_j)$. Now by (6.5), we have

$$M_{i,k} + M_{j,k} = r(R_i) - l(R_k) + 1 + \chi(i > k) + r(R_k) - l(R_j) + 1 + \chi(k > j) = |R_k| + 1, \quad (6.26)$$

so the pair (R_i, R_j) is strict. □

Proposition 6.1.39. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a minimal noncommuting path and suppose that for every $1 \le t' \le n-1$, the pair $(R_{t'}, R_{t'+1})$ is not strict. Then we have the following.

1. If $l(R_{t+1}) > l(R_t)$, then $M_{t,t+1} = 0$ and $l(R_{t+1}) = r(R_t) + 1$.

2. If $l(R_{t+1}) > l(R_t)$ and $l(R_t) < l(R_{t-1})$, then $R_{t+1} \precnsim R_{t-1}$.

3. If $l(R_{t+1}) < l(R_t)$ and $l(R_t) > l(R_{t-1})$, then $R_{t-1} \precnsim R_{t+1}$.

Proof.

1. By Corollary 6.1.5, Part 2 we cannot have $M_{t,t+1} = \min\{|R_t|, |R_{t+1}|\}$, and if $0 < M_{t,t+1} < \min\{|R_t|, |R_{t+1}|\}$, then the pair (R_t, R_{t+1}) would be strict, so we must have $M_{t,t+1} = 0$ and $l(R_{t+1}) = r(R_t) + 1$.

2. By the previous part, we must have $M_{t,t+1} = 0$. By Proposition 6.1.17, Part 2, we must have $R_{t-1} \prec R_{t+1}$ or $R_{t+1} \prec R_{t-1}$. However, if $R_{t-1} \prec R_{t+1}$, then because $M_{t-1,t} > 0$ by Corollary 6.1.5, Part 2, we would have $M_{t-1,t} + M_{t-1,t+1} \ge |R_{t-1}| + 1$ and the pair (R_t, R_{t+1}) would be strict, so we must have $R_{t+1} \precnsim R_{t-1}$.

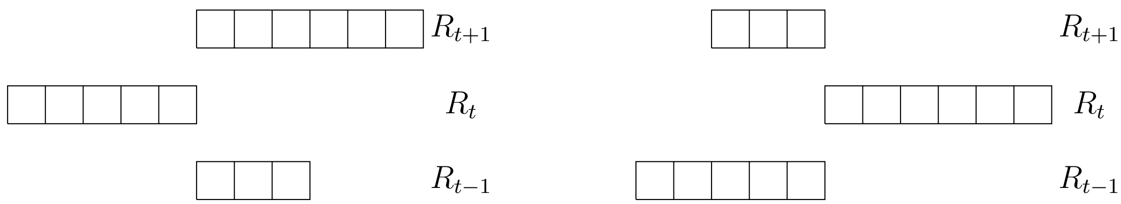
3. By rotating, this follows from the previous part.

□

Example 6.1.40. Figure 6.11 illustrates the contradictions that we deduce in the proofs of Parts 2 and 3 of Proposition 6.1.39. If $R_{t-1} \prec R_{t+1}$ as on the left, then the pair (R_t, R_{t+1}) would be strict, which is how we concluded that $R_{t+1} \precnsim R_{t-1}$. If $R_{t+1} \prec R_{t-1}$ as on the right, then the pair (R_{t-1}, R_t) would be strict, which is how we concluded that $R_{t-1} \precnsim R_{t+1}$.

It will also be beneficial to make the following definition.

Figure 6.11: Forbidden configurations in a minimal noncommuting path with no strict pairs



Definition 6.1.41. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a horizontal-strip. A *strict sequence* of $\boldsymbol{\lambda}$ is a sequence of rows $(R_{j_1}, \ldots, R_{j_k})$ such that $k \geq 2$, $j_1 < \cdots < j_k$, $M_{j_t, j_{t'}} = 0$ for every $1 \leq t < t' \leq k$, and there is some h with $h < j_1$ or $h > j_k$ for which

$$M_{j_t,h} > 0 \text{ for all } 1 \leq t \leq k \text{ and } M_{j_1,h} + \cdots + M_{j_k,h} \geq |R_h| + 1. \tag{6.27}$$

Note that if a pair of rows (R_i, R_j) is a strict sequence, then it meets the second condition of being a strict pair. Conversely, if a pair of rows (R_i, R_j) meets the second condition of being a strict pair, then it is a strict sequence.

Example 6.1.42. In Figure 6.12, we have $M_{t,t'} = 0$ for every $1 \leq t < t' \leq 6$, $M_{t,7} > 0$ for $1 \leq t \leq 6$, and $M_{1,7} + \cdots + M_{6,7} = 2 + 5 + 2 + 4 + 3 + 3 = 19 = |R_7| + 1$, so $(R_1, \ldots, R_6)$ is a strict sequence. Informally, our next Proposition will show that every strict sequence looks very much like this example. Because $M_{t,t'} = 0$ for $1 \leq t < t' \leq 6$, the weighted graph normally would not know about the relationship between these rows. However, the presence of the row R_7 glues these rows together and means that the weighted graph data forces the adjacent rows not to commute.

Remark 6.1.43. Because we define a strict sequence using the weighted graph data, it is preserved under isomorphisms. To be specific, if $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ and $\boldsymbol{\mu} = (S_1, \ldots, S_n)$ are horizontal-strips with $\varphi : \Pi(\boldsymbol{\lambda}) \xrightarrow{\sim} \Pi(\boldsymbol{\mu})$, then if $(R_{j_1}, \ldots, R_{j_k})$ is a strict sequence, we can cycle to assume that $\varphi_h > \varphi_{j_t}$ for every $1 \leq t \leq k$. Because $M_{j_t, j_{t'}}(\boldsymbol{\lambda}) = 0$ for $1 \leq t < t' \leq k$, the integers $l(S_{\varphi_{j_t}})$ for $1 \leq t \leq k$ must be distinct, so we can let $\sigma : \{1, \ldots, k\} \to \{1, \ldots, k\}$ be the permutation that sorts them in increasing order, in other words

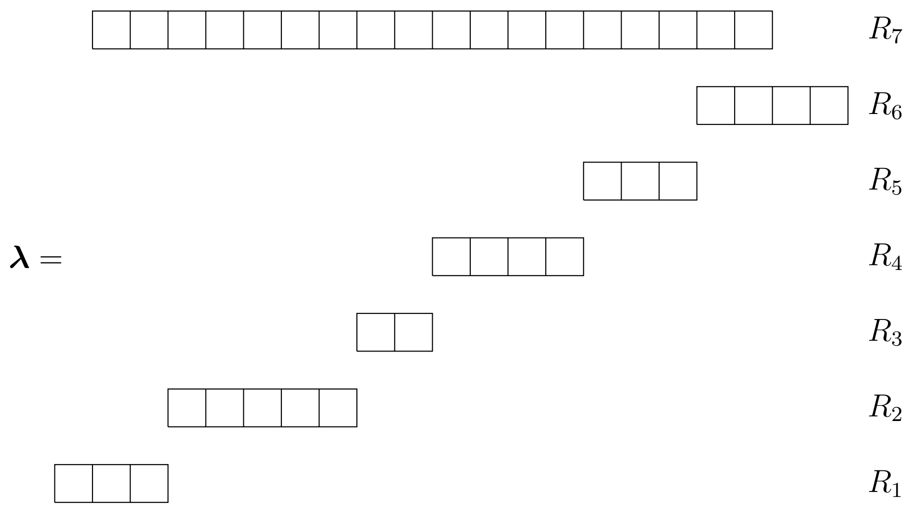
$$l(S_{\varphi_{j_{\sigma_1}}}) < l(S_{\varphi_{j_{\sigma_2}}}) < \cdots < l(S_{\varphi_{j_{\sigma_k}}}). \tag{6.28}$$

Then the sequence $(S_{\varphi_{j_{\sigma_1}}}, \ldots, S_{\varphi_{j_{\sigma_k}}})$ is strict.

Proposition 6.1.44. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a horizontal-strip with a sequence of rows $(R_{j_1}, \ldots, R_{j_k})$ with $k \geq 2$, $j_1 < \cdots < j_k$, $M_{j_t, j_{t'}} = 0$ for every $1 \leq t < t' \leq k$, and there is some h with $h < j_1$ or $h > j_k$ for which $M_{j_t,h} > 0$ for every $1 \leq t \leq k$. Then this sequence is strict if and only if $l(R_{j_{t+1}}) = r(R_{j_t}) + 1$ for every $1 \leq t \leq k - 1$ and

$$l(R_{j_1}) + \chi(j_1 > h) \leq l(R_h) \leq r(R_h) + \chi(h > j_k) \leq r(R_{j_k}). \tag{6.29}$$

Figure 6.12: An example of a strict sequence



Proof. By Proposition 6.1.4 Part 2, because all of the $M_{j_t,j_{t'}}$ are zero, the integers $l(R_{j_t})$ for $1 \le t \le k$ are distinct, so let $\sigma : \{1, \dots, k\} \to \{j_1, \dots, j_k\}$ sort the rows R_{j_t} so that $l(R_{j_t})$ is increasing. Then because the $M_{j_t,j_{t'}}$ are zero we have

$$l(R_{\sigma_1}) < r(R_{\sigma_1})+1+\chi(\sigma_1 > \sigma_2) \le l(R_{\sigma_2}) < r(R_{\sigma_2})+1+\chi(\sigma_2 > \sigma_3) \le \cdots \le l(R_{\sigma_k}). \quad (6.30)$$

Because $M_{\sigma_1,h} > 0$, we must have $l(R_h) \le r(R_{\sigma_1}) + 1 \le l(R_{\sigma_2})$ and because $M_{\sigma_k,h} > 0$, we must have $r(R_h) \ge l(R_{\sigma_k}) - 1 \ge r(R_{\sigma_{k-1}})$. We now have that

$$\begin{aligned}
M_{\sigma_1,h} &\le r(R_{\sigma_1}) - l(R_h) + 1 + \chi(\sigma_1 > h) \le l(R_{\sigma_2}) - l(R_h) + \chi(\sigma_1 > h) - \chi(\sigma_1 > \sigma_2) \quad (6.31)\\
M_{\sigma_2,h} &= |R_{\sigma_2}| = r(R_{\sigma_2}) - l(R_{\sigma_2}) + 1 \le l(R_{\sigma_3}) - l(R_{\sigma_2}) - \chi(\sigma_2 > \sigma_3)\\
&\vdots\\
M_{\sigma_{k-1},h} &= |R_{\sigma_{k-1}}| = r(R_{\sigma_{k-1}}) - l(R_{\sigma_{k-1}}) + 1 \le l(R_{\sigma_k}) - l(R_{\sigma_{k-1}}) - \chi(\sigma_{k-1} > \sigma_k)\\
M_{\sigma_k,h} &\le r(R_h) - l(R_{\sigma_k}) + 1 + \chi(h > \sigma_k).
\end{aligned}$$

Also note that $\chi(\sigma_1 > h) - \chi(\sigma_1 > \sigma_2) - \cdots - \chi(\sigma_{k-1} > \sigma_k) + \chi(h > \sigma_k) \le 1$ with equality only if $\sigma_1 < \cdots < \sigma_k$. Therefore, by summing the above equations, we have

$$\sum_{t=1}^{k} M_{\sigma_t,h} \le r(R_h) - l(R_h) + 1 + 1 = |R_h| + 1, \quad (6.32)$$

so the sequence is strict if and only if we have equality everywhere, meaning that $\sigma_1 < \cdots < \sigma_k$, $l(R_{j_{t+1}}) = r(R_{j_t}) + 1$ for $1 \le t \le k-1$, and (6.29) holds. □

Remark 6.1.45. This proof shows that if $l(R_{j_1}) < \cdots < l(R_{j_k})$ and the $M_{j_t,j_{t'}} = 0$, then in fact $M_{j_1,h} + \cdots + M_{j_k,h} \leq |R_h| + 1$ for all h, so we could replace the condition $M_{j_1,h} + \cdots + M_{j_k,h} \geq |R_h| + 1$ with the equivalent condition $M_{j_1,h} + \cdots + M_{j_k,h} = |R_h| + 1$.

Proposition 6.1.46. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a horizontal-strip with a strict sequence of rows $(R_{j_1}, \ldots, R_{j_k})$. Suppose that there is some $j_t < x < j_{t+1}$ with $l(R_x) = l(R_{j_{t+1}})$ and $R_{j_{t+1}} \precnsim R_x$. Then one of the following holds.

1. The sequence $(R_{j_1}, \ldots, R_{j_t}, R_x)$ is strict.
2. There is a shorter strict sequence of the form $(R_{j_1}, \ldots, R_{j_t}, R_x, R_{j_{t'}}, \ldots, R_{j_k})$ for some $t' \geq t + 2$.
3. There is a strict pair $(R_x, R_{j_{t'+1}})$ for some $t' \geq t + 1$.

Example 6.1.47. Informally, Proposition 6.1.46 describes a situation like the one in Figure 6.13. The sequence $(R_{j_1}, \ldots, R_{j_6})$ is the strict sequence from Example 6.1.42. The rows R_{x_1}, R_{x_2}, and R_{x_3} illustrate the three possibilities described in Proposition 6.1.46. Informally, because R_{x_1} extends past R_{j_6}, the sequence $(R_{j_1}, R_{j_2}, R_{x_1})$ is strict. Because $R_{x_2} = R_{j_3} \cup R_{j_4} \cup R_{j_5}$, it can replace these rows to produce the shorter strict sequence $(R_{j_1}, R_{j_2}, R_{x_2}, R_{j_6})$. Finally, because R_{x_3} ends between the rows R_{j_4} and R_{j_5}, it results in the strict pair (R_{x_3}, R_{j_5}).

Proof of Proposition 6.1.46. By the definition of a strict sequence, there is some h with $h < j_1$ or $h > j_k$ for which $M_{j_t,h} > 0$ for all $1 \leq t \leq k$ and $\sum_{t=1}^{k} M_{j_t,h} \geq |R_h| + 1$, and by cycling we may assume without loss of generality that $h > j_k$, so that $l(R_{j_1}) \leq l(R_h) < r(R_h) + 1 \leq r(R_{j_k})$ by (6.29). Noting that $R_{j_{t+1}} \precnsim R_x$, let $t + 1 \leq t' \leq k$ be maximal such that $R_{j_{t'}} \prec R_x$. If $t' = k$, then we must have $r(R_x) \geq r(R_{j_k}) \geq r(R_h) + 1$ by Proposition 6.1.13 and now the sequence $(R_{j_1}, \ldots, R_{j_t}, R_x)$ is strict by Proposition 6.1.44, so the first possibility holds and we may now assume that $t' \leq k - 1$. By maximality of t', we have that $R_{j_{t'+1}} \not\prec R_x$. If $M_{x,j_{t'+1}} = 0$, then we must have $r(R_x) = r(R_{j_{t'}}) = l(R_{j_{t'+1}}) - 1$ and the sequence $(R_{j_1}, \ldots, R_{j_t}, R_x, R_{j_{t'+1}}, \ldots, R_{j_k})$ is strict by Proposition 6.1.44. Also note that because $R_x \not\prec R_{j_{t+1}}$, we must have $t' > t + 1$ so this strict sequence is indeed shorter and the second possibility holds. Finally, if $M_{x,j_{t'+1}} > 0$, then because $l(R_x) = l(R_{j_{t+1}}) < l(R_{j_{t'+1}})$, we have $R_x \not\prec R_{j_{t'+1}}$, and by maximality of t we have $R_{j_{t'+1}} \not\prec R_x$, so $0 < M_{x,j_{t'+1}} < \min\{|R_x|, |R_{j_{t'+1}}|\}$, the pair $(R_x, R_{j_{t'+1}})$ is strict, and the third possibility holds. □

We now describe the structure of a minimal noncommuting path with no strict pairs.

Proposition 6.1.48. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a minimal noncommuting path with $l(R_1) < l(R_n)$ and $R_1 \not\leftrightarrow R_n$ and suppose that for every $1 \leq t \leq n - 1$, the pair (R_t, R_{t+1}) is not strict. Then one of the following holds.

1. One of the sequences $(R_1, \ldots, R_{n-1})$, $(R_2, \ldots, R_{n-1})$, or $(R_2, \ldots, R_n)$ is strict.
2. We have $n = 4$, $l(R_2) = l(R_4) = r(R_1) - 1 = r(R_3) - 1$, $R_4 \precnsim R_2$, and $R_1 \precnsim R_3$.

Figure 6.13: A strict sequence with a row R_x as in Proposition 6.1.46

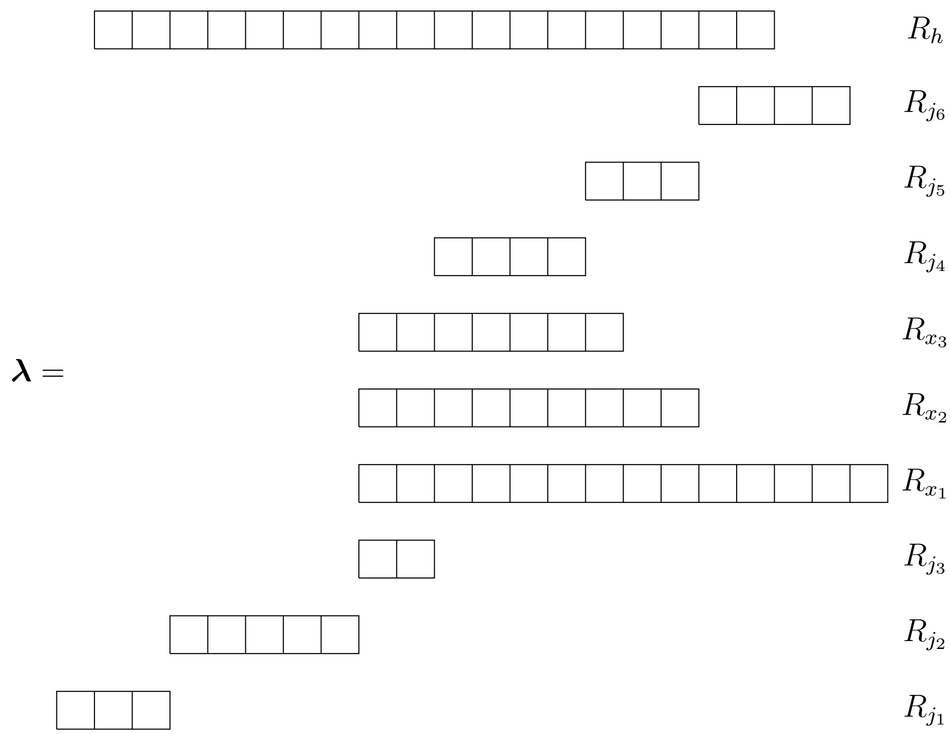
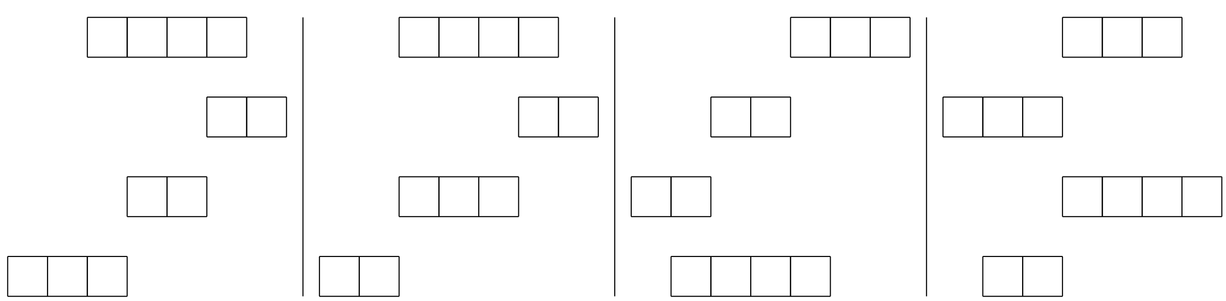


Figure 6.14: Some minimal noncommuting paths with no internal strict pairs



Example 6.1.49. Informally, Proposition 6.1.48 tells us that a minimal noncommuting path with no strict pairs (other than possibly (R_1, R_n)) must look like one of the examples in Figure 6.14.

Proof of Proposition 6.1.48. We consider several cases.

Case 0: We have $n = 3$.

In this case, we have $R_1 \nleftrightarrow R_2$ and $R_2 \nleftrightarrow R_3$. Because $l(R_1) < l(R_3)$, we must have either $l(R_1) < l(R_2)$, in which case the pair (R_1, R_2) is strict by Proposition 6.1.38, or $l(R_2) < l(R_3)$, in which case the pair (R_2, R_3) is strict by Proposition 6.1.38, a contradiction.

We may now assume that $n \geq 4$, so it remains to consider the several possibilities outlined in Lemma 6.1.26.

Case 1: There is no $i \geq 3$ for which $R_1 \precnsim R_i$ and there is no $j \leq n-2$ for which $R_n \precnsim R_j$.

Case 1a: We have $l(R_2) > l(R_1)$, $l(R_{n-1}) < \cdots < l(R_2)$, and $l(R_n) > l(R_{n-1})$.
Because $l(R_2) > l(R_1)$ and $l(R_3) < l(R_2)$, we have $R_1 \precnsim R_3$ by Proposition 6.1.39, contradicting our hypothesis.

Case 1b: We have $l(R_{n-1}) > \cdots > l(R_1)$, $R_t \prec R_n$ for $2 \leq t \leq n-2$, and $l(R_n) < l(R_{n-1})$.
By Proposition 6.1.39, we must have $l(R_{t+1}) = r(R_t) + 1$ for $1 \leq t \leq n-2$. Now by Proposition 6.1.44, if $M_{1,n} > 0$ then the sequence $(R_1, \ldots, R_{n-1})$ is strict and if $M_{1,n} = 0$ then the sequence $(R_2, \ldots, R_{n-1})$ is strict, so the first possibility holds.

Case 1c: We have $l(R_2) < l(R_1)$, $l(R_n) > \cdots > l(R_2)$, and $R_t \prec R_1$ for $3 \leq t \leq n-1$.
By rotating, the conclusion follows from Case 1b.

We now assume that $R_1 \precnsim R_i$ for some minimal $i \geq 3$ or $R_n \precnsim R_j$ for some maximal $j \leq n-2$. By rotating, we may assume that $R_1 \precnsim R_i$ for some minimal $i \geq 3$. Note that by Corollary 6.1.5, Part 2, we cannot have $R_1 \prec R_n$, so we have $i \leq n-1$. It remains to consider the cases where $i = n-1$ and where $i \leq n-2$.

Case 2: We have $R_1 \precnsim R_i$ for some minimal $i \geq 3$, and in fact $i = n-1$.

Case 2a: We have $l(R_{i-1}) < \cdots < l(R_1)$ and $l(R_i) > l(R_{i-1})$.
By Proposition 6.1.18, Part 2, we have $R_{i-2} \prec R_i$, but because $l(R_{i-1}) < l(R_{i-2})$ and $l(R_i) > l(R_{i-1})$, we have $R_i \precnsim R_{i-2}$ by Proposition 6.1.39, a contradiction.

Case 2b: We have $l(R_{i-1}) > \cdots > l(R_1)$, $R_t \prec R_n$ for $2 \leq t \leq i-1$, and $l(R_i) < l(R_{i-1})$.
Because $l(R_i) < l(R_{i-1})$ and $l(R_n) > l(R_i)$, by Proposition 6.1.39, Part 2 we have $R_n \precnsim R_{i-1}$, contradicting our hypothesis that $R_t \prec R_n$ for $2 \leq t \leq i-1$.

Case 2c: We have $n = 4$, $l(R_2) > l(R_1)$, $l(R_3) < l(R_2)$, and $R_4 \precnsim R_2$.
By Proposition 6.1.13, Proposition 6.1.4, and Proposition 6.1.39, Part 1, we must have $r(R_1) + 1 = l(R_2) \leq l(R_4) \leq r(R_1) + 1$, so $l(R_2) = l(R_4)$, and we must have $l(R_4) - 1 = r(R_3) \geq r(R_1) = l(R_2) - 1 = l(R_4) - 1$, so $r(R_3) = r(R_1)$. In particular, we have $M_{1,n} = 0$.

By Proposition 6.1.39, Parts 2 and 3, we must have $R_4 \not\precsim R_2$ and $R_1 \not\precsim R_3$ and the second possibility holds.

Case 3: We have $R_i \not\precsim R_i$ for some minimal $3 \leq i \leq n-2$.

If $l(R_{i-1}) < l(R_{i-2})$ and $l(R_i) > l(R_{i-1})$, then by Proposition 6.1.18, Part 2, we have $R_{i-2} \prec R_i$, but by Proposition 6.1.39, Part 2 we have $R_i \not\precsim R_{i-2}$, a contradiction. Therefore, by Lemma 6.1.26, we must have $i = j = 3$, $l(R_2) > l(R_1)$, $R_n \not\precsim R_2$, and $l(R_3) < l(R_2)$. Similarly, by rotating, we must have $i = j = n-2$, so $n = 5$, $l(R_5) > l(R_4)$, $R_1 \not\precsim R_4$, and $l(R_4) < l(R_3)$. However, by Proposition 6.1.18, Parts 1 and 2, we have $R_2 \prec R_4$ and $R_4 \prec R_2$, which is impossible because $n \geq 5$. $\square$

We now describe another operation that we can perform on a horizontal-strip while preserving similarity. We can think of it as a *local rotation.*

Lemma 6.1.50. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ and suppose that $l(R_i) = r(R_{i-1}) + 1$ for some $2 \leq i \leq n$. Assume that $\boldsymbol{\lambda}$ satisfies the inductive hypothesis (6.9) in Lemma 6.1.10. Let $I = \{1, \ldots, i-2, i+1, \ldots, n\}$ and define the four disjoint subsets of I

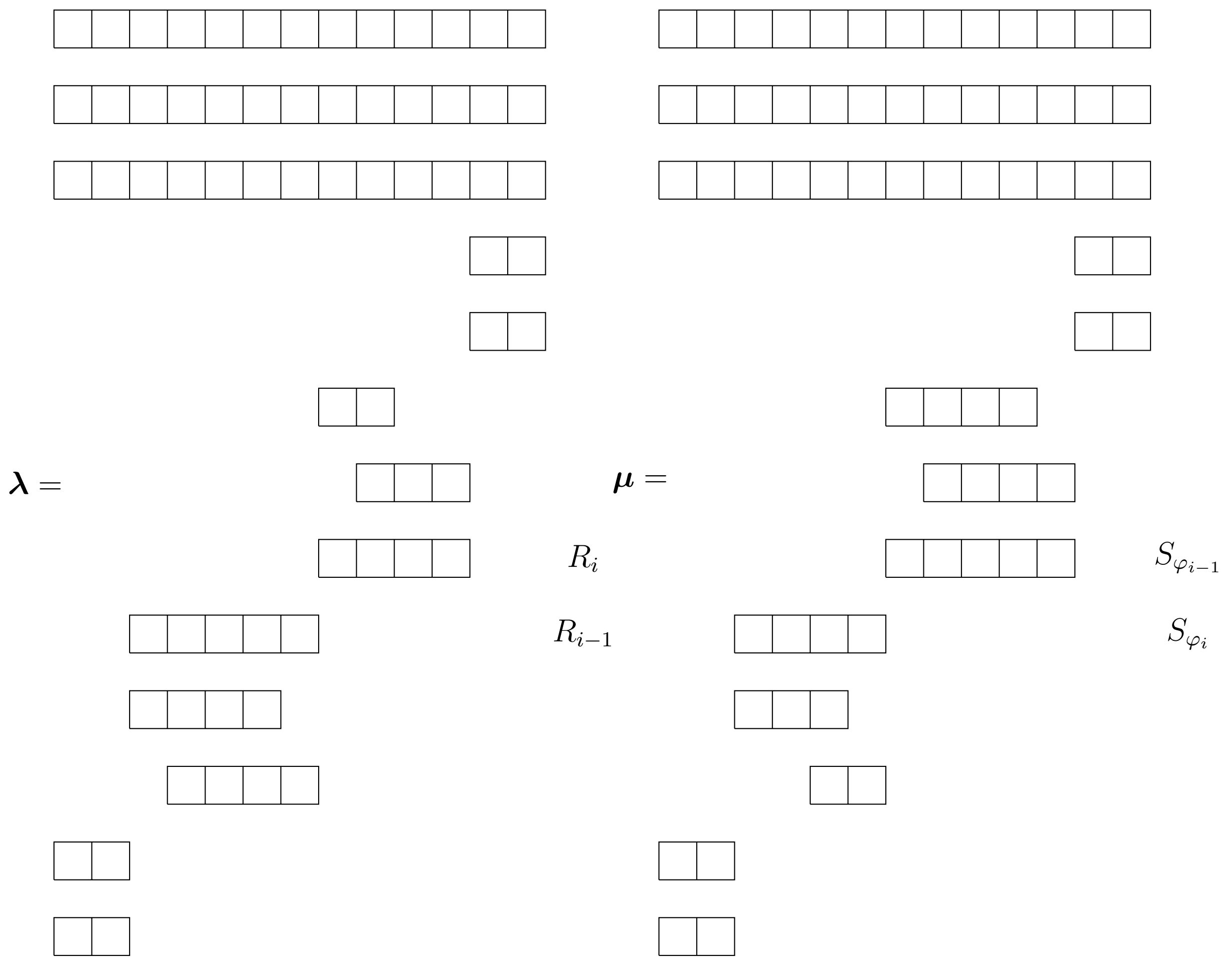
$$\begin{aligned}
A &= \{t \in I : \ M_{i-1,t} = M_{i,t} = 0\}, \\
B &= \{t \in I : \ R_{i-1} \prec R_t, \ R_i \prec R_t\}, \\
C_{i-1} &= \{t \in I : \ R_t \prec R_{i-1}, \ M_{i,t} = 0, \ M_{a,t} = 0 \text{ for all } a \in A, \ R_t \prec R_b \text{ for all } b \in B\}, \\
C_i &= \{t \in I : \ R_t \prec R_i, \ M_{i-1,t} = 0, \ M_{a,t} = 0 \text{ for all } a \in A, \ R_t \prec R_b \text{ for all } b \in B\}.
\end{aligned}$$

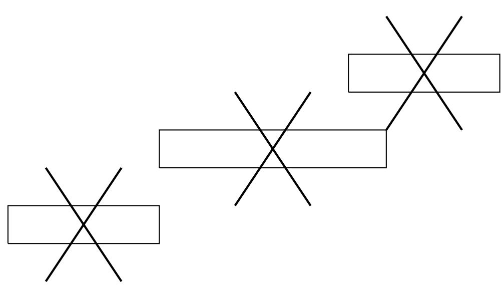
Let $C = C_{i-1} \cup C_i \cup \{i-1, i\}$ and suppose that $A \cup B \cup C = \{1, \ldots, n\}$, in other words every row of $\boldsymbol{\lambda}$ falls into one of these categories. Then there is a horizontal-strip $\boldsymbol{\mu} = (S_1, \ldots, S_n) \in \mathcal{S}(\boldsymbol{\lambda})$ and $\varphi : \Pi(\boldsymbol{\lambda}) \xrightarrow{\sim} \Pi(\boldsymbol{\mu})$ with $l(S_{\varphi_t}) = l(R_t)$ for all $t \in A \cup B$ and $l(S_{\varphi_i}) = l(R_{i-1})$.

Example 6.1.51. Figure 6.15 illustrates a situation where we can apply Lemma 6.1.50. The rows below $\boldsymbol{\lambda}$ with crosses signify rows that cannot be present because the condition $A \cup B \cup C = \{1, \ldots, n\}$ requires that every row of $\boldsymbol{\lambda}$ be either disjoint from R_{i-1} and R_i, contained in R_{i-1}, contained in R_i, or containing both. Lemma 6.1.50 allows us to locally rotate the six rows of C to produce the similar horizontal-strip $\boldsymbol{\mu}$. Although our proof constructs this specific $\boldsymbol{\mu}$, we will only need that $l(S_{\varphi_t}) = l(R_t)$ for all $t \in A \cup B$ and $l(S_{\varphi_i}) = l(R_{i-1})$.

Proof of Lemma 6.1.50. Informally, we will first use commuting and cycling to bring the rows of C together. To be specific, we first claim that if $t \in C_i$ and $t' \notin C_i$ with $i < t' < t$, then $R_t \leftrightarrow R_{t'}$. Because $t \in C_i$, we have $R_t \prec R_i$ and $M_{i-1,t} = 0$, so by Proposition 6.1.13 we have $l(R_i) \leq l(R_t) \leq r(R_t) \leq r(R_i)$. Now if $t' \in A$, then $M_{i-1,t'} = M_{i,t} = M_{t,t'} = 0$, so either $r(R_{t'}) < l(R_{i-1}) - 1 < l(R_t) - 1$ and $R_t \leftrightarrow R_{t'}$ by Proposition 6.1.4, Parts 2 and 3, or $l(R_{t'}) > r(R_i) \geq r(R_t) \geq l(R_t)$, but now we cannot have $R_t \not\leftrightarrow R_{t'}$ by Corollary 6.1.5, Part 2. If $t' \in B$, then $R_{t'} \prec R_{i-1}$, so by Proposition 6.1.13 we have $l(R_{t'}) \leq l(R_{i-1}) - 1 < l(R_t)$,

Figure 6.15: A horizontal-strip $\boldsymbol{\lambda}$ and a local rotation $\boldsymbol{\mu}$





but now we cannot have $R_t \not\leftrightarrow R_{t'}$ by Corollary 6.1.5, Part 2. If $t' \in C_{i-1}$, then because $R_{t'} \prec R_{i-1}$ and $M_{i,t'} = 0$, we have $r(R_{t'}) < l(R_i) - 1 \leq l(R_t) - 1$, so $R_t \leftrightarrow R_{t'}$ by Proposition 6.1.4, Part 1. This establishes our claim.

Therefore, by cycling and commuting, we may assume that $C_i = \{i+1, \ldots, y\}$ for some y and similarly, by considering a rotation of $\boldsymbol{\lambda}$, we may assume that $C_{i-1} = \{x, \ldots, i-2\}$ for some x. In particular, we have $l(R_{i-1}) \leq l(R_t) \leq r(R_t) \leq r(R_{i-1})$ for every $t \in C_{i-1}$ and $l(R_i) \leq l(R_t) \leq r(R_t) \leq r(R_i)$ for every $t \in C_i$. To summarize, we have

$$\boldsymbol{\lambda} = (R_1, \ldots, R_{x-1}, R_x, \ldots, R_{i-2}, R_{i-1}, R_i, R_{i+1}, \ldots, R_y, R_{y+1}, \ldots, R_n), \tag{6.33}$$

where $A \cup B = \{1, \ldots, x-1\} \cup \{y+1, \ldots, n\}$, $C_{i-1} = \{x, \ldots, i-2\}$, and $C_i = \{i+1, \ldots, y\}$.

Now let $N = l(R_{i-1}) + r(R_i)$ and define the horizontal-strip $\boldsymbol{\mu} = (S_1, \ldots, S_n)$ by $S_t = R_t$ if $t < x$ or $t > y$, and $S_t = N - R_{x+y-t}$ otherwise, that is

$$\boldsymbol{\mu} = (R_1, \ldots, R_{x-1}, N - R_y, \ldots, N - R_i, N - R_{i-1}, \ldots, N - R_x, R_{y+1}, \ldots, R_n), \tag{6.34}$$

and define $\varphi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ by $\varphi_t = t$ if $t < x$ or $t > y$ and $\varphi_t = x + y - t$ otherwise. Indeed, we have $l(S_{\varphi_t}) = l(R_t)$ for all $t \in A \cup B$ and $l(S_{\varphi_i}) = \ell(N - R_i) = l(R_{i-1}) + r(R_i) - r(R_i) = l(R_{i-1})$. We claim that $\varphi : \Pi(\boldsymbol{\lambda}) \xrightarrow{\sim} \Pi(\boldsymbol{\mu})$. We have $|R_t| = |S_{\varphi_t}|$ for every $1 \leq t \leq n$, so it remains to check that the edge weights are preserved. If $t, t' \in A \cup B$, then the relative positions of R_t and $R_{t'}$ have not changed, so indeed $M_{t,t'}(\boldsymbol{\lambda}) = M_{\varphi_t,\varphi_{t'}}(\boldsymbol{\mu})$. If $t, t' \in C$, then this follows because $M(R_t, R_{t'}) = M(N - R_{t'}, N - R_t)$. Now suppose that $t \in A \cup B$ and $t' \in C$. We have either $t' < i - 1$ and $R_{t'} \prec R_{i-1}$ or $t' > i$ and $R_{t'} \prec R_i$, so in either case we have

$$l(R_{i-1}) \leq l(R_{t'}) \leq r(R_{t'}) \leq r(R_i), \text{ so } l(R_{i-1}) \leq \ell(N - R_{t'}) \leq r(N - R_{t'}) \leq r(R_i). \tag{6.35}$$

Also note that $\chi(t > i) = \chi(t > i - 1) = \chi(t > t') = \chi(t > x + y - t')$. Now if $t \in A$, we have either

$$l(R_t) > r(R_i) + \chi(i > t) \geq r(N - R_{t'}) + \chi(i > x + y - t') \text{ or} \tag{6.36}$$

$$r(R_t) < l(R_{i-1}) - \chi(t > i) \leq \ell(N - R_{t'}) - \chi(t > x + y - t'), \tag{6.37}$$

so in either case, we have $M_{\varphi_t,\varphi_{t'}}(\boldsymbol{\mu}) = 0$. Similarly, if $t \in B$, we have

$$l(R_t) \leq l(R_{i-1}) + \chi(i - 1 > t) \leq \ell(N - R_{t'}) + \chi(x + y - t' > t) \text{ and} \tag{6.38}$$

$$r(R_t) \geq r(R_i) - \chi(t > i) \geq r(N - R_{t'}) - \chi(t > x + y - t'), \tag{6.39}$$

so we have $N - R_{t'} \prec R_t$ and $M_{\varphi_t,\varphi_{t'}}(\boldsymbol{\mu}) = |S_{\varphi_{t'}}|$.

Finally, we show that $G_{\boldsymbol{\lambda}}(\boldsymbol{x};q) = G_{\boldsymbol{\mu}}(\boldsymbol{x};q)$. Define the horizontal-strips

$$\boldsymbol{\lambda}' = (R_1, \dots, R_i, R_{i-1}, \dots, R_n), \tag{6.40}$$
$$\boldsymbol{\lambda}'' = (R_1, \dots, R_{i-1} \cup R_i, R_{i-1} \cap R_i, \dots, R_n), \tag{6.41}$$
$$\boldsymbol{\mu}' = (R_1, \dots, N - R_y, \dots, N - R_{i-1}, N - R_i, \dots, N - R_x, \dots, R_n), \text{ and} \tag{6.42}$$
$$\boldsymbol{\mu}'' = (R_1, \dots, (N - R_i) \cup (N - R_{i-1}), (N - R_i) \cap (N - R_{i-1}), \dots, R_n). \tag{6.43}$$

Because Proposition 4.1.18 describes exactly how to derive the weighted graphs $\Pi(\boldsymbol{\lambda}')$ and $\Pi(\boldsymbol{\lambda}'')$ from $\Pi(\boldsymbol{\lambda})$, we have that $\Pi(\boldsymbol{\lambda}') \cong \Pi(\boldsymbol{\mu}')$ and $\Pi(\boldsymbol{\lambda}'') \cong \Pi(\boldsymbol{\mu}'')$. We also have $n(\boldsymbol{\lambda}'') < n(\boldsymbol{\lambda})$, $n(\boldsymbol{\lambda}') = n(\boldsymbol{\lambda})$, and $M(\boldsymbol{\lambda}') > M(\boldsymbol{\lambda})$, so because $\boldsymbol{\lambda}$ satisfies (6.9) by hypothesis, we have that $G_{\boldsymbol{\lambda}'}(\boldsymbol{x};q) = G_{\boldsymbol{\mu}'}(\boldsymbol{x};q)$ and $G_{\boldsymbol{\lambda}''}(\boldsymbol{x};q) = G_{\boldsymbol{\mu}''}(\boldsymbol{x};q)$. Finally, by (4.27), we have

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x};q) = \frac{1}{q}G_{\boldsymbol{\lambda}'}(\boldsymbol{x};q) + \frac{q-1}{q}G_{\boldsymbol{\lambda}''}(\boldsymbol{x};q) = \frac{1}{q}G_{\boldsymbol{\mu}'}(\boldsymbol{x};q) + \frac{q-1}{q}G_{\boldsymbol{\mu}''}(\boldsymbol{x};q) = G_{\boldsymbol{\mu}}(\boldsymbol{x};q). \tag{6.44}$$

This completes the proof. □

The hypothesis of Lemma 6.1.50 that $A \cup B \cup C = \{1, \dots, n\}$ is a little technical so it will be convenient to rephrase it as follows.

Proposition 6.1.52. Let $\boldsymbol{\lambda} = (R_1, \dots, R_n)$ be a horizontal-strip with $l(R_i) = r(R_{i-1}) + 1$ and define the sets A, B, C, and I as in Lemma 6.1.50. If the following hold for every $t \in I$, then we have $A \cup B \cup C = \{1, \dots, n\}$.

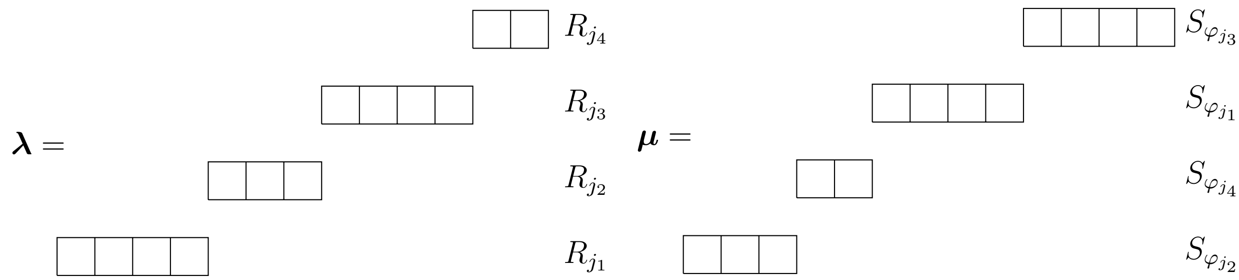
1. If $M_{i-1,t} > 0$ and $M_{i,t} > 0$, then $R_{i-1} \prec R_t$ and $R_i \prec R_t$.
2. If $M_{i-1,t} > 0$ and $M_{i,t} = 0$, then $R_t \prec R_{i-1}$.
3. If $M_{i-1,t} = 0$ and $M_{i,t} > 0$, then $R_t \prec R_i$.
4. If $R_t \prec R_{i-1}$, then $M_{i,t} = 0$, $M_{a,t} = 0$ for all $a \in A$, and $R_t \prec R_b$ for all $b \in B$.
5. If $R_t \prec R_i$, then $M_{i-1,t} = 0$, $M_{a,t} = 0$ for all $a \in A$, and $R_t \prec R_b$ for all $b \in B$.

Proof. Let $t \in I$. We need to show that $t \in A \cup B \cup C_{i-1} \cup C_i$. The integers $M_{i-1,t}$ and $M_{i,t}$ are either zero or nonzero. If $M_{i-1,t} = M_{i,t} = 0$, then $t \in A$. If $M_{i-1,t} > 0$ and $M_{i,t} > 0$, then by (1) we have $t \in B$. If $M_{i-1,t} > 0$ and $M_{i,t} = 0$, then by (2) and (4) we have $t \in C_{i-1}$. If $M_{i-1,t} = 0$ and $M_{i,t} > 0$, then by (3) and (5) we have $t \in C_i$. □

The next Lemma is very technical but it is the key idea that uses local rotation to extend Corollary 6.1.37 to strict sequences.

Lemma 6.1.53. Let $\boldsymbol{\lambda} = (R_1, \dots, R_n)$ be a horizontal-strip with a sequence $(R_{j_1}, \dots, R_{j_k})$ with $k \geq 2$, $j_1 < \cdots < j_k$, $l(R_{j_{t+1}}) = r(R_{j_t}) + 1$ for $1 \leq t \leq k-1$, and suppose that there is

Figure 6.16: An example where we can apply Lemma 6.1.53



no noncommuting path in $\boldsymbol{\lambda}$ from R_{j_t} to $R_{j_{t+1}}$ for any $1 \leq t \leq k-1$. Assume that $\boldsymbol{\lambda}$ satisfies (6.9). Let $\boldsymbol{\mu} = (S_1, \ldots, S_n)$ and $\varphi : \Pi(\boldsymbol{\lambda}) \xrightarrow{\sim} \Pi(\boldsymbol{\mu})$ be such that

$$\text{for some permutation } \sigma : \{1, \ldots, k\} \to \{1, \ldots, k\} \text{ we have } l(S_{\varphi_{j_{\sigma_{t+1}}}}) = r(S_{\varphi_{j_{\sigma_t}}}) + 1 \quad (6.45)$$

for every $1 \leq t \leq k-1$. Then there exists a good substitute for $(\boldsymbol{\lambda}, \boldsymbol{\mu})$.

Example 6.1.54. Figure 6.16 illustrates a situation where Lemma 6.1.53 applies. Informally, the condition (6.45) asks that these rows in $\boldsymbol{\lambda}$ still link end to end in $\boldsymbol{\mu}$, although they may be permuted. In this example, we have $\sigma_1 = 2$, $\sigma_2 = 4$, $\sigma_3 = 1$, and $\sigma_4 = 3$.

Remark 6.1.55. If $(R_{j_1}, \ldots, R_{j_k})$ is a strict sequence of $\boldsymbol{\lambda}$, then $k \geq 2$, $j_1 < \cdots < j_k$, and by Proposition 6.1.44, $l(R_{j_{t+1}}) = r(R_{j_t}) + 1$ for $1 \leq t \leq k-1$. Moreover, if $\boldsymbol{\mu} = (S_1, \ldots, S_n)$ and $\varphi : \Pi(\boldsymbol{\lambda}) \xrightarrow{\sim} \Pi(\boldsymbol{\mu})$, then by Remark 6.1.43 and by cycling $\boldsymbol{\mu}$ if necessary, there will be a permutation $\sigma : \{1, \ldots, k\} \to \{1, \ldots, k\}$ with $(S_{\varphi_{j_{\sigma_1}}}, \ldots, S_{\varphi_{j_{\sigma_k}}})$ a strict sequence and therefore $l(S_{\varphi_{j_{\sigma_{t+1}}}}) = r(S_{\varphi_{j_{\sigma_t}}}) + 1$ for every $1 \leq t \leq k-1$. Therefore, a strict sequence satisfies the hypothesis (6.45) of Lemma 6.1.53.

Remark 6.1.56. Informally, the strategy will be to apply Lemma 6.1.50 to perform a series of local rotations to permute the rows of $\boldsymbol{\lambda}$ to match those of $\boldsymbol{\mu}$. We will be able to perform these local rotations unless some other row R_t of $\boldsymbol{\lambda}$ violates some condition of Proposition 6.1.52, forcing certain rows of $\boldsymbol{\lambda}$ to link end to end. However, in this case, these rows of $\boldsymbol{\lambda}$ will be a proper subset of rows that satisfies (6.45) and we can use induction to reason about these rows.

Proof of Lemma 6.1.53. We use induction on k. If $k = 2$, then because $l(R_{j_2}) = r(R_{j_1}) + 1$ and by hypothesis there is no noncommuting path in $\boldsymbol{\lambda}$ from R_{j_1} to R_{j_2}, we can use Lemma 6.1.16 to replace $\boldsymbol{\lambda}$ with a similar horizontal-strip as necessary to assume that $j_2 = j_1 + 1$, and then by (6.45) we have either $l(S_{\varphi_{j_2}}) = r(S_{\varphi_{j_1}}) + 1$ or $l(S_{\varphi_{j_1}}) = r(S_{\varphi_{j_2}}) + 1$, so $S_{\varphi_{j_1}} \not\leftrightarrow S_{\varphi_{j_2}}$ and the result follows from Corollary 6.1.30. So we now assume that $k \geq 3$ and that the result holds for $2 \leq k' \leq k-1$. Note that if $J \subseteq \{1, \ldots, k\}$ is an interval with $2 \leq |J| \leq k-1$

and such that $\sigma^{-1}(J) = \{1 \leq t' \leq k : \sigma_{t'} \in J\} \subseteq \{1, \ldots, k\}$ is an interval, then the sequence of rows $(R_{j_t} : t \in J)$ satisfies (6.45) and we are done by our induction hypothesis on k. In particular, if $\sigma_1 = 1$ or $\sigma_k = 1$, then we can take $J = \{2, \ldots, k\}$, and if $\sigma_1 = k$ or $\sigma_k = k$, then we can take $J = \{1, \ldots, k-1\}$, so we may assume that

$$2 \leq \sigma_1, \sigma_k \leq k - 1, \tag{6.46}$$

and in particular, we may assume that $k \geq 4$.

We now continue to use our induction hypothesis to make several additional simplifying assumptions. For $1 \leq t \leq n$ such that $t \neq j_{t'}$ for any $1 \leq t' \leq k$, consider the sets

$$E_t = \{1 \leq t' \leq k : R_{j_{t'}} \prec R_t\} \text{ and } F_t = \{1 \leq t \leq k : M_{j_{t'},t}(\boldsymbol{\lambda}) > 0\}. \tag{6.47}$$

Note that $E_t \subseteq F_t$. Also, if $t_1 < t_2 < t_3$ and $t_1, t_3 \in F_t$, then by Proposition 6.1.4, Parts 1 and 3, we have

$$l(R_t) \leq r(R_{j_{t_1}}) + 1 \leq l(R_{j_{t_2}}) \leq r(R_{j_{t_2}}) \leq l(R_{j_{t_3}}) - 1 \leq r(R_t), \tag{6.48}$$

so by Proposition 6.1.13 we have $R_{t_2} \prec R_t$ and $t_2 \in E_t$, so E_t and F_t are intervals in $\{1, \ldots, k\}$ and $|E_t| \geq |F_t| - 2$. Similarly, consider the sets

$$E'_t = \{1 \leq t' \leq k : S_{\varphi_{j_{\sigma_{t'}}}} \prec S_{\varphi_t}\} \text{ and } F'_t = \{1 \leq t' \leq k : M_{\varphi_{j_{\sigma_{t'}}},\varphi_t}(\boldsymbol{\mu}) > 0\}. \tag{6.49}$$

Note that $E'_t \subseteq F'_t$, $E'_t = \sigma^{-1}(E_t)$, $F'_t = \sigma^{-1}(F_t)$ and as before, if $t_1 < t_2 < t_3$ and $t_1, t_3 \in F'_t$, then by Proposition 6.1.4, Parts 1 and 3, we have

$$l(S_{\varphi_t}) \leq r(S_{\varphi_{j_{\sigma_{t_1}}}}) + 1 \leq l(S_{\varphi_{j_{\sigma_{t_2}}}}) \leq r(S_{\varphi_{j_{\sigma_{t_2}}}}) \leq l(S_{\varphi_{j_{\sigma_{t_3}}}}) - 1 \leq r(S_{\varphi_t}), \tag{6.50}$$

so by Proposition 6.1.13, we have $S_{\varphi_{j_{\sigma_{t_2}}}} \prec S_{\varphi_t}$ and $t_2 \in E'_t$, so E'_t and F'_t are intervals in $\{1, \ldots, k\}$. Therefore, if $2 \leq |F_t| \leq k - 1$, then taking $J = F_t$ above we are done by our induction hypothesis on k. Similarly, if $|F_t| = k$ and $|E_t| \leq k - 1$, then because $|E_t| \geq k - 2 \geq 2$, taking $J = E_t$ above we are done by our induction hypothesis on k. This means that we may assume that

$$\text{if } |F_t| \geq 2, \text{ then } E_t = F_t = \{1, \ldots, k\}. \tag{6.51}$$

Now suppose that $F_t = \{t'\}$. Our goal is to show that we may assume that $R_t \prec R_{j_{t'}}$, that $M_{a,t}(\boldsymbol{\lambda}) = 0$ for every a with $M_{a,j_{t'}}(\boldsymbol{\lambda}) = 0$, and $R_t \prec R_b$ for every b with $R_{j_{t'}} \prec R_b$.

Suppose that $R_t \not\prec R_{j_{t'}}$. If $2 \leq t' \leq k - 1$, then $M_{j_{t'-1},t}(\boldsymbol{\lambda}) = 0$ and $M_{j_{t'+1},t}(\boldsymbol{\lambda}) = 0$ by definition of F_t and by Proposition 6.1.4, Parts 1 and 3, we would have

$$l(R_{j_{t'}}) = r(R_{j_{t'-1}}) + 1 \leq l(R_t) \leq r(R_t) \leq l(R_{j_{t'+1}}) - 1 = r(R_{j_{t'}}) \tag{6.52}$$

and therefore $R_t \prec R_{j_{t'}}$ by Proposition 6.1.13. This means that we must have $t' = 1$ or k and for the same reason, $\sigma_{t'} = 1$ or k, but this contradicts our assumption (6.46). Therefore, we must have $R_t \prec R_{j_{t'}}$.

Now suppose that there is some a with $M_{a,j_{t'}}(\boldsymbol{\lambda}) = 0$ but $M_{a,t}(\boldsymbol{\lambda}) > 0$. Because $R_t \prec R_{j_{t'}}$, we have $M_{j_{t'},t}(\boldsymbol{\lambda}) + M_{a,t}(\boldsymbol{\lambda}) \geq |R_t| + 1$, so either $j_{t'} < a$ and the pair $(R_{j_{t'}}, R_a)$ is strict, or $a < j_{t'}$ and the pair $(R_a, R_{j_{t'}})$ is strict. In either case, by Proposition 6.1.35 we have $R_{j_{t'}} \not\leftrightarrow R_a$, and by Corollary 6.1.5, Part 3, we have either

$$l(R_a) = r(R_{j_{t'}}) + 1 = l(R_{j_{t'+1}}) \text{ or } l(R_{j_{t'}}) = r(R_a) + 1. \tag{6.53}$$

In particular, if $t' = 1$, then either $F_a = \emptyset$ or $\{2\}$, if $2 \leq t' \leq k-1$, then either $F_a = \{t'-1\}$ or $\{t'+1\}$, and if $t' = k$, then either $F_a = \emptyset$ or $\{k-1\}$. Similarly, F'_a is either empty, in which case $\sigma_{t'} = 1$ or k, or F'_a and F'_t consist of consecutive singletons. Therefore, if $F_a = \emptyset$, then this contradicts (6.46), and otherwise, taking $J = F_t \cup F_a$ we are done by our induction hypothesis on k.

Next, let us suppose that for some b we have $R_{j_{t'}} \prec R_b$ but $R_t \not\prec R_b$. If $2 \leq t' \leq k-1$, then $M_{j_{t'-1},t}(\boldsymbol{\lambda}) = 0$ and $M_{j_{t'+1},t}(\boldsymbol{\lambda}) = 0$ by definition of F_t and by Proposition 6.1.4, Parts 2 and 3, we would have

$$l(R_{j_{t'}}) = r(R_{j_{t'-1}}) + 1 \leq l(R_t) \leq r(R_t) \leq l(R_{j_{t'+1}}) - 1 = r(R_{j_{t'}}). \tag{6.54}$$

Now if $R_{j_{t'}} \subseteq R_b$, we would have $R_t \subseteq R_{j_{t'}} \subseteq R_b$ and $R_t \prec R_b$ by Proposition 6.1.13, so we must have either $R_{j_{t'}} \subseteq R_b^+$ and $l(R_b) \leq l(R_{j_{t'}}) - 1 = r(R_{j_{t'-1}})$, or $R_{j_{t'}} \subseteq R_b^-$ and $r(R_b) \geq r(R_{j_{t'}}) + 1 = l(R_{j_{t'+1}})$. However, this means that either $\{t'-1, t'\} \subseteq F_b$ or $\{t', t'+1\} \subseteq F_b$, so by (6.51) we must have $F_b = \{1, \ldots, k\}$, so $M_{j_{t'-1},t}(\boldsymbol{\lambda}) > 0$ and $M_{j_{t'+1},t}(\boldsymbol{\lambda}) > 0$, and by Proposition 6.1.13, we have

$$l(R_b) \leq r(R_{j_{t'-1}}) + 1 = l(R_{j_{t'}}) \leq l(R_t) \leq r(R_t) \leq r(R_{j_{t'}}) = l(R_{j_{t'+1}}) - 1 \leq r(R_b) \tag{6.55}$$

and $R_t \prec R_b$ by Proposition 6.1.13 after all. Therefore, if $R_{j_{t'}} \prec R_b$ but $R_t \not\prec R_b$, we must have $t' = 1$ or k and for the same reason, $\sigma_{t'} = 1$ or k, but this contradicts our assumption (6.46). To summarize, we may assume that

$$\begin{aligned} &\text{if } |F_t| \geq 2, \text{ then } E_t = F_t = \{1, \ldots, k\}, \text{ and if } F_t = \{t'\}, \text{ then } R_t \prec R_{j_{t'}}, \\ &M_{a,t}(\boldsymbol{\lambda}) = 0 \text{ whenever } M_{a,j_{t'}}(\boldsymbol{\lambda}) = 0, \text{ and } R_t \prec R_b \text{ whenever } R_{j_{t'}} \prec R_b. \end{aligned} \tag{6.56}$$

Let t_0 and t_1 be such that $t_0 < t_1$ and $\{t_0, t_1\} = \{\sigma_1, \sigma_2\}$, and let $x = j_{t_0}$ and $y = j_{t_1}$. Note that we have $x < y$, $l(R_y) \geq r(R_x) + 1$, and $S_{\varphi_x} \not\leftrightarrow S_{\varphi_y}$. We use induction on $l(R_y) - r(R_x)$. If $l(R_y) - r(R_x) = 1$, then $t_1 = t_0 + 1$ and because there is no noncommuting path in $\boldsymbol{\lambda}$ from R_x to R_y, we may use Lemma 6.1.16 to assume that $y = x + 1$. In this case, it now follows from Corollary 6.1.30 that there exists a good substitute for $(\boldsymbol{\lambda}, \boldsymbol{\mu})$. So we now assume that

$l(R_y) - r(R_x) \geq 2$, so in particular $t_1 \geq t_0 + 2$. Because there is no noncommuting path in $\boldsymbol{\lambda}$ from $R_{j_{t_1-1}}$ to R_y, we may use Lemma 6.1.16 to assume that $y = j_{t_1-1} + 1$.

Our plan now is to apply Lemma 6.1.50 to the rows R_{y-1} and R_y to replace $\boldsymbol{\lambda}$ with a similar horizontal-strip for which $l(R_y)$ has decreased and $r(R_x)$ has not changed, so that we will be done by our induction hypothesis on $l(R_y)-r(R_x)$. It remains to check the conditions of Proposition 6.1.52.

1. If $M_{y-1,t}(\boldsymbol{\lambda}) > 0$ and $M_{y,t}(\boldsymbol{\lambda}) > 0$, then $|F_t| \geq 2$, so by (6.56) we have $E_t = \{1, \ldots, k\}$, so $R_{y-1} \prec R_t$ and $R_y \prec R_t$.

2. If $M_{y-1,t}(\boldsymbol{\lambda}) > 0$ and $M_{y,t}(\boldsymbol{\lambda}) = 0$, then $t_1 - 1 \in F_t$ and $t_1 \notin F_t$, so by (6.56) we must have $F_t = \{t_1 - 1\}$ and then $R_t \prec R_{y-1}$.

3. If $M_{y-1,t}(\boldsymbol{\lambda}) = 0$ and $M_{y,t}(\boldsymbol{\lambda}) > 0$, then $t_1 - 1 \notin F_t$ and $t_1 \in F_t$, so by (6.56) we must have $F_t = \{t_1\}$ and $R_t \prec R_y$.

4. If $R_t \prec R_{y-1}$, then $t_1 - 1 \in F_t$. By Proposition 6.1.4, Part 1, and because $x < y$, we cannot have both $M_{x,t}(\boldsymbol{\lambda}) > 0$ and $M_{y,t}(\boldsymbol{\lambda}) > 0$ because then

$$\begin{aligned} l(R_t) &\leq r(R_x) + \chi(x > t) \text{ and } r(R_t) \geq l(R_y) - \chi(t > y), \text{ so} \\ |R_t| &= r(R_t) - l(R_t) + 1 \geq l(R_y) - r(R_x) + 1 - \chi(t > y) - \chi(x > t) \\ &= r(R_{y-1}) + 1 - l(R_{y-1}) + 1 + 1 - \chi(t > y) - \chi(x > t) > |R_{y-1}|, \end{aligned} \tag{6.57}$$

contradicting $R_t \prec R_{y-1}$ by Proposition 6.1.13. Therefore, either $t_0 \notin F_t$ or $t_1 \notin F_t$, so by (6.56) we must have $F_t = \{t_1 - 1\}$ and $M_{y,t}(\boldsymbol{\lambda}) = 0$. Moreover, if $a \in A$, then in particular $M_{a,y}(\boldsymbol{\lambda}) = 0$, so by (6.56) we have $M_{a,t}(\boldsymbol{\lambda}) = 0$, and if $b \in B$, then in particular we have $R_{y-1} \prec R_b$, so by (6.56) we have $R_t \prec R_b$ as well.

5. If $R_t \prec R_y$, then $t_1 \in F_t$. By Proposition 6.1.13 and because $x < y$, we cannot have $M_{x,t}(\boldsymbol{\lambda}) > 0$ because then

$$\begin{aligned} l(R_t) &\leq r(R_x) + \chi(x > t) \leq l(R_{y-1}) - 1 + \chi(x > t) \leq r(R_{y-1}) - 1 + \chi(x > t) \\ &= l(R_{y-1}) - 1 - 1 + \chi(x > t) \leq l(R_y) - 1 - \chi(t > y) < l(R_y) - \chi(t > y), \end{aligned} \tag{6.58}$$

contradicting $R_t \prec R_y$ by Proposition 6.1.13. Therefore, $t_0 \notin F_t$, so by (6.56) we must have $F_t = \{t_1\}$ and $M_{y-1,t}(\boldsymbol{\lambda}) = 0$. Moreover, if $a \in A$, then in particular $M_{a,y-1}(\boldsymbol{\lambda}) = 0$, so by (6.56) we have $M_{a,t}(\boldsymbol{\lambda}) = 0$, and if $b \in B$, then in particular we have $R_y \prec R_b$, so by (6.56) we have $R_t \prec R_b$ as well.

This concludes our verification of the conditions of Proposition 6.1.52. Therefore, the result follows by Lemma 6.1.50 and our induction hypothesis on $l(R_y) - r(R_x)$. □

We now generalize Corollary 6.1.37 to the case where $\boldsymbol{\lambda}$ has a pair of strict rows that are not necessarily adjacent.

Corollary 6.1.57. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a horizontal-strip that satisfies (6.9). If $\boldsymbol{\lambda}$ has a strict sequence $(R_{j_1}, \ldots, R_{j_k})$ such that the pairs $(R_{i'}, R_{j'})$ are not strict for $j_1 \leq i' < j' \leq j_k$, then $\boldsymbol{\lambda}$ is good.

Proof. Because $\boldsymbol{\lambda}$ has a strict sequence $(R_{j_1}, \ldots, R_{j_k})$ such that the pairs $(R_{i'}, R_{j'})$ are not strict for $j_1 \leq i' < j' \leq j_k$, we may assume that $(R_{j_1}, \ldots, R_{j_k})$ is such a strict sequence with $j_k - j_1$ minimal, and among such strict sequences, with k minimal. Now if there is a minimal noncommuting path in $\boldsymbol{\lambda}$ from R_{j_t} to $R_{j_{t+1}}$ for any $1 \leq t \leq k-1$, then by Proposition 6.1.48, there is either a strict sequence between R_{j_t} and $R_{j_{t+1}}$, contradicting minimality of $j_k - j_1$, or there is some $j_t < x < j_{t+1}$ with $l(R_x) = l(R_{j_{t+1}})$ and $R_{j_{t+1}} \precsim\!\!\!\!/\; R_x$, but in that case by Proposition 6.1.46 there is either a strict sequence between nearer rows, contradicting minimality of $j_k - j_1$, there is a shorter strict sequence from R_{j_1} to R_{j_k}, contradicting minimality of k, or there is a strict pair, contradicting our hypothesis. Therefore, there is no minimal noncommuting path in $\boldsymbol{\lambda}$ from R_{j_t} to $R_{j_{t+1}}$ for any $1 \leq t \leq k-1$.

Now let $\boldsymbol{\mu} = (S_1, \ldots, S_n)$ and $\varphi : \Pi(\boldsymbol{\lambda}) \xrightarrow{\sim} \Pi(\boldsymbol{\mu})$. By cycling, we may assume without loss of generality that $\varphi_h > \varphi_{j_t}$ for every $1 \leq t \leq k$. Because $M_{j_t, j_{t+1}}(\boldsymbol{\lambda}) = 0$ for $1 \leq t \leq k-1$, the integers $l(S_{\varphi_{j_t}})$ for $1 \leq t \leq k$ must be distinct, so let $\sigma : \{1, \ldots, k\} \to \{1, \ldots, k\}$ be the permutation that sorts them in increasing order, in other words

$$l(S_{\varphi_{j_{\sigma_1}}}) < l(S_{\varphi_{j_{\sigma_2}}}) < \cdots < l(S_{\varphi_{j_{\sigma_k}}}). \tag{6.59}$$

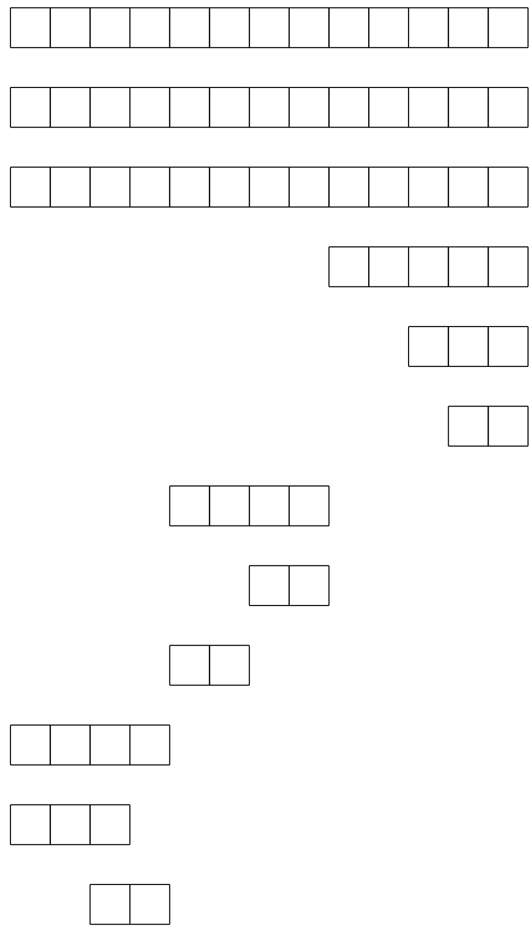
Now the sequence $(S_{\varphi_{j_{\sigma_1}}}, \ldots, S_{\varphi_{j_{\sigma_k}}})$ is strict, so by Lemma 6.1.53 there exists a good substitute for $(\boldsymbol{\lambda}, \boldsymbol{\mu})$. □

Corollary 6.1.58. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a horizontal-strip that satisfies (6.9). If $\boldsymbol{\lambda}$ has a pair of strict rows, then $\boldsymbol{\lambda}$ is good.

Proof. Because $\boldsymbol{\lambda}$ has a pair of strict rows, we may assume that (R_i, R_j) is a strict pair with $j - i$ minimal, in other words the pairs $(R_{i'}, R_{j'})$ are not strict for $i \leq i' < j' \leq j$ with $j' - i' < j - i$. Also note that if there is no minimal noncommuting path in $\boldsymbol{\lambda}$ from R_i to R_j, then by Lemma 6.1.16 we may replace $\boldsymbol{\lambda}$ with a similar horizontal-strip as necessary to assume that $j = i + 1$, in which case we are done by Corollary 6.1.37. Therefore, we may assume that there is a minimal noncommuting path in $\boldsymbol{\lambda}$ from R_i to R_j. By Proposition 6.1.48, there is either a strict sequence $(R_{j_1}, \ldots, R_{j_k})$ in $\boldsymbol{\lambda}$ such that the pairs $(R_{i'}, R_{j'})$ are not strict for $j_1 \leq i' < j' \leq j_k$, in which case we are done by Corollary 6.1.57, or we have $M_{i,j} = 0$ and there is $i < x < j$ with $l(R_x) = l(R_j)$ and $R_j \precsim\!\!\!\!/\; R_x$. However, in the latter case, because the pair (R_i, R_j) is strict, then since $M_{i,j} = 0$ we must have $M_{i,k} + M_{j,k} = |R_k| + 1$ for some k, but now $M_{i,k} + M_{x,k} \geq M_{i,k} + M_{j,k} = |R_k| + 1$ so the pair (R_i, R_x) is strict, contradicting minimality of $j - i$. □

Corollary 6.1.59. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a horizontal-strip that satisfies (6.9). If $\boldsymbol{\lambda}$ has a strict sequence $(R_{j_1}, \ldots, R_{j_k})$, then $\boldsymbol{\lambda}$ is good.

Figure 6.17: A nesting horizontal-strip



Proof. By Corollary 6.1.58, we may assume that $\boldsymbol{\lambda}$ has no strict pairs, after which the result follows by Corollary 6.1.57. □

By Corollary 6.1.58 and Corollary 6.1.59, we may assume in completing the proof of Lemma 6.1.10 that there are no strict pairs or strict sequences in $\boldsymbol{\lambda}$ or any similar horizontal-strip. It will be convenient to make the following definition.

Definition 6.1.60. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a horizontal-strip. We say that $\boldsymbol{\lambda}$ is *nesting* if for every $1 \leq i < j \leq n$ we have either $M_{i,j} = 0$, $R_i \prec R_j$, or $R_j \prec R_i$, and if $M_{i,j} = 0$, then $M_{i,k} + M_{j,k} \leq |R_k|$ for every k.

Example 6.1.61. An example of a nesting horizontal-strip is given in Figure 6.17. Informally, every pair of rows is either disjoint or one is contained in the other, up to possibly shifting by one cell.

We can now summarize Corollary 6.1.58 as follows.

Corollary 6.1.62. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a horizontal-strip that satisfies (6.9). If $\boldsymbol{\lambda}$ is not nesting, then $\boldsymbol{\lambda}$ is good.

Proof. If $\boldsymbol{\lambda}$ is not nesting, then there is some $1 \leq i < j \leq n$ with either $0 < M_{i,j} < \min\{|R_i|, |R_j|\}$ or $M_{i,j} = 0$ and $M_{i,k} + M_{j,k} \geq |R_k| + 1$ for some k. By rotating, we may assume that $l(R_i) < l(R_j)$ so that the pair (R_i, R_j) is strict, and then the result follows from Corollary 6.1.58. □

We now explore some properties of nesting horizontal-strips.

Proposition 6.1.63. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a nesting horizontal-strip.

1. The pairs (R_i, R_j) are not strict for $1 \leq i < j \leq n$.
2. If $R_i \prec R_j$ and $R_j \prec R_k$, then $R_i \prec R_k$. In other words, the relation $\prec$ is transitive on the rows of $\boldsymbol{\lambda}$.
3. If $R_i \prec R_j$ and $M_{j,k} = 0$, then $M_{i,k} = 0$.
4. We cannot have $i < x < y < j$ with $l(R_x) = l(R_j) = r(R_i) - 1 = r(R_y) - 1$, $R_j \precnsim R_x$, and $R_1 \precnsim R_y$.

Proof.

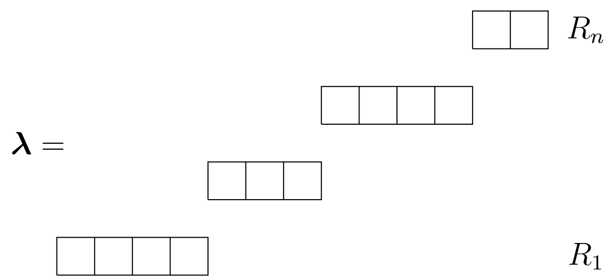
1. This follows directly from the definitions of strictness and nesting.
2. Because $M_{i,j} + M_{j,k} = |R_i| + |R_j| \geq |R_j| + 1$, then by definition of nesting we cannot have $M_{i,k} = 0$, so we must have $R_i \prec R_k$ or $R_k \prec R_i$. If $R_k \prec R_i$, then by Proposition 6.1.13 we have $|R_k| \leq |R_i| \leq |R_j| \leq |R_k|$, so $|R_i| = |R_k|$ and in fact $R_i \prec R_k$ as well.
3. Because $M_{j,k} = 0$, by definition of nesting we must have $M_{i,j} + M_{i,k} = |R_i| + M_{i,k} \leq |R_i|$, so we must have $M_{i,k} = 0$.
4. Because the conditions $R_j \precnsim R_x$ and $R_1 \precnsim R_y$ imply that $|R_x| \geq 2$ and $|R_y| \geq 2$, then we would have $M_{x,y} = 1 < \min\{|R_x|, |R_y|\}$, contradicting that $\boldsymbol{\lambda}$ is nesting.

□

Proposition 6.1.64. Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a nesting minimal noncommuting path with $l(R_1) < l(R_n)$, $M_{1,n} = 0$, and $R_1 \leftrightarrow R_n$. Then we have $l(R_{t+1}) = r(R_t) + 1$ for every $1 \leq t \leq n - 1$.

Example 6.1.65. Informally, Proposition 6.1.64 states that if $\boldsymbol{\lambda}$ is nesting, a minimal noncommuting path must look like the example in Figure 6.18.

Figure 6.18: A minimal noncommuting path in a nesting horizontal-strip



Proof of Proposition 6.1.64. If $l(R_{n-1}) > l(R_n)$, then because $l(R_1) < l(R_n)$, we must have $l(R_{t-1}) < l(R_t)$ for some maximal $2 \leq t \leq n-1$. But then we must have $R_{t-1} \not\precsim R_{t+1}$ by Proposition 6.1.39 and therefore $R_1 \prec R_{t+1}$ by Proposition 6.1.18, Part 1, which is impossible by Proposition 6.1.13 because $l(R_1) < l(R_n) \leq l(R_{t+1})$. Therefore, we must have $l(R_{n-1}) < l(R_n)$, and more specifically, because $M_{n-1,n} < \min\{|R_{n-1}|, |R_n|\}$ by Corollary 6.1.5, Part 2 and because $\boldsymbol{\lambda}$ is nesting, we must have $M_{n-1,n} = 0$ and $r(R_{n-1}) = l(R_n) - 1 > r(R_1)$. Similarly, by rotating, we must have $l(R_2) = r(R_1) + 1 < l(R_n)$. Note that if $n = 3$, then this proves our claim, so we now use induction on $n \geq 4$.

If $l(R_1) < l(R_{n-1})$, then because $R_1 \leftrightarrow R_{n-1}$ by minimality of the noncommuting path and therefore $M_{1,n-1} = 0$ by Proposition 6.1.4, we have by our induction hypothesis that $l(R_{t+1}) = r(R_t) + 1$ for every $1 \leq t \leq n-2$, which proves our claim. Similarly, by rotating, we are done if $r(R_2) < r(R_n)$. Therefore, the only remaining case to consider is when $l(R_{n-1}) \leq l(R_1)$ and $r(R_2) \geq r(R_n)$. However, we would have $R_1 \prec R_{n-1}$ and $R_n \prec R_2$ by Proposition 6.1.13, and

$$l(R_{n-1}) \leq l(R_1) \leq r(R_1) + 1 = l(R_2) \leq r(R_{n-1}) = l(R_n) - 1 \leq r(R_n) \leq r(R_2), \tag{6.60}$$

and therefore $M_{2,n-1} > 0$ by Proposition 6.1.4 Parts 2 and 3. Because $\boldsymbol{\lambda}$ is nesting, we must have either $R_2 \prec R_{n-1}$ or $R_{n-1} \prec R_2$, but if $R_2 \prec R_{n-1}$, we have $M_{2,n-1} + M_{2,n} = |R_2| + |R_n| \geq |R_2| + 1$, and if $R_{n-1} \prec R_2$, we have $M_{1,n-1} + M_{2,n-1} = |R_1| + |R_{n-1}| \geq |R_{n-1}| + 1$, a contradiction in either case. This completes the proof. □

We are at long last ready to prove Lemma 6.1.10, which implies Theorem 3.1.8. The strategy is as follows. We will use commuting and cycling to rewrite $\boldsymbol{\lambda}$ so that the bottom two rows are in the desired form. If the corresponding two rows S_i and S_j of $\boldsymbol{\mu}$ do not commute, then we are done by Corollary 6.1.30, and if there is no minimal noncommuting path between them, then we can again use commuting and cycling to bring them closer together until they do not commute. If there is a minimal noncommuting path $(S_i = S_{i_1}, \ldots, S_{i_k} = S_j)$, then because we may assume that $\boldsymbol{\mu}$ is nesting, Proposition 6.1.64 specifies the structure of these rows. In this case, we will use an argument similar to that of the proof of Lemma 6.1.53 to locally rotate pairs of rows $(S_{i_{t-1}}, S_{i_t})$ of $\boldsymbol{\mu}$ to again bring S_j toward S_i until they do not commute.

Proof of Lemma 6.1.10. We first note that by Corollary 6.1.59 and Corollary 6.1.62, we may assume that $\boldsymbol{\lambda}$ is nesting and has no strict pairs or strict sequences. We will first show that there is a similar horizontal-strip $\boldsymbol{\lambda}' = (R'_1, \ldots, R'_n) \in \mathcal{S}(\boldsymbol{\lambda})$ with $l(R'_1) < l(R'_2)$ and $R'_1 \not\leftrightarrow R'_2$. Because $n(\boldsymbol{\lambda}) - M(\boldsymbol{\lambda}) \geq 1$, by (6.7) we have $M_{i,j}(\boldsymbol{\lambda}) < \min\{|R_i|, |R_j|\}$ for some $1 \leq i < j \leq n$, and because $\boldsymbol{\lambda}$ is nesting, we must in fact have $M_{i,j}(\boldsymbol{\lambda}) = 0$. By rotating and cycling, we may assume without loss of generality that $i = 1$ and $l(R_1) < l(R_j)$, and then $l(R_j) \geq r(R_1) + 1$ by Proposition 6.1.4, Parts 2 and 3.

Suppose that $l(R_j) - r(R_1) = 1$, so that $R_i \not\leftrightarrow R_j$. If there is a minimal noncommuting path in $\boldsymbol{\lambda}$ from R_1 to R_j, then by Proposition 6.1.48, either $\boldsymbol{\lambda}$ has a strict sequence, contradicting our assumption, or there is $1 < x < y < j$ with $l(R_x) = l(R_j) = r(R_i) - 1 = r(R_y) - 1$, $R_j \precsim\!\!\!\!/\; R_x$, and $R_1 \precsim\!\!\!\!/\; R_y$, contradicting that $\boldsymbol{\lambda}$ is nesting by Proposition 6.1.63, Part 4. Therefore, there is no minimal noncommuting path in $\boldsymbol{\lambda}$ from R_1 to R_j, so by Lemma 6.1.16 we can find our desired horizontal-strip $\boldsymbol{\lambda}'$. We now suppose that $l(R_j) - r(R_1) \geq 2$, which means that $R_1 \leftrightarrow R_j$ by Proposition 6.1.4, Part 1, and we use induction on $l(R_j) - r(R_i)$.

If there is a minimal noncommuting path $(R_1 = R_{i_1}, \ldots, R_{i_k} = R_j)$ in $\boldsymbol{\lambda}$ from R_1 to R_j, then because $R_1 \leftrightarrow R_j$ we have by Proposition 6.1.64 that $l(R_{i_2}) - r(R_1) = 1$ and we can repeat the above argument with the rows R_1 and R_{i_2}. If there is no minimal noncommuting path in $\boldsymbol{\lambda}$ from R_1 to R_j, then by commuting and cycling we have $(R_1, \ldots, R_{j-1}, R_{j+1}, \ldots, R_n, R_j^-) \in \mathcal{S}(\boldsymbol{\lambda})$ and $l(R_j^-) - r(R_1) < l(R_j) - r(R_1)$, so we are done by our induction hypothesis on $l(R_j) - r(R_1)$. Therefore, there is indeed a horizontal-strip $\boldsymbol{\lambda}' = (R'_1, \ldots, R'_n) \in \mathcal{S}(\boldsymbol{\lambda})$ with $l(R'_1) < l(R'_2)$ and $R'_1 \not\leftrightarrow R'_2$.

We will now strengthen the conditions on our choice of $\boldsymbol{\lambda}'$. Consider the set

$$\mathcal{S}^*(\boldsymbol{\lambda}) = \{\boldsymbol{\lambda}' = (R'_1, \ldots, R'_n) \in \mathcal{S}(\boldsymbol{\lambda}) : \ l(R'_1) < l(R'_2),\ R'_1 \not\leftrightarrow R'_2\} \tag{6.61}$$

and for $\boldsymbol{\lambda}' = (R'_1, \ldots, R'_n) \in \mathcal{S}^*(\boldsymbol{\lambda})$, define the integer

$$h(\boldsymbol{\lambda}') = |\{3 \leq t \leq n : \ R'_1 \prec R'_t,\ R'_2 \prec R'_t\}|. \tag{6.62}$$

Because we have shown that the set $\mathcal{S}^*(\boldsymbol{\lambda})$ is nonempty and because we have a uniform bound $h(\boldsymbol{\lambda}') \leq n - 2$, we may let $\boldsymbol{\lambda}' = (R'_1, \ldots, R'_n) \in \mathcal{S}^*(\boldsymbol{\lambda})$ be such that $h(\boldsymbol{\lambda}')$ is maximal, and among those, with $|R'_2|$ maximal. Let $\boldsymbol{\mu} = (S_1, \ldots, S_n)$ and $\varphi : \Pi(\boldsymbol{\lambda}') \xrightarrow{\sim} \Pi(\boldsymbol{\mu})$, and note that again by Corollary 6.1.59 and Corollary 6.1.62 we may assume that $\boldsymbol{\mu}$ is nesting and has no strict pairs or strict sequences.

Let $i = \varphi_1$ and $j = \varphi_2$, and note that by cycling and rotating we may assume that $i < j$ and $l(S_i) < l(S_j)$, and then because $M_{i,j}(\boldsymbol{\mu}) = 0$, we have $l(S_j) \geq r(S_i) + 1$

by Proposition 6.1.4, Parts 2 and 3. If $l(S_j) - r(S_i) = 1$, then $S_i \not\leftrightarrow S_j$ by Proposition 6.1.4, Part 3, so there exists a good substitute for $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ by Corollary 6.1.30 and we would be done. We now suppose that $l(S_j) - r(S_i) \geq 2$, which means that $S_i \leftrightarrow S_j$ by Proposition 6.1.4, Part 1, and we use induction on $l(S_j) - r(S_i)$. If there is no minimal noncommuting path in $\boldsymbol{\mu}$ from S_i to S_j, then by commuting and cycling we have $(S_i, \ldots, S_{j-1}, S_{j+1}, \ldots, S_n, S_1^-, \ldots, S_j^-) \in \mathcal{S}(\boldsymbol{\mu})$ and $l(S_j^-) - r(S_i) < l(S_j) - r(S_i)$, so we are done by our induction hypothesis on $l(S_j) - r(S_i)$. Therefore, we may assume that there is a minimal noncommuting path $(S_i = S_{i_1}, \ldots, S_{i_k} = S_j)$ in $\boldsymbol{\mu}$ from S_i to S_j. Because $\boldsymbol{\mu}$ is nesting and $S_i \leftrightarrow S_j$, we have that $l(S_{i_{t+1}}) = r(S_{i_t})+1$ for every $1 \leq t \leq k-1$ by Proposition 6.1.64. Also note that if there is a minimal noncommuting path in $\boldsymbol{\mu}$ from S_{i_t} to $S_{i_{t+1}}$ for any $1 \leq t \leq k-1$, then by Proposition 6.1.48 either $\boldsymbol{\mu}$ has a strict sequence, contradicting our assumption, or there is $1 < x < y < j$ with $l(S_x) = l(S_j) = r(S_1) - 1 = r(S_y) - 1$, $S_j \precnsim S_x$, and $S_1 \precnsim S_y$, contradicting that $\boldsymbol{\mu}$ is nesting by Proposition 6.1.63, Part 4. Therefore, there is no minimal noncommuting path in $\boldsymbol{\mu}$ from S_{i_t} to $S_{i_{t+1}}$ for any $1 \leq t \leq k-1$,

We now make the following useful observation. For every row R'_t of $\boldsymbol{\lambda}'$ with $R'_1 \prec R'_t$ and $R'_2 \prec R'_t$, we have $S_i \prec S_{\varphi_t}$ and $S_j \prec S_{\varphi_t}$ and therefore by Proposition 6.1.13 we have

$$l(S_{\varphi_t}) \leq l(S_1) + 1 \leq r(S_1) + 1 = l(S_{i_2}) \leq r(S_{i_{k-1}}) = l(S_j) - 1 \leq r(S_j) - 1 \leq r(S_{\varphi_t}) \quad (6.63)$$

and therefore $S_{i_{t'}} \prec S_{\varphi_t}$ for every $1 \leq t' \leq k$.

Now suppose that there is some t with $S_{i_{k-1}} \precnsim S_t$ and $M_{j,t}(\boldsymbol{\mu}) = 0$. We must have $r(S_t) \leq l(S_j) - 1 = r(S_{i_{k-1}})$ and by Proposition 6.1.13, $l(S_t) \leq l(S_{i_{k-1}}) - 1 = r(S_{i_{k-2}})$, so $M_{i_{k-2},t}(\boldsymbol{\mu}) > 0$. Because $\boldsymbol{\lambda}$ is nesting, this means that either $S_{i_{k-2}} \prec S_t$ or $S_t \prec S_{i_{k-2}}$, but if $S_t \prec S_{i_{k-2}}$ we would have $M_{i_{k-2},t}(\boldsymbol{\mu}) + M_{i_{k-1},t}(\boldsymbol{\mu}) = |S_t| + |S_{i_{k-1}}| \geq |S_t| + 1$ and the pair $(S_{i_{k-2}}, S_{i_{k-1}})$ would be strict, a contradiction by Proposition 6.1.63, Part 1. Therefore, we must have $S_{i_{k-2}} \prec S_t$. Because there is no minimal noncommuting path in $\boldsymbol{\mu}$ from $S_{i_{k-2}}$ to $S_{i_{k-1}}$, we may cycle and use Lemma 6.1.16 to replace $\boldsymbol{\mu}$ with a similar horizontal-strip to assume that $i_{k-2} = 1$ and $i_{k-1} = 2$, but now we would have $h(\boldsymbol{\mu}) > h(\boldsymbol{\lambda}')$ because S_t is counted only by $h(\boldsymbol{\mu})$, contradicting maximality of $h(\boldsymbol{\lambda}')$. Therefore, we may assume that

$$\text{there is no } t \text{ with } S_{i_{k-1}} \precnsim S_t \text{ and } M_{j,t}(\boldsymbol{\mu}) = 0. \quad (6.64)$$

Similarly, now suppose that there is some t with $S_j \precnsim S_t$ and $M_{i_{k-1},t}(\boldsymbol{\mu}) = 0$. Because there is no minimal noncommuting path in $\boldsymbol{\mu}$ from $S_{i_{k-1}}$ to S_j, we may cycle and use Lemma 6.1.16 to replace $\boldsymbol{\mu}$ with a similar horizontal-strip to assume that $i_{k-1} = 1$ and $j = 2$. Because $S_j \precnsim S_t$ and $M_{i_{k-1},t}(\boldsymbol{\mu}) = 0$, by Proposition 6.1.13, we have $l(S_t) = l(S_j)$ and $|S_t| > |S_j|$. Now if there is a minimal noncommuting path in $\boldsymbol{\mu}$ from S_1 to S_t, then as before, by Proposition 6.1.48 either $\boldsymbol{\mu}$ has a strict sequence, contradicting our assumption, or we contradict that $\boldsymbol{\mu}$ is nesting. Therefore, there is no minimal noncommuting path in $\boldsymbol{\mu}$ from S_1 to S_t, so by Lemma 6.1.16 we may replace $\boldsymbol{\mu}$ by a similar horizontal-strip to instead assume that $t = 2$. We also note that for every row $S_{t'}$ of $\boldsymbol{\mu}$ with $S_1 \prec S_{t'}$ and $S_j \prec S_{t'}$, we must have $M_{t,t'}(\boldsymbol{\mu}) > 0$ and therefore either $S_t \prec S_{t'}$ or $S_{t'} \prec S_t$. However, if $S_{t'} \prec S_t$, then we would have $M_{1,t'}(\boldsymbol{\mu}) + M_{t,t'}(\boldsymbol{\mu}) = |S_1| + |S_{t'}| \geq |S_{t'}| + 1$ and the pair (S_1, S_t) would be strict, a contradiction, and therefore $S_t \prec S_{t'}$. However, we now have $h(\boldsymbol{\mu}) \geq h(\boldsymbol{\lambda}')$ and $|S_2| > |S_j| = |R_2'|$, contradicting either the maximality of $h(\boldsymbol{\lambda}')$ or the maximality of $|R_2'|$. Therefore, we may assume that

$$\text{there is no } t \text{ with } S_j \precnsim S_t \text{ and } M_{i_{k-1},t}(\boldsymbol{\mu}) = 0. \tag{6.65}$$

Because there is no minimal noncommuting path in $\boldsymbol{\mu}$ from $S_{i_{k-1}}$ to S_j, we may use Lemma 6.1.16 to replace $\boldsymbol{\mu}$ by a similar horizontal-strip to assume that $j = i_{k-1} + 1$ and without changing $l(S_j) - r(S_i)$. Our plan is now to apply Lemma 6.1.50 to the rows S_{j-1} and S_j to replace $\boldsymbol{\mu}$ by a similar horizontal-strip for which $l(S_j)$ has decreased and $r(S_i)$ has not changed, so that we will be done by our induction hypothesis on $l(S_j) - r(R_i)$. It remains to check the conditions of Proposition 6.1.52.

1. If $M_{j-1,t}(\boldsymbol{\mu}) > 0$ and $M_{j,t}(\boldsymbol{\mu}) > 0$, then because $\boldsymbol{\mu}$ is nesting we must have $S_{j-1} \prec S_t$ or $S_t \prec S_{j-1}$. If $S_t \prec S_{j-1}$, then $M_{j-1,t}(\boldsymbol{\mu}) + M_{j,t}(\boldsymbol{\mu}) = |S_t| + M_{j,t}(\boldsymbol{\mu}) \geq |S_t| + 1$ and the pair (S_{j-1}, S_j) would be strict, contradicting that $\boldsymbol{\mu}$ is nesting. Therefore, we must have $S_{j-1} \prec S_t$ and similarly we must have $S_j \prec S_t$.

2. If $M_{j-1,t}(\boldsymbol{\mu}) > 0$ and $M_{j,t}(\boldsymbol{\mu}) = 0$, then because $\boldsymbol{\mu}$ is nesting we must have $S_{j-1} \prec S_t$ or $S_t \prec S_{j-1}$, but by (6.64) we cannot have $S_{j-1} \precnsim S_t$, so we must have $S_t \prec S_{j-1}$.

3. If $M_{j-1,t}(\boldsymbol{\mu}) = 0$ and $M_{j,t}(\boldsymbol{\mu}) > 0$, then because $\boldsymbol{\mu}$ is nesting we must have $S_j \prec S_t$ or $S_t \prec S_j$, but by (6.65) we cannot have $S_j \precnsim S_t$, so we must have $S_t \prec S_j$.

4. If $S_t \prec S_{j-1}$, then by Proposition 6.1.63, Part 3, we must have $M_{j,t}(\boldsymbol{\mu}) = 0$ and $M_{a,t}(\boldsymbol{\mu}) = 0$ for every $a \in A$. Moreover, if $b \in B$, then $S_{j-1} \prec S_b$ and therefore $S_t \prec S_b$ because by Proposition 6.1.63, Part 2, the relation $\prec$ is transitive.

5. If $S_t \prec S_j$, then by Proposition 6.1.63, Part 3, we must have $M_{j-1,t}(\boldsymbol{\mu}) = 0$ and $M_{a,t}(\boldsymbol{\mu}) = 0$ for every $a \in A$. Moreover, if $b \in B$, then $S_j \prec S_b$ and therefore $S_t \prec S_b$ because by Proposition 6.1.63, Part 2, the relation $\prec$ is transitive.

This concludes our verification of the conditions of Proposition 6.1.52. Therefore, the result follows by Lemma 6.1.50 and our induction hypothesis on $l(S_j) - r(S_i)$. This completes the proof of Lemma 6.1.10, and therefore our proof of Theorem 3.1.8. □

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