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Rapid Communication

The decoupling of damped linear systems in configuration and state spaces

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ABSTRACT

It has recently been reported that any viscously damped linear system can be decoupled in the configuration space by a real, nonlinear, time-dependent transformation. The purpose of this rapid communication is to provide a few clarifying remarks about the decoupling operation. It is shown that, for homogeneous systems, the time-dependent configuration-space decoupling transformation is real, linear and time-invariant when cast in state space. In addition, the configuration–space transformation generates a diagonalizing structure-preserving transformation. In non-homogeneous systems, both the configuration and associated state transformations are nonlinear and depend continuously on the excitation. An example is given of a linear system that can be decoupled in configuration but not in state space.

1. Introduction

The equation of motion of an n-degree-of-freedom viscously damped linear system can be written as

$$\mathbf{M}\ddot{\mathbf{q}}+\mathbf{C}\dot{\mathbf{q}}+\mathbf{K}\mathbf{q}=\mathbf{f}(t), \tag{1}$$

where for passive systems the mass matrix $\mathbf{M}$, the damping matrix $\mathbf{C}$, and the stiffness matrix $\mathbf{K}$ are real, symmetric and positive definite of order n. The Lagrangian coordinate $\mathbf{q}$ and the excitation $\mathbf{f}(t)$ are real n-dimensional column vectors. Unless $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are diagonal, Eq. (1) is coupled, i.e., the ith component equation involves not only q_i and its derivatives but also other coordinates and their derivatives as well. Coordinate coupling presents a considerable challenge to system analysis and design.

When $\mathbf{C}=\mathbf{0}$, Eq. (1) can be readily decoupled by modal analysis, which utilizes a real congruence transformation to diagonalize $\mathbf{M}$ and $\mathbf{K}$ simultaneously. If $\mathbf{C}\neq\mathbf{0}$, the system is said to be classically damped if it can still be decoupled by modal analysis. Rayleigh [1] showed that proportional damping, for which $\mathbf{C}=\alpha\mathbf{M}+\beta\mathbf{K}$, is a particular case of classical damping. Subsequently, Caughey and O'Kelly [2] established that $\mathbf{C}\mathbf{M}^{-1}\mathbf{K}=\mathbf{K}\mathbf{M}^{-1}\mathbf{C}$ is a necessary and sufficient condition under which a system is classically damped. There is, of course, no particular reason why this condition should be satisfied. To be sure, Eq. (1) can always be recast as a first-order equation of dimension $2n$ in state space. If the eigenvalue problem associated with the resulting state equation is non-defective, the state equation can be decoupled by complex modal analysis [3,4]. Upon decoupling, however, the (complex) state variables can no longer be identified as displacements and velocities. Physical insight is thus greatly diminished. This is an important reason why configuration-space decoupling, if possible, is truly preferred.

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In two recent papers [5,6], modal analysis was extended to decouple any damped linear systems in configuration space. The extension is based upon the intuitive process of phase synchronization, which shifts the phase angles in each non-classically damped mode of vibration so as to transform it into a classical mode. The overall decoupling transformation is real, invertible, nonlinear and time-dependent in configuration space. How does this time-dependent transformation operate in state space? What is the state-space interpretation of phase synchronization? A basic objective of the present paper is to answer these questions. The exposition is organized as follows. In Section 2, the time-dependent decoupling transformation associated with phase synchronization is concisely reviewed. This configuration-space decoupling transformation is reformulated in state space in Section 3, with an emphasis on homogeneous systems. In Section 4, it is shown that the configuration-space decoupling transformation also generates a diagonalizing structure-preserving transformation in state space. Two examples are given in Section 5, one of which involves a classically damped system that cannot be decoupled by complex modal analysis in state space. Finally, a summary of findings is provided in Section 6.

2. Decoupling by phase synchronization in configuration space

Based upon a consideration of the physics of damping [5,6], phase synchronization generates a real and invertible transformation that converts Eq. (1) into

$$\ddot{\mathbf{p}}+\mathbf{D}_1\dot{\mathbf{p}}+\mathbf{\Omega}_1\mathbf{p}=\mathbf{g}(t) \tag{2}$$

for which $\mathbf{D}_1$, $\mathbf{\Omega}_1$ are real and diagonal. The diagonal elements of $\mathbf{D}_1$, $\mathbf{\Omega}_1$ are simple combinations of the eigenvalues of the quadratic eigenvalue problem [7–9]

$$(\mathbf{M}\lambda^2+\mathbf{C}\lambda+\mathbf{K})\mathbf{v}=\mathbf{0}. \tag{3}$$

There are competing methods for solving the above equation, and many software packages offer built-in functions to tackle Eq. (3). For example, MATLAB provides the "polyeig" function for solving polynomial eigenvalue problems. While the number of flops (floating point operations) required for solution of $\mathbf{Ku}=\lambda\mathbf{Mu}$ is $14n^3$, solution of Eq. (3) involves about $216n^3$ flops. If each repeated eigenvalue possesses a full complement of independent eigenvectors, Eq. (3) is non-defective. Under this condition

$$\mathbf{g}(t)=\mathbf{T}_1^T\mathbf{f}(t)+\mathbf{T}_2^T\dot{\mathbf{f}}(t), \tag{4}$$

$$\mathbf{q}(t)=\mathbf{T}_1\mathbf{p}(t)+\mathbf{T}_2\dot{\mathbf{p}}(t)-\mathbf{T}_2\mathbf{T}_2^T\mathbf{f}(t)=\left(\mathbf{T}_1+\mathbf{T}_2\frac{\mathrm{d}}{\mathrm{d}t}\right)\mathbf{p}(t)-\mathbf{T}_2\mathbf{T}_2^T\mathbf{f}(t), \tag{5}$$

where $\mathbf{T}_1$, $\mathbf{T}_2$ are real square matrices whose elements are simple combinations of the eigenvalues and corresponding eigenvectors of Eq. (3). In the configuration space, Eq. (5) is a nonlinear time-dependent transformation. A flowchart for fast decoupling has been presented [6]. The above decoupling process and associated equations reduce to modal analysis if Eq. (1) is undamped or classically damped.

2.1. Simplifying assumptions

Although Eqs. (4) and (5) have been extended to defective systems [5,6] so that Eq. (1) can be decoupled without restrictions, this type of generality will be suppressed in the present paper. Unless otherwise stated, it will be assumed that (a) all eigenvalues of Eq. (3) are complex (with non-zero imaginary parts) and distinct, and (b) $\mathbf{f}(t)=\mathbf{0}$. These assumptions are made to streamline the presentation and, as explained later on, they can be readily relaxed.

2.2. Phase synchronization of free vibration

A homogeneous system can be decoupled by simply compensating for the phase drifts caused by viscous damping in each damped mode. If the $2n$ eigenvalues of Eq. (3) are complex, then the eigenvalues λ_j and the corresponding eigenvectors $\mathbf{v}_j$ occur in n pairs of complex conjugates. Let

$$\lambda_j=\alpha_j+i\omega_j,\quad \mathbf{v}_j=\begin{bmatrix} r_{j1}e^{-i\varphi_{j1}} & r_{j2}e^{-i\varphi_{j2}} & \cdots & r_{jn}e^{-i\varphi_{jn}}\end{bmatrix}^T, \tag{6}$$

where α_j, ω_j, r_{jk} and φ_{jk} are real parameters for j, $k=1,\ldots,n$. In phase synchronization, the kth element of the jth damped mode is shifted by φ_{jk}/ω_j. The overall result is the conversion of Eq. (1) into Eq. (2) for which

$$\mathbf{D}_1=-\mathrm{diag}[\lambda_j+\overline{\lambda}_j]=-\mathrm{diag}[2\alpha_1,2\alpha_2,\ldots,2\alpha_n], \tag{7}$$

$$\mathbf{\Omega}_1=\mathrm{diag}[\lambda_j\overline{\lambda}_j]=\mathrm{diag}[\alpha_1^2+\omega_1^2,\alpha_2^2+\omega_2^2,\ldots,\alpha_n^2+\omega_n^2]. \tag{8}$$

In addition, when $\mathbf{f}(t)=\mathbf{0}$, Eq. (5) becomes a time-shifting transformation defined by

$$\mathbf{q}(t)=\sum_{j=1}^{n}\mathrm{diag}[p_j(t-\varphi_{j1}/\omega_j),p_j(t-\varphi_{j2}/\omega_j),\ldots,p_j(t-\varphi_{jn}/\omega_j)]\mathbf{z}_j, \tag{9}$$

where $\mathbf{z}_j = \begin{bmatrix} r_{j1}e^{\alpha_j\varphi_{j1}/\omega_j} & r_{j2}e^{\alpha_j\varphi_{j2}/\omega_j} & \cdots & r_{jn}e^{\alpha_j\varphi_{jn}/\omega_j} \end{bmatrix}^T$. Each eigenvector $\mathbf{v}_j$ of Eq. (3) can only be determined up to an arbitrary multiplicative constant. For convenience, the magnitude of $\mathbf{v}_j$ may be fixed in accordance with

$$2\lambda_j\mathbf{v}_j^T\mathbf{M}\mathbf{v}_j + \mathbf{v}_j^T\mathbf{C}\mathbf{v}_j = 2i\omega_j. \tag{10}$$

The above normalization reduces to the familiar mass normalization $\mathbf{v}_j^T\mathbf{M}\mathbf{v}_k = \delta_{jk}$ for an undamped or classically damped system [10]. If the magnitude and sign of $\mathbf{v}_j$ are fixed, then Eq. (9) is uniquely defined. In contrast, the coefficients $\mathbf{D}_1$, $\mathbf{\Omega}_1$ of the decoupled system, defined in Eqs. (7) and (8), are independent of $\mathbf{v}_j$ and they remain valid even if $\mathbf{f}(t)\neq\mathbf{0}$.

3. State-space formulation of phase synchronization

What is the state-space version of the time-dependent decoupling transformation (9)? Since physical insight is diminished due to (complex) state transformations, it would be laborious to recast and interpret in state space every equation associated with phase synchronization. This is however not necessary. An efficient reformulation is provided if a trial state-space version of Eq. (9) is first surmised through intuition. The trial version is then validated by rigorous mathematics.

3.1. Derivation of reformulated transformation

Under the assumption that the eigenvalues λ_j of Eq. (3) are complex and distinct, the free response of system (1) is

$$\mathbf{q} = \sum_{j=1}^{n}(a_j\mathbf{v}_j e^{\lambda_j t} + \overline{a}_j\overline{\mathbf{v}}_j e^{\overline{\lambda}_j t}) = \mathbf{V}e^{\mathbf{\Lambda}t}\mathbf{a} + \overline{\mathbf{V}}e^{\overline{\mathbf{\Lambda}}t}\overline{\mathbf{a}}, \tag{11}$$

where a_j are constants depending on initial conditions and $\mathbf{a} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}^T$. In addition, $\mathbf{\Lambda}$ and $\mathbf{V}$ are defined in terms of λ_j and $\mathbf{v}_j$ in Eq. (6) by

$$\mathbf{\Lambda} = \text{diag}[\lambda_1,\lambda_2,\ldots,\lambda_n], \quad \mathbf{V} = [\mathbf{v}_1|\mathbf{v}_2|\cdots|\mathbf{v}_n]. \tag{12}$$

The state of Eq. (1) is then given by

$$\begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \mathbf{V} & \overline{\mathbf{V}} \\ \mathbf{V}\mathbf{\Lambda} & \overline{\mathbf{V}\mathbf{\Lambda}} \end{bmatrix} \begin{bmatrix} e^{\mathbf{\Lambda}t} & \mathbf{0} \\ \mathbf{0} & e^{\overline{\mathbf{\Lambda}}t} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \overline{\mathbf{a}} \end{bmatrix}. \tag{13}$$

In free vibration, system (1) can be physically excited into a form of oscillation in which all components perform oscillations with the same frequency ω_j. This is referred to as a damped mode of vibration, and the mode with frequency ω_j can be independently excited by rendering a_k=0 for $k\neq j$. From Eq. (11), the mode with frequency ω_j can be written as

$$\mathbf{s}_j(t) = a_j\mathbf{v}_j e^{\lambda_j t} + \overline{a}_j\overline{\mathbf{v}}_j e^{\overline{\lambda}_j t} = 2e^{\alpha_j t}\text{Re}[a_j\mathbf{v}_j e^{i\omega_j t}]. \tag{14}$$

In system (2), free vibration with frequency ω_j can only be generated by the jth decoupled equation, which has the form

$$\ddot{p}_j - (\lambda_j + \overline{\lambda}_j)\dot{p}_j + \lambda_j\overline{\lambda}_j p_j = 0. \tag{15}$$

Phase synchronization shifts the phase angles of the elements of $\mathbf{s}_j(t)$ but the process does not disturb the coefficients a_j. Hence the solution of Eq. (15) should be

$$p_j = a_j e^{\lambda_j t} + \overline{a}_j e^{\overline{\lambda}_j t} = 2e^{\alpha_j t}\text{Re}[a_j e^{i\omega_j t}]. \tag{16}$$

The state of system (2) is therefore given by

$$\begin{bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda} & \overline{\mathbf{\Lambda}} \end{bmatrix} \begin{bmatrix} e^{\mathbf{\Lambda}t} & \mathbf{0} \\ \mathbf{0} & e^{\overline{\mathbf{\Lambda}}t} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \overline{\mathbf{a}} \end{bmatrix}. \tag{17}$$

If Eqs. (13) and (17) are combined, the state transformation

$$\begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \mathbf{V} & \overline{\mathbf{V}} \\ \mathbf{V}\mathbf{\Lambda} & \overline{\mathbf{V}\mathbf{\Lambda}} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda} & \overline{\mathbf{\Lambda}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{bmatrix} = \mathbf{T}\begin{bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{bmatrix} \tag{18}$$

is obtained. It can be checked that the transformation matrix $\mathbf{T}$ is real and nonsingular. Thus the time-dependent configuration-space transformation (9) becomes a linear time-invariant transformation (18) when cast in state space.

This surprising result, surmised through intuition, can be readily validated. In free vibration, the state-space versions of Eqs. (1) and (2) are given, respectively, by

$$\begin{bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} = \mathbf{A} \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix}, \tag{19}$$

$$\begin{bmatrix} \dot{\mathbf{p}} \\ \ddot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{\Omega}_1 & -\mathbf{D}_1 \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{bmatrix} = \mathbf{B} \begin{bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{bmatrix}, \tag{20}$$

where $\mathbf{D}_1$, $\mathbf{\Omega}_1$ are defined in Eqs. (7) and (8). Observe that the quadratic eigenvalue problems associated with Eqs. (1) and (2) have the same eigenvalues with the same multiplicities. In addition, Eq. (3) and the matrix **A** in Eq. (19) have identical eigenvalues, and the same is true for the quadratic eigenvalue problem associated with Eq. (2) and the matrix **B** in Eq. (20). As a consequence, **A**, **B** have the same eigenvalues, i.e., they are isospectral, and each is diagonalizable because λ_j are assumed distinct. From linear algebra, two diagonalizable matrices are isospectral if and only if they are similar. It can be checked by direct manipulations that

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{B}, \tag{21}$$

where **T** is defined in Eq. (18). Thus Eq. (18) converts Eq. (19) into Eq. (20) through a similarity transformation. The state-space version of Eq. (9) is indeed Eq. (18).

While Eq. (9) decouples Eq. (1) in configuration space, Eq. (18) does not decouple the state-space version of system (1) because **B** is not diagonal. Rather, Eq. (18) operates in such a way that Eq. (19) is converted into Eq. (20), from which the decoupled system (2) is extracted.

3.2. Relaxation of assumptions

Subject to the simplifying assumptions of Section 2.1, the time-dependent decoupling transformation (9) becomes a linear time-invariant transformation in state space. It can be shown that the same observation is true for free vibration under real, complex, or repeated eigenvalues, as long as Eq. (3) is non-defective. If there exist $2r \le 2n$ distinct real eigenvalues, there is an equivalence class of $(2r)!/2^r r!$ different forms of $\mathbf{D}_1$, $\mathbf{\Omega}_1$ associated with the real eigenvalues [6]. For each member of this equivalence class, the corresponding time-dependent configuration-space decoupling transformation is equivalent to a linear time-invariant state transformation. When Eq. (3) is defective, Jordan sub-matrices appear in many formulas associated with decoupling [6]. As a result, both the configuration-space decoupling transformation and its state-space version are time-dependent.

If $\mathbf{f}(t) \neq \mathbf{0}$, the nonlinear configuration-space decoupling transformation (5) depends continuously on the excitation $\mathbf{f}(t)$. Consequently, its reformulated state-space version also involves $\mathbf{f}(t)$. If the eigenvalues of Eq. (3) are complex and distinct, it can be shown that [6]

$$\begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \mathbf{V} & \overline{\mathbf{V}} \\ \mathbf{V}\mathbf{\Lambda} & \overline{\mathbf{V}\mathbf{\Lambda}} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda} & \overline{\mathbf{\Lambda}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{bmatrix} - \begin{bmatrix} (\overline{\mathbf{V}}-\mathbf{V})(\overline{\mathbf{\Lambda}}-\mathbf{\Lambda})^{-2}(\overline{\mathbf{V}}-\mathbf{V})^T\mathbf{f}(t) \\ (\overline{\mathbf{V}\mathbf{\Lambda}}-\mathbf{V}\mathbf{\Lambda})(\overline{\mathbf{\Lambda}}-\mathbf{\Lambda})^{-2}(\mathbf{V}-\overline{\mathbf{V}})^T\mathbf{f}(t) \end{bmatrix} \tag{22}$$

which is a direct extension of Eq. (18). Finally, phase synchronization can be used to decouple systems with symmetric or non-symmetric coefficients, provided that **M** is nonsingular. The observations in this section remain valid when **M**, **C** and **K** are not symmetric. The decoupling of non-symmetric systems will be taken up in a future paper.

4. Decoupling and structure-preserving transformations

A traditional approach to decoupling, as emphasized by Lancaster [7–9], is to address the problem as a reduction of the quadratic pencil $Q(\lambda)=\mathbf{M}\lambda^2+\mathbf{C}\lambda+\mathbf{K}$. Garvey and others [11–14] recently proposed the powerful notion of structure-preserving transformations in diagonalizing $Q(\lambda)$. First, $Q(\lambda)$ is cast in state space as a linear pencil in the form [13]

$$L(\lambda) = \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \lambda + \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix}. \tag{23}$$

A real equivalence transformation $\{\mathbf{U}_L, \mathbf{U}_R\}$ is then sought which preserves the block structure of the coefficients of $L(\lambda)$ in such a way that

$$\mathbf{U}_L^T L(\lambda) \mathbf{U}_R = \begin{bmatrix} \mathbf{C}_D & \mathbf{M}_D \\ \mathbf{M}_D & \mathbf{0} \end{bmatrix} \lambda + \begin{bmatrix} \mathbf{K}_D & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_D \end{bmatrix}, \tag{24}$$

where $\mathbf{M}_D$, $\mathbf{C}_D$, $\mathbf{K}_D$ are real diagonal matrices of order n. Such an equivalence transformation is referred to as a diagonalizing structure-preserving transformation and, if available, it decouples Eq. (1) in the configuration space.

Suppose a diagonalizing structure-preserving transformation $\{\mathbf{U}_L, \mathbf{U}_R\}$ has been determined. In free vibration, Eq. (1) may be expressed in state space as

$$\begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} = \mathbf{0}. \tag{25}$$

Define a real state transformation by

$$\begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} = \mathbf{U}_R \begin{bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{bmatrix}. \tag{26}$$

Then Eq. (25) can be transformed into

$$\begin{bmatrix} \mathbf{C}_D & \mathbf{M}_D \\ \mathbf{M}_D & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{p}} \\ \ddot{\mathbf{p}} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_D & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_D \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{bmatrix} = \mathbf{0} \tag{27}$$

from which the decoupled second-order equation

$$\mathbf{M}_D \ddot{\mathbf{p}} + \mathbf{C}_D \dot{\mathbf{p}} + \mathbf{K}_D \mathbf{p} = \mathbf{0} \tag{28}$$

is obtained. Structure-preserving transformation is a powerful concept, and numerical algorithms to compute $\{\mathbf{U}_L, \mathbf{U}_R\}$ have been studied [14–16]. However, these algorithms are generally quite restrictive [14].

It is now asserted that the time-dependent transformation (9) generates a diagonalizing structure-preserving transformation in state space. Indeed, if the eigenvalues λ_j of Eq. (3) are complex and distinct, then

$$\mathbf{U}_L = \mathbf{U}_R = \mathbf{T} = \begin{bmatrix} \mathbf{V} & \overline{\mathbf{V}} \\ \mathbf{V}\mathbf{\Lambda} & \overline{\mathbf{V}\mathbf{\Lambda}} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda} & \overline{\mathbf{\Lambda}} \end{bmatrix}^{-1}, \tag{29}$$

where the eigenvectors $\mathbf{v}_j$ are still normalized in accordance with Eq. (10). It can be verified by direct manipulations that $\mathbf{M}_D$=$\mathbf{I}$, $\mathbf{C}_D$=$\mathbf{D}_1$, and $\mathbf{K}_D$=$\mathbf{\Omega}_1$ in Eq. (27). Again, this observation remains true under more general conditions. It can be shown that Eq. (9) always generates a diagonalizing structure-preserving transformation as long as Eq. (3) is non-defective. In addition, there are systems that can be decoupled by phase synchronization but not by structure-preserving transformations. Example 4 in [6] is such a system. If $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are non-symmetric, the related structure-preserving transformation is an equivalence but not congruence transformation.

5. Illustrative examples

It is widely believed that complex modal analysis can readily decouple in state space any classically damped system. A counter-example is provided in this section. A second example illustrates the theoretical development presented earlier.

Example 1. A two-degree-of-freedom system of the form (1) is defined by $\mathbf{M}$=$\mathbf{I}$,

$$\mathbf{K} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad \mathbf{C} = 2\sqrt{\mathbf{K}} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}. \tag{30}$$

This damped system is already in a decoupled form, and both modal analysis and phase synchronization reduce to identity transformation in configuration space. The eigenvalues of the state companion matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \\ 0 & -4 & 0 & -4 \end{bmatrix} \tag{31}$$

are λ_1= -1 and λ_2= -2 and each is repeated. However, there is only one eigenvector $\begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}^T$ associated with λ_1 and also only one eigenvector $\begin{bmatrix} 0 & 1 & 0 & -2 \end{bmatrix}^T$ associated with λ_2. Therefore, $\mathbf{A}$ is defective and cannot be diagonalized. As a result, the system in this example cannot be decoupled by complex modal analysis in state space. A generalization is obvious: a classically damped multi-degree-of-freedom system cannot be decoupled by complex modal analysis in state space if one or more degrees are critically damped.

There should not be any confusion about the role played by structure-preserving transformations: they are state-space transformations aiming at decoupling systems in the configuration space. From Eq. (29), a diagonalizing structure-preserving transformation for this example is given by $\mathbf{U}_L$=$\mathbf{U}_R$=$\mathbf{I}$.

Example 2. Consider a non-classically damped system governed by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} 0.5 & -0.1 \\ -0.1 & 1 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{32}$$

Solution of the quadratic eigenvalue problem (3) yields

$$\lambda_1=\overline{\lambda}_3=\alpha_1+i\omega_1=-0.25+0.97i,\quad \mathbf{v}_1=\overline{\mathbf{v}}_3=\begin{bmatrix} r_{11}e^{-i\varphi_{11}} \\ r_{12}e^{-i\varphi_{12}} \end{bmatrix}=\begin{bmatrix} 1.00e^{-i0.0002} \\ -0.03e^{-i1.49} \end{bmatrix}, \tag{33}$$

$$\lambda_2=\overline{\lambda}_4=\alpha_2+i\omega_2=-0.50+1.93i,\quad \mathbf{v}_2=\overline{\mathbf{v}}_4=\begin{bmatrix} r_{21}e^{-i\varphi_{21}} \\ r_{22}e^{-i\varphi_{22}} \end{bmatrix}=\begin{bmatrix} 0.07e^{-i1.66} \\ -1.00e^{-i3.14} \end{bmatrix}, \tag{34}$$

where the eigenvectors have been normalized in accordance with Eq. (10). Phase synchronization converts Eq. (32) into Eq. (2), for which

$$\mathbf{D}_1=-\mathrm{diag}[\lambda_j+\overline{\lambda}_j]=\mathrm{diag}[0.50,\ 1.00], \tag{35}$$

$$\mathbf{\Omega}_1=\mathrm{diag}[\lambda_j\overline{\lambda}_j]=\mathrm{diag}[1.00,\ 3.99], \tag{36}$$

and $\mathbf{g}(t)=0$. The configuration-space decoupling transformation (9) becomes

$$\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}=\begin{bmatrix} \sum_{j=1}^{2} r_{j1}e^{\alpha_j\varphi_{j1}/\omega_j}p_j(t-\varphi_{j1}/\omega_j) \\ \sum_{j=1}^{n} r_{j2}e^{\alpha_j\varphi_{j2}/\omega_j}p_j(t-\varphi_{j2}/\omega_j) \end{bmatrix}=\begin{bmatrix} 1.00p_1(t-0.002)+0.04p_2(t-0.86) \\ -0.02p_1(t-1.53)-0.44p_2(t-1.63) \end{bmatrix}. \tag{37}$$

From Eq. (18), the state-space version of Eq. (37) is given by

$$\begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}=\begin{bmatrix} 1.00 & -0.02 & -0.00 & -0.04 \\ 0.01 & 1.00 & 0.04 & 0.00 \\ 0.00 & 0.14 & 1.00 & 0.01 \\ -0.04 & -0.00 & -0.01 & 1.00 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix}=\mathbf{T}\begin{bmatrix} p_1 \\ p_2 \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix}. \tag{38}$$

The matrix $\mathbf{T}$ above defines a diagonalizing structure-preserving transformation $\{\mathbf{U}_L,\ \mathbf{U}_R\}$ with $\mathbf{U}_L=\mathbf{U}_R=\mathbf{T}$. This transformation converts Eq. (25) into Eq. (27) for which $\mathbf{M}_D=\mathbf{I}$, $\mathbf{C}_D=\mathbf{D}_1$, and $\mathbf{K}_D=\mathbf{\Omega}_1$.

6. Conclusions

Several observations about the decoupling of damped linear systems in configuration and state spaces are summarized in the following remarks:

1. In non-defective homogeneous systems, the real, time-dependent, configuration-space decoupling transformation due to phase synchronization is real, linear and time-invariant when cast in state space. In addition, the configuration-space decoupling transformation generates a diagonalizing structure-preserving transformation. Neither the state-space transformation due to phase synchronization nor the structure-preserving transformation decouples the state equation associated with a second-order system.
2. In non-homogeneous systems, both the configuration-space decoupling transformation and associated state transformation are nonlinear and depend continuously on the excitation.
3. There are damped linear systems that can be decoupled by modal analysis or phase synchronization in configuration space but not by complex modal analysis in state space.

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