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The decoupling of damped linear systems in free or forced vibration

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ABSTRACT

The purpose of this paper is to extend classical modal analysis to decouple any viscously damped linear system in non-oscillatory free vibration or in forced vibration. Based upon an exposition of how exponential decay in a system can be regarded as imaginary oscillations, the concept of damped modes of imaginary vibration is introduced. By phase synchronization of these real and physically excitable modes, a time-varying transformation is constructed to decouple non-oscillatory free vibration. When time drifts caused by viscous damping and by external excitation are both accounted for, a time-varying decoupling transformation for forced vibration is derived. The decoupling procedure devised herein reduces to classical modal analysis for systems that are undamped or classically damped. This paper constitutes the second and final part of a solution to the "classical decoupling problem." Together with an earlier paper, a general methodology that requires only the solution of a quadratic eigenvalue problem is developed to decouple any damped linear system.

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1. Introduction

It has long been recognized that coordinate coupling in damped linear systems is a considerable barrier to analysis and design. In the absence of damping, a linear system possesses classical normal modes, which constitute a coordinate transformation that decouples the undamped system. This process of decoupling the equation of motion of a dynamical system is a time-honored procedure termed modal analysis. A damped linear system cannot be decoupled by modal analysis unless it also possesses a full set of classical normal modes, in which case the system is said to be classically damped. Rayleigh [1] showed that a system is classically damped if its damping matrix is a linear combination of its inertia and stiffness matrices. Classical damping is routinely assumed in applications.

Practically speaking, classical damping means that energy dissipation is almost uniformly distributed throughout the system. In general, there is no reason why this condition should be satisfied and thus damped linear systems cannot be decoupled by modal analysis [2–6]. In addition, it has been shown [7] that no time-invariant linear transformations in the configuration space can decouple all damped systems. Even partial decoupling, i.e. simultaneous transformation of the coefficient matrices of the equation of motion to upper triangular forms, cannot be ensured with time-invariant linear transformations [8]. As a consequence, any universal decoupling transformation in the configuration space must be time-varying or nonlinear. In an earlier paper [9], it was shown that a non-classically damped system in oscillatory free vibration can be transformed into one with classical damping by tuning its damped modes of vibration. This technique, referred to as phase synchronization, generates a real time-varying transformation that decouples the system in configuration space. Furthermore, the decoupling procedure reduces to modal analysis under classical damping. The term phase

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synchronization has been used in such areas as circuit theory and stochastic systems to mean different things. In the present context, it refers to the process of synchronizing the phase angles in a non-classically damped mode so as to transform it into a classical mode.

The purpose of this paper is to build upon phase synchronization to decouple any viscously damped linear system in non-oscillatory free vibration and in forced vibration. The organization of this paper is as follows. In Section 2, the problem of decoupling is formulated and assumptions made only for gaining physical insight are discussed. Phase synchronization of oscillatory free vibration is concisely surveyed in Section 3. This survey sets up the terminology and notation used throughout the paper. The concept of damped modes of imaginary vibration is introduced in Section 4. By synchronizing these physically excitable modes, a time-varying transformation is constructed to decouple non-oscillatory free vibration. In Section 5, both oscillatory and non-oscillatory systems driven by external forces are considered. When time drifts caused by viscous damping and by external excitation are both accounted for, a decoupling transformation for forced vibration is derived. It is also shown that the decoupling procedure devised herein is a direct generalization of classical modal analysis. In Section 6, four examples are given to illustrate the process of decoupling. A summary of major findings is provided in Section 7. This paper constitutes the second and final part of a general methodology to decouple any damped linear system.

2. Problem statement

The equation of motion of an n-degree-of-freedom viscously damped linear system can be written as

$$\mathbf{M}\ddot{\mathbf{q}}+\mathbf{C}\dot{\mathbf{q}}+\mathbf{K}\mathbf{q}=\mathbf{f}(t), \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are real square matrices of order n. The generalized coordinate $\mathbf{q}$ and the excitation $\mathbf{f}(t)$ are real n-dimensional vectors. For passive systems, $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are symmetric and positive definite. Unless the three coefficient matrices are diagonal, Eq. (1) is coupled. Coupling is not an inherent property of a system but depends on the generalized coordinates used. The "classical decoupling problem" is concerned with the transformation of Eq. (1) into

$$\ddot{\mathbf{p}}+\mathbf{D}_1\dot{\mathbf{p}}+\mathbf{\Omega}_1\mathbf{p}=\mathbf{g}(t), \tag{2}$$

where $\mathbf{D}_1$, $\mathbf{\Omega}_1$ are real diagonal matrices of order n, and $\mathbf{p}$ and $\mathbf{g}(t)$ are also real. This is a well-trodden problem that has attracted the attention of many researchers in the past century. Over the years, various types of decoupling approximation were employed in the analysis of damped systems [10–19]. Different indices of coupling were also introduced to quantify coordinate coupling [20–28]. However, a solution to the "classical decoupling problem" has not been reported in the open literature.

When $\mathbf{f}(t)=\mathbf{0}$, the decoupling problem is mathematically equivalent to the problem of simultaneous conversion of $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ into diagonal forms. A traditional approach, as emphasized by Lancaster [29–33], is to address this problem as a reduction of quadratic pencils of matrices. Garvey and others [34–38] recently diagonalized a class of matrix pencils by transformations in state space. To be sure, the problem of simultaneous diagonalization of matrices can also be interpreted from other perspectives [39–42]. As mentioned earlier, any universal decoupling transformation in the configuration space, if it exists, must be at least time-varying or even nonlinear.

2.1. The quadratic eigenvalue problem

Associated with Eq. (1) is the quadratic eigenvalue problem [29,31,33]

$$(\mathbf{M}\lambda^2+\mathbf{C}\lambda+\mathbf{K})\mathbf{v}=\mathbf{0}. \tag{3}$$

There are $2n$ eigenvalues λ_j but there cannot be more than n linearly independent eigenvectors $\mathbf{v}_j$, where $j=1,\ldots,2n$. Since $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are real, any complex eigenvalues and the corresponding eigenvectors must occur in complex conjugate pairs. In addition, the real parts of all eigenvalues must be negative because energy is dissipated by damping.

If the eigenvalues are distinct, the free response of system (1) is a linear combination of the form

$$\mathbf{q}=\sum_{j=1}^{2n} c_j\mathbf{v}_j e^{\lambda_j t}, \tag{4}$$

where c_j are $2n$ constants to be obtained from initial conditions. Clearly, the free response is purely non-oscillatory if all λ_j are negative. The response is (partially) non-oscillatory if some λ_j are real and the rest complex. A method to decouple oscillatory free vibration, for which all λ_j are complex, has already been provided [9]. Thus the present paper has a two-fold objective. First, a transformation is constructed to decouple Eq. (1) when $\mathbf{f}(t)=\mathbf{0}$ and when some λ_j are negative. Second, a general methodology is developed to decouple Eq. (1) in general setting, i.e. when $\mathbf{f}(t)\neq\mathbf{0}$ and when λ_j occur in any combination.

2.2. Assumptions and possible relaxation

In non-oscillatory free vibration, it will of course be assumed that some eigenvalues of the quadratic eigenvalue problem (3) are real. For both free and forced vibrations, it will be assumed that the $2n$ eigenvalues λ_j of Eq. (3) are distinct. This assumption is made to streamline the introduction of new concepts and it can be readily relaxed. There will be little change to the exposition of decoupling under the less restrictive condition that eigenvectors associated with repeated eigenvalues are independent. For example, Eq. (4) remains the free response of system (1) as long as there is a full complement of independent eigenvectors.

When an eigenvalue is repeated m times and a full complement of m independent eigenvectors cannot be found, the eigenvalue problem (3) is said to be defective. As an example, Eq. (3) must be defective if any eigenvalue is repeated more than n times [29]. Decoupling of systems with defective eigenvalue problems is of a purely theoretical nature but is still relatively straightforward. However, physical insight is obscured due to the occurrence of Jordan sub-matrices in the corresponding equations. A numerical example will be provided to indicate how defective systems can be decoupled. It should be recalled that when **M**, **C** and **K** are randomly chosen (uniformly distributed in bounded continuous domains), the probability that Eq. (3) being defective is zero [9]. Should the methodology expounded in this paper be accepted for use, a thorough treatment of defective problems will probably be more deserving.

3. Preliminaries in decoupling

To set up the necessary terminology and notation, the decoupling of classically damped systems and non-classically damped systems in purely oscillatory free vibration will be concisely surveyed.

3.1. Classical modal analysis and associated inadequacy

Since **M**, **K** are positive definite, solution of the symmetric eigenvalue problem

$$\mathbf{K}\mathbf{u} = \lambda\mathbf{M}\mathbf{u} \tag{5}$$

generates n positive eigenvalues λ_i and n real eigenvectors $\mathbf{u}_i$ that are orthogonal with respect to either **M** or **K**. Upon mass normalization such that $\mathbf{u}_i^{\mathrm{T}}\mathbf{M}\mathbf{u}_j = \delta_{ij}$, the eigenvectors constitute the columns of a modal matrix $\mathbf{U} = [\mathbf{u}_1|\mathbf{u}_2|\cdots|\mathbf{u}_n]$. Define a modal transformation by

$$\mathbf{q} = \mathbf{U}\mathbf{p}. \tag{6}$$

In terms of the principal coordinate **p**, Eq. (1) takes the canonical form:

$$\ddot{\mathbf{p}} + \mathbf{D}\dot{\mathbf{p}} + \mathbf{\Omega}\mathbf{p} = \mathbf{U}^{\mathrm{T}}\mathbf{f}(t), \tag{7}$$

where $\mathbf{\Omega} = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$ is the spectral matrix and $\mathbf{D} = \mathbf{U}^{\mathrm{T}}\mathbf{C}\mathbf{U}$ is the modal damping matrix. A system is classically damped if it can be decoupled by classical modal analysis, whereby **D** is diagonal. A necessary and sufficient condition [43] for classical damping is $\mathbf{C}\mathbf{M}^{-1}\mathbf{K} = \mathbf{K}\mathbf{M}^{-1}\mathbf{C}$. There is, of course, no particular reason why this condition should be satisfied. In general, a damped linear system cannot be decoupled by classical modal analysis.

Classical modal analysis utilizes real coordinate transformations and is amenable to physical interpretation. Foss and others [44–47] extended classical modal analysis to a process of complex modal analysis in the state space to treat non-classically damped systems. However, complex modal analysis still cannot decouple all damped linear systems. A condition of non-defective eigenvectors in the state space must be satisfied in order for complex modal analysis to achieve complete decoupling. As an example, complex modal analysis fails to decouple even a classically damped system if one or more degrees of freedom are critically damped. Furthermore, upon state-space transformation it is generally not possible to classify the $2n$ state variables as displacements and velocities. Physical insight is thus greatly diminished. Perhaps it is fair to say that decoupling in configuration space renders decoupling in state space unnecessary; the state of a system can always be obtained from displacements and their time derivatives.

3.2. Phase synchronization of oscillatory free vibration

Suppose all eigenvalues of Eq. (3) are complex and distinct. The eigenvalues λ_j and the corresponding eigenvectors $\mathbf{v}_j$ occur in n pairs of complex conjugates. Let

$$\lambda_j = \alpha_j + \mathrm{i}\omega_j, \tag{8}$$

$$\mathbf{v}_j = [r_{j1}\mathrm{e}^{-\mathrm{i}\varphi_{j1}} \;\; r_{j2}\mathrm{e}^{-\mathrm{i}\varphi_{j2}} \;\; \cdots \;\; r_{jn}\mathrm{e}^{-\mathrm{i}\varphi_{jn}}]^{\mathrm{T}}, \tag{9}$$

where α_j, ω_j, r_{jk} and φ_{jk} are real parameters for $j, k = 1, \ldots, n$. The $2n$ complex conjugate eigensolutions $\mathbf{v}_j\mathrm{e}^{\lambda_j t}$ are sometimes referred to as the complex modes in configuration space [29,48,49]. Two eigensolutions $\mathbf{v}_j\mathrm{e}^{\lambda_j t}$ and $\overline{\mathbf{v}}_j\mathrm{e}^{\overline{\lambda}_j t}$ combine to

generate a non-classically damped mode of vibration defined by the linear combination [9]

$$\mathbf{s}_j(t) = a_j\mathbf{v}_j e^{(\alpha_j+i\omega_j)t} + \overline{a}_j\overline{\mathbf{v}}_j e^{(\alpha_j-i\omega_j)t} = C_j e^{\alpha_j t}\begin{bmatrix} r_{j1}\cos(\omega_j t-\theta_j-\varphi_{j1}) \\ r_{j2}\cos(\omega_j t-\theta_j-\varphi_{j2}) \\ \vdots \\ r_{jn}\cos(\omega_j t-\theta_j-\varphi_{jn}) \end{bmatrix}, \tag{10}$$

where a_j is an arbitrary constant which, in polar form, may be expressed as $2a_j = C_j e^{-i\theta_j}$. The real parameters C_j and θ_j are determined by initial conditions. Each damped mode $\mathbf{s}_j(t)$ is real and physically excitable.

If suitable phase shifts are introduced into each damped mode $\mathbf{s}_j(t)$ so that $\varphi_{j1} = \varphi_{j2} = \cdots = \varphi_{jn} = 0$, then various components of the system vibrate in a synchronous manner, passing through their equilibrium positions at the same instant of time. In other words, the system is transformed into one with classical damping. A basic objective of a process termed phase synchronization is to do just that [9]. Upon phase synchronization, the resulting classically damped system can be decoupled by modal analysis. When $\mathbf{f}(t) = \mathbf{0}$ and all eigenvalues λ_j of Eq. (3) are complex and distinct, Eq. (1) can always be converted into Eq. (2), for which

$$\mathbf{D}_1 = -\mathrm{diag}[\lambda_j + \overline{\lambda}_j] = -\mathrm{diag}[2\alpha_1, 2\alpha_2, \ldots, 2\alpha_n], \tag{11}$$

$$\mathbf{\Omega}_1 = \mathrm{diag}[\lambda_j\overline{\lambda}_j] = \mathrm{diag}[\alpha_1^2+\omega_1^2, \alpha_2^2+\omega_2^2, \ldots, \alpha_n^2+\omega_n^2], \tag{12}$$

and $\mathbf{g}(t) = \mathbf{0}$. The free response $\mathbf{q}(t)$ of Eq. (1) can be recovered from the free response $\mathbf{p}(t)$ of Eq. (2) by

$$\mathbf{q}(t) = \sum_{j=1}^{n} \mathrm{diag}[p_j(t-\varphi_{j1}/\omega_j), p_j(t-\varphi_{j2}/\omega_j), \ldots, p_j(t-\varphi_{jn}/\omega_j)]\mathbf{z}_j, \tag{13}$$

where

$$\mathbf{z}_j = [r_{j1}e^{\alpha_j\varphi_{j1}/\omega_j} \;\; r_{j2}e^{\alpha_j\varphi_{j2}/\omega_j} \;\; \cdots \;\; r_{jn}e^{\alpha_j\varphi_{jn}/\omega_j}]^{\mathrm{T}}. \tag{14}$$

The real time-varying decoupling transformation (13), obtained through solution of only the quadratic eigenvalue problem (3), can be cast in state space in the form

$$\begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda} & \overline{\mathbf{\Lambda}} \end{bmatrix}\begin{bmatrix} \mathbf{V} & \overline{\mathbf{V}} \\ \mathbf{V\Lambda} & \overline{\mathbf{V}}\,\overline{\mathbf{\Lambda}} \end{bmatrix}^{-1}\begin{bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{bmatrix}, \tag{15}$$

where $\mathbf{V}$ and $\mathbf{\Lambda}$ are given in terms of λ_j and $\mathbf{v}_j$ in Eqs. (8) and (9) by

$$\mathbf{V} = [\mathbf{v}_1|\mathbf{v}_2|\cdots|\mathbf{v}_n], \tag{16}$$

$$\mathbf{\Lambda} = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]. \tag{17}$$

It can be readily shown that the overall transformation matrix in Eq. (15) is real and invertible. Unlike Eq. (13), the displacement and velocities can no longer be separated in Eq. (15). Initial conditions of $\mathbf{q}(t)$ and $\mathbf{p}(t)$ can be connected by putting $t = 0$ in either Eq. (13) or (15). Perhaps it should be emphasized that Eq. (15) does not decouple the first-order state equation associated with the free vibration of Eq. (1). Rather, it transforms the state equation in such a way that when a second-order equation is extracted from the new state equation, the extracted equation is Eq. (2) with coefficients defined by Eqs. (11), (12) and with $\mathbf{g}(t) = \mathbf{0}$.

Finally, each eigenvector of Eq. (3) can only be determined up to an arbitrary multiplicative constant. The eigenvectors $\mathbf{v}_j$ may be normalized for convenience in accordance with

$$2\lambda_j\mathbf{v}_j^{\mathrm{T}}\mathbf{M}\mathbf{v}_j + \mathbf{v}_j^{\mathrm{T}}\mathbf{C}\mathbf{v}_j = 2i\omega_j, \tag{18}$$

which implies that $2\overline{\lambda}_j\overline{\mathbf{v}}_j^{\mathrm{T}}\mathbf{M}\overline{\mathbf{v}}_j + \overline{\mathbf{v}}_j^{\mathrm{T}}\mathbf{C}\overline{\mathbf{v}}_j = -2i\omega_j$, for $\overline{\mathbf{v}}_j$. The above normalization reduces to normalization with respect to the mass matrix $\mathbf{M}$ for an undamped or classically damped system [6]. A flowchart depicting the decoupling of oscillatory free vibration is shown in Fig. 1, for which $c = n$, $r = 0$, $\mathbf{f}(t) = \mathbf{g}(t) = \mathbf{0}$, and Eq. (69) coincides with Eq. (13). Phase synchronization can also be used to decouple defective systems [9], in which case the phase angles between different Jordan sub-matrices are shifted. It has been shown that the method of phase synchronization is a direct generalization of classical modal analysis.

4. The mechanics of decoupling of non-oscillatory free vibration

To begin, consider purely non-oscillatory free vibration for which all eigenvalues λ_j of Eq. (3) are real and distinct. The corresponding eigenvectors $\mathbf{v}_j$ can all be chosen real. By convention, the eigenvalues will be arranged in an increasing order such that $\lambda_1 < \lambda_2 < \cdots < \lambda_{2n} < 0$. Can phase synchronization [9], applicable to oscillatory free vibration, be modified to decouple non-oscillatory free vibration? Where are the phase angles to synchronize? What is the meaning of a damped mode of vibration when λ_j are real? These questions will be examined in an organized manner.

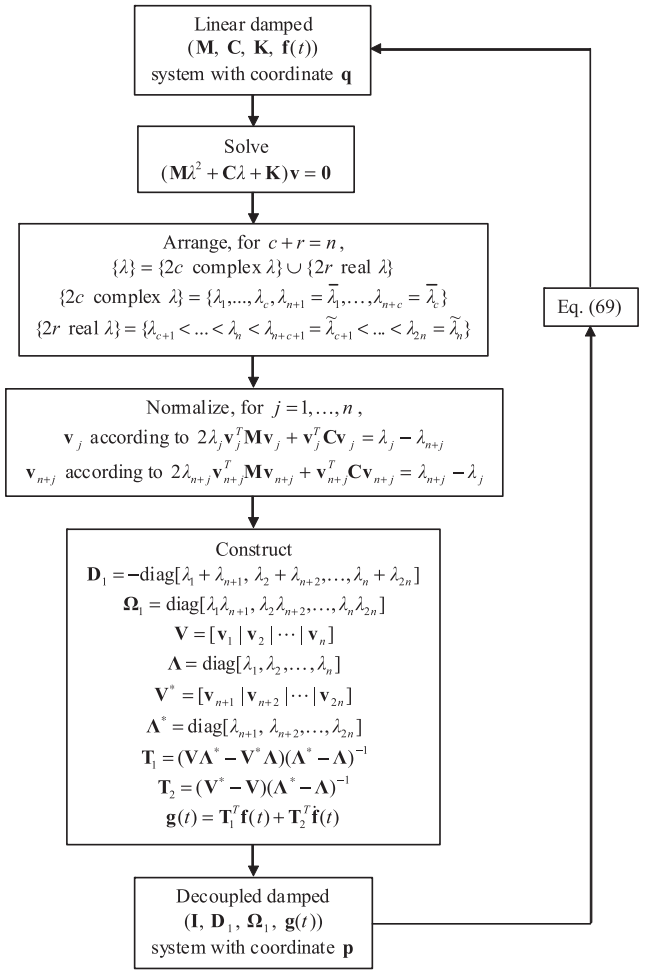

Fig. 1. Flowchart for decoupling and for response calculation of damped linear systems, for which primary–secondary pairing of the real eigensolutions is used. All required parameters are obtained through solution of a quadratic eigenvalue problem.

4.1. The concept of imaginary vibration

From a strict mathematical viewpoint, there need not be day and night distinction between oscillatory and non-oscillatory behaviors. Real exponential decay of a damped system can be regarded as oscillations with an imaginary frequency. To demonstrate this, consider a single-degree-of-freedom system in non-oscillatory free vibration. If $n=1$, the coefficient matrices in Eq. (1) become positive scalars m, c and k. The two real and distinct eigenvalues may be written as

$$\lambda_1, \lambda_2 = \frac{1}{2m}(-c \mp \sqrt{c^2-4mk}) = \alpha \pm \mathrm{i}\omega, \tag{19}$$

where $c^2-4mk>0$ and

$$\alpha = -\frac{c}{2m}, \quad \omega = \frac{\mathrm{i}\sqrt{c^2-4mk}}{2m}. \tag{20}$$

In terms of complex trigonometric functions [50], the free response of Eq. (1) is

$$q(t) = c_1 \mathrm{e}^{\lambda_1 t} + c_2 \mathrm{e}^{\lambda_2 t} = C\mathrm{e}^{\alpha t}\cos(\omega t - \varphi). \tag{21}$$

The constants c_1, c_2 or C, φ depend on initial conditions. It can be verified that

$$C^2 = q^2(0) + \left(\frac{\dot{q}(0)-\alpha q(0)}{\omega}\right)^2, \quad \tan\varphi = \frac{\dot{q}(0)-\alpha q(0)}{\omega q(0)}. \tag{22}$$

In Eq. (21), real exponential decay has been represented as oscillations with an imaginary frequency ω, for which the amplitude C and phase angle φ may not be real.

Since imaginary vibration exhibits itself physically as real exponential decay, its characterization can be subtle at times. For example, if the initial displacement and velocity are such that

$$q(0) = \mathrm{i}\left(\frac{\dot{q}(0)-\alpha q(0)}{\omega}\right), \tag{23}$$

then $C=0$ and $\tan\varphi = 1/\mathrm{i}$. The last expression implies that $\tanh \mathrm{i}\varphi = 1$ and, therefore, $\mathrm{i}\varphi = \infty$. Under condition (23), Eq. (21) becomes

$$q(t) = (C\mathrm{e}^{\alpha t})\cosh(\mathrm{i}\omega t - \mathrm{i}\varphi) = 0\cdot\infty. \tag{24}$$

The above indeterminate form can be evaluated by recognizing that $q(t)$ represents real exponential decay. It can be checked, under condition (23), that $c_1 = 0$ and $c_2 = q(0)$. Thus Eq. (24) is equivalent to

$$q(t) = q(0)\mathrm{e}^{\lambda_2 t} = q(0)\mathrm{e}^{(\alpha-\mathrm{i}\omega)t}. \tag{25}$$

Finally, it is not necessary to investigate the infinite nature of the phase angle φ when $\tan\varphi = 1/\mathrm{i}$. In stereographic projection of the unit sphere onto the complex plane, all positions of infinity on the complex plane are equivalent as they are identified with the north pole [51].

4.2. Real quadratic conjugation

Were the two eigenvalues λ_1, λ_2 in Eq. (19) complex, they would be complex conjugates of each other. Since λ_1, λ_2 are real, they may be referred to as real quadratic conjugates using the notation $\lambda_2 = \tilde{\lambda}_1$. It will be evident that real quadratic conjugation is an important concept in interpreting imaginary oscillations. In order to take advantage of the algebra of complex variables, real quadratic conjugates will be expressed in complex notation. A pair of real quadratic conjugates, say d and $\tilde{d}$, are defined as two real roots of a common quadratic equation. In analogy to Eq. (19), one can write in rectangular form $d, \tilde{d} = a \pm \mathrm{i}b$, where a is real and b is imaginary. In polar form, $d, \tilde{d} = r\mathrm{e}^{\mp \mathrm{i}\theta}$, where the amplitude r and phase angle θ are defined by

$$r^2 = d\tilde{d}, \quad \theta = \frac{\mathrm{i}}{2}\ln\left(\frac{d}{\tilde{d}}\right). \tag{26}$$

Although $d, \tilde{d}$ are real, the amplitude r may be real or imaginary. On the other hand, the phase angle θ is either imaginary or complex with $-\pi/2$ as the real part such that [50]

$$\theta = -\frac{\pi}{2} + \frac{\mathrm{i}}{2}\ln\left|\frac{d}{\tilde{d}}\right|. \tag{27}$$

Similar to the evaluation of Eq. (24), polar representations of real quadratic conjugates should be interpreted appropriately in limiting cases ($\tilde{d} = 0$ for example). Finally, any complex number $z = a + \mathrm{i}b$ has a unique complex conjugate $\overline{z} = a - \mathrm{i}b$ but real quadratic conjugation need not be unique. In fact, any two real numbers c_1, c_2 form a pair of real quadratic conjugates because they may be regarded as the roots of the same quadratic equation.

Real quadratic conjugation generates imaginary oscillations in much the same fashion that complex conjugation generates real oscillations. Let $c_1 = \tilde{c}_2 = r\mathrm{e}^{-\mathrm{i}\theta}$ in Eq. (21). It can be readily shown that

$$c_1\mathrm{e}^{\lambda_1 t} + c_2\mathrm{e}^{\lambda_2 t} = r\mathrm{e}^{-\mathrm{i}\theta}\mathrm{e}^{(\alpha+\mathrm{i}\omega)t} + r\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{(\alpha-\mathrm{i}\omega)t} = 2r\mathrm{e}^{\alpha t}\cos(\omega t - \theta), \tag{28}$$

where ω is an imaginary frequency. An upshot is that Eq. (28) can be extended to higher dimensions. Let $\mathbf{c}_1$, $\mathbf{c}_2$ be two real column vectors of order n. The jth elements of $\mathbf{c}_1$, $\mathbf{c}_2$ are two real numbers which may be regarded as a pair of real quadratic conjugates. Thus it is permissible to write

$$\mathbf{c}_1 = \tilde{\mathbf{c}}_2 = [r_1\mathrm{e}^{-\mathrm{i}\theta_1} \;\; r_2\mathrm{e}^{-\mathrm{i}\theta_2} \;\; \cdots \;\; r_n\mathrm{e}^{-\mathrm{i}\theta_n}]^{\mathrm{T}}. \tag{29}$$

It follows from Eq. (28) that

$$\mathbf{c}_1\mathrm{e}^{\lambda_1 t} + \mathbf{c}_2\mathrm{e}^{\lambda_2 t} = \mathbf{c}_1\mathrm{e}^{\lambda_1 t} + \tilde{\mathbf{c}}_1\mathrm{e}^{\tilde{\lambda}_1 t} = 2\mathrm{e}^{\alpha t}\begin{bmatrix} r_1\cos(\omega t-\theta_1) \\ r_2\cos(\omega t-\theta_2) \\ \vdots \\ r_n\cos(\omega t-\theta_n) \end{bmatrix}. \tag{30}$$

4.3. Damped modes of imaginary vibration

If the eigenvalues λ_j of Eq. (3) are complex, a real damped mode of vibration is formed by pairing $\mathbf{v}_j\mathrm{e}^{\lambda_j t}$ with $\overline{\mathbf{v}}_j\mathrm{e}^{\overline{\lambda}_j t}$ as depicted in Eq. (10). This natural pairing scheme breaks down when all λ_j are real because the eigensolutions are also real. However, as is evident in Eq. (30), any two real eigensolutions may be regarded as real quadratic conjugates of each other

and they can be paired to generate a damped mode of imaginary vibration. There are

$$N=\frac{\binom{2n}{2}\binom{2n-2}{2}\binom{2n-4}{2}\cdots\binom{2}{2}}{n!}=\frac{(2n)!}{2^n n!} \tag{31}$$

different ways to pair the real eigensolutions by real quadratic conjugation. Indeed, there are N different ways to decouple Eq. (1) when all λ_j are real. One such scheme, termed primary–secondary pairing, will be devised for the construction of an algorithm for decoupling.

Among the $2n$ real eigenvalues $\lambda_1<\lambda_2<\cdots<\lambda_{2n}<0$, the n largest eigenvalues are referred to as primary eigenvalues and the n smallest eigenvalues are termed secondary eigenvalues. As established by Lancaster [29,33], there are n linearly independent eigenvectors associated with the primary eigenvalues and likewise for the secondary eigenvalues. In addition, there is usually a gap between the sets of primary and secondary eigenvalues. Identify the primary eigenvalues as real quadratic conjugates of the secondary eigenvalues so that

$$\lambda_1<\cdots<\lambda_j<\cdots<\lambda_n<\lambda_{n+1}=\tilde{\lambda}_1<\cdots<\lambda_{n+j}=\tilde{\lambda}_j<\cdots<\lambda_{2n}=\tilde{\lambda}_n<0. \tag{32}$$

One version of the fundamental theorem of algebra states that the characteristic polynomial associated with Eq. (3) possesses n real quadratic factors. These quadratic factors are not unique, and Eq. (32) simply states that λ_j, λ_{n+j} are real roots of the same quadratic factor. The corresponding eigenvectors $\mathbf{v}_j$, $\mathbf{v}_{n+j}$ may also be regarded as real quadratic conjugates and, similar to Eq. (29), one can write $\mathbf{v}_j=\tilde{\mathbf{v}}_{n+j}=[r_{j1}\mathrm{e}^{-\mathrm{i}\varphi_{j1}}\ \ r_{j2}\mathrm{e}^{-\mathrm{i}\varphi_{j2}}\ \ \cdots\ \ r_{jn}\mathrm{e}^{-\mathrm{i}\varphi_{jn}}]^{\mathrm{T}}$ for $j=1,\ldots,n$. Express the real eigenvalues in rectangular form such that $\lambda_j,\tilde{\lambda}_j=\alpha_j\pm\mathrm{i}\omega_j$ where, in the decoupling algorithm based upon primary–secondary pairing, α_j, ω_j are computed from

$$\alpha_j=\tfrac{1}{2}(\tilde{\lambda}_j+\lambda_j)=\tfrac{1}{2}(\lambda_{n+j}+\lambda_j), \tag{33}$$

$$\omega_j=\frac{\mathrm{i}}{2}(\tilde{\lambda}_j-\lambda_j)=\frac{\mathrm{i}}{2}(\lambda_{n+j}-\lambda_j). \tag{34}$$

In analogy to Eq. (10), construct a damped mode of imaginary vibration by pairing $\mathbf{v}_j\mathrm{e}^{\lambda_j t}$ with $\tilde{\mathbf{v}}_j\mathrm{e}^{\tilde{\lambda}_j t}=\mathbf{v}_{n+j}\mathrm{e}^{\lambda_{n+j}t}$ in the real linear combination

$$\mathbf{s}_j(t)=a_j\mathbf{v}_j\mathrm{e}^{\lambda_j t}+b_j\mathbf{v}_{n+j}\mathrm{e}^{\lambda_{n+j}t}=a_j\mathbf{v}_j\mathrm{e}^{\lambda_j t}+\tilde{a}_j\tilde{\mathbf{v}}_j\mathrm{e}^{\tilde{\lambda}_j t}=C_j\mathrm{e}^{\alpha_j t}\begin{bmatrix} r_{j1}\cos(\omega_j t-\theta_j-\varphi_{j1})\\ r_{j2}\cos(\omega_j t-\theta_j-\varphi_{j2})\\ \vdots\\ r_{jn}\cos(\omega_j t-\theta_j-\varphi_{jn})\end{bmatrix}, \tag{35}$$

where, once again, a_j, b_j may be regarded as real quadratic conjugates and, in polar form $2a_j=2\tilde{b}_j=C_j\mathrm{e}^{-\mathrm{i}\theta_j}$. Note that Eq. (35) becomes identical to Eq. (10) if real quadratic conjugation is replaced by complex conjugation. Although C_j, ω_j, θ_j, r_{jk} and φ_{jk} may not be real, $\mathbf{s}_j(t)$ in Eq. (35) is always real. Indeed, each damped mode of imaginary vibration $\mathbf{s}_j(t)$ can be physically excited with the real initial conditions

$$\mathbf{q}(0)=a_j\mathbf{v}_j+\tilde{a}_j\tilde{\mathbf{v}}_j=C_j\begin{bmatrix} r_{j1}\cos(\theta_j+\varphi_{j1})\\ r_{j2}\cos(\theta_j+\varphi_{j2})\\ \vdots\\ r_{jn}\cos(\theta_j+\varphi_{jn})\end{bmatrix}, \tag{36}$$

$$\dot{\mathbf{q}}(0)=\lambda_j a_j\mathbf{v}_j+\tilde{\lambda}_j\tilde{a}_j\tilde{\mathbf{v}}_j=\alpha_j C_j\begin{bmatrix} r_{j1}\cos(\theta_j+\varphi_{j1})\\ r_{j2}\cos(\theta_j+\varphi_{j2})\\ \vdots\\ r_{jn}\cos(\theta_j+\varphi_{jn})\end{bmatrix}+\omega_j C_j\begin{bmatrix} r_{j1}\sin(\theta_j+\varphi_{j1})\\ r_{j2}\sin(\theta_j+\varphi_{j2})\\ \vdots\\ r_{jn}\sin(\theta_j+\varphi_{jn})\end{bmatrix}. \tag{37}$$

Furthermore, it can be observed from Eq. (4) that purely non-oscillatory free response of Eq. (1) is simply a superposition of n damped modes of imaginary vibration.

4.4. Phase synchronization of imaginary vibration

It has been explained that oscillatory free vibration has no divine rights in its functional representation as a superposition of sinusoidal oscillations. Every formula established for oscillatory free vibration can be suitably applied to purely non-oscillatory free vibration with the use of imaginary frequencies. The upshot is that the methodology of decoupling by phase synchronization [9], developed for oscillatory systems, can be applied to non-oscillatory free vibration if a pairing scheme is chosen for the real eigensolutions. Phase synchronization of a non-oscillatory system amounts to introducing complex phase shifts to each damped mode in Eq. (35) so that $\varphi_{j1}=\varphi_{j2}=\cdots=\varphi_{jn}=0$ for $j=1,\ldots,n$. A system possessing these synchronized modes is classically damped. Basically, all equations for decoupling oscillatory free

vibration can be carried over to purely non-oscillatory free vibration provided that complex conjugation is suitably replaced by real quadratic conjugation. In particular, the real decoupled system (2) is defined by

$$\mathbf{D}_1 = -\mathrm{diag}[\lambda_j + \tilde{\lambda}_j] = -\mathrm{diag}[\lambda_j + \lambda_{n+j}] = -\mathrm{diag}[2\alpha_1, 2\alpha_2, \ldots, 2\alpha_n], \tag{38}$$

$$\mathbf{\Omega}_1 = \mathrm{diag}[\lambda_j \tilde{\lambda}_j] = \mathrm{diag}[\lambda_j \lambda_{n+j}] = \mathrm{diag}[\alpha_1^2 + \omega_1^2, \alpha_2^2 + \omega_2^2, \ldots, \alpha_n^2 + \omega_n^2], \tag{39}$$

where $\mathbf{g}(t) = \mathbf{0}$.

Once the solution $\mathbf{p}(t)$ of the decoupled system (2) is obtained, the free response $\mathbf{q}(t)$ of Eq. (1) can be recovered by using Eq. (13), which always generates a real $\mathbf{q}(t)$. Again, Eq. (13) can be cast as a state-space transformation

$$\begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda} & \tilde{\mathbf{\Lambda}} \end{bmatrix} \begin{bmatrix} \mathbf{V} & \tilde{\mathbf{V}} \\ \mathbf{V\Lambda} & \tilde{\mathbf{V}}\tilde{\mathbf{\Lambda}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{bmatrix}, \tag{40}$$

where $\mathbf{V}$, $\mathbf{\Lambda}$ are still defined by Eqs. (16) and (17). Unlike Eq. (13), $\mathbf{q}(t)$ and $\mathbf{p}(t)$ are connected at the same instant in Eq. (40) but displacements and velocities can no longer be separated. Initial conditions between $\mathbf{q}(t)$ and $\mathbf{p}(t)$ are connected either by

$$\mathbf{q}(0) = \sum_{j=1}^{n} \mathrm{diag}[p_j(-\varphi_{j1}/\omega_j), p_j(-\varphi_{j2}/\omega_j), \ldots, p_j(-\varphi_{jn}/\mathrm{i}\omega_j)]\mathbf{z}_j, \tag{41}$$

$$\dot{\mathbf{q}}(0) = \sum_{j=1}^{n} \mathrm{diag}[\dot{p}_j(-\varphi_{j1}/\mathrm{i}\omega_j), \dot{p}_j(-\varphi_{j2}/\mathrm{i}\omega_j), \ldots, \dot{p}_j(-\varphi_{jn}/\mathrm{i}\omega_j)]\mathbf{z}_j, \tag{42}$$

or by putting t=0 in Eq. (40). Observe that the exact solution $\mathbf{p}(t)$ of the decoupled system (2) is given by

$$p_j(t) = \frac{\tilde{\lambda}_j p_j(0) - \dot{p}_j(0)}{\tilde{\lambda}_j - \lambda_j} \mathrm{e}^{\lambda_j t} - \frac{\lambda_j p_j(0) - \dot{p}_j(0)}{\tilde{\lambda}_j - \lambda_j} \mathrm{e}^{\tilde{\lambda}_j t} \tag{43}$$

for $j = 1, \ldots, n$. If Eq. (43) is substituted into (13), an analytical expression of $\mathbf{q}(t)$ can be obtained. If λ_j are complex, it has been shown [9] that the calculation of oscillatory free response by phase synchronization is more efficient than direct numerical integration. Since every decoupling formula valid for complex λ_j can be suitably carried over to real λ_j, the improvement in efficiency in response calculation is comparable when λ_j are all real or all complex. The issue of efficiency will be revisited under forced vibration.

In the solution of Eq. (3), each eigenvector can only be determined up to an arbitrary multiplicative constant. The real eigenvectors $\mathbf{v}_j$ may be normalized for convenience in accordance with, for example

$$2\lambda_j \mathbf{v}_j^{\mathrm{T}} \mathbf{M} \mathbf{v}_j + \mathbf{v}_j^{\mathrm{T}} \mathbf{C} \mathbf{v}_j = 2\mathrm{i}\omega_j = \lambda_j - \lambda_{n+j}, \tag{44}$$

which implies that $2\tilde{\lambda}_j \tilde{\mathbf{v}}_j^{\mathrm{T}} \mathbf{M} \tilde{\mathbf{v}}_j + \tilde{\mathbf{v}}_j^{\mathrm{T}} \mathbf{C} \tilde{\mathbf{v}}_j = -2\mathrm{i}\omega_j = \lambda_{n+j} - \lambda_j$ for $\tilde{\mathbf{v}}_j$. The above normalization reduces to normalization with respect to the mass matrix $\mathbf{M}$ for an undamped or classically damped system. Any normalization can only specify the magnitude of $\mathbf{v}_j$ but its sign is still arbitrary. With or without normalization the equations of phase synchronization are not unique although the decoupling procedure is always valid. This issue of non-uniqueness should not be surprising since a similar situation exists in classical modal analysis: the modal transformation (6) is not unique because any column $\mathbf{u}_j$ of the modal matrix $\mathbf{U}$ may be replaced by $-\mathbf{u}_j$. Decoupling by phase synchronization of damped modes of imaginary vibration is a direct generalization of classical modal analysis. A flowchart depicting the decoupling and response calculation of purely non-oscillatory free vibration is shown in Fig. 1, for which $c = 0$, $r = n$, $\mathbf{f}(t) = \mathbf{g}(t) = \mathbf{0}$, and Eq. (69) coincides with Eq. (13). All parameters required for decoupling and response calculation are obtained through solution of the quadratic eigenvalue problem (3).

Thus far phase synchronization of imaginary vibration has been presented in terms of complex trigonometric functions. The entire exposition can instead be given in terms of hyperbolic functions using relationships such as $\cos(\mathrm{i}x) = \cosh x$ and $\sin(\mathrm{i}x) = \mathrm{i}\sinh x$. However, complex parameters still appear here and there. For this and other reasons, the hyperbolic version of the decoupling of purely non-oscillatory free vibration will not be given.

4.5. Duality and non-uniqueness

As previously shown, oscillatory and non-oscillatory behaviors are theoretically equivalent. Exponential decay can simply be regarded as vibration with an imaginary frequency and there is a degree of duality between real and imaginary oscillations. Yet, when λ_j are complex, there is a unique way of pairing the eigensolutions to form the damped modes of real oscillations: $\mathbf{v}_j \mathrm{e}^{\lambda_j t}$ is paired with its complex conjugate $\overline{\mathbf{v}}_j \mathrm{e}^{\overline{\lambda}_j t}$ as depicted in Eq. (10). This natural pairing scheme breaks down when λ_j are real. There are N different ways to pair the real eigensolutions by real quadratic conjugation, where N is given by Eq. (31). A flowchart for decoupling oscillatory free vibration [9] cannot be used for non-oscillatory free vibration unless an explicit pairing scheme is devised for the real eigensolutions. The primary–secondary pairing scheme shown in Fig. 1 is optimal in the sense that, on average, real eigensolutions with well separated decay exponents are paired.

Independently of the chosen pairing scheme, the method of phase synchronization is always applicable. As long as the eigenvalues are distinct, Eq. (43) implies that all N pairing schemes are comparable in such a way that every decoupled degree of freedom is overdamped. A numerical example will be given to illustrate the use of different pairing schemes.

Why is there a natural and unique way of pairing the eigensolutions only when λ_j are complex? What is the extent of duality between real and imaginary oscillations? Theoretically speaking, the pairing of $\mathbf{v}_j e^{\lambda_j t}$ with $\overline{\mathbf{v}}_j e^{\overline{\lambda}_j t}$ when λ_j are complex is mandated by the constraint that Eq. (1) is real and one also insists on real and physically excitable modes of vibration. If Eq. (1) and its damped modes could be complex, pairing by complex conjugation would break down. In this case there would also be N different pairing schemes. Since the objective of this paper is to develop a practical method for decoupling damped linear systems, this issue of duality will not be pursued further.

4.6. Free vibration with mixed eigenvalues

In (partially) non-oscillatory vibration, both real and complex eigenvalues λ_j of Eq. (3) occur simultaneously. Free vibration with mixed eigenvalues can be decoupled by dividing its eigensolutions into two groups: one associated with the real eigenvalues and the other with complex eigenvalues. The complex eigensolutions are treated by complex conjugation and the real ones by real quadratic conjugation. Let $2c$ eigenvalues be complex and $2r=2(n-c)$ be real. Separate the distinct eigenvalues into two disjoint sets so that

$$\{\lambda\}=\{2c \text{ complex } \lambda\}\cup\{2r \text{ real } \lambda\}. \tag{45}$$

The complex eigenvalues occur as c pairs of complex conjugates and the real eigenvalues can be classified into primary and secondary eigenvalues. Enumerate the eigenvalues in such a way that

$$\{2c \text{ complex } \lambda\}=\{\lambda_1,\ldots,\lambda_c,\lambda_{n+1}=\overline{\lambda}_1,\ldots,\lambda_{n+c}=\overline{\lambda}_c\}, \tag{46}$$

$$\{2r \text{ real } \lambda\}=\{\lambda_{c+1}<\cdots<\lambda_n<\lambda_{n+c+1}=\tilde{\lambda}_{c+1}<\cdots<\lambda_{2n}=\tilde{\lambda}_n\}. \tag{47}$$

The above indexing means that the first c eigenvalues are complex and the next $r=n-c$ are the real secondary eigenvalues. These are followed by c complex conjugates of the first set and finally by r real primary eigenvalues. This issue of indexing may be important in computer programming. It permits the normalization Eqs. (18) and (44) to be combined in a streamlined fashion.

Simultaneous phase synchronization of the damped modes of real and imaginary vibration produces a decoupled system (2) for which

$$\mathbf{D}_1=-\mathrm{diag}[\lambda_j+\overline{\lambda}_j,\lambda_k+\tilde{\lambda}_k]=-\mathrm{diag}[\lambda_1+\lambda_{n+1},\lambda_2+\lambda_{n+2},\ldots,\lambda_n+\lambda_{2n}], \tag{48}$$

$$\mathbf{\Omega}_1=\mathrm{diag}[\lambda_j\overline{\lambda}_j,\lambda_k\tilde{\lambda}_k]=\mathrm{diag}[\lambda_1\lambda_{n+1},\lambda_2\lambda_{n+2},\ldots,\lambda_n\lambda_{2n}]. \tag{49}$$

In the configuration space, the decoupling transformation is still given by Eq. (13). In state space, the decoupling transformation is

$$\begin{bmatrix}\mathbf{p}(t)\\ \dot{\mathbf{p}}(t)\end{bmatrix}=\begin{bmatrix}\mathbf{I} & \mathbf{I}\\ \mathbf{\Lambda} & \mathbf{\Lambda}^*\end{bmatrix}\begin{bmatrix}\mathbf{V} & \mathbf{V}^*\\ \mathbf{V\Lambda} & \mathbf{V}^*\mathbf{\Lambda}^*\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{q}(t)\\ \dot{\mathbf{q}}(t)\end{bmatrix}, \tag{50}$$

where $\mathbf{V}$, $\mathbf{\Lambda}$ are still defined by Eqs. (16) and (17). In addition, $\mathbf{V}^*$ and $\mathbf{\Lambda}^*$ are given by

$$\mathbf{V}^*=[\overline{\mathbf{v}}_1|\cdots|\overline{\mathbf{v}}_c|\tilde{\mathbf{v}}_{c+1}|\cdots|\tilde{\mathbf{v}}_n]=[\mathbf{v}_{n+1}|\mathbf{v}_{n+2}|\cdots|\mathbf{v}_{2n}], \tag{51}$$

$$\mathbf{\Lambda}^*=\mathrm{diag}[\overline{\lambda}_1,\ldots,\overline{\lambda}_c,\tilde{\lambda}_{c+1},\ldots,\tilde{\lambda}_n]=\mathrm{diag}[\lambda_{n+1},\lambda_{n+2},\ldots,\lambda_{2n}]. \tag{52}$$

Clearly, Eq. (50) is a generalization of Eq. (15) or (40). A flowchart depicting the decoupling of any damped system in free vibration is shown in Fig. 1, for which $\mathbf{f}(t)=\mathbf{g}(t)=\mathbf{0}$ and Eq. (69) coincides with Eq. (13). While only the eigenvalues λ_j of Eq. (3) are required to construct the decoupled system, the eigenvectors $\mathbf{v}_j$ are needed to generate the decoupling transformation.

It has been assumed that the eigenvalues of Eq. (3) are distinct. This assumption can be relaxed effortlessly if there is a full complement of independent eigenvectors associated with each repeated eigenvalue. When the multiplicity of a repeated eigenvalue is greater than the corresponding number of independent eigenvectors, Eq. (3) is defective. Whether defective or non-defective, system (1) can always be decoupled into the form (2), where $\mathbf{D}_1$, $\mathbf{\Omega}_1$ are given by Eqs. (48) and (49). It is the decoupling transformation (13) or (50) that increases in complexity for defective systems. As mentioned earlier, the probability that Eq. (3) being defective is zero [9]. Nonetheless, defective systems were considered in certain situations [31,32,42,52–54]. A numerical example will be provided to indicate how defective systems can be decoupled.

5. Decoupling of forced vibration

It has been shown that a damped system in free vibration can be decoupled if suitable phase shifts are introduced into each damped mode of vibration so that all components are either in phase or out of phase. This process is termed phase synchronization, and its purpose is to compensate for the time drifts caused by viscous damping. The required phase shifts are real for oscillatory free vibration; they are either imaginary or complex with $-\pi/2$ as the real part for purely non-oscillatory free vibration. In any case the time drifts are constant; that is why the time-varying decoupling transformation (13) is merely time-shifting. In the presence of an external excitation $\mathbf{f}(t)$, the interplay between these time drifts and $\mathbf{f}(t)$ generates a genuinely time-varying decoupling transformation for forced vibration.

Postulate that system (1) is decoupled into the form (2), where the diagonal matrices $\mathbf{D}_1, \mathbf{\Omega}_1$ are still given by Eqs. (48) and (49). What is the relationship between $\mathbf{f}(t)$ and $\mathbf{g}(t)$? How is the decoupling transformation (13) generalized? While it is possible to investigate these issues in the configuration space, with the theoretical framework that has been set up it is more efficient to perform additional manipulations in state space. To streamline the presentation, forced vibration with complex eigenvalues will first be considered. This will be followed by a general setting in which both real and complex eigenvalues occur simultaneously.

5.1. State space analysis of systems with complex eigenvalues

Suppose all eigenvalues λ_j of Eq. (3) are complex and distinct. Cast Eq. (1) in state space in the symmetric form

$$\begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \mathbf{f}(t) \\ \mathbf{0} \end{bmatrix} \tag{53}$$

for complex modal analysis [44,45]. Since λ_j are distinct, the quadratic eigenvalue problem (3) is non-defective, which implies that the state-space eigenvalue problem

$$\begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \mathbf{w} = -\lambda \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \mathbf{w} \tag{54}$$

is also non-defective because the eigenvectors of Eqs. (3) and (54) are connected by $\mathbf{w}_j = [\mathbf{v}_j^{\mathrm{T}} \;\; \lambda_j \;\; \mathbf{v}_j^{\mathrm{T}}]^{\mathrm{T}}$. If the generalized eigenvalue problem (54) is non-defective, the two symmetric coefficient matrices of Eq. (53) can be diagonalized simultaneously by a congruence transformation. This is the essence of complex modal analysis.

Define a complex modal transformation by

$$\begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \mathbf{V} & \overline{\mathbf{V}} \\ \mathbf{V}\mathbf{\Lambda} & \overline{\mathbf{V}}\overline{\mathbf{\Lambda}} \end{bmatrix} \mathbf{x} = \mathbf{S}_1 \mathbf{x}, \tag{55}$$

where $\mathbf{V}$, $\mathbf{\Lambda}$ are given by Eqs. (16) and (17), and $\mathbf{x}$ is a $2n$-dimensional column vector. It is assumed that the eigen vectors $\mathbf{v}_j$ of Eq. (3) have been normalized in accordance with Eq. (18) so that $\mathbf{V}$ reduces to the modal matrix $\mathbf{U}$ of Eq. (6) if system (1) is undamped or classically damped. Substitute Eq. (55) into (53) and pre-multiply the resulting equation by $\mathbf{S}_1^{\mathrm{T}}$ to obtain

$$\begin{bmatrix} \mathbf{\Lambda}-\overline{\mathbf{\Lambda}} & \mathbf{0} \\ \mathbf{0} & \overline{\mathbf{\Lambda}}-\mathbf{\Lambda} \end{bmatrix} \dot{\mathbf{x}} + \begin{bmatrix} (\overline{\mathbf{\Lambda}}-\mathbf{\Lambda})\mathbf{\Lambda} & \mathbf{0} \\ \mathbf{0} & (\mathbf{\Lambda}-\overline{\mathbf{\Lambda}})\overline{\mathbf{\Lambda}} \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{V}^{\mathrm{T}}\mathbf{f} \\ \overline{\mathbf{V}}^{\mathrm{T}}\mathbf{f} \end{bmatrix}, \tag{56}$$

which is a system possessing diagonal coefficient matrices in state space. Although Eq. (53) has been decoupled in state space, it is no longer possible to classify the $2n$ state variables of $\mathbf{x}$ as displacements and velocities. Thus physical insight is greatly diminished.

5.2. Transformation of driving forces

Define a $2n$-dimensional vector $[\mathbf{p}_1^{\mathrm{T}} \;\; \mathbf{p}_2^{\mathrm{T}}]^{\mathrm{T}}$ by

$$\begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda} & \overline{\mathbf{\Lambda}} \end{bmatrix} \mathbf{x}, \tag{57}$$

where $\mathbf{I}$ is the identity matrix of order n. By inversion,

$$\mathbf{x} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda} & \overline{\mathbf{\Lambda}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix} = \begin{bmatrix} (\overline{\mathbf{\Lambda}}-\mathbf{\Lambda})^{-1}\overline{\mathbf{\Lambda}} & (\mathbf{\Lambda}-\overline{\mathbf{\Lambda}})^{-1} \\ (\mathbf{\Lambda}-\overline{\mathbf{\Lambda}})^{-1}\mathbf{\Lambda} & (\overline{\mathbf{\Lambda}}-\mathbf{\Lambda})^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix} = \mathbf{S}_2 \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix}. \tag{58}$$

Substitute Eq. (58) into (56) and then pre-multiply the resulting equation by $\mathbf{S}_2^{\mathrm{T}}$ to obtain

$$\begin{bmatrix} \mathbf{D}_1 & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{p}}_1 \\ \dot{\mathbf{p}}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{\Omega}_1 & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{T}_1^{\mathrm{T}}\mathbf{f} \\ \mathbf{T}_2^{\mathrm{T}}\mathbf{f} \end{bmatrix}, \tag{59}$$

where $\mathbf{T}_1$ and $\mathbf{T}_2$ are given by

$$\mathbf{T}_1=(\mathbf{V}\overline{\mathbf{\Lambda}}-\overline{\mathbf{V}}\mathbf{\Lambda})(\overline{\mathbf{\Lambda}}-\mathbf{\Lambda})^{-1}=2\mathrm{Re}\left[\frac{\overline{\lambda}_1}{\overline{\lambda}_1-\lambda_1}\mathbf{v}_1\left|\frac{\overline{\lambda}_2}{\overline{\lambda}_2-\lambda_2}\mathbf{v}_2\right|\cdots\left|\frac{\overline{\lambda}_n}{\overline{\lambda}_n-\lambda_n}\mathbf{v}_n\right.\right], \tag{60}$$

$$\mathbf{T}_2=(\overline{\mathbf{V}}-\mathbf{V})(\overline{\mathbf{\Lambda}}-\mathbf{\Lambda})^{-1}=2\,\mathrm{Re}\left[\frac{1}{\lambda_1-\overline{\lambda}_1}\mathbf{v}_1\left|\frac{1}{\lambda_2-\overline{\lambda}_2}\mathbf{v}_2\right|\cdots\left|\frac{1}{\lambda_n-\overline{\lambda}_n}\mathbf{v}_n\right.\right]. \tag{61}$$

The upper and lower halves of Eq. (59) are

$$\dot{\mathbf{p}}_2+\mathbf{D}_1\dot{\mathbf{p}}_1+\mathbf{\Omega}_1\mathbf{p}_1=\mathbf{T}_1^{\mathrm{T}}\mathbf{f}, \tag{62}$$

$$\dot{\mathbf{p}}_1-\mathbf{p}_2=\mathbf{T}_2^{\mathrm{T}}\mathbf{f}. \tag{63}$$

Eliminate $\mathbf{p}_2$ from the above two equations to get

$$\ddot{\mathbf{p}}_1+\mathbf{D}_1\dot{\mathbf{p}}_1+\mathbf{\Omega}_1\mathbf{p}_1=\mathbf{T}_1^{\mathrm{T}}\mathbf{f}+\mathbf{T}_2^{\mathrm{T}}\dot{\mathbf{f}}. \tag{64}$$

When Eqs. (2) and (64) are compared, it becomes obvious that $\mathbf{p}_1$ can be identified with $\mathbf{p}$. Therefore, the relationship between $\mathbf{f}(t)$ and $\mathbf{g}(t)$ is

$$\mathbf{g}(t)=\mathbf{T}_1^{\mathrm{T}}\mathbf{f}(t)+\mathbf{T}_2^{\mathrm{T}}\dot{\mathbf{f}}(t). \tag{65}$$

Observe that $\mathbf{T}_1$, $\mathbf{T}_2$, and hence $\mathbf{g}(t)$ can be readily obtained upon solution of the quadratic eigenvalue problem (3).

5.3. Decoupling transformations

Upon substitution of Eq. (65) in (2), forced vibration with complex eigenvalues is decoupled in the configuration space. What is the relationship between the solution $\mathbf{p}(t)$ of Eq. (2) and solution $\mathbf{q}(t)$ of the original system (1)? To answer this question, first combine Eqs. (55) and (57) to obtain

$$\begin{bmatrix}\mathbf{q}\\ \dot{\mathbf{q}}\end{bmatrix}=\begin{bmatrix}\mathbf{V} & \overline{\mathbf{V}}\\ \mathbf{V}\mathbf{\Lambda} & \overline{\mathbf{V}}\overline{\mathbf{\Lambda}}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{I}\\ \mathbf{\Lambda} & \overline{\mathbf{\Lambda}}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{p}_1\\ \mathbf{p}_2\end{bmatrix}. \tag{66}$$

When the overall transformation matrix in the above expression is evaluated, Eq. (66) becomes

$$\begin{bmatrix}\mathbf{q}\\ \dot{\mathbf{q}}\end{bmatrix}=\begin{bmatrix}(\mathbf{V}\overline{\mathbf{\Lambda}}-\overline{\mathbf{V}}\mathbf{\Lambda})(\overline{\mathbf{\Lambda}}-\mathbf{\Lambda})^{-1} & (\overline{\mathbf{V}}-\mathbf{V})(\overline{\mathbf{\Lambda}}-\mathbf{\Lambda})^{-1}\\ (\mathbf{V}-\overline{\mathbf{V}})\mathbf{\Lambda}\overline{\mathbf{\Lambda}}(\overline{\mathbf{\Lambda}}-\mathbf{\Lambda})^{-1} & (\overline{\mathbf{V}\mathbf{\Lambda}}-\mathbf{V}\mathbf{\Lambda})(\overline{\mathbf{\Lambda}}-\mathbf{\Lambda})^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{p}_1\\ \mathbf{p}_2\end{bmatrix}. \tag{67}$$

The upper half of Eq. (67) is

$$\mathbf{q}=\mathbf{T}_1\mathbf{p}_1+\mathbf{T}_2\mathbf{p}_2. \tag{68}$$

Recall that $\mathbf{p}_1$ has been identified with $\mathbf{p}$. It follows by using Eq. (63) that

$$\mathbf{q}=\mathbf{T}_1\mathbf{p}+\mathbf{T}_2\dot{\mathbf{p}}-\mathbf{T}_2\mathbf{T}_2^{\mathrm{T}}\mathbf{f}. \tag{69}$$

The above expression represents a time-varying decoupling transformation in the configuration space. A closer examination of Eq. (69) reveals that it consists of two parts. The first part, $\mathbf{T}_1\mathbf{p}+\mathbf{T}_2\dot{\mathbf{p}}$, accounts for time drifts caused by viscous damping in free vibration. The second part, $\mathbf{T}_2\mathbf{T}_2^{\mathrm{T}}\mathbf{f}$, accounts for time drifts induced by the excitation $\mathbf{f}(t)$. When $\mathbf{f}(t)=\mathbf{0}$, Eq. (69) reduces to $\mathbf{q}=\mathbf{T}_1\mathbf{p}+\mathbf{T}_2\dot{\mathbf{p}}$ which, by direct manipulations, is the same as Eq. (13).

To cast Eq. (69) in state space, simply rewrite Eq. (66) in the form

$$\begin{bmatrix}\mathbf{p}(t)\\ \dot{\mathbf{p}}(t)\end{bmatrix}=\begin{bmatrix}\mathbf{I} & \mathbf{I}\\ \mathbf{\Lambda} & \overline{\mathbf{\Lambda}}\end{bmatrix}\begin{bmatrix}\mathbf{V} & \overline{\mathbf{V}}\\ \mathbf{V}\mathbf{\Lambda} & \overline{\mathbf{V}}\overline{\mathbf{\Lambda}}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{q}(t)\\ \dot{\mathbf{q}}(t)\end{bmatrix}+\begin{bmatrix}\mathbf{0}\\ \mathbf{T}_2^{T}\mathbf{f}(t)\end{bmatrix}. \tag{70}$$

The above invertible transformation generalizes Eq. (15) from free to forced vibration. Initial conditions of $\mathbf{p}(t)$ in Eq. (2) and $\mathbf{q}(t)$ in the original system (1) can be connected by putting $t=0$ in Eq. (70). A flowchart depicting the decoupling of forced vibration with complex eigenvalues is shown in Fig. 1, for which $c=n$ and $r=0$.

5.4. Forced vibration with mixed eigenvalues

Suppose real and complex eigenvalues of Eq. (3) occur simultaneously. As in free vibration, forced vibration with mixed eigenvalues can be decoupled by dividing its eigensolutions into two groups: one associated with the real eigenvalues and the other with complex eigenvalues. Let $2c$ eigenvalues be complex and $2r=2(n-c)$ be real. Separate the distinct eigenvalues in accordance with Eq. (45). Index the $2c$ complex conjugate eigenvalues in a manner depicted in Eq. (46). Index the $2r$ real eigenvalues according to Eq. (47). Upon decoupling, system (1) is transformed into Eq. (2), with $\mathbf{D}_1$, $\mathbf{\Omega}_1$ given by Eqs. (48) and (49).

To compute the excitation $\mathbf{g}(t)$ in the decoupled system (2), normalize the complex eigenvectors by Eq. (18) and the real eigenvectors be Eq. (44). Construct $\mathbf{V}$, $\mathbf{\Lambda}$ by Eqs. (16), (17) and $\mathbf{V}^*$ and $\mathbf{\Lambda}^*$ by Eqs. (51) and (52). Define

$$\mathbf{T}_1 = (\mathbf{V}\mathbf{\Lambda}^* - \mathbf{V}^*\mathbf{\Lambda})(\mathbf{\Lambda}^* - \mathbf{\Lambda})^{-1}, \tag{71}$$

$$\mathbf{T}_2 = (\mathbf{V}^* - \mathbf{V})(\mathbf{\Lambda}^* - \mathbf{\Lambda})^{-1}. \tag{72}$$

Using $\mathbf{T}_1$, $\mathbf{T}_2$ as given above, the driving forces $\mathbf{f}(t)$ and $\mathbf{g}(t)$ are still connected by Eq. (65). In addition, the form of the decoupling transformations (69), (70) remains unchanged for mixed eigenvalues as long as $\mathbf{T}_1$, $\mathbf{T}_2$ are defined by Eqs. (71) and (72) and $\overline{\mathbf{V}}$, $\overline{\mathbf{\Lambda}}$ are replaced by $\mathbf{V}^*$, $\mathbf{\Lambda}^*$, respectively. A flowchart depicting the decoupling of any damped linear system is shown in Fig. 1 when Eq. (3) is non-defective. If Eq. (1) is to be decoupled under the constraint that eigenvalues of Eq. (3) remain invariant, then Eq. (2) as prescribed in Fig. 1 must be the decoupled system, unique up to an equivalence class (because of non-unique pairing of the real eigensolutions).

5.5. Reduction to classical modal analysis

The decoupling methodology developed herein is a direct generalization of classical modal analysis. Without loss of generality, assume that all eigenvectors are normalized in accordance with either Eqs. (18) or (44). Suppose all eigenvalues λ_j of Eq. (3) are complex. It can be shown that, when system (1) is undamped or classically damped, $\mathbf{V} = \overline{\mathbf{V}} = \mathbf{U}$, where $\mathbf{V}$ is given by Eq. (16) and $\mathbf{U}$ is the classical modal matrix of Eq. (6). It follows from Eqs. (60) and (61) that $\mathbf{T}_1 = \mathbf{U}$ and $\mathbf{T}_2 = \mathbf{0}$. Thus Eq. (65) simplifies to $\mathbf{g}(t) = \mathbf{U}^{\mathrm{T}}\mathbf{f}(t)$. In addition, Eq. (69) simplifies to $\mathbf{q} = \mathbf{U}\mathbf{p}$, which is exactly the classical modal transformation. Therefore the decoupled system (2) becomes identical to Eq. (7).

To the other extreme, assume that all eigenvalues λ_j are real. If system (1) is classically damped, the sets of eigenvectors associated with the primary and secondary eigenvalues are each identical to the set of classical normal modes. Among the N different ways to pair the eigensolutions by real quadratic conjugation, where N is given by Eq. (31), there exists a pairing scheme such that $\mathbf{V} = \tilde{\mathbf{V}} = \mathbf{U}$. Subsequent reduction to classical modal analysis occurs in the same manner as for complex eigenvalues. Note that the pairing scheme that imposes the condition $\mathbf{V} = \tilde{\mathbf{V}} = \mathbf{U}$ is not necessarily the same as the primary–secondary pairing scheme defined by Eq. (32). In the case of real eigenvalues, reduction of phase synchronization to classical modal analysis is not as neat as for complex eigenvalues.

When real and complex eigenvalues of Eq. (3) occur simultaneously, the decoupling procedure expounded earlier obviously remains a direct generalization of modal analysis. There exists a pairing scheme for the real eigensolutions such that $\mathbf{V} = \mathbf{V}^* = \mathbf{U}$. Using this pairing scheme, Eqs. (65) and (69) simplify to $\mathbf{g}(t) = \mathbf{U}^{\mathrm{T}}\mathbf{f}(t)$ and $\mathbf{q} = \mathbf{U}\mathbf{p}$ for systems that are undamped or classically damped. There remains a final question. What if the eigenvectors are not normalized in accordance with Eqs. (18) or (44)? Reduction of the method of phase synchronization to classical modal analysis remains valid except that multiplicative constants occur here and there. For example, $\mathbf{V}$ and $\mathbf{U}$ may differ by a constant matrix multiplier instead of being equal.

5.6. Efficiency of solution by decoupling

System solution is probably not the most important reason for decoupling. It is the possibility, for example, of modal reduction (using the real damped modes) and of an investigation of energy distribution among independent coordinates that would make decoupling worthwhile. Nevertheless, it may still be instructive to examine the efficiency of solution of Eq. (1) by decoupling, when both real and complex λ_j occur at the same time. It will be assumed that the excitation $\mathbf{f}(t)$ and response $\mathbf{q}(t)$ are sufficiently smooth (adequate if twice differentiable). One measure of the performance of an algorithm is the number of flops (floating point operations) required to evaluate the response at m points within a given time window. The flops associated with two procedures are compared. (a) In direct numerical integration, a standard procedure is to recast the second-order Eq. (1) in state space as a first-order system of dimension $2n$. The state equation is then discretized, and the resulting difference equation is solved by matrix computations [42]. The estimate of flops of this standard procedure for response calculation at m instants is [9,55–57]

$$N_1 = 160n^3 + 16mn^2, \tag{73}$$

where n is the number of degrees of freedom and $m \gg n$ in general. (b) In solving Eq. (1) by decoupling, Eq. (2) is obtained through solution of Eq. (3) and evaluation of Eq. (65). Each independent decoupled system in Eq. (2) is then solved numerically at m instants with the same algorithm used in procedure (a). Subsequently, Eq. (69) is employed to compute the response $\mathbf{q}(t)$. The estimate of flops is [9,55–57]

$$N_2 = 10mn^2 + 16mn + 213n^3 + 4n^2. \tag{74}$$

The variations of N_1 and N_2 with n are illustrated in Fig. 2 for a window containing $m = 10^6$ instants. It is observed that response calculation by decoupling generally reduces the flops and economizes on both core memory and computing time. In fact, Fig. 2 is rather conservative because N_2 has been estimated by using the same sampling time in the integration of all decoupled equations. If an optimal sampling time can be individually chosen for each decoupled equation, N_2 may

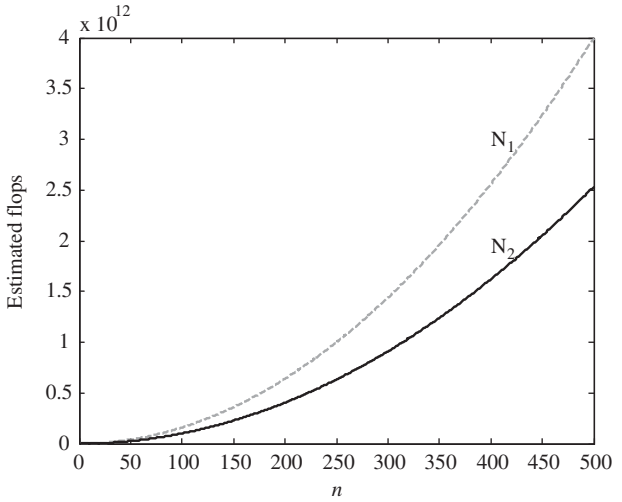

Fig. 2. Comparison of efficiency in response calculation under forced vibration by direct numerical integration N_1 (– – –) and by decoupling N_2 (——). Estimated flops to evaluate the response at $m=10^6$ instants are plotted against the degree of freedom n.

decrease substantially. Moreover, each decoupled equation may be solved exactly in many applications in terms of elementary functions (rather than convolution integrals).

On the other hand, the efficiency of response calculation by decoupling depends on the size of the time window. In addition, validity of Eqs. (73) and (74) requires that the excitation $\mathbf{f}(t)$ and response $\mathbf{q}(t)$ be sufficiently smooth. Distributional excitation such as an impulse and weak solutions (less than twice differentiable) are excluded [58,59]. Thus Fig. 2 should be interpreted as indicative rather than absolute in the comparison of efficiency.

6. Illustrative examples

The decoupling of systems possessing both real and complex eigenvalues, and executing free or forced oscillations, are considered. A defective system will also be decoupled by phase synchronization.

Example 1. The choice of pairing schemes and damped modes of imaginary vibration are examined. Consider a system governed by Eq. (1), with

$$\mathbf{M}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad \mathbf{C}=\begin{bmatrix}4 & -1\\ -1 & 8\end{bmatrix},\quad \mathbf{K}=\begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix},\quad \mathbf{f}(t)=\mathbf{0}, \tag{75}$$

and initial conditions

$$\mathbf{q}(0)=[1\ \ -1]^{\mathrm{T}},\quad \dot{\mathbf{q}}(0)=[1\ \ 1]^{\mathrm{T}}. \tag{76}$$

This system is non-classically damped and it cannot be decoupled by modal analysis. Solution of the quadratic eigenvalue problem (3) yields

$$\lambda_1=-4-\sqrt{14}<\lambda_2=-2-\sqrt{2}<\lambda_3=-2+\sqrt{2}<\lambda_4=-4+\sqrt{14}. \tag{77}$$

Instead of primary–secondary pairing as prescribed in Eq. (32), assign $\lambda_4=\tilde{\lambda}_1$, $\lambda_3=\tilde{\lambda}_2$. Using the first parts of Eqs. (33) and (34), $\lambda_1=\alpha_1+\mathrm{i}\omega_1=-4+\mathrm{i}(\mathrm{i}\sqrt{4})$, $\lambda_2=\alpha_2+\mathrm{i}\omega_2=-2+\mathrm{i}(\mathrm{i}\sqrt{2})$. It can also be checked that

$$\mathbf{v}_1=\begin{bmatrix}r_{11}\mathrm{e}^{-\mathrm{i}\varphi_{11}}\\ r_{12}\mathrm{e}^{-\mathrm{i}\varphi_{12}}\end{bmatrix}=\begin{bmatrix}0.59\mathrm{e}^{-\mathrm{i}(-0.85\mathrm{i})}\\ -0.42\mathrm{e}^{-\mathrm{i}(0.85\mathrm{i})}\end{bmatrix}=\begin{bmatrix}0.25\\ -0.98\end{bmatrix},\quad \mathbf{v}_4=\tilde{\mathbf{v}}_1=\begin{bmatrix}r_{11}\mathrm{e}^{-\mathrm{i}\varphi_{11}}\\ r_{12}\mathrm{e}^{-\mathrm{i}\varphi_{12}}\end{bmatrix}=\begin{bmatrix}1.39\\ -0.18\end{bmatrix}, \tag{78}$$

$$\mathbf{v}_2=\begin{bmatrix}r_{21}\mathrm{e}^{-\mathrm{i}\varphi_{21}}\\ r_{22}\mathrm{e}^{-\mathrm{i}\varphi_{22}}\end{bmatrix}=\begin{bmatrix}0.59\mathrm{e}^{-\mathrm{i}(0.44\mathrm{i})}\\ -0.42\mathrm{e}^{-\mathrm{i}(0.44\mathrm{i})}\end{bmatrix}=\begin{bmatrix}0.92\\ 0.27\end{bmatrix},\quad \mathbf{v}_3=\tilde{\mathbf{v}}_2=\begin{bmatrix}r_{21}\mathrm{e}^{-\mathrm{i}\varphi_{21}}\\ r_{22}\mathrm{e}^{-\mathrm{i}\varphi_{22}}\end{bmatrix}=\begin{bmatrix}0.38\\ 0.65\end{bmatrix}, \tag{79}$$

where the eigenvectors are normalized in accordance with Eq. (44). From Eq. (35), the two modes of imaginary vibration are given by

$$\mathbf{s}_1(t)=C_1\mathrm{e}^{-4t}\begin{bmatrix}0.59\cos(\mathrm{i}\sqrt{14}t-\theta_1+0.85\mathrm{i})\\ -0.42\cos(\mathrm{i}\sqrt{14}t-\theta_1-0.85\mathrm{i})\end{bmatrix}, \tag{80}$$

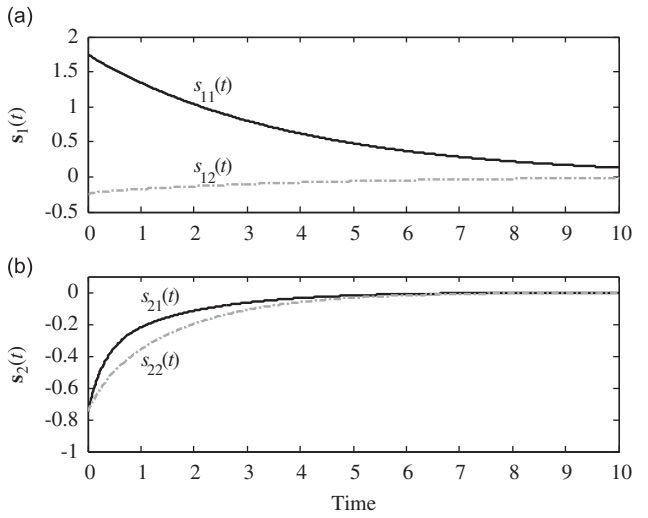

Fig. 3. Damped modes of imaginary vibration of Example 1: (a) first mode $\mathbf{s}_1(t)$ with first element $s_{11}(t)$ (——) and second element $s_{12}(t)$ (– · –); and (b) second mode $\mathbf{s}_2(t)$ with first element $s_{21}(t)$ (——) and second element $s_{22}(t)$ (– · –).

$$\mathbf{s}_2(t) = C_2 \mathrm{e}^{-2t} \begin{bmatrix} 0.59\cos(\mathrm{i}\sqrt{2}t - \theta_2 - 0.44\mathrm{i}) \\ 0.42\cos(\mathrm{i}\sqrt{2}t - \theta_2 + 0.44\mathrm{i}) \end{bmatrix}. \tag{81}$$

The free response is a superposition of these two real damped modes. In fact, the constants C_j, θ_j can be determined by imposing the initial conditions to yield $C_1\mathrm{e}^{-\mathrm{i}\theta_1} = 0.36\mathrm{e}^{-1.95}$ and $C_2\mathrm{e}^{-\mathrm{i}\theta_2} = -1.26\mathrm{e}^{-0.44}$. As shown in Fig. 3, the two modes of imaginary vibration indeed represent real exponential decay. By phase synchronization of $\mathbf{s}_1(t)$ and $\mathbf{s}_2(t)$, the system can be converted into the decoupled system (2), in which

$$\mathbf{D}_1 = -\mathrm{diag}[\lambda_j + \tilde{\lambda}_j (j = 1, 2)] = \mathrm{diag}[8, 4], \tag{82}$$

$$\mathbf{\Omega}_1 = \mathrm{diag}[\lambda_j \tilde{\lambda}_j (j = 1, 2)] = \mathrm{diag}[2, 2], \tag{83}$$

and $\mathbf{g}(t) = \mathbf{0}$. From Eq. (40), the initial conditions of the decoupled system are

$$\mathbf{p}(0) = [1.28 \;\; -1.39]^{\mathrm{T}}, \quad \dot{\mathbf{p}}(0) = [-0.52 \;\; 1.96]^{\mathrm{T}} \tag{84}$$

The solution $\mathbf{q}(t)$ of the original $(\mathbf{M}, \mathbf{C}, \mathbf{K}, \mathbf{0})$ system can be recovered from solution $\mathbf{p}(t)$ of the decoupled $(\mathbf{I}, \mathbf{D}_1, \mathbf{\Omega}_1, \mathbf{0})$ system by Eq. (13), in which

$$\mathbf{Z} = [\mathbf{z}_1 | \mathbf{z}_2] = \begin{bmatrix} 1.48 & 0.32 \\ -0.17 & 0.78 \end{bmatrix}. \tag{85}$$

Both $\mathbf{p}(t)$ and $\mathbf{q}(t)$ are plotted in Fig. 4. It can be checked that $\mathbf{q}(t)$, whether generated by decoupling or by direct numerical integration of the original equation of motion, is the same.

In the above computation, primary–secondary pairing as prescribed in Eq. (32) was not used. If one insists on primary–secondary pairing so that $\lambda_3 = \tilde{\lambda}_1$ and $\lambda_4 = \tilde{\lambda}_2$, then Eqs. (33) and (34) indicate that $\lambda_1 = \alpha_1 + \mathrm{i}\omega_1 = -4.16 + \mathrm{i}(3.58)$ and $\lambda_2 = \alpha_2 + \mathrm{i}\omega_2 = -1.84 + \mathrm{i}(1.58)$. By Eq. (35),

$$\mathbf{s}_1(t) = -0.25\mathrm{i}\mathrm{e}^{-4.16t} \begin{bmatrix} 0.39\cos(\mathrm{i}3.58t + \pi/2 + 1.59\mathrm{i} + 0.45\mathrm{i}) \\ 1.00\mathrm{i}\cos(\mathrm{i}3.58t + \pi/2 + 1.59\mathrm{i} + \pi/2 + 0.04\mathrm{i}) \end{bmatrix}, \tag{86}$$

$$\mathbf{s}_2(t) = 1.72\mathrm{i}\mathrm{e}^{-1.84t} \begin{bmatrix} 0.94\cos(\mathrm{i}1.58t + \pi/2 + 0.81\mathrm{i} - 0.04\mathrm{i}) \\ -0.18\mathrm{i}\cos(\mathrm{i}1.58t + \pi/2 + 0.81\mathrm{i} + \pi/2 - 045\mathrm{i}) \end{bmatrix}, \tag{87}$$

under the same initial conditions given in Eq. (76). A different decoupled $(\mathbf{I}, \mathbf{D}_1, \mathbf{\Omega}_1, \mathbf{0})$ system is obtained, since in this case

$$\mathbf{D}_1 = \mathrm{diag}[8.33, \;\; 3.67], \quad \mathbf{\Omega}_1 = \mathrm{diag}[4.54, \;\; 0.88]. \tag{88}$$

The two decoupled systems, defined by Eqs. (82), (83) and by Eq. (88), are both overdamped. With primary–secondary pairing, initial conditions of the decoupled system are

$$\mathbf{p}(0) = [-0.59 \;\; 1.55]^{\mathrm{T}}, \quad \dot{\mathbf{p}}(0) = [0.16 \;\; 0.81]^{\mathrm{T}} \tag{89}$$

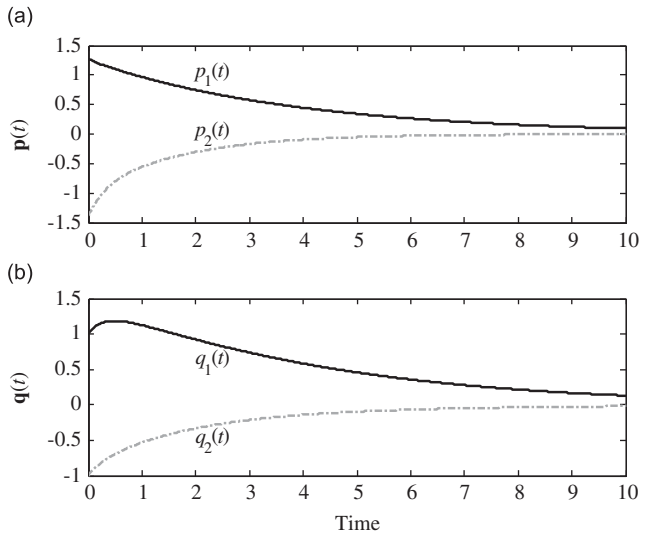

Fig. 4. Free response of coupled and decoupled systems in Example 1: (a) free response $\mathbf{p}(t)$ of decoupled $(\mathbf{I}, \mathbf{D}_1, \mathbf{\Omega}_1, \mathbf{0})$ system with first element $p_1(t)$ (——) and second element $p_2(t)$ (– · –); and (b) free response $\mathbf{q}(t)$ of the original $(\mathbf{M}, \mathbf{C}, \mathbf{K}, \mathbf{0})$ system with first element $q_1(t)$ (——) and second element $q_2(t)$ (– · –).

The decoupling transformation (13) that generates solution $\mathbf{q}(t)$ of the original (**M**, **C**, **K**, **0**) system from the decoupled solution is also different since

$$\mathbf{Z} = [\mathbf{z}_1 | \mathbf{z}_2] = \begin{bmatrix} 0.66 & 0.90 \\ 1.01 - 0.27\mathrm{i} & -0.10 + 0.03\mathrm{i} \end{bmatrix} \tag{90}$$

As far as response calculation is concerned, the choice of pairing scheme is rather immaterial.

Example 2. Forced vibration is decoupled and the transformation of driving force is examined. Consider a mass-spring-damper system governed by an equation of the type (1), with

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0.7 & -0.1 \\ -0.1 & 0.2 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} \cos t \\ \sin 2t \end{bmatrix}, \tag{91}$$

and initial conditions

$$\mathbf{q}(0) = [1 \;\; 2]^{\mathrm{T}}, \quad \dot{\mathbf{q}}(0) = [-1 \;\; 1]^{\mathrm{T}}. \tag{92}$$

Since $\mathbf{C}\mathbf{M}^{-1}\mathbf{K} \neq \mathbf{K}\mathbf{M}^{-1}\mathbf{C}$, the system is non-classically damped. Solution of the quadratic eigenvalue problem (3) yields two pairs of complex conjugate eigensolutions prescribed by

$$\lambda_1 = -0.18 + 1.00\mathrm{i}, \quad \mathbf{v}_1 = \begin{bmatrix} 0.74\mathrm{e}^{-\mathrm{i}7.38^\circ} \\ -0.72\mathrm{e}^{-\mathrm{i}172.51^\circ} \end{bmatrix}, \tag{93}$$

$$\lambda_2 = -0.27 + 1.68\mathrm{i}, \quad \mathbf{v}_2 = \begin{bmatrix} -0.73\mathrm{e}^{-\mathrm{i}167.13^\circ} \\ -0.73\mathrm{e}^{-\mathrm{i}12.68^\circ} \end{bmatrix}. \tag{94}$$

This system can be converted into the decoupled form (2), for which

$$\mathbf{D}_1 = -\mathrm{diag}[\lambda_j + \overline{\lambda}_j (j = 1, 2)] = \mathrm{diag}[0.36,\ 0.54], \tag{95}$$

$$\mathbf{\Omega}_1 = \mathrm{diag}[\lambda_j \overline{\lambda}_j (j = 1, 2)] = \mathrm{diag}[1.03,\ 2.90]. \tag{96}$$

Using Eqs. (60) and (61)

$$\mathbf{T}_1 = \begin{bmatrix} 0.72 & 0.74 \\ 0.73 & -0.69 \end{bmatrix}, \quad \mathbf{T}_2 = \begin{bmatrix} -0.09 & 0.10 \\ 0.09 & 0.10 \end{bmatrix}. \tag{97}$$

It follows from Eq. (65) that the transformed driving force is given by

$$\mathbf{g}(t) = \begin{bmatrix} 0.72\cos t + 0.18\cos 2t + 0.09\sin t + 0.73\sin 2t \\ 0.74\cos t + 0.20\cos 2t - 0.10\sin t - 0.69\sin 2t \end{bmatrix}. \tag{98}$$

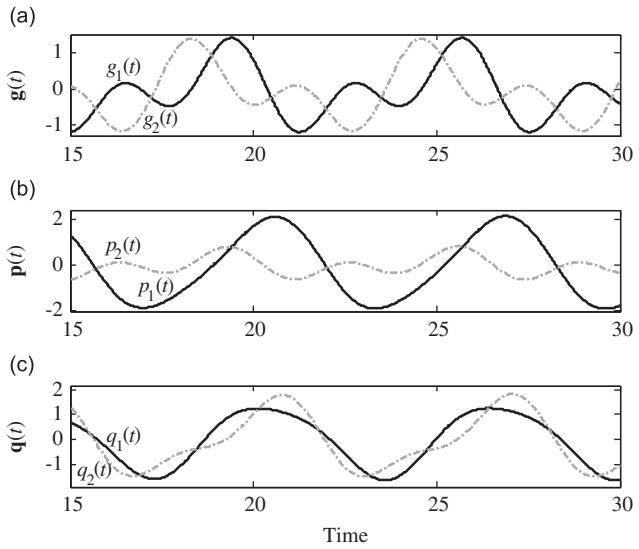

Fig. 5. Forced response and excitation of Example 2 with a smooth $\mathbf{f}(t)$ as defined in Eq. (91): (a) excitation $\mathbf{g}(t)$ of the decoupled $(\mathbf{I}, \mathbf{D}_1, \boldsymbol{\Omega}_1, \mathbf{g}(t))$ system with first element $g_1(t)$ (——) and second element $g_2(t)$ (–·–·); (b) steady-state response $\mathbf{p}(t)$ of the decoupled $(\mathbf{I}, \mathbf{D}_1, \boldsymbol{\Omega}_1, \mathbf{g}(t))$ system with first element $p_1(t)$ (——) and second element $p_2(t)$ (– – –); and (c) steady-state response $\mathbf{q}(t)$ of the original $(\mathbf{M}, \mathbf{C}, \mathbf{K}, \mathbf{f}(t))$ system with first element $q_1(t)$ (——) and second element $q_2(t)$ (–·–·).

Initial conditions of the decoupled ($\mathbf{I}$, $\mathbf{D}_1$, $\boldsymbol{\Omega}_1$, $\mathbf{g}(t)$) system are

$$\mathbf{p}(0) = [2.32 \;\; -0.71]^{\mathrm{T}}, \quad \dot{\mathbf{p}}(0) = [-0.43 \;\; -1.61]^{\mathrm{T}}. \tag{99}$$

The decoupled system can be readily solved and solution $\mathbf{q}(t)$ of the original ($\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$, $\mathbf{f}(t)$) system can be recovered from $\mathbf{p}(t)$ by Eq. (69). Steady-state behaviors of $\mathbf{g}(t)$, $\mathbf{p}(t)$ and $\mathbf{q}(t)$ are shown in Fig. 5. It can be checked that $\mathbf{q}(t)$, whether generated by decoupling or by direct numerical integration, is the same.

It would appear that an implicit assumption in Eq. (65) is that $\mathbf{f}(t)$ be differentiable. This is not the case. To illustrate how discontinuous driving forces may be transformed by decoupling, superpose a square wave on the excitation in Eq. (91) to get

$$\mathbf{f}(t) = \begin{bmatrix} \cos t \\ \sin 2t \end{bmatrix} + \begin{bmatrix} 9 \\ -3 \end{bmatrix} \mathrm{sgn}\left(\sin \frac{t}{3}\right). \tag{100}$$

Using distributional derivative [58,59] of $\mathbf{f}(t)$, the excitation $\mathbf{g}(t)$ can be obtained from Eq. (65). As shown in Fig. 6, discontinuities in $\mathbf{f}(t)$ give rise to delta distributions in $\mathbf{g}(t)$. Under the excitation of Eq. (100), initial conditions of the decoupled system as deduced from Eq. (70) are

$$\mathbf{p}(0) = [2.32 \;\; -0.71]^{\mathrm{T}}, \quad \dot{\mathbf{p}}(0) = [-0.43 \;\; -1.61]^{\mathrm{T}}. \tag{101}$$

The distributional response $\mathbf{p}(t)$ of the decoupled ($\mathbf{I}$, $\mathbf{D}_1$, $\boldsymbol{\Omega}_1$, $\mathbf{g}(t)$) system is also shown in Fig. 6. The response $\mathbf{q}(t)$ of the original ($\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$, $\mathbf{f}(t)$) system can still be recovered readily from $\mathbf{p}(t)$ by Eq. (69).

Example 3. A four-degree-of-freedom system possessing both real and complex eigenvalues is decoupled. In Eq. (1), let $\mathbf{M} = \mathbf{I}$:

$$\mathbf{C} = \begin{bmatrix} 0.1 & -0.1 & 0 & 0 \\ -0.1 & 0.2 & -0.1 & 0 \\ 0 & -0.1 & 0.2 & -0.1 \\ 0 & 0 & -0.1 & 1.35 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1.1 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} te^{-0.3t} \sin 2t. \tag{102}$$

Assume zero initial conditions $\mathbf{q}(0) = \mathbf{0}$, $\dot{\mathbf{q}}(0) = \mathbf{0}$. This system is non-classically damped. Solution of the quadratic eigenvalue problem (3) yields $c = 3$ pairs of complex conjugates and $r = 1$ pair of real quadratic conjugates prescribed by

$$\lambda_1 = -0.33 + 0.60\mathrm{i}, \quad \mathbf{v}_1 = \begin{bmatrix} -0.62e^{-\mathrm{i}28.29^\circ} \\ -0.50e^{-\mathrm{i}56.43^\circ} \\ -0.58e^{-\mathrm{i}112.71^\circ} \\ -0.97e^{-\mathrm{i}154.37^\circ} \end{bmatrix}, \tag{103}$$

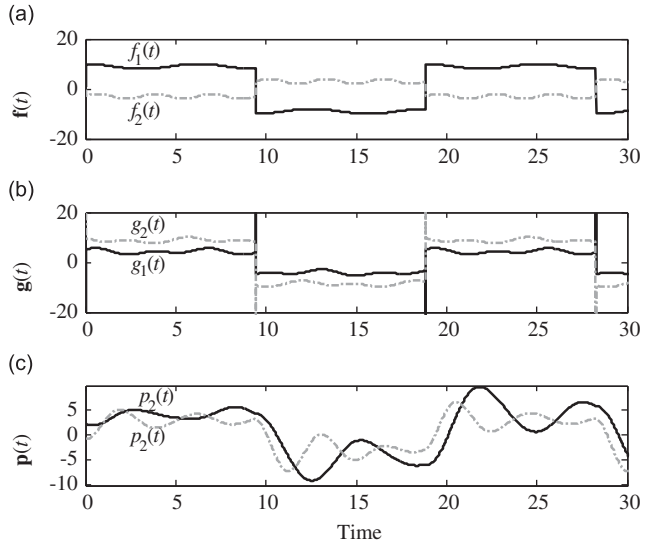

Fig. 6. Forced response and excitation of Example 2 with a discontinuous $\mathbf{f}(t)$ as given by Eq. (100): (a) excitation $\mathbf{f}(t)$ of the original $(\mathbf{M},\mathbf{C},\mathbf{K},\mathbf{f}(t))$ system with first element $f_1(t)$ (——) and second element $f_2(t)$ (– · –); (b) excitation $\mathbf{g}(t)$ of the decoupled $(\mathbf{I},\mathbf{D}_1,\boldsymbol{\Omega}_1,\mathbf{g}(t))$ system with first element $g_1(t)$ (——) and second element $g_2(t)$ (– · –); and (c) forced response $\mathbf{p}(t)$ of the decoupled $(\mathbf{I},\mathbf{D}_1,\boldsymbol{\Omega}_1,\mathbf{g}(t))$ system with first element $p_1(t)$ (——) and second element $p_2(t)$ (– · –).

$$\lambda_2 = -0.20 + 1.31\mathrm{i}, \quad \mathbf{v}_2 = \begin{bmatrix} 0.55\mathrm{e}^{-\mathrm{i}9.57^\circ} \\ 0.44\mathrm{e}^{-\mathrm{i}167.83^\circ} \\ -0.71\mathrm{e}^{-\mathrm{i}16.01^\circ} \\ -0.47\mathrm{e}^{-\mathrm{i}132.32^\circ} \end{bmatrix}, \tag{104}$$

$$\lambda_3 = -0.19 + 1.81\mathrm{i}, \quad \mathbf{v}_3 = \begin{bmatrix} 0.30\mathrm{e}^{-\mathrm{i}7.00^\circ} \\ -0.71\mathrm{e}^{-\mathrm{i}4.55^\circ} \\ -0.64\mathrm{e}^{-\mathrm{i}177.08^\circ} \\ 0.21\mathrm{e}^{-\mathrm{i}130.64^\circ} \end{bmatrix}, \tag{105}$$

$$\lambda_4 = -0.21 + \mathrm{i}(\mathrm{i}0.07), \quad \mathbf{v}_4 = \begin{bmatrix} 0.59\mathrm{e}^{-0.06} \\ 0.62\mathrm{e}^{-0.04} \\ 0.67\mathrm{e}^{0.02} \\ 0.76\mathrm{e}^{0.09} \end{bmatrix}. \tag{106}$$

From Eqs. (71) and (72),

$$\mathbf{T}_1 = \begin{bmatrix} -0.38 & 0.53 & 0.30 & 0.70 \\ -0.05 & -0.45 & -0.70 & 0.68 \\ 0.52 & -0.66 & 0.64 & 0.64 \\ 1.11 & 0.37 & -0.16 & 0.57 \end{bmatrix}, \quad \mathbf{T}_2 = \begin{bmatrix} 0.49 & -0.07 & -0.02 & 0.53 \\ 0.71 & -0.07 & 0.03 & 0.31 \\ 0.90 & 0.15 & 0.02 & -0.17 \\ 0.70 & 0.27 & -0.09 & -0.93 \end{bmatrix}. \tag{107}$$

It can be checked that the decoupled ($\mathbf{I}$, $\mathbf{D}_1$, $\boldsymbol{\Omega}_1$, $\mathbf{g}(t)$) system is given by

$$\mathbf{D}_1 = -\mathrm{diag}[\lambda_j + \overline{\lambda}_j (j = 1,2,3), \lambda_4 + \tilde{\lambda}_4] = \mathrm{diag}[0.65, 0.40, 0.39, 0.41], \tag{108}$$

$$\boldsymbol{\Omega}_1 = \mathrm{diag}[\lambda_j \overline{\lambda}_j (j = 1,2,3), \lambda_4 \tilde{\lambda}_4] = \mathrm{diag}[0.46, 1.75, 3.32, 0.04], \tag{109}$$

$$\mathbf{g}(t) = \mathrm{e}^{-0.3t} \begin{bmatrix} 1.4t\cos(2t) + (0.70 + 0.90t)\sin(2t) \\ 0.54t\cos(2t) + (0.27 + 0.29t)\sin(2t) \\ -0.18t\cos(2t) - (0.09 + 0.13t)\sin(2t) \\ -1.86t\cos(2t) + (-0.93 + 0.85t)\sin(2t) \end{bmatrix}, \tag{110}$$

with initial conditions $\mathbf{p}(0) = \mathbf{0}$, $\dot{\mathbf{p}}(0) = \mathbf{0}$. The decoupled system can be readily solved and solution $\mathbf{q}(t)$ of the original ($\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$, $\mathbf{f}(t)$) system can be recovered from $\mathbf{p}(t)$ by Eq. (69). As shown in Fig. 7, $\mathbf{g}(t)$, $\mathbf{p}(t)$ and $\mathbf{q}(t)$ are all oscillatory. It can be verified that $\mathbf{q}(t)$, whether generated by decoupling or by direct numerical integration, is the same.

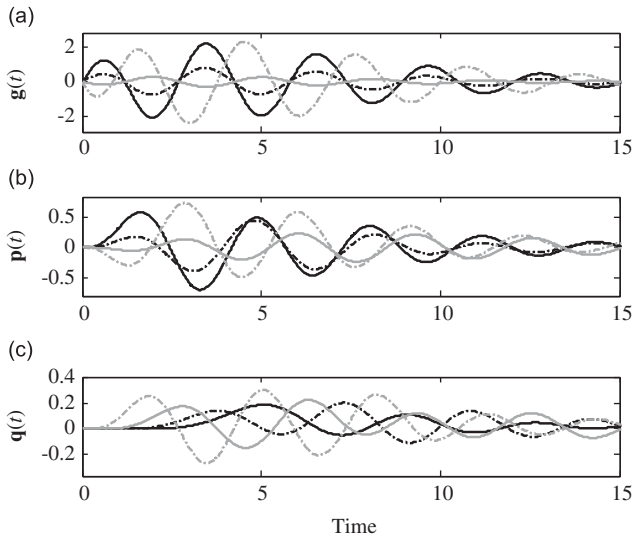

Fig. 7. Forced response and excitation of Example 3: (a) excitation $\mathbf{g}(t)$ of the decoupled $(\mathbf{I},\mathbf{D}_1,\boldsymbol{\Omega}_1,\mathbf{g}(t))$ system with elements $g_1(t)$ (——), $g_2(t)$ (– · –), $g_3(t)$ (——) and $g_4(t)$ (– · · –); (b) forced response $\mathbf{p}(t)$ of the decoupled $(\mathbf{I},\mathbf{D}_1,\boldsymbol{\Omega}_1,\mathbf{g}(t))$ system with elements $p_1(t)$ (——), $p_2(t)$ (– · –), $p_3(t)$ (——) and $p_4(t)$ (– · · –); and (c) forced response $\mathbf{q}(t)$ of the original $(\mathbf{M},\mathbf{C},\mathbf{K},\mathbf{f}(t))$ system with elements $q_1(t)$ (——),$q_2(t)$ (– · –),$q_3(t)$ (——) and $q_4(t)$ (– · · –).

Example 4. A defective system in forced vibration is decoupled. Consider a non-classically damped system governed by an equation of the type (1), with $\mathbf{M}=\mathbf{I}$

$$\mathbf{C}=\frac{1}{3}\begin{bmatrix}4 & -\sqrt{5}\\ -\sqrt{5} & 8\end{bmatrix},\quad \mathbf{K}=\begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix},\quad \mathbf{f}(t)=\frac{\cos 2t}{t+1}\begin{bmatrix}1\\ 2\end{bmatrix}, \tag{111}$$

and zero initial conditions $\mathbf{q}(0)=\mathbf{0}$, $\dot{\mathbf{q}}(0)=\mathbf{0}$. Solution of the quadratic eigenvalue problem (3) yields two repeated eigenvalues

$$\lambda_1,\overline{\lambda}_1=\lambda_2,\overline{\lambda}_2=-1\pm \mathrm{i} \tag{112}$$

each of multiplicity 2. There is only one eigenvector $\mathbf{v}_1=[-0.08-0.25\mathrm{i}\ \ -0.19]^{\mathrm{T}}$ associated with $\lambda_1=\lambda_2$ and also only one eigenvector $\overline{\mathbf{v}}_1$ associated with $\overline{\lambda}_1=\overline{\lambda}_2$. Hence this system is defective.

As discussed earlier, this defective system can still be decoupled into the form (2), with $\mathbf{D}_1$, $\boldsymbol{\Omega}_1$ given by Eqs. (11) and (12). In the present example, it can be verified that [9]

$$\mathbf{D}_1=2\mathbf{I},\quad \boldsymbol{\Omega}_1=2\mathbf{I}. \tag{113}$$

To determine the transformed excitation $\mathbf{g}(t)$ and a decoupling transformation, it will be shown how Eqs. (65) and (69) can be updated. Suppose all eigenvalues of Eq. (3) are complex and there are only $d<n$ pairs of complex conjugate eigenvalues. Let m_k be the multiplicity of λ_k. Then m_k is also the multiplicity of $\overline{\lambda}_k$ so that $m_1+m_2+\cdots+m_d=n$. If $\mathbf{J}_k$ is a Jordan block of order m_k associated with $\lambda_k=\alpha_k+\mathrm{i}\omega_k$, $\overline{\mathbf{J}}_k$ must be a Jordan block associated with $\overline{\lambda}_k=\alpha_k-\mathrm{i}\omega_k$. Let

$$\mathbf{V}_k=[\mathbf{v}_1^k|\mathbf{v}_2^k|\cdots|\mathbf{v}_{m_k}^k] \tag{114}$$

be a matrix of order $n\times m_k$ whose columns are made up of eigenvectors and generalized eigenvectors $\mathbf{v}_j^k$ that constitute a Jordan chain of length m_k associated with λ_k. Construct the following square matrices of order n by

$$\mathbf{V}=[\mathbf{V}_1|\mathbf{V}_2|\cdots|\mathbf{V}_d],\quad \mathbf{J}=\mathrm{diag}[\mathbf{J}_1,\mathbf{J}_2,\cdots,\mathbf{J}_d],\quad \boldsymbol{\Lambda}=\mathrm{diag}[\mathbf{J}],\quad \mathbf{N}=\mathbf{J}-\boldsymbol{\Lambda}. \tag{115}$$

For a defective system with complex eigenvalues, it can be shown that transformation (65) is modified to

$$\mathbf{g}(t)=\dot{\mathbf{g}}_1(t)+\mathbf{D}_1\mathbf{g}_1(t)+\mathbf{g}_2(t), \tag{116}$$

where $\mathbf{g}_1(t)$ and $\mathbf{g}_2(t)$ are obtained from

$$\begin{bmatrix}\mathbf{g}_1(t)\\ \mathbf{g}_2(t)\end{bmatrix}=\begin{bmatrix}\mathrm{e}^{-Nt} & \mathrm{e}^{-Nt}\\ \boldsymbol{\Lambda}\mathrm{e}^{-Nt} & \overline{\boldsymbol{\Lambda}}\mathrm{e}^{-\mathbf{N}t}\end{bmatrix}\begin{bmatrix}\mathbf{V} & \overline{\mathbf{V}}\\ \mathbf{VJ} & \overline{\mathbf{V}}\overline{\mathbf{J}}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{0}\\ \mathbf{f}(t)\end{bmatrix}. \tag{117}$$

In addition, the decoupling transformation (69) is modified to

$$\mathbf{q}=\mathbf{T}_1(t)\mathbf{p}+\mathbf{T}_2(t)\dot{\mathbf{p}}-\mathbf{T}_2(t)\mathbf{g}_1, \tag{118}$$

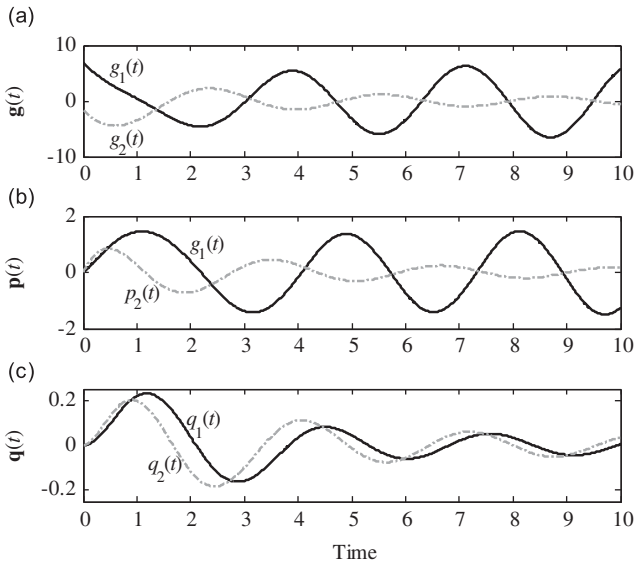

Fig. 8. Forced response and excitation associated with the defective system of Example 4: (a) excitation $\mathbf{g}(t)$ of the decoupled $(\mathbf{I}, \mathbf{D}_1, \boldsymbol{\Omega}_1, \mathbf{g}(t))$ system with first element $g_1(t)$ (——) and second element $g_2(t)$ (– · –); (b) forced response $\mathbf{p}(t)$ of the decoupled $(\mathbf{I}, \mathbf{D}_1, \boldsymbol{\Omega}_1, \mathbf{g}(t))$ system with first element $p_1(t)$ (——) and second element $p_2(t)$ (– · –); and (c) forced response $\mathbf{q}(t)$ of the original $(\mathbf{M}, \mathbf{C}, \mathbf{K}, \mathbf{f}(t))$ system with first element $q_1(t)$ (——) and second element $q_2(t)$ (– · –).

where $\mathbf{T}_1(t)$, $\mathbf{T}_2(t)$ are real and invertible matrices given by

$$\mathbf{T}_1(t) = (\mathbf{V}e^{Nt}\overline{\boldsymbol{\Lambda}} - \overline{\mathbf{V}}e^{Nt}\boldsymbol{\Lambda})(\overline{\boldsymbol{\Lambda}} - \boldsymbol{\Lambda})^{-1} = 2\,\mathrm{Re}[\mathbf{V}e^{Nt}\overline{\boldsymbol{\Lambda}}(\overline{\boldsymbol{\Lambda}} - \boldsymbol{\Lambda})^{-1}], \tag{119}$$

$$\mathbf{T}_2(t) = (\overline{\mathbf{V}} - \mathbf{V})e^{Nt}(\overline{\boldsymbol{\Lambda}} - \boldsymbol{\Lambda})^{-1} = 2\,\mathrm{Re}[\mathbf{V}e^{Nt}(\boldsymbol{\Lambda} - \overline{\boldsymbol{\Lambda}})^{-1}]. \tag{120}$$

Initial conditions of the original and decoupled systems are connected by

$$\begin{bmatrix} \mathbf{p}(0) \\ \dot{\mathbf{p}}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \boldsymbol{\Lambda} & \overline{\boldsymbol{\Lambda}} \end{bmatrix} \begin{bmatrix} \mathbf{V} & \overline{\mathbf{V}} \\ \mathbf{V}\mathbf{J} & \overline{\mathbf{V}\mathbf{J}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{q}(0) \\ \dot{\mathbf{q}}(0) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{g}_1(0) \end{bmatrix} \tag{121}$$

where $\mathbf{g}_1(0) = [1.58\ \ 3.79]^{\mathrm{T}}$ in this example. If Eq. (3) is non-defective, $\mathbf{N} = \mathbf{0}$ and all equations developed for defective systems reduce to those for non-defective systems. In the present example, $d = 1$ and it can be verified that

$$\mathbf{V} = \begin{bmatrix} -0.08 - 0.25\mathrm{i} & 0.50 - 0.58\mathrm{i} \\ 0.19 & 0.19\mathrm{i} \end{bmatrix}, \quad \mathbf{J} = \begin{bmatrix} -1 + \mathrm{i} & 1 \\ 0 & -1 + \mathrm{i} \end{bmatrix}, \tag{122}$$

$$\mathbf{T}_1(t) = \begin{bmatrix} -0.33 & -0.33t - 0.08 \\ 0.19 & 0.19t + 0.19 \end{bmatrix}, \quad \mathbf{T}_2(t) = \begin{bmatrix} -0.25 & -0.25t - 0.58 \\ 0 & 0.19 \end{bmatrix}. \tag{123}$$

The driving force $\mathbf{g}(t)$, solution $\mathbf{p}(t)$ of the decoupled $(\mathbf{I}, \mathbf{D}_1, \boldsymbol{\Omega}_1, \mathbf{g}(t))$ system and $\mathbf{q}(t)$ of the original $(\mathbf{M}, \mathbf{C}, \mathbf{K}, \mathbf{f}(t))$ system are shown in Fig. 8. It can be checked that $\mathbf{q}(t)$, whether generated by decoupling or by direct numerical integration, is the same.

7. Conclusions

Classical modal analysis has been extended to decouple any viscously damped linear system in non-oscillatory free vibration and in forced vibration. The extension utilizes phase synchronization to compensate for time drifts caused by viscous damping and external excitation. Together with an earlier paper on the decoupling of oscillatory free vibration [9], a solution to the "classical decoupling problem" has been provided. The decoupling methodology developed herein possesses ample physical insight and it also lends itself to numerical computations. Major findings of this paper are summarized in the following statements.

1. Phase synchronization of damped modes in exponential decay is based upon the concept of imaginary vibration. Unlike oscillatory free vibration, there are N different ways to decouple purely non-oscillatory free vibration, where N is given by Eq. (31).

2. In free or forced vibration, all parameters required for the construction of decoupled systems and decoupling transformations are obtained through the solution of a quadratic eigenvalue problem.
3. Any viscously damped linear system (no restrictions) can be completely decoupled by phase synchronization. A flowchart outlining the decoupling procedure is given in Fig. 1. If Eq. (3) is defective (with probability zero for a randomly chosen eigenvalue problem), only the driving force $\mathbf{g}(t)$ and Eq. (69) need to be updated.
4. While damped linear systems are decoupled in the configuration space, transformations of initial conditions are prescribed in state space (otherwise initial values are connected at different time instants due to phase synchronization).
5. The methodology of phase synchronization is a direct generalization of classical modal analysis.

To streamline the introduction of new concepts (such as damped modes of imaginary vibration), most formulas have been established with the assumption that eigenvalues of the quadratic eigenvalue problem (3) are distinct. These formulas remain unchanged when Eq. (3) is non-defective (each repeated eigenvalue possesses a full complement of independent eigenvectors). A system for which Eq. (3) is defective can still be decoupled by phase synchronization. However, physical insight is obscured due to the occurrence of Jordan sub-matrices in the corresponding equations. Although only four numerical examples are presented, extensive simulations using the flowchart of Fig. 1 have been performed to support the validity of phase synchronization.

System decoupling plays a fundamental role not only in linear vibrations but also in such diverse areas as quantum mechanics, mathematical economics, and computational science. It not only provides an efficient means of evaluating the system response but also greatly facilitates qualitative analysis. Among other things, it is hoped that this paper would point to directions along which further research efforts should be made. One such direction is obvious. The symmetry of $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ has not been used directly in phase synchronization. Thus the method devised in this paper may be further extended to decouple certain systems with non-symmetric coefficient matrices. The study of other issues, such as energy distribution among the independent decoupled coordinates, numerical algorithms for decoupling, and modal reduction using the real damped modes of vibration, is also worthwhile in a subsequent course of investigation.

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