LBL-16994 c.2
Preprint

Lawrence Berkeley Laboratory
UNIVERSITY OF CALIFORNIA

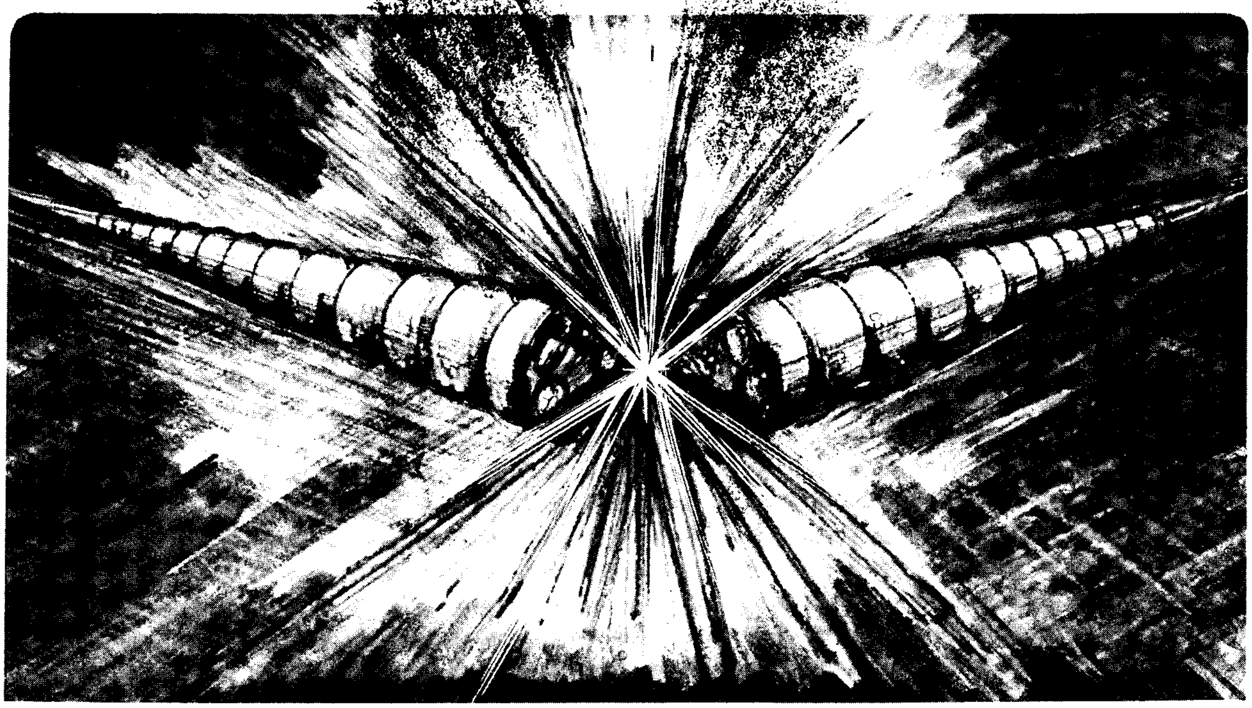
Accelerator & Fusion Research Division

Submitted to Physics Letters A

DISSIPATIVE HAMILTONIAN SYSTEMS: A UNIFYING PRINCIPLE

A.N. Kaufman

November 1983



Prepared for the U.S. Department of Energy under Contract DE-AC03-76SF00098

Submitted to Physics Letters A LBL-16994

Dissipative Hamiltonian Systems: A Unifying Principle*

Allan N. Kaufman

Physics Department and Lawrence Berkeley Laboratory
University of California
Berkeley, CA 94720

28 November 1983

Abstract

The concept of Hamiltonian system is generalized to include a wide class of dissipative processes. Evolution of any observable is generated jointly by a Hamiltonian, with an entropy-conserving Poisson bracket, and an entropy, with an energy-conserving dissipative bracket. This approach yields many of the standard kinetic equations, such as those representing particle collisions, three-wave interactions, and wave-particle resonances.

* This work was supported by the Director, Office of Energy Research, of the U.S. Department of Energy, under Contract No. DE-AC03-76SF00098.

A common feature of many of the standard kinetic equations of evolution is that their dissipative terms conserve energy while they monotonically increase entropy. Such equations also include entropy-conserving terms, which are generated by a Hamiltonian and Poisson bracket (PB).

It is now recognized [1-4] that associated with a PB is a special set of observables, denoted Casimirs, whose PB with any observable vanishes. Thus they are invariant for any Hamiltonian. (They play a crucial role in Arnold's stability method [3], and in foliating Poisson manifolds into symplectic leaves. [4]) Such a Casimir is the entropy.

In this paper we introduce the analogous concept of dissipative bracket (DB), and an associated set of observables, whose DB with any observable vanishes. These are the dissipative invariants, such as energy and momentum.

In analogy to the PB $\{,\}$, which is bilinear, acts as a derivative, and is antisymmetric and Jacobi, we introduce the DB $\langle \, , \, \rangle$, which is bilinear, acts as a derivative, and is symmetric and positive semi-definite. A dissipative Hamiltonian system is equipped with a PB, a DB, and two functions generating its evolution, its Hamiltonian H and its entropy S.

The governing equation of evolution for any observable A is defined as

$$\dot{A} = \{A, H\} + \langle A, S \rangle. \tag{1}$$

From the properties stated above, we have

$$0 = \{S, H\} = \langle H, S \rangle, \tag{2}$$

i.e., entropy is a Casimir, energy is a dissipative invariant; and

$$\{H, H\} = 0, \qquad \langle S, S\rangle \geq 0. \tag{3}$$

It follows immediately that $\dot{H} = 0$, $\dot{S} \geq 0$, which we recognize as the first and second laws of thermodynamics.

We illustrate this concept by applying it to three standard kinetic equations: [5-10] (A) the Landau kinetic equation, used in plasma physics; (B) the wave-kinetic equation for resonant triads; (C) the wave-particle-resonance kinetic equation, a generalization of the quasilinear theory of plasma physics.

In example (A) the dynamical variable is the distribution $f(z)$ (one species for simplicity) on particle phase space $z = (\underline{r}, \underline{p})$. The system PB on observables $A(f)$ is given by the Lie-Poisson formula: [11-13]

$$\{A_1, A_2\} = \int d^6z\ f(z)\ [\delta A_1/\delta f(z),\ \delta A_2/\delta f(z)], \tag{4}$$

where [,] is the particle bracket:

$$[a_1, a_2] = \partial a_1/\partial \underline{r}\cdot\partial a_2/\partial \underline{p} - \partial a_1/\partial \underline{p}\cdot\partial a_2/\partial \underline{r}\ . \tag{5}$$

We choose a form for the DB so as to satisfy the requirements discussed above, and to represent pair collisions:

$$\langle A_1, A_2\rangle = \frac{1}{2}\int d^6z \int d^6z'\ f(z)\ f(z') \int d^3k\ \alpha_{\underline{k}}(z,z')\delta(\Delta_{\underline{k}}H)(\Delta_{\underline{k}}A_1)(\Delta_{\underline{k}}A_2), \tag{6}$$

where $\Delta_{\underline{k}}$ is a linear differential operator chosen below. The delta-function guarantees that $\langle H,A\rangle = 0$, for all A, so that energy is a dissipative invariant for any S.

We note that the DB (6) is indeed positive semi-definite, provided that α (representing the short-range interaction) is positive everywhere. It is manifestly symmetric and bilinear, and acts as a derivative on A_1 and A_2.

In order that the total momentum $\underline{P} = \int d^6z\, f(z)\underline{p}$ be a dissipative invariant, we choose

$$\Delta_{\underline{k}}\, A(f) = \underline{k}\cdot\frac{\partial}{\partial\underline{p}}\, \delta A/\delta f(z) - \underline{k}\cdot\frac{\partial}{\partial\underline{p}'}\, \delta A/\delta f(z'). \tag{7}$$

We observe that indeed $\langle \underline{P}, A\rangle = 0$ for any A, since $\Delta_{\underline{k}}\underline{P}$ vanishes identically.

We relate the (self-consistent) particle Hamiltonian to the system Hamiltonian H(f): [13]

$$h(z;f) \equiv \delta H/\delta f(z), \tag{8}$$

and note that, since $\partial h/\partial\underline{p} = \dot{\underline{r}}$, we have

$$\Delta_{\underline{k}}\, H(f) = \underline{k}\cdot(\dot{\underline{r}}-\dot{\underline{r}}')\ . \tag{9}$$

The system Hamiltonian H(f) need not be specified further; it may be relativistic and may include long-range interaction. (However, if a magnetic field acts on the colliding particles, a generalization of (7) is to be used.)

All that remains is a choice for the entropy functional S(f), a Casimir which we adopt from Boltzmann:

$$S(f) = -\int d^6z\, f(z)\, \ln f(z). \tag{10}$$

It is now completely straightforward to deduce the evolution equation for f(z), using (1), (4), (6), (7), (8), (9), (10); we obtain

$$\frac{\partial f(z)}{\partial t} = - [f(z), h(z;f)]$$

$$- \frac{\partial}{\partial \underline{p}} \cdot \int d^6z' \int d^3k \; \underline{k} \; \underline{k} \; \alpha_{\underline{k}}(z,z') \; \delta(\underline{k}\cdot\dot{\underline{r}} - \underline{k}\cdot\dot{\underline{r}}')\cdot(\frac{\partial}{\partial \underline{p}'} - \frac{\partial}{\partial \underline{p}}) \; f(z) \; f(z'). \tag{11}$$

This is the standard Landau kinetic equation, [14] including the Vlasov evolution [f, h]. The coupling coefficient α (symmetric in z, z') is to be obtained from the underlying reversible system.

For example (B), we proceed largely by analogy. The dynamical variable is the action density I(y) (one wave-branch for simplicity) on ray phase space $y = (\underline{x},\underline{k})$. The system PB is [15,13]

$$\{A_1, A_2\} = \int d^6y \; I(y) \; [\delta A_1/\delta I(y), \; \delta A_2/\delta I(y)], \tag{12}$$

with $[a_1, a_2] = \partial a_1/\partial \underline{x} \cdot \partial a_2/\partial \underline{k} - \partial a_1/\partial \underline{k} \cdot \partial a_2/\partial \underline{x}$. (13)

With a possibl y nonlinear system Hamiltonian H(I), we introduce the (self-consistent) wave frequency: [15,13]

$$\omega(y;I) \equiv \delta H(I)/\delta I(y), \tag{14}$$

which is the ray Hamiltonian $\omega(\underline{x},\underline{k})$, assumed positive here. We then choose the differential operator $\Delta_{y,y',y''}$:

$$\Delta_{y,y',y''} \; A(I) \equiv \delta A/\delta I(y) - \delta A/\delta I(y') - \delta A/\delta I(y''), \tag{15}$$

so as to represent local resonant interaction of wave triads:

$$\Delta_{y,y',y''} H(I) = \omega(\underline{k},\underline{x}) - \omega(\underline{k}',\underline{x}') - \omega(\underline{k}'',\underline{x}''), \tag{16}$$

$$\Delta_{y,y',y''} \underline{P}(I) = \underline{k} - \underline{k}' - \underline{k}'', \tag{17}$$

with $\underline{P}(I) = \int d^6y \; I(y) \; \underline{k}$, the total wave momentum.

Thus the DB choice:

$$\langle A_1, A_2 \rangle = \frac{1}{2}\int d^6y \int d^6y' \int d^6y'' \; I(y)I(y')I(y'') \; \alpha(y,y',y'') \; \delta(\Delta H)\delta(\Delta \underline{P})(\Delta A_1)(\Delta A_2) \tag{18}$$

guarantees that wave energy and momentum are dissipative invariants. The coupling α is positive, symmetric in (y', y''), comes from the underlying Hamiltonian theory, and includes approximate spatial locality factors $\delta^3(\underline{x}-\underline{x}') \; \delta^3(\underline{x}-\underline{x}'')$.

With the standard expression for wave entropy [5,6]

$$S(I) = \int d^6y \; \ln I(y), \tag{19}$$

which, like (10), is a Casimir, we again straightforwardly obtain the evolution of $I(y)$:

$$\frac{\partial I(y)}{\partial t} = - [I(y), \omega(y)]$$

$$+ \frac{1}{2} \int d^6y' \int d^6y'' \; \alpha(y,y',y'') \; \delta(\underline{k}-\underline{k}'-\underline{k}'') \; \delta(\omega-\omega'-\omega'')(I'I'' - I(I'+I''))$$

$$- \int d^6y' \int d^6y'' \; \alpha(y',y,y'') \; \delta(\underline{k}'-\underline{k}-k'') \; \delta(\omega'-\omega-\omega'')(II'' - I'(I+I'')). \tag{20}$$

This generalizes the standard wave kinetic equation [5,6] to nonuniform nonlinear media. (In (20), ω' denotes $\omega(y')$, I' denotes $I(y')$, etc.)

As the final example (C), we now couple the waves and particles resonantly, with reference to an unmagnetized plasma. (Again, the magnetized case is a generalization of the formulas below.) With the dynamical variables $f(z)$, $I(y)$, we choose the total PB on observables $A(f,I)$ to be the sum of the respective brackets (4) and (12), with [,] defined for functions in the respective phase spaces z, y.

Now f represents the oscillation-center distribution [16], so that

$$K(z;\, f,I) \equiv \delta H(f,I)/\delta f(z) \tag{21}$$

is the ponderomotive Hamiltonian [16,13]. We still have $\dot{\underline{r}} = \partial K/\partial \underline{p}$, but $\underline{r}$ is now the oscillation-center position. The wave frequency [13]

$$\omega(y;\, f,I) \equiv \delta H(f,I)/\delta I(y) \tag{22}$$

now depends explicitly on the evolving oscillation-center distribution; thus (22) is the local dispersion relation.

The differential operator $\Delta_{y,z}$ is chosen to make the total momentum

$$\underline{P}(f,I) = \int d^6z\ f(z)\underline{p} + \int d^6y\ I(y)\underline{k} \tag{23}$$

a dissipative invariant. Thus

$$\Delta_{y,z}\, A(f,I) \equiv \delta A/\delta I(y) - \underline{k}\cdot\frac{\partial}{\partial \underline{p}}\, \delta A/\delta f(z) \tag{24}$$

yields $\Delta_{y,z} \underline{P} = 0$. From (21 and (22), we obtain the local Landau resonance:

$$\Delta_{y,z} H(f,I) = \omega(y;f,I) - \underline{k} \cdot \underline{\dot{r}}(z;f,I) \tag{25}$$

We now choose the simplest form for the DB:

$$\langle A_1, A_2 \rangle = \int d^6z \int d^6y \; f(z)\, I(y)\, \alpha(z,y)\, \delta(\Delta H)\, (\Delta A_1)(\Delta A_2), \tag{26}$$

with α positive, representing the coupling strength. The total entropy S(f,I) is the sum of (10) and (19). Again, the equations of evolution follow straightforwardly:

$$\frac{\partial I(y)}{\partial t} = - [I(y), \omega(y;f,I)]_y + \int d^6z \; \alpha(z,y)\, \delta(\omega - \underline{k}\cdot\underline{\dot{r}})\, (f(z) - I(y)\, \underline{k}\cdot\partial f(z)/\partial\underline{p}), \tag{27}$$

$$\frac{\partial f(z)}{\partial t} = - [f(z), K(z;f,I)]_z + \frac{\partial}{\partial \underline{p}} \cdot \int d^6y \; \alpha(z,y)\, \delta(\omega - \underline{k}\cdot\underline{\dot{r}})\, \underline{k}\, (-f(z) + I(y)\, \underline{k}\cdot\partial f(z)/\partial\underline{p}). \tag{28}$$

In these coupled equations, the [,] terms represent the self-consistent wave propagation and oscillation-center evolution, including ponderomotive effects. The dissipative terms independent of I represent Cerenkov emission and radiation reaction. The terms linear in I represent Landau damping/growth and quasilinear diffusion. We note the common coupling [17] for these dissipative effects.

With these three examples expressed as dissipative Hamiltonian systems, we may expect that many other processes can be similarly represented. Examples that come to mind are (at the kinetic level) spontaneous and induced wave scattering by particles [13], bremsstrahlung and collisional damping, and (at the fluid level) resistivity, viscosity, thermal conductivity, diffusion, and thermal equilibration. Preliminary studies indicate that this is indeed the case; results will be published elsewhere [18]. It is hoped that this unifying principle will lead to new stability theorems for dissipative Hamiltonian systems.

References

1. R. G. Littlejohn, "Singular Poisson Tensors," in Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems, eds. M. Tabor and Y. M. Treve (AIP Conf. Proc. 88, 1982), 47-66.

2. E. C. G. Sudarshan and N. Mukunda, Classical Dynamics: A Modern Perspective (Wiley 1974).

3. D. D. Holm, J. E. Marsden, T. Ratiu, and A. Weinstein, "Nonlinear Stability Conditions and a Priori Estimates for Barotropic Hydrodynamics," Physics Letters 98A (1983) 15-21.

4. A. Weinstein, "Stability of Poisson-Hamilton Equilibria," in Fluids and Plasmas: Geometry and Dynamics, ed. J. Marsden (Am. Math. Soc. Providence, R.I., in press).

5. R. C. Davidson, Methods in Nonlinear Plasma Theory (Academic Press 1972).

6. A. Hasegawa, Plasma Instabilities and Nonlinear Effects (Springer-Verlag 1975).

7. S. Ichimaru, Basic Principles of Plasma Physics (Benjamin, Reading 1973).

8. B. B. Kadomtsev, Plasma Turbulence (Academic Press 1965).

9. R. Z. Sagdeev and A. A. Galeev, Nonlinear Plasma Theory. (Benjamin 1969).

10. V. N. Tsytovich, Theory of Turbulent Plasma (Consultants Bureau, New York 1977).

11. J. Marsden and A. Weinstein, "The Hamiltonian Structure of the Maxwell-Vlasov Equations," Physica 4D (1982) 394-406.

12. P. J. Morrison, "The Maxwell-Vlasov Equations as a continuous Hamiltonian System", Phys. Lett. 80A (1980) 383-386.

13. A. N. Kaufman, "Natural Poisson Structures of Nonlinear Plasma Dynamics," Physica Scripta T2/2 (1982) 517-521.

14. E. Frieman, "A Kinetic Equation for Spatially Inhomogeneous Systems with Weak Coupling," Nuclear Fusion: 1962 Supp. Part 2, 487-489.

15. S. W. McDonald and A. N. Kaufman, "Hamiltonian Kinetic Theory of Plasma Ponderomotive Processes," in Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems, eds. M. Tabor and Y. M. Treve, (AIP Conf. Proc. 88, 1982), 117-120.

16. R. L. Dewar, "Oscillation Center Quasilinear Theory," Phys. Fluids 16 (1973) 1102-1107.

17. A. N. Kaufman, "Resonant Interactions Between Particles and Normal Modes in a Cylindrical Plasma," Phys. Fluids 14 (1971) 387-397.

18. P. J. Morrison reports similar results for fluid models (to be published).

TECHNICAL INFORMATION DEPARTMENT
LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720