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Authors

Paarporn, Keith
Chandan, Rahul
Alizadeh, Mahnoosh
[et al.](#)

Publication Date

2020-09-01

Peer reviewed

Characterizing the interplay between information and strength in Blotto games

Keith Paarporn, Rahul Chandan, Mahnoosh Alizadeh, and Jason R. Marden

Abstract—In this paper, we investigate informational asymmetries in the Colonel Blotto game, a game-theoretic model of competitive resource allocation between two players over a set of battlefields. The battlefield valuations are subject to randomness. One of the two players knows the valuations with certainty. The other knows only a distribution on the battlefield realizations. However, the informed player has fewer resources to allocate. We characterize unique equilibrium payoffs in a two battlefield setup of the Colonel Blotto game. We then focus on a three battlefield setup in the General Lotto game, a popular variant of the Colonel Blotto game. We characterize the unique equilibrium payoffs and mixed equilibrium strategies. We quantify the value of information - the difference in equilibrium payoff between the asymmetric information game and complete information game. We find information strictly improves the informed player’s performance guarantee. However, the magnitude of improvement varies with the informed player’s strength as well as the game parameters. Our analysis highlights the interplay between strength and information in adversarial environments.

I. INTRODUCTION

In adversarial interactions, informational asymmetries may provide an advantage to one competitor over the other. Adversarial contests appear in a wide range of applications, such as political campaigns [1], [2], security of cyber-physical systems [3], and competitive advertising [4]. Rigorous studies on the interplay between asymmetric information and strategic decision-making in adversarial settings has received attention in the recent literature [5], [6]. Though these works analyze general zero-sum game settings, another popular framework for analyzing such contests is the Colonel Blotto game. Two players strategically allocate their limited resources over a finite set of battlefields. A player secures a battlefield if it has allocated more resources to it than the opponent. Each player aims to secure as many battlefields as possible. The goal of the present paper is to quantify the performance improvements one player in the Blotto game may experience when it possesses system-level information while its opponent does not.

Though simply posed, the Colonel Blotto game features highly complex strategies. The established literature on the Colonel Blotto game and its variants primarily focuses on characterizing mixed-strategy Nash equilibria, due to non-existence of pure equilibria in most cases of interest. First

introduced by Borel in 1921 [7], researchers have incrementally contributed to this body of work over the last one hundred years [8], [9]. In 1950, Gross and Wagner [8] provided an equilibrium solution of the two battlefield case with asymmetric values and resources, as well as for n homogeneous battlefields and symmetric resources. In 2006, the work of Roberson [9] generalized the solution to n battlefields and asymmetric forces by leveraging the solutions of all-pay auctions and the theory of copulas [10].

There recently has been renewed interest in the Colonel Blotto game, along with its many variants and extensions [2], [11]–[15]. Notably, results for heterogeneous battlefield valuations have appeared in Colonel Blotto as well as the General Lotto variant, which admits a more tractable analysis [13], [14]. Blotto games have also received attention in engineering application areas such as the security of cyber-physical systems [3] and the formation of large-scale networks [16], [17]. The vast majority of these studies assume the players have complete information about the opposing player’s resource budget and the values of each battlefield.

Few works have considered the Blotto or Lotto game with incomplete information. In [18], the authors study a Blotto game in which players have incomplete information about the other’s resource budgets. In [19], the players are subject to incomplete information about the battlefield valuations. In both of these works, all players are equally uninformed about the parameters, and hence symmetric Bayes-Nash equilibria are provided. To the best of our knowledge, one-shot Blotto games where the players possess asymmetric information has not received attention in the literature.

In this paper, we formulate a Bayesian game framework in which one player is completely informed and the other does not receive any side information about the battlefield valuations (Section III). Our main contributions characterize unique equilibrium payoffs and strategies in representative scenarios of this framework. We first analyze the Colonel Blotto game with two battlefields (Section IV). With three battlefields, we are able to solve for unique mixed-strategy equilibria in a representative General Lotto game under the same informational asymmetry (Section V). We do this by leveraging established results on all-pay auctions with asymmetric information [20] (Section VI). We can then quantify the difference between the equilibrium payoff in this game and the scenario where both players are uninformed.

These characterizations allow us to determine conditions under which the informed player has an advantage in the Lotto game, despite having fewer resources to allocate. As an illustrative analysis, we consider a scenario where

K. Paarporn (kpaarporn@ucsb.edu), R. Chandan (rchandan@ucsb.edu), M. Alizadeh (alizadeh@ucsb.edu), and J. R. Marden (jrmarden@ece.ucsb.edu) are with the Department of Electrical and Computer Engineering at the University of California, Santa Barbara, CA.

This work is supported by UCOP Grant LFR-18-548175, ONR grant #N00014-17-1-2060, and NSF grant #ECCS-1638214.

both players are uninformed and the option of purchasing information with a fraction of its budget is available only to the weaker player. We quantify a measure of informational value determined by the largest proportion of resources the player is willing to give up in exchange for information (Section V).

II. MODEL

Two players, I and U , allocate their non-atomic forces across n battlefields simultaneously. In each battlefield $j \in \{1, \dots, n\}$, the player that sends more forces wins the battlefield and receives payoff v^j , while the losing player receives $-v^j$. In the case of a tie on a battlefield, both players receive zero payoff. The utility to each player is the sum of payoffs across all battlefields. The players have a budget on their forces, $X_I, X_U > 0$. We assume $X_I < X_U$. The budgets and battlefield valuations $\mathbf{v} := \{v^j\}_{j=1}^n$ are common knowledge. For $z \in \{I, U\}$, a pure strategy is a non-negative vector $\mathbf{x}_z = (x_z^1, \dots, x_z^n) \in \mathbb{R}_+^n$. We call \mathbf{x}^z an *allocation*. The payoff to player z is

$$u_z(\mathbf{x}_z, \mathbf{x}_{-z}; \mathbf{v}) := \sum_{j=1}^n v^j \text{sgn}(x_z^j - x_{-z}^j), \quad (1)$$

where we follow the convention $\text{sgn}(0) = 0$. This defines a zero-sum game since the payoff to player $-z$ is the negative of the above. A mixed strategy for player z is an n -variate distribution function F_z on \mathbb{R}_+^n . That is, an allocation \mathbf{x}_z is drawn from the distribution F_z . A distribution F_z has n univariate marginal (cumulative) distributions $\{F_z^j\}_{j=1}^n$, specifying the allocation distributions to each battlefield. Extending the definition of (1) to mixed strategies, the expected payoff for each player can be expressed as

$$\begin{aligned} u_z(F_z, F_{-z}; \mathbf{v}) &= \int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^n} \sum_{j=1}^n v^j \text{sgn}(x_z^j - x_{-z}^j) dF_z dF_{-z} \\ &= \sum_{j=1}^n v^j \int_0^\infty (2F_{-z}^j - 1) dF_z^j. \end{aligned} \quad (2)$$

A. The Colonel Blotto game

In the Colonel Blotto game, a feasible allocation \mathbf{x}_z for player $z \in \{I, U\}$ lies in the set

$$B(X_z) := \left\{ \mathbf{x}_z \in \mathbb{R}_+^n : \sum_{j=1}^n x_z^j = X_z \right\}. \quad (\text{BC})$$

Furthermore, the support of a mixed strategy F_z is contained in $B(X_z)$. Thus, an allocation \mathbf{x}_z drawn from F_z belongs to $B(X_z)$ with probability one. We denote the set of all such feasible mixed strategies with $\mathcal{B}(X_z)$. We refer to the Colonel Blotto game with $\text{CB}(X_I, X_U, \mathbf{v})$.

B. The General Lotto game

The General Lotto game relaxes the support constraint in the Colonel Blotto game, requiring each player's allocation

budget to be met only in expectation with respect to their marginal distributions. That is,

$$\mathcal{L}(X_z) := \left\{ F_z : \sum_{j=1}^n \mathbb{E}_{F_z^j}[x_z^j] = X_z \right\} \quad (\text{LC})$$

is the set of such feasible mixed strategies. Note that both the budget constraint (LC) and expected payoff (2) only depend on the univariate marginal distributions and not the joint n -variate distribution F_z . Hence, a player's choice amounts to selecting n independent univariate marginals that satisfy (LC). We specify General Lotto game with $\text{GL}(X_I, X_U, \mathbf{v})$.

III. BLOTTO AND LOTTO GAMES WITH ASYMMETRIC INFORMATION

Using a standard Bayesian games framework, we wish to model a situation in which the battlefield valuations are subject to randomness. Suppose there are m possible sets of battlefield valuations, labeled by the states $\Omega = \{\omega_1, \dots, \omega_m\}$. The state ω_i is realized with probability $p_i > 0$, and the probability vector $\mathbf{p} \in \Delta(\Omega)$ satisfying $\sum_{i=1}^m p_i = 1$ is common knowledge to the players. Given state ω_i is realized, battlefield j is worth $v_i^j > 0$, and we denote V as the $m \times n$ matrix with elements v_i^j .

We consider the setting where player I observes the true state realization, and player U does not receive any side information about the realization¹. We call U the "uninformed" player. Specifically, player I has m distinct types $T_I := \{t_1, \dots, t_m\}$, receiving type t_i whenever state ω_i is realized. That is, I knows the realization with certainty. We refer to T_I as its type space. Player U has a single type, and hence infers the realization (and I 's type) according to the common prior \mathbf{p} . The information structure is known to both players.

A strategy for player I consists of n -variate distributions $\{F_I(t_i)\}_{i=1}^m$, one for each type t_i , $i = 1, \dots, m$. We refer to the collection $F_I = \{F_I(t_i)\}_{i=1}^m$ as I 's strategy. Player U 's strategy F_U is a single n -variate distribution. For the Lotto game, $F_I(t_i) \in \mathcal{L}(X_I)$ for each t_1, \dots, t_m , and $F_U \in \mathcal{L}(X_U)$. For the Blotto game, they belong to $\mathcal{B}(X_I)$ and $\mathcal{B}(X_U)$, respectively. We denote $\{F_I^j(t_i)\}_{j=1}^n$ as the univariate marginals of $F_I(t_i)$ for each $t_i \in T_I$, and F_U^j the univariate marginals of F_U . All of these components specify a Lotto or Blotto game with incomplete information, which we denote as $\text{GL}(X_I, X_U, V, \mathbf{p})$ and $\text{CB}(X_I, X_U, V, \mathbf{p})$, respectively. The expected utility of player U is

$$\pi_U(F_U, F_I) := \sum_{i=1}^m p_i u_U(F_U, F_I(t_i), \omega_i) \quad (3)$$

where with some abuse of notation, we use ω_i to refer to the battlefield valuation set in state i . The ex-interim utility for player I , given type t_i is the expected payoff

$$\pi_I(F_I(t_i), F_U|t_i) := u_I(F_I(t_i), F_U, \omega_i). \quad (4)$$

¹A more general framework of asymmetric information may be formulated with Bayesian games. That is, arbitrary partial information structures may be assigned to the players. Since our focus is on one informed and one uninformed player, we leave such generalizations to future work.

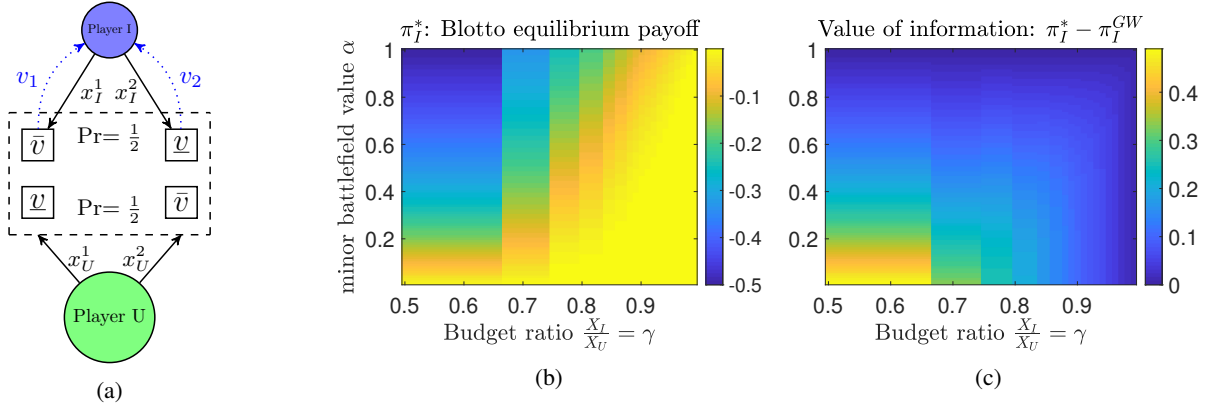


Fig. 1: (a) A diagram of the Colonel Blotto game with two battlefields and asymmetric budgets and information. There are two possible battlefield valuation sets, each realized with probability $\frac{1}{2}$. Player I has fewer resources than player U, but knows the true realization. (b) The equilibrium payoff (8) to the informed player in the Blotto game $\text{CB}(X_I, X_U, V, \frac{1}{2})$, where $\bar{v} = 1$ and $\underline{v} = \alpha \in (0, 1)$. It is negative when $\gamma \in (\frac{1}{2}, 1)$ and $\alpha \in (0, 1)$. (c) The value of information, $\pi_I^* - \pi_I^{GW}$, is non-negative. Information offers the most improvement for low budget (γ near $\frac{1}{2}$) and when there is higher priority in the diagonal battlefields (α low).

A Bayes-Nash equilibrium (BNE) is a strategy profile (F_I^*, F_U^*) satisfying

$$F_I^*(t_i) \in \arg \max_{F_I(t_i)} \pi_I(F_I(t_i), F_U^*|t_i), \forall i = 1, \dots, m$$

$$\text{and } F_U^* \in \arg \max_{F_U} \pi_U(F_U, F_I^*). \quad (5)$$

An equivalent BNE condition is that (F_I^*, F_U^*) satisfies the best-response correspondences in ex-ante payoffs, given that each $p_i > 0$ [21]. That is, $F_U^* \in \arg \max_{F_U} \pi_U(F_U, F_I^*)$ and $F_I^* \in \arg \max_{\{F_I(t_i)\}_{i=1}^m} \pi_I(F_I, F_U^*)$, where I 's ex-ante payoff is

$$\pi_I(F_I, F_U) := \sum_{i=1}^m p_i \pi_I(F_I(t_i), F_U|t_i). \quad (6)$$

We note that since the underlying complete information game is zero-sum, the games with asymmetric information are also zero-sum with respect to ex-ante utilities (3), (6). This allows us to speak of a unique equilibrium ex-ante payoff $\pi_I^* = -\pi_U^*$ of the Blotto and Lotto games with asymmetric information.

IV. COLONEL BLOTTO RESULTS

In the following structural result, we characterize sufficient conditions on the budget ratio $\gamma := \frac{X_I}{X_U}$ such that the stronger player U is guaranteed a positive equilibrium payoff regardless of the information asymmetry.

Proposition 1. Assume $\frac{X_I}{X_U} < 1$. Consider the game $\text{CB}(X_I, X_U, V, \mathbf{p})$ where $V \in m \times n$, i.e. there are n battlefields and m possible state realizations. A sufficient condition for which $\pi_U^* \geq 0$ is

$$\begin{cases} \frac{X_I}{X_U} < \frac{2}{n}, & \text{if } n \text{ is even} \\ \frac{X_I}{X_U} < \frac{2}{n+1}, & \text{if } n \text{ is odd} \end{cases} \quad (7)$$

Proof. Denote $E_j = \sum_{i=1}^m p_i v_i^j$ as the prior expected value of battlefield j . Without loss of generality, assume the n battlefields are ordered according to $E_1 \geq E_2 \geq \dots \geq E_n$.

For n even, consider player U 's deterministic strategy that places $2X_I/n$ to the first $n/2$ battlefields, and the remaining resources allocated arbitrarily. This guarantees U attains at least half of the total available value. That is, U 's security value for this strategy is at least zero. Similar reasoning applies to the case of n odd. ■

For two battlefields, the weaker resource player I never wins the game, despite having better information. The following analysis focuses on the two battlefield case of $\text{CB}(X_I, X_U, V, \mathbf{p})$ under the uniform prior $\mathbf{p} = \frac{1}{2}\mathbb{1}_2$. Although I never wins, we quantify how information improves its payoff relative to when both players are uninformed. A solution for a general prior $\mathbf{p} = [p, 1-p]$ remains a challenge in this analysis, due to the complexity of finding suitable equilibrium distributions. Hence for the rest of this section, we assume $p = \frac{1}{2}$.

A. Two battlefields case

If $\frac{X_I}{X_U} < \frac{1}{2}$, player U can secure both battlefields regardless of what I does. We thus restrict our attention to the regime $\frac{X_I}{X_U} \in (\frac{1}{2}, 1)$. We state the main result of this section, which characterizes the equilibrium payoff to I . Here, we assume a symmetric structure on the battlefields $V = \frac{1}{\bar{v} + \underline{v}} \begin{bmatrix} \bar{v} & \underline{v} \\ \underline{v} & \bar{v} \end{bmatrix}$ for $\bar{v} > \underline{v} > 0$. Figure 1a illustrates the setup of this game.

Theorem 1. Let $\frac{X_I}{X_U} \in (\frac{1}{2}, 1)$, and assume $p = \frac{1}{2}$. Define $q = \lfloor \frac{X_U}{X_U - X_I} \rfloor$. Then the ex-ante equilibrium payoff to the informed player I is

$$\pi_I^* = \begin{cases} - \left(2 \sum_{k=0}^{\frac{q-1}{2}} (\bar{v}/\underline{v})^k - 1 \right)^{-1} & \text{if } q \text{ odd} \\ - \frac{\underline{v}}{\bar{v} + \underline{v}} \left(\sum_{k=0}^{\frac{q}{2}-1} (\bar{v}/\underline{v})^k \right)^{-1} & \text{if } q \text{ even} \end{cases} \quad (8)$$

We provide the proof in the Appendix, which details a set of equilibrium mixed strategies. When both players are uninformed, i.e. both have a single type, we have a complete information game where the two battlefield valuations are

their expected values, $\frac{1}{2}$. The equilibrium payoff is then given by an application of Gross and Wagner's solution [8], which yields $\pi_I^{\text{GW}} := -\frac{1}{q}$. The following result verifies the equilibrium payoff improves when I obtains information.

Corollary 1. We have $\pi_I^* > \pi_I^{\text{GW}}$. That is, information strictly improves the equilibrium payoff for I .

Proof. Consider the q even case. Since $\frac{\bar{v}}{\bar{v}+\underline{v}} < \frac{1}{2}$ and $\frac{\bar{v}}{\underline{v}} > 1$ for $\bar{v} > \underline{v}$, from (8) we get $-\frac{\underline{v}}{\bar{v}+\underline{v}} \left(\sum_{k=0}^{\frac{q}{2}-1} (\bar{v}/\underline{v})^k \right)^{-1} > -\frac{1}{q}$. Similar arguments apply in the q odd case. ■

We may quantify a value of information in this setting as the payoff difference $\pi_I^* - \pi_I^{\text{GW}}$. Setting $\bar{v} = 1$ and $\underline{v} = \alpha \in (0, 1)$, we plot the equilibrium values π_I^* in Figure 1, as well as the value of information. In the two battlefield Blotto game, an informed player still cannot gain an advantage, as $\pi_I^* < 0$ for all parameters $(\alpha, \gamma) \in (0, 1) \times (0, 1)$. Information offers the most payoff improvement when I 's budget is low and the minor battlefield α is worth less (Fig. 1c).

In the next section, we consider the three battlefield case in the General Lotto game. In general, equilibrium solutions for $n \geq 3$ heterogeneous battlefields have not been characterized in the Colonel Blotto game, due to the complexity of finding suitable copulas for the marginal distributions [2]. The Lotto constraint relaxation allows for analytic tractability in cases of more than two heterogeneous battlefields [14].

V. RESULTS ON GENERAL LOTTO

We restrict our attention to a representative three-battlefield Lotto game $\text{GL}(X_I, X_U, V_{\alpha\beta}, \mathbf{p})$, where $\mathbf{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and $V_{\alpha\beta} = \frac{1}{1+\alpha+\beta} \begin{bmatrix} 1 & \alpha & \beta \\ \beta & 1 & \alpha \\ \alpha & \beta & 1 \end{bmatrix}$. where $1 > \alpha \geq \beta > 0$. In each set of battlefields (rows), the total valuation is normalized to one. Though this formulation may be quite specific, the representative game serves as an illustrative and tractable scenario highlighting the informational asymmetry between players. Intuitively, for small α, β values, the valuable battlefield is worth 1 but sits at a different location in each realization. An informed player would be able to take the most advantage by focusing its resources on this battlefield without wasting resources on the other battlefields.

A. Main result: characterization of equilibrium payoff

Theorem 2. Let $\gamma = \frac{X_I}{X_U} \leq 1$. Then player I 's equilibrium payoff, $\pi_I^*(\alpha, \beta, \gamma)$, for $\alpha, \beta, \gamma \in (0, 1)$ takes the values

$$\begin{aligned} & \frac{3\gamma}{1+\alpha+\beta} - 1, & \text{if } \gamma \in (0, \frac{1}{3}] \\ & \frac{1}{1+\alpha+\beta} \left[\left(1 - \frac{1}{3\gamma}\right) (3\gamma\alpha + (1-\alpha)) + 1 \right] - 1, & \text{if } \gamma \in (\frac{1}{3}, \frac{2}{3}] \end{aligned} \quad (9)$$

and

$$\frac{1}{1+\alpha+\beta} \left[2 - \frac{1}{3\gamma} + \alpha \left(2 - \frac{1}{\gamma}\right) + 3\beta\gamma \left(1 - \frac{2}{3\gamma}\right) \right] - 1 \quad (10)$$

if $\gamma \in (\frac{2}{3}, 1]$.

A plot of π_I^* is shown in Figure 2a for the special case $\alpha = \beta$. We provide the details of the proof in Section VI, where we draw upon known results in asymmetric information

all-pay auctions. An iterative algorithm is formulated in [20] to construct mixed equilibrium strategies. To verify that the set of constructed strategies indeed is a BNE of $\text{GL}(X_I, X_U, V_{\alpha\beta}, \frac{1}{3}\mathbf{1}_3)$, the constraint (LC) must be met.

B. The value of information in the Lotto game

In the following analysis, we consider a scenario in which both players are uninformed, and the option to purchase information with a fraction of its budget is available to player I . We quantify a value of information as the equilibrium payoff gain or loss in purchasing information, and specify the maximal cost I is willing to pay before it experiences a payoff loss. In the case both players are uninformed, we may use the algorithm of [20] to solve the Lotto game. In this setting, we arrive at the equilibrium payoff $\gamma - 1$ for I . We first highlight some immediate consequences of Theorem 2.

Corollary 2. We have that $\pi_I^*(\alpha, \beta, \gamma) > \gamma - 1$ for $(\alpha, \beta, \gamma) \in (0, 1)^3$. That is, information strictly improves the equilibrium payoff for I . Additionally, π_I^* is strictly decreasing in α and β , and strictly increasing in γ .

Proof. The first claim follows from comparing the expressions (9) and (10) to $\gamma - 1$. The second claim follows by showing the respective signs of partial derivatives hold. ■

We also characterize the parameter region in which an informed I wins the game for the special case $\alpha = \beta$.

Corollary 3. Fix a budget ratio γ . Then $\pi_I^*(\alpha, \alpha, \gamma) > 0$ if and only if $\alpha < \frac{\frac{1}{3}-\gamma}{3\gamma^2-4\gamma+\frac{1}{3}}$.

Proof. Player I cannot win in the first region $\gamma \in (0, \frac{1}{3}]$, as $\pi_I^* \leq 0$. We find that the second expression of (9) is equivalent to (10) when $\beta = \alpha$. We then solve the equation $\pi_I^*(\alpha, \alpha, \gamma) = 0$ for α , from which we obtain the result. ■

We now provide a general quantity of describing the value of information. In particular, we consider a scenario where both players are uninformed. Player I , with the budget ratio γ , has an opportunity to purchase information with a fraction c_I of its budget. The value of information quantifies the equilibrium payoff gain or loss in purchasing the information at the budget fraction cost c_I . Formally, the value of information is the quantity

$$\text{Vol}(\alpha, \gamma) := \pi_I^*(\alpha, \alpha, (1 - c_I)\gamma) - (\gamma - 1). \quad (11)$$

An instance of the value of information is plotted in Figure 2b, when the cost is $c_I = \frac{1}{5}$. We note there is a regime in which information is not worth the cost, i.e. $\text{Vol}(\alpha, \gamma) < 0$.

We also seek to find the highest cost on information that I is willing to pay before it experiences an equilibrium payoff loss. To quantify this cost, let γ_e be defined as the value that satisfies $\pi_I^*(\alpha, \alpha, \gamma_e) = \gamma - 1$. Such a value is unique and well-defined for any α , since $\pi_I^*(\alpha, \alpha, \gamma) > \gamma - 1$ and is strictly increasing in γ , by Corollary 2. Then $C_I(\alpha, \gamma) := \frac{\gamma - \gamma_e}{\gamma}$ is the largest fraction of resources I can give up. That is, for all $c_I \in [0, C_I(\alpha, \gamma)]$, we have $\pi_I^*(\alpha, \alpha, (1 - c_I)\gamma) \geq \gamma - 1$, with equality if and only if $c_I = C_I(\alpha, \gamma)$. Then,

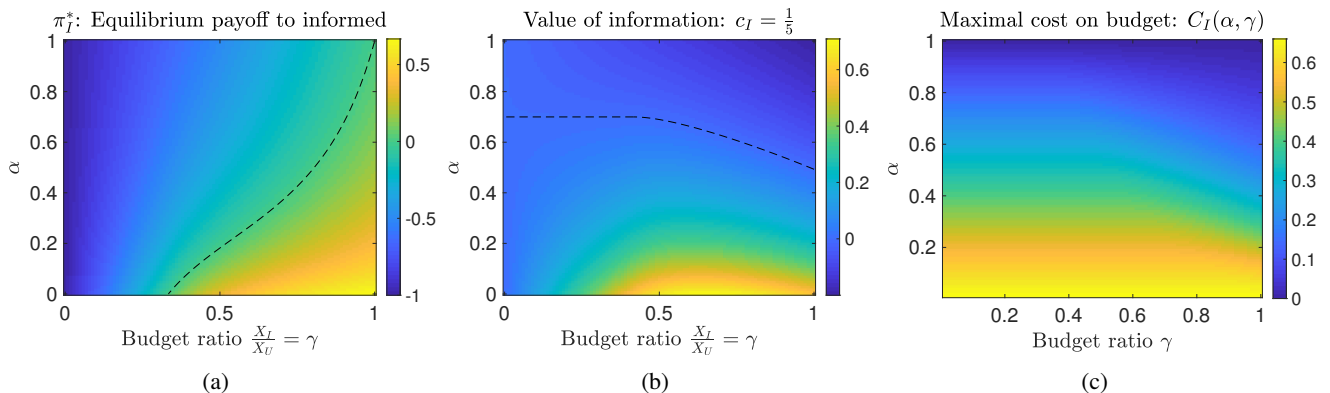


Fig. 2: (a) The equilibrium payoff π_I^* to the informed player in the Lotto game $GL(X_I, X_U, V_{\alpha, \beta}, \frac{1}{3}\mathbb{1}_3)$, under the special case $\alpha = \beta$. The dashed line are the points (γ, α) for which $\pi_I^* = 0$. That is, for parameters below the line, the informed player “wins” the game and “loses” for parameters above the line (see Corollary 3). (b) The value of information, or payoff gain or loss to player I from purchasing information at the cost $c_I = \frac{1}{5}$. The dashed black line in Figure 2b indicates the parameters for which $\text{VoI}(\alpha, \gamma) = 0$. (c) The maximal cost (12), or the largest fraction of its resource budget γ that player I is willing to give up in exchange for information before it experiences a payoff loss. Information is more valuable in lower α ranges - when the battlefield rewards are concentrated at diverse locations.

Corollary 4. Fix a budget ratio γ . Then

$$C_I(\alpha, \gamma) = \begin{cases} \frac{1}{\gamma}(\gamma - \gamma_e(\alpha, \gamma)) & \text{if } (\alpha, \gamma) \in B \\ 1 - \frac{1}{3c_\alpha} & \text{if } (\alpha, \gamma) \notin B \end{cases} \quad (12)$$

where $\gamma_e(\alpha, \gamma) := \frac{-2(1-\alpha)c_\alpha - \gamma + \sqrt{(2(1-\alpha)c_\alpha - \gamma)^2 + 4c_\alpha^2 \alpha(1-\alpha)}}{6\alpha c_\alpha}$, $c_\alpha := \frac{1}{1+2\alpha}$, and $B := \{(\alpha, \gamma) \in (0, 1)^2 : \gamma_e(\alpha, \gamma) \geq \frac{1}{3}\}$.

Proof. We solve the equation $\pi_I^*(\alpha, \alpha, \gamma_e) = \gamma - 1$ for γ_e . Recall the second expression of (9) is equivalent to (10) when $\beta = \alpha$. The resulting value is valid only if $\gamma_e > \frac{1}{3}$. If not, we solve for γ_e using the first entry of (9). ■

Figure 2c plots $C_I(\alpha, \gamma)$. As an example, when $\gamma = 1$ and $\alpha = 0$, player I can exchange up to $\frac{2}{3}$ of its budget for information and still obtain a payoff greater than $\gamma - 1 = 0$. When α is high, player I cannot afford to give up resources, as information is less valuable in this regime.

VI. PROOF OF THEOREM 2

We show that the equilibria in the Lotto game $GL(X_I, X_U, V, \mathbf{p})$ coincides with equilibria of n independent all-pay auctions with asymmetric information and valuations. The work of Siegel [20] provides an iterative algorithm for which to construct equilibrium mixed strategies in the all-pay auction. We leverage this procedure to construct proposed equilibrium distributions in the Lotto game, and verify that they satisfy the constraint (LC).

A. Two-player all-pay auctions with asymmetric information and valuations

Here, we specialize the setup of [20] to our setting of an informed player I and an uninformed player U . Assume the two players compete in an all-pay auction over an indivisible good, where the players I and U have the same type spaces as before. When type $t_i \in T_I$ is realized ($i \in \{1, \dots, m\}$) with probability $p_i > 0$, player I 's valuation of the good is

$\nu_I(t_i)$ and player U 's valuation is $\nu_U(t_i)$. Assume the types in T_I are ordered such that

$$\nu_I(t_i) \geq \nu_I(t_k) \quad (13)$$

whenever $i < k$, $i, k \in \{1, \dots, m\}$. In the two player all-pay auction with asymmetric information and valuations, player I selects the distributions of bids $F_I : T_I \times \mathbb{R}_+ \rightarrow [0, 1]$ contingent on its type. We write $F_I(t_i)$ to refer to the distribution of bids in type i , $F_I(t_i, x)$ to refer to its value at $x \geq 0$, and f_I to refer to its density function. As player U has a single type, it selects a single distribution of bids, $F_U : \mathbb{R}_+ \rightarrow [0, 1]$. The best-response problems for each player are

$$\begin{aligned} \max_{F_I(t_i)} \sum_{i=1, \dots, m} p_i \int_0^\infty [\nu_I(t_i) F_U - x] dF_I(t_i) \\ \max_{F_U} \sum_{i=1}^m p_i \int_0^\infty [\nu_U(t_i) F_I(t_i) - x] dF_U. \end{aligned} \quad (14)$$

B. Algorithm of [20]

An iterative procedure is formulated in [20] to construct equilibrium mixed strategies satisfying the correspondences (14). We note that this procedure handles general player information structures in the two-player all-pay auction. In this paper, we restrict our attention to its application in our informed-uninformed setting. In particular, the constructed marginals are proven to be piecewise constant functions with finite support, with the possibility of having point masses placed at zero. We do not give further details of this algorithm due to space limitations and for ease of exposition. We refer the reader to [20] for a general proof that the output distributions satisfy (14).

C. Connection of General Lotto to all-pay auctions

In the game $GL(X_I, X_U, V, \mathbf{p})$, player U 's Lagrangian at both ex-ante and interim levels may be written as

$$\sum_{j=1}^n \max_{F_U^j} \left[\sum_{i=1}^m p_i \int_0^\infty \left(\frac{2v_i^j}{\lambda_U} F_U^j(t_i) - x \right) dF_U^j \right] \quad (15)$$

where $\lambda_U > 0$ is the Lagrange multiplier on U 's expected budget constraint (LC), and we have removed constant additive and multiplicative terms in the expression that do not depend on the decision variables $\{F_U^j\}_{j=1}^n$. In a similar fashion, player I 's Lagrangian maximization at the interim level may be written

$$\max_{F_I^j(t_i)} \lambda_I^i \sum_{j=1}^n \int_0^\infty \left(\frac{2v_i^j}{\lambda_I^i} F_U^j - x \right) dF_I^j(t_i) \quad (16)$$

where the multiplier $\lambda_I^i > 0$ corresponds to the budget constraint (LC) in type t_i . When two sets of battlefields contain the same valuations, we can deduce equivalence between the corresponding Lagrange multipliers.

Lemma 1. Consider the game $GL(X_I, X_U, V, \mathbf{p})$. Suppose the rows i and k of V have the same elements, each with identical multiplicities. Then the equilibrium ex-interim payoff to player I for type t_i is equivalent to that of type t_k . Furthermore, $\lambda_I^i = \lambda_I^k$.

Proof. An equivalent formulation of (16) in type i is

$$\max_{F_I(t_i) \in \mathcal{L}(X_I)} \pi_I(F_I, F_U | t_i) \quad (17)$$

The corresponding problem for type t_k is identical to the above, because the valuations in both rows are the same (possibly with some permutation of indices j), and the optimization for player U remains (15). This shows the equivalence of interim equilibrium payoffs.

To show $\lambda_I^i = \lambda_I^k$, let F_I^* solve (17) for both types t_i and t_k . Any allocation $x_j^I \in \mathbb{R}_+$ to battlefield j in the support of F_I^{j*} solves the one-dimensional problem $\max_x \frac{2v_i^j}{\lambda_I^i} F_U^{j*} - x$ as well as $\max_x \frac{2v_k^j}{\lambda_I^k} F_U^{j*} - x$. Therefore, the first-order necessary condition for optimality that holds is $\lambda_I^i = 2v_i^j f_U^{j*}(x_j^I) = \lambda_I^k$. ■

To solve for a BNE, we write the best-response correspondences at the ex-ante level. Player U 's ex-ante best-response problem is given by (15), while Player I 's ex-ante optimization problem may be written

$$\sum_{j=1}^n \max_{F_I^j(t_i)} \left[\sum_{i=1}^m \lambda_I^i \int_0^\infty \left(\frac{2v_i^j p_i}{\lambda_I^i} F_U^j - x \right) dF_I^j(t_i) \right] \quad (18)$$

after removing additive constants. Under the conditions $\lambda_I^i = \lambda_I > 0$ for all $i = 1, \dots, m$ (by Lemma 1), each battlefield

j is an independent all-pay auction, whose problem is

$$\max_{F_U^j} \left[\sum_{i=1}^m p_i \int_0^\infty \left(\frac{2v_i^j}{\lambda_U} F_I^j(t_i) - x \right) dF_U^j \right] \quad (19)$$

$$\max_{F_I^j(t_i)} \left[\sum_{i=1}^m p_i \int_0^\infty \left(\frac{2v_i^j p_i}{\lambda_I} F_U^j - x \right) dF_I^j(t_i) \right] \quad (20)$$

which coincides with (14) with auction valuations $\nu_U(t_i) = \frac{2v_i^j}{\lambda_U}$ and $\nu_I(t_i) = \frac{2v_i^j p_i}{\lambda_I}$. Here, we have multiplied each maximization problem of by $p_i > 0$, which does not change the optimal solutions. In this setting, mixed-strategy equilibria of the Lotto game are equivalent to that of n independent two-player all-pay auctions with asymmetric information and valuations.

We may then apply the algorithm of [20] to construct equilibrium distributions F_U^j and $\{F_I^j(t_i)\}_{i=1}^m$ for each battlefield j . The constructed distributions are functions of the known parameters as well as the Lagrange multipliers. If there exists unique multipliers $\lambda_I^*, \lambda_U^* > 0$ such that the Lotto constraints (LC) is met for all types $\{t_i\}_{i=1}^m$ for I and for type t_U for U , then it is clear the n constructed strategy profiles constitute a BNE for $GL(X_I, X_U, V, \mathbf{p})$. Indeed, we have applied the algorithm to obtain a set of distributions, and verified there is a unique $\lambda_I^*, \lambda_U^* > 0$ such that (LC) is satisfied. The details of this calculation are outlined as follows.

Proof of Theorem 2. In $GL(X_I, X_U, V_{\alpha\beta}, \frac{1}{3}\mathbb{1}_3)$, we deduce from Lemma 1 that $\lambda_I^i = \lambda_I$ for all $i = 1, 2, 3$. Hence we need only apply the algorithm of [20] to a single ‘‘column’’ of the game to obtain all marginals of the BNE since the valuations are identical in each battlefield, i.e. the correspondences (19), (20) are the same for all battlefields j , with a permutation of indices $i \in \{1, 2, 3\}$.

Denote the constructed marginal distributions as F_U for player U and $\{F_I^d, F_I^\alpha, F_I^\beta\}$ for player I , where ‘‘d’’ is for the diagonal battlefield value 1. We find that depending on whether $\frac{\lambda_U}{\lambda_I} \geq 1$, $\frac{\lambda_U}{\lambda_I} \in (\frac{1}{2}, 1)$, or $\frac{\lambda_U}{\lambda_I} \in (\frac{1}{2}, \frac{1}{3})$, the algorithm determines three distinct sets of marginal distributions for I and U . In each case, there are unique $\lambda_I, \lambda_U > 0$ such that the constraint (LC) is met. For brevity, we illustrate the calculation for one such case, as the other two follow similar methods. If $\frac{\lambda_U}{\lambda_I} \geq 1$, the algorithm of [20] gives

$$F_U(x) = \frac{3\lambda_I}{2c} x, \quad x \in \left[0, \frac{2c}{3\lambda_I} \right] \quad (21)$$

$$\begin{aligned} F_I^d(x) &= \frac{3\lambda_U}{2c} x + 1 - \frac{\lambda_U}{\lambda_I}, & x \in \left[0, \frac{2c}{3\lambda_I} \right] \\ F_I^\alpha(x) &= 1, & x \geq 0 \\ F_I^\beta(x) &= 1, & x \geq 0 \end{aligned} \quad (22)$$

The constraint (LC) requires that $\mathbb{E}_{F_I^d} + \mathbb{E}_{F_I^\alpha} + \mathbb{E}_{F_I^\beta} = X_I$ and $3\mathbb{E}_{F_U} = X_U$, from which we obtain the unique solutions $\lambda_I = \frac{1}{X_U(1+\alpha+\beta)}$ and $\lambda_U = 3\gamma\lambda_I$. This solution implies the budget ratio satisfies $\gamma \in (0, \frac{1}{3})$. Using these marginals, we

calculate the equilibrium ex-ante payoff (6) as

$$\frac{2}{1+\alpha+\beta} \left[\int_0^\infty F_U dF_I^d + \alpha \int_0^\infty F_U dF_I^\alpha + \beta \int_0^\infty F_U dF_I^\beta \right] - 1. \quad (23)$$

From this, we obtain (9). The other cases $\frac{\lambda_I}{\lambda_U} \in (\frac{1}{2}, 1)$ and $\frac{\lambda_U}{\lambda_I} \in (\frac{1}{2}, \frac{1}{3})$ correspond to the budget ranges $\gamma \in [\frac{1}{3}, \frac{2}{3})$ and $\gamma \in [\frac{2}{3}, 1)$, respectively.

For completeness, we present the marginals for the other two cases. When $\gamma \in [\frac{1}{3}, \frac{2}{3})$, the unique solution of the multipliers are $\lambda_I = \frac{1}{X_U(1+\alpha+\beta)} \left[\frac{1-\alpha}{9\gamma^2} + \alpha \right]$ and $\lambda_U = 3\gamma\lambda_I$. Denote the intervals $I_1 = \left[0, \frac{2c}{3} \left(\frac{\alpha}{\lambda_I} - \frac{\alpha}{\lambda_U} \right) \right]$ and $I_2 = \left[\frac{2c}{3} \left(\frac{\alpha}{\lambda_I} - \frac{\alpha}{\lambda_U} \right), \frac{2c}{3} \left(\frac{\alpha}{\lambda_I} + \frac{1-\alpha}{\lambda_U} \right) \right]$. Then the equilibrium marginals are

$$F_U(x) = \begin{cases} \frac{3\lambda_I x}{2\alpha c} & x \in I_1 \\ \frac{3\lambda_I}{2c} x + (1 - \frac{\lambda_I}{\lambda_U})(1 - \alpha) & x \in I_2 \end{cases} \quad (24)$$

$$\begin{aligned} F_I^d(x) &= \frac{3\lambda_U}{2c} x - \alpha \left(\frac{\lambda_U}{\lambda_I} - 1 \right) & x \in I_2 \\ F_I^\alpha(x) &= \frac{3\lambda_U}{2\alpha c} x + 2 - \frac{\lambda_U}{\lambda_I} & x \in I_1 \\ F_I^\beta(x) &= 1, & x \geq 0 \end{aligned} \quad (25)$$

When $\gamma \in [\frac{2}{3}, 1)$, the unique solution of the multipliers are $\lambda_I = \frac{c}{X_U} \left[\beta + \frac{1}{9\gamma^2} (1 + 3\alpha - 4\beta) \right]$ and $\lambda_U = 3\gamma\lambda_I$. Denote the intervals $I_1 = \left[0, \frac{2c}{3} \left(\frac{\beta}{\lambda_I} - \frac{2\beta}{\lambda_U} \right) \right]$, $I_2 = \left[\frac{2c}{3} \left(\frac{\beta}{\lambda_I} - \frac{2\beta}{\lambda_U} \right), \frac{2c}{3} \left(\frac{\beta}{\lambda_I} + \frac{\alpha - 2\beta}{\lambda_U} \right) \right]$, and $I_3 = \left[\frac{2c}{3} \left(\frac{\beta}{\lambda_I} + \frac{\alpha - 2\beta}{\lambda_U} \right), \frac{2c}{3} \left(\frac{\beta}{\lambda_I} + \frac{\alpha - 2\beta + 1}{\lambda_U} \right) \right]$. Then the equilibrium marginals are calculated to be

$$F_U(x) = \begin{cases} \frac{3\lambda_I}{2\beta c} x & x \in I_1 \\ \frac{3\lambda_I}{2\alpha c} x + \left(1 - 2\frac{\lambda_I}{\lambda_U} \right) \left(1 - \frac{\beta}{\alpha} \right) & x \in I_2 \\ \frac{3\lambda_I}{2c} x + (1 - \beta) + (2\beta - \alpha - 1) \frac{\lambda_I}{\lambda_U} & x \in I_3 \end{cases} \quad (26)$$

$$\begin{aligned} F_I^d(x) &= \frac{3\lambda_U}{2c} x - \beta \left(\frac{\lambda_U}{\lambda_I} - 2 \right) - \alpha & x \in I_3 \\ F_I^\alpha(x) &= \frac{3\lambda_U}{2\alpha c} x - \frac{\beta}{\alpha} \left(\frac{\lambda_U}{\lambda_I} - 2 \right) & x \in I_2 \\ F_I^\beta(x) &= \frac{3\lambda_U}{2\beta c} x + 3 - \frac{\lambda_U}{\lambda_I} & x \in I_1 \end{aligned} \quad (27)$$

VII. CONCLUSION

In this paper, we extended the Colonel Blotto and General Lotto games to a setting where players have asymmetric information about the valuations of the battlefields. We focused on the case when one player is completely informed about the valuations but has fewer resources to allocate, and the other player is uninformed. Our analysis on the two battlefield case in the Colonel Blotto game shows an informed player still cannot defeat its opponent in a mixed-strategy Nash equilibrium. We find a three battlefield scenario presents enough complexity such that the informed player in the General Lotto game can attain the advantage for certain parameters.

A direction of future research involves generalizing the connection of the all-pay auctions with asymmetric information to the General Lotto game. This will allow us to

investigate General Lotto games where the players hold arbitrary information structures.

APPENDIX

Proof of Theorem 1. Let $d := X_U - X_I$ and $q := \lfloor \frac{X_U}{X_U - X_I} \rfloor$ so that $X_U = qd + r$, where $0 \leq r < d$. Denote a delta mass function centered at $y \in \mathbb{R}$ by δ_y . Define $c := \bar{v}/v$. We prove the Theorem by proposing a set of mixed strategy distributions $F_I^* = \{F_I^*(t_1), F_I^*(t_2)\}$, and F_U^* each satisfying (BC), and showing the strategy F_z^* is a best-response to F_{-z}^* , $z = I, U$. For brevity, we prove the case when q is odd, as the even case provides similar mixed strategies and follows similar arguments. Let $e \in (r, d)$, and consider the strategies

$$F_U^* = \frac{1}{s_A} \left(\sum_{k=1}^{\frac{q-1}{2}} c^{\frac{q+1}{2}-k} \delta_{e+(k-1)d} + \delta_{e+\frac{q-1}{2}d} \right. \quad (28)$$

$$\left. + \sum_{k=1}^q c^{k-\frac{q+1}{2}} \delta_{e+(k-1)d} \right)$$

$$F_I^*(t_1) = \frac{1}{s_B} \left(\frac{vc^{\frac{q+1}{2}+1}}{\bar{v}+v} \delta_{\frac{q-1}{2}d} + \sum_{k=\frac{q-1}{2}+1}^{q-1} c^{q-1-k} \delta_{kd} \right) \quad (29)$$

$$F_I^*(t_2) = \frac{1}{s_B} \left(\sum_{k=0}^{\frac{q-1}{2}-1} c^k \delta_{kd} + \frac{vc^{\frac{q-1}{2}}}{\bar{v}+v} \delta_{\frac{q-1}{2}d} \right)$$

where

$$s_A := 1 + 2 \sum_{k=1}^{\frac{q-1}{2}} c^k, \quad s_B := \frac{vc^{\frac{q-1}{2}}}{\bar{v}+v} + \sum_{k=0}^{\frac{q-1}{2}-1} c^k \quad (30)$$

are normalizing factors. Before proceeding, we make a few remarks. These strategies are similar in nature to the strategies provided in Gross & Wagner [8]. There, the authors show that a strategy composed of equally spaced delta functions with geometrically decreasing weights equalizes the payoff of the other player, and vice versa. In a similar fashion, the strategies (28) and (29) equalize the ex-ante payoffs in certain intervals of the players' allocation space. Any allocation in these intervals give a best-response to the other player's equilibrium strategy.

Let $x_U \in [0, X_U]$ be an allocation to battlefield 1, leaving $X_U - x_U$ to battlefield 2. The payoff $u_U(x_U, F_I^*(t_1), \omega_1)$ (2) of any allocation x_U against $F_I^*(t_1)$ in battlefield set 1 is

$$\begin{aligned} & \frac{1}{s_B} \left[\sum_{k=\frac{q-1}{2}+1}^{q-1} c^{q-1-k} (\bar{v} \text{sgn}(x_U - kd) + v \text{sgn}((k+1)d - x_U)) \right] \\ & + \frac{1}{s_B} \left[\frac{vc^{\frac{q-1}{2}}}{\bar{v}+v} (\bar{v} \text{sgn}(x_U - \frac{q-1}{2}d) + v \text{sgn}(\frac{q+1}{2}d - x_U)) \right] \end{aligned} \quad (31)$$

For notational purposes, let $H_U(\bar{v}, v, x_U)$ denote the above quantity. Then against $F_I^*(t_2)$, $u_U(x_U, F_I^*(t_2), \omega_2) = H_U(v, \bar{v}, x_U)$ in battlefield set 2. Given $p = \frac{1}{2}$, the ex-ante utility is $\pi_U(x_U, F_I^*) = \frac{1}{2} u_U(x_U, F_I^*(t_1), \omega_1) + \frac{1}{2} u_U(x_U, F_I^*(t_2), \omega_2)$. After some algebra, we arrive at

$$\pi_U(x_U, F_I^*) = \begin{cases} \frac{\bar{v}}{s_B} & \text{if } x_U \in (0, qd] \\ 0 & \text{if } x_U \in (qd, X_U] \end{cases} \quad (32)$$

Any mixed strategy F_U with support on the interval $(0, qd)$ is a best-response to F_I^* , and hence $F_U^* \in \text{BR}_U(F_I^*)$. Now, any pure allocation $x_I^1 \in [0, X_I]$ of player I against F_U^* in type t_1 gives the ex-interim utility $\pi_I(x_I^1, F_U^*|t_1)$ (4), as

$$\begin{aligned} & \frac{-1}{s_A} \left[\sum_{k=1}^{\frac{q-1}{2}} c^{\frac{q+1}{2}-k} (\bar{v} \text{sgn}(e+(k-1)d-x_I^1) - \underline{v} \text{sgn}(e+(k-2)d-x_I^1)) \right. \\ & \left. + \left(\bar{v} \text{sgn}\left(e+\frac{q-1}{2}d-x_I^1\right) - \underline{v} \text{sgn}\left(e+\left(\frac{q-1}{2}-1\right)d-x_I^1\right) \right) \right] \\ & + \sum_{k=\frac{q+1}{2}+1}^q c^{k-\frac{q+1}{2}} (\bar{v} \text{sgn}(e+(k-1)d-x_I^1) - \underline{v} \text{sgn}(e+(k-2)d-x_I^1)) \end{aligned} \quad (33)$$

For notational purposes, let $H_I(\bar{v}, \underline{v}, x_I^1)$ denote the above quantity. One can list all possible values this takes as a function of x_I^1 (not shown due to space constraint). This is increasing in x_I^1 , and attains the maximum value when $x_I^1 \in (e + (\frac{q-1}{2} - 1)d, X_I)$, giving $\max_{x_I^1} \pi_I(x_I^1, F_U^*|t_1) = -\frac{\underline{v}}{s_A}(1+c)$.

Any pure allocation $x_I^2 \in [0, X_I]$ of player I against F_U^* in type t_2 gives ex-interim utility $\pi_I(x_I^2, F_U^*|t_2) = H_I(\underline{v}, \bar{v}, x_I^2)$. Through a similar analysis, we find the maximal value $-\frac{\underline{v}}{s_A}(1+c)$ is attained for $x_I^2 \in (0, e + \frac{q-1}{2}d)$. Hence, any mixed strategy $F_I(t_1)$ with support on $(e + (\frac{q-1}{2} - 1)d, X_I)$ and $F_I(t_2)$ with support on $(0, e + \frac{q-1}{2}d)$ is a best-response to F_U^* . Consequently, $F_I^* \in \text{BR}_I(F_U^*)$. The quantities $\frac{\underline{v}}{s_B}$ and $\frac{\underline{v}}{s_A}(1+c)$ coincide, giving the value of the game. ■

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