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Interactions between computability theory and set theory

by

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Committee in charge:

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Professor Leo Harrington
Professor Wesley Holliday

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Interactions between computability theory and set theory

To Emma

Contents

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Chapter 1

Introduction

This thesis explores connections between computability theory and set theory. The bulk of this thesis focuses on extending ideas and techniques from computability theory to higher set-theoretic levels — *higher reverse mathematics* (chapters 2 and 3) and *uncountable computable structure theory* (chapters 4, 5, and 6). We also look at applications of set theory to computability theory, either by directly answering computability-theoretic questions via set-theoretic considerations (chapter 8) or by bringing set-theoretic constructions into contact with classical computability theory (chapter 7). Finally, we also look at results which stretch from the computability-theoretic to the set-theoretic — specifically, determinacy principles (chapter 2).

Below we give a summary of the results in this thesis. Each individual chapter is self-contained, but background knowledge in set theory and computability theory is helpful; we recommend [35] and [13] respectively, whose notation we follow. Small amounts of proof theory (conservative extensions) and basic model theory (o-minimality) appear in chapters 2 and 4, but are covered as needed, and we assume no background outside of a standard introduction to mathematical logic such as [50].

1.1 Higher reverse mathematics

In the first part of this thesis, we look at higher reverse mathematics — roughly speaking, the study of the effective content of theorems of mathematics which cannot easily be expressed in the language of second-order arithmetic. Chapter 1 — which consists of work published as "Transfinite recursion in higher reverse mathematics" [68] — gives an introduction to the subject, introduces a base theory RCA_0^3 for third-order reverse mathematics, and studies analogues of the system ATR_0 at higher types. We show, for example, that the comparability of well-orderings of sets of reals is a very weak principle, relatively speaking, and that Σ_1^2-separation for functionals implies clopen determinacy for reals.

The main result of chapter 2 is the separation of two determinacy principles. In 1977, John Steel [73] showed that clopen and open determinacy are equivalent over RCA_0, despite

their different computability-theoretic properties (e.g. clopen games have relatively hyper-arithmetic winning strategies); the culprit is the high complexity of the predicate "clopen," which is Π_1^1 complete. We show that once we pass to a context where clopen games are relatively easier to identify, the principles separate: we define clopen and open determinacy principles for games played on $\mathbb{R}$, and construct a model $\mathcal{M}$ separating them.

Theorem 1.1.1. *Over RCA_0^3, clopen determinacy for reals is strictly weaker than open determinacy for reals.*

The construction of $\mathcal{M}$ uses a variation of a notion of forcing with tagged trees introduced by Steel [74]. Leaving aside the technical details, we let G be a certain generic tree, labelled with ordinals. Elements of $\mathcal{M}$ are given by names which depend on G in a "bounded" way: for a functional to be in $\mathcal{M}$, we demand that it be the evaluation of some name which respects one of a prescribed family of equivalence relations on the forcing, $\mathbb{P}$, used to produce G. A name ν for a functional respects an equivalence relation $\approx$ if whenever $p \approx q$, $r \in \mathbb{R}$, and $k \in \omega$, we have

$$p \Vdash \nu(r) = k \iff q \Vdash \nu(r) = k.$$

In classical Steel forcing, we instead look at functions which are hyperarithmetic relative to G; this has roughly the same effect, but the higher-type analogue of "hyperarithmetic" is not well-behaved, hence our more abstract approach. Of crucial importance is the countable closure of $\mathbb{P}$, which is used to control the second-order part of the model and in the verification that clopen determinacy holds in $\mathcal{M}$, to show that no clopen games of high rank enter $\mathcal{M}$; this has no analogue in classical Steel-forcing arguments.

From the proof of Borel determinacy (see [52]), we should expect a connection between Theorem 1.1.1 and determinacy principles for higher Borel levels of games on ω. Indeed, Sherwood Hachtman [25] answered a question posed in an early draft of [68] by constructing a *canonical* separating model. Let θ be the least ordinal such that L_θ satisfies "$\mathcal{P}(\omega)$ exists and for every well-founded tree T of height ω, there is a map $\rho\colon T \to ON$ with $\rho(x) < \rho(y)$ whenever $x \supsetneq y$;" Hachtman showed (in the course of his broader analysis of θ) that the structure $(\omega, \mathbb{R}^{L_\theta}, (\omega^\mathbb{R})^{L_\theta})$ also satisfies clopen determinacy for reals but not open determinacy for reals.

In chapter e, we present some further results in higher reverse mathematics of a more technical nature. First, we show that RCA_0^3 is a conservative subtheory of Kohlenbach's RCA_0^ω. The main technical obstacle in this proof is that the desired term model has to be defined in a slightly subtle way; however, no major difficulties emerge. We then move on to choice principles in the context of higher reverse mathematics. Looking back at the results of chapter 1, two applications of the axiom of choice were relevant: that the reals are well-orderable (to get the Kleene-Brouwer order of a tree $\subseteq \mathbb{R}^{<\omega}$, and that real-indexed families of nonempty sets of reals have choice functions (to pass from a quasistrategy to a strategy).

We show that a well-ordering $\prec$ of the reals does not imply that real-indexed families of sets of reals have choice functions, over RCA_0^3; this requires a somewhat technical construction, since we have to avoid Π_1^1-comprehension for functionals which would allow us to

produce selection functions by picking the $\prec$-least real in each set (in particular, this makes this essentially a reverse mathematical result, as opposed to a statement about strength over ZF). Our construction is inspired by *truth-table reducibility*, and bears a certain thematic resemblance to the construction of the model $\mathcal{M}$ above: again we take a forcing extension of the universe, and let our model consist of the functionals which have names which only depend on "bounded information" about the generic. The notion of bounded information we use, however, is radically different: in particular, it is not framed (and does not seem frameable) in terms of equivalence relations on the forcing, but rather in terms of how many queries a name is allowed to make to the generic object before its value on a real is computed.

1.2 Computability in generic extensions

In the second part of this thesis, consisting of chapters 4 through 6, we switch from examining *theorems* to examining *structures*. We look at the behavior of countable structures in generic extensions — first the existence (or not) of copies of them in the ground model, and later their computability-theoretic properties.

We begin in chapter 4 by looking at structures which exist and are countable in some generic extension of the universe. (This is joint with Julia Knight and Antonio Montalban [41].)

Definition 1 ([41]). *A generically presentable structure is a pair $(\nu, \mathbb{P})$ where $\mathbb{P}$ is a forcing notion and ν is a $\mathbb{P}$-name for a structure with domain ω, such that ν names the same structure in every extension by $\mathbb{P}$; formally, such that*

$$\Vdash_{\mathbb{P}^2} \nu[G_0] \cong \nu[G_1].$$

We are interested in when generically presentable structures have copies; that is, when there is some $\mathcal{A} \in V$ such that $\Vdash_{\mathbb{P}} \mathcal{A} \cong \nu[G]$. We show that the effects of $\mathbb{P}$ on cardinals is crucial:

Theorem 1.2.1 ([41]). *Suppose $(\nu, \mathbb{P})$ is a generically presentable structure and forcing with $\mathbb{P}$ does not make ω_2^V countable. Then:*

- *$(\nu, \mathbb{P})$ has a copy $\mathcal{A}$ in V.*

- *If additionally forcing with $\mathbb{P}$ does not make ω_1^V countable, then $\mathcal{A}$ may be taken to be countable (in V).*

Theorem 1.2.2 ([41]). *Suppose $\mathbb{P}$ makes ω_2^V countable. Then there is a generically presentable structure $(\nu, \mathbb{P})$ which has no copy in V.*

Generically presentable structures were independently and shortly later introduced by Itay Kaplan and Saharon Shelah [36], who also proved the above two theorems.

We use Theorem 1.2.1 to give a new proof of a theorem of Harrington (unpublished) on the Scott ranks of counterexamples to Vaught's conjecture:

Corollary 1.2.3 (Harrington). *Suppose T is a counterexample to Vaught's conjecture. Then T has models of size $\aleph_1$ with Scott rank arbitrarily high below ω_2.*

Harrington's theorem was independently re-proved by John Baldwin, Sy-David Friedman, Michael Koerwien, and Chris Laskowski [6] at around the same time, and also by Paul Larson [46] about a year earlier; their proofs are similar thematically, yet appear sufficiently different to be called distinct proofs.

We also show that rigid generically presentable structures have copies in V:

Theorem 1.2.4 ([41]). *Let $(\nu, \mathbb{P})$ be a generically presentable structure. If $\Vdash_{\mathbb{P}}$ "$\nu[G]$ is rigid", then $(\nu, \mathbb{P})$ has a copy in V.*

This is less directly related with the rest of this thesis, however.

In the course of examining generically presentable structures, we introduce *generic Muchnik reducibility* as a method for comparing the complexity of uncountable structures without leaving the context of Turing reducibility (i.e. without using α-recursion or similar generalizations). Classical Muchnik reducibility is the natural way of comparing countable structures: $\mathcal{A} \leq_w \mathcal{B}$ if every copy of $\mathcal{B}$ with domain ω computes a copy of $\mathcal{A}$ with domain ω.

Definition 2 (S.). *Suppose $\mathcal{A}$ and $\mathcal{B}$ are structures of arbitrary cardinality. We say $\mathcal{A}$ is generically Muchnik reducible to $\mathcal{B}$ — and write $\mathcal{A} \leq_w^* \mathcal{B}$ — if $V[G] \models \mathcal{A} \leq_w \mathcal{B}$ for some generic extension $V[G]$ of the universe in which $\mathcal{A}$ and $\mathcal{B}$ become countable.*

By Shoenfield absoluteness, this is well-behaved; in particular, we may replace "some generic extension" with "every generic extension" without changing the definition, and $\leq_w^*$ restricted to countable structures is just $\leq_w$. We prove some basic results about generic Muchnik reducibility, including:

Proposition 1.2.5 ([41]). *If $\mathcal{A}$ is a countable structure, then $\mathcal{A} \leq_w^* (\omega_1; <)$ if and only if there is some countable ordinal α such that $\mathcal{A} \leq_w^* (\alpha; <)$.*

Proposition 1.2.6 ([41]). *$(\omega_1; <) <_w^* (\mathbb{R}; +, \times)$, strictly.*

Connecting with generic presentability, we show that the un-copied structure of Theorem 1.2.2 can be taken to be relatively simple:

Proposition 1.2.7 ([41]). *If forcing with $\mathbb{P}$ makes ω_2^V countable, then there is a generically presentable $(\nu, \mathbb{P})$ with no copy in V such that $\Vdash_{\mathbb{P}} \nu[G] \leq_w^* (\omega_2^V; <)$.*

Following the results of chapter r, some interest emerged in generic Muchnik reducibility. The next work on the subject was by Greg Igusa and Julia Knight [32] and independently by Rod Downey, Noam Greenberg, and Joe Miller [12]; they showed (via different proofs) that Cantor space is strictly weaker, in terms of generic Muchnik reducibility, than the field of real numbers, contrary to the expectations of the author. Chapter 5, which is joint work with

Greg Igusa and Julia Knight [33], follows up on this work: we study reducts and expansions of the field of real numbers $\mathcal{R}$. We prove that even very weak reducts of $\mathcal{R}$, such as $\mathbb{R}$ equipped with the order relation and with constants naming each rational or with predicates for the rational half-open intervals, are already as computationally powerful as $\mathcal{R}$; in the other direction, expanding $\mathcal{R}$ by any analytic function adds no computational power:

Theorem 1.2.8. *Let* $f \colon \mathbb{R}^n \to \mathbb{R}$ *be analytic. Then* $(\mathbb{R}; +, \times) \equiv_w^* (\mathbb{R}; +, \times, f)$.

Shortly after [33] was written, Theorem 1.2.8 was greatly improved by Uri Andrews, Julia Knight, Rutger Kuyper, Steffen Lempp, Joe Miller, and Maria Soskova [2]; they showed that in fact any expansion of $\mathcal{R}$ by continuous functions is generically Muchnik equivalent to $\mathcal{R}$ itself, sidestepping the use of o-minimality. (However, o-minimality comes back into the picture when we pay attention to what parameters are needed for the computation.)

Chapter 6 (which is joint work in progress with Greg Igusa [34]) is an appendix to the work already mentioned above. In it, we continue the study of structures computing every real. We show that by contrast adding continuous functions to Cantor space can result in a complexity jump up to $\mathcal{R}$, which is strictly generically Muchnik above Cantor space; this was already known by Andrews, Knight, Kuyper, Lempp, Miller, and Soskova, but we provide further examples and some results on when continuous functions have this behavior. In particular, we show that functions which individually are "tame" can together result in a jump in complexity. We also look at ultrafilters as (uncountable) structures, and show that every nonprincipal ultrafilter is generically Muchnik above $\mathcal{R}$, and that generic Muchnik reducibility restricted to ultrafilters refines the Rudin-Keisler ordering. Finally, we exhibit an example of a minimally complicated structure computing every real, and show that there is a structure computable from Baire space which does not compute Cantor space (so that there is not an "hourglass" phenomenon).

1.3 Further results

The final part of this thesis — chapters 7 and 8 — consists of more miscellaneous and partial results. In chapter 7, which is joint work with Uri Andrews, Mingzhong Cai, and David Diamondstone [1], we look at operations on Turing ideals arising from ultrafilters on ω. Given a Turing ideal $\mathcal{I}$ and a non-principal ultrafilter U, let $U(\mathcal{I})$ be the set of all U-limits of sequences of reals in $\mathcal{I}$. We give a complete characterization of the possible values of $U(\mathcal{I})$ for a fixed countable Turing ideal $\mathcal{I}$:

Theorem 1.3.1 ([1])**.** *Let* $\mathcal{I}$ *be a countable Turing ideal and* $\mathcal{J} \subseteq \mathcal{P}(\omega)$. *The following are equivalent:*

- *There is a nonprincipal ultrafilter* U *such that* $U(\mathcal{I}) = \mathcal{J}$.

- $\mathcal{J}$ *is a countable Scott set containing* X' *for every* $X \in \mathcal{I}$.

This yields a new purely combinatorial proof of the classical result in reverse mathematics that WKL_0 is strictly weaker than ACA_0; in particular, this argument avoids the Low Basis Theorem.

Consistently relative to large cardinals, I show that Theorem 1.3.1 can fail for uncountable Turing ideals (the statement of this result appears without proof in [1]). We also look at a natural notion of lowness associated to this class of operations: a set X is *ultrafilter-low* if for some nonprincipal ultrafilter U, $U(REC) = U(deg(X))$. We show that every computably traceable X, and every X which is bounded by a 2-generic, is U-low. A large number of open questions remain, including:

- Is there an ultrafilter U such that $U(\mathcal{I})$ is arithmetically closed for every Turing ideal $\mathcal{I}$? We show that there is no U such that $U(\mathcal{I}) = ARITH(\mathcal{I})$ for all $\mathcal{I}$, but this does not answer the question.

- When is a real ultrafilter-low? In particular, are there Δ_2^0 ultrafilter-low reals? And, are sufficiently *random* reals ultrafilter-low?

- Is there a real X such that $U(deg(X)) = U(REC)$ for *every* nonprincipal ultrafilter U? Such a real would have to be Δ_2^0 by Theorem 1.3.1 above.

Finally, in chapter 8 we return to computable structure theory, specifically the computability-theoretic aspects of ordinals. On a broad scale, ordinals are extremely well-understood from the computability-theoretic point of view; for instance, (generic) Muchnik reducibility of ordinals is completely understood. However, once we look at listing classes of ordinals, or at computing ordinals in a "uniform" manner, things become considerably more complicated.

We begin with a result motivated by Montalban's [56], which introduced a class of *copy-diagonalize games* associated to classes of structures. We show that there is a reasonably-definable class of ordinals whose associated copy-diagonalize game is undetermined. We then use a similar argument to show that, in the presence of sufficient determinacy, there is a countable set B of countable ordinals which is "difficult to list," in the following sense: that for any noncomputable ordinal α, there is a real r such that r does not compute a copy of α, but r join any enumeration of B does.

We then turn to a finer view of the relative complexity of ordinals than is provided by Muchnik reductions. We look at *uniform computations*, specifically, *Medvedev reducibility*: if $\mathcal{A}$ and $\mathcal{B}$ are countable structures, $\mathcal{A}$ is *Medvedev reducible to* $\mathcal{B}$ — and write $\mathcal{A} \leq_s \mathcal{B}$ — if there is some e such that, whenever B is a copy of $\mathcal{B}$ with domain ω, Φ_e^B is a copy of $\mathcal{A}$ with domain ω. The question we attack is:

Question 1. *What does the Medvedev degree structure of countable ordinals look like?*

Note that the *Muchnik* degree structure of ordinals is completely understood: $\alpha \leq_w \beta$ if and only if $\alpha < \beta^+$, where β^+ is the least admissible ordinal greater than β (see e.g. [66]).

This was a question originally studied by Joel David Hamkins and Zhenhao Li [26]. They asked, in particular, whether there are *Medvedev-incomparable* ordinals. We show that the answer is yes, in a strong sense:

Theorem 1.3.2. *There is a club of countable ordinals which are pairwise Medvedev incomparable.*

The proof uses set-theoretic arguments. In the course of the proof, we are led to define the *Medvedev ordinal variance* of a structure: $Var(\mathcal{A})$ is the transitive collapse of (e.g., the order type of the set of) the ordinals Medvedev reducible to $\mathcal{A}$. We in fact show:

Proposition 1.3.3. *There is a countable θ_{club} such that club-many ordinals α satisfy*

$$Var(\alpha) = \theta_{club}.$$

By extending Medvedev reducibility to uncountable structures a la generic Muchnik reducibility, we show that perhaps surprisingly, the Medvedev ordinal variance is bounded strictly below ω:

Proposition 1.3.4. *There is a countable ordinal θ_{sup} such that $Var(\alpha) \leq \theta_{sup}$ for all ordinals α.*

This result assumes set-theoretic hypotheses beyond ZFC, for generic absoluteness. We leave open the relationship between these two bounds:

Question 2. *Is $\theta_{sup} = \theta_{club}$?*

We also appropriately extend these results to counterexamples to Vaught's conjecture, following a line of work by Montalban (see e.g. [54]). These are of course very coarse results, and their proofs apply to any natural numbers-indexed relation which is sufficiently definable; in particular, assuming Projective Determinacy, this includes all projectively definable relations.

Finally, we show that in general, Medvedev reductions do not reach as high as Muchnik reductions:

Theorem 1.3.5. *For all but countably many countable ordinals α, there is an ordinal β with $\alpha < \beta < \alpha + \alpha$ such that $\beta \not\leq_s \alpha$.*

We are unable to answer, however, whether this phenomenon already happens at ω_1^{CK}:

Question 3. *Does $\omega_1^{CK} \geq_s \alpha$ for all $\alpha < \omega_2^{CK}$?*

This section is work in progress, and we hope that in the near future we will be able to produce a complete picture of the theory of Medvedev degrees of ordinals.

We conclude with a result on the Muchnik complexity of countable sets of ordinals, under large cardinal assumptions; while this is not directly related to the topic of Medvedev complexity of ordinals, it shares a thematic tone, and may be useful down the road as a technical tool.

Chapter 2

Higher reverse mathematics, 1/2

The work presented in this chapter appeared in [68].

2.1 Introduction

The question

> "What role do incomputable sets play in mathematics?"

has been a central theme in modern logic for almost as long as modern logic has existed. Six years before Alan Turing formalized the notion of computability, van der Waerden [77] showed that the splitting set of a field is not uniformly computable from the field; put another way, van der Waerden demonstrated the *necessity* of certain incomputable sets for Galois theory. Other results, especially Turing's solution to the Entscheidungsproblem and the solution by Davis, Matiyasevitch, Putnam, and Robinson of Hilbert's Tenth Problem, established the incomputability of particular sets of natural numbers of interest. In 1975, Friedman [18] initiated the axiomatic study of this question, dubbed "Reverse Mathematics."

Reverse mathematics requires the choice of both a common language in which to express all analyzed theorems, and a base theory in that language over which all equivalences and non-implications are to be proved. The natural choice of language is that of second-order arithmetic, since it is in this language that computability-theoretic principles are most naturally expressed. The base theory is taken to be RCA_0, a precise definition of which is contained in [71]; as a base theory, RCA_0 is justified by the fact that it captures exactly "computable" mathematics, in the sense that the ω-models of RCA_0 are precisely the Turing ideals. One notable feature of reverse mathematics is the existence of the "Big Five," five subtheories of second-order arithmetic — RCA_0, WKL_0, ACA_0, ATR_0, and $\Pi^1_1\text{-}\mathsf{CA}_0$ — each of which is "robust," in the sense that the same theory results when small changes are made to its exact statement (or to the precise coding mechanisms used), and which correspond to the exact strength, over RCA_0, of the vast majority of theorems studied by reverse mathematics.

However, there is a significant amount of classical mathematics, including parts of measure theory and most of general topology, which resists any natural coding into the language of second-order arithmetic. This was already recognized by Friedman in [19]. Somewhat later, Victor Harnik [27] developed a higher-order version of RCA_0 in order to study the axiomatic strength of various results from stability theory. At the time, however, the higher-order program failed to draw mathematical attention comparable to that of second-order reverse mathematics.

Recently, however, there has been a return to this subject. The framework of finite types — in which objects of arbitrary finite order, such as sets of sets of reals, are treated directly — has begun to emerge as a natural setting for a higher reverse mathematics, following Ulrich Kohlenbach's paper on the subject [43].[1] Kohlenbach expands the language of second-order arithmetic to all finite types, and extends the system RCA_0 to include a version of primitive recursion for arbitrary finite-type functionals. The resulting system, RCA_0^ω, is a proof-theoretically natural conservative extension of RCA_0. (From the point of view of computability theory, however, the choice of base theory may not be so clear; see the discussion at the end of this paper.)

Work on reverse mathematics in finite types has so far proceeded along one or the other of two general avenues: the analysis of classical theorems about objects not naturally codeable within second-order arithmetic, such as ultrafilters or general topological spaces ([31], [44], [76]), or the analysis of higher-type "uniformizations" of classical theorems of second-order arithmetic ([43], [67]). The present paper instead looks at the higher-type analogues of theorems studied by classical reverse mathematics, focusing in particular on what old patterns hold or fail and what new patterns emerge.[2]

One natural question along these lines is the following: to what extent do the robust subsystems of second-order arithmetic have robust analogues at higher types? It is this question which the present paper addresses, focusing on the system ATR_0. In the classical case, much of the robustness of ATR_0 comes from the fact that being a well-ordering is Π_1^1-complete. For instance, this is what drives the method of "pseudohierarchies" by which ill-founded linear orders which appear well-founded, such as those constructed in [28], are used to prove a large number of equivalences at the level of ATR_0; see [71]. Moving up a type, however, changes the situation completely: since we can code an infinite sequence of reals by a single real, the class of well-orderings of subsets of $\mathbb{R}$ is again Π_1^1, instead of being Π_1^2 complete. This causes the entire method of pseudohierarchies to break down, and raises doubt that the higher-type analogues of various theorems classically equivalent to ATR_0 are still equivalent.

We begin by presenting in section 2 a base theory, RCA_0^3, which is essentially equivalent to, yet simpler to use than, RCA_0^ω. We then study the complexity over RCA_0^3 of several higher-type analogues of several principles classically equivalent to ATR_0: comparability

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[1]Although it is by no means the only one — see [70] for an approach via α-recursion theory instead, and also [23] for a closely-related α-recursive structure theory. Shore also suggests other approaches which could be interesting, such as via E-recursion or the computation theory of Blum-Shub-Smale.

[2]This is also the approach taken in [70], there with respect to α-recursion rather than finite types.

of well-orderings, clopen determinacy, open determinacy, Σ_1^1 separation, and definition by recursion along a well-founded tree. In section 2, we prove some basic implications and nonimplications. At the bottom of this hierarchy lies the principle asserting the comparability of well-orderings of sets of reals, which we show is remarkably weak at higher types relative to the other principles; above clopen determinacy, a higher-type version of the separation principle Σ_1^1-Sep. We also examine the role of the axiom of choice in higher determinacy principles.

The main result of this paper, to which section 3 is devoted, concerns the two determinacy principles. In classical reverse mathematics, clopen determinacy fails in HYP, the model consisting of the hyperarithmetic sets, despite hyperarithmetic clopen games having hyperarithmetic winning strategies, since the method of pseudohierarchies allows us to construct games which are "hyperarithmetically clopen" but are undetermined in HYP. This method, as noted above, is no longer valid at higher types, while the complexities of winning strategies for clopen games on reals can still be bounded by a transfinite iteration of an appropriate jump-like operator. This suggests that at higher types, open determinacy becomes strictly stronger than clopen determinacy; using an uncountable version of Steel's tagged tree forcing, we show that this is indeed the case.

Background and Conventions

We refer the reader to [45] for the relevant background in set theory; for descriptive set theory, [59] and [38] are the standard sources. For background on reverse mathematics, see [71]. Finally, for background in finite types, as well as the various computability-theoretic concerns which arise in higher-type settings, see [48].

There are several notational conventions we adopt for simplicity. Throughout, we use $\mathbb{R}$ to refer to the Baire space, the set of functions from ω to ω; this is because, during the main result, ordinals will be used as tags, and for this reason a symbol other than "ω^ω" is preferable. If σ is a nonempty finite string, we write σ^- for the immediate $\preccurlyeq$-predecessor of σ, and if f is an infinite string we write f^- for the string $n \mapsto f(n+1)$.

When writing formulas in many-sorted logic, we use the convention that the first time a variable occurs it is decorated with the appropriate sort symbol; for example,

$$\exists x^1 \forall y^0 (xy = 2)$$

is the statement "There is a function from naturals to naturals which is identically 2." (See section 2.1 for a discussion of types.) If φ is a sentence, then $[\![\varphi]\!]$ is the truth value of φ: 1 if φ holds, and 0 if φ does not. We will denote the constant function $n \mapsto i$ by $\underline{i}$.

If $\Sigma, \Pi \colon A^{<\omega} \to A$, we write $\Sigma \otimes \Pi$ for the element of A^ω built by alternately applying Σ and Π:

$$\Sigma \otimes \Pi = \langle \Sigma(\langle\rangle), \Pi(\langle \Sigma(\langle\rangle)\rangle), \Sigma(\langle \Sigma(\langle\rangle), \Pi(\langle \Sigma(\langle\rangle)\rangle)\rangle), ... \rangle.$$

We write $(\Sigma \otimes \Pi)_k$ for the length-k initial segment of $\Sigma \otimes \Pi$. A game is said to be a *win for player X* if that player has a winning strategy. A *quasistrategy* for a game played on a set

A (so, viewed as a subtree of $A^{<\omega}$) is a multi-valued map from $A^{<\omega}$ to A; a quasistrategy is said to be *winning* if each element of A^ω which is compatible with the quasistrategy is a win for the corresponding player.

Finally, our main theorem 2.3.2 rely heavily on the method of set-theoretic forcing. For completeness, we present here a brief summary of this method; for details and proofs, see chapter VII of [45].

Given a model V of ZFC and a poset $\mathbb{P} \in V$, a *filter* is a subset F of $\mathbb{P}$ which is closed upwards, and such that any two elements of F have a common lower bound in F; a set $D \subseteq \mathbb{P}$ is *dense* if every element of $\mathbb{P}$ has a lower bound in D. The $\mathbb{P}$-*names* are defined inductively to be the sets $\{(p_i, \gamma_i) : i \in I\}$ of pairs with first coordinate an element of the partial order $\mathbb{P}$, and second coordinate a $\mathbb{P}$-name. If G is a filter meeting every dense subset of $\mathbb{P}$ which is in V — that is, G is $\mathbb{P}$-*generic over* V — and γ is a $\mathbb{P}$-name, we let $\gamma[G] = \{\theta[G] : \exists p \in G((p, \theta) \in \gamma)\}$ (this is of course a recursive definition). Crucially, the definition of $\gamma[G]$ is made inside V, although G will itself will never be in V.

We then define the *generic extension of V by G* to be

$$V[G] = \{\gamma[G] : \gamma \text{ is a } \mathbb{P}\text{-name in } V\}.$$

If $V[G] \models \varphi$ whenever $p \in G$, we write $p \Vdash \varphi$; the relation $\Vdash$ is the *forcing relation* given by $\mathbb{P}$. The essential properties of set-theoretic forcing are that the generic extension $V[G]$ is a model of ZFC; that the forcing relation is definable in the ground model; and that any statement true in the generic extension is forced by some condition in the generic filter. These are Theorems VII.4.2, VII.3.6(1), and VII.3.6(2) of [45], respectively.

Additionally, the forcing used in the proof of 2.3.2 will be *countably closed*:

Definition 3. $\mathbb{P}$ *is* countably closed *if any chain of countably many conditions* $... \leq p_2 \leq p_1 \leq p_0$ *has a common strengthening* $p \leq p_0, p_1, p_2, ...$

Countable closure yields a strong restriction on how a forcing notion can alter the set-theoretic universe, which will be crucial in 2.3.2:

Fact 2.1.1. *If* $\mathbb{P}$ *is countably closed and* $X \in V$, *then forcing with* $\mathbb{P}$ *adds no new countable subsets of* X. *In particular, forcing with a countably closed* $\mathbb{P}$ *adds no new reals.*

2.2   Reverse mathematics beyond type 1

We begin this section by developing a framework for reverse mathematics in higher types; we then define the various higher-type versions of ATR$_0$ we will consider in this paper, and prove some basic separations and equivalences.

The base theory

We begin by making precise the notion of a finite type.[3]

Definition 4. *The* finite types *are defined as follows:*

- *0 is a finite type;*

- *if σ, τ are finite types, then so is $\sigma \to \tau$; and*

- *only something required to be a finite type by the above rules is a finite type.*

We denote the set of all finite types by FT.

    The intended interpretation of finite types is as a hierarchy of functionals, with type 0 representing the "atomic" objects — here, natural numbers, or more generally elements of some first-order model of an appropriate theory of arithmetic — and type $\sigma \to \tau$ representing the set of maps from the set of objects of type σ to the set of objects of type τ.

    Within the finite types is the special subclass ST of standard finite types, *defined inductively as follows: 0 is a standard type, and if σ is a standard type, then so is $\sigma \to 0$. The standard types are for simplicity identified with natural numbers: $0 \to 0$ is denoted by "1," $(0 \to 0) \to 0$ by "2," etc.*

The appeal of the finite-type framework to reverse mathematics is extremely compelling: the use of finite types lets us talk directly about objects that previously required extensive coding to treat in reverse mathematics, or could not be treated at all. For example, a topological space with cardinality $\leq \beth_i$ (where $\beth_0 = \aleph_0$ and $\beth_{i+1} = 2^{\beth_i}$) can be directly represented as a pair of functionals (F^i, G^{i+1}) corresponding to the characteristic functions of the underlying set and the collection of open subsets. Usually, this representation is even natural. In [43], Kohlenbach developed a base theory for reverse mathematics in all the finite types at once, RCA_0^ω.

However, working with all finite types at once is cumbersome. First, morally speaking, all finite-type functionals are equivalent to functionals of finite standard type via appropriate pairing functions; second, arbitrarily high types are rarely directly relevant. For that reason, we will use a base theory RCA_0^3, defined below, which only treats functionals of types 0, 1, and 2. In a subsequent paper, we will show that our theory is essentially equivalent to Kohlenbach's; specifically, RCA_0^ω is a conservative extension of RCA_0^3.

Definition 5. L^3 *is the many-sorted first order language, consisting of the following:*

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[3]The one oddity of working with types is that the natural formalization is via many-sorted first-order logic, as opposed to ordinary first-order logic. In many-sorted logic, each element of the model and each variable symbol is labelled by one of a fixed collection of sorts; similarly, function, constant, and relation symbols in the signature must be appropriately labelled with sorts. When there are infinitely many sorts — as is the case with Kohlenbach's RCA_0^ω, but not our RCA_0^3 — the resulting logic is subtlely different from single-sorted first-order logic; however, these differences shall not be relevant here. For a careful introduction to many-sorted logic, see Chapter VI of [51].

- *Sorts s_0, s_1, s_2, with corresponding equality predicates $=_0, =_1, =_2$. We will identify sort s_i with type i; recall that the objects of type 0, 1, and 2 are intended to be natural numbers, reals, and maps from reals to naturals, respectively.*

- *On the sort s_0, the usual signature of arithmetic: two binary functions*

$$+, \times \colon s_0 \times s_0 \to s_0,$$

  a binary relation

$$< \subseteq s_0 \times s_0,$$

  and two constants

$$0, 1 \in s_0.$$

- *Application operators $\cdot_0, \cdot_1$ with*

$$\cdot_0 \colon s_1 \times s_0 \to s_0, \quad \cdot_1 \colon s_2 \times s_1 \to s_0.$$

  These operators will generally be omitted; e.g., Fx or $F(x)$ instead of $F \cdot_1 x$ or $\cdot_1(F, x)$.

- *A binary operation*

$$* \colon s_2 \times s_1 \to s_1$$

  and a binary operation

$$\frown \colon s_0 \times s_1 \to s_1.$$

The additional operations $*$ and $\frown$ allow coding which in Kohlenbach's setting is handled through functionals of non-standard type. Axioms which completely determine $*$ and $\frown$ are given in Definition 6, below. We will abuse notation slightly and use $\frown$ to denote both the concatenation of strings, and the specific L^3-symbol, as no confusion will arise. Throughout this paper, "L^3-term" will mean "L^3-term with parameters."

Finally, the syntactic classes Σ_i^0 and Π_i^0 are defined for L^3 as follows:

- *A formula φ is in Σ_0^0 if and only if it has only bounded quantifiers over type 0 objects and no occurrences of $=_1$ or $=_2$. (Note that arbitrary parameters, however, are allowed.)*

- *A formula φ is in Π_{i+1}^0 if*

$$\varphi \equiv \forall x^0 \theta(x),$$

  where $\theta \in \Sigma_i^0$.

- *A formula φ is in Σ_{i+1}^0 if*

$$\varphi \equiv \exists x^0 \theta(x),$$

  where $\theta \in \Pi_i^0$.

The higher syntactic classes Σ_i^1, Σ_j^2, etc. are defined in the analogous way, with lower-type quantifiers being "for free" as usual.

The base theory for third-order reverse mathematics which we will use in this paper, RCA_0^3, is then defined as follows:

Definition 6. *RCA_0^3 is the L^3-theory consisting of the following axioms:*

1. *Σ_1^0-induction and the ordered semiring axioms, P^-, for the type 0 objects.*

2. *Extensionality axioms for the type 1 and 2 objects:*

$$\forall F^1, G^1(\forall x^0(Fx = Gx) \iff F =_1 G) \quad and \quad \forall F^2, G^2(\forall x^1(Fx = Gx) \iff F =_2 G)$$

3. *The Δ_1^0 comprehension[4] schemes for type 1 and 2 objects:*

$$\{\forall x^0 \exists! y^0 \varphi(x, y) \to \exists f^1 \forall x^0(\varphi(x, f(x))): \varphi \in \Sigma_1^0\}$$

   and

$$\{\forall x^1 \exists! y^0 \varphi(x, y) \to \exists F^2 \forall x^1(\varphi(x, F(x))): \varphi \in \Sigma_1^0\}.$$

   (The notation "$\exists!$" is shorthand for "there exists exactly one.") Recall that Σ_1^0 formulas may have arbitrary parameters.

4. *Finally, the following axioms defining $\frown$ and $*$:*

$$\forall k^0, r^1, n^0[(k \frown r)(n + 1) = r(n) \wedge (k \frown r)(0) = k],$$

   and

$$\forall F^2, r^1, k^0[(F * r)(k) = F(k \frown r)].$$

Before continuing further, it is worth explaining the definitions of $\frown$ and $*$. The first axiom just says that $\frown$ is the usual concatenation operation, appending a natural number to the beginning of a string of natural numbers. The second describes a way to turn type-2 functionals into type-$(1 \to 1)$ functionals, and is slightly more complicated. In order to vew a functional F as a map from reals to reals, we first replace an input real r by the infinite sequence of reals $(0 \frown r, 1 \frown r, ...)$, and then apply F to each of the reals in this sequence in turn; this yields a sequence of natural numbers — that is, a real — $(F(0 \frown r), F(1 \frown r), ...)$. This real is $F * r$.

This particular definition of $*$ is merely a technical device, and could be replaced with any of a number of similar constructions; the important point is that we have a way of interpreting a single real r as a sequence of reals, and that by applying a type-2 functional to each real in that sequence we can view the functional as a map from $\mathbb{R}$ to $\mathbb{R}$ instead of a map from $\mathbb{R}$ to ω.

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[4]There are several equivalent formulations of these, including as choice principles; we choose the following presentation, since it seems the most natural. Since we work with functionals which take values in ω instead of with sets (= functionals with values in $\{0, 1\}$), these schemes do look more complicated than the usual Δ_1^0-comprehension scheme in RCA_0; however, the intuition behind them — that if exactly one of an effective collection of existential sentences holds, then we can effectively find which one holds — is the same, and it is straightforward to show that our schemes are equivalent to their "dual" versions in terms of Π_1^0 formulas. For this reason, we slightly abuse terminology and call these schemes "Δ_1^0-comprehension."

Convention 2.2.1. *Throughout this paper, if $M \models \mathsf{RCA}_0^3$ we will write M_0, M_1, M_2 for the type-0, -1, and -2 parts of M, respectively.*

Note that if $(M_0, M_1, M_2; *_0, \frown_0), (M_0, M_1, M_2; *_1, \frown_1) \models \mathsf{RCA}_0^3$, then in fact

$$(M_0, M_1, M_2; *_0, \frown_0) = (M_0, M_1, M_2; *_1, \frown_1);$$

that is, models of RCA_0^3 are determined by their 0-, 1-, and 2-type objects, and it is enough to specify these types to specify the full model. Despite this, the symbols $\frown$ and $*$ are necessary for RCA_0^3 in order for the comprehension schemes to have full force (given that we avoid objects of non-standard type). As evidence of this, the following two facts are easy to prove, yet crucially rely on comprehension over Δ_1^0 formulas involving $\frown$ and $*$:

Fact 2.2.2. RCA_0^3 *proves each of the following statements:*

1. *For each type-2 functional F, there is a real r such that*

$$\forall s^1, n^0[\forall k^0(s(k) = n) \to r(n) = F(s)].$$

2. *For each type-2 functional F, there is a type-2 functional G such that*

$$G(\langle a_0, a_1, a_2, ..., a_n, ...\rangle) = F(\langle a_0, a_2, a_4, ..., a_{2n}, ...\rangle)$$

Proof. For (1), first note that the type-2 comprehension scheme gives us a functional I such that $\forall r^1[I(r) = r(1)]$, and hence

$$\forall r^1, k^0[\forall i^0(I * (k \frown r)(i) = k)].$$

Now our desired real r can be defined by

$$r(k) = F(I * (k \frown \underline{0})),$$

which exists by the type-1 comprehension scheme.

For (2), let H be the type-2 functional defined by the quantifier-free formula $H(r) = r(2r(0) + 1)$; then the desired G is defined by the quantifier-free formula

$$G(r) = k \iff F(H * r) = k,$$

and so again is guaranteed to exist by the type-1 comprehension scheme. $\square$

It can be shown that neither (1) nor (2) is provable if we restrict the Δ_1^0 comprehension schemes to formulas not involving $*$ and $\frown$. Essentially, $*$ and $\frown$ are the price we pay for a base theory which closely resembles RCA_0 and has reasonable models.

To drive this last point home, we end this section by presenting some natural models of RCA_0^3:

Example 2.2.3. *Let $\mathcal{I}$ be a Turing ideal; that is, $\mathcal{I}$ is closed under the join $\oplus$ and is closed downwards under Turing reducibility. Then the smallest ω-model of RCA_0^3 containing precisely the reals in $\mathcal{I}$ is*

$$\mathcal{S}_{\mathcal{I}} = (\omega, \mathcal{I}, \{r \mapsto \varphi_e^{r \oplus s}(0) : s \in \mathcal{I} \text{ and } \varphi_e^{r \oplus s}(0) \downarrow \text{ for every } r \in \mathcal{I}\}).$$

We will call such a pair (e, s) a Turing code *for the map $r \mapsto \varphi_e^{r \oplus s}(0)$.*

Proof. Any ω-model of RCA_0^3 whose real part is $\mathcal{I}$ must be at least as large as $\mathcal{S}_{\mathcal{I}}$, so it is enough to show that $\mathcal{S}_{\mathcal{I}} \models \mathsf{RCA}_0^3$. Axioms (1), (2), and (4) are immediate; it only remains to show that the comprehension schema are satisfied.

We focus on the type-2 case; the type-1 case is identical. Intuitively, we should be able to compute the value of any functional defined according to the Δ_1^0-comprehension scheme effectively from the real parameters and Turing codes for the type-2 parameters involved, since the value of the functional is determined by an effective collection of Σ_1^0 sentences. The only possible difficulty could arise from the new symbols, $\frown$ and $*$. To ensure that these pose no problems, we first observe that we can — uniformly in a Turing code for a functional F and a natural number $c \in \omega$ — find a Turing code for the map $r \mapsto F(c^\frown r)$. The case of $*$ is slightly more interesting, but still poses no problems. By a straightforward induction on n, if $F_1, ..., F_n \in \mathcal{S}_{\mathcal{I}}$, then there is some $e \in \omega$ and $s \in \mathcal{I}$ such that, for every $r \in \mathcal{I}$ and $i \in \omega$, we have

$$\Phi_e^{r \oplus s}(i) = F_1 * (F_2 * (... * (F_n * r)))(i).$$

It now follows by a tedious but straightforward induction on formula complexity that we can effectively compute the values of any type-2 functional defined in a Δ_1^0 fashion from parameters in $\mathcal{S}_{\mathcal{I}}$, uniformly in the real parameters and in Turing codes for the type-2 parameters. But this yields a Turing code for the functional so defined, which is therefore already in $\mathcal{S}_{\mathcal{I}}$. $\qquad\square$

Corollary 2.2.4. *The structure $\mathcal{C} = (\omega, \mathbb{R}, \{f \colon \mathbb{R} \to \omega : f \text{ is continuous}\})$ — when interpreted as an L^3-structure in the natural way — is the smallest model of RCA_0^3 containing all the reals.*

Example 2.2.5. *Recall that the class of* Borel *sets is the smallest class of subsets of $\mathbb{R}$ containing the open sets which is closed under complementation and countable unions. Note that if a set is Borel, then this is witnessed by a well-founded tree whose terminal nodes are open intervals with rational endpoints, and whose other nodes correspond either to complementation or countable union; we will call the smallest rank of such a witnessing tree the* Borel rank *of the set. Then:*

- *a map $f \colon \mathbb{R} \to \omega$ is* Borel measurable *if $f^{-1}(i)$ is Borel for every $i \in \omega$, and*

- *a map $f \colon \mathbb{R} \to \mathbb{R}$ is* Borel measurable *if $f^{-1}(X)$ is Borel for every Borel $X \subseteq \mathbb{R}$.*

We let $\mathcal{B}$ be the three-sorted structure $(\omega, \mathbb{R}, \{F \colon \mathbb{R} \to \omega : F \text{ is Borel measurable}\})$. Since $\mathcal{B}$ contains all the reals, each of the symbols in the language of RCA_0^3 has a natural interpretation in $\mathcal{B}$; in particular, $\mathcal{B}$ is closed under the operations $\frown$ and $*$. Then $\mathcal{B} \models \mathsf{RCA}_0^3$.

We will use this model in a separation result later (2.2.15).

Proof. Since the first- and second-order parts of $\mathcal{B}$ are ω and $\mathbb{R}$, $\mathcal{B}$ is clearly closed under $\frown$ and $*$, and satisfies parts (1), (2), and (4) of the axioms of RCA_0^3, as well as the Δ_1^0-comprehension scheme for type-1 objects. So it only remains to show that $\mathcal{B}$ satisfies the comprehension scheme for type-2 functionals.

There are multiple ways to show that $\mathcal{B}$ satisfies the Δ_1^0-comprehension scheme for type-2 functionals. First, we observe that if $X \subseteq \mathbb{R}$ is Borel and $k \in \omega$, then $X_k = \{r : k^\frown r \in X\}$ is also Borel; this is proved by a straightforward induction on the Borel rank of X, which we omit.

Now suppose $Y \colon \mathbb{R} \to \omega$ is Δ_1^0 relative to some type-1 parameters $r_0, ..., r_m$ and some type-2 parameters $F_0, ..., F_n$. That is, there is a Σ_0^0-formula $\psi(x^0, y^1, z^0)$ with only the displayed variables, which does not involve equality between type-1 or type-2 terms, with some type-1 parameters $r_0, ..., r_m$ and some type-2 parameters $F_0, ..., F_n$, such that

$$Y(r) = k \iff \exists x^0 \psi(x, r, k);$$

we will show that Y is Borel-measurable, and hence in $\mathcal{B}$.

To see this, fix $i \in \omega$ and consider the set of reals $X = Y^{-1}(i)$; we must show that X is Borel. Note that since

$$X = \bigcup_{j \in \omega} \{s : \psi(j, s, i)\},$$

so it is enough to show that the sets $X_j = \{s : \psi(j, s, i)\}$ are each Borel. So fix $j \in \omega$. The set X_j is a Boolean combination of sets of the form $X_{j,\alpha} = \{s : \alpha(j, r, i)\}$ for α atomic; so it is enough to show that each such set is Borel. So fix such an atomic α. Since ψ cannot involve any instances of equality of type-1 or -2 objects, α must have the form $\mathfrak{t}_0 = \mathfrak{t}_1$, for terms $\mathfrak{t}_0, \mathfrak{t}_1$ of type 0. It is easy to see that $X_{j,\alpha}$ is Borel if for each $k \in \{0, 1\}$ and $c \in \omega$, $\{s : \mathfrak{t}_k(s) = c\}$ is Borel.

We will now be finished if we can show that, if $c \in \omega$ and $\mathfrak{t}$ is a term with one free type-1 variable y^1, real parameters $r_0, ..., r_m$, and type-2 parameters $F_0, ..., F_n$, then $\{s : \mathfrak{t}(s) = c\}$ is Borel. This is proved by induction on the complexity of the term. We omit most of the induction, since it is straightforward, and prove the only difficult part:

Claim: if $F_0, ..., F_n$ are Borel, then the map $r \mapsto F_0 * (F_1 * (... * (F_n * r)))$ is Borel.

Proof of claim. It suffices to show that if F is Borel-measurable and $X \subseteq \mathbb{R}$ is Borel, then the set $P_X = \{r : F * r \in X\}$ is also Borel; that is, $r \mapsto F * r$ is Borel-measurable. This is proved by induction on the Borel rank of X. If X is open, then if $r \in P_X$ there is some finite initial segment σ of $F * r$ such that if s is any real extending σ, we have $r \in X$. So P_X contains the set $C_\sigma = \{s : \sigma \prec F * s\}$. But that set is the intersection of finitely many sets of the form $\{s : F(c^\frown s) = i\}$, which are Borel since F is Borel-measurable; so C_σ is Borel.

Moreover, every element of P_X is contained in some C_σ, of which there are only countably many; so P_X is Borel.

Since taking preimages commutes with unions and complementation, the rest of the induction is immediate.

$\square$

The remaining induction is uneventful.

$\square$

Higher-type analogues of ATR_0

In what follows, we treat higher-type determinacy principles, and towards that end some definitions are necessary. We study games of length ω on $\mathbb{R}$ — that is, players I and II alternate playing real numbers, building an ω-sequence of reals. We identify both clopen games and open games with their underlying *game trees*, which are subtrees of $\mathbb{R}^{<\omega}$: thus, we identify clopen games with well-founded trees — players alternate playing reals, moving further along the tree, and the first player to be unable to play and stay on the tree loses — and we identify open games with trees — players alternate playing reals, and player I (Open) wins if and only if the play ever leaves the tree. There are several reasonable ways to encode game trees $\subseteq \mathbb{R}^{<\omega}$ as type-2 functionals and finite strings of reals as individual reals, and the specific choice of coding is unimportant. We will assume such a coding method in the background, so that we may for instance apply a type-2 functional to a node on a subtree of $\mathbb{R}^{<\omega}$; there will be no subtleties in this regard.

When discussing plays, however, things become more complicated. If Σ, Π are strategies, then the kth stage in the play $\Sigma \otimes \Pi$, $(\Sigma \otimes \Pi)_k$ — or rather, a real coding $(\Sigma \otimes \Pi)_k$ — is defined as follows. There is a functional F, whose existence is guaranteed by the comprehension scheme, such that $F * (k^\frown r)$ is the kth "row" of r; specifically, F is defined by

$$s \mapsto s(2 + \langle s(0), s(1) \rangle).$$

We say that a real r *codes* $(\Sigma \otimes \Pi)_k$ if

- $F * (0^\frown r) = \underline{0}$,

- $\forall 0 < 2j + 1 \le k[F * ((2j+1)^\frown r) = \Sigma * (F * ((2j)^\frown r))]$, and

- $\forall 0 < 2j + 2 \le k[F * ((2j+2)^\frown r) = \Pi * (F * ((2j+1)^\frown r))]$;

similarly, we say that r codes the whole play $\Sigma \otimes \Pi$ if r codes $(\Sigma \otimes \Pi)_k$ for all k. This definition lets us refer to the play $\Sigma \otimes \Pi$ inside the language of RCA_0^3; and we use, e.g., "$(\Sigma \otimes \Pi)_k \notin T$" as shorthand for "there is a real r coding $(\Sigma \otimes \Pi)_k$, and $r \notin T$."

There is a subtlety here, which arises due to a particular weakness in the base theory RCA_0^3. (The end of this paper addresses the foundational aspects of this; for now, we simply treat it as it affects us.) RCA_0^3 is too weak to guarantee the existence of a real coding the

whole play $\Sigma \otimes \Pi$. This is a consequence of Hunter's proof[5] ([31], Theorem 2.5) that the theory

$$\mathsf{RCA}_0^3 + \mathcal{E}_1 := \mathsf{RCA}_0^3 + \text{``}\exists J^2 \forall x^1 (J * x = x')\text{''}$$

is conservative over ACA_0: if Σ and Π are each the operator J described above, then "$\Sigma \otimes \Pi$ exists" implies "$0^{(\omega)}$ exists," so that sentence cannot be a consequence of $\mathsf{RCA}_0^3 + \mathcal{E}_1$, let alone RCA_0^3 itself.

This can be salvaged in general by altering the base theory; and in fact, since this same subtlety arises in other ways, this is a reasonable course of action — see the end of Section 4 of this paper. In our case, however, all potential difficulties are handled by the strength of the principles we consider. For example, in the definition of clopen and open determinacy, we use a strong definition of "winning strategy:" e.g., a strategy Σ for Open in an open game is winning if for every strategy Π for Closed, there is a real coding some stage $(\Sigma \otimes \Pi)_k$ of the game by which Σ has won. This builds into the statements of the theorems we examine all the strength we need to perform the intuitively natural calculations involving stages of games.

The end result is that, although we cannot meaningfully talk about the play of a game $\Sigma \otimes \Pi$ directly within RCA_0^3, the principles we study in this paper happen to have enough power to allow us to do so. As an example of this, it is easy to see that each of the principles introduced in Definition 7 below imply that at most one player has a winning strategy in an open or clopen game; however, this is not provable in the base theory RCA_0^3 alone.

Consider the following four theorems, all equivalent to ATR_0 over RCA_0:

- *Comparability of well-orderings*: If X, Y are well-orders with domain $\subseteq \mathbb{N}$, then there is an embedding from one into the other.

- *Clopen determinacy*: Every well-founded subtree of $\omega^{<\omega}$, viewed as a clopen game, is determined.

- *Open determinacy*: Every subtree of $\omega^{<\omega}$, viewed as an open game, is determined.

- Σ_1^1 *separation*: If $\varphi(A)$ is a Σ_1^1 sentence (possibly with parameters) with a single free set variable, and $X = (X_i)_{i \in \omega}$ is an array of sets such that

$$\forall k \in \omega \exists j \in 2(\neg \varphi(X_{\langle k, j \rangle})),$$

then there is some set Y such that

$$\forall k \in \omega(\neg \varphi(X_{\langle k, Y(k) \rangle})).$$

These each have reasonable higher-type analogues, each of which is a theorem of ZFC:

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[5]Originally formulated for Kohlenbach's theory RCA_0^ω, but immediately adaptable to RCA_0^3.

Definition 7. *Over* RCA_0^3*, we define the following principles:*

- *The comparability of well-orderings of reals,* **CWO**: *If X, Y are well-orderings with domain $\subseteq \mathbb{R}$, then there is an embedding from one into the other.*

- *Clopen determinacy for reals,* $\Delta_1^{\mathbb{R}}$*-***Det**: *for every tree $T \subseteq \mathbb{R}^{<\omega}$ which is well-founded, viewed as a clopen game, either there is a winning strategy for player I:*

$$\exists \Sigma \colon \mathbb{R}^{<\omega} \to \mathbb{R}, \forall \Pi \colon \mathbb{R}^{<\omega} \to \mathbb{R}[\exists k \in \omega((\Sigma \otimes \Pi)_{2k+1} \in T \wedge (\Sigma \otimes \Pi)_{2k+2} \notin T)];$$

or there is a winning strategy for player II:

$$\exists \Pi \colon \mathbb{R}^{<\omega} \to \mathbb{R}, \forall \Sigma \colon \mathbb{R}^{<\omega} \to \mathbb{R}[\exists k \in \omega((\Sigma \otimes \Pi)_{2k} \in T \wedge (\Sigma \otimes \Pi)_{2k+1} \notin T)].$$

- *Open determinacy for reals,* $\Sigma_1^{\mathbb{R}}$*-***Det**: *for every tree $T \subseteq \mathbb{R}^{<\omega}$, viewed as an open game, either there is a winning strategy for player I (Open):*

$$\exists \Sigma \colon \mathbb{R}^{<\omega} \to \mathbb{R}, \forall \Pi \colon \mathbb{R}^{<\omega} \to \mathbb{R}[\exists k \in \omega((\Sigma \otimes \Pi)_k \notin T)];$$

or there is a winning strategy for player II (Closed):

$$\exists \Pi \colon \mathbb{R}^{<\omega} \to \mathbb{R}, \forall \Sigma \colon \mathbb{R}^{<\omega} \to \mathbb{R}[\forall k \in \omega((\Sigma \otimes \Pi)_k \in T)].$$

- *The Σ_1^2-separation principle,* Σ_1^2*-***Sep**: *If $\varphi(f^2)$ is a Σ_1^2-formula with a single type-2 free variable, and $X = (X_\eta)_{\eta \in \mathbb{R}}$, $Y = (Y_\eta)_{\eta \in \mathbb{R}}$ are real-indexed collections of type-2 functionals[6] such that*

$$\neg \exists x^1 (\varphi(X_x) \wedge \varphi(Y_x)),$$

then there is some type-2 object F such that

$$\forall x^1 [\varphi(X_x) \to F(x) = 1 \quad and \quad \varphi(Y_x) \to F(x) = 0].$$

*(Note that, strictly speaking, Σ_1^2-***Sep** *is an infinite scheme, as opposed to a single sentence.) It is these principles which we choose to study in this paper. The remainder of this section is devoted to the simpler parts of their analysis; the separation of clopen and open determinacy for reals is the subject of the following section.*

Note that the determinacy principles above are not provable in ZF alone, whereas **CWO** and Σ_1^2-**Sep** are, so in order to analyze these principles properly we need some version of the axiom of choice:

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[6]A real-indexed set of type-2 functionals $(Z_s)_{s \in \mathbb{R}}$ is coded by the type-2 functional

$$\hat{Z} \colon r \mapsto Z_{P_0 * r}(P_1 * r),$$

where P_0, P_1 correspond to the left and right projections of a reasonable pairing function $\mathbb{R}^2 \cong \mathbb{R}$.

Definition 8. *Let $\langle \cdot, \cdot \rangle$ be an appropriate pairing function on $\mathbb{R}$. The* selection principle *for* $\mathbb{R}$, *SF, is the assertion that for every $\mathbb{R}$-indexed set of nonempty sets of reals has a selection functional; that is, for every type-2 functional F — interpreted as the $\mathbb{R}$-indexed set of reals*

$$\{\{s \in \mathbb{R} : F(\langle r, s \rangle) = 1\} : r \in \mathbb{R}\}$$

— there is a type-2 functional G satisfying

$$\forall r^1 (F(\langle r, G * r \rangle) = 1).$$

Now we turn to the implications. Clearly $\Sigma_1^{\mathbb{R}}$-Det implies $\Delta_1^{\mathbb{R}}$-Det. A more interesting implication is the following:

Fact 2.2.6. *Over RCA_0^3, we have*

$$\Sigma_1^2\text{-}Sep + SF \to \Delta_1^{\mathbb{R}}\text{-}Det.$$

Proof. This is somewhat involved. We begin with three technical results, which are of independent interest:

Fact 2.2.7 (Comprehension). *$RCA_0 + \Sigma_1^2$-Sep implies Δ_1^2-comprehension for type-2 functionals for each n: given $n \in \omega$ and any Σ_1^2 formula φ with one type-1 variable which is equivalent to a Π_1^2 formula, there is a functional F such that*

$$F(r) = 1 \iff \varphi(r), \quad F(r) = 0 \iff \neg\varphi(r)$$

Proof. Apply Σ_1^2-Sep to the pair $(\varphi, \neg\varphi)$. $\square$

Fact 2.2.8 (Iteration). *$RCA_0^3 + \Sigma_1^2$-Sep proves that given a functional F and a real r, we can form the iteration sequence $(r, F * r, F * (F * r), ...)$. Formally, for every type-2 functional F there is a type-2 functional G such that — for every r — we have*

$$G * (n^\frown r) = F * (F * (...(F * r)))$$

(where the right hand side contains n applications of "$F*$").

In particular, note that this implies that given any strategies Σ_0 and Σ_1, the real coding their entire play $\Sigma_0 \otimes \Sigma_1$ exists.

Proof. Fix a real r. Let $\chi(x^1, y^0, z^1)$ be the formula asserting that x is a real whose first y rows are of the form $z, F * z, F * (F * z), ...$ Then let $\varphi(w^1)$ be the formula

$$\exists s(\chi(s, w(1), w^{--}), s(w(0) = 0))$$

and let ψ be the formula

$$\forall s(\chi(s, w(1), w^{--}) \to s(w(0) = 1))$$

Clearly φ and ψ are Σ_1^2 and $\varphi \wedge \psi$ is inconsistent, so we may apply Σ_1^2-Sep. This yields the desired G. $\square$

Proposition 2.2.9 (Paths from subtrees). *$RCA_0^3 + \Sigma_1^2$-Sep+SF proves that a tree $T \subseteq \mathbb{R}^{<\omega}$ is well-founded if and only if it has no nonempty subtrees with no terminal nodes.*

Proof. Clearly a witness to T being ill-founded yields a subtree with no terminal nodes. In the other direction, suppose T is well-founded but has a nonempty subtree S with no terminal nodes. By SF and 2.2.7, there is a functional F such that if σ is a predecessor of an element of S, then $F(\sigma) \in S$ is an extension of σ. Fix $\sigma_0 \in S$ and let $\sigma_{i+1} = F(\sigma_i)$ for $i > 0$; by 2.2.8, the sequence $(\sigma_0, \sigma_1, ...)$ exists. $\square$

We now return to the proof of 2.2.6. Let $T \subseteq \mathbb{R}^{<\omega}$ be a well-founded tree, viewed as a clopen game; we will show that T is determined.

Definition 9. *For $\sigma \in T$, let $T[\sigma] = \{\rho : \sigma \preccurlyeq \rho \in T\}$. A U-tree for σ is a functional $F : T[\sigma] \to \{Safe, Unsafe\}$ satisfying the following properties:*

1. *$ev(\sigma) = Unsafe$;*

2. *$ev(\tau) = Safe \iff$ there is some immediate extension ρ of τ such that $\rho \in T$ and $ev(\rho) = Unsafe$.*

An S-tree for σ is a pair (ρ, F) such that $\rho \in T$ is an immediate extension of σ and F is a U-tree for ρ. We let $\varphi_U(\sigma)$ and $\varphi_S(\sigma)$ be the sentences, "There is a U-tree for σ" and "There is an S-tree for σ," respectively; note that both φ_U and φ_S are Σ_1^2.

Intuitively, the existence of a U-tree for σ indicates that σ is *unsafe*, that is, the game G_σ is a win for player II. Similarly, the existence of an S-tree for σ provides one bit of information towards a winning strategy for player I in G_σ.

Lemma 2.2.10. *No $\sigma \in T$ satisfies $\varphi_U(\sigma) \wedge \varphi_S(\sigma)$.*

Proof. Otherwise, let T_0 and T_1 be U- and S-trees for σ; then the set of nodes ρ on which $T_0(\rho) \neq T_1(\rho)$ forms a nonempty subtree of T with no terminal nodes, contradicting the wellfoundedness of T via 2.2.9. $\square$

By 2.2.10 we can apply Σ_1^2-Sep to get a functional $ev : T \to \{Safe, Unsafe\}$ such that

- $\varphi_U(\sigma) \to ev(\sigma) = Unsafe$,

- $\varphi_S(\sigma) \to ev(\sigma) = Safe$.

It is now enough to show that either ev is a U-tree for $\langle \rangle$, or there is some length-1 string $\langle a \rangle \in T$ such that the restriction $ev_{\langle a \rangle}$ to $T[\langle a \rangle]$ is a U-tree for $\langle a \rangle$. To see that this is sufficient, suppose ev is a U-tree for $\langle \rangle$ (the other case is identical). Then by SF there is a strategy for player II such that, if $|\sigma|$ is odd and $ev(\sigma) = Safe$, then $ev(\sigma^\frown \Sigma(\sigma))$ is on T and is marked $Unsafe$ by ev; this strategy can clearly never lose, so by 2.2.8 Σ is a winning strategy.

Definition 10. *Say that a node σ of T is* bad *if one of the following conditions holds:*

1. $ev(\sigma) = Safe$ but for every immediate extension ρ of σ we have $ev(\rho) = Safe$, or

2. $ev(\sigma) = Unsafe$ but there is some immediate extension ρ of σ such that $ev(\rho) = Unsafe$.

Lemma 2.2.11. *If σ is bad, then there is some proper extension of σ which is bad.*

Proof. Suppose σ is bad but no proper extension of σ is bad.

If σ is bad via case (1), then the map

$$ev' \colon T[\sigma] \to \{Safe, Unsafe\} \colon \rho \mapsto \begin{cases} ev(\rho) & \text{if } \rho \neq \sigma \\ Unsafe & \text{if } \rho = \sigma \end{cases}$$

is a U-tree for σ whose existence follows from 2.2.7; but this contradicts the definition of ev.

If σ is bad via case (2), then we can similarly construct an S-tree for σ, again contradicting the definition of ev. $\square$

Corollary 2.2.12. *There are no bad nodes of T.*

Proof. Suppose otherwise. The set of bad nodes exists by 2.2.7; by 2.2.11 and 2.2.9, this contradicts the well-foundedness of T. $\square$

But now we are done: since T has no bad nodes, either ev is a U-tree for $\langle \rangle$, or — letting σ be some length-1 node of T which satisfies $ev(\sigma) = Unsafe$ — ev_σ is a U-tree for σ, and as observed above either possibility allows us to produce a winning strategy for T. $\square$

Note that at the close of the proof, we conclude that in fact for every $\sigma \in T$ we have $\varphi_U(\sigma) \iff \neg\varphi_S(\sigma)$; yet since the proof of this fact itself goes through Σ_1^2-Sep, the slightly weaker theory SF$+\Delta_1^2$-comprehension for type-2 functionals does not seem to imply $\Delta_1^{\mathbb{R}}$-Det.

To compliment 2.2.6, we show that the assumption of SF cannot be removed:

Fact 2.2.13. *Over RCA_0^3, $\Delta_1^{\mathbb{R}}$-Det implies SF.*

Proof. Let F be an instance of SF, viewed as an $\mathbb{R}$-indexed family of sets of reals $\{F_r\}_{r \in \mathbb{R}}$. Consider the game in which player I plays a real r, and player II wins if and only if they immediately play a real $s \in F_r$. A winning strategy cannot exist for player I, and any winning strategy for player II immediately yields a selection functional for F. $\square$

2.2.6 and 2.2.13 together raise the question of the role that variants of the axiom of choice might play in higher-order versions of ATR$_0$. This will be treated in more detail in a forthcoming paper; for now, we introduce one final principle, which is close to the classical statement of ATR$_0$ itself and which captures exactly the choiceless part of $\Delta_1^{\mathbb{R}}$-Det:

Definition 11. *For a tree $T \subseteq \mathbb{R}^{<\omega}$ and a node $\sigma \in T$ we let $T_\sigma = \{\tau : \sigma^\frown\tau \in T\}$, and if $F\colon T \to \omega$ we let $F_\sigma\colon \tau \mapsto F(\sigma^\frown\tau)$. Σ_1^1 rank-recursion on $\mathbb{R}$, denoted "BR," is then the scheme asserting that for any tree $T \subseteq \mathbb{R}^{<\omega}$ which does not contain a nonempty subtree with no terminal nodes (recall that, absent choice, this is a strengthening of well-foundedness) and every Σ_1^1 formula Σ_1^1-formula $\varphi(Y^2, Z^2)$ with only the displayed free variables, there is a type-2 functional F with range $\subseteq \{0,1\}$ such that, for $\sigma \in T$,*

$$F(\sigma) = 1 \iff \varphi(F_\sigma, T_\sigma).$$

Theorem 2.2.14. *Over RCA_0^3, we have $\mathsf{BR+SF} \iff \Delta_1^\mathbb{R}\text{-}\mathsf{Det}$.*

Proof. $\Delta_1^\mathbb{R}\text{-}\mathsf{Det} \to \mathsf{BR+SF}$: we have already observed (2.2.13) that $\Delta_1^\mathbb{R}\text{-}\mathsf{Det}$ implies SF. To show that $\Delta_1^\mathbb{R}\text{-}\mathsf{Det}$ implies BR, given a well-founded T and an appropriate formula φ, consider the following well-founded game. First, player I chooses some $\sigma \in T$; then, player II responds by playing either "Safe" or "Unsafe." The game then continues by playing the clopen game T_σ, with player II going first if she chose "Safe" and player II going second if she chose "Unsafe." Clearly only player II can have a winning strategy, and any winning strategy computes the desired h by setting $h(\sigma) = 0$ if the winning strategy for II tells II to play "Unsafe" if I plays σ.

In the other direction, given a clopen game G, use BR with the formula

$$\varphi(Y^2, Z^2) \equiv \exists a^1 (a \in Z \text{ and } Y(a) = 0)$$

(recall that Y is meant to stand for F_σ and Z for T_σ). The resulting function h then computes a winning quasistrategy for G: if $h(\sigma) = 0$, then σ is a loss for whoever's turn it is, and one player or the other can win by ensuring that their opponent always plays from nodes marked 0 by h. SF then lets us pass from this winning quasistrategy to a geniune winning strategy for G. $\qquad\square$

We end this section by presenting a straightforward separation result — the first instance of divergence from the standard reverse-mathematical picture. Given the low complexity of wellfoundedness at higher types, it is reasonable to expect that CWO is quite weak relative to the higher-type determinacy principles. This is, in fact, true:

Lemma 2.2.15. *Over RCA_0^3, CWO does not imply $\Delta_1^\mathbb{R}\text{-}\mathsf{Det}$.*

Proof. We will show that in fact the model $\mathcal{B}$ generated by the Borel sets, defined in 2.2.5, satisfies CWO but not $\Delta_1^\mathbb{R}\text{-}\mathsf{Det}$. Showing that $\mathcal{B} \models \mathsf{CWO}$ is straightforward: any uncountable Borel set of reals contains a perfect subset, and there is no Borel well-ordering of $\mathbb{R}$. These facts follow from Borel determinacy ([38], Theorem 20.5), and together imply that all Borel well-orderings are countable. It then follows that any two Borel well-orderings are comparable by a boldface Σ_2^0 embedding, so $\mathcal{B} \models \mathsf{CWO}$.

To show that $\mathcal{B} \models \neg\Delta_1^\mathbb{R}\text{-}\mathsf{Det}$, fix some analytic (that is, boldface Σ_1^1) set $X \subseteq \mathbb{R}$ which is not Borel. Let $T \subseteq \omega^{<\omega}$ be a tree such that

$$X = \{a \in \mathbb{R} : \exists b \in \mathbb{R}((\langle a(i), b(i)\rangle)_{i\in\omega} \in [T])\};$$

such a tree is guaranteed to exists since X is Σ_1^1, and since $\mathcal{B}$ contains all reals we have that $T \in \mathcal{B}$. Now consider the game $\mathcal{G}$ which proceeds as follows:

- Player I plays a real a.

- Player II guesses whether $a \in X$ or not.

- If player II guesses "yes," then player II must also play a real b; player I then plays a natural number k; the game is now over, and player I wins if and only if $(\langle a(i), b(i) \rangle)_{i<k} \notin T$.

- If player I guesses "no," then player I plays a real b, player II plays a natural number k; the game is now over, and this time player II wins if and only if $(\langle a(i), b(i) \rangle)_{i<k} \notin T$.

Informally, player I is challenging player II to correctly compute X, and the tree T is used to evaluate whether II's guess was correct. This is a clopen game, and viewed as a subtree of $\mathbb{R}^{<\omega}$ it is clearly Borel, so $\mathcal{G} \in \mathcal{B}$.

However, this game is undetermined in $\mathcal{B}$. To see this, note that since $\mathcal{B}$ contains every real, a strategy in $\mathcal{B}$ is winning in $\mathcal{B}$ if and only if it is actually winning, since otherwise any play defeating it would be coded by a real and hence exist in $\mathcal{B}$. So if $\mathcal{B}$ satisfies $\Delta_1^{\mathbb{R}}$-Det, then $\mathcal{B}$ must contain an actual winning strategy for $\mathcal{G}$; but X is Borel relative to any winning strategy for $\mathcal{G}$ (since such a strategy Σ must be a strategy for player II, and must have the property that $\Sigma(\langle a \rangle) = 1 \iff a \in X$). Since $\mathcal{B}$ consists precisely of the Borel functionals, if $\mathcal{G}$ were determined in $\mathcal{B}$ then X would have to be Borel, which is a contradiction. $\qquad\square$

Note that Borel instances of $\Delta_1^{\mathbb{R}}$-Det can be constructed whose winning strategies are much more complex than Σ_1^1; so CWO is in fact *far* weaker than $\Delta_1^{\mathbb{R}}$-Det.

2.3 Separating clopen and open determinacy

In this section we construct a model M of $\mathsf{RCA}_0^3 + \Delta_1^{\mathbb{R}}\text{-Det} + \neg\Sigma_1^{\mathbb{R}}\text{-Det}$, using a variation of Steel's tagged tree forcing; see [74], and also [57] and [61]. Throughout this section, we work over a transitive ground model V of ZFC+CH.

Remark 2.3.1. *Recently, Sherwood Hachtman [25] has developed an alternate proof of this result; using methods from inner model theory, he shows that the smallest initial segment of Goedel's constructible universe L which is a model of $\mathsf{RCA}_0^3 + \Delta_1^{\mathbb{R}}$-Det does not satisfy $\Sigma_1^{\mathbb{R}}$-Det. More precisely, he shows that if θ is the least ordinal such that $(\omega, \mathbb{R}, \omega^{\mathbb{R}})^{L_\theta} \models \mathsf{RCA}_0^3 + \Delta_1^{\mathbb{R}}$-Det, then $(\omega, \mathbb{R}, \omega^{\mathbb{R}})^{L_\theta} \models \neg\Sigma_1^{\mathbb{R}}$-Det.*

The general picture of classical Steel forcing is as follows. Conditions are well-founded trees, with additional information representing rank, ordered by extension (with certain restrictions). The full generic object is an infinite, ill-founded tree, whose nodes are labelled with their ranks in the tree, together with a collection of distinguished paths. The model

built from this generic is gotten by looking at all sets hyperarithmetic relative to the tree and finitely many of the paths; in particular, the ordinal labels are forgotten. This loss of information is crucial.

In our case, our conditions will be countable, ill-founded trees with additional information, ordered appropriately. The generic object will be, as in the classical case, a tree whose nodes are labelled essentially with their rank. This tree can be viewed as the game tree of an open game; this open game, which is classically a win for player II (Closed), will exist but be undetermined in our model. The difficult portion of the proof is ensuring that the model we build satisfies $\Delta_1^{\mathbb{R}}$-Det. Rather than use a higher-order notion of hyperarithmeticity (see below), we construct our model out of those functionals which depend on the generic tree only in a limited way; see Definition 14.

The idea behind the game we construct is as follows. Consider the clopen game G_α, for α an ordinal, in which players I and II alternately build decreasing sequences of ordinals less than α, and the first player whose sequence terminates loses. Clearly player II wins this game, since all she has to do is consistently play slightly larger ordinals than what player I plays.

$$G_\alpha: \quad \begin{array}{c|cccc} \text{Player I} & \alpha_0 & \alpha_1 < \alpha_0 & & \cdots \\ \hline \text{Player II} & & \beta_0 & & \beta_1 < \beta_0 & \cdots \end{array}$$

Now there is a natural open game, $\mathcal{O}_\alpha$, associated to G_α. $\mathcal{O}_\alpha$ has the same rules as G_α, except that on player I's turn, she can give up and start over, playing an arbitrary ordinal below α. If she does this, then player II gets to play an arbitrary ordinal below α as well. After a restart, play then continues as normal, until player II loses or player I restarts again. Player I (Open) wins if player II's sequence ever reaches zero; player II (Closed) wins otherwise.

$$\mathcal{O}_\alpha: \quad \begin{array}{c|cccc} \text{Player I (Open)} & \alpha_0 & \alpha_1 & & \cdots \\ \hline \text{Player II (Closed)} & & \beta_0 & \beta_1 & \cdots \end{array} \qquad (\forall i, \alpha_{i+1} < \alpha_i \rightarrow \beta_{i+1} < \beta_i)$$

Essentially, $\mathcal{O}_\alpha$ is gotten by "pasting together" ω-many copies of G_α, one after the other, and player II must win *all* of these clopen sub-games in order to win $\mathcal{O}_\alpha$. This is still a win for player II, but in a more complicated fashion. In particular, if player II happened to not be able to directly see the ordinals player I played, but was only able to see the underlying game tree itself, she would need quite a lot of transfinite recursion to be able to figure out what move to play next - seemingly more than she would need to win G_α, since there is much more "noise" in the structure of $\mathcal{O}_\alpha$. This is roughly the situation we create in the construction below. We will define a forcing notion which adds a tree $T_G \subseteq \mathbb{R}^{<\omega}$. This tree can be viewed as an open game on $\mathbb{R}$ of length ω in the usual manner. In the full generic extension, this game will be identical to the game $\mathcal{O}_{\omega_2^V}$ — that is, the game tree of $\mathcal{O}_{\omega_2^V}$ will be isomorphic to T_G in the full generic extension — but the function which assigns to nodes of T_G their ordinal ranks will be extremely complicated.

There are several differences between our construction and Steel's tagged tree forcing, however. Most importantly, our forcing is countably closed. Countable closure is an extremely powerful condition, which we use throughout this argument but especially in 2.3.9; at the same time, countable closure also adds a layer of complexity to the proof of the retagging lemma, an important combinatorial property of Steel-type forcings, which usually follows from well-foundedness of the trees underlying the forcing conditions. In our case, the proof of the retagging lemma uses a much weaker "local well-foundedness" property. Additionally, there is an important shift in how we define the desired substructure of the full generic extension. In classical Steel forcing, the desired substructure is defined by first picking out specific elements of the generic extension — usually paths through a certain tree — and then closing under hyperarithmetic reducibility; the proof then continues by showing that every element of the resulting model depends only on "bounded" information about the generic. In our case, we start at the end, and simply consider the part of the generic extension depending on the generic in a "bounded" way. This is both clearer and more flexible a method than the standard approach; also, higher-type analogue of the hyperarithmetic sets — the so-called "hyperanalytic" sets — is more complicated to work with. See [60] for a definition of this analogue, as well as an account of some early difficulties faced in its study.

Constructing the model

The forcing we use in this section is the following:

Definition 12. *Let $\omega_2^* = \omega_2 \cup \{\infty\}$, ordered by taking the usual order on ω_2 and setting $\infty > x$ for all $x \in \omega_2^*$ (including $\infty > \infty$). $\mathbb{P}$ is the forcing consisting of all partial maps $p \colon \subseteq \mathbb{R}^{<\omega} \to \omega_2^* \times \omega_2^*$ satisfying the following conditions, ordered by reverse inclusion:*

- *$dom(p)$ is a countable subtree of $\mathbb{R}^{<\omega}$ with $p(\langle\rangle) \downarrow = (\infty, \infty)$ (the game starts with player Open moving, and no meaningful tags);*

- *$\sigma \in dom(p) \to [(|\sigma| = 2k+1 \wedge p(\sigma^-)_1 = p(\sigma)_1) \vee (|\sigma| = 2k \wedge p(\sigma^-)_0 = p(\sigma)_0)]$ (player Open is playing $p(\sigma)_0$, Closed is playing $p(\sigma)_1$, and on a given turn exactly one of these values changes);*

- *if $p_1(\sigma) = 0$, then no extension of σ is in the domain of p (if Closed ever hits 0, she loses); and*

- *$\sigma^\frown\langle a, b\rangle \in dom(p)$, $|\sigma| = 2k$, $\infty \neq p(\sigma)_0 > p(\sigma^\frown\langle a, b\rangle)_0 \to p(\sigma)_1 > p(\sigma^\frown\langle a, b\rangle)_1$ (as long as player Open has not just played an ∞, or failed to play less than her previous play, Closed's next play has to be less than her previous play).*

Note that the way this last condition is phrased allows $p(\sigma)_1$ to be anything when $p(\sigma)_0 = \infty$, for $|\sigma| = 2k$, since we have $\infty > \infty$. Also, if $|\sigma| = 2k$ and $p(\sigma^-)_1 = \infty$, then $p(\sigma)_1$ can be anything.

From this point on, we fix a filter $G \subseteq \mathbb{P}$ which is $\mathbb{P}$-generic over V.

The main difference between our forcing $\mathbb{P}$ and Steel forcing is that $\mathbb{P}$ is countably closed (recall 2.1.1). The immediate use of countable closure is that it lets us completely control the type-1 objects in our model; later, we will use countable closure in a more subtle way, to show that no well-orderings of reals of length $\geq \omega_2^V$ are in our model, even though such well-orderings will exist in the full generic extension (Lemma 2.3.9).

As with Steel forcing, we have a retagging notion:

Definition 13. *For $p, q \in \mathbb{P}$ and $\alpha \in \omega_2$, we say that q is an α-retagging of p, and write $p \approx_\alpha q$, if*

- $dom(p) = dom(q)$;

- *for $\sigma \in dom(p), i \in 2$ we have*

$$p(\sigma)_i < \alpha \to q(\sigma)_i = p(\sigma)_i$$

and

$$p(\sigma)_i \geq \alpha \to q(\sigma)_i \geq \alpha.$$

These retagging relations let us define the set of names which depend on the generic in a "bounded" way:

Definition 14. *Let ν be a name for a type-2 functional, that is, a map $\mathbb{R} \to \omega$, and suppose $\alpha \in \omega_2$. Then ν is α-stable if for all $a \in \mathbb{R}, k \in \omega$, we have*

$$\forall p, q \in \mathbb{P}[p \approx_\alpha q, p \Vdash \nu(a) = k \to q \Vdash \nu(a) = k.]$$

Finally, we can define our desired model:

Definition 15. *Fix G $\mathbb{P}$-generic over V. M is defined inductively to be the L^3-structure*

$$M = (\omega, \mathbb{R}, \{\nu[G] : \exists \alpha < \omega_2(\nu \text{ is } \alpha\text{-stable})\}).$$

The purpose of this section is to prove

Theorem 2.3.2. $M \models RCA_0^3 + \Delta_1^{\mathbb{R}}\text{-}Det + \neg\Sigma_1^{\mathbb{R}}\text{-}Det.$

We begin with two simple properties of the model M.

Definition 16. *T_G is the underlying tree of G; that is,*

$$T_G = \{\sigma \in \mathbb{R}^{<\omega} : \exists p \in G(\sigma \in dom(p))\}.$$

Fact 2.3.3.   *1. $\mathcal{P}(\omega^\omega) \cap V \subset M_2$.*

2. $T_G \in M_2$.

Proof. (1) follows from the fact that canonical names for sets in V do not depend on the poset $\mathbb{P}$, and are hence 0-stable. For (2), the only way to force $\sigma \notin T_G$ is to have some $p \in G$, $\tau \prec \sigma$ such that $p(\tau)_1 = 0$, so it follows that the canonical name for T_G is 1-stable. $\qquad\square$

We can now prove the first non-trivial fact about M: that it does not satisfy open determinacy for reals. Specifically, we will show that T_G, viewed as an open game, is undetermined in M.

The first step is the following:

Lemma 2.3.4. $V[G] \models T_G$ *is a win for Closed.*

Proof. By a straightforward density argument, if G is generic, then whenever $|\sigma| = 2k+1$, $p \in G$, and $p(\sigma)_1 = \infty$, there is some $q \in G$ and $a \in \mathbb{R}$ such that $q(\sigma^\smallfrown\langle a\rangle)_1 = \infty$. It follows that the strategy

$$\Pi(\sigma) = \text{ the } \leq_W\text{-least } a \text{ such that } \exists p \in G(p(\sigma^\smallfrown\langle a\rangle)_1 = \infty)$$

is winning for Closed. $\qquad\square$

The indeterminacy of T_G in M then follows from a two-part argument: strategies for Open can be defeated using 2.3.4 and the countable closure of $\mathbb{P}$, and stable strategies for Closed can be defeated by pulling the rug out from under her:

Lemma 2.3.5. $M \models \neg\Sigma_1^{\mathbb{R}}$-*Det.*

Proof. Consider the open game corresponding to T_G (in which player I is Open). Recall that T_G is in M and T_G is "really" a win for player Closed by 2.3.3 and 2.3.4, respectively; we claim that this game is undetermined in M.

Suppose Σ is a strategy for player Open in M. Consider the tree of game-states "allowed" by Σ:

$$A_\Sigma = \{\sigma \in T_G \colon \exists\Pi(\sigma \prec \Sigma \otimes \Pi)\}.$$

Since T_G is actually a win for Closed, the tree A_Σ must be ill-founded. Let $f \in V[G]$ be a path through T_G. Then $f \in V$, since $\mathbb{P}$ is countably closed and f can be coded by a single real. But then within V, we can construct a strategy Π which defeats Σ by playing along f:

$$\tau \prec f \rightarrow \Pi(\tau) = f(|\tau|), \quad \tau \not\prec f \rightarrow \Pi(\tau) = 0.$$

Since Π exists in V, $\Pi \in M_2$; so T_G is not a win for Open in M.

Now suppose Π is a strategy for player Closed in M, and suppose (towards a contradiction) that

$$p \Vdash \nu \text{ is a winning strategy in } T_G$$

where ν is an α-stable name for Π, $\alpha \in \omega_2$. We can find

- $q \leq p$,

- $a \in \mathbb{R} - \{c \colon \langle c \rangle \in dom(p)\}$,

- $b \in \mathbb{R}$, and

- $\beta > \alpha$

such that $\langle a, b \rangle \in dom(q)$, $q(\langle a \rangle) = (\beta, \infty)$, and $q \Vdash \nu(\langle a \rangle) = b$. Now since $q \leq p$ and p forces that Π wins, we must have $q(\langle a, b \rangle) = (\beta, \gamma)$ with $\gamma > \beta$; so $\gamma > \alpha$. But then we can find a $\hat{q} \approx_\alpha q$ such that $\hat{q} \leq p$ and $\hat{q}(\langle a, b \rangle) = (\hat{\beta}, \hat{\gamma})$ for some $\hat{\beta} > \hat{\gamma}$. But then $\hat{q}$ forces that there is some finite play extending $\langle a, b \rangle$ which is a win for Open; and since every possible finite play exists in M, this contradicts the assumption that ν was forced to be a name for a winning strategy.   □

To analyze M further, we require the analogue of Steel's retagging lemma for our forcing:

Lemma 2.3.6. *[Retagging] Suppose $\alpha < \omega_2$ has uncountable cofinality, $p \approx_\alpha q$, $r \leq q$, and $\gamma < \alpha$. Then there is some $\hat{r} \leq p$ with $\hat{r} \approx_\gamma r$.*

Proof. This is a straightforward combinatorial construction. It is worth noting, however, that Steel's retagging lemma is proved using the fact that conditions in Steel forcing are (essentially) well-founded trees. Of course, are conditions are not well-founded, so we must be slightly more subtle: the heart of this proof is the realization that conditions in $\mathbb{P}$, though not well-founded, are "locally well-founded" in a precise sense. Intuitively, when deciding how to tag a given node of r', we only need to look at a well-founded piece of the domain of r; using the ranks of these well-founded pieces as parameters gives us enough "room" for the natural construction to go through.

Formally, we proceed as follows. Since α has uncountable cofinality, we can find a $\tilde{\gamma}$ such that $\gamma < \tilde{\gamma} < \alpha$ and $\tilde{\gamma}$ is larger than every $r(\sigma)_i$ and $p(\tau)_i$ ($i \in \{0,1\}, \sigma, \tau \in \mathbb{R}^{<\omega}$) which is less than α.

For $\sigma \in dom(r) - dom(p)$, let

$$T_\sigma = \{\tau \colon \sigma^\frown \tau \in dom(r) \wedge \forall \rho \prec \tau (|\sigma^\frown \rho| \text{ odd } \to \infty \neq r(\sigma^\frown \rho^-)_0 > r(\sigma^\frown \rho)_0)\}$$

be the set of ways to extend σ within $dom(r)$ which according to r don't involve player Open restarting after σ, and note that for each $\sigma \in dom(r) - dom(p)$ the tree T_σ is well-founded. Also, let N be the set of nodes of $dom(r)$ that are new (that is, not in $dom(p)$) but don't follow any new restarts by player Open:

$$\{\sigma \in dom(r) - dom(p) \colon \forall \tau \preceq \sigma (\tau \in dom(r) - dom(p), |\tau| \text{ odd} \to r(\tau^-)_0 > r(\tau)_0 \neq \infty)\}.$$

The idea is that we really only need to focus on nodes in N: nodes in $dom(p)$ have already had their tags determined, and nodes not in $N \cup dom(p)$ will have no constraints on their tags coming from p at all, since they must follow a restart by Open. In order to define the value of $\hat{r}$ on some node σ in N, though, we need an upper bound on how large N is above

σ to keep from running out of ordinals prematurely; this is provided by taking the rank of T_σ.

Formally, we build the retagged condition as follows. Recalling that $V \models \mathsf{ZFC}$, fix in V a well-ordering of $\mathbb{R}^{<\omega}$, and via that ordering let $rk(S)$ be the rank of S for $S \subseteq \mathbb{R}^{<\omega}$ a well-founded tree. Then we define $\hat{r}$ as follows:

$$\hat{r}(\sigma) = \begin{cases} \uparrow, & \text{if } \sigma \notin dom(r), \\ p(\sigma), & \text{if } \sigma \in dom(p), \\ r(\sigma), & \text{if } \sigma \notin (N \cup dom(p)), \\ (\min\{\tilde{\gamma} + rk(T_\sigma), r(\sigma)_0\}, \hat{r}(\sigma^-)_1), & \text{if } \sigma \in N \text{ and } |\sigma| \text{ is odd}, \\ (\hat{r}(\sigma^-)_0, \min\{\tilde{\gamma} + rk(T_\sigma), r(\sigma)_1\}), & \text{if } \sigma \in N \text{ and } |\sigma| \text{ is even}. \end{cases}$$

It is readily checked that $\hat{r} \in \mathbb{P}$ — the assumption on $\tilde{\gamma}$ being used here to show that the coordinates of $\hat{r}$ are decreasing when the corresponding coordinates of r drop from $\geq \alpha$ to $< \alpha$ — and that $\hat{r} \leq p$ and $\hat{r} \approx_\gamma r$ (in fact, $\hat{r} \approx_{\tilde{\gamma}} r$). □

As a straightforward application of the retagging lemma, we can now show that M is a model of RCA_0^3:

Lemma 2.3.7. $M \models RCA_0^3$.

Proof. P^-, the extensionality axioms, the axioms defining $*$ and $\frown$, and comprehension for reals are all trivially satisfied, the last of these since M contains precisely the reals in V and $V \models \mathsf{ZFC}$. Only the comprehension scheme for type-2 functionals is nontrivial. We will prove that *arithmetic* comprehension for type-2 functionals holds in M, since this proof is no harder than the proof for Δ_1^0 comprehension.

Intuitively, we will show that functionals defined in an arithmetic way depend, value-by-value, on only countably many bits of information. From this, and the countable closure of our forcing, we will be able to find stable names for such functionals.

Let $\varphi(X^1, y^0)$ be an arithmetic (that is, Σ_n^0 for some $n \in \omega$; recall Definition 5) formula such that for each $a \in \mathbb{R}$ there is precisely one $k \in \omega$ with

$$M \models \varphi(a, k).$$

Since each natural number is definable, we can assume φ has no type-0 parameters. Let $(F_i)_{i<n}$ be the type-2 parameters used in φ, let $(s_j)_{j<m}$ be the type-1 parameters used in φ, and let ν_i be an α-stable name for F_i; since each F_i has a stable name, and there are only finitely many F_i, we can find some large enough $\alpha < \omega_2$ so that such names exist. Note that we can work directly with the s_j, as opposed to just dealing with their names, since our forcing adds no new reals.

For $a \in \mathbb{R}$, let $\mathcal{C}_a$ be any countable set of names for reals such that

- $\mathcal{C}_a$ contains a name for a and each s_j;

- whenever a name μ is in $\mathcal{C}_a$ and $k \in \omega$, $\mathcal{C}_a$ contains a name ν such that $\Vdash \nu = k^\frown \mu$; and

- whenever μ is in $\mathcal{C}_a$ and $i < n$, $\mathcal{C}_a$ contains a name μ' such that $\Vdash \mu' = \nu_i * \mu$.

Although we have not been completely precise in defining the sets $\mathcal{C}_a$, it is clear that the definition above is effective in the sense that a suitable set of sets of names $\{\mathcal{C}_a : a \in \mathbb{R}\}$ exists in the ground model, V.

The key fact about the $\mathcal{C}_a$ is that, by construction, they determine the truth value of the formula φ at a: that is, the truth value of $\varphi(a, k)$ depends only on the values of the F_i on the reals named by elements of $\mathcal{C}_a$. Formally,

$$\forall \mu \in \mathcal{C}_a \exists k \in \mathbb{R}[(p \Vdash \mu = k) \wedge (q \Vdash \mu = k)] \to \forall l \in \omega[(p \Vdash \varphi(a, l)) \iff (q \Vdash \varphi(a, l))].$$

Now let ν be a name for the functional defined by φ. We will show that ν is $(\alpha + \omega_1)$-stable. Let $r \in \mathbb{R}$ and $p, q \in \mathbb{P}$ such that $p \approx_{\alpha + \omega_1} q$ and $p \Vdash \nu(r) = k$. Let

$$D_r = \{t \in \mathbb{P} : \forall \mu \in \mathcal{C}_r \exists s \in \mathbb{R}(t \Vdash \mu = s)\}$$

be the set of conditions which decide the value of each name in $\mathcal{C}_r$. Since $\mathcal{C}_r$ is countable, and $\mathbb{P}$ is countably closed, the set D_r is dense. Now suppose towards contradiction that $q \not\Vdash \nu(r) = k$. Then since D_r is dense, we can find some $q' \leq q$ such that

$$q' \in D_r \quad \text{and} \quad q' \Vdash \nu(r) = l$$

for some natural $l \neq k$. By the retagging lemma, there is some $p' \leq p$ such that $p' \approx_\alpha q'$; but since each of the ν_i are α-stable, we must have

$$\forall i < n, t \in \mathbb{R}, \mu \in \mathcal{C}_r[(q' \Vdash \mu = t) \iff (p' \Vdash \mu = t)].$$

But since the truth value of $\varphi(r, k)$ depends only on the values of the $\mathcal{C}_r$, this contradicts the fact that $p' \leq p$ and $p \Vdash \nu(r) = k$. $\qquad\square$

Clopen determinacy in M

Showing that M satisfies clopen determinacy for reals, however, requires a more delicate proof. Intuitively, given a stable name for a clopen game, we ought to be able to inductively construct a stable name for a winning (quasi)strategy in that game by just iterating the retagging lemma in the right way. However, since the rank of a stable name is required to be $< \omega_2$, we cannot iterate the retagging lemma ω_2-many times, so we need all clopen games in M to have rank $< \omega_2$. This cannot be derived from the retagging lemma alone; instead, we need to look at particular subposets of $\mathbb{P}$:

Definition 17. *For $\alpha < \omega_2$, $\mathbb{P}_\alpha$ is the subposet of $\mathbb{P}$ defined by*

$$\mathbb{P}_\alpha = \{p \in \mathbb{P} : \forall \sigma \in dom(p), i \in 2(p(\sigma)_i < \alpha \vee p(\sigma)_i = \infty)\}.$$

Conditions in $\mathbb{P}_\alpha$ will turn out to satisfy a slightly stronger retagging property with respect to $\approx_\alpha$ — the projecting lemma, below — than conditions in general, and this will be used to prove that this forcing adds no stable well-orderings of reals longer than any in the ground model. Note that this is false for unstable well-orderings; in particular, forcing with $\mathbb{P}$ collapses ω_2 in the full generic extension.

Definition 18. *For $p \in \mathbb{P}$, $\alpha < \omega_2$, we let the α-projection of p,*

$$p^\alpha \colon dom(p) \to (\alpha \cup \{\infty\}) \times (\alpha \cup \{\infty\}),$$

be the map given by

$$\forall \sigma \in dom(p), i \in 2, \quad p^\alpha(\sigma)_i = \begin{cases} p(\sigma)_i & \text{if } p(\sigma)_i < \alpha \\ \infty & \text{otherwise.} \end{cases}$$

Lemma 2.3.8. *[Projecting] For all $p \in \mathbb{P}$, $\alpha < \omega_2$, we have:*

1. *$p^\alpha \in \mathbb{P}_\alpha$;*

2. *$p^\alpha \approx_\alpha p$;*

3. *$p \leq q \to p^\alpha \leq q^\alpha$;*

4. *$|\mathbb{P}_\alpha|^V = \aleph_1$; and*

5. *$\mathbb{P}_\alpha$ is countably closed.*

Proof. For (1), note that since we set $\infty > \infty$, the map

$$x \mapsto \begin{cases} x & \text{if } x < \alpha \\ \infty & \text{otherwise} \end{cases}$$

satisfies $x < y \iff \pi(x) < \pi(y)$. So as long as p is in $\mathbb{P}$, the projection p^α will not contain any illegal instances of the second coordinate increasing (which is the only possible obstacle to being a condition), and so will also be in $\mathbb{P}$ - and clearly if $p^\alpha \in \mathbb{P}$, then $p^\alpha \in \mathbb{P}_\alpha$.

(2) and (3) are immediate consequences of (1). Property (3) shows that we can allow $\gamma = \alpha$ in the retagging lemma above if p is assumed to be in $\mathbb{P}_\alpha$, and that we can take $\hat{r}$ to be in $\mathbb{P}_\alpha$ as well in that case.

For (4), note that elements of $\mathbb{P}_\alpha$ can be coded by countable subsets of $\mathbb{R} \times \omega_1$; the result then follows since $V \models \mathsf{CH}$.

Finally, for (5), let $(p_i)_{i \in \omega}$ be a sequence of conditions in $\mathbb{P}_\alpha$ with $p_{i+1} \leq p_i$. Then since $\mathbb{P}$ is countably closed, we have some $q \in \mathbb{P}$ with $q \leq p_i$ for all $i \in \omega$; but then $q^\alpha \in \mathbb{P}_\alpha$ by (1), and since each $p_i \in \mathbb{P}_\alpha$, we have $p_i^\alpha = p_i$ and hence $q^\alpha \leq p_i$ by (3). $\square$

This lemma helps provide us with explicit upper bounds on the lengths of type-2 well-orderings in M, via the construction below. We can use this result to provide a bound on the lengths of well-orderings in M, which in turn allows the induction necessary for showing clopen determinacy to go through.

Lemma 2.3.9. *[Bounding] Suppose ν is a stable name for a well-ordering of $\mathbb{R}$ (that is, $\Vdash \nu$ is a well-ordering of $\mathbb{R}$). Then there is some ordinal $\lambda < \omega_2$ such that*

$$\Vdash \nu \preccurlyeq \lambda.$$

That is, ω_2 is not collapsed in a stable way by forcing with $\mathbb{P}$.

Proof. Suppose ν is an α-stable name for a well-ordering of a set of reals. The proof takes place around the subposet $\mathbb{P}_\alpha$. For a sequence of reals $\bar{a} = \langle a_0, ..., a_n \rangle$ and a condition $p \in \mathbb{P}$, say that p is *adequate* for $\bar{a}$, and write $Ad(p, \bar{a})$, if p forces that $\bar{a}$ is a descending sequence through ν:

$$p \Vdash a_0 >_\nu ... >_\nu a_n.$$

Note that since ν is α-stable, p is adequate for $\bar{a}$ if and only if p^α is adequate for $\bar{a}$, by (2) of the previous lemma.

In order to bound the size of ν in any generic extension, we create in the ground model an approximation to the tree of descending sequences through ν, as follows:

$$\mathcal{T}_\nu = \{\langle (p_i, a_i) \rangle_{i < n} \colon p_i \in \mathbb{P}_\alpha \wedge \forall i < j < n (p_j \le p_i \wedge Ad(p_j, \langle a_0, ..., a_{i-1} \rangle))\}.$$

Elements of $\mathcal{T}_\nu$ are potential descending sequences, together with witnesses to their possibility. Now since ν is a name for a well-ordering, we must have that $\mathcal{T}_\nu$ is well-founded. Otherwise, we would have a sequence of condition/real pairs, $\langle (p_i, a_i) \rangle_{i \in \omega}$, which build an infinite descending sequence through ν, that is,

$$p_{i+1} \le p_i, \quad p_{i+2} \Vdash a_i >_\nu a_{i+1}.$$

But then a common strengthening $q \le p_i$, which exists by the countable closure of $\mathbb{P}_\alpha$, would create an infinite descending chain in ν; and this contradicts the assumption that $\Vdash \nu$ is well-founded.

Additionally, $|\mathcal{T}_\nu| = \aleph_1$, since $\mathcal{T}_\nu \subseteq (\mathbb{P}_\alpha \times \mathbb{R})^{<\omega}$ and $|\mathbb{P}_\alpha| = \aleph_1$ by Lemma 2.3.8(4). Fixing in V a bijection between ω_1 and $\mathcal{T}_\nu$ we can take the Kleene-Brouwer ordering $\mathcal{L}_\nu$ of $\mathcal{T}_\nu$. Since $\mathcal{T}_\nu$ is well-founded, this is a well-ordering; below, we will show that in fact

$$\Vdash \nu \preccurlyeq \mathcal{L}_\nu.$$

Let

$$K_\nu^G = \{\langle a_0, ..., a_n \rangle \colon a_0 >_{\nu[G]} ... >_{\nu[G]} a_n\}$$

be the tree of descending sequences through $\nu[G]$ in $V[G]$, and fix a well-ordering $\leq_W$ of $\mathbb{P}_\alpha$ in V. For $\bar{a} \in K_\nu^G$, we define a condition in $\mathbb{P}_\alpha$ by recursion as follows:

$$h(\bar{a}) = \text{ the } \leq_W\text{-least } p \in \mathbb{P}_\alpha \text{ such that } p \leq h(\bar{b}) \text{ for all } \bar{b} \prec \bar{a} \text{ and } Ad(p, \bar{a}).$$

(Note that by the previous lemma and the fact that ν is α-stable, h is defined for all $\bar{a} \in K_\nu^G$.) An embedding from K_ν^G into $\mathcal{T}_\nu$ can then be defined:

$$e \colon K_\nu^G \to \mathcal{T}_\nu \colon \langle a_i \rangle_{i<n} \mapsto \langle (h(\langle a_0, ..., a_i \rangle), a_i) \rangle_{i<n}.$$

It follows that $\nu[G] \preccurlyeq \mathcal{L}_\nu$, as desired. $\square$

Now we are finally ready to prove that M satisfies clopen determinacy. For simplicity, this proof is broken into three pieces. First, we show that the rank of a node in a clopen game can be determined in an α-stable way, for appropriately large α. Then we define a set which encodes the rank of nodes in a clopen game, as well as which player these nodes are winning for, and show that this set is similarly well-behaved. Finally, we use this to give stable names for winning strategies in clopen games which themselves have stable names — and this will suffice to show that $\Delta_1^{\mathbb{R}}$-Det holds in M. Unfortunately, the first two steps in this proof is exceedingly tedious, as we require more and more room to retag conditions, but the intuition is that of a straightforward induction.

Fix in V a well-ordering $\leq_W$ of $\mathbb{R}$. Using this well-ordering, we can define the rank $rk(T)$ of a well-founded tree $T \subset \mathbb{R}^{<\omega}$ in the usual way; and for $\sigma \in T$, we let $rk_T(\sigma) = rk(\{\tau : \sigma^\frown\tau \in T\})$. If ν is a name for a well-founded tree, then $rk(\nu)$ and $rk_\nu(\sigma)$ are the standard names for $rk(\nu[G])$ and $rk_{\nu[G]}(\sigma)$.

Lemma 2.3.10. *Let ν be a β-stable name for a well-founded subtree of $\mathbb{R}^{<\omega}$, $p \in \mathbb{P}$, $\gamma < \omega_2$, and $\sigma \in \mathbb{R}^{<\omega}$ such that*
$$p \Vdash rk_\nu(\sigma) = \gamma,$$

and suppose $q \approx_{\beta+\omega_1(\gamma 2+2)} p$; then

$$q \Vdash rk_\nu(\sigma) = \gamma.$$

Proof. By induction on γ. For $\gamma = 0$, suppose q is a counterexample to the claim; then we can find $r \leq q$ and $a \in \mathbb{R}$ such that

$$r \Vdash \sigma^\frown\langle a \rangle \in \nu.$$

Now by the retagging lemma, we can find some $\hat{r} \leq p$ such that $\hat{r} \approx_\beta r$. Since ν is β-stable, we have

$$r \Vdash \sigma^\frown\langle a \rangle \in \nu,$$

which contradicts the assumption on p.

Now suppose the lemma holds for all $\gamma < \theta$, and let $p \Vdash rk_\nu(\sigma) = \theta$; then

$$p \Vdash \forall a \in \mathbb{R}(\sigma^\frown\langle a \rangle \in \nu \to rk_\nu(\sigma^\frown\langle a \rangle) < \theta).$$

Suppose towards a contradiction that

$$q \approx_{\beta+\omega_1(\theta 2+2)} p \quad \text{and} \quad q \not\Vdash rk_\nu(\sigma) = \theta;$$

then there is some $r \leq q$, $a \in \mathbb{R}$ such that

$$r \Vdash \sigma^\frown\langle a \rangle \in \nu \wedge rk_\nu(\sigma^\frown\langle a \rangle) \geq \theta.$$

By the retagging lemma we get some $\hat{r} \leq p$ such that $\hat{r} \approx_{\beta+\omega_1(\theta 2+1)} r$, and since ν is β-stable we have $\hat{r} \Vdash \sigma^\frown\langle a \rangle \in \nu$. Since $\hat{r} \leq p$, and $p \Vdash rk_\nu(\sigma) = \theta$, we must be able to find some $\delta < \theta$ and $s \leq \hat{r}$ such that $s \Vdash rk_\nu(\sigma^\frown\langle a \rangle) = \delta$; using the retagging lemma a second time, we can get some $\hat{s} \leq r$ such that $\hat{s} \approx_{\beta+\omega_1(\delta 2+2)} s$. But then by the induction hypothesis $s \Vdash rk_\nu(\sigma^\frown\langle a \rangle) = \delta$, contradiction the assumption on r. $\square$

Definition 19. *If $T \subset \mathbb{R}^{<\omega}$ is a well-founded tree, thought of as a clopen game, a node σ on T is* safe *if the corresponding clopen game*

$$T^\sigma = \{\tau : \sigma^\frown\tau \in T\}$$

is a win for player I. For ν be a β-stable name for a well-founded subtree of $\mathbb{R}^{<\omega}$ with rank $< \alpha$ for some $\alpha < \omega_2$ (see Lemma 2.3.9), let Δ_ν be a name for the set which encodes rank and safety of nodes on ν:

$$\Delta_\nu[G] := \{(\sigma, \delta, i) : \sigma \in \nu[G] \quad \text{and} \quad rk_{\nu[G]}(\sigma) = \delta \quad \text{and} \quad i = [\![\sigma \text{ is safe in } \nu[G]]\!] \}.$$

We will show that Δ_ν is well-behaved, in the sense of stability, and use this to give a stable name for a winning strategy for ν.

Lemma 2.3.11. *Let ν be a β-stable name for a well-founded subtree of $\mathbb{R}^{<\omega}$ of rank $< \alpha$; and for simplicty, let $\kappa = \beta + \omega_1(\alpha 2 + 2)$. If $p \Vdash (\sigma, \delta, i) \in \Delta_\nu$, and $q \approx_{\kappa+\omega_1(\delta 2+2)} p$, then $q \Vdash (\sigma, \delta, i) \in \Delta_\nu$.*

Proof. Suppose not. Let δ be the least ordinal such that for some σ, i there are conditions p, q such that

- $q \approx_{\kappa+\omega_1(\delta 2+2)} p$,

- $p \Vdash (\sigma, \delta, i) \in \Delta_\nu$, and

- $q \not\Vdash (\sigma, \delta, i) \in \Delta_\nu$.

There are two cases. If $\delta = 0$, then we must have $i = 0$; since ν is β-stable, there can be no condition below q which adds a child of σ to ν (since then we can use the retagging lemma to force this below p, which already forces that σ is terminal in ν), and so $q \Vdash (\sigma, 0, 0) \in \Delta_\nu$.

So suppose $\delta > 0$. Since $p \Vdash rk_\nu(\sigma) = \delta$, by the previous lemma we have $q \Vdash rk_\nu(\sigma) = \delta$; so q just disagrees on whether σ is safe, which means we must be able to find some $r \leq q$ such that

$$r \Vdash (\sigma, \delta, 1 - i) \in \Delta_\nu.$$

By the retagging lemma we can find an $\hat{r} \leq p$ such that $\hat{r} \approx_{\kappa + \omega_1(\delta 2 + 1)} r$.

Now the proof breaks into two subcases based on whether $i = 0$ or $i = 1$. We treat the first case; the proofs are essentially identical.

We have $r \approx_{\kappa + \omega_1(\delta 2 + 1)} \hat{r}$ and $r \Vdash (\sigma, \delta, 1) \in \Delta_\nu$. Since r thinks σ is safe, r must think there is some immediate successor of σ which is unsafe. That is, we can find $s \leq r$, $\theta < \delta$, and $a \in \mathbb{R}$ such that $s \Vdash (\sigma^\frown\langle a \rangle, \theta, 0) \in \Delta_\nu$; by retagging again we can find

$$\hat{s} \leq \hat{r}, \hat{s} \approx_{\kappa + \omega_1(\theta 2 + 2)} s,$$

which by our assumption on δ means that

$$\hat{s} \Vdash \sigma^\frown\langle a \rangle \in \nu \text{ and is unsafe.}$$

But $\hat{s} \leq \hat{r} \leq p$ and p believes σ is unsafe, which means p believes σ has no safe extensions - a contradiction. $\qquad\square$

Finally, we are ready to show that ν is determined in M:

Corollary 2.3.12. *Let ν be a β-stable name for a well-founded subtree of $\mathbb{R}^{<\omega}$, viewed as a clopen game, with $rk(\nu) < \alpha < \omega_2$ for some limit ordinal α. Then there is an $(\beta + \omega_1(\alpha 4 + 5))$-stable name for a (type-2 functional coding a) winning strategy for ν.*

Proof. (Note that requiring α to be a limit is a benign hypothesis, as we can always make α larger if necessary; this assumption is just made to simplify some ordinal arithmetic below.) Recall that $\leq_W$ is a well-ordering of $\mathbb{R}$ in V. Let μ be a name for the type-2 functional which encodes the strategy picking out the $\leq_W$-least winning move at any given stage:

$$\mu[G](n^\frown\sigma) = \begin{cases} a(n) & \text{if } a \text{ is the } \leq_W\text{-least real such that } \exists \beta < \alpha[(\sigma^\frown\langle a \rangle, \beta, 0) \in \Delta_\nu], \\ 0 & \text{if no such real } s \text{ exists.} \end{cases}$$

For simplicity, we assume that $\Vdash$ "no string containing a '0' is on ν," so that there is no ambiguity in this definition. Clearly μ yields a winning strategy for whichever player wins ν.

All that remains to show is that μ is stable. Let $\lambda = (\beta + \omega_1(\alpha 4 + 5))$, fix σ and a, and let $p \approx_\lambda q$ are conditions in $\mathbb{P}$ such that $p \Vdash \mu(\sigma) = a$. We can find some $r \leq q$ and some

b such that $q' \Vdash \mu(\sigma) = b$; we'll show that $b = a$, and so we must have had $q \Vdash \mu(\sigma) = a$ already.

There are two cases:

Case 1: $a = 0$. Suppose towards a contradiction that $b \neq 0$. Since $a = 0$, we have $p \Vdash \forall \delta < \alpha, \forall b \in \mathbb{R}[(\sigma^\frown \langle b \rangle, \delta, 0) \notin \Delta_\nu]$. Let $s \leq r$ and $\delta < \alpha$ be such that $s \Vdash (\sigma^\frown \langle b \rangle, \delta, 0) \in \Delta_\nu$; by the retagging lemma, there is $p' \leq p$ with $p' \approx_{\beta + \omega_1(\alpha 4 + 4)} s$, which by Lemma 2.3.11 is impossible.

Case 2: $a \neq 0$. By identical logic as in the previous case, we must have $b \neq 0$; suppose towards contradiction that $b \neq a$. With two applications of the retagging lemma, we can find ordinals $\delta_0, \delta_1 < \alpha$ and conditions $p' \leq p$, $r' \leq r$ such that

- $p' \approx_{\beta + \omega_1(\alpha 4 + 2)} r'$,

- $p' \Vdash (\sigma^\frown \langle a \rangle, \delta_0, 0) \in \Delta_\nu$, and

- $r' \Vdash (\sigma^\frown \langle b \rangle, \delta_1, 0) \in \Delta_\nu$.

By Lemma 2.3.11, we have $r' \Vdash (\sigma^\frown \langle a \rangle, \delta_0, 0) \in \Delta_\nu$ and $p' \Vdash (\sigma^\frown \langle b \rangle, \delta_1, 0) \in \Delta_\nu$ as well. Also note that we have $p' \Vdash \mu(\sigma) = a$, $r' \Vdash \mu(\sigma) = b$, since $p' \leq p$ and $r' \leq r \leq q$. Now since $a \neq b$, either $a <_W b$ or $b <_W a$, and so either way we have a contradiction.

This completes the proof. $\square$

Since $M_1 = \mathbb{R}$, M computes well-foundedness of subtrees of $\mathbb{R}^{<\omega}$ correctly; so by 2.3.9, it then follows that every clopen game in M has a winning strategy in M. Together with 2.3.5 and 2.3.7, this completes the proof of Theorem 2.3.2.

2.4 Conclusion

In this paper, we have sought to understand how the passage to higher types affects mathematical constructions related to the system ATR_0; given both the sheer number of such constructions, and the relative youth of higher-order reverse mathematics, this remains necessarily incomplete. We close by mentioning three particular directions for further research we find most immediately compelling:

- Despite the analysis provided by this paper, there are still basic questions remaining unaddressed. It is unclear what is the relationship between $\Sigma_1^{\mathbb{R}}\text{-}\mathsf{Det}$ and $\Sigma_1^2\text{-}\mathsf{Sep}$. We suspect that these principles are incomparable, but separations at this level are unclear: for example, it is open even whether $\Sigma_1^{\mathbb{R}}\text{-}\mathsf{Det}$ implies the Π_2^2-comprehension principle for type-2 functionals, although the answer is almost certainly that it does not.

For that matter, in this paper we have focused entirely on the strengths of third-order theorems relative to other third-order theorems; their strength relative to second-order principles has been completely unexplored. For instance, it is entirely possible, albeit unlikely, that $\Delta_1^{\mathbb{R}}$-Det and $\Sigma_1^{\mathbb{R}}$-Det have the same second-order consequences.

- One interesting aspect of the shift to higher types we have not touched on at all is the extra structure available in higher-type versions of classical theorems. Given a Π_2^1 principle

$$\varphi \equiv \forall X^1 \exists Y^1 \theta(X, Y),$$

we can take its higher-type (so prima facie Π_2^2) analogue

$$\varphi^* \equiv \forall F^2 \exists G^2 \theta^*(F, G).$$

Now, individual reals are topologically uninteresting, but passing to a higher type changes the situation considerably. Specifically, we can consider *topologically restricted* versions of φ^*: given a pointclass Γ, let

$$\varphi^*[\Gamma] \equiv \forall F^2 \in \Gamma \exists G^2 \theta(F, G).$$

The relevant example is restricted forms of determinacy: the principles $\Delta_1^{\mathbb{R}}$-Det$[\Gamma]$ (resp., $\Sigma_1^{\mathbb{R}}$-Det$[\Gamma]$) assert determinacy for clopen (resp., open) games whose underlying trees when viewed as sets of reals are in Γ. In particular, the system $\Sigma_1^{\mathbb{R}}$-Det$[Open]$ is extremely weak, at least by the standards of higher-type determinacy theorems: it is equivalent over RCA_0^3 to the classical system ATR_0.

The techniques used in the proof of Theorem 2.3.2 are topologically badly behaved. In particular, they tell us nothing about the restricted versions $\Delta_1^{\mathbb{R}}$-Det$[\Gamma]$ and $\Sigma_1^{\mathbb{R}}$-Det$[\Gamma]$. With some work the argument of this paper might extend to showing that $\Delta_1^{\mathbb{R}}$-Det$[\Gamma] \not\vdash \Sigma_1^{\mathbb{R}}$-Det$[\Gamma]$ over RCA_0^3, for reasonably large pointclasses Γ, but not immediately; and certainly a detailed understanding of which restricted forms of open determinacy for reals are implied by which restricted forms of clopen determinacy will require substantially new ideas. This finer structure seems to allow a rich connection between classical descriptive set theory and higher reverse mathematics, and is worth investigating.

- Finally, there is a serious foundational question regarding the base theory for higher-order reverse mathematics. The language of higher types is a natural framework for reverse mathematics, as explained at the beginning of section 2.1; however, the specific base theory RCA_0^ω is not entirely justified from a computability-theoretic point of view. While proof-theoretically natural, it does not necessarily capture "computable higher-type mathematics." The most glaring exmaple of this concerns the Turing jump operator. In the theory RCA_0^ω, the existence of a functional corresponding to the jump operator

$$\mathfrak{J}^{1 \to 1} \colon f \mapsto f'$$

is conservative over ACA_0 ([31], Theorem 2.5). However, intuitively we can compute the ωth jump (and much more) of a given real by iterating $\mathfrak{J}$; thus, given a model M of RCA_0^ω, there may be algorithms using only parameters from M and effective operations which compute reals not in M. From a computability theoretic point of view, then, RCA_0^ω may be an unsatisfactorily weak base theory.

Of course, this discussion hinges on what, precisely, "computability" means for higher types. A convincing approach is given in [40], justified by arguments by Kleene and others (see especially [21]) similar in spirit to Turing's original informal argument. It is thus desirable — at least for higher-type reverse mathematics motivated by computability theory, as opposed to proof theory — to have a base theory corresponding to full Kleene recursion.[7] We will address these, and other, aspects of the base theory issue in a future paper. However, the search for the "right" base theory is very fertile mathematical ground, drawing on and responding to foundational ideas from proof theory, generalized recursion theory, and even set theory, and deserves attention from many corners and active debate.

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[7]It should be noted that the separations 2.2.15 and 2.3.2 in this paper do not suffer from the choice of base theory. This is because — by a straightforward, albeit tedious, induction — Kleene computability from a type-2 object satisfies the following *countable use condition*: if F is a given type-2 object, and φ_e^F is a type-2 object computed from F, then for each real r there is a countable set of reals C_r such that

$$\forall G^2(G \upharpoonright C_r = F \upharpoonright C_r \to \varphi_e^F(r) = \varphi_e^G(r)).$$

The models in 2.2.15 and 2.3.2 then satisfy this stronger theory by essentially the same argument as in 2.3.7.

Chapter 3

Higher reverse mathematics, 2/2

The work in this chapter originally appeared in an early draft of [68]; it remains work in progress.

3.1 The strength of RCA_0^3

In this section we review Kohlenbach's original base theory RCA_0^ω, and show that it is equivalent to our base theory RCA_0^3 in a precise sense.

Definition 20. *Let L^ω be the many-sorted language consisting of*

- *a sort t_σ for each finite type $\sigma \in FT$,*

- *application operators*
$$\cdot_{\sigma,\rho} \colon t_{\sigma \to \rho} \times t_\sigma \to t_\rho$$
  for all finite types,

- *the signature of arithmetic for the type-0 functionals, and*

- *equality predicates $=_\sigma$ for each $\sigma \in FT$.*

RCA_0^ω is the L^ω-theory consisting of the following axioms:

- *The ordered semiring axioms, P^-, for the type-0 objects, and extensionality axioms for all the finite types;*

- *the schemata*
$$\exists \Pi^{\sigma \to (\tau \to \tau)} \forall X^\sigma, Y^\tau (\Pi X Y = Y)$$
  and
$$\exists \Sigma^{(\sigma \to (\rho \to \tau)) \to ((\sigma \to \rho) \to (\sigma \to \tau))} \forall X^{\sigma \to (\rho \to \tau)}, Y^{\sigma \to \rho}, Z^\sigma (((\Sigma X) Y) Z = (X Z)(Y Z))$$
  defining the $\mathcal{K}$- and $\mathcal{S}$-combinators, respectively;

- *the axiom $\mathcal{R}_0$ asserting the existence of a primitive recursion functional, which for clarity we will always denote R_0:*

$$\exists R_0^{0 \to ((0 \to 1) \to (0 \to 0))} \forall x^0, g^{0 \to 1}, k^0 (R_0(x, g)(0) = x \wedge R_0(x, g)(k+1) = g(R_0(x, g)(k)), k);$$

  and

- *the choice scheme $\mathsf{qf\text{-}AC}^{1,0}$, which consists — for each quantifier-free formula $\varphi(X^1, y^0)$ in only the displayed free variables, containing no equality predicate of type $\neq 0$ — of the axiom*

$$\forall X^1 \exists y^0 \varphi(X, y) \Rightarrow \exists F^2 \forall X^1 \varphi(X, F(X)).$$

We will prove that RCA_0^ω is a conservative extension of RCA_0^3.[1]

To begin, we need some basic results about pairing higher-type objects.

Definition 21. *Fix a pairing operator $\langle \cdot, \cdot \rangle$ on natural numbers. In a slight abuse of notation, for reals x, y we let $\langle x, y \rangle$ be the real gotten by pairing x and y pointwise:*

$$\langle x, y \rangle \colon a \mapsto \langle x(a), y(a) \rangle.$$

For a finite sequence $\bar{c}$ of objects which are either all reals or all naturals, let $\langle \bar{c} \rangle$ be the usual coding of $\bar{c}$ by repeated use of the appropriate-type pairing operator $\langle \cdot, \cdot \rangle$, associating to the right.

For a a real, we let $\tilde{a} = a$; for $b \in \omega$, we let

$$\tilde{b} = \text{``}n \mapsto b.\text{''}$$

For $\bar{c}$ a finite sequence of objects which are each either reals or naturals, let

$$\langle \bar{c} \rangle_{\mathbb{R}} = \langle \tilde{c}_0, \tilde{c}_1, \ldots \rangle.$$

We write π_i for the projection map onto the ith coordinate; both RCA_0^ω and RCA_0^3 prove the existence of the relevant projection functionals. (In the case of RCA_0^3, a real-valued projection $\pi_i(\langle \overline{w} \rangle_{\mathbb{R}})$ is given by $F_i * (\langle \overline{w}_{\mathbb{R}} \rangle)$ for a certain functional F_i.) Throughout, we use the pairing functions and projection maps in formulas putatively in the language L^3 or L^ω, even though those symbols are not in either language, when it is clear that no expressive power is added.

We begin with the easier result: that RCA_0^3 is a subtheory of RCA_0^ω.

Lemma 3.1.1. *Whenever*

$$N = (N_\sigma)_{\sigma \in FT} \models \mathsf{RCA}_0^\omega,$$

we have $(N_0, N_1, N_2) \models \mathsf{RCA}_0^3$ (with the symbols $\frown$ and $*$ interpreted in the obvious way).

---

[1]Since the language of RCA_0^ω does not include the symbols "$*$" and "$\frown$," it is technically better to say that RCA_0^3 is a conservative extension of a subtheory of RCA_0^ω; however, since this will not be an issue, we ignore this point going forward.

Proof. The proof that N contains functionals of type $2 \to (1 \to 1)$ and $0 \to (1 \to 1)$ corresponding to $*$ and $\frown$, respectively, is not hard. It is somewhat tedious, however — for example, constructing a term corresponding to $\frown$ requires a definition by cases, and relies on the functional R_0 — and so we omit it.

Since RCA_0^ω proves P^-, extensionality, and Σ_1^0-induction,[2] it now suffices to show that RCA_0^ω proves the Δ_1^0 comprehension schemata for type 1 and 2 objects; since the former follows in turn from the latter and a bit of coding, we just need to prove Δ_1^0 comprehension for type 2 objects in RCA_0^ω.

Definition 22. *An L^3-formula $\varphi(\overline{x})$ with parameters from N and only type-1 and type-0 variables is* representable *if there is some type-2 functional $F_\varphi \in N$ such that*

$$N \models F_\varphi(\langle \overline{a} \rangle_{\mathbb{R}}) = 1 \iff \varphi(\overline{a}).$$

Sublemma 3.1.2. *All Σ_0^0 formulas are representable.*

Proof of claim. By induction on the number of bounded quantifiers. Note that the representable formulas are closed under negation, so we need only consider one kind of bounded quantifier.

The base case follows immediately from $\mathsf{qf\text{-}AC}^{1,0}$: if $\varphi(r)$ has no quantifiers, then

$$\psi(r,k) \equiv (k = 0 \wedge \neg\varphi(r)) \vee (k = 1 \wedge \varphi(r))$$

is a quantifier-free formula, and applying $\mathsf{qf\text{-}AC}^{1,0}$ to ψ yields a representing functional for φ.

For the induction step, it is enough to show that

$$\varphi(r^1) \equiv \exists x^0 < F^2(r)(G^2(\langle x, r \rangle) = 1)$$

is representable whenever $F, G \in N$ are type-2 parameters. The key tool here is the primitive recursion operator, R_0. Using R_0, we can define a functional H whose value on a real r is computed by starting with 0, cycling through all naturals less than $F(r)$ and incrementing each time we encounter a solution to $G(\langle -, r \rangle) = 1$; that is,

$$H(r) = 0 \iff \forall x < F(r), G(\langle x, r \rangle) = 0.$$

Rigorously, we let H be the type-2 functional defined by

$$\lambda r^1 . R_0(0, \lambda x^0 . (\pi_0(x) + G(\langle \pi_1(x), r \rangle)))(F(r)),$$

and note that H is clearly in N. Then the representing functional we desire is simply

$$I : r \mapsto [\![H(r) > 0]\!],$$

whose existence in N follows from applying the axiom $\mathsf{qf\text{-}AC}^{1,0}$ to the formula

$$\varphi(r,k) \equiv (k = 0 \wedge H(r) = 0) \vee (k = 1 \wedge H(r) > 0).$$

This finishes the proof of the sub-lemma. □

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[2]See [43].

Now we can prove the full Δ_1^0 comprehension scheme in RCA_0^ω, as follows. Let

$$\varphi(X^1, y^0) \equiv \exists z^0 \theta(X^1, y^0, z^0)$$

be a Σ_1^0 formula satisfying the hypothesis of the comprehension scheme, with $\theta \in \Sigma_0^0$. By the lemma, let $F \in N$ be the functional such that

$$F(\langle A^1, b^0, c^0 \rangle_{\mathbb{R}}) = 1 \iff N \models \theta(A, b, c).$$

Now consider the formula

$$\psi(X^1, w^0) \equiv w = \langle s, t \rangle \wedge F(\langle X, s, t \rangle_{\mathbb{R}}) = 1;$$

applying $\mathsf{qf\text{-}AC}^{1,0}$ to ψ yields a functional $G \in N$ of type 2, and φ is represented by the functional

$$X^1 \mapsto \pi_0(G(X)),$$

which is clearly in N. $\qquad\square$

The other half of the equivalence is a conservativity result:

Theorem 3.1.3. *RCA_0^ω is conservative over RCA_0^3, in the following sense: given any model $(M_0, M_1, M_2; ^\frown, *)$ of RCA_0^3, there is a model $N = (N_\sigma)_{\sigma \in FS}$ of RCA_0^ω with the same first-, second-, and third-order parts and corresponding application operators:*

$$M_0 = N_0, M_1 = N_1, M_2 = M_2, \quad \cdot_0^M = \cdot_{0,0}^N, \cdot_1^M = \cdot_{1,0}^N.$$

Proof. Let $M = (M_0, M_1, M_2) \models \mathsf{RCA}_0^3$. Define the set of *$\lambda$-terms over M* as follows:

Definition 23. *Fix $M \models \mathsf{RCA}_0^3$. The set of λ-terms over M is defined inductively as follows:*

- *If t is an L^3-term of type σ, then t is a λ-term of type σ.*

- *If t, s are λ-terms of types $\sigma \to \tau$ and σ respectively, then $((\mathsf{t})(\mathsf{s}))$ is a λ-term of type τ.*

- *If t is a λ-term of type σ, x is a variable of type ρ, and the expression "λx" does not occur in t, then $\lambda x.(\mathsf{t})$ is a λ-term of type $\rho \to \sigma$.*

Free and bound variables are defined as usual. A λ-term $\lambda x^\sigma.\theta(x)$ is intended to denote the map $a \mapsto \theta(a)$, and so the set of (appropriate equivalence classes of) λ-terms is meant to define a type-structure. For simplicity, we will refer to λ-terms over M simply as "λ-terms."

We let T be the set of all λ-terms. We say that $\mathsf{t} \in T$ is closed *if t has no free variables, that is, if each variable appearing in t is within the scope of a λ.*

Now, let $\equiv_{ext}$ be the smallest equivalence relation on T satisfying the following:

- *If a, b are L^3-terms and $M \models a = b$, then $a \equiv_{ext} b$.*

- *For all λ-terms $\mathfrak{s}$ of type σ in which the variable $x^σ$ does not appear and all λ-terms $\mathfrak{t}$, we have*

$$((\lambda x^σ.\mathfrak{t})(\mathfrak{s})) \equiv \mathfrak{t}[\mathfrak{s}/x].$$

  (Recall that $\rceil[a/b]$ denotes the expression gotten by replacing each occurence of a by b in $\rceil$.)

- *If $\mathfrak{t}, \mathfrak{s}$ are λ-terms of type (σ → τ) such that*

$$\mathfrak{t}(\mathfrak{a}) \equiv_{ext} \mathfrak{s}(\mathfrak{a})$$

  for all $\mathfrak{a}$ of type σ, then $\mathfrak{t} \equiv_{ext} \mathfrak{s}$.

- *If $\mathfrak{t}, \mathfrak{s}$ are λ-terms of type 1 and $\mathfrak{t}(a) \equiv_{ext} \mathfrak{s}(a)$ for all $a \in M_0$, then $\mathfrak{t} \equiv_{ext} \mathfrak{s}$.*

- *If $\mathfrak{t}, \mathfrak{s}$ are λ-terms of type 2 and $\mathfrak{t}(a) \equiv_{ext} \mathfrak{s}(a)$ for all $a \in M_1$, then $\mathfrak{t} \equiv_{ext} \mathfrak{s}$.*

Our desired model, N, will be built out of these $\equiv_{ext}$-classes. The first step towards an analysis of $\equiv_{ext}$-classes of λ-terms is the following classical result, here stated in a form slightly weaker than usual but more directly useful for our purposes:

Definition 24. *A λ-term $\mathfrak{t}$ is in* normal form *if it contains no subterm of the form $((\lambda x.\mathfrak{s})(\mathfrak{u}))$*

Theorem 3.1.4. *[Normal Form Theorem)] For each λ-term $\mathfrak{t}$, there is a λ-term $\mathfrak{s}$ in normal form such that*

$$\mathfrak{t} \equiv_{ext} \mathfrak{s}.$$

See, e.g., section 4.3 of [22] for a proof. The value of the normal form theorem is that it allows us to focus our attention on only nicely-behaved λ-terms. The relevant nice behavior is captured in the following lemma:

Lemma 3.1.5. *Let $\mathfrak{t}$ be a λ-term in normal form. Then:*

1. *Every subterm of $\mathfrak{t}$ is in normal form.*

2. *If $\mathfrak{t}$ has standard type (that is, type 0, 1, 2, etc.), then every subterm of $\mathfrak{t}$ also has standard type.*

3. *If $\mathfrak{t}$ is of type 0 or 1, then all bound variables in $\mathfrak{t}$ are of type 0.*

4. *If $\mathfrak{t}$ is of type 2, then $\mathfrak{t}$ contains at most one bound variable of type 1, and all other bound variables are of type 0.*

5. *If $\mathfrak{t}$ is of type 0, 1, or 2, then every subterm of $\mathfrak{t}$ is of type 0, 1, or 2.*

Proof. (1) is immediate. For (2), suppose that $\mathfrak{t}$ has standard type, and suppose $\mathfrak{t}$ has subterms of non-standard type. Note that, by induction, every λ-term of non-standard type has form either $\mathfrak{s}_0(\mathfrak{s}_1)$ or $\lambda x^\sigma.\mathfrak{s}_0$. Now let $\mathfrak{s}$ be the minimal leftmost λ-subterm of $\mathfrak{t}$ which has nonstandard type; that is, let $\mathfrak{s}$ be the unique subterm of $\mathfrak{t}$ such that (i) no subterm of $\mathfrak{t}$ containing any characters to the left of $\mathfrak{s}$ is of nonstandard type, and (ii) $\mathfrak{s}$ is the shortest subterm of $\mathfrak{t}$ with property (i). Then $\mathfrak{s}$ clearly cannot be of the form $\mathfrak{s}_0(\mathfrak{s}_1)$, since then $\mathfrak{s}_0$ would also need to be of nonstandard type and would then contradict the minimality of $\mathfrak{s}$ among leftmost nonstandard-type λ-terms; so $\mathfrak{s}$ is of the form $\lambda x^\sigma.\mathfrak{u}$. But since there is no λ-term of nonstandard type occuring to the left of $\mathfrak{s}$, and only λ-terms of nonstandard type can be applied to $\mathfrak{s}$ on the left, we must have that $\mathfrak{s}$ is bound by an application on the right. This immediately contradicts the normality of $\mathfrak{t}$, by (1).

For (3), suppose $\mathfrak{t}$ contains a bound variable y^σ of type $\sigma \neq 0$. The subterm in which y appears bound, $\lambda y.\mathfrak{s}$, then has type $\sigma \to \tau$ for some $\tau \in FT$. As in the proof of (2), let $\mathfrak{u}$ be the minimal leftmost subterm of $\mathfrak{t}$ which contains $\lambda y.\mathfrak{s}$ and has type $\neq 0$; since $\mathfrak{t}$ is in normal form, $\mathfrak{u}$ must be of the form

$$\lambda x_0^{\sigma_0}.\lambda x_1^{\sigma_1}...\lambda x_n^{\sigma_n}.\lambda y^\sigma.\mathfrak{s}.$$

Now since $\mathfrak{t}$ has type 0 or 1, $\mathfrak{u}$ must be on the left or right side of an application. Since $\mathfrak{t}$ is in normal form, $\mathfrak{u}$ must be on the right side of an application; that is, $\mathfrak{t}$ contains a subterm of the form $((\mathfrak{v})(\mathfrak{u}))$. But $\mathfrak{v}$ cannot be a parameter from M, since the type of $\mathfrak{u}$ has height at least 2, so $\mathfrak{v}$ is of the form $\lambda z.\mathfrak{w}$; so $((\mathfrak{v})(\mathfrak{u}))$ is not in normal form, contradicting (1).

(4) follows similarly to (3). Since $\mathfrak{t}$ has type 2, $\mathfrak{t}$ may have the form $\lambda y^1.\mathfrak{s}$, in which case it contains at least one bound variable of type 1; but then $\mathfrak{s}$ has type 0, and so by (3) y is the only bound variable of type $\neq 0$ occuring in $\mathfrak{t}$.

(5) follows the pattern of the previous parts. Towards contradiction, consider the minimal leftmost subterm $\mathfrak{s}$ of $\mathfrak{t}$ of type $n > 2$; then no functional can apply to $\mathfrak{s}$ on the left, and binding $\mathfrak{s}$ with a λ would result in a subterm of nonstandard type, contradicting (2), so $\mathfrak{s}$ must be on the left of an application. But by minimality $\mathfrak{s}$ has the form $\lambda x^1.\mathfrak{u}$, so this contradicts the normality of $\mathfrak{t}$. $\qquad\square$

Definition 25. *Let T_i^* be the set of all closed λ-terms of type i, for $i \in \{0, 1, 2\}$. For $i \in 3$, let $e_i \colon M_i \to T_i^*$ be the map*

$$e_i \colon a \mapsto [a]_{ext}.$$

Our immediate goal is to show that the e_i are bijections. The remainder of the conservativity proof will then follow easily. Injectivity is straightforward; to show surjectivity, we use the following construction:

Definition 26. *Fix a type-1 variable y^1. Let $\mathcal{T}$ be the set of ordered pairs $(\mathfrak{t}, S)$, where $\mathfrak{t}$ is a λ-term in normal form of type 0 or 1 containing no free variables of nonzero type besides possibly y, and $S = (x_0, ..., x_m)$ is a list of type-0 variables including all those occurring in $\mathfrak{t}$.*

For $(\mathfrak{t}, (x_i)_{i<n}) \in \mathcal{T}$, say that $F \in M_2$ codes $(\mathfrak{t}, (x_i)_{i<n})$ if

- *either* $\mathfrak{t}$ *is of type 0, and for all* $a_0, ..., a_{n-1} \in M_0, b \in M_1$, *we have*

$$F(a_0 {}^\frown ... {}^\frown a_{n-1} {}^\frown b) \equiv_{ext} \mathfrak{t}(a_0/x_0, ..., a_{n-1}/x_{n-1}, b/y);$$

- *or* $\mathfrak{t}$ *is of type 1, and for all* $a_0, ..., a_{n-1} \in M_0, b \in M_1$, *we have*

$$F * (a_0 {}^\frown ... {}^\frown a_{n-1} {}^\frown b) \equiv_{ext} \mathfrak{t}(a_0/x_0, ..., a_{n-1}/x_{n-1}, b/y).$$

The key lemma is the following:

Lemma 3.1.6. *Every* $(\mathfrak{t}, (x_i)_{i<n}) \in \mathcal{T}$ *is coded by some* $F \in M_2$.

Proof. We will prove the lemma by induction on the complexity of $\mathfrak{t}$; note that by Lemma 3.1.5, if $(\mathfrak{t}, (x_i)_{i<n}) \in \mathcal{T}$, then $(\mathfrak{t}, (x_i)_{i<n}) \in \mathcal{T}$ for all subterms $\mathfrak{s}$ of $\mathfrak{t}$, so an induction is possible. For this induction, we make the following abbreviations. For π a permutation of n for $n \in \omega$, we let R_π be the type-2 functional satisfying

$$\forall r \in M_1, i \in M_0 \colon (R_\pi * r)(i) = \begin{cases} r(\pi(i)) & \text{if } i < n, \\ r(i) & \text{if } n \leq i. \end{cases}$$

For $n \in \omega$, we let P_n be the type-2 functional satisfying

$$\forall r \in M_1, i \in M_0 \colon (P_n * r)(i) = r(n + i).$$

The existence of such functionals in M_2 is an easy consequence of the type-2 comprehension scheme. Finally, recall (5) the definition of the language L^3, as well as our convention that "L^3-term" means "L^3-term with parameters." The induction then proceeds as follows:

Fix $(\mathfrak{t}, (x_i)_{i<n}) \in \mathcal{T}$, and suppose that for all subterms of $\mathfrak{t}$, and all appropriate lists of variables, the result holds.

For $\mathfrak{t}$ an L^3-term, the comprehension scheme for type-2 functionals gives us the desired F immediately. If $\mathfrak{t}$ has type 0, apply comprehension to the formula

$$\Phi(u^1, v^0) \equiv v = \mathfrak{t}[u(0)/x_0, ..., u(n-1)/x_{n-1}, (P_n * u)/y],$$

and if $\mathfrak{t}$ has type 1, apply comprehension to the formula

$$\Psi(u^1, v^0) \equiv v = \mathfrak{t}[u(1)/x_0, ..., u(n)/x_{n-1}, (P_{n+1} * u)/y](u(0)).$$

Clearly the so-defined F codes $(\mathfrak{t}, (x_i)_{i<n})$.

If $\mathfrak{t}$ is of the form $\mathfrak{s}_0 + \mathfrak{s}_1$, by induction let F_0, F_1 represent $(\mathfrak{s}_0, (x_i)_{i<n})$ and $(\mathfrak{s}_1, (x_i)_{i<n})$ respectively. Then

$$F \colon u^1 \mapsto F_0(u) + F_1(u),$$

whose existence is again guaranteed by comprehension, clearly codes $(\mathfrak{t}, (x_i)_{i<n})$. (Multiplication and successor are handled identically.)

If $\mathfrak{t} = \lambda z^0.\mathfrak{s}$, note that $\mathfrak{t}$ necessarily has type 1. By induction let G represent $(\mathfrak{s}, (z, x_0, ..., x_{n-1}))$. Then G also codes $(\mathfrak{t}, (x_i)_{i<n})$.

If $\mathfrak{t} = \mathfrak{s_0}(\mathfrak{s_1})$, then by 3.1.5(5) $\mathfrak{s_0}$ has type either 1 or 2. If $\mathfrak{s_0}$ has type 1 (and so $\mathfrak{s_1}$ has type 0), let F_0 and F_1 represent $(\mathfrak{s_0}, (x_i)_{i<n})$ and $(\mathfrak{s_1}, (x_i)_{i<n})$ respectively. Then

$$F: u^1 \mapsto (F_0 * u)(F_1(u))$$

codes $(\mathfrak{t}, (x_i)_{i<n})$, and is guaranteed to exist by comprehension.

Finally, suppose $\mathfrak{t} = \mathfrak{s_0}(\mathfrak{s_1})$ and $\mathfrak{s_0}$ has type 2. Note that $\mathfrak{s_0}$ cannot be of the form $\lambda y^1.\mathfrak{u}$, since $\mathfrak{t}$ is in normal form. Similarly, if $\mathfrak{s_0}$ were of the form $\mathfrak{u}(\mathfrak{v})$, then $\mathfrak{u}$ would have to have non-standard type, and this would contradict 3.1.5(2). This leaves as the only possibility that $\mathfrak{s_0}$ is a single type-2 parameter, that is, $\mathfrak{t}$ is of the form

$$F(\mathfrak{s_1})$$

for some parameter $F \in M_2$. By induction, let G code $\mathfrak{s_1}$; then the functional H defined by

$$H: r \mapsto F(G * r)$$

is in M by comprehension, and codes $\mathfrak{t}$.

Since these are all the cases which can arise, this completes the induction. $\square$

As an immediate corollary, we get the surjectivity of e_2:

Corollary 3.1.7. *For each closed λ-term $\mathfrak{t}$ of type 2, there is some $F \in M_2$ such that $F \equiv_{ext} \mathfrak{t}$.*

Proof. By 3.1.4, we can assume $\mathfrak{t}$ is in normal form; then either $\mathfrak{t}$ is an L^3-term, in which case we are done, or $\mathfrak{t}$ has the form

$$\mathfrak{t} = \lambda y^1.\mathfrak{s}.$$

Applying Lemma 3.1.6 to $\mathfrak{s}$, we get an $F \in M_2$ such that for all $b \in M_1$, $F(b) = \mathfrak{s}[b/y]$; and so $\mathfrak{t} \equiv_{ext} F$. $\square$

This then passes to e_1 and e_0:

Corollary 3.1.8. *For each closed λ-term $\mathfrak{t}$ of type 0 (type 1), there is an $a \in M_0$ ($b \in M_1$) such that $\mathfrak{t} \equiv_{ext} a$ ($\mathfrak{t} \equiv_{ext} b$).*

Proof. For the type 1 case, let $\lambda x^0.\mathfrak{s}$ be a λ-term of type 1; now consider the type-2 term $\mathfrak{t'} = \lambda y^1.\mathfrak{s}[y(0)/x]$; by Corollary 4.14, $\mathfrak{t'} \equiv_{ext} F$ for some $F \in M_2$. But then consider the real $f: k^0 \mapsto F(k\char`\^\overline{0})$, which is in M_1 by comprehension; clearly $f \equiv_{ext} \mathfrak{t}$ since $F \equiv_{ext} \mathfrak{t'}$, so we are done.

The type 0 case follows similarly. Let $\mathfrak{t}$ be a λ-term of type 0, and consider the type-2 term $\mathfrak{t'} = \lambda y^1.\mathfrak{t}$. Taking $F \in M_2$ such that $F \equiv_{ext} \mathfrak{t'}$, we must have $F(\overline{0}) \equiv_{ext} \mathfrak{t}$; but $F(\overline{0}) \in M_0$. $\square$

Now consider the following L^ω-structure N_M, defined as follows:

- $N_\sigma = T_\sigma^*$.

- Application in N is defined as

$$\cdot_{\sigma,\rho} \colon ([\mathfrak{t}]_{ext}, [\mathfrak{s}]_{ext}) \mapsto [((\mathfrak{t})(\mathfrak{s}))]_{ext}.$$

- The arithmetic functions are transferred from M_0 to N_0 in the obvious way; for example,

$$+^N \colon ([\mathfrak{a}]_{ext}, [\mathfrak{b}]_{ext}) \mapsto [\mathfrak{a} + \mathfrak{b}]_{ext},$$

  etc.

- The relation $<^N$ is defined by

$$<^N = \{([a]_{ext}, [b]_{ext}) : a, b \in M_0, M \models a < b\}.$$

Lemma 3.1.9. $N_M \models \mathsf{RCA}_0^\omega$.

Proof. Extensionality and P^- for the type-0 functionals hold trivially. The Π and Σ combinators are easily expressed as λ-terms, and so the corresponding axioms are satisfied.

The primitive recursion axiom, $\mathcal{R}_0$, takes a bit more work to express as a λ-term. By Σ_1^0 induction in M, for any real $r^1 \in M_1$ and any naturals $a^0, b_0 \in M_0$, there is a (natural number coding a) primitive recursive derivation for

$$R_0(a, r)(b) = k$$

for a unique $k \in M_0$. This lets us apply the Δ_1^0 comprehension scheme for type-2 functionals, and so it follows that there is an $F \in M_2$ such that $F(a^{0\frown}b^{0\frown}r^1) = k^0$ if and only if there is a code for a primitive recursive derivation of $R_0(r, a)(b) = k$. Now consider the λ-term

$$\mathfrak{t} := \lambda x^0.\lambda r^1.\lambda y^0.(F(x^\frown y^\frown r));$$

by definition of F, $[\mathfrak{t}]_{ext}$ clearly witnesses the axiom $\mathcal{R}_0$ in N.

Finally, the choice principle $\mathsf{qf\text{-}AC}^{1,0}$ requires a bit of work: an appropriate quantifier-free formula Φ may contain high-type parameters, in which case the comprehension scheme in M does not directly apply. Instead, we have to essentially lower the types of the parameters, using the coding provided by Lemma 3.1.6.

Let $\Phi(y^1, x^0)$ be a quantifier-free formula in the displayed free variables, containing no occurrences of $=_\sigma$ for $\sigma \neq 0$. Then since Φ contains no higher-type equality predicates, every maximal term in Φ must have type 0. Let $\mathfrak{t}_0, ..., \mathfrak{t}_m$ be a list of these maximal terms, so that Φ is a Boolean combination of formulas of the form $\mathfrak{t}_i = \mathfrak{t}_j$ or $\mathfrak{t}_i < \mathfrak{t}_j$ for $i, j \leq m$.

Note that each $\mathfrak{t}_i$ has free variables from among $\{y^1, x^0\}$. Thus, by Lemma 3.1.6, we can find functionals $F_0, ..., F_m \in M_2$ such that $F_i(a^\frown b) \equiv_{ext} \mathfrak{t}_i[a/x, b/y]$ for every $a \in M_0, b \in M_1$.

Let $\hat{\Phi}$ be the quantifier-free formula gotten from Φ by replacing each t_i by $F_i(x^\frown y)$ for each $i \leq m$; then applying comprehension to $\hat{\Phi}$ yields a type-2 functional $G \in M_2$, and it is easily checked that

$$\forall b \in N_1(\Phi(b, [G]_{ext}(b))).$$

This completes the proof. □

Theorem 3.1.3 is thus proved. □

3.2　Choice principles

We end this chapter by examining the two choice principles which were implicitly used in the separation of clopen and open determinacy in Chapter 1:

- WO: "There is a well-ordering of the reals." This was used to pass from a tree $\subseteq \mathbb{R}^{<\omega}$ to its Kleene-Brouwer ordering.

- SF: "Every real-indexed family of nonempty sets of reals has a choice function." This was used to pass from a quasistrategy to a strategy.

Each of these is readily formulated in the language of RCA_0^3. We show that WO and SF are incomparable over RCA_0^3, and that $\mathsf{RCA}_0^3+\mathsf{WO}$ is a conservative extension of ACA_0.

We begin by showing that WO and SF are incomparable over RCA_0^3. One direction is easy:

Lemma 3.2.1. $\mathsf{RCA}_0^3+\mathsf{SF}\not\models\mathsf{WO}$.

Proof. Let C be the set of all continuous functions (in some model of ZFC) from $\mathbb{R}$ to ω; we will see that

$$\mathcal{C} = (\omega, \mathbb{R}, C) \models \mathsf{RCA}_0^3+ \mathsf{SF}+ \neg\mathsf{WO}.$$

Immediately, we have $\mathcal{C} \models \neg\mathsf{WO}$, since there is no continuous well-ordering of the reals; equally immediately, all axioms of RCA_0^3 except the Δ_1^0-comprehension scheme for type-2 objects hold in $\mathcal{C}$. To show that the comprehension scheme also holds, the key step is showing that any functional defined by a Δ_1^0-formula with continuous functionals as parameters is again continuous; this is an easy yet tedious induction on formula complexity, so we omit it.

Finally, we must show that $\mathcal{C}$ satisfies SF. To see this, suppose F is an instance of SF, that is, F is a type-2 functional such that for every real a, there is some real b such that $F(\langle a, b\rangle) = 1$. Now for $a \in \mathbb{R}$, let $\sigma_a \in \omega^{<\omega}$ be the lexicographically least string such that for all reals $\hat{a}, b$, if $\hat{a} \upharpoonright |\sigma_a| = a \upharpoonright |\sigma_a|$ and $\sigma_a \prec b$, then $F(\hat{a}, b) = 1$; such a string exists, since F is continuous and is an instance of SF. More importantly, the map $a \mapsto \sigma_a$ is continuous. From this, it follows that the function

$$g\colon \mathbb{R} \to \omega\colon r \mapsto \begin{cases} \sigma_{r^-}(r(0)) & \text{if } r(0) < |\sigma_{r^-}|, \\ 0 & \text{otherwise.} \end{cases}$$

is continuous and satisfies $F(a, g(a)) = 1$ for all reals a; so we are done.[3]   □

The other separation is more complicated. To the best of our knowledge, there is no natural model of $\mathsf{WO}+\neg\mathsf{SF}$ as there is of $\mathsf{SF}+\neg\mathsf{WO}$, so we have to build one. Towards this end, we begin with an appropriate model W of ZF in which there is no well-ordering of $\mathbb{R}$, and adjoin a well-ordering of $\mathbb{R}$ by forcing. Of course, this also means that in the generic extension, real-indexed sets of reals have selection functions, so the full model $(\omega, \mathbb{R}, \omega^{\mathbb{R}})^{W[G]}$ does not separate SF from WO. Instead, we look at the restricted model

$$(\omega, \mathbb{R}^V, \{\nu[G] : \nu \in N\})$$

for a class N of well-behaved names for type-2 functionals, chosen so that the generic well-ordering of $\mathbb{R}$ winds up in the model, but selection functions for real-indexed nonempty sets of reals do not in general. This is a variation on the basic idea of "symmetric submodels" which are used to produce models of ZF in which the axiom of choice fails in controlled ways (see [35], pp. 221-223). The proof of our main result in the following section is also a variation on this basic idea.

Theorem 3.2.2. $\mathsf{RCA}_0^3 + \mathsf{WO} \not\models \mathsf{SF}$.

Proof. We take as the ground model for our forcing argument some

$$W \models \mathsf{ZF} + \mathsf{DC} + \text{``The reals are not well-ordered;''}$$

the equiconsistency of this theory with ZF itself was proved by Feferman [16]. In W, let

$$\mathbb{P} = \{p : p \text{ is a countable partial injective function from } \mathbb{R} \text{ to } \omega_1\}.$$

First, note that $\mathbb{P}$ is indeed countably closed, so the reals in the generic extension are precisely the reals in W. We will use this implicitly in what follows. For X a set, we let $[X]^\omega$ denote the set of countable subsets of X. We now define, for $n \in \omega + 1$, the *n-countable names* inductively as follows:

- A *0-code* is a pair $c = (c_0, c_1)$, with $c_0 : \mathbb{R} \to [\mathbb{R}]^\omega$ and $c_1 : \mathbb{P} \to \omega$. If ν is a name for a map $\mathbb{R} \to \omega$ and c is a 0-code, we say that c is *good for ν* if

$$\forall p \in \mathbb{P}, a \in \mathbb{R}[c_0(a) \subseteq dom(p) \Rightarrow p \Vdash \nu(a) = c_1(p)].$$

  For $q \in \mathbb{P}$, 0-code c is *ν-good below q* if

$$\forall p \leq q \in \mathbb{P}, a \in \mathbb{R}[c_0(a) \subseteq dom(p) \Rightarrow p \Vdash \nu(a) = c_1(p)].$$

---

[3]Although this model does have the desired properties, it satisfies SF in a rather unsatisfying way; a more interesting separating model is given by the projective functions, under appropriate large cardinal axioms.

- Suppose that the set C_n of n-codes has already been defined, as well as the notions "ν-good" and ν-good below p" for n-codes. An $(n+1)$-*code* is a pair $c = (c_0, c_1)$ with $c_0 \colon \mathbb{R} \to [\mathbb{R}]^\omega$ and $c_1 \colon \mathbb{P} \to C_n$. If ν is a name for a map $\mathbb{R} \to \omega$ and c is an $(n+1)$-code, we say that c is ν-*good* if

$$\forall a \in \mathbb{R}, p \in \mathbb{P}[c_0(a) \subseteq dom(p) \Rightarrow c_1(p) \text{ is } \nu\text{-good below } p];$$

and for $q \in \mathbb{P}$, we say that c is ν-*good below* q if

$$\forall a \in \mathbb{R}, p \leq q \in \mathbb{P}[c_0(a) \subseteq dom(p) \Rightarrow c_1(p) \text{ is } \nu\text{-good below } p].$$

- A name ν for a map $\mathbb{R} \to \omega$ is n-*countable* if there is some n-code c which is ν-good.

- ν is ω-*countable* if ν is n-countable for some $n \in \omega$.

- Finally, a name μ for a map $\mathbb{R} \to \mathbb{R}$ is n- or ω-*countable* if the name ν for the map $r \mapsto \mu(r^-)(r(0))$ is n- or ω-countable.

The intuition is that the value of an ω-countable name is determined by conditions with large enough domains, mostly regardless of where the elements are sent. This could certainly be pushed past ω, but finite countability is enough for our purposes.

We can now define our target model: Letting G be $\mathbb{P}$-generic over W, we set

$$M = (\omega, \mathbb{R}^V = \mathbb{R}^{V[G]}, \{\nu[G] : \nu \text{ is } \omega\text{-countable}\}).$$

Finally, we can finish our proof by showing that $M \models \mathsf{RCA}_0^3 + \mathsf{WO} + \neg\mathsf{SF}$, as follows:

- $M \models \mathsf{WO}$. This is immediate: the canonical name for the well-ordering

$$\prec_G = \{(a, b) : G(a) < G(b)\}$$

(viewing G as a map $\mathbb{R} \to \omega_1$) is clearly 0-countable, since to determine whether $G(a) < G(b)$ just depends on $G(a)$ and $G(b)$.

- $M \not\models \mathsf{SF}$. Our counterexample is $\prec_G$, defined above. Let ν be n-countable. Fix $p \in \mathbb{P}$; we will find a real a and a condition $q \leq p$ such that

$$q \Vdash \neg(a \prec_G \nu(a)).$$

Let $a = \sup(dom(p)) + 1$, and let $\hat{p}$ be any condition $\leq p$ such that $a \in dom(\hat{p})$ and $G(a) - ran(\hat{p})$ is infinite. By induction on n, we can "fill in" the holes in $ran(\hat{p})$ with the reals required to decide $\nu(a)$; that is, we can find $\hat{q} \leq \hat{p}$ such that $\sup(ran(\hat{q})) = \hat{q}(a)$, $\hat{q} \Vdash \nu(a) = b$ for some real b, and $ran(\hat{q})$ is a *proper* subset of $\hat{q}(a)$. If $b \in dom(\hat{q})$, we take $q = \hat{q}$; if not, we let q be any extension of $\hat{q}$ with $\sup(ran(q)) = q(a)$ and $b \in dom(q)$. Either way, the result is a condition, q, such that $q \Vdash \nu(a) = b$ but $q(b) < q(a)$, so ν is not a selection function for $\prec_G$.

- $M \models \mathsf{RCA}_0^3$. All axioms except the Δ_1^0-comprehension scheme for type-2 objects are trivially satisfied, since M is an ω-model containing all the reals. To show that the comprehension scheme holds, note that by a straightforward induction, if ν_0 and ν_1 are m- and n-countable names for maps $\mathbb{R} \to \mathbb{R}$ then the name for their composition $\nu_0 \circ \nu_1$ is $(m + n)$-countable. From this, it immediately follows that any Δ_1^0 expression θ with ω-countable parameters defines an ω-countable functional: let m be such that all parameters in θ are m-countable, and let k be the length of θ; then the functional defined by θ is mk-countable.

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Chapter 4

Computable structures in generic extensions

The work in this chapter appeared as [41], and is joint with Julia Knight and Antonio Montalban; it appears here with their permission.

4.1 Introduction

In computable structure theory, one studies the complexity of structures using techniques from computability theory. Almost all of this work concerns countable structures; much less is known about the complexity of uncountable structures. However, the computability theory of uncountable structures has received more attention in the last few years. (See for instance the proceedings volume of the conference Effective Mathematics of the Uncountable [24].) One idea for studying the complexity of an uncountable structure that seems new is to consider what happens to the structure when its domain is made countable.

Before making this idea more concrete, we recall the notion of *Muchnik reducibility* between countable structures. This is the standard way in computable structure theory to say that one structure is more complicated than another, in the sense that it harder to compute.

Definition 4.1.1. *Given countable structures $\mathcal{A}$ and $\mathcal{B}$ we say that $\mathcal{A}$ is* Muchnik reducible *to $\mathcal{B}$, and we write $\mathcal{A} \leq_w \mathcal{B}$, if, from any copy of $\mathcal{B}$, we can compute a copy of $\mathcal{A}$.*

On its face, this notion is limited to countable structures. However, by examining generic extensions of the set-theoretic universe, V, we can extend it further:

Definition 4.1.2 (Schweber)**.** *For a pair of structures $\mathcal{A}$ and $\mathcal{B}$, not necessarily countable in V, we say that $\mathcal{A}$ is* generically Muchnik reducible *to $\mathcal{B}$, and we and write $\mathcal{A} \leq_w^* \mathcal{B}$, if for any generic extension $V[G]$ of the set theoretic universe V in which both structures are countable, we have*

$$V[G] \models \mathcal{A} \leq_w \mathcal{B}.$$

In Section 4.2, we will prove the basic properties of this reducibility. We will show that it coincides with Muchnik reducibility on countable structures; i.e., if $\mathcal{A}$ and $\mathcal{B}$ are countable, then $\mathcal{A} \leq_w \mathcal{B}$ if and only if $\mathcal{A} \leq_w^* \mathcal{B}$ (Corollary 4.2.5). More generally, we do not need to consider all the generic extensions that make $\mathcal{A}$ and $\mathcal{B}$ countable — this is a consequence of Shoenfield absoluteness (Theorem 4.2.1), a general principle about forcing. We will prove that for any two such generic extensions if $\mathcal{A} \leq_w \mathcal{B}$ holds in one, then it holds in the other (Lemma 4.2.3). This shows that generic Muchnik reducibility is a very absolute, and hence, natural, notion of computability-theoretic complexity.

We will also show that the equivalence $\equiv_w^*$, induced from the reducibility $\leq_w^*$, respects $\mathcal{L}_{\infty\omega}$-elementary equivalence. In Section 4.2, we will also exhibit some examples of this reducibility. For instance, we show that the countable structures generically Muchnik reducible to the linear order ω_1 are precisely those Muchnik reducible to some countable well-ordering, and we identify two natural structures — $\mathcal{W}$ and $\mathcal{R}$, the powerset of ω and the field of real numbers — each of which lies above every countale structure in the generic Muchnik reducibility. We show that $\mathcal{W} \leq_w^* \mathcal{R}$; recently Igusa and Knight [32] have shown that $\mathcal{R} \not\leq_w^* \mathcal{W}$, so these two structures are fundamentally different.

Closely related to generic reducibility is *generic presentability*. Intuitively, a countable structure $\mathcal{A}$ is *generically presentable* if there is some forcing notion $\mathbb{P}$ such that any forcing extension by $\mathbb{P}$ always contains a copy of $\mathcal{A}$. We will be interested in when generically presentable structures already have copies in the real universe. To be precise, we define:

Definition 27. *A* generically presentable structure *is a pair* $(\mathbb{P}, \nu)$, *where* $\mathbb{P}$ *is a forcing notion and* ν *is a* $\mathbb{P}$-name, *such that*

$$\Vdash_{\mathbb{P}} \text{``}\nu[G] \text{ is a structure with domain } \omega\text{''} \quad and \quad \Vdash_{\mathbb{P}\times\mathbb{P}} \text{``}\nu[G_0] \cong \nu[G_1].\text{''}$$

We say $(\mathbb{P}, \nu)$ *is* generically presented by $\mathbb{P}$. *When the forcing notion is clear from context, we will abbreviate "$(\mathbb{P}, \nu)$" by "ν," or abuse notation and use notation for classical structures ("$\mathcal{A}$," "$\mathcal{B}$," etc.) instead. If $(\mathbb{P}, \nu)$ is a generically presentable structure and $\mathbb{Q}$ is another forcing notion, we say $(\mathbb{P}, \nu)$ is* generically presentable by $\mathbb{Q}$ *if there is a generically presentable structure $(\mathbb{Q}, \mu)$ such that*

$$\Vdash_{\mathbb{P}\times\mathbb{Q}} \text{``}\nu[G_0] \cong \mu[G_1],\text{''}$$

and we will elide the distinction between such a pair of generically presentable structures when no confusion will result. Note that every actual structure may be thought of as a generically presented structure.

Remark 4.1.3. *After submitting, we learned that at around the same time, generic presentability was independently being studied by two other groups. Itay Kaplan and Saharon Shelah, addressing a question of Jindrich Zapletal, defined generic presentability and gave alternate proofs of our Theorems 4.3.14 and 4.3.18. Separately, Paul Larson [46] studied the Scott analysis of structures (see Section 4.3); since — roughly — a structure is generically presentable if and only if its Scott sentence exists, his work yields proofs of our Theorems 4.3.14 and 4.4.1.*

In Section 4.2 below we give an alternate approach to generic presentability, via countable models of set theory.

Remark 4.1.4. *Usually, a* copy *of a structure $\mathcal{A}$ is just a structure $\mathcal{B}$ which is isomorphic to $\mathcal{A}$. However, in this paper we will find ourselves studying structures which may not yet exist, or copies of structures in larger universes, so it is worth making precise what we mean by "copy." In this paper, we will primarily use the word "copy" in two ways:*

- *If $\mathcal{A}$ is a structure in V, we will often want to consider copies of $\mathcal{A}$ with domain ω. Although these will not exist in V if $\mathcal{A}$ is uncountable, they will exist in generic extensions; we will use the term "ω-copy" (of $\mathcal{A}$) to refer to a copy of $\mathcal{A}$ with domain ω, which may live in a generic extension of the universe.*

- *Separately, we will also want to ask whether a generically presentable structure is already present, up to isomorphism, in V. Towards that end, if $\mathcal{A} \in V$ is an actual structure and $\mathcal{B} = (\mathbb{P}, \nu)$ is a generically presentable structure, we say that $\mathcal{A}$ is a* copy *of $\mathcal{B}$ if $\Vdash_\mathbb{P} \mathcal{A} \cong \nu[G]$.*

Although these two uses of the word "copy" are somewhat at odds, we will be careful to make clear at each point what notion of "copy" is meant.

Convention 4.1.5. *For simplicity, as is standard in set theory, we will frequently abuse notation by referring to generic extensions $V[G]$ of the universe V as if they exist rather than writing everything out in terms of names.*

We will be interested in examining when a generically presentable structure already exists — that is, when it has a copy (or an ω-copy) in the ground model V. It is well-known that if a *set S* is in $V[G]$ for every $\mathbb{P}$-generic G, then S must belong to V already (Solovay [72], see Theorem 4.2.23 below for a precise statement and proof). However, the situation for isomorphic copies of a given structure is more complicated. There are cases in which the analogous fact is true, and there are cases in which it is not. This paper is devoted to analyzing this situation.

In particular, we are interested in the interaction between generic presentability and generic Muchnik reducibility. Generic Muchnik reducibility can be extended to generically presentable structures in a natural way — if $\mathcal{A}$ and $\mathcal{B}$ are generically presentable structures (or one is generically presentable and the other is an actual structure, or etc.), then $\mathcal{A} \leq_w^* \mathcal{B}$ if and only if, whenever $\mathbb{P}$ is a forcing presenting both $\mathcal{A}$ and $\mathcal{B}$, we have $\Vdash_\mathbb{P} \mathcal{A} \leq_w \mathcal{B}$. Now if $\mathcal{A} \leq_w^* \mathcal{B}$, then $\mathcal{B}$ contains all the information necessary to build $\mathcal{A}$ — up to a certain amount of genericity. To what extent is this genericity actually necessary? Ted Slaman formulated this question as follows:

Main Question 1 (Slaman)**.** *Suppose $\mathcal{A}$ is a generically presentable structure and $\mathcal{A} \leq_w^* \mathcal{B}$ for some actual structure $\mathcal{B} \in V$. Is there a copy of $\mathcal{A}$ in V?*

This can be rephrased as a question about inner models, as follows. Suppose $\mathcal{A} \leq_w^* \mathcal{B}$ with $\mathcal{B}$ in Gödel's constructible universe, L; must we have some $\mathcal{C} \cong \mathcal{A}$ (in V) with $\mathcal{C} \in L$? Note that if $\mathcal{A} \leq_w^* \mathcal{B}$ with $\mathcal{B} \in L$, then there is a generically presentable structure $(\mathbb{P}, \nu) \in L$ such that — in V — we have $\Vdash_\mathbb{P} \nu[G] \cong \mathcal{A}$, so this really is a special case of the previous question. Of course, L may be replaced with any inner model of ZFC, or even much less than ZFC.

We begin by studying the role of forcing-theoretic properties in generic presentability. We prove:

Theorem 4.1.6. *Any structure generically presentable by a forcing notion that does not make ω_2 countable has a copy (not necessarily with domain ω) in V.*

This theorem yields as a corollary a partial positive answer to Slaman's question.

Corollary 4.1.7. *If $\mathcal{A}$ is a generically presentable structure which is $\leq_w^* \mathcal{B}$ for some actual structure $\mathcal{B} \in V$ with cardinality $\leq \aleph_1$, then $\mathcal{A}$ has a copy in V. Alternately, from an inner model perspective, we have that if $\mathcal{B}$ lives in L and, within L, has size $\aleph_1^L$, then $\mathcal{A}$ has a copy in L.*

We also give a new proof of the following result of Harrington.

Theorem 4.1.8 (Harrington)**.** *If T is a counter-example to Vaught's conjecture, then it has models of arbitrarily high Scott rank below ω_2.*

On the other hand, these positive results cannot be extended much further: making ω_2 countable *always* introduces a structure with universe ω that does not have a copy in V, and that moreover has low complexity as measured by the generic Muchnik reducibility. This provides an exact dichotomy among structures generically presentable, and a negative answer to Slaman's question in general.

Theorem 4.1.9. *There is a generically presentable structure $\mathcal{M}$, which is presented by any notion of forcing that makes ω_2 countable, but which has no copy in V. Moreover, this $\mathcal{M}$ is generically Muchnik reducible to the ordering $(\omega_2, <)$.*

We close with a *structural* approach to the question: what properties ensure that generic presentability implies existence in the ground model? We show that this occurs at least when the structures involved are as "set-like" as possible, in the sense of being rigid — that is, having no non-trivial automorphisms. In Section 4.4, we show the following:

Theorem 4.1.10. *Suppose $\mathcal{A}$ is rigid and is generically presentable. Then there is an isomorphic copy of $\mathcal{A}$ already in V.*

4.2   Generic reducibility

Basic properties

The key result for analyzing generic presentability and generic reducibility is the Shoenfield Absoluteness Theorem (see [35]). The version we state below is slightly weaker than the actual theorem, but it is all we will need here:

Theorem 4.2.1 (Shoenfield). *Suppose φ is a Π_2^1 sentence, with real parameters. Then, for every forcing extension W of V, $V \models \varphi \iff W \models \varphi$.*

An easy fact about (countable) Muchnik reducibility of structures is the following.

Observation 4.2.2. *Basic facts about $\leq_w$ are invariant under forcing. Specifically, we have the following.*

1. *The relation "$\leq_w$" is Π_2^1.*

2. *For countable $\mathcal{A}$, the predicate "$\geq_w \mathcal{A}$" is Π_1^1 in a Scott sentence of $\mathcal{A}$.*

Together with Theorem 4.2.1, this implies that much of the theory of $\leq_w^*$ is absolute. In particular, we have the next lemma.

Lemma 4.2.3. *Fix arbitrary structures $\mathcal{M}, \mathcal{N}$ in V. If there is some generic extension in which $\mathcal{M}$ and $\mathcal{N}$ are countable and $\mathcal{M} \leq_w \mathcal{N}$, then $\mathcal{M} \leq_w^* \mathcal{N}$.*

Proof. Suppose otherwise. Then there must exist posets $\mathbb{P}_0$ and $\mathbb{P}_1$ in V such that forcing with either collapses both $\mathcal{M}$ and $\mathcal{N}$,

$$\Vdash_{\mathbb{P}_0} \mathcal{M} \leq_w \mathcal{N} \quad \text{and} \quad \Vdash_{\mathbb{P}_1} \mathcal{M} \not\leq_w \mathcal{N}.$$

Let $G = H_0 \times H_1$ be $\mathbb{P}_0 \times \mathbb{P}_1$-generic over V. Let $\mathcal{M}_0$ and $\mathcal{N}_0$ be reals in $V[H_0]$ coding copies of $\mathcal{M}$ and $\mathcal{N}$ with domain ω, and let $\mathcal{M}_1$ and $\mathcal{N}_1$ be reals in $V[H_1]$ coding copies of $\mathcal{M}$ and $\mathcal{N}$ with domain ω. Then, in $V[H_0]$, $\mathcal{M}_0 \leq_w \mathcal{N}_0$, while in $V[H_1]$, $\mathcal{M}_1 \not\leq_w \mathcal{N}_1$. By Shoenfield's absoluteness, this is still true in $V[H_0][H_1]$. This gives us a contradiction because, in $V[H_0][H_1]$, $\mathcal{M}_0$ is isomorphic to $\mathcal{M}_1$ and $\mathcal{N}_0$ to $\mathcal{N}_1$. $\square$

Remark 4.2.4. *For κ an infinite cardinal, the partial order $Col(\kappa, \omega)$ of finite sequences of ordinals $< \kappa$, ordered in the natural way, collapses κ to ω. This forcing notion is (a special case of) the Levy collapse. By 4.2.3, we may always assume that the forcings we consider are Levy collapses for κ at least as large as each structure under consideration.*

As an immediate corollary of Lemma 4.2.3, we get the following.

Corollary 4.2.5. *For structures $\mathcal{A}, \mathcal{B}$ countable in V, we have $\mathcal{A} \leq_w \mathcal{B}$ if and only if $\mathcal{A} \leq_w^* \mathcal{B}$.*

Potential isomorphism

Generic Muchnik reducibility also has strong connections with infinitary logic.

Definition 4.2.6. *Let $\mathcal{L}$ be a language; that is, a set of relation and operation symbols.*

- *$\mathcal{L}_{\infty\omega}$ is the collection of formulas obtained from the atomic $\mathcal{L}$-formulas by closing under arbitrary set-sized Boolean combinations and single instances of quantification. See [39] for a treatment of the basic properties of $\mathcal{L}_{\infty\omega}$.*

- *For structures $\mathcal{A}, \mathcal{B}$ of arbitrary cardinality, we say that $\mathcal{A}$ is $\mathcal{L}_{\infty\omega}$-elementary equivalent to $\mathcal{B}$, and we write $\mathcal{A} \equiv_{\infty\omega} \mathcal{B}$, if the structures satisfy the same $\mathcal{L}_{\infty\omega}$ sentences.*

There is a structural characterization of $\equiv_{\infty\omega}$, due to Carol Karp:

Definition 4.2.7. *Suppose I is a set of partial maps. We say that I has the back-and-forth property — equivalently, I is a back-and-forth system — if $\langle \emptyset, \emptyset \rangle \in I$ and for every $\langle \bar{a}, \bar{b} \rangle \in I$,*

1. *$\bar{a}$ and $\bar{b}$ satisfy the same atomic formulas,*

2. *for every $c \in \mathcal{A}$, there is $d \in \mathcal{B}$ such that $\langle \bar{a}c, \bar{b}d \rangle \in I$, and*

3. *for every $d \in \mathcal{B}$, there is $c \in \mathcal{A}$ such that $\langle \bar{a}c, \bar{b}d \rangle \in I$.*

An I with the back-and-forth property is called a back-and-forth system between $\mathcal{A}$ and $\mathcal{B}$.

Theorem 4.2.8 ([37])**.** *$\mathcal{A} \equiv_{\infty\omega} \mathcal{B}$ iff there is a back-and-forth system between $\mathcal{A}$ and $\mathcal{B}$.*

It is then not hard to see that for $\mathcal{A}$ and $\mathcal{B}$ countable, we have that $\mathcal{A} \equiv_{\infty\omega} \mathcal{B}$ if and only if $\mathcal{A} \cong \mathcal{B}$. Additionally, Karp's characterization shows that $\equiv_{\infty\omega}$ is absolute with respect to forcing. Clearly $\equiv_{\infty\omega}$ is upwards absolute. To show downwards absoluteness, let ν be a name for a back-and-forth system between $\mathcal{A}$ and $\mathcal{B}$ in some forcing extension by $\mathbb{P}$; then $I = \{\langle \bar{a}, \bar{b} \rangle : \exists p \in \mathbb{P}(p \Vdash \langle \bar{a}, \bar{b} \rangle \in \nu))\}$ is a back-and-forth system between $\mathcal{A}$ and $\mathcal{B}$. Thus, for possibly uncountable structures $\mathcal{A}$ and $\mathcal{B}$, we have $\mathcal{A} \equiv_{\infty\omega} \mathcal{B}$ iff $\mathcal{A} \cong \mathcal{B}$ once they are made countable:

Lemma 4.2.9 (Essentially Barwise [8])**.** *The following are equivalent:*

1. *$\mathcal{A} \equiv_{\infty\omega} \mathcal{B}$,*

2. *in every generic extension where $\mathcal{A}$ and $\mathcal{B}$ are countable, $\mathcal{A} \cong \mathcal{B}$,*

3. *in some generic extension where $\mathcal{A}$ and $\mathcal{B}$ are countable, $\mathcal{A} \cong \mathcal{B}$.*

As an immediate corollary, we have the following.

Corollary 4.2.10. *$\mathcal{A} \equiv_{\infty\omega} \mathcal{B}$ implies $\mathcal{A} \equiv_w^* \mathcal{B}$.*

This lets us connect $\equiv_{\infty\omega}$-equivalence and generic Muchnik reducibility in a strong way:

Lemma 4.2.11. *Let $\mathcal{A} \in V$ be a structure. The following are equivalent:*

1. *$\mathcal{A} \leq_w^* \mathcal{B}$ for some countable structure $\mathcal{B}$.*

2. *$\mathcal{A} \equiv_w^* \mathcal{B}$ for some countable structure $\mathcal{B}$.*

3. *$\mathcal{A} \equiv_{\infty\omega} \mathcal{B}$ for some countable structure $\mathcal{B}$.*

Proof. Clearly (3) implies (2) and (2) implies (1). To see that (1) implies (3), suppose $\mathcal{A} \leq_w^* \mathcal{C}$ for $\mathcal{C}$ countable, let C be an ω-copy of $\mathcal{C}$ in V, and let $V[G]$ be a generic extension in which $\mathcal{A}$ is countable. Then in $V[G]$, there is some index e such that, for the eth Turing machine Φ_e, $\Phi_e^C \cong \mathcal{A}$. This means that in V, Φ_e^C must be total, and so Φ_e^C is a copy of $\mathcal{A}$ which lives in V; that is, the structure $\mathcal{B} = \Phi_e^C$ is $\mathcal{L}_{\infty\omega}$-equivalent to $\mathcal{A}$. $\square$

Examples

We present below some examples of uncountable structures whose complexity in terms of $\leq_w^*$ we have been able to analyze.

Example 4.2.12. *Let $\mathcal{U}$ be the structure with domain $\omega \sqcup \mathcal{P}(\omega)$, with signature consisting of only the $\in$-relation on $\omega \times \mathcal{P}(\omega)$.*

Proposition 4.2.13. *$\mathcal{U} \equiv_w^* 0$, where 0 is the empty structure.*

Proof. We will show there is a computable structure $\mathcal{S}$ that is $\equiv_{\infty\omega}$-equivalent to $\mathcal{U}$. By the absoluteness of $\equiv_{\infty\omega}$, we will then have that in any generic extension that makes $\mathcal{U}$ countable, $\mathcal{S}$ and $\mathcal{U}$ are still $\equiv_{\infty\omega}$-equivalent, and, hence, isomorphic.

We note that the orbit in $\mathcal{U}$ of a tuple of sets $\bar{X}$ is determined by the cardinalities of the Boolean combinations of the sets X_i. To guarantee that we have a back-and-forth family of finite partial isomorphisms, we let $\mathcal{S}$ consist of ω together with a family of sets P having the following properties:

- P is an algebra of sets; i.e., it is closed under union, intersection, and complement,

- P includes all finite sets,

- if $X \in P$ is infinite, then there are disjoint $Y, Z \in P$, both infinite, such that $Y \cup Z = X$.

We can easily find such an $\mathcal{S}$ which is computable. We could, for example, take the family of primitive recursive sets. $\square$

Similarly, the field of complex numbers is essentially computable.

Example 4.2.14. *Let $\mathcal{C} = (\mathbb{C}; +, \times)$. This is $\equiv_{\infty\omega}$-equivalent to the algebraically closed field of countably infinite transcendence degree and characteristic zero. By a well-known result of Rabin [62], this has a computable copy. Then $\mathcal{C}$ has minimal complexity; that is, $\mathcal{C}$ has a computable copy in every generic extension in which it is countable.*

If we consider a variant of $\mathcal{U}$ in which the elements of ω have names, we reach the opposite end of the complexity spectrum:

Example 4.2.15. *Let $\mathcal{W}$ be the expansion of $\mathcal{U}$ to include the successor relation on ω. Then any ω-copy (4.1.4) of $\mathcal{W}$ computes every real in the ground model V, so given any countable $\mathcal{A} \in V$ we have $\mathcal{A} \leq_w^* \mathcal{W}$.*

The situation is the same with respect to the real numbers.

Example 4.2.16. *The field of real numbers $\mathcal{R} = (\mathbb{R}; +, \times)$ is, like $\mathcal{W}$, maximally complicated with respect to countable structures: for every countable structure $\mathcal{A}$, we have $\mathcal{A} \leq_w^* \mathcal{R}$. To see this, suppose $V[G]$ is a generic extension in which $\mathcal{R}$ has an ω-copy, R. First, note that the standard ordering $<_{\mathbb{R}}$ is defined both by an existential formula and by a universal formula, and so after collapse the corresponding relation on any ω-copy of R is computable relative to that copy.*

Now fix a real in the ground model $b \in \mathcal{R}$ and let $\hat{b} \in$ R be the corresponding element of the ω-copy. Since $<_R$ is computable from the atomic diagram of R and there is a uniform effective procedure for identifying each rational number in R, the cut corresponding to $\hat{b}$ is also computable from the atomic diagram of R; thus, every real in the ground model is computable from the atomic diagram of R. Since R was an arbitrary ω-copy of $\mathcal{R}$ in an arbitrary generic extension, it follows that $\mathcal{R} \geq_w^* \mathcal{A}$ for every countable $\mathcal{A} \in V$.

We would now like to compare the structures $\mathcal{R}$ and $\mathcal{W}$ under $\leq_w^*$. It is easy to show the following.

Proposition 4.2.17. $\mathcal{R} \geq_w^* \mathcal{W}$

Proof. We can use the elements of $\mathcal{R}$ in the interval $[0, 1)$ to enumerate the subsets of ω in V. To each real r in the interval, we associate the set A_r consisting of those n such that the nth term in the binary expansion of r is 1. Minimal care has to be taken for double binary representations: if we assume no binary expansion ends up in an infinite string of 1s, we then need to add those sets. $\square$

Recently, Igusa and Knight [32] have shown that this reduction is strict. However, this relies crucially on the fact that $\mathcal{R}$ is not very saturated (specifically, that $\mathcal{R}$ is Archimedean). For example, for an elementary extension $\mathcal{M}$ of $\mathcal{R}$ that is ω-saturated, we have $\mathcal{W} \geq_w^* \mathcal{M}$. More generally, we have the following.

Proposition 4.2.18. *Let $\mathcal{M}$ be an ω-saturated model of a complete elementary first order theory T. Then $\mathcal{W} \geq_w^* \mathcal{M}$.*

Proof. Assume without loss of generality that T is decidable — we may make this assumption since every real can be computed uniformly from the atomic diagram of a single parameter in $\mathcal{W}$ (specifically, itself). Macintyre and Marker [49] showed that for an enumeration $\mathcal{E}$ of a Scott set $\mathcal{S}$, and an elementary first order theory T in $\mathcal{S}$, $\mathcal{E}$ computes the complete diagram of a recursively saturated model of T realizing exactly the types in $\mathcal{S}$ that are consistent with T. After we collapse the cardinal so that $\mathcal{W}$ becomes countable, it computes an enumeration $\mathcal{E}$ of the Scott set $\mathcal{S}$ consisting of the subsets of ω in $\mathcal{W}$. Now, the theory of $\mathcal{M}$ is in $\mathcal{S}$, and the types realized in $\mathcal{M}$ are exactly those in $\mathcal{S}$ that are consistent with T. Then the result of Macintyre and Marker yields a recursively saturated model realizing exactly these types. This model is isomorphic to the collapse of $\mathcal{M}$. $\qquad\square$

Finally, uncountable well-orderings live strictly between the two extremes.

Example 4.2.19. *The linear order $\omega_1 = (\omega_1, <)$ computes — that is, is generically Muchnik above — precisely those countable structures which are Muchnik reducible to some countable well-ordering. One direction is obvious; in the other direction, suppose $\mathcal{A} \leq_w^* \omega_1$ is countable, and let $V[G]$ be a forcing extension in which ω_1 is countable. Then $V[G]$ satisfies "$\mathcal{A}$ is Muchnik reducible to a countable well-ordering," which is $\mathbf{\Sigma}_2^1$ via 4.2.2, and so already true in V by Shoenfield absoluteness.*

Proposition 4.2.20. $\mathcal{R} >_w^* \omega_1$ *and* $\mathcal{W} >_w^* \omega_1$, *strictly.*

Proof. To see that $\mathcal{R} \not\leq_w^* \omega_1$, fix some non-computable real $r \in \mathcal{R}$. Then the cut corresponding to r, and hence r itself, is computable in any ω-copy $\mathtt{R}$ of $\mathcal{R}$ in any generic extension since the ordering relation is both Σ_1 and Π_1. On the other hand, by a result of Richter [64], the only sets computable in all copies of a countable linear ordering are the computable sets, so in any generic extension in which ω_1 is countable there will be ω-copies of ω_1 whose atomic diagrams do not compute r.

To see that $\omega_1 \leq_w^* \mathcal{R}$, suppose $V[G]$ is a generic extension in which $\mathcal{R}$ is countable, and let $\mathtt{R} \in V[G]$ be a copy of $\mathcal{R}$ with domain ω. Now $\mathtt{R}$ computes an enumeration of the sets coded by the cuts in $\mathcal{R}$—the reals in V. Some of the reals code linear orderings. For an ordering r coded in $\mathtt{R}$, if r is not a well ordering, this is witnessed by a decreasing sequence d, also coded in $\mathtt{R}$. A countable well ordering in V is isomorphic to a countable ordinal, so it stays well ordered in $V[G]$. Using $\mathtt{R}''$, we get an ω-sequence of well-orderings: For $a \in \mathtt{R}$, we take the ordering coded by a, if this is a well ordering, and otherwise, we have a finite ordering. The result is an ordering of type ω_1^V. Now, we apply in $V[G]$ the theorem saying that, for any set X and any linear order $\mathcal{L}$, if X'' computes a copy of $\mathcal{L}$ then X computes a copy of $\omega \cdot \mathcal{L}$ ([3], Theorem 9.11). Since $\omega_1^V \cong \omega \cdot \omega_1^V$, our $\mathtt{R}$ computes a copy of ω_1^V.

The proof that $\mathcal{W} >_w^* \omega_1$ is identical. $\qquad\square$

Generic presentability

In this section we elaborate on the concept of generic presentability.

Recall the definition of generic presentability:

Definition 28. *A* generically presentable structure *is a pair* $(\mathbb{P}, \nu)$, *where* $\mathbb{P}$ *is a forcing notion and* ν *is a* $\mathbb{P}$-*name, such that*

$$\Vdash_{\mathbb{P}} \text{``} \nu[G] \text{ is a structure with domain } \omega \text{''} \quad \text{and} \quad \Vdash_{\mathbb{P} \times \mathbb{P}} \text{``} \nu[G_0] \cong \nu[G_1]. \text{''}$$

Remark 4.2.21. *It may be helpful to note that for any generically presentable structure* $\mathcal{A} = (\mathbb{P}, \nu)$, *there is some cardinal* λ *such that, for any* $\kappa \geq \lambda$, $\mathcal{A}$ *is presented by the Levy collapse* $Col(\kappa, \omega)$. *To see this, take* $\lambda = 2^{|\mathbb{P}|}$. *Then forcing with* $Col(\kappa, \omega)$ *for* $\kappa \geq \lambda$ *will in turn make the set of dense subsets of* $\mathbb{P}$ *countable, at which point we can construct a generic filter through* $\mathbb{P}$.

Although this is the definition we will use throughout this paper, it will be useful to note that it can be relativized to arbitrary models of ZFC:

Definition 29. *For a model* M *of* ZFC, *a* generically presentable structure over M *is a pair* $(\mathbb{P}, \nu) \in M$, *where* $\mathbb{P}$ *is a forcing notion in* M *and* ν *is a* $\mathbb{P}$-*name in* M, *such that*

$$M \models \quad [\Vdash_{\mathbb{P}} \text{``} \nu[G] \text{ is a structure with domain } \omega \text{''} \quad \text{and} \quad \Vdash_{\mathbb{P} \times \mathbb{P}} \text{``} \nu[G_0] \cong \nu[G_1] \text{''}].$$

The value of this relativization is the following. Often it is useful to imagine that the set-theoretic universe in which we work is actually countable, and lives inside a larger universe. For instance, this perspective means that the generic filters implicit in forcing arguments have to exist, reducing the need to talk about names directly. The following result shows that generic presentability has an equivalent and perhaps simpler definition if we adopt this viewpoint:

Proposition 4.2.22. *Suppose* M *is a countable transitive model of* ZFC *and* $\mathcal{A}$ *is a structure in the real universe,* V. *Then the following are equivalent:*

1. *There is a generically presented structure over* M, $(\mathbb{P}, \nu)$, *such that for every* G *which is* $\mathbb{P}$-*generic over* M *we have* $V \models \text{``} \nu[G] \cong \mathcal{A}. \text{''}$

2. *There is a forcing notion* $\mathbb{P}$ *in* M *such that, for every* G *which is* $\mathbb{P}$-*generic over* M, *we have a structure* $\mathcal{B} \in M[G]$ *such that* $V \models \text{``} \mathcal{A} \cong \mathcal{B}. \text{''}$

Proof. Clearly (1) implies (2). To show (2) implies (1), let $\mathbb{P}$ be a poset such that every generic extension of M by $\mathbb{P}$ contains a copy of $\mathcal{A}$ (as seen in V). Let G and H be mutually $\mathbb{P}$-generic over M, and let ν and μ be names for copies of $\mathcal{A}$ in $M[G]$ and $M[H]$, respectively. Since G and H are mutually generic, there is some $(p, q) \in G \times H$ such that $(p, q) \Vdash_{\mathbb{P} \times \mathbb{P}} \nu[G_0] \cong \mu[G_1]$. This means that $(p, p) \Vdash_{\mathbb{P} \times \mathbb{P}} \nu[G_0] \cong \nu[G_1]$ by considering the condition (p, q, p) in the *triple* product $\mathbb{P} \times \mathbb{P} \times \mathbb{P}$. Letting $\mathbb{Q} = \{q \in \mathbb{P} : q \leq p\}$ and $\hat{\nu}$ be the natural restriction of ν to $\mathbb{Q}$, we have that $(\mathbb{Q}, \hat{\nu})$ is a generically presented structure over M which is as desired. $\square$

An argument similar to the proof of 4.2.22 shows that the analogue of generic presentability for *sets* is trivial:

Theorem 4.2.23 (Solovay). *If a set is present in two mutually generic extensions, then it was already present in the ground model. Formally:*

- *(Internal version) If $\mathbb{P}$ is a forcing notion, $p, q \in \mathbb{P}$, and ν, μ are $\mathbb{P}$ names such that $(p, q) \Vdash_{\mathbb{P}^2} \nu[G_0] = \mu[G_1]$, then there is some set S such that $p \Vdash \nu[G] = S$.*

- *(External version) If M is a countable transitive model of ZFC, $\mathbb{P}$ is a forcing notion in M, G, H are mutually $\mathbb{P}$-generic filters over M, and $X \in M[G] \cap M[H]$, then $X \in M$.*

Proof. We will prove (2) only, since the proofs are similar. Suppose $M, \mathbb{P}, G, H, X$ are as hypothesized with X of minimal rank, so $X \subseteq A$ for some $A \in M$. Let $\mu, \nu \in M$ be $\mathbb{P}$-names such that $\nu[G] = \mu[H] = X$, and let $(p, q) \in G \times H$ be such that $(p, q) \Vdash_{\mathbb{P}^2} \nu[G_0] = \mu[G_1]$. Suppose towards contradiction there is some $a \in A$ such that $p \not\Vdash a \in \nu$ and $p \not\Vdash a \notin \nu$, and suppose $X(a) = i$; then picking $r \leq p$ with $r \Vdash \nu[G_0](a) = 1 - i$ and $s \leq q$ with $s \Vdash \mu[G_1](a) = i$ (which must exist since $X(a) = i$) yields absurdity. So p already decides membership of each element of A in X, and hence $X = \{a : p \Vdash a \in \nu\} \in M$. $\qquad\square$

Remark 4.2.24. *Note that this argument breaks down completely when we look at structures-up-to-isomorphism instead of sets-up-to-equality, essentially because structures, unlike sets, do not have unique representations. Broadly speaking, in order to adapt this argument to show that a generically presentable structure $\mathcal{A}$ has a copy in the ground model V we need to argue that there is a way to build up $\mathcal{A}$ explicitly from its small substructures. Although this is not always possible, the following model-theoretic perspective will be useful for producing positive results: to any structure we may associate a "Scott sentence," an infinitary first-order sentence which characterizes the structure and is defined in a suitably absolute fashion. Moreover, if a structure $\mathcal{A}$ is countable, then its Scott sentence provides a reasonably effective recipe for building a copy of $\mathcal{A}$ — specifically, since the satisfiability of $\mathcal{L}_{\omega_1\omega}$-sentences is absolute, if a model of set theory contains the Scott sentence of $\mathcal{A}$ as an $\mathcal{L}_{\omega_1\omega}$-sentence then that model contains a copy of $\mathcal{A}$ itself. Intuitively, we are motivated to claim that* a structure is generically presentable if and only if its Scott sentence already exists. *As written of course this is vague, but it is an important intuition for the arguments given in Section 3.*

4.3 Generic presentability and ω_2

In this section and the next, we address the question "when do generically presentable structures have copies in V?" This section focuses on a forcing-theoretic aspect of the question. For which forcing notions $\mathbb{P}$ do we have copies in V for all structures generically presentable by $\mathbb{P}$ with universe ω? Surprisingly, this is entirely determined by how $\mathbb{P}$ affects cardinals: ω_2 remains uncountable after forcing with $\mathbb{P}$ if and only if every structure generically presentable by $\mathbb{P}$ on ω has a copy in V.

As a consequence of proving the left-to-right direction of this result, we also give a new proof of the result due to Harrington that counterexamples to Vaught's conjecture must

have models of arbitrarily high Scott rank in ω_2. The right-to-left direction follows from a construction of Laskowski and Shelah [47].

Scott Analysis.

We begin by reviewing the *Scott analysis* of a structure. Scott [69] proved that for every countable structure $\mathcal{A}$, there is an infinitary sentence σ of $\mathcal{L}_{\omega_1\omega}$ such that the countable models of σ are exactly the isomorphic copies of $\mathcal{A}$. Such a sentence is called a *Scott sentence*.

There are several definitions of *Scott rank* in the literature (see, in particular, [7], [3], [53], [9], and [55]). The definitions give slightly different values. However, all of the definitions assign countable Scott ranks to countable structures. In general, the complexity of the Scott sentence is only a little greater than the Scott rank of the structure. If one definition assigns a computable ordinal Scott rank, then the other definitions do as well, and then there is a Scott sentence that is Σ_α, for some computable ordinal α. The definition that we give below is the one used by Sacks [65]. We begin by defining a family of definable expansions of $\mathcal{A}$.

Definition 4.3.1. *For each α, we define a fragment $\mathfrak{L}_\alpha^{\mathcal{A}}$ of $\mathcal{L}_{\infty,\omega}$ as follows:*

- *Let $\mathfrak{L}_0^{\mathcal{A}}$ consist of the elementary first order formulas.*

- *Given $\mathfrak{L}_\alpha^{\mathcal{A}}$, for each complete non-principal type $\Phi(x) \subseteq \mathfrak{L}_\alpha^{\mathcal{A}}$ realized in $\mathcal{A}$, add the formula $\bigwedge \Phi(x)$ to $\mathfrak{L}_{\alpha+1}^{\mathcal{A}}$, and close under finite logical connectives and first-order quantifiers.*

- *At limit levels, take unions.*

For each α there is a natural way to expand $\mathcal{A}$ to a $\mathfrak{L}_\alpha^{\mathcal{A}}$-structure $\mathcal{A}_\alpha$; we will abuse notation by omitting the subscript, since no confusion will arise.

At some step α, $\mathcal{A}$ becomes $\mathfrak{L}_\alpha^{\mathcal{A}}$-atomic, in the sense that all $\mathfrak{L}_\alpha^{\mathcal{A}}$-types are principal.

Definition 4.3.2. *The* Scott rank *of $\mathcal{A}$, $sr(\mathcal{A})$, is the least ordinal α such that $\mathcal{A}$ is an $\mathfrak{L}_\alpha^{\mathcal{A}}$-atomic structure.*

Lemma 4.3.3. *If $\mathcal{A}$ is generically presentable, then, for every ordinal β, $\mathfrak{L}_\beta^{\mathcal{A}} \in V$.*

Proof. First, let us remark that we can code the formulas in $\mathfrak{L}_\beta^{\mathcal{A}}$ by sets: for instance, we code an infinitary conjunction of formulas ψ_i by a pair, the first element being a code that means "conjunction" and the second element being the set of codes for the formulas ψ_i — say, by defining $code(\bigwedge_{i \in I} \psi_i(x)) = \langle 17, \{code(\psi_i(x)) : i \in I\}\rangle$. This is quite standard, so we let the reader fill in the details.

The one important detail is that we are not coding infinitary conjunctions using sequences of formulas, but using sets where the order of the formulas does not matter. The key point is that if we have different presentations of a structure $\mathcal{A}$, the types realized in each

presentation are the same as sets. We can then prove by induction on β, that $\mathfrak{L}_\beta^A$ is a set that is independent of the presentation of $\mathcal{A}$. Since $\mathcal{A}$ is generically presentable, say by a forcing notion $\mathbb{P}$, the language $\mathfrak{L}_\beta^A$ belongs to all $\mathbb{P}$-forcing extensions of V, and so by Solovay's Theorem 4.2.23, we get that $\mathfrak{L}_\beta^A$ belongs to V. $\qquad\square$

Definition 4.3.4. *Given a structure $\mathcal{A}$, let $\hat{\mathcal{L}}$ be the language containing a relation symbol for each formula in $\mathfrak{L}_{sr(\mathcal{A})}^{\mathcal{A}}$ (the* Morleyization *of $\mathfrak{L}_{sr(\mathcal{A})}^{\mathcal{A}}$), and let $\hat{\mathcal{A}}$ be the natural expansion of $\mathcal{A}$ to the language $\hat{\mathcal{L}}$. Note that if $\mathcal{A}$ is generically presentable, then $\hat{\mathcal{L}} \in V$ since $\mathfrak{L}_{sr(\mathcal{A})}^{\mathcal{A}} \in V$.*

Notice that $\hat{\mathcal{A}}$ is atomic in a very strong way: each $\hat{\mathcal{L}}$-type is generated by a quantifier-free $\hat{\mathcal{L}}$-formula.

Remark 4.3.5. *Throughout this section we will tacitly assume that $\mathcal{L}$ (and hence $\hat{\mathcal{L}}$ as well) is no larger than $\mathcal{A}$; that is, that the statement "$|\mathcal{L}| \leq |\mathcal{A}|$" is true in every forcing extension by $\mathbb{P}$ (where $\mathbb{P}$ is a forcing generically presenting $\mathcal{A}$). This assumption is used, for example, in 4.3.7 below, and is necessary for straightforwardly applying the facts about amalgamation we will prove in section 4.3. Note that this assumption holds for the vast majority of natural structures.*

Lemma 4.3.6. *If $\mathcal{A}$ is generically presentable, then so is $\hat{\mathcal{A}}$.*

Proof. We already showed that $\mathfrak{L}_{sr(\mathcal{A})}^{\mathcal{A}} \in V$, so $\hat{\mathcal{L}} \in V$. There is only one way to expand $\mathcal{A}$ to the $\hat{\mathcal{L}}$-structure $\hat{\mathcal{A}}$. So, $\hat{\mathcal{A}}$ has a presentation with domain ω in every generic extension of V where $\mathcal{A}$ does. $\qquad\square$

Proposition 4.3.7. *Suppose $\mathcal{A}$ is generically presentable by a forcing notion that does not collapse ω_1. Then $\mathcal{A}$ has a copy in V with domain ω.*

Proof. Intuitively, the Scott sentence of $\mathcal{A}$ must lie in V, and since ω_1 is not collapsed we can reconstruct $\mathcal{A}$ from its Scott sentence.

In detail, let $\mathbb{P}$ be a forcing notion that does not collapse ω_1, and for which $\mathcal{A}$ is generically presentable. Since $\hat{\mathcal{A}}$ is generically presentable, and $\hat{\mathcal{L}} \in V$, we have that $Th_{\hat{\mathcal{L}}}(\hat{\mathcal{A}})$, the $\hat{\mathcal{L}}$-theory of $\hat{\mathcal{A}}$, is a set of $\hat{\mathcal{L}}$ sentences that belongs to all $\mathbb{P}$-generic exensions. Thus, $Th_{\hat{\mathcal{L}}}(\hat{\mathcal{A}}) \in V$.

In all of these extensions, $\hat{\mathcal{L}}$ is countable (because $\mathcal{A}$ is), and, hence, $\hat{\mathcal{L}}$ cannot be uncountable in V. Otherwise, there would be an injection from ω_1 into $\hat{\mathcal{L}}$, and since $\mathbb{P}$ does not collapse ω_1, $\hat{\mathcal{L}}$ would stay uncountable in $V[G]$.

Now, in each of these generic extensions, $\hat{\mathcal{A}}$ is the unique countable atomic model of $Th_{\hat{\mathcal{L}}}(\hat{\mathcal{A}})$. The existence of such a model is a Σ_1^1 statement with $Th_{\hat{\mathcal{L}}}(\hat{\mathcal{A}})$ as parameter. By absoluteness, this must be true in V too, and by the uniqueness of $\hat{\mathcal{A}}$ in $V[G]$, this model must be isomorphic to $\hat{\mathcal{A}}$. $\qquad\square$

Keeping ω_2 uncountable.

We now turn to the *Fraïssé limit* construction, first used in [17]:

Definition 4.3.8. *Fix a relational language $\mathcal{L}$. For an $\mathcal{L}$-structure $\mathcal{B}$, we denote by $\mathbb{K}_{\mathcal{B}}$ the set of (structures isomorphic to) finite substructures of $\mathcal{B}$, and we call $\mathbb{K}_{\mathcal{B}}$ the* age *of $\mathcal{B}$. For $\mathbb{K}$ a set of finite structures and $\mathcal{A}$ a structure, we say that $\mathcal{A}$ is the* Fraïssé limit *of $\mathbb{K}$ if $\mathbb{K}_{\mathcal{A}} = \mathbb{K}$ and the set of isomorphisms between finite substructures of $\mathcal{A}$ has the back-and-forth property.*

Convention 4.3.9. *When we speak of the cardinality of an age, we will mean the cardinality of the age modulo isomorphism, that is, the number of isomorphism types of finite structures in that age.*

It is clear from the definition that if $\mathcal{A}$ and $\mathcal{B}$ are countable Fraïssé limits for the same age $\mathbb{K}$, then $\mathcal{A} \cong \mathcal{B}$. A given age may have non-isomorphic uncountable Fraïssé limits. For example, if $\mathbb{K}$ is the set of finite linear orderings, the Fraïssé limits are the dense linear orderings without endpoints, and there are many — in fact, $2^{\aleph_1}$ many, the most possible — non-isomorphic ones of cardinality $\aleph_1$.

Lemma 4.3.10. *If $\mathcal{A}$ is generically presentable, then $\mathbb{K}_{\mathcal{A}} \in V$.*

Proof. This follows from Solovay's Theorem 4.2.23: $\mathbb{K}_{\mathcal{A}}$ is a set of finite structures that is independent of the presentation of $\mathcal{A}$. $\qquad\square$

Using the same argument as in Proposition 4.3.7, we get a bound on the size of $\hat{\mathcal{L}}$ and $\mathbb{K}_{\hat{\mathcal{A}}}$:

Corollary 4.3.11. *If $\mathcal{A}$ is generically presentable by a forcing not making ω_2 countable, then $\hat{\mathcal{L}}$ and $\mathbb{K}_{\hat{\mathcal{A}}}$ have size $\leq \aleph_1$ in V.*

Fraïssé [17] proved that if $\mathbb{K}$ is a countable set of finite structures satisfying the Hereditary Property (HP), the Joint Embedding Property (JEP) and the Amalgamation Property (AP), then it has a Fraïssé limit (see 6.1 of [29] for definitions). The next lemma says that this is still the case when $\mathbb{K}$ has size $\aleph_1$. The earliest reference we know is Delhomme, Pouzet, Sagi, and Sauer [11, Corollary 2, p. 1378]. We give the proof because we want to make clear that the result does not automatically generalize to ages of size $> \aleph_1$; and indeed, we will see in the next subsection that there is an age of size $\aleph_2$ with no limit (Corollary 4.3.19).

Lemma 4.3.12. *Let $\mathbb{K}$ be a family of $\aleph_1$ finite structures on a relational language $\mathcal{L}$ of size $\leq \aleph_1$. If $\mathbb{K}$ has HP, JEP, and AP, then there is a Fraïssé limit $\mathcal{A}$ with age $\mathbb{K}$.*

Proof. The key is the following:

> *Claim:* Suppose we have embeddings $\mathcal{A} \to \mathcal{B}$ and $\mathcal{A} \to \mathcal{C}$ where $\mathcal{A}, \mathcal{B} \in \mathbb{K}$ and $\mathcal{C}$ is countable and its age is a subset of $\mathbb{K}$. Then there is a countable structure $\mathcal{D}$, whose age is a subset of $\mathbb{K}$, and which amalgamates these embeddings.

To prove the claim, write $\mathcal{C}$ as the union of an increasing sequence $\{\mathcal{C}_n : n \in \omega\}$ where each $\mathcal{C}_n \in \mathbb{K}$, and with $\mathcal{C}_0 = \mathcal{A}$. Let $\mathcal{D}_0 = \mathcal{B}$, and note that we have an embedding from $\mathcal{C}_0$ to $\mathcal{D}_0$. Given $\mathcal{D}_n$, by induction we will have an embedding from $\mathcal{C}_n$ into $\mathcal{D}_n$, and $\mathcal{D}_n$ will be an element of $\mathbb{K}$; and by definition we have an embedding from $\mathcal{C}_n$ into $\mathcal{C}_{n+1}$. We then form $\mathcal{D}_{n+1}$ by amalgamating the embeddings $\mathcal{C}_n \to \mathcal{C}_{n+1}$ and $\mathcal{C}_n \to \mathcal{D}_n$ within $\mathbb{K}$. The direct limit $\mathcal{D}$ of the $\mathcal{D}_i$ is the desired amalgamation.

Now we prove the lemma. Suppose $\mathbb{K}$ is such a family of finite structures. There is a sequence $(\mathcal{A}_\xi)_{\xi \in \omega_1}$ of structures such that:

- $\xi_0 < \xi_1 \to \mathcal{A}_{\xi_0} \subseteq \mathcal{A}_{\xi_1}$;

- each $\mathcal{A}_\xi$ is countable and its age is a subset of $\mathbb{K}$; and

- for every $\xi \in \omega_1$ and $\mathcal{B}, \mathcal{C} \in \mathbb{K}$ and every pair of embeddings $\mathcal{B} \to \mathcal{C}$ and $\mathcal{B} \to \mathcal{A}_\xi$, there is $\gamma > \xi$ and an embedding $\mathcal{C} \to \mathcal{A}_\gamma$ compatible with the inclusion $\mathcal{A}_\xi \to \mathcal{A}_\gamma$.

The union $\mathcal{A}$ of the $\mathcal{A}_\xi$ clearly has age $\mathbb{K}$. It is clear from the construction that the set of finite partial isomorphisms has the back-and-forth property. $\qquad\square$

Note that the limit $\mathcal{A}$ constructed above need not be $\aleph_1$-homogeneous or unique.

Corollary 4.3.13. *Let $\mathcal{B}$ be an $\mathcal{L}$-structure that lives in an extension of the universe and is ω-homogeneous in the sense that the family of isomorphisms between finite substructures has the back-and-forth property. Suppose $\mathcal{B}$ is generically presentable, and $|\mathbb{K}_\mathcal{B}|, |\mathcal{L}| \leq \aleph_1$ in V. Then in V there is a structure $\mathcal{L}_{\infty\omega}$-equivalent to $\mathcal{B}$.*

Proof. Since $\mathcal{B}$ is generically presentable, we have that $\mathbb{K}_\mathcal{B} \in V$ by Lemma 4.3.10. Since $\mathcal{B}$ is ω-homogeneous, $\mathbb{K}_\mathcal{B}$ has HP, JEP and AP in any model where $\mathcal{B}$ lives; since these properties are absolute, we conclude that $\mathbb{K}_\mathcal{B}$ has these properties in V. Since $|\mathbb{K}_\mathcal{B}| \leq \aleph_1$ and $|\mathcal{L}| \leq \aleph_1$ in V, by Lemma 4.3.12 we have that $\mathbb{K}_\mathcal{B}$ has a Fraïssé limit $\mathcal{F}$ in V. In a generic extension presenting $\mathcal{B}$, the age $\mathbb{K}_\mathcal{B}$—and, hence, the Fraïssé limit $\mathcal{F}$—will be countable. Then $\mathcal{F} \cong \mathcal{B}$, by the uniqueness of countable Fraïssé limits, so $\mathcal{F}$ is the required structure $\mathcal{L}_{\infty\omega}$-equivalent to $\mathcal{B}$ which lives in V. $\qquad\square$

We are now ready to prove the main positive result of this section.

Theorem 4.3.14. *Suppose $\mathcal{A}$ is generically presentable by a forcing notion $\mathbb{P}$ that does not make ω_2 countable. Then there is a copy of $\mathcal{A}$ in V, with cardinality at most $\aleph_1$ in V.*

More precisely, if $(\mathbb{P}, \nu)$ is a generically presentable structure and $\mathbb{P}$ does not make ω_2 countable, then there is a copy $\mathcal{B} \in V$ of $(\mathbb{P}, \nu)$, with $|\mathcal{B}| \leq \aleph_1$.

Proof. Let $\mathcal{L}$ be the language of $\mathcal{A}$. Since $\mathcal{A}$ is generically presentable, by Lemmas 4.3.3 and 4.3.6 we know that $\hat{\mathcal{L}}$ is in V and $\hat{\mathcal{A}}$ is generically presentable. Consider some generic extension $V[G]$ by a forcing which generically presents $\mathcal{A}$ and which does not make ω_2 countable. Using in $V[G]$ the fact that Scott ranks of countable structures are countable,

since ω_2^V is still uncountable in $V[G]$ the language $\hat{\mathcal{L}}$ has size $\leq \aleph_1$ in V. This implies that $\mathbb{K}_{\hat{\mathcal{A}}}$ is in V by Lemma 4.3.10 and has size $\leq \aleph_1$ in V by Corollary 4.3.11. Now, we can apply Corollary 4.3.13 to get a copy of $\hat{\mathcal{A}}$ which lives in V (of course, $\hat{\mathcal{A}}$ need not be countable in V). Intuitively, we now want to take the reduct of this copy to $\mathcal{L}$, but $\hat{\mathcal{L}}$ need not include $\mathcal{L}$ (for instance, if two $\mathcal{L}$-symbols have the same interpretation); instead, from $\hat{\mathcal{A}}$ we can now "decode" the correct interpretations of each of the symbols in $\mathcal{L}$, and thus produce a copy of $\mathcal{A}$ itself. $\qquad\square$

Note that Theorem 4.3.14 does not directly imply Proposition 4.3.7, since the latter concludes that the generically presentable structure in question has a *countable* copy in V.

We may apply Theorem 4.3.14 to prove the following.

Theorem 4.3.15 (Harrington, unpublished)**.** *If T is a counterexample to Vaught's Conjecture, then for each $\beta < \omega_2$, T has a model of size $\aleph_1$ with Scott rank $\geq \beta$.*

Proof. Recall that if T is a counterexample to Vaught's conjecture it has countable models of arbitrary Scott rank below ω_1. Being a counterexample to Vaught's conjecture is a Π_2^1 property ([58]; see also [65], Proposition 5.1) and hence absolute. Let $\mathbb{P} = \omega_1^{<\omega}$ be the usual Levy collapse of ω_1 and let G be $\mathbb{P}$-generic. Note that $\mathbb{P}$ is *homogeneous* in the following sense: the partial orders $\mathbb{P}$ and $\{q \in \mathbb{P} : q \leq p\}$ are isomorphic for any $p \in \mathbb{P}$. Since T is a counterexample to Vaught's conjecture, in $V[G]$ we have a countable model $\mathcal{B}$ of Scott rank $\alpha \geq \beta$. We claim that $\mathcal{B}$ is generically presentable over V by $\mathbb{P}$, that is, that there is a generically presentable structure $(\mathbb{P}, \mu)$ such that $V[G] \models \mu[G] \cong \mathcal{B}$. This would give us the claimed result: since $\mathbb{P}$ does not collapse ω_2, by Theorem 4.3.14, we would have a copy of $\mathcal{B}$ of size $\aleph_1$ in V, and since Scott rank is absolute, this copy is as wanted.

So fix a $\mathbb{P}$-generic G and a name $\nu \in V$ for a structure $\mathcal{B}$ in $V[G]$ which is (in $V[G]$) a countable model of T with Scott rank $\geq \alpha$, and suppose towards contradiction that $\mathcal{B}$ is *not* generically presentable in the sense of the previous paragraph. This will let us produce a size-continuum set of countable models of T of bounded Scott rank, thus contradicting the assumption that T is a counterexample to Vaught's conjecture.

We proceed as follows. First, suppose without loss of generality that

$$\Vdash \text{``}\nu[G] \models T \text{ and } sr(\nu[G]) = \alpha\text{;''}$$

we can make this assumption since *some* condition in G must force this, and $\mathbb{P}$ is homogeneous so we may take that condition to be the empty condition. We now claim that whenever H_0, H_1 are mutually $\mathbb{P}$-generic, we have $\nu[H_0] \not\cong \nu[H_1]$. This immediately follows from the assumption that $\mathcal{B}$ is not generically presentable — otherwise, taking an H_0, H_1 mutually generic with $\nu[H_0] \cong \nu[H_1]$, we must have some $p_i \in H_i$ such that $(p_0, p_1) \Vdash_{\mathbb{P} \times \mathbb{P}} \nu[G_0] \cong \nu[G_1]$. Since $\mathbb{P}$ is homogeneous we may assume $p_0 = p_1 = \emptyset$; but then this contradicts our assumption that $\mathcal{B}$ is not generically presentable.

So we have that mutually generic filters through $\mathbb{P}$ yield non-isomorphic models of T of Scott rank α. Now, consider a forcing notion $\mathbb{Q}$ that adds perfectly many mutually $\mathbb{P}$-generics. (This is quite standard: for instance let $\mathbb{Q}$ be the set of finite partial maps from $2^{<\omega}$ to $\omega_1^{<\omega}$ and then obtain the $\mathbb{P}$ generics by concatenating the $\omega_1^{<\omega}$-strings along each path in 2^ω.) After forcing with $\mathbb{Q}$, by the arguments above we obtain continuum many pairwise-nonisomorphic countable models of T, each of Scott rank $\alpha < \omega_1$. Since being a counterexample to Vaught's conjecture is absolute, this is a contradiction. $\square$

Remark 4.3.16. *Recently, Baldwin, S.-D. Friedman, Koerwien, and Laskowski [4] have given a new proof of Harrington's result using similar genericity arguments; their proof uses a generic version of the Morley tree, which they show is invariant across forcing extensions.*

Finally, we can use Theorem 4.3.14 to give a partial positive answer to Slaman's question:

Corollary 4.3.17. *Suppose $\mathcal{A}$ is a generically presentable structure with $\mathcal{A} \leq_w^* \mathcal{B}$ for some $\mathcal{B} \in V$ with cardinality $\leq \aleph_1$. Then $\mathcal{A}$ has a copy in V.*

Proof. Let $\mathbb{P}$ be a forcing notion that collapses ω_1 while keeping ω_2 uncountable, such as $\mathbb{P} = \omega_1^{<\omega}$. Let $V[G]$ be a generic extension by $\mathbb{P}$. Then $\mathcal{B}$ is countable in $V[G]$, and, a fortiori, there is a copy of $\mathcal{A}$ in $V[G]$. It follows that $\mathcal{A}$ is $\mathbb{P}$-generically presentable. Then by Theorem 4.3.14, there is a copy of $\mathcal{A}$ in V. $\square$

Collapsing ω_2 to ω.

We close this section by presenting a strong negative result, coming from a construction due to Shelah and Laskowski [47]. Throughout the rest of this section, we abbreviate the linear order $(\omega_2, <)$ by "ω_2."

Theorem 4.3.18. *There is a structure $\mathcal{A}$, generically presentable by any forcing making ω_2 countable, but with no copy in V.*

Proof. Laskowski and Shelah [47] gave an example of an elementary first order theory T, in a countable language, such that:

1. The language has a sort W such that, for every model $\mathcal{M}$ of T and every subset $A \subseteq \mathsf{W}^{\mathcal{M}}$, $T(A)$ has an atomic model if and only if $|A| \leq \aleph_1$.

2. T has a countable model $\mathcal{M}_0$ such that $\mathsf{W}^{\mathcal{M}_0}$ is totally indiscernible in the sense that any permutation of $\mathsf{W}^{\mathcal{M}_0}$ extends to an automorphism of $\mathcal{M}_0$. Furthermore, $\mathcal{M}_0$ is atomic over $\mathsf{W}^{\mathcal{M}_0}$.

For $\mathcal{C}$ a countable structure, let $\mathcal{M}_{\mathcal{C}}$ be the two-sorted structure with one sort corresponding to a copy of $\mathcal{C}$, one sort corresponding to a copy of $\mathcal{M}_0$, and with a function symbol f providing a bijection between $\mathcal{C}$ and $\mathsf{W}_0^{\mathcal{M}}$. Since the elements of $\mathsf{W}^{\mathcal{M}_0}$ are totally indiscernible, any two choices of f yield isomorphic structures, so $\mathcal{M}_{\mathcal{C}}$ is well-defined.

Now consider the "structure" $\mathcal{M}_{\omega_2}$ which lives in any extension of the universe where ω_2 is countable. Thus, $\mathcal{M}_{\omega_2}$ is generically presentable by $Col(\omega_2, \omega)$. However, there is no copy of $\mathcal{M}_{\omega_2}$ in V: Since if the first sort is really ω_2, of size $\aleph_2$, then in the second sort, the predicate W has size $\aleph_2$. But, by the assumption on $\mathcal{M}_0$, $\mathcal{M}_{\mathcal{C}}$ is always atomic over $\mathcal{C}$ (a fact that is absolute), and by the assumption on T, $T(\mathsf{W}^{\mathcal{M}_{\omega_2}})$ has no atomic models. $\square$

The structure of Laskowski and Shelah also provides a counterexample to a natural extension of Lemma 4.3.12.

Corollary 4.3.19. *There is an age S of size $\aleph_2$ with the Hereditary, Joint Embedding, and Amalgamation properties but for which there is no Fraïssé limit.*

Proof. Consider the theory $T(A) = Th(\mathcal{M}_0, a_{a \in A})$, where $A = A^{\mathcal{M}}$ has size $\aleph_2$. The principal types are dense, but $T(A)$ has no atomic model. We add predicate symbols for the principal types. For $B \subseteq A$ of size up to $\aleph_1$, there is an atomic model of the corresponding theory $T(B) = Th(\mathcal{M}_0, a_{a \in B})$. Let K consist of the finite substructures of the atomic models of the theories $T(B)$. In total, what we have is appropriate to be the age for an atomic model of $T(A)$. That is, we have the Hereditary, Joint Embedding, and Amalgamation properties (essentially [47], pg. 3). However, any Fraïssé limit of S would yield an atomic model of $T(A)$, so the Fraïssé limit cannot exist. $\square$

4.4   Generically presentable rigid structures

In the previous section, we gave a complete characterization of those posets $\mathbb{P}$ with the property that every structure generically presentable by $\mathbb{P}$ has a copy already in the ground model. In this section, we examine the dual question: what properties of *structures* ensure that generic presentability implies the existence of a copy in the ground model? Specifically, we extend Solovay's Theorem 4.2.23 to structures that are sufficiently "set-like:"

Theorem 4.4.1. *If a generically presentable structure is rigid, then it has a copy in the ground model.*
More precisely, suppose $(\mathbb{P}, \nu)$ is a generically presentable structure such that $\Vdash_{\mathbb{P}}$ "$\nu[G]$ has no nontrivial automorphisms." Then $(\mathbb{P}, \nu)$ has a copy in V.

Proof. We assume the language $\mathcal{L}$ of the rigid generically presentable structure $\mathcal{N} = (\mathbb{P}, \nu)$ is relational. On $\omega \times \mathbb{P}$, we define the relation $\equiv$ as follows:

$$(a, p) \equiv (b, q) \iff (p, q) \Vdash_{\mathbb{P}^2} \text{"}\{(a, b)\} \text{ extends to an isomorphism } \nu[\dot{g}_0] \cong \nu[\dot{g}_1].\text{"}$$

If $(a, p) \equiv (a, p)$, we say a is *stable* in p, and we write $\mathbb{M}$ for the set $\{(a, p) : a$ is stable in $p\}$.

Lemma 4.4.2. *The relation $\equiv$ is an equivalence relation on $\mathbb{M}$.*

Proof. Symmetry is clear, and reflexivity is immediate from the definition of $\mathbb{M}$. For transitivity, suppose $(a, p) \equiv (b, q) \equiv (c, r)$, and let $G_0 \times G_1$ be $\mathbb{P}^2$-generic over V with $p \in G_0, r \in G_1$; and let H be $\mathbb{P}$-generic over $V[G_0 \times G_1]$-generic, with $q \in H$. Then clearly in $V[G_0 \times G_1][H]$, there is an isomorphism between $\nu[G_0]$ and $\nu[G_1]$ taking a to c; but this is a Σ_1^1 property, and so already true in $V[G_0 \times G_1]$. Thus, $(a, p) \equiv (c, r)$. $\qquad\square$

Now let M be the set of $\equiv$-classes of elements of $\mathbb{M}$. The basic properties of M, which parallel the properties of ages needed for Fraïssé constructions, are:

Lemma 4.4.3. *For $p \in \mathbb{P}, a \in \omega$,*

1. ***(Extension)*** *if a is stable in p and $q \leq p$, then a is stable in q and $(a, p) \equiv (a, q)$; and*

2. ***(Genericity)*** *there is some $q \leq p$ with a stable in q.*

Proof. (1): That a is stable in q is immediate from the definition of stability. To see that $(a, p) \equiv (a, q)$, note that any pair of generics H_0, H_1 witnessing the failure of $(a, p) \equiv (a, q)$ would also witness the instability of (a, p).

(2): Consider the condition $(p, p) \in \mathbb{P}^2$. By our assumption on ν, there must be some condition $(q, q') \leq (p, p)$ and $a' \in \omega$ such that

$$(q, q') \Vdash_{\mathbb{P}^2} \{(a, a')\} \text{ extends to an isomorphism } \nu[\dot{g}_0] \cong \nu[\dot{g}_1].$$

It now follows that a is stable in q: given $G_0 \times G_1$ $\mathbb{P}^2$-generic over V extending (q, q), fix some H which is $\mathbb{P}$-generic over $V[G_0 \times G_1]$ with $q' \in H$. Then in $V[G_0 \times G_1][H]$ there is an isomorphism between $\nu[G_0]$ and $\nu[G_1]$ extending $\{(a, a)\}$; but this is a Σ_1^1 fact, so already true in $V[G_0 \times G_1]$. $\qquad\square$

Finally, the following result is where rigidity is used. Intuitively, rigidity plays the role in our proof that ω-homogeneity plays in standard Fraïssé limit constructions.

Lemma 4.4.4. ***(Simultaneity)*** *Suppose $p, q \in \mathbb{P}$ and $i_1, ..., i_n \colon \subseteq \omega \to \omega$ are partial maps in V with disjoint domains which are each forced by (p, q) in $\mathbb{P}^2$ to extend to isomorphisms $j_1, ..., j_n \colon \nu[G_0] \cong \nu[G_1]$. Then*

$$(p, q) \Vdash_{\mathbb{P}^2} \bigcup_{1 \leq j \leq n} i_j \text{ extends to an isomorphism } \nu[G_0] \cong \nu[G_1].$$

Note that this result immediately implies the seemingly stronger result in which disjointness of domains is not assumed.

Proof. We will prove the lemma in the case where $n = 2$, $p = q$, $i_1 = \{(a, a)\}$ and $i_2 = \{(b, b)\}$ for some distinct $a, b \in \omega$; the general result is no different. Note that the assumption on i_j in this case means just that a and b are stable in p.

Let $G_0 \times G_1$ be $\mathbb{P}^2$-generic extending (p, p). Then, forced by (p, p), there are isomorphisms $j_1, j_2 : \nu[G_0] \cong \nu[G_1]$ with $j_1(a) = a$ and $j_2(b) = b$. Consider the map $j = j_1 \circ j_2^{-1}$. This is an automorphism of $\nu[G_1]$, and hence by rigidity must be the identity; so $j_1(b) = b$, since $j_2^{-1}(b) = b$ by assumption on j_2. But then j_1 is an isomorphism extending $\{(a, a), (b, b)\}$, so (p, p) forces that there is an isomorphism between $\nu[G_0]$ and $\nu[G_1]$ extending $\{(a, a), (b, b)\}$. $\square$

Now we come to the body of the proof of Theorem 4.4.1. We can turn $\mathbb{M}$ into an $\mathcal{L}$-structure, $\mathcal{M}$, as follows: writing (a, p) for the equivalence class of $(a, p) \in \mathbb{M}$, for each n-ary relation symbol $R \in \mathcal{L}$ we let $R^{\mathcal{M}}$ be the set of tuples $((\mathbf{a}_1, p_1), \ldots, (\mathbf{a}_n, p_n))$ such that

$$\exists q \in \mathbb{P}, c_1, ..., c_n \text{ stable in } q \; (\forall i \leq n[(\mathbf{a}_i, p_i) = (\mathbf{c}_i, q)] \;\wedge\; q \Vdash \text{``}\nu \models R(c_1, ..., c_n)\text{''}).$$

Informally, this definition ensures that each relation R holds whenever it ought to hold; we will also need the converse result, that each R fails whenever it ought to fail, and this is where Simultaneity will come in.

Lemma 4.4.5. *Let G be $\mathbb{P}$-generic over V. Then $V[G] \models \nu[G] \cong \mathcal{M}$.*

Proof. For $a \in \nu[G]$, let $Stab_a^G = \{p \in G : (a, p) \in \mathbb{M}\}$. Then for every $p, q \in Stab_a^G$, we must have $(a, p) \equiv (a, q)$: since $p, q \in G$, there must be a common strengthening $r \leq p, q$; by 4.4.3(1), we have $(a, p) \equiv (a, r)$ and $(a, q) \equiv (a, r)$, and hence $(a, p) \equiv (a, q)$ by transitivity. So the set $\{(a, p) : p \in Stab_a^G\}$ is contained in a single $\equiv$-class, and hence corresponds to a single element of $\mathcal{M}$.

Consider the map $i : \nu[G] \to \mathcal{M} : a \mapsto \{a\} \times Stab_a^G$; We claim that i is an isomorphism. Surjectivity is an immediate consequence of genericity (Lemma 4.4.3(2)), and injectivity follows from the rigidity of $\nu[G]$.

Finally, we must show that i is a homomorphism. Let R be a relation symbol in $\mathcal{L}$ and $\bar{a} \in \nu[G]$. First, suppose $\nu[G] \models R(\bar{a})$. Let $p \in G$ be such that $p \in \bigcap_{a \in \bar{a}} Stab_a^G$ and $p \Vdash \nu[G] \models R(\bar{a})$. Then p witnesses that $\mathcal{M} \models R(i(\bar{a}))$. Conversely, suppose $\mathcal{M} \models R(i(\bar{a}))$ and fix $p \in \bigcap_{a \in \bar{a}} Stab_a^G$. Then we must have some $q \in \mathbb{P}$ and $\bar{c}$ stable in q such that $(c_i, q) \equiv (a_i, p)$ for each i and $q \Vdash R(\bar{c})$. But then by simultaneity (Lemma 4.4.4) we must have $p \Vdash R(\bar{a})$. $\square$

This finishes the proof of Theorem 4.4.1. $\square$

Chapter 5

Expansions and reducts of $\mathbb{R}$

The work in this section appeared as [33], and is joint with Greg Igusa and Julia Knight; it appears here with their permission.

5.1 Introduction

The behavior of structures in generic extensions of the universe has been studied from a number of different angles; for example, Baldwin, Laskowski, and Shelah [5] studied the conditions under which non-isomorphic structures may become isomorphic, and Knight, Montalban, and Schweber [41] (and independently Kaplan and Shelah [**KS14**–**IK14″**]) studied structures existing in every generic extension of the universe by some forcing. In the latter example, general results about such "generically presentable" structures led to a new proof of a result of Harrington saying that if T is a counterexample to Vaught's Conjecture, then T has models of cardinality $\aleph_1$ with arbitrarily large Scott ranks less than ω_2. (There are now at least three new proofs of this result. In addition to the one in [41], there is one by Baldwin, S.-D. Friedman, Koerwien, and Laskowski [6] and one by Larson [46]; these other proofs do not use generically presentable structures directly, but do use related ideas.)

The present paper continues the general theme of studying structures in generic extensions. We examine the *computability-theoretic* properties of structures in generic extensions, and in particular its connections with tameness in model theory. In [41], the third author defined a notion that lets us compare the computing power of structures of any cardinality:

Definition 30 (Schweber)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be structures in V (of any cardinality). We say that $\mathcal{A} \leq_w^* \mathcal{B}$ if in a generic extension $V(G)$ in which both $\mathcal{A}$ and $\mathcal{B}$ are countable, every copy of $\mathcal{B}$ computes a copy of $\mathcal{A}$.*

In [41], there are a few examples comparing familiar structures. In particular, it is shown that $\mathcal{W} \leq_w^* \mathcal{R}$, where $\mathcal{R}$ is the ordered field of real numbers, and $\mathcal{W}$ is a structure representing the power set of ω, coded as $\mathcal{W} = (P(\omega) \cup \omega, P(\omega), \omega, \in, S)$, where S is the successor relation on ω. In computability, these two structures are sometimes identified; both are referred to

as "the reals". Of course, they are not the same structure: $\mathcal{R}$ is a field, while $\mathcal{W}$ is just a family of subsets of ω.

Let $\mathcal{R}^*$ be an ω-saturated extension of $\mathcal{R}$. In [32], it is shown that $\mathcal{R}^* \equiv^*_w \mathcal{W}$ and that $\mathcal{R} \not\leq^*_w \mathcal{R}^*$, so $\mathcal{R} \not\leq^*_w \mathcal{W}$. In the proof, we note that $\mathcal{R}$ is a residue field section of $\mathcal{R}^*$. After collapse, $\mathcal{R}^*$ is no longer ω-saturated, but it is recursively saturated, and it realizes just the types in the Scott set that is the old $P(\omega)$. We show that for a countable recursively saturated real closed field K, with residue field k, some copy of K does not compute a copy of k. The proof of this involves a reduction. It is shown that if every copy of K computes a copy of k, then the set $FT(K)$ consisting of finite elements that are not infinitesimally close to any algebraic element must be defined in K by a computable Σ_2 formula. It is then shown that $FT(K)$ has no such definition.

In the present paper, we consider further structures related to the reals. Let $\mathcal{R}_{exp} = (\mathcal{R}, exp)$. We show that $\mathcal{R}_{exp} \equiv^*_w \mathcal{R}$. More generally, if f is total analytic on $\mathcal{R}$ and $\mathcal{R}_f = (\mathcal{R}, f)$, then $\mathcal{R}_f \equiv^*_w \mathcal{R}$. The process of generalizing our proof from the first example to the latter example also allows us to prove a number of results found independently by Downey, Greenberg, and Miller [] showing that a number of seemingly weaker structures are also equivalent to $\mathcal{R}$ under $\equiv^*_w$.

The structure $\mathcal{W}$ represents Cantor space. It is clearly equivalent to $\mathcal{C} = (2^\omega, (R_n)_{n \in \omega})$, where $R_n f$ iff $f(n) = 1$. To represent Baire space, we may take $\mathcal{B} = (\omega^\omega, (R_{n,k})_{n,k \in \omega})$, where $R_{n,k}(f)$ iff $f(n) = k$. This structure is also equivalent to $\mathcal{R}$ [12].

In Section 2, we show that $\mathcal{R}_{exp} \equiv^*_w \mathcal{R}$. The proof combines ideas from computability (jumps and effective guessing strategies), computable structure theory (definability by computable infinitary formulas), and model theory (o-minimality). In Section 3, we generalize the result from Section 2 to show that for any expansion $\mathcal{M}$ of a very weak base structure, if $\mathcal{M}$ is o-minimal and has definable Skolem functions, then $\mathcal{M} \equiv^*_w \mathcal{R}$. In Section 4, we apply the result from Section 3 to show that the other structures we are interested in are all equivalent to $\mathcal{R}$: the expansions $\mathcal{R}_f \equiv^*_w \mathcal{R}$, where f is analytic, the reduct $\mathcal{R}^+$, and the structure $\mathcal{B}$ representing Baire space. In the remainder of the introduction, we give some background on o-minimality.

We end with a cautionary remark. If $\mathcal{A}$ is an expansion of $\mathcal{B}$ such that $\mathcal{A} \equiv^*_w \mathcal{B}$, it may not be the case that (in an appropriate generic extension) every copy of $\mathcal{B}$ computes a copy of $\mathcal{A}$ *together with an isomorphism between the copy of $\mathcal{B}$ and the reduct of the copy of $\mathcal{A}$*. Indeed, that is the case with expansions of $\mathcal{R}$: for example, the functions which $\mathcal{R}$ can compute in this sense are precisely the piecewise algebraic functions.

o-minimality

Definition 31. *A structure $\mathcal{M}$ with a dense linear ordering on the universe is o-minimal if each set definable by an elementary first order formula (with parameters) is a finite union of intervals (possibly trivial) with endpoints in $\mathcal{M}$.*

The following is well-known [42].

Proposition 5.1.1. *If T is the elementary first order theory of an o-minimal structure $\mathcal{M}$, then all models of T are o-minimal.*

We say that T is an *o-minimal theory*.

Examples.

1. $\mathcal{R}$ is o-minimal. Tarski [75] proved that $Th(\mathcal{R})$ is decidable. In the proof, Tarski gave an effective elimination of quantifiers. There is an algorithm (familiar to every school child) for deciding the truth of the quantifier-free sentences. As a side result, Tarski stated the fact that in $\mathcal{R}$, and the other models of the theory, the definable sets are finite unions of intervals.

2. $\mathcal{R}^+$ is o-minimal. It is clear from the definition that any reduct of an o-minimal structure that includes the ordering is o-minimal.

3. $\mathcal{R}_{sin}$ is not o-minimal—think of the set of zeroes of $sin(x)$.

4. $\mathcal{R}_{exp}$ is o-minimal. Wilkie [78] showed that $T_{exp} = Th(\mathcal{R}_{exp})$ is model complete; i.e., if $\mathcal{M}_1$ and $\mathcal{M}_2$ are models of T_{exp}, with $\mathcal{M}_1 \subseteq \mathcal{M}_2$, then $\mathcal{M}_1 \prec \mathcal{M}_2$. By results of Khovanskii [30], it follows that the theory is o-minimal. Ressayre [63] gave another proof of model completeness.

5. $\mathcal{R}_{an}$ is o-minimal, where this is the expansion of $\mathcal{R}$ with the restrictions f_I of analytic functions f to compact intervals $I = [a, b]$. More precisely, f_I is the total function that agrees with f on I and has value 0 otherwise. By results of van den Dries [14], building on work of Gabrielov [20], $\mathcal{R}_{an}$ is o-minimal.

We will use the following facts. The first is due to van den Dries [15].

Proposition 5.1.2. *Any o-minimal expansion of $\mathcal{R}^+$ has definable Skolem functions.*

The second fact is due to Pillay [42].

Proposition 5.1.3. *For an o-minimal theory with definable Skolem functions, definable closure is a good closure notion, satisfying the Exchange Property—if a is definable from $\bar{b}, c$ and not from $\bar{b}$, then c is definable from $\bar{b}, a$.*

This means that independence, basis, and dimension are well-defined. The third fact is also due to Pillay [42].

Proposition 5.1.4. *For an independent tuple $\bar{b}$ in an o-minimal structure $\mathcal{A}$, if $\varphi(\bar{x})$ is a finitary formula true of $\bar{b}$, then there is an open box B around $\bar{b}$, with vertices having coordinates in $\mathcal{A}$, such that $\varphi(\bar{x})$ is valid on B.*

Remark. If $\mathcal{A}$ is an Archimedean model of $Th(\mathcal{R})$ (or $Th(\mathcal{R}^+)$), and $\bar{a}$ is a tuple in an open box B with vertices having coordinates in $\mathcal{A}$, then there is another open box $B^* \subseteq B$ such that $\bar{a} \in B^*$ and B^* has vertices with rational coordinates. We refer to B^* as a *rational box*.

5.2 $\mathcal{R}_{exp} \equiv_w^* \mathcal{R}$

In this section, our goal is to prove the following.

Theorem 5.2.1. $\mathcal{R}_{exp} \equiv_w^* \mathcal{R}$.

Here is a brief overview of the proof. Clearly, $\mathcal{R} \leq_w^* \mathcal{R}_{exp}$. We show that $\mathcal{R}_{exp} \leq_w^* \mathcal{R}$. Let T_{exp} be the elementary first order theory of $\mathcal{R}_{exp}$. After collapse, let K be a copy of $\mathcal{R}$ with universe a subset of ω.

Being isomorphic to $\mathcal{R}$, K can be expanded to a structure K_{exp} satisfying T_{exp}. We will show that this expansion is unique. Then, using definability of algebraic independence, we will show that Δ_2^0 relative to K, we can find a basis for K_{exp}. We then use a computable approximation to this basis in a finite injury priority construction in order to construct a copy of K_{exp} using K.

Expanding K to a model of T_{exp}

We first show that for a countable Archimedean real closed ordered field K with an added function f satisfying T_{exp}, the expansion is unique, and the function exp^K is defined by a computable Π_1 formula. The same is true if we substitute for exp an arbitrary continuous function.

Lemma 5.2.2 (Uniqueness)**.** *Let f be a continuous function on the reals, and let $T_f = Th(\mathcal{R}, f)$. If K is an Archimedean real closed ordered field, there is at most one expansion (K, f^K) satisfying T_f. Moreover, for the function f^K is defined by a computable Π_1 formula with a real parameter. Hence, it is Δ_2^0 relative to K.*

Proof. To prove uniqueness, we identify K with a subfield of $\mathcal{R}$. Let $a \in K$. For each open interval I containing $f(a)$, and having rational endpoints, there is an open interval J containing a, also with rational endpoints, such that f (as a function on $\mathcal{R}$) maps J to I. For each such pair of intervals I and J, with rational endpoints, there is a sentence in T_f saying that f maps J into I. Then the function f^K must map J to I in K. This implies that $f^K(a)$ in K must match $f(a)$ in $\mathcal{R}$. This proves uniqueness.

We show that there is a computable Π_1 formula with the meaning $y = f(x)$. We have $y = f(x)$ iff for all pairs of rational intervals J and I such that T_f contains the sentence saying that $f : J \to I$, if $x \in J$, then $y \in I$. This gives a definition that is the conjunction of finitary quantifier-free formulas over a set that is c.e. relative to T_f. We can replace this by a c.e. conjunction involving a real parameter $r \in [0, 1]$, where r, in its preferred binary expansion, has 1 in the kth place iff k is the Gödel number of a sentence of T_{exp}.

Let $c_k(u)$ be a finitary quantifier-free formula saying of $u \in [0, 1]$ that its preferred binary expansion has 1 in the kth place. For *all* k, we define a finitary quantifier-free formula $\rho_k(u, x, y)$. If k is the Gödel number of a sentence saying that $f : J \to I$, then $\rho_k(u, x, y)$ says $c_k(u) \to (x \in I \to y \in J)$, and if k is not the Gödel number of such a sentence,

then $\rho_k(u, x, y) = \top$. Then the computable Π_1 formula $\bigwedge_k \rho_k(r, x, y)$ holds just in case $y = f(x)$. $\qquad\square$

Remark. Not all Archimedean real closed ordered fields can be expanded to models of T_{exp}. In particular, since $e = exp(1)$ is transcendental, the ordered field of real algebraic numbers cannot be expanded in this way.

By Proposition 5.1.4, for a tuple $\bar{a}$ that is independent in $\mathcal{R}_{exp}$, each formula $\varphi(\bar{x})$ true of $\bar{a}$ is valid on an open box B around $\bar{a}$, with vertices having rational coordinates. We need the converse of this.

Lemma 5.2.3. *Let $\bar{a}$ be a tuple of reals. Suppose that for every formula $\varphi(\bar{x})$ true of $\bar{a}$ in $\mathcal{R}_{exp}$, there is an open box B around $\bar{a}$, with vertices having rational coordinates, such that T_{exp} contains the sentence saying that $\varphi(\bar{x})$ is valid on B. Then $\bar{a}$ is independent in $\mathcal{R}_{\exp}$.*

Proof. Suppose not. Say a_k is defined from $a_1, \cdots, a_{k-1}$ in $\mathcal{R}_{\exp}$. Let $\varphi(\bar{x})$ be a formula saying that x_k is defined in this way from $x_1, \cdots, x_{k-1}$. This cannot be valid on an open box. $\qquad\square$

Independence relations on $\mathcal{R}_{exp}$

Definition 32. *Suppose K is an Archimedean real closed ordered field with an expansion (K, exp) satisfying T_{exp}. Let $IND_n(K)$ be the set of n-tuples in K that are independent in (K, exp).*

We show that the relations $IND_n(\mathcal{R})$ are defined in $\mathcal{R}$ by computable sequences of computable Π_2 and computable Σ_2 formulas. The computable Π_2 definitions are easy.

Lemma 5.2.4 (Computable Π_2 definition of IND_n). *For each n, we can effectively find a computable Π_2 definition of IND_n, with a parameter r coding T_{exp}.*

Proof. We have $\bar{a} \in IND_n$ iff for each formula $\varphi(\bar{x})$, there is an open box B around $\bar{a}$, with vertices having rational coordinates, such that T_{exp} contains one of the sentences $(\forall \bar{x} \in B)\varphi(\bar{x})$ or $(\forall \bar{x} \in B)\neg\varphi(\bar{x})$. We can express this as a computable Π_2 formula. Let $c_k(u)$ be the formula saying that the kth place in the preferred binary expansion of u is 1. For each formula φ in the appropriate variables, and each rational box B, let $k(\varphi, B)$ be the Gödel number of the sentence saying $(\forall \bar{x} \in B)\varphi(\bar{x})$. We have $\bar{a} \in IND_n$ iff

$$\bigwedge_\varphi \bigvee_B \left(\bar{a} \in B \ \& \ (c_{k(\varphi, B)}(r) \vee c_{k(\neg\varphi, B)}(r))\right),$$

where the conjunction is over all φ with appropriate variables, and the disjunction is over all rational boxes B. This is computable Π_2, with the parameter r, as required. $\qquad\square$

The computable Σ_2 definition for the relation IND_n is less obvious.

Lemma 5.2.5 (Computable Σ_2 definition of IND_n). *For each n, we can effectively find a computable Σ_2 definition of IND_n, with a parameter r coding $T_{\exp}$.*

Proof. Fix n. Let $(\varphi_m(\bar{x}))_{m \in \omega}$ be a computable list of formulas in the variables $\bar{x}$, in the language of $\mathcal{R}_{exp}$. We build a tree $\mathcal{T}$, computable in T_{exp}, consisting of finite sequences of rational boxes $B_1, B_2, \ldots, B_s$ such that $B_1 \supseteq B_2 \supseteq \ldots \supseteq B_s$ and for each k, one of the sentences $(\forall \bar{x} \in B_{k+1})\varphi_k(\bar{x})$ or $(\forall \bar{x} \in B_{k+1})\neg\varphi_k$ is in T_{exp}. By Proposition 5.1.4 and Lemma 5.2.3, $\bar{a} \in IND_n$ iff there is a path $\pi = B_1, B_2, \ldots$ through $\mathcal{T}$ such that for each s, $\bar{a}$ is in the box B_s. We must be sure that this definition can be expressed by a computable Σ_2 formula in the language of real closed ordered fields, with the parameter r.

Claim: There is a computable Π_1 formula, with parameter r, saying that x codes a path through $\mathcal{T}$.

Proof of Lemma. The preferred binary expansion of x gives the characteristic function f_x of a set $S_x \subseteq \omega$. The set S_x may be finite, although it cannot be co-finite. We consider a path through $\mathcal{T}$ to be a set S with the following properties.

1. all elements of S are (codes for) finite sequences $(B_1, \ldots, B_s)$ in $\mathcal{T}$,

2. if $(B_1, \ldots, B_s, B_{s+1})$ is in S, then so is $(B_1, \ldots, B_s)$,

3. if two sequences in S have length s, then they are equal,

4. S is infinite.

We show that there are computable Π_1 formulas saying that S_x has each of the four properties above.

For Property 1, we say that S_x has no elements not in $\mathcal{T}$. Since $\mathcal{T}$ is computable in T_{exp} and T_{exp} is coded by r, there is a c.e. set C of pairs (σ, k), with $\sigma \in 2^{<\omega}$, such that $k \notin \mathcal{T}$ iff for some $(\sigma, k) \in C$, r *agrees with* σ, where this means that the preferred binary expansion of r extends σ; i.e., $c_k(r)$ holds for $\sigma(k) = 1$ and $\neg c_k(r)$ holeds for $\sigma(k) = 0$. To get a computable Π_1 formula saying that S_x has no elements not in $\mathcal{T}$, we take the c.e. conjunction over $(\sigma, k) \in C$ of formulas saying that if r agrees with σ, then $\neg c_k(x)$. This is computable Π_1 with parameter r.

For Property 2, we must say that S_x (a set of codes for finite sequences), is closed under initial segments. We take the conjunction over pairs (k, k') such that for some s, k is the code for a sequence of length $s + 1$ and k' is the code for the initial segment of length k, of formulas saying $c_k(x) \rightarrow c_{k'}(x)$. This is computable Π_1, with no parameter.

For Property 3, we say that if two sequences in S_x have length s, then they are equal. We take the conjunction of formulas saying $\neg c_k(x)$ & $c_{k'}(x)$, over all pairs (k, k') coding distinct sequences of the same length. This is computable Π_1, with no parameter.

For Property 4, we must say that S_x is infinite. We recall that the elements x of $[0, 1)$ that code finite sets are just the dyadic rationals. We have a computable Π_1 formula saying that r is not equal to any of these rationals.

Putting the four statements together, we have a computable Π_1 formula, with parameter r, saying that x codes a path through $\mathcal{T}$. □

Knowing that x codes a path through $\mathcal{T}$, we want a computable Π_1 formula saying that an n-tuple $\bar{u}$ lies in the boxes on this path. We take the conjunction over k coding a finite sequence of rational boxes $(B_1, \ldots, B_s)$ of the formulas saying $c_k(x) \to \bar{u} \in B_s$. To say that $\bar{u}$ is independent, we have a computable Σ_2 formula saying that there exists x such that S_x is a path through $\mathcal{T}$ and $\bar{u}$ lies in the boxes corresponding to this path. □

Thanks to the computable Π_2 and computable Σ_2 definitions, we know that for any copy K of $\mathcal{R}$, the relations $IND_n(K)$ are Δ_2^0 relative to K, uniformly in n.

Lemma 5.2.6 (Basis). *Suppose K is a copy of $\mathcal{R}$. Then we have a sequence $b_1, b_2, \ldots$, Δ_2^0 relative to K, and forming a basis for $K_{\exp}$.*

Proof. Applying a procedure that is Δ_2^0 relative to K, we run through the elements, and we use the relations IND_n to choose a basis. We let b_1 be first satisfying $IND_1(u_1)$, we let b_2 be first such that (b_1, b_2) satisfies $IND_2(u_1, u_2)$, etc. □

To complete the proof that $\mathcal{R}_{exp} \leq_w^* \mathcal{R}$, we show the following.

Proposition 5.2.7 (Enumerating the complete diagram of the expansion). *After collapse, let $K \cong \mathcal{R}$. Then there is a is a copy $\mathcal{C}$ of $\mathcal{K}_{\exp}$ with complete diagram computable in K.*

Proof. Let $b_1, b_2, \ldots$ be a basis for $K_{\exp}$ that is Δ_2^0 relative to K, determined as in the previous lemma. Guessing at this basis, and using T_{exp}, we enumerate the complete diagram of a copy $\mathcal{C}$ of K_{exp}. The universe of $\mathcal{C}$ will be ω, which we think of as a set of constants. We fix a computable enumeration of the sentences $\varphi(\bar{c})$, where $\varphi(\bar{x})$ is a formula in the language of $\mathcal{R}_{exp}$ and $\bar{c}$ is a tuple of constants. We suppose that the language includes symbols for the definable Skolem functions. We fix a computable enumeration of terms $\tau(\bar{c})$, where $\tau(\bar{x})$ is a term and $\bar{c}$ is a tuple of constants. We enumerate the complete diagram of $\mathcal{C}$ in stages. Let δ_s be the set of sentences enumerated by stage s.

The set δ_s includes sentences saying that the constants mentioned are all distinct. We start with $\delta_0 = \emptyset$, and $\delta_s \subseteq \delta_{s+1}$. We will arrange that for each sentence $\varphi(\bar{c})$, one of $\pm\varphi(\bar{c})$ is in δ_s for some s. We will also arrange that for each term $\tau(\bar{c})$, some sentence of the form $\tau(\bar{c}) = c'$ appears in δ_s for some s. To determine an isomorphism f from $\mathcal{C}$ onto K_{exp}, it is enough to determine $f^{-1}(b_n)$ for all n, since the rest of the elements are definable from the basis. We have the following requirements.

R$_n$: Determine $f^{-1}(b_n)$.

At stage s, we have tentatively mapped some constants $\bar{d}_s$ to a tuple $\bar{v}_s$ in K which we believe to be an initial segment of the basis $b_1, b_2, \ldots$. In δ_s, we have mentioned the constants $\bar{d}_s$, plus some further constants $\bar{c}_s$. Each $c_i \in \bar{c}_s$ has been given a definition $\tau_i(\bar{d}_s)$, and the the sentence $c_i = \tau_i(\bar{d})$ is in δ_s. We will maintain the condition that what we have said in δ_s about $\bar{d}$ is valid on a rational box B_s around $\bar{v}_s$. We must make this precise.

Let $\chi_s(\bar{d}_s, \bar{c}_s)$ be the conjunction of the sentences in δ_s. Let $\bar{u}$ be a tuple of variables corresponding to $\bar{d}_s$. We suppose that these variables do not appear in the sentences of δ_s. Let $\chi_s^*(\bar{u})$ be the formula obtained from $\chi_s(\bar{d}_s, \bar{c}_s)$ by replacing each $d_i \in \bar{d}$ by the corresponding variable u_i, and replacing each $c_i \in \bar{c}_s$ by $\tau_i(\bar{u})$, where c_i is defined to be $\tau_i(\bar{d}_s)$. Note that $\chi_s^*(\bar{u})$ has conjuncts saying that the terms u_i and $\tau_i(\bar{u})$ are all distinct.

Now, $\chi_s^*(\bar{u})$ expresses what we have said about $\bar{d}_s$ in δ_s. We say how to check that this is true on a rational box B_s around $\bar{v}_s$. We write $\chi_s^*(B_s)$ for the sentence saying $(\forall \bar{u} \in B_s)\chi_s^*(\bar{u})$. We check that $\bar{v}_s \in B_s$ and that $\chi_s^*(B_s) \in T_{exp}$. We can check that $\bar{v} \in B_s$ using K. We can check, using the real that codes T_{exp}, that the sentence $\chi_s^*(B_s) \in T_{exp}$.

At stage $s+1$, if our stage s guess $\bar{v}_s$ at the initial segment $\bar{b}_s$ of the basis seems correct, then $\bar{v}_{s+1} = \bar{v}_s, v'$, where v' appears to be the next element of the basis. If at stage s our guess $\bar{v}_s$ at $\bar{b}_s$ changes, then $\bar{v}_{s+1}$ is the restriction of $\bar{v}_s$ to the part that still seems to be an initial segment of the basis. In this case, the elements of $\bar{d}_s$ tentatively mapped to the elements of $\bar{v}_s$ that are not in $\bar{v}_{s+1}$ will be remapped to definable elements, and at all future stages will be treated as part of the $\bar{c}$. In the event that at a later stage some elements of $\bar{v}_s$ appear to return to the basis, we will create new constants to map to those elements.

At stage $s + 1$, assuming that our stage $\bar{v}_{s+1}$ has a new element v', we map to a new constant d' to it. We put into δ_{s+1} sentences saying that d is not equal to any element of $\bar{d}_s$ or $\bar{c}_s$. We decide the next sentence φ that mentions only the constants from $\bar{d}_s, \bar{c}_s$. Also, for the next term $\tau(\bar{d}_s)$ not already given a name, add a sentence $c = \tau(\bar{d}_s)$, where c is either in $\bar{c}_s$ or the first constant not yet mentioned. The lemmas below guarantee that we can do this all of this, while maintaining the condition that what we have said in δ_{s+1} about $\bar{d}_{s+1}$ is valid on a rational box around $\bar{v}_{s+1}$. We need some terminology.

Definition 33. *We say that $(\delta, \bar{d}; \bar{c})$ is a* good triple *if*

1. *δ is a finite set of sentences with constants split into disjoint sets $\bar{d}$, and $\bar{c}$,*

2. *δ includes sentences saying that the constants are all distinct,*

3. *for each $c \in \bar{c}$, δ includes a sentence $\tau(\bar{d}) = c$.*

For a good triple $(\delta, \bar{d}; \bar{c})$, a *test formula* $\chi^*(\bar{u})$ is obtained in the way we obtained $\chi_s^*(\bar{u})$ from δ_s above.

Definition 34. *For a good triple $(\delta, \bar{d}; \bar{c})$, we say that $\chi^*(\bar{u})$ is a* test formula *if it is obtained by the following steps.*

1. *Let χ be the conjunction of δ, let $\bar{u}$ be a sequence of new variables corresponding to $\bar{d}$.*

2. Let $\chi^*(\bar{u})$ be the formula obtained from χ by replacing d_i by u_i and replacing c_i by $\tau_i(\bar{u})$, where $c_i = \tau_i(\bar{d})$ is a sentence of δ defining c_i in terms of $\bar{d}$.

For our construction, at stage s, we will have a good triple $(\delta_s, \bar{d}_s; \bar{c}_s)$ with a test formula χ_s^* that is valid on a rational box B_s, so that the sentence $\chi_s^*(B_s)$ is in T_{exp}. Moreover, we will have f tentatively mapping $\bar{d}_s$ to $\bar{v}_s \in K$, where $\bar{v}_s \in B_s$. We believe that $\bar{v}_s$ is an initial segment of the basis.

Lemma 5.2.8. *Let $(\delta, \bar{d}; \bar{c})$ be a good triple with test formula $\chi^*(\bar{u})$ valid on a rational box B containing an independent tuple $\bar{b}$. Let b' be a further element independent over $\bar{b}$. Let δ' be the result of adding to δ sentences saying of a new constant d that it is not equal to any mentioned in δ. Then $(\delta', \bar{d}, d; \bar{c})$ is a good triple, with test formula $\chi'^*(\bar{u}, u')$ that is valid on a rational box B' around $\bar{b}, b'$. (We may suppose that the projection of B' on the initial coordinates, omitting the last one, is contained in B.)*

Lemma 5.2.9. *Let $(\delta, \bar{d}; \bar{c})$ be a good triple with test formula $\chi^*(\bar{u})$ valid on a rational box B containing an independent tuple $\bar{b}$. Let φ be a sentence with constants among $\bar{d}, \bar{c}$. There is a good triple $(\delta', \bar{d}; \bar{c})$, where δ' is the result of adding $\pm\varphi$ to δ, with test formula $\chi'^*(\bar{u})$ valid on a rational box $B' \subseteq B$ containing $\bar{b}$.*

Lemma 5.2.10. *Let $(\delta, \bar{d}; \bar{c})$ be a good triple with test formula $\chi^*(\bar{u})$ valid on a rational box B containing an independent tuple $\bar{b}$. For a term $\tau(\bar{d})$, there is a good triple $(\delta', \bar{d} : \bar{c}')$, with a test formula $\chi'^*(\bar{u})$ valid on a rational box $B' \subseteq B$ containing $\bar{b}$, where where δ' and $\bar{c}'$ satisfy one of the following:*

1. *δ' is the result of adding to δ a sentence $c = \tau(\bar{d})$, for some $c \in \bar{c}$, and $\bar{c}' = \bar{c}$,*

2. *δ' is the result of adding to δ a sentence $c' = \tau(\bar{d})$, where c' is new, along with sentences saying that c' is not equal to any of the constants in $\bar{d}, \bar{c}$, and $\bar{c}'$ is $\bar{c}, c'$.*

In our construction, it may be that at stage $s+1$, our guess at the initial segment of the basis changes. Then $\bar{v}_{s+1}$ is the restriction of $\bar{v}_s$ to the part that seems correct. We must give the extra elements of $\bar{d}_s$ definitions in terms of $\bar{d}_{s+1}$. The following lemma says that we can do this.

Lemma 5.2.11. *Let $(\delta, \bar{d}, d'; \bar{c})$ be a good triple with test formula $\chi^*(\bar{u}, u')$ valid on a rational box B containing a tuple $\bar{b}, b'$, where $\bar{b}$ is independent. There is a good triple $(\delta', \bar{d}; \bar{c}, d')$ with test formula $\chi(\bar{u}, u')$ valid on a rational box containing $\bar{b}$, where δ' is the result of adding to δ a sentence $d' = \tau(\bar{d})$. We may take B' to be the projection of B on the initial coordinates, omitting the one that corresponds to u'.*

Proof of Lemma. The box B is a cross product of rational intervals. Say that I is the interval corresponding to the coordinate u', and take $q \in I$. There is a term τ in our language naming q. Let δ' be the result of adding to δ the defining sentence $d' = \tau$, and modifying the definitions $c_i = \tau(\bar{u}, u')$, by replacing u' by τ. We have in δ sentences saying

that d' is distinct from all constants in $\bar{d}, \bar{c}$. The formulas of δ are valid on B, and they guarantee that for $\bar{u} \in B'$, nothing in I can be equal to any u_i. Also, for c_i with a definition $c_i = \tau(\bar{d})$ in δ, for $\bar{u} \in B'$, nothing in I can be equal to $\tau(\bar{u})$. $\qquad\square$

We begin at stage 0 with the good triple $(\emptyset, \emptyset, \emptyset)$. Our guess at an initial segment of the basis is $\emptyset$, and f is not defined on any elements. Suppose at stage s, our guess at an initial segment of the basis is $\bar{v}_s$, we have the good triple $(\delta_s, \bar{d}_s; \bar{c}_s)$, with test formula $\chi_s^*(\bar{u})$ valid on a box B_s around $\bar{v}_s$, and we have f mapping $\bar{d}_s$ to $\bar{v}_s$.

We must say what happens at stage $s + 1$. Supposing $\bar{v}_s$ still appears to be an initial segment of the basis, and that v' is the next element of the basis, we consider letting $\bar{v}_{s+1} = \bar{v}_s, v'$ and extending f to map a new constant d' to v', and letting $\bar{d}_{s+1} = \bar{d}_s, d'$. Assuming that we can find an appropriate rational box on which the test formula is valid, we let δ_{s+1} be an extension of δ_s, with some sentences added as follows:

Step 1 We add sentences saying that d' is not equal to anything in $\bar{d}_s, \bar{c}_s$.

Step 2 We add one of the sentences $\pm\varphi$, where φ is the first sentence on our list that involves only constants from $\bar{d}_s, \bar{c}_s$,

Step 3 For the first term $\tau(\bar{d}_s)$ such that δ_s does not include a defining sentence $c = \tau(\bar{d})$, we add such a sentence. Here c may be an element of $\bar{d}_s$, or $\bar{c}_s$ or a new constant.

Lemma 5.2.8 says that we can carry out Step 1, finding a rational box on which the appropriate test formula is valid, provided that our guess the initial segment of the basis is correct. Lemma 5.2.9 says that we can carry out Step 2, provided that our guess at the initial segment of the basis is correct. Lemma 5.2.10 says that we can carry out Step 3, provided that our guess at the initial segment of the basis is correct.

Running our approximations ahead, either $\bar{v}_s$ will no longer seem to be an initial segment of the basis, or else we will arrive at $\bar{v}_{s+1}$ the result of adding a single element to $\bar{v}_s$ and a good triple $(\delta_{s+1}, \bar{d}_{s+1}, \bar{c}_{s+1})$, carrying out all three steps, with a test formula that is valid on an appropriate rational box B_{s+1} containing $\bar{v}_{s+1}$. We do not add to the diagram unless this happens.

If it appears that $\bar{v}_s$ is not an initial segment of the basis, then we apply Lemma 5.2.11 finitely many times, to give definitions to the elements of $\bar{d}_s$ that are mapped to the elements of $\bar{v}_s$ that are not in $\bar{v}_{s+1}$. This lemma tells us how to arrive at an appropriate next good triple and a rational box B_{s+1}. If those elements of $\bar{v}_s$ later return to our approximation for $\bar{b}$, the construction will create new elements that will be mapped to those elements.

Eventually, our guess at the initial segment of the basis of length n is correct. Say this happens at stage s. The initial segment of the basis of length n is $\bar{v}_s$, and for all stages $t \geq s$, the stage t version of f will map $\bar{d}_s$ to $\bar{v}_s$. What we say about $\bar{d}_s$ is true about $\bar{v}_s$. Taking the limit, f gives pre-images to all elements of the basis. Each element of our $\mathcal{C}$ that is not the pre-image of a basis element under f has a definition in terms of some elements that pre-images of the basis elements. We have arranged that if $\varphi(\bar{d}, \bar{c})$ is in δ_s, where $f(\bar{d}) = \bar{v}$

is part of the basis, and c_i has been given a definition $\tau_i(\bar{d})$ in δ_s, then $\varphi(\bar{v}, \bar{\tau}(\bar{v}))$ is true in K_{exp}. Thus, f is an isomorphism. This completes the proof of Proposition 5.2.7. $\qquad\square$

5.3   Generalizing

In the previous section, we showed that $\mathcal{R}_{exp} \leq_w^* \mathcal{R}$. In this section, we generalize. We replace $\mathcal{R}$ by an apparently very weak structure, and we replace $\mathcal{R}_{exp}$ by an arbitrary o-minimal expansion $\mathcal{M}$ of the weak structure having definable Skolem functions. For the weak structure, we consider two possibilities. The first is $\mathcal{R}_{\mathbb{Q}}$, with just has the ordering on the real numbers, plus constants for the rationals. This structure is weak in terms of what is definable by elementary first order formulas. The second structure, $\mathcal{R}_{int}$, with the ordering on the reals, plus the intervals $[q, q')$, where $q < q'$ are dyadic rationals, is even weaker than $\mathcal{R}_{\mathbb{Q}}$ in terms of what is definable by elementary first order formulas.

Our proof is phrased most naturally with $\mathcal{R}_{\mathbb{Q}}$ used as the weak structure for our general result. In [] Downey, Greenberg, and Miller prove that $\mathcal{R}_{int} \equiv_w^* \mathcal{R}$ using a similar proof that was found independently. (They actually use $\mathcal{B}$, which can easily be shown to be equivalent to $\mathcal{R}_{int}$.) In the next section, we mention how our proof of Theorem 5.3.1 can be modified to use $\mathcal{R}_{int}$ as the weak structure.

Theorem 5.3.1. *Let $\mathcal{M}$ be an o-minimal expansion of $\mathcal{R}_{\mathbb{Q}}$ with definable Skolem functions. Then $\mathcal{M} \leq_w^* \mathcal{R}_{\mathbb{Q}}$. In fact, after collapse, every copy K of $\mathcal{R}_{\mathbb{Q}}$ computes the complete diagram of a copy of $\mathcal{M}$.*

We will imitate the proof that $\mathcal{R}_{exp} \leq_w^* \mathcal{R}$. Let $T_{\mathcal{M}} = Th(\mathcal{M})$. We split the proof of Theorem 5.3.1 into a sequence of lemmas, following the outline from the previous section. The greatest difference is in the lemma below. In the previous section, the proof of Lemma 5.2.2, on uniqueness of the expansion, did not use o-minimality, just the fact that exp is a continuous function.

Lemma 5.3.2 (Uniqueness)**.** *For $K \cong \mathcal{R}_{\mathbb{Q}}$, there is a unique expansion $K_{\mathcal{M}}$ to a model of $T_{\mathcal{M}}$.*

Proof. Since $\mathcal{M}$ is o-minimal, with definable Skolem functions, definable closure is a good closure notion. Since $K \cong \mathcal{R}_{\mathbb{Q}}$, there is at least one expansion of K to a model of $T_{\mathcal{M}}$, say K_1. Let $b_1, b_2, \ldots$ be a basis for K_1. Suppose K_2 is another expansion of K to a model of $T_{\mathcal{M}}$. Suppose $\varphi(\bar{x})$ is true in K_1 of a basis tuple $\bar{b}$. Since $\bar{b}$ is independent in K_1, by Proposition 5.1.4 there is a rational box B around $\bar{b}$ such that the sentence Since $\mathcal{M}$ is o-minimal, with definable Skolem functions, definable closure is a good closure notion. Since $(\forall \bar{x} \in B)\varphi(\bar{x})$ is in $T_{\mathcal{M}}$. Then $\varphi(\bar{x})$ must be true of $\bar{b}$ in K_2. For an element c that is not in the basis, we have a definition of c from basis elements in K_1, say $c = \tau(\bar{b})$. Let c' be $\tau(\bar{b})$ in K_2. We can show that $c = c'$. If c is in a rational interval I, then the formula saying $\tau(\bar{x}) \in I$ is true of $\bar{b}$ in K_1. This formula is also true of $\bar{b}$ in K_2, so $c' \in I$. This shows that $K_1 = K_2$. $\qquad\square$

We write $K_{\mathcal{M}}$ for the unique expansion of K to a model of $T_{\mathcal{M}}$. Let IND_n be the set of n-tuples in K that are independent in the expansion $K_{\mathcal{M}}$. The next lemma is the analogue of Lemma 5.2.4.

Lemma 5.3.3 (Computable Π_2 definition of IND_n). *For each n, we can effectively find a computable Π_2 definition of IND_n, with a real parameter r coding $T_{\mathcal{M}}$.*

Proof. We have $\bar{b}$ in IND_n iff for all formulas $\varphi(\bar{x})$ in the appropriate variables, there is a rational box B around $\bar{b}$ such that $T_{\mathcal{M}}$ contains one of the sentences $(\forall \bar{x} \in B)\varphi(\bar{x})$ or $(\forall \bar{x} \in B)\neg\varphi(\bar{x})$. We express this as a computable Π_2 formula, just as we did in the proof of Lemma 5.2.4. $\square$

The next lemma is the analogue of Lemma 5.2.5.

Lemma 5.3.4 (Computable Σ_2 definition of IND_n). *For each n, we can effectively find a computable Σ_2 definition of IND_n, with a real parameter r coding $T_{\mathcal{M}}$.*

Proof. For a fixed tuple of variables $\bar{x}$, let $(\varphi_n(\bar{x}))_{n \in \omega}$ be a computable list of the formulas in the language of $\mathcal{M}$ with free variables $\bar{x}$. Let $\mathcal{T}$ be the tree of finite sequences of rational boxes $(B_1, B_2, \ldots, B_s)$ such that $B_1 \supseteq B_2 \supseteq \ldots \supseteq B_s$ such that for each k, one of the sentences $(\forall \bar{x} \in B_{k+1})\varphi_k(\bar{x})$ or $(\forall \bar{x} \in B_{k+1})\neg\varphi_k(\bar{x})$ is in $T_{\mathcal{M}}$. The tree is $\mathcal{T}$ computable in $T_{\mathcal{M}}$. We have $\bar{b}$ in IND_n iff there is a path π through $\mathcal{T}$ such that $\bar{b}$ is in all of the boxes associated with π.

The computable Σ_2 definition of $IND_n(\bar{u})$ says that there exists x coding a path through $\mathcal{T}$ such that $\bar{u}$ is in all of the boxes associated with the path. As in the proof of Lemma 5.2.5, we have a computable Π_1 formula (with a parameter for $T_{\mathcal{M}}$) saying that x codes a path through $\mathcal{T}$, and another computable Π_1 formula saying that $\bar{u}$ lies in the boxes corresponding to the path coded by x. $\square$

The next lemma is the analogue of Lemma 5.2.6.

Lemma 5.3.5 (Basis). *There is a basis $b_1, b_2, \ldots$ for $K_{\mathcal{M}}$ that is Δ_2^0 relative to K.*

Proof. We proceed exactly as in the proof of Lemma 5.2.6. Since the relations IND_n are Δ_2^0 relative to K (uniformly), we apply a procedure Δ_2^0 relative to K to run through the elements of K in order, adding a given element to our basis just in case it is independent of those previously added. $\square$

The next lemma is the analogue of Proposition 5.2.7. This will complete the proof of Theorem 5.3.1.

Lemma 5.3.6 (Enumerating the complete diagram of the expansion). *Any copy of K computes the complete diagram of a copy of the expansion $K_{\mathcal{M}}$.*

Proof. Let $b_1, b_2, \ldots$ be a basis for $K_{\mathcal{M}}$ that is Δ_2^0 relative to K. Guessing at the basis, we enumerate the complete diagram of a copy of $K_{\mathcal{M}}$. We maintain the condition that if we have tentatively mapped $\bar{d}$ to $\bar{v}$ in K which we believe to be an initial segment $\bar{b}$ of the basis, and we have defined other elements $\bar{c}$ in terms of $\bar{d}$, then the theory $T_{\mathcal{M}}$ guarantees that what we have said about $\bar{d}$ is true on a rational box around $\bar{v}$. $\square$

Corollary 5.3.7. $\mathcal{R}_{\mathbb{Q}} \equiv_w^* \mathcal{R}$

Proof. It is easy to see that $\mathcal{R}_{\mathbb{Q}} \leq_w^* \mathcal{R} \leq (\mathcal{R}, (q)_{q \in \mathbb{Q}})$. By the main result of this section, $(\mathcal{R}, (q)_{q \in \mathbb{Q}}) \leq_w^* \mathcal{R}_{\mathbb{Q}}$. $\square$

5.4 The structure $\mathcal{R}_{int}$

Recall that $\mathcal{R}_{int}$ has just the real numbers, with the ordering, and unary predicates $P_{q,q'}$ for the intervals $[q, q')$ with dyadic rational endpoints. This can be thought of as the minimal structure that is able to recover the (preferred) binary expansions of the real numbers, as each initial segment of the binary expansion of a number corresponds exactly to the number being in a half-open interval of this sort. For instance, knowing that the binary expansion of x begins 0.10 corresponds exactly to knowing that $x \in [\frac{1}{2}, \frac{3}{4})$.

We could have substituted $\mathcal{R}_{int}$ in Theorem 5.3.1, with some modifications that we will mention shortly. However, these modifications would have forced us to re-verify many of the results of Section 2, and it is already proved in [**t**]hat $\mathcal{B} \equiv \mathcal{R}$, so we do not discuss the modifications in detail.

The first modification is to Lemma 5.3.4. The dyadic rationals are definable without quantifiers in $\mathcal{R}_{\mathbb{Q}}$, but in $\mathcal{R}_{int}$, they are Π_1 definable. (We have that $x = \frac{1}{2}$ iff $(x \in [\frac{1}{2}, 1)) \wedge (\forall y \in [\frac{1}{2}, 1))(x \leq y)$.) Thus, in $\mathcal{R}_{int}$ we do not have a computable Π_1 formula saying that x is not a dyadic rational, so to ensure that x codes an infinite set, we instead use the complement of the set that x would normally code. Every number has infinitely many zeroes in its binary expansion, so this ensures that every x codes an infinite set.

The second modification is simpler but more cumbersome to verify. Every use of open rational intervals and open rational boxes needs to be replaced with half-open dyadic rational intervals and boxes, and then every proof and Lemma needs to be re-verified.

Remark. If we wish to use this modified version of Theorem 5.3.1 to prove that $\mathcal{R}_{\mathbb{Q}} \leq_w^* \mathcal{R}_{int}$, it is both relevant and somewhat amusing to notice that $\mathcal{R}_{\mathbb{Q}}$ is, in fact, o-minimal, and that it has a very strange notion of algebraicity: The algebraic closure of any set of elements is simply that set together with all of $\mathbb{Q}$. Thus, a basis for $\mathcal{R}_{\mathbb{Q}}$ is simply the entire set of irrationals of $\mathbb{R}$.

We conclude this section with a proof that $\mathcal{R}_{int} \equiv_w^* \mathcal{B}$, which will allow us to use the result from [12] to prove that $\mathcal{R}_{int} \equiv \mathcal{R}$.

Lemma 5.4.1. $\mathcal{B} \leq_w^* \mathcal{R}_{int}$.

Proof. Given a copy K of $\mathcal{R}_{int}$, we can enumerate the preferred binary expansions of the reals in the interval $[0,1)$. For each such real, we get a function $f \in 2^\omega$ such that f has infinitely many 0's. Given such an f, we pass to a function $g \in \omega^\omega$, where $g(0)$ is the number of 1's before the first 0, and for $k > 0$, $g(k)$ is the number of 1's between the kth 0 and the $(k+1)^{st}$. This gives a copy of $\mathcal{B}$. $\square$

Lemma 5.4.2. $\mathcal{R}_I \leq_w^* \mathcal{B}$.

Proof. Given a copy of $\mathcal{B}$, we can enumerate the functions $g \in \omega^\omega$. From each g, we pass effectively to a function $f \in 2^\omega$ such that $g(0)$ is the number of 1's before the first 0 in f, and $g(k+1)$ is the number of 1's between the $(k+1)^{st}$ and $(k+2)^{nd}$ 0's in f. The functions $f \in 2^\omega$ are just the preferred binary expansions of reals in the interval $[0,1)$. The ordering on these reals corresponds to the lexicographic ordering on the functions f. For each dyadic rational q, we give a name in which we mark the first in the infinite sequence of 0's. For a function f that is the preferred binary expansion of a real r in the interval $[0,1)$, we cannot effectively determine whether $r = q$. However, for a pair $q < q' \in D$, we can effectively determine whether $r \in [q, q')$. We have a copy of the restriction of $\mathcal{R}_I$ to the interval $[0,1)$. For the full structure, we take pairs (z, f), where $z \in \mathbb{Z}$ and $f \in 2^\omega$ has infinitely many 0's. We take the lexicographic ordering on these pairs. The full set of dyadic rationals consists of the elements $z + q$, for q with a special name. We can determine membership in intervals with these endpoints. This gives a copy of $\mathcal{R}_I$. $\square$

Using the result of Downey, Greenberg, and Miller [12], we may conclude that $\mathcal{R}_{int}$ is equivalent to our other structures.

Proposition 5.4.3. $\mathcal{R}_{int} \equiv_w^* \mathcal{R}$.

Proof. We have just shown that $\mathcal{R}_{int} \equiv_w^* \mathcal{B}$. In [12], it is shown that $\mathcal{B} \equiv_w^* \mathcal{R}$. $\square$

5.5 Applying the general results

In this section, we apply the results from the previous section to show that various structures are equivalent to $\mathcal{R}$ in computing power. We begin with $\mathcal{R}_f = (\mathcal{R}, f)$, where f is analytic. In Section 2, we considered the case where f is the exponential function. In this case, $\mathcal{R}_f$ was *o*-minimal, but if f is the sine function, then $\mathcal{R}_f$ is not *o*-minimal.

Proposition 5.5.1. *Let* f *be analytic on* $\mathcal{R}$. *Then* $\mathcal{R}_f \equiv_w^* \mathcal{R}$.

Proof. Clearly, $\mathcal{R} \leq_w^* \mathcal{R}_f$. Let $\mathcal{R}_{bounded\, f}$ be the expansion of $\mathcal{R}$ by the family of functions f_z, for $z \in \mathbb{Z}$, where

$$f_z(x) = \begin{cases} f(x) & \text{if } x \in [z, z+1] \\ 0 & \text{otherwise} \end{cases}$$

Lemma 5.5.2. $\mathcal{R}_f \leq_w^* \mathcal{R}_{bounded\, f}$.

Proof. Let $(K, (f_z^K)_{z\in\omega})$ be a copy of $\mathcal{R}_{bounded\,f}$. We define f^K such that (K, f^K) is isomorphic to $\mathcal{R}_f$. Given $a \in K$, we can find, effectively in the field K, the integer z that is the "floor" of a; i.e., $z \le a < z + 1$. Then $f^K(a) = f_z^K(a)$. $\square$

We can now complete the proof of Proposition 5.5.1. It is clear that $\mathcal{R}_{bounded\,f} \le_w^* (\mathcal{R}_{bounded\,f}, (q)_{q\in\mathbb{Q}})$. Using Theorem 5.3.1, we get $(\mathcal{R}_{bounded\,f}, (q)_{q\in\mathbb{Q}}) \le_w^* \mathcal{R}_\mathbb{Q} \le_w^* \mathcal{R}$. This shows that $\mathcal{R}_f \le_w^* \mathcal{R}$. $\square$

Next, we reprove a result from [12]. Consider the reduct $\mathcal{R}^+$ of $\mathcal{R}$, without multiplication, but including addition, the ordering, and the constants 0 and 1.

Proposition 5.5.3. $\mathcal{R} \equiv_w^* \mathcal{R}^+$.

Proof. By Theorem 5.3.1, we have $\mathcal{R} \le_w^* \mathcal{R}_\mathbb{Q}$. It is easy to see that $\mathcal{R}_\mathbb{Q} \le_w^* (\mathcal{R}^+, (q)_{q\in\mathbb{Q}}) \le_w^* \mathcal{R}^+ \le_w^* \mathcal{R}$. $\square$

We also consider one example that is fairly simple, but that has a different flavor from our other examples. Let $(r_n)_{n\in\omega}$ be any sequence of elements of $\mathcal{R}$, and consider $(\mathcal{R}, (r_n)_{n\in\omega})$, the expansion of $\mathcal{R}$ with constants for those elements.

Proposition 5.5.4. $\mathcal{R} \equiv_w^* (\mathcal{R}, (r_n)_{n\in\omega})$.

Proof. Let $\mathcal{M}$ be the expansion of $\mathcal{R}_\mathbb{Q}$ with all of the structure of $\mathcal{R}$ and the constants r_n. We have $\mathcal{R} \le_w^* \mathcal{M} \le_w^* \mathcal{R}_\mathbb{Q} \le_w^* \mathcal{R}$. We use Theorem 5.3.1 for the second reduction. The others are clear. $\square$

5.6 Arbitrary continuous f

Recently, Andrews, Knight, Kuyper, Lempp, Miller, and Soskova [2] have extended the results of this chapter to *all* continuous functions: they showed that if f is continuous, then $\mathcal{B} \equiv_w^* (\mathbb{R}, +, \times, f)$. The key result they used was that if $\mathcal{I}$ is a jump ideal, then any enumeration of $\mathcal{I}$ computes a *running jump enumeration* of $\mathcal{I}$:

Theorem 5.6.1 ([2]). *Let $\mathcal{I}$ be a countable jump ideal. For any listing $\mathcal{E} = \{E_i : i \in \omega\}$ of the functions in $\mathcal{I}$, $\mathcal{E}$ computes a sequence of pairs $\mathcal{S} = \{(D_i, J_i) : i \in \omega\}$ such that*

- $\{D_i : i \in \omega\} = \{E_j : j \in \omega\}$ *(of course, possibly out of order), and*

- $J_i = (\bigoplus_{j\le i} D_j)'$.

Moreover, the computation of $\mathcal{S}$ from $\mathcal{E}$ can be done uniformly in $0'$.

This avoids the use of o-minimality to control injury. However, once we look at the parameters necessary to perform the relevant generic Muchnik reductions, o-minimality appears to return to relevance. *Generic Medvedev reducibility* (the uniform version of generic Muchnik reducibility) is defined analogously to $\le_w^*$:

Definition 35. *For structures* $\mathcal{A}, \mathcal{B}$ *(not necessarily countable), we say* $\mathcal{A}$ *is* generically Medvedev reducible *to* $\mathcal{B}$ — *and write* $\mathcal{A} \leq_s^* \mathcal{B}$ — *if for some e, there is a generic extension* $V[G]$ *in which* $\mathcal{A}, \mathcal{B}$ *are countable and*

$$V[G] \models \textit{For every } \omega\textit{-copy B of } \mathcal{B}, \ \Phi_e^B \cong \mathcal{A}.$$

($\leq_s^*$ will be reintroduced in chapter 8.) As with generic Muchnik reducibility, we may replace "there is a generic extension" with "in every generic extension." Most Muchnik reductions occurring in nature are not Medvedev reductions, but become Medvedev reductions when the top structure is expanded by a finite sequence of constants. This is true in the cases examined here. The results of this chapter show that given any continuous function f and a real code r for f, we have

$$(\mathcal{R}, r, 0') \geq_s^* (\mathbb{R}; +, \times, f)$$

. By contrast the results of this chapter show that given any *analytic* function f and a real code r for the *full first-order theory* of $(\mathbb{R}; +, \times, f)$, we have

$$(\mathcal{R}, r) \geq_s^* (\mathbb{R}; +, \times, f),$$

that is, $0'$ is no longer necessary. In case the theory of f is reasonably simple — as is conjectured to be the case for example with the exponential — this is a strong improvement.

Of course, the question remains whether this is actually a real difference:

Question 4. *Is there a continuous function* $f \colon \mathbb{R} \to \mathbb{R}$ *such that*

$$(\mathcal{R}, r) \not\geq_s^* (\mathbb{R}; +, \times, f)$$

(where r is the canonical real coding the full first-order theory of f)?

Chapter 6

Notes on structures computing every real

The work in this chapter is joint with Greg Igusa, and is work in progress; it appears here with his permission.

6.1 Introduction

We continue to study the structures $\mathcal{A}$ which compute every real, in the sense that if $V[G]$ is a generic extension in which $\mathcal{A}$ is countable and $r \in V$ is a ground real, then $V[G] \models$ "Every ω-copy of $\mathcal{A}$ iscomputes r." We first take a brief look at expansions of Cantor space by continuous functions which strictly increase its complexity, in contrast to the situation for $\mathcal{R}$. We then turn to ultrafilters, viewed as structures in a natural way, and show that $\leq_w^*$ restricted to ultrafilters refines the Rudin-Keisler ordering. Finally, we show that there is a $\leq_w^*$-least structure computing every real, and that there is a structure strictly above Cantor space but not below Baire space.

6.2 Functions on Cantor space

In the previous chapter, we proved that expansions of $\mathcal{R}$ by *analytic* functions do not lead to an increase in generic Muchnik complexity. Shortly after the work in that chapter was done, Andrews, Knight, Kuyper, Lempp, Miller, and Soskova [2] extended this to all *continuous* functions. However, the anIK14alogous result fails badly for Cantor space. In this section, we briefly discuss some expansions of Cantor space by continuous functions which yield strictly more complex structures — specifically, which are generically Muchnik equivalent to $\mathcal{R}$.

It is immediate that, if I have an expansion of Cantor space in which the set of infinite reals is relatively intrinsically computably enumerable (r.i.c.e.) in every generic extension in which the structure is countable, then this structure is generically Muchnik above Baire space: identify each infinite Cantor space real with its principal function. With slightly

more work, we can see that the same is true if the infinite reals are *co*-r.i.c.e. More generally, enumerating any countable dense set of reals results in Baire space:

Lemma 6.2.1. *Suppose $\mathcal{M}$ is an expansion of Cantor space such that there is a countable dense set D of Cantor space reals, which is c.e. in every copy of $\mathcal{M}$ (after collapse). Then $\mathcal{M} \geq_w^* \mathcal{R}$.*

Proof. First, without loss of generality assume D does not contain either of the endpoints of Cantor space (the all-0 or all-1 reals). We'll look at Cantor space as a linear order in the natural way: $x < y$ if on the first bit of difference n, $x(n) < y(n)$.

Before collapse (that is, in V), we can organize D as follows:

- We first enumerate (via a greedy algorithm) a sequence of elements of D forming an ω-chain with limit 1: $x_0 < x_1 < ... < 1$.

- As this is going on, we do the same to each x_i: so we enumerate e.g. a sequence $x_3 < x_{(0,4)} < x_{(1,4)} < x_{(2,4)} < ... < x_4$.

- Let X be the set of all reals we ever generate this way (so an element of X has the form x_σ for some finite sequence σ). Via back-and-forth, we have $X = D$.

- Now fix as a parameter a Cantor space real r coding the array X. Since the above was conducted entirely in V, such a real exists.

Now look at a copy C of $\mathcal{M}$ after collapse. Given any real x not in D, we can associate a Baire space real $f(x)$ using r:

- $f(x)(0)$ is the least n_0 such that $x < x_{(n_0)}$.

- $f(x)(1)$ is the least n_1 such that $x < x_{(f(x)(0), n_1)}$.

- Etc.

Of course, for a real in D, this process breaks - but since D is c.e. in C, we can use an appropriate injury argument to handle that occasion. When we realize that a real we are looking at is in D, we replace it with a real not yet in D, which agrees with our original real on sufficiently many digits, and continue the process. This gives an enumeration of all maps $\omega \to \omega$ in V; that is, a copy of $\mathcal{B}$. $\qquad \square$

Remark 6.2.2. *This generalizes to countable D which are somewhere dense. However, countable nowhere dense sets of reals, or uncountable sets of reals, might yield structures in between $\mathcal{W}$ and $\mathcal{R}$. Specifically, for $A \subseteq 2^\omega$, let $\mathcal{W}_A$ be the structure $\mathcal{W}$ together with a unary predicate for A. Then it is currently possible that for some A we have $\mathcal{W} <_w^* \mathcal{W}_A <_w^* \mathcal{R}$.*

By a similar (slightly easier, in fact, since no injury is necessary) argument, co-enumerating any countable dense set of Cantor space reals yields a structure computing Baire space.

Remark 6.2.3. *A similar result can be proved for certain structures associated to some more general metric spaces.*

As an application of the lemma above, we show:

Proposition 6.2.4. *There is a continuous $F\colon 2^\omega \to 2^\omega$ such that $\mathcal{W}_F \equiv^*_w \mathcal{R}$.*

Proof. Let F be the left shift operator: $F(g)(n) = g(n+1)$. Then the set of sequences with finitely many 1s is r.i.c.e. in $\mathcal{W}_F$: g has only finitely many 1s iff some finite iterate of F applied to g yields $\bar{0}$. By Lemma 6.2.1, $\mathcal{W}_F \equiv^*_w \mathcal{R}$. $\qquad\square$

While there is essentially only one left shift operator, there are many distinct right shift operators: for $\sigma \in 2^{<\omega}$, let $S_\sigma\colon 2^\omega \to 2^\omega$ be defined by $S_\sigma(g)(n) = g(n - |\sigma|)$, if $n \geq |\sigma|$, and $S_\sigma(g)(n) = \sigma(n)$ otherwise.

Proposition 6.2.5. *There are continuous F_0, F_1 such that $\mathcal{W}_{F_0} \equiv^*_w \mathcal{W}_{F_1} \equiv^*_w \mathcal{W}$ but $\mathcal{W}_{\{F_0,F_1\}} \equiv^*_w \mathcal{R}$.*

Proof. Let $F_0 = S_{\langle 0 \rangle}$, $F_1 = S_{\langle 1 \rangle}$. The set of sequences with finitely many 1s is r.i.c.e. in $\mathcal{W}_{\{F_0,F_1\}}$, since a real has only finitely many 1s iff it is gotten from $\bar{0}$ by applying F_0 and F_1 in some (finite) combination; so by Lemma 6.2.1, $\mathcal{W}_{\{F_0,F_1\}} \equiv^*_w \mathcal{R}$.

However, each reduct $\mathcal{W}_{F_0}$ and $\mathcal{W}_{F_1}$ is generically Muchnik equivalent to $\mathcal{W}$. We prove this for $\mathcal{W}_{F_0}$, the proof for $\mathcal{W}_{F_1}$ being identical. $\qquad\square$

6.3 Ultrafilters

In this section we examine ultrafilters on ω from the point of view of generic Muchnik reducibility. There is a natural way to view an ultrafilter as a structure in a finite language; we show that when so viewed, every ultrafilter computes every real, an ultrafilter is non-principal iff it lies above $\mathcal{R}$ iff it is not equivalent to $\mathcal{W}$, and that the generic Muchnik order on ultrafilters refines the Rudin-Keisler ordering.

Definition 36. *For U an ultrafilter on ω, let I_U be the structure with two sorts, "points" (corresponding to elements of ω) and "sets" (corresponding to elements of U). The language of I_U consists of the unary successor operation, S, on the sort of points, and the membership relation E on points $\times$ sets.*

For efficiency, we identify U and I_U below. An immediate starting point is the observation that ultrafilters compute every real. Indeed, every ultrafilter computes Cantor space:

Proposition 6.3.1. *If U is an ultrafilter, then $U \geq^*_w \mathcal{W}$.*

Proof. Every element of Cantor space is either an element of U, or the complement of an element of U; and exactly one of these holds. So from an enumeration of U we can easily produce an enumeration of $\mathcal{W}$. $\qquad\square$

In general, structures which compute every real in a "uniform" way will compute Cantor space.

In comparison with Baire space, things become slightly more complicated. It is easy to show that no nonprincipal ultrafilter is generically Muchnik reducible to Baire space (and that principal ultrafilters are generically Muchnik equivalent to Cantor space); in the opposite direction, it is unclear whether every ultrafilter computes Baire space. However, a broad class of them do:

Proposition 6.3.2. *Every Ramsey ultrafilter computes* $\mathcal{R}$.

Proof. Given a Ramsey (hence nonprincipal) ultrafilter U, the set of principal functions of elements of U is a set of total maps $\omega \to \omega$. In [2], it is shown that (in an appropriate generic extension) from any enumeration E_1 of the reals (viewed as elements of Cantor space) in V, together with an enumeration E_2 of a dominating family of functions (viewed as elements of Baire space) in V, we can compute Baire space. The proof is completed by noting that Ramsey ultrafilters yield dominating families: given any Ramsey ultrafilter U and $f \colon \omega \to \omega$, there is a $u \in U$ such that the principal function of u is everywhere greater than f. (To prove this, consider a coloring of pairs of natural numbers in which each natural has eventual color 0 but is colored 1 with a "large" initial segment.) □

Note that this approach fails badly for ultrafilters in general: consider any ultrafilter all of whose elements have positive upper density. However, we can extend this result to all ultrafilters which lie above any Ramsey ultrafilter, in an appropriate sense:

Definition 37. *If U, V are ultrafilters on ω, then U is* Rudin-Keisler reducible to V — *written "$U \leq_{RK} V$" — if there is some $f \colon \omega \to \omega$ such that $f^{-1}(X) \in V$ iff $X \in U$.*

Proposition 6.3.3. *If U, V are ultrafilters on ω with $U \leq_{RK} V$, then $U \leq_w^* V$.*

Proof. Let $f \colon \omega \to \omega$ be a Rudin-Keisler reduction from U to V. Since V computes every real, we may assume we have f as a parameter. In a generic extension where V is countable, we build a copy of U in stages. First, we get an enumeration $\{r_i : i \in \omega\}$ of all ground-model reals from V: the r_is are the sets in V, and their complements. It will now be enough to construct a listing of those ground-model reals which are in U (since it's easy to pass from such a list to a copy of U).

This is done by a priority argument, albeit a somewhat degenerate one. Let $\{v_i : i \in \omega\}$ be a listing of the reals in V. Given a ground model real r_i, say r_i *looks good* at stage s if for some $n < s$ we have $f^{-1}(r_i \restriction s) \restriction s \subseteq v_n$ — that is, if v_n looks like a witness to the statement "$f^{-1}(r_i) \in V$." Call such an n a *size witness* for r_i at stage s.

We will build a list of reals $s_{\langle i,j \rangle}$ as follows. Intuitively, $s_{\langle i,j \rangle}$ will be equal to r_i unless j fails to be a size witness for r_i. Formally, we let $s_{\langle i,j \rangle}(k) = 0$ iff

- j is a size witness for r_i at stage k, and

- $r_i(k) = 0$;

and $s_{\langle i,j \rangle} = 1$ otherwise.

It is easy to check that $\mathfrak{S} = \{s_{i,j} : i, j \in \omega\}$ is exactly U. We then effectively pass to an *injective* sub-enumeration of $\mathfrak{S}$, and use that to build a copy of U.   □

Combining the previous two results yields:

Corollary 6.3.4. *If $U \leq_{RK} V$ and U is Ramsey, then $V \geq_w^* \mathcal{R}$.*

This leaves two natural questions:

Question 5. *Does $\leq_w^*$ agree with $\leq_{RK}$ on ultrafilters?*

Question 6. *Does every ultrafilter compute Baire space?*

6.4   Degrees of structures computing all reals

In this section we prove a couple basic results about the generic Muchnik degrees of structures which compute every real. We show that there is a least such structure, and that there is a structure computing every real which lies strictly above Cantor space, $\mathcal{W}$, but does not compute the field of reals, $\mathcal{R}$; we also show that this latter structure is not computable from $\mathcal{R}$.

Definition 38. *A* computed real *is a set X of triples $(m, n, k) \in \omega^3$ such that*

- *$(m, n, k), (m, n', k') \in X$ implies $n = n', k = k'$.*

- *If $(m, n, k) \in X$ and $m' < m$, then there are n', k' such that $(m', n', k') \in X$.*

The interpretation of a computed real is as a partial map $\omega \to \omega$ whose domain is an initial segment of ω, and on input m outputs n after exactly k many steps if $(m, n, k) \in X$.

To each computed real X we associate a structure $S(X)$ as follows. The language of $S(X)$ consists of two sorts (or unary predicates) U and W which partition the structure; each sort consists of a countable infinity of elements. On the U sort, we have a unary function symbol S so that $(U^{S(X)}, S)$ is a copy of ω with successor; from now on we identify elements of U with the corresponding natural numbers. Finally, there is a ternary relation $E \subseteq U \times U \times W$ such that:

- *$S(X) \models \exists w \in W (E(m, n, v))$ iff for some k, $(m, n, k) \in X$.*

- *For each $w \in W$, there are at most one pair $m, n \in U$ such that $E(m, n, w)$.*

- *For infinitely many $w \in W$, there are no $m, n \in U$ with $E(m, n, w)$.*

Essentially, $S(X)$ *is the structure associated to X by forgetting when computations in X converged, but without taking projections (so there is no implicit jump). In the opposite direction, to an infinite (so, total) computed real X let $\pi(X) = \{(m, n) : \exists k((m, n, k) \in X)\}$ be the projection of X to a real.*

Finally, $\mathfrak{P}$ consists of the disjoint union of infinitely many copies of $S(X)$, for each computed real X in the ground model V.

Clearly $\mathfrak{P}$ computes every real. What is less obvious is that it is the minimal such structure under generic Muchnik reducibility:

Proposition 6.4.1. *Suppose $\mathcal{A}$ computes every real. Then $\mathcal{A} \geq_w^* \mathfrak{P}$.*

Proof. Let $\mathbb{P}$ be the Cohen forcing associated to $\mathcal{A}$: that is, a condition in $\mathbb{P}$ is a finite injective sequence of elements of $\mathcal{A}$. Since $\mathcal{A}$ computes every real, a fortiori $\mathbb{P}$-generic copies of $\mathcal{A}$ compute every real. In an appropriate generic extension $V[G]$, let A be an ω-copy of $\mathcal{A}$. Let the *norm* of a condition p, denoted $|p|$, be the maximum of the union of its domain and range (viewed as subsets of ω) if $p \neq \emptyset$, and 0 otherwise. To each pair $(e, p) \in \omega \times \mathbb{P}$, we (computably in A) associate a computed real $X_{(e,p)}$ as follows: $(m, n, k) \in X_{(e,p)}$ iff there is some $q \leq p$ such that

- $|q| = k$ and $\Phi_e^q(m)[k] \downarrow = n$,

- for any $q' \leq p$ with $|q'| < k$, $\Phi_e^q(m)[|q'|] \uparrow$, and

- for all $\hat{m} \leq m$ and $q_0, q_1 \leq p$ with $|q_0| = |q_1| = k$, we have

$$\neg[\Phi_e^{q_0}(\hat{m})[k] \downarrow \neq \Phi_e^{q_0}(\hat{m})[k] \downarrow].$$

(That is, there are not two conditions of norm $\leq k$ which reveal possible disagreement below p.)

Note that there are only finitely many conditions of a fixed norm, so this is effective relative to A. Now build a structure $\mathfrak{S}$ consisting of the disjoint union of infinitely many copies of each $S(X_{(e,p)})$; we will see that $\mathfrak{S} \cong \mathfrak{P}$.

First, note that the structure associated to each finite computed real appears (infinitely often) in $\mathfrak{S}$: each finite computed real is computable, so corresponds to some $S(X_{(e,p)})$ where e ignores the parameter p. Moreover, if r is a real in the ground model, then — since every sufficiently generic copy of $\mathcal{A}$ computes r — there must be an (e, p) such that $\pi(X_{(e,p)}) = r$: let p be a condition forcing $\Phi_e^G = r$.

So we need only show that only ground model reals appear. Fix $(e, p) \in \omega \times \mathbb{P}$. If $X_{(e,p)}$ is finite, then there is nothing to show. If $X_{(e,p)}$ is infinite, then we have

$$\pi(X_{(e,p)}) = \{(m, n) : \exists q \leq p((q \Vdash \Phi_e^G(m) = n))\}.$$

But this set is in the ground model. So we are done. □

The exact relationship between $\mathfrak{P}$ and more natural structures remains somewhat unclear. In particular:

Question 7. *is* $\mathfrak{P} \equiv^*_w \mathcal{W}$*?*

Turning to the interval between $\mathcal{W}$ and $\mathcal{R}$, although we cannot yet construct a structure of strictly intermediate complexity, we show that — assuming the continuum hypothesis — there is not an "hourglass" phenomenon:

Proposition 6.4.2 (ZFC+CH). *There is a structure $\mathcal{A}$ such that $\mathcal{A} >^*_w \mathcal{W}$ but $\mathcal{A} \not\geq^*_w \mathcal{R}$.*

Proof. Let $\mathcal{A}$ be the disjoint union of $\mathcal{R}^*$ (a size-continuum countably saturated real closed field) and the linear order $(\omega_2, <)$. Clearly $\mathcal{A} \geq^*_w \mathcal{W}$; we need only show that $\mathcal{A} \not\geq^*_w \mathcal{R}$.

By the same arguments as in [32], if $\mathcal{A} \geq^*_w \mathcal{R}$ then the set FT of elements of the $\mathcal{R}^*$-part of $\mathcal{A}$ which are infinitely close to a finite transcendental real would have (in V) an infinitary Σ_2 definition over $\mathcal{A}$. However, let $\mathcal{E}$ be an $\mathcal{L}_{\omega_1\omega}$-elementary substructure of $\mathcal{A}$ containing the whole $\mathcal{R}^*$-part of $\mathcal{A}$, whose ordinal part is isomorphic to ω_1^V. Then the same infinitary Σ_2 definition defines FT in $\mathcal{E}$. However, in Chapter 3 we showed that any structure computing every real is generically Muchnik above $(\omega_1, <)$, so $\mathcal{E} \leq^*_w \mathcal{W}$; this contradicts the fact that $\mathcal{W} \not\geq^*_w \mathcal{R}$. $\square$

Question 8. *Can we remove CH from the above argument?*

Proposition 6.4.3 (ZFC). *The structure $\mathcal{A}$ above is not generically Muchnik reducible to $\mathcal{R}$.*

Proof. It is enough to show that $\mathcal{R} \not\geq^*_w (\omega_2, <)$, which is true for cardinality reasons (CH) together with the fact that $(\omega_2, <)$ is rigid:

- Let $V[H]$ be a generic extension where ω_2^V is made countable.

- Let $\mathbb{P}$ be the Cohen forcing associated to $\mathcal{R}$; essentially, $\mathbb{P} = \mathcal{R}^{<\omega}$. Note that $\mathbb{P}$ exists in V.

- Let G be a $\mathbb{P}$-generic filter *over* $V[H]$, and let R_G be the associated ω-copy of $\mathcal{R}$. Note that G is also $\mathbb{P}$-generic over V, and in $V[G]$ ω_2^V is still uncountable.

- By genericity over $V[H]$, there must be some $p \in G$ and $e \in \omega$ such that p forces that $\Phi_e^{R_G}$ is a copy of $(\omega_2, <)$. But then in $V[G]$, $\Phi_e^{R_G}$ is a certain countable structure.

- However, since ω_2 is still uncountable in $V[G]$, we cannot have $\Phi_e^{R_G} \cong (\omega_2^V, <)$: isomorphism in a forcing extension is just elementary equivalence over $\mathcal{L}_{\infty\omega}$, and $\mathcal{L}_{\infty\omega}$-equivalent ordinals are equal. So we are done.

$\square$

Question 9. *Is there a structure strictly between (in the sense of generic Muchnik reducibility) Cantor space and Baire space?*

Chapter 7

Limit computability and ultrafilters

The work here appeared as [1], and is joint with Uri Andrews, Mingzhong Cai, and David Diamondstone; it appears here with their permission.

7.1 Introduction

If X is a set of natural numbers, we can view X as a countable array of sets in a natural way:

Definition 39. *Let $X \subseteq \omega$. Then the we let $X_i = \{j : \langle i, j \rangle \in X\}$ and $X^j = \{i : \langle i, j \rangle \in X\}$ be the ith column and jth row of X, respectively.*

We can then consider the eventual behavior of each row of X. In particular, in case every row of X is finite or cofinite — that is, if $\lim_s(X^j(s))$ exists for every j — then we can define the *limit* of X as

$$\lim(X) = \{j : \lim_s(X^j(s)) = 1\} = \{j : X^j \text{ cofinite}\}.$$

If $Z = \lim(Y)$ and $Y \leq_T X$, we say Z is *limit computable* relative to X.

Shoenfield showed that $A \leq_T X'$ if and only if $A = \lim(Y)$ for some $Y \equiv_T X$. While this is only one of many characterizations of the jump, limit computability is of particular interest because it suggests a wide class of generalizations: given any notion of "generalized limit," we can consider the collection of sets which are *generalized limit computable* relative to a given X. These in turn yield generalized jump operators, that take a set X to the collection of sets which are generalized limit computable relative to X.

In this paper, we investigate limit computability along (nonprincipal) ultrafilters. For each ultrafilter $\mathcal{U}$, we introduce a function $\delta_{\mathcal{U}}$ taking each Turing degree $\mathbf{a}$ to the collection of sets "$\mathcal{U}$-limit computable" in members of $\mathbf{a}$. Besides establishing its basic properties, we characterize the possible values of $\delta_{\mathcal{U}}(\mathbf{a})$, define a notion of "lowness for ultrafilters" and study the question of characterizing these degrees, and examine the ordering on ultrafilters induced by the construction $\mathcal{U} \mapsto \delta_{\mathcal{U}}$.

We recall the definition of an ultrafilter:

Definition 7.1.1. *A set $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is an* ultrafilter *if it satisfies the following properties:*

1. $\omega \in \mathcal{A}$, $\emptyset \notin \mathcal{A}$.

2. If $X \in \mathcal{A}$ and $X \subseteq Y \subseteq \omega$, then $Y \in \mathcal{A}$.

3. If $X, Y \in \mathcal{A}$, then $X \cap Y \in \mathcal{A}$.

4. For every $X \subseteq \omega$, $X \in \mathcal{A}$ or $(\omega - X) \in \mathcal{A}$.

Additionally, an ultrafilter is nonprincipal*if it contains no finite set. Although the existence of nonprincipal ultrafilters is not provable in* ZF *alone, it follows from the axiom of choice that there are $2^{2^{\aleph_0}}$-many ultrafilters on ω, the maximum number possible.*

Throughout this paper, we will always write "ultrafilter" to mean "nonprincipal ultrafilter."

Using the fourth ultrafilter axiom, we can take the limit along any ultrafilter of any sequence $(X_i)_{i \in \omega}$ of sets, and for nonprincipal ultrafilters this notion of limit agrees with the classical one when each X_i is finite or cofinite. Taking limits along an ultrafilter then yields the notion of limit computability along an ultrafilter, which in turn yields a class of operators on Turing degrees.

Formally, we proceed as follows. We begin by defining the limit, along an ultrafilter, of an array of reals:

Definition 7.1.2. *For a sequence of sets $X = (X_i)_{i \in \omega}$ and an ultrafilter $\mathcal{U}$, we let*

$$\lim_{\mathcal{U}}((X_i)_{i \in \omega}) = \{j : \{i : j \in X_i\} \in \mathcal{U}\}\} = \{j : X^j \in \mathcal{U}\}.$$

Note that, as in the case of classical limit computability, each column X_i functions as an approximation to the limit set $\lim_{\mathcal{U}}(X)$, and dually each row X^j determines the jth bit of $\lim_{\mathcal{U}}(X)$.

We can now define the maps, $\delta_{\mathcal{U}}$:

Definition 7.1.3. *Fix an ultrafilter $\mathcal{U}$. For a Turing degree $\mathbf{a}$, we let*

$$\delta_{\mathcal{U}}(\mathbf{a}) = \{\lim_{\mathcal{U}}((X_i)_{i \in \omega}) : (X_i)_{i \in \omega} = X \in \mathbf{a}\}.$$

Remark 7.1.4. *Note that $\delta_{\mathcal{U}}(\mathbf{a}) = \{\lim_{\mathcal{U}}((X_i)_{i \in \omega}) : (X_i)_{i \in \omega} = X \leq_T \mathbf{a}\}$: for $X \leq_T \mathbf{a}$, if we fix some set $Y \in \mathbf{a}$, then replacing the 0th column of X by Y results in an array of degree $\mathbf{a}$ whose rows have the same $\mathcal{U}$-limits as those of X.*

It is the maps $\delta_{\mathcal{U}}$ and their images, especially $\delta_{\mathcal{U}}(\mathbf{0})$, which are the subject of this article. We call maps of the form $\delta_{\mathcal{U}}$ *ultrafilter jumps*.

We begin by establishing basic closure properties of sets of the form $\delta_{\mathcal{U}}(\mathbf{a})$; this culminates in the following characterization, which is our main result. Recall that a *Scott set* is a collection of reals closed under Turing reducibility and join, and which contains an infinite tree $T \subseteq 2^{<\omega}$ only if it also contains an infinite path through T.

Theorem 7.1.5. *For a Turing degree* $\mathbf{a}$*, the following are equivalent:*

- $\mathfrak{S}$ *is a countable Scott set containing* $\mathbf{a}'$*.*

- *There is some ultrafilter* $\mathcal{U}$ *such that* $\delta_{\mathcal{U}}(\mathbf{a}) = \mathfrak{S}$*.*

Next, we look at how a single ultrafilter jump can behave with respect to different degrees. We call a degree $\mathbf{a}$ *u-low* if there is some ultrafilter $\mathcal{U}$ such that $\delta_{\mathcal{U}}(\mathbf{a}) = \delta_{\mathcal{U}}(\mathbf{0})$. Using techniques similar to those in the proof of the main theorem, we show the following:

Theorem 7.1.6. *If* $\mathbf{a}$ *is bounded by a 2-generic or is computably traceable, then* $\mathbf{a}$ *is u-low. Conversely, any degree which computes a* DNR_2 *or is high is not u-low.*

We then turn our attention to the structure on the class of all ultrafilters provided by the construction $\mathcal{U} \mapsto \delta_{\mathcal{U}}$. Our main result in this direction is that the partial order induced by this construction is related to a classical reducibility notion on ultrafilters:

Definition 7.1.7. *For ultrafilters* $\mathcal{U}, \mathcal{V}$*, we write "*$\mathcal{U} \leq \mathcal{V}$*" if* $\delta_{\mathcal{U}}(\mathbf{a}) \subseteq \delta_{\mathcal{V}}(\mathbf{a})$ *for all degrees* $\mathbf{a}$ *on some cone, and "*$\mathcal{U} \equiv \mathcal{V}$*" if* $\mathcal{U} \leq \mathcal{V}$ *and* $\mathcal{V} \leq \mathcal{U}$*.*

Theorem 7.1.8. *The partial order on ultrafilters induced by* $\leq$ *is a quotient of the Rudin-Keisler ordering of ultrafilters on* ω*.*

We also show that the operation of composition of ultrafilter jumps is captured by a binary operation on ultrafilters:

Theorem 7.1.9. *There is a binary operation* $*$ *such that for every pair of ultrafilters* $\mathcal{U}$ *and* $\mathcal{V}$*, we have*

$$\delta_{\mathcal{U}} \circ \delta_{\mathcal{V}} = \delta_{\mathcal{U}*\mathcal{V}}.$$

This operation is immediately seen to be compatible with the ordering, $\leq$, so that we have the structure of a partially ordered semigroup.

Finally, we end by presenting two directions for further research. Additionally, throughout this paper we raise a number of questions arising from the theorems above, which remain open.

Throughout this paper, we will need the following pair of basic combinatorial facts:

Definition 7.1.10. *A collection* $\{X_i : i \in I\}$ *of sets is* free *if every finite Boolean combination is infinite. In particular, each* X_i *and its complement must be infinite and the* X_i *must be distinct.*

Fact 1. *Suppose $\{X_i : i \in I\}$ is free, and $J \subseteq I$. Then there is an ultrafilter $\mathcal{U}$ with $\{i \in I : X_i \in \mathcal{U}\} = J$.*

Fact 2. *We can effectively find large free sets. Specifically, there is a total Φ_e such that*

$$\{\Phi_e^X : X \subseteq \omega\}$$

is a free set.

To prove Fact 1, by freeness every finite intersection of elements of $\{X_j : j \in J\} \cup \{\overline{X}_i : i \notin J\}$ is infinite, and so there is an ultrafilter containing $\{X_j : j \in J\} \cup \{\overline{X}_i : i \notin J\}$.

To prove Fact 2, construct a computable function $\iota : 2^{<\omega} \to 2^{<\omega}$ which

- builds reals along paths: $\sigma \prec \tau \iff \iota(\sigma) \prec \iota(\tau)$, and

- forces all Boolean combinations to be large: for every $I \subseteq 2^n$, the set

$$\{j : \forall \sigma \in 2^n (\iota(\sigma)(j) = 1 \iff \iota(\sigma) \in I)\}$$

has size at least n.

We then let $\Phi_e^X = \iota(X)$. (Note that, in fact, we have $\Phi_e^X \equiv_T X$.)

Our notation and terminology are mostly standard, except for our notation for rows and columns (see Definition 39). For background on computability theory and set theory, we refer to [13] and [35], respectively. For background on ultrafilters, see [10].

Finally, a word of reassurance: since ultrafilters usually arise in the context of set theory, it is reasonable to worry that answers to questions about the maps $\delta_\mathcal{U}$ may be independent of ZFC. However, since the action of $\delta_\mathcal{U}$ on a degree $\mathbf{a}$ is determined by countably much information about $\mathcal{U}$, most relevant questions are at worst Π_2^1, and hence set-theoretically absolute (see chapter 25 of [35]). Indeed, with two exceptions, set theory will not be a serious concern in this article. The exceptions are proposition 7.5.1 — where we examine properties of a natural ordering of ultrafilters arising from the construction $\mathcal{U} \mapsto \delta_\mathcal{U}$ — and section 7.6, where we mention a set-theoretic direction for further research.

7.2 Basic Properties of $\delta_\mathcal{U}$

In the previous section, we motivated the study of the functions $\delta_\mathcal{U}$ by drawing a comparison with the Turing jump. We begin this section by elaborating on that analogy. The following lemma shows that each function $\delta_\mathcal{U}$ dominates the Turing jump in a completely uniform way:

Lemma 7.2.1. *There are Turing functionals $\Phi_{e_0}, \Phi_{e_1}, \Phi_{e_2}$ witnessing the following (for every ultrafilter $\mathcal{U}$):*

1. $\delta_{\mathcal{U}}$ grows at least as fast as the Turing jump: for every $Y = \lim f(x, s)$, we have $\lim_{\mathcal{U}}(\Phi_{e_0}^f) = Y$.

2. $\delta_{\mathcal{U}}$ strictly dominates the Turing jump: for every set X, we have $\lim_{\mathcal{U}}(\Phi_{e_1}^X) \notin \Delta_2^0(X)$.

3. For every set X, we have $\lim_{\mathcal{U}}(\Phi_{e_2}^X) \not\leq_T \lim_{\mathcal{U}}(X)$, that is, $\delta_{\mathcal{U}}(\deg(X))$ has no top element.

Proof. (1) follows from the relativized limit lemma. Suppose f is a total X-computable function such that

$$\forall x, \lim_{s \to \infty} f(x, s) \downarrow = Y(x).$$

Let Φ_{e_0} be defined by

$$\Phi_{e_0}^f(\langle i, j \rangle) = f(j, i).$$

Then since $\mathcal{U}$ contains all cofinite sets we have $\lim_{\mathcal{U}}(\Phi_{e_0}^f) = Y$.

For (2), say that a set Z has the *limit property* if for all j, $\lim_{i \to \infty} Z(\langle i, j \rangle)$ exists. To prove part (ii) we need only construct a $Z \leq_T X$ such that for all nonprincipal $\mathcal{U}$ and all $\hat{Z} \leq_T X$ with the limit property, we have $\lim_{\mathcal{U}}(Z) \neq \lim_{\mathcal{U}}(\hat{Z})$. To do this, we proceed as follows. For $e, s \in \omega$, let

$$n_{e,s} = \max\{j : \Phi_e^X(\langle j, e \rangle)[s] \downarrow\}, \quad v_{e,s} = \Phi_e^X(\langle n_{e,s}, e \rangle)$$

(with the convention that $v_{e,s} = 0$ if $n_{e,s}$ is undefined). Now let Z be defined by

$$Z(\langle k, e \rangle) = 1 - v_{e,k},$$

and note that $Z \leq_T X$. The proof of (iii) is completed by noting that whenever Φ_e^X is the characteristic function of a set with the limit property, then

$$\lim_{k \to \infty} Z(\langle k, e \rangle) \downarrow = 1 - \lim_{k \to \infty} \Phi_e^X(\langle k, e \rangle),$$

so $\lim_{\mathcal{U}}(Z)(e) = 1 - \lim_{\mathcal{U}}(\Phi_e^X)(e)$, and hence $\lim_{\mathcal{U}}(Z)$ is not Δ_2^0. This construction, moreover, is effective, so we get the desired index e_1.

The proof of (3) is similar to that of (2). $\qquad\square$

Lemma 7.2.1 raises the problem of classifying the possible images of $\delta_{\mathcal{U}}(\mathbf{a})$.

Lemma 7.2.2. $\delta_{\mathcal{U}}(\mathbf{a})$ *is a Turing ideal, that is, closed under $\oplus$ and $\leq_T$.*

Proof. Closure under $\oplus$ follows from the fact that

$$\lim_{\mathcal{U}}(\{A_i\}_{i \in \omega}) \oplus \lim_{\mathcal{U}}(\{B_i\}_{i \in \omega}) = \lim_{\mathcal{U}}(\{A_i \oplus B_i\}_{i \in \omega}).$$

To show that $\delta_{\mathcal{U}}(\mathbf{a})$ is closed under $\leq_T$, fix $A = (A_i)_{i \in \omega}$ and suppose $\Phi_e^{\lim_{\mathcal{U}}(A)} = B$. Then let

$$C_i = \{j : \Phi_e^{A_i}(j)[i] \downarrow = 1\}$$

and let $C = (C_i)_{i\in\omega}$. We claim that $\lim_{\mathcal{U}}(C) = B$. To see this, fix $k \in \omega$. There is some initial segment $\sigma \prec \lim_{\mathcal{U}}(A)$ such that $\Phi_e^\sigma(k) \downarrow$; since ultrafilters are closed under finite intersections, for $\mathcal{U}$-many i we have $\sigma \prec A_i$, and for cofinitely many i we have $i > |\sigma|$. Together, these facts imply that for $\mathcal{U}$-many i we have $C_i(k) = \Phi_e^\sigma(k) = B(k)$, which in turn implies $\lim_{\mathcal{U}}(C) = B$. $\qquad\square$

In fact, an even stronger closure property is satisfied:

Proposition 7.2.3. *For every ultrafilter $\mathcal{U}$ and degree $\mathbf{a}$, $\delta_{\mathcal{U}}(\mathbf{a})$ is a Scott set. In fact, as in 7.2.1 this is uniform: there is a single $e \in \omega$ such that for all X and $\mathcal{U}$, we have*

$$\lim_{\mathcal{U}}(X) \text{ is an infinite subtree of } 2^{<\omega} \quad \Rightarrow \quad \lim_{\mathcal{U}}(\Phi_e^X) \text{ is a path through } \lim_{\mathcal{U}}(X).$$

Proof. The intuition behind this proof is that a tree T in $\delta_{\mathcal{U}}(\mathbf{a})$ must be "named" by a sequence of trees $(X_i)_{i\in\omega}$ in $\mathbf{a}$, which — if T is to be infinite — must have arbitrarily long paths. By producing a sequence of increasingly long paths through this sequence of trees, we produce a sequence of sets in $\mathbf{a}$ which $\mathcal{U}$ sends to an infinite path through the named tree. Note that this is intuitively the same argument as for closure under Turing reducibility.

The details are as follows. Suppose $X = (X_i)_{i\in\omega} \in \mathbf{a}$ is such that $T = \lim_{\mathcal{U}}(X)$ is an infinite subtree of $2^{<\omega}$. First, we can assume without loss of generality that each column X_i is also a tree (i.e., downwards closed). To see this, let Y_i be the downwards-closed part of X_i, and let $Y = (Y_i)_{i\in\omega} \in \mathbf{a}$. Since $Y \subseteq X$ we have $\lim_{\mathcal{U}}(Y) \subseteq \lim_{\mathcal{U}}(X)$ — in fact, $\lim_{\mathcal{U}}(Y) = \lim_{\mathcal{U}}(X)$ — and $\lim_{\mathcal{U}}(Y)$ is clearly a tree; so any path we build through $\lim_{\mathcal{U}}(Y)$ will also be a path through T.

So assume X is a sequence of trees. Then X computes a sequence $P = (f_i)_{i\in\omega}$ of sets $f_i \subseteq X_i$ such that f_i is a finite path through X_i of maximal length $\leq i$ (the "$\leq i$" is required to make this search effective). We claim that $\lim_{\mathcal{U}}(P)$ is an infinite path through T.

Clearly $\lim_{\mathcal{U}}(P) \subseteq T$, is closed downwards, and is a path in T (that is, any two elements are comparable); so it is enough to show that $\lim_{\mathcal{U}}(P)$ is infinite. Towards a contradiction, suppose $\sigma \in \lim_{\mathcal{U}}(P)$ of length n is terminal. Then since ultrafilters are closed under finite intersections, we have that for $\mathcal{U}$-many i, $f_i \succcurlyeq \sigma$. Moreover, by definition of P, for all but n-many i, we have

$$|f_i| \leq n \iff ht(X_i) \leq n.$$

Together these imply that for $\mathcal{U}$-many i, X_i has height at most n, and so $\lim_{\mathcal{U}}(X)$ has height at most n as well, which is a contradiction.

By examining the argument above, it is clear that this is a uniform construction, that is, that the construction of P is uniformly computable in X and does not depend on $\mathcal{U}$. $\quad\square$

Remark 7.2.4. *Lemma 7.2.3 yields an alternate proof of the classical result in reverse mathematics that the theory WKL_0 is strictly weaker than the theory ACA_0 (see chapter VIII of [71]), as follows: via a greedy algorithm we can construct an ultrafilter $\mathcal{U}$ such that the set $\{e : W_e \in \mathcal{U}\}$ is Δ_4^0; this ensures that $\delta_{\mathcal{U}}(\mathbf{0})$ consists entirely of Δ_4^0 sets, and so is not arithmetically closed. This is genuinely different from the standard proof, which follows from*

iterating the Low Basis Theorem. In particular, neither lowness nor iterated forcing are used in the proof of 7.2.3.

In the following section, we will show that the converse of 7.2.3 holds: given any countable Scott set $\mathfrak{S}$ containing $\mathbf{0}'$, there is a nonprincipal ultrafilter $\mathcal{U}$ such that $\delta_{\mathcal{U}}(\mathbf{0}) = \mathfrak{S}$, and more generally if $\mathbf{a}' \in \mathfrak{S}$ then we can find a $\mathcal{U}$ with $\delta_{\mathcal{U}}(\mathbf{a}) = \mathfrak{S}$.

7.3 Building Scott sets

We now completely characterize the possible images of ultrafilter jumps by proving the converse of 7.2.3. This does not provide a characterization of the maps $\delta_{\mathcal{U}}$, however, since we only determine the possible *local* behaviors of those maps. However, in the next section we do make progress towards this goal, by studying what sorts of simultaneous behaviors can be realized by ultrafilter jumps.

Theorem 7.3.1. *Let $\mathbf{a}$ be a degree, and let $\mathcal{I}$ be a countable Scott set containing $\mathbf{a}'$. Then $\mathcal{I} = \delta_{\mathcal{U}}(\mathbf{a})$ for some nonprincipal ultrafilter $\mathcal{U}$.*

Proof of 7.3.1. Call a pair (A, B) with $A \in \mathbf{a}$ and $B \in \mathcal{I}$ an *axiom*; informally, we interpret (A, B) as meaning "A is mapped to B by $\lim_{\mathcal{U}}$." Precisely, for C a set of axioms, say that an ultrafilter $\mathcal{U}$ *satisfies* C if $\lim_{\mathcal{U}}(A) = B$ whenever $(A, B) \in C$. Since every family of sets, all of whose finite intersections are infinite, can be extended to a nonprincipal ultrafilter, satisfiability has a purely *combinatorial* definition: if $\mathcal{A} = \{(A_i, B_i) : i \in I\}$ is a set of axioms, we say $\mathcal{A}$ is *consistent* if for every $F \subseteq I$ finite and $n \in \omega$, the intersection

$$[\bigcap_{\substack{j \in F \\ m < n \\ B_j(m)=1}} (A_j)^m] \quad \cap \quad [\bigcap_{\substack{j \in F \\ m < n \\ B_j(m)=0}} \overline{(A_j)^m}]$$

is infinite. Equivalently, $\mathcal{A}$ is consistent if and only if there is a nonprincipal ultrafilter satisfying $\mathcal{A}$.

Remark 7.3.2. *In 7.4.5 we will consider a different notion of consistency — instead of "A gets mapped to B," our commitments will have the form "A and B get mapped to the same set."*

Fix $\mathcal{I} = \{Y_i : i \in \omega\}$, and let $\mathbf{a} = \{X_i : i \in \omega\}$; we will build the desired ultrafilter in stages. We will build a consistent set of axioms C such that (i) for every $A \in \mathbf{a}$ there is some $B \in \mathcal{I}$ with $(A, B) \in C$, and (ii) for every $B \in \mathcal{I}$ there is some $A \in \mathbf{a}$ such that $(A, B) \in C$. We handle (i) at even stages, and (ii) at odd stages:

- In (i), in deciding where to map a set $A \in \mathbf{a}$ we run the risk of contradicting already-enumerated axioms $(A_i, B_i)_{i < k}$ — for example, if the fifth rows of A and A_0 are identical, then the fifth bit of B must be $B_0(5)$. To find a $B \in \mathcal{I}$ to which it is "safe" to map A, it turns out to be equivalent to find a path through a certain infinite binary tree computable in the jump of the (finitely many) axioms built so far.

- In (ii), an apparent difficulty is posed by the fact that $\mathbf{a}$ cannot "see" the commitment we have already made, since right components of axioms lie outside $\mathbf{a}$; however, this turns out not to matter. Suppose $B \in \mathcal{I}$ and $(A_i, B_i)_{i<k}$ is consistent; then if A is "sufficiently different" from the A_is, the set $(A_i, B_i)_{i<k} \cup \{(A, B)\}$ is also consistent. So in deciding what should be mapped to B, we ignore B entirely, and simply choose some A which is sufficiently different from the sets we have enumerated on the left, so far.

Formally, we proceed as follows:

Even case. Suppose that we have $C_{2s} = \{(A_i, B_i) : i < 2s\}$, and that C_{2s} is consistent, and consider the set $X_s \in \mathbf{a}$. We will find a $B \in \mathcal{I}$ such that $C_{2s} \cup \{(X_s, B)\}$ is consistent. Let $D = \{(A_i)^j : i < 2s, B_i(j) = 1\} \cup \{\mathbb{N} - (A_i)^j : i < 2s, B_i(j) = 0\}$; intuitively, D is the collection of sets we have guaranteed are in the ultrafilter so far. Write $D = \{D_k : k \in \omega\}$, and note that this can be done effectively in $\bigoplus_{i<s} B_i := \hat{B} \in \mathcal{I}$. Say that $\sigma \in 2^{<\omega}$ is *temporarily consistent* if $|\sigma| = n$ and $\forall m < n$,

- $\sigma(m) = 1 \Rightarrow |(X_s)^m \cap (\bigcap_{j<n} D_j)| \geq n$, and

- $\sigma(m) = 0 \Rightarrow |(\mathbb{N} - (X_s)^m) \cap (\bigcap_{j<n} D_j)| \geq n$;

note that $\hat{B} \oplus X_s'$ can uniformly decide whether a $\sigma \in 2^{<\omega}$ is temporarily consistent. Let $T \subseteq 2^{<\omega}$ be the tree of temporarily consistent nodes; since C_{2s} is consistent by induction, T is infinite, and since $\mathcal{I}$ is a Scott ideal containing X_s' and $\hat{B} \in \mathcal{I}$ there is some $B \in \mathcal{I}$ whose characteristic function is a path through T. Then $C_{2s} \cup \{(X_s, B)\}$ is consistent, so let $C_{2s+1} = C_{2s} \cup \{(X_s, B)\}$.

Odd case. Suppose that we have a consistent set of axioms $C_{2s+1} = \{(A_i, B_i) : i < 2s+1\}$, and consider the set $Y_s \in \mathcal{I}$; we need to find some $A \in \mathbf{a}$ such that $C_{2s+1} \cup \{(A, Y_s)\}$ is consistent. Our main difficulty is that the condition C_{2s+1} we have built so far is not $\mathbf{a}$-computable — in $\mathbf{a}$, we can only see $\{A_i : i < 2s+1\}$ — so in order to guarantee consistency we will need to ensure that the axiom (A, Y) is consistent with any possible consistent set of axioms with left coordinates from among the A_i ($i < 2s + 1$). To do this, we use a modification of 2:

Definition 40. *A set X is* free over *a family of sets $Z = \{Z_i : i \in \omega\}$ if every finite Boolean combination of elements of Z, which is infinite, has infinite intersection with both X and $\omega - X$.*

Lemma 7.3.3. *We can find free sets in a uniformly effective manner. Specifically, there is an e such that for all $Z = \{Z_i : i \in \omega\}$, Φ_e^Z is free over $\{(Z_i)^j : i, j \in \omega\}$.*

Proof. We need to build X such that for every set B which can be written as a Boolean combination of finitely many elements of Z, either B is finite or both $B \cap X$ and $B \cap \overline{X}$ are infinite. Let $(B_i)_{i \in \omega}$ be a list of all Boolean combinations of elements of Z, with each

combination occurring infinitely often, such that for all i, $B_{2i} = B_{2i+1}$; note that such a B can be chosen recursively in Z. At stage 0, set $p_0 = \emptyset$ and say that all i *await attention*. At stage s, suppose we have defined a string $p_s \in 2^{<\omega}$ with length s. Say that j *requires attention* if $j < s$, and at the beginning of stage s, j awaits attention, and $s \in B_j$. Let i be the least number which requires attention, and let $p_{s+1} = p_s^\frown \langle 1 \rangle$ if i is even and $p_{s+1} = p_s^\frown \langle 0 \rangle$ if i is odd. From now on, say that i *is satisfied*, and move on to stage $s+1$ - at the beginning of which all j which were satisfied at the beginning of stage s remain satisfied, i is satisfied, and all other requirements await attention.

Let $X = \bigcup p_s$. To see that X has the desired property, first note by induction that for each $j \in \omega$, either B_j is finite or there is some stage s by which j is satisfied. Now, for B a finite Boolean combination of elements of Z which is infinite, let $I = \{j : B_j = B\} = \{j_0, j_0 + 1, j_1, j_1 + 1, ...\}$. Each time j_i is satisfied, a new element is added to $B \cap X$; each time $j_i + 1$ is satisfied, a new element is added to $B \cap \overline{X}$. So both $B \cap X$ and $B \cap \overline{X}$ are infinite. $\qquad\square$

To finish the proof of Theorem 7.3.1, we iterate Lemma 7.3.3 to build an $X \in \mathbf{a}$ such that for each $k \in \omega$, X^k is free over $\{A_i : i < 2s+1\} \cup \{X^j : j < k\}$; we then take $C_{2s+2} = C_{2s+1} \cup \{(A, X)\}$. $\qquad\square$

Having completely classified the sets of the form $\delta_{\mathcal{U}}(\mathbf{a})$ in terms of $\mathbf{a}$, we now face the question of classifying ultrafilter jumps themselves:

Question 1. *What conditions on a function $f \colon \{Turing\ degrees\} \to \{Scott\ sets\}$ ensure that $f = \delta_{\mathcal{U}}$ for some $\mathcal{U}$?*

One interesting special case is the following:

Question 2. *Is there a $\mathcal{U}$ such that $\delta_{\mathcal{U}}(\mathbf{a})$ is always arithmetically closed?*

This is partly motivated by Remark 7.2.4, which suggests that there may be further interaction between the study of the maps $\delta_{\mathcal{U}}$ and reverse mathematics.

Currently it is not clear how to approach this type of problem, largely because constructing ultrafilter jumps "to order" is quite difficult. We make some technical progress in this direction, however, in the following section, in which we study what *simultaneous* behaviors can be realized by ultrafilter jumps.

7.4   Lowness notions

Theorem 7.3.1 allows us to control the value of $\delta_{\mathcal{U}}(\mathbf{a})$ for a fixed degree $\mathbf{a}$; however, it says nothing about what *simultaneous* behaviors can occur.

First of all, it is obvious that if $\mathbf{b} \geq \mathbf{a}$, then $\delta_{\mathcal{U}}(\mathbf{b}) \supset \delta_{\mathcal{U}}(\mathbf{a})$, and so one particularly interesting question is the following: for what degrees $\mathbf{a}$ is there an ultrafilter $\mathcal{U}$ such that

$\delta_{\mathcal{U}}(\mathbf{a}) = \delta_{\mathcal{U}}(\mathbf{0})$? We will call such a degree *u-low*, and we will call a real *u-low* if it belongs to a *u*-low degree.

It is easy to see that $\mathbf{0}'$ is not *u*-low: by a standard diagonalization argument, $\mathbf{0}'$ computes an array $A = (A_i)_{i \in \omega}$ such that the *e*th row of the *e*th computable array has no agreement with A_e. More precisely:

Proposition 7.4.1. *If* $\mathbf{a}$ *contains a DNR$_2$ real, then* $\mathbf{a}$ *is not u-low.*

Proof. Such a degree $\mathbf{a}$ contains a set A such that for every e, if Φ_e is total then

$$\Phi_e(\langle i, e \rangle) \neq A(\langle i, e \rangle).$$

It follows that we can never have $\lim_{\mathcal{U}}(A) = \lim_{\mathcal{U}}(C)$ for any computable array C, so we are done. $\square$

As an aside, note that this rules out the most natural possible positive answer to Question 2:

Corollary 7.4.2. *No ultrafilter jump* $\delta_{\mathcal{U}}$ *is the "arithmetic closure" operator; that is, for every* $\mathcal{U}$ *there is some* $\mathbf{a}$ *such that* $\delta_{\mathcal{U}}(\mathbf{a}) \neq ARITH(\mathbf{a})$.

However, this does not rule out the existence of ultrafilters which are arithmetically closed in pathological ways, so Question 2 remains open.

In addition, high degrees are not *u*-low. Recall that a degree $\mathbf{a}$ is high if $\mathbf{a}' \geq \mathbf{0}''$.

Proposition 7.4.3. *If* $\mathbf{a}$ *is high, then* $\mathbf{a}$ *is not u-low.*

Proof. By Martin's Lemma, such a degree $\mathbf{a}$ computes a *dominant* function f which dominates every computable function. Using f we can compute a set A such that

$$\Phi_e(\langle i, e \rangle) \neq A(\langle i, e \rangle)$$

is true cofinitely often for each $e \in Tot$, i.e., for each e such that Φ_e is total. So as in the DNR$_2$ case we are done. $\square$

In light of Propositions 7.4.1 and 7.4.3, it is reasonable to ask whether *any* nonzero degree is *u*-low. In fact, many degrees are *u*-low, including every 2-generic and every computably traceable degree. We begin with a basic combinatorial lemma:

Lemma 7.4.4. *Suppose* $\{A_i : i \in \omega\}$ *and* $\{X_i : i \in \omega\}$ *are collections of sets of natural numbers. Then the following are equivalent:*

1. *There is an ultrafilter* $\mathcal{U}$ *such that for all* i, $\lim_{\mathcal{U}}(A_i) = \lim_{\mathcal{U}}(X_i)$. *(Note that here we think of each* A_i *and* X_i *as an array of sets, so they will each have their own rows* $(A_i)^j, (A_i)^j$ *and columns* $(A_i)_k, (X_i)_k$.)

2. *For every n, k, there is some $m > k$ such that for every $i, j < n$, we have*

$$(A_i)^j(m) = (X_i)^j(m).$$

If $C = \{(A_i, X_i) : i \in \omega\}$ is a collection of pairs of sets such that the above conditions hold, we call $\mathcal{C}$ a *consistent system*; note that this is a different sense of consistency that that used in 7.3.1.

Proof. (2) $\Rightarrow$ (1): Suppose condition (2) holds. Then letting $D_{i,j} = \{x : (A_i)^j(x) = (X_i)^j(x)\}$ be the set on which the jth rows of A_i and X_i agree, we have that $\mathcal{D} = \{D_{i,j} : i, j \in \omega\}$ has the finite intersection principle. Any ultrafilter $\mathcal{U} \supset \mathcal{D}$ witnesses (1), so we are done.

The proof of (1)$\Rightarrow$(2) is similar. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 7.4.5. *Every real bounded by a 2-generic is u-low.*

Recall that a real f is 2-generic if (when viewed as a filter in the poset $2^{<\omega}$) it *meets or avoids* every Σ_2^0 subset of $2^{<\omega}$: if $A \subseteq 2^{<\omega}$ is Σ_2^0 and $f \cap A = \emptyset$, then $\exists \tau \prec f (\forall \sigma \succ \tau, \sigma \notin A)$.

Proof. Fix G 2-generic; we will construct a $\mathcal{U}$ such that $\delta_{\mathcal{U}}(deg(G)) = \delta_{\mathcal{U}}(\mathbf{0})$. Let

$$Tot_G = \{e_0 < e_1 < ...\} = \{e : \Phi_e^G \text{ is total}\}.$$

For $i \in \omega$, let t_i be the first condition in G such that $t_i \Vdash$ "$\Phi_{e_j}^G$ is total" for every $j \leq i$; note that such conditions exist since G is 2-generic. This is the only point in the proof where full 2-genericity is required. (We do not need to take the *least* such conditions, but we do need the t_is to be successively stronger conditions: $t_0 \geq t_1 \geq ....$) Let $\mathbb{P} = \{p_j : j \in \omega\}$ be a listing of Cohen conditions.

We will construct recursive sets X_i such that there is an ultrafilter which maps X_i and $\Phi_{e_i}^G$ to the same set. These X_i will be defined column-by-column, with each column making an increasingly strong guess as to the corresponding column of $\Phi_{e_i}^G$. The complexity of the construction comes from the fact that these guesses must be made effectively, and also must cohere with each other; this second requirement is the reason for having X_i take into account the Φ_{e_j} with $j < i$ in the construction below. Note that the X_i are individually recursive, but the array $(X_i)_{i \in \omega}$ need not be recursive.

Construction 7.4.6. *We define the sets X_i ($i \in \omega$) as follows:*

1. *For $p_k \not\leq t_i$, the kth column of X_i is empty: $\{\langle j, k \rangle : j \in \omega\} \cap X = \emptyset$.*

2. *For $p_k \leq t_i$, we define a sequence of conditions $q_0, ..., q_i$ as follows:*

   - q_0 *is the lexicographically least condition $\leq p_k$ such that*

   $$\forall m < k, \Phi_{e_0}^{q_0}(\langle m, k \rangle) \downarrow .$$

- q_{j+1} *is the lexicographically least condition* $\leq q_j$ *such that*

$$\forall m < k, \Phi_{e_{j+1}}^{q_{j+1}}(\langle m, k \rangle) \downarrow.$$

Note that such q_j *exist since* $p_k \leq t_i \leq t_{i-1} \leq \ldots \leq t_0$. *We then define the kth column of* X_i *as by*

$$(X_i)_k = \{m : m < k \wedge \Phi_{e_i}^{q_i}(m) \downarrow = 1\}.$$

We claim that there is an ultrafilter $\mathcal{U}$ such that $\lim_{\mathcal{U}}(\Phi_{e_i}^G) = \lim_{\mathcal{U}}(X_i)$ for every i. By Lemma 7.4.4, it is enough to show that the pair of sequences

$$\{\Phi_{e_i}^G : i \in \omega\}, \{X_i : i \in \omega\}$$

satisfies the property 7.4.4(1).

To show this, fix n, k, and consider the set of conditions

$$E_{n,m} = \{p \in \mathbb{P} : \exists k > m (\forall i, j < n, (\Phi_{e_i}^p)^j(k) \downarrow = (X_i)^j(k))\}.$$

Each $E_{n,m}$ is Σ_1^0; we will show that G meets each $E_{n,m}$.

It will be enough to show that $E_{n,m}$ is dense below t_n — the 2-genericity of G, together with the fact that $E_{n,m}$ is Σ_1^0, means that G must then meet $E_{n,m}$. Towards this, we fix some condition $p \leq t_n$. There must be some k such that $k > m$ and $p_k \leq p$. Since $p_k \leq p \leq t_n$, the kth column of X_n was constructed according to step (2) of Construction 7.4.6. Let $q_n \leq p_k$ be the nth condition as defined in the construction of X_n. By the construction of X_n and the fact that $k > m$, we have, for every $i < n$,

$$\Phi_{e_i}^{q_n}(\langle m, k \rangle) = X_i(\langle m, k \rangle);$$

so $q_n \in E_{n,m}$. $\square$

The analogous question for measure remains unsolved.

Question 3. *Are sufficiently random reals u-low?*

By a similar argument to the proof of Theorem 7.4.5, we can show that another important computability-theoretic property implies u-lowness:

Theorem 7.4.7. *Computably traceable implies u-low.*

Recall that a degree $\mathbf{a}$ is computably traceable if for every $h \in \mathbf{a}$, there is a computable j such that $h(n) \in D_{j(n)}$ and $|D_{j(n)}| \leq 2^n$ for every n, where D_e is the canonical finite set coded by e. Note that since there are computably traceable degrees which are not 2-generic and vice versa, Theorems 7.4.7 and 7.4.5 compliment each other.

Proof. Let $\{A_i : i \in \omega\}$ be a list of all sets of degree $\leq_T \mathbf{a}$; as in the proof of 7.4.5, we will construct a collection $\{X_i : i \in \omega\}$ of recursive sets such that there is some ultrafilter $\mathcal{U}$ satisfying $\lim_{\mathcal{U}}(X_i) = \lim_{\mathcal{U}}(A_i)$ for every i. This ultrafilter will then satisfy $\delta_{\mathcal{U}}(\mathbf{a}) = \delta_{\mathcal{U}}(\mathbf{0})$.

To construct the X_i, we work in stages. Each X_i will have associated with it three functions: the *interval* function f_i, the *block* function g_i, and the *guessing* function h_i. We view A_i and X_i as arrays in the usual way, so that $A_i, X_i \subseteq \omega^2$; in order to construct X_i, we partition the full array ω^2 into "blocks," and partition the nth block into 2^n-many "intervals," and define X_i on each interval separately.

The functions g_i and f_i tell us how to perform this construction: $g_i(n)$ is the number of columns in the nth block, and $f_i(m)$ is the number of columns in the mth interval. (Recall that each block will be partitioned into exponentially-many intervals.) Now we let h_i be a computable function such that for every k, $D_{h_i(k)}$ is a finite set of size 2^k listing the possible behaviors of A_i on the (finitely many) values in the kth block and above the diagonal $\{\langle s, s \rangle : s \in \omega\}$; the existence of such an h_i is guaranteed by the assumption that $\mathbf{a}$ is computably traceable. We then define X_i so that X_i agrees with A_i on at least one interval in each block, by predicting A_i's behavior on the tth interval using the tth element of $D_{h_i(k)}$.

This describes the process for building a single X_i. To ensure that agreements between the X_is and the A_is occur across is, we make intervals of X_{i+1} correspond to blocks of A_i; this guarantees that the collection of pairs $\{(A_i, X_i) : i \in \omega\}$ forms a consistent system (see 7.4.4).

Precisely, the construction is the following:

- At stage 0 we have $f_0 \colon x \mapsto 1$ and $g_0 \colon x \mapsto 2^x$.

- At stage $i+1$, blocks from stage i become intervals and the new blocks are exponentially large collections of intervals. That is, $f_{i+1} = g_i$, $g_{i+1}(0) = f_{i+1}(0)$, and

$$g_{i+1}(n+1) = \sum_{j=2^{n+1}-1}^{2^{n+2}-1} f_{i+1}(j).$$

The h_i are then computable maps such that for every x, (the canonical code for) the finite set

$$A_i \upharpoonright \{\langle m, n \rangle : m \leq n \text{ and } \sum_{t=0}^{x-1} g_i(t) \leq n < \sum_{t=0}^{x} g_i(t)\}$$

is an element of $D_{h_i(x)} = \{s_1^{i,x} < s_2^{i,x} < ... < s_{2^x}^{i,x}\}$.

We then let X_i be defined by copying the set coded by $s_m^{i,x}$ on the mth interval in the xth block. It is easy to see that for every j, there is at least one interval in the jth block such that X_i and A_i agree on the first j-many rows. Since our construction nests blocks at level i inside intervals at level $i + 1$, it is now easy to see by Lemma 7.4.4 that the family $\{(A_i, X_i) : i \in \omega\}$ is consistent. $\square$

The above results provide a wide swath of u-low degrees. However, our knowledge of u-lowness is still very incomplete. The following question remains open:

Question 4. *Is there an exact characterization of u-lowness in terms of classical computability-theoretic properties?*

Less ambitiously, note that no Δ_2^0 degree is either 2-generic or computably traceable, and so the following question remains open:

Question 5. *Is there a nonzero Δ_2^0 real which is u-low?*

Finally, while investigating u-lowness, an even stronger notion of weakness with respect to ultrafilters arises. We say a degree $\mathbf{a}$ is *u-trivial* if $\delta_{\mathcal{U}}(\mathbf{a}) = \delta_{\mathcal{U}}(\mathbf{0})$ for *every* ultrafilter $\mathcal{U}$.

Question 6. *Is there a nonzero u-trivial degree?*

Any u-trivial degree must be low: if X is not low, then there is a Scott set containing $\emptyset'$ and not containing X', and by Theorem 7.3.1 there is an ultrafilter $\mathcal{U}$ such that $\delta_{\mathcal{U}}(deg(X)) \neq \delta_{\mathcal{U}}(REC)$. In particular, a positive answer to Question 6 would yield a strong positive answer to 5.

7.5 Comparing ultrafilters

We now turn to what the construction of the maps $\delta_{\mathcal{U}}$ can tell us about the set of ultrafilters. We begin by defining a natural preorder arising from these maps, and then turn to a natural associated algebraic (semigroup) structure; we end by presenting a connection with a classical structure on ultrafilters, the *Rudin-Keisler order*. This section is self-contained, but for background and further information on the Rudin-Keisler order, and orderings on ultrafilters in general, see [10], especially chapters 9 and 16.

Definition 41. *For $\mathcal{U}, \mathcal{V}$ ultrafilters, let $\mathcal{U} \leq \mathcal{V}$ if for some degree $\mathbf{b}$, we have $\delta_{\mathcal{U}}(\mathbf{a}) \subseteq \delta_{\mathcal{V}}(\mathbf{a})$ for all $\mathbf{a} \geq_T \mathbf{b}$; that is, $\mathcal{U} \leq \mathcal{V}$ if $\delta_{\mathcal{V}}$ dominates $\delta_{\mathcal{U}}$ on a cone. We write $\mathcal{D}_{ult}$ for the resulting partial order on (equivalence classes of) ultrafilters.*

Note that $\mathcal{U} < \mathcal{V}$ does *not* imply that, on a cone, $\delta_{\mathcal{U}}(\mathbf{a}) \subsetneq \delta_{\mathcal{V}}(\mathbf{a})$. Indeed, it is not clear whether such a situation ever occurs.

Question 7. *Are there $\mathcal{U}, \mathcal{V}$ such that $\delta_{\mathcal{U}}(\mathbf{a}) \subsetneq \delta_{\mathcal{V}}(\mathbf{a})$ for all $\mathbf{a}$ (on a cone)?*

Proposition 7.5.1. *$\mathcal{D}_{ult}$ is ω_1-directed: given any ω_1-sized collection $\{\mathcal{U}_\eta : \eta \in \omega_1\}$ of ultrafilters, there is a $\mathcal{V}$ with $\mathcal{U}_\eta < \mathcal{V}$ for every η.*

Proof. We use Fact 2 to construct an ultrafilter $\mathcal{V}$ which dominates each $\mathcal{U}_\eta$ on a cone. Let $h : \mathbb{R} \to \omega_1$ be a function such that for each $\alpha \in \omega_1$, the set $\{r : h(r) > \alpha\}$ contains a

cone, and which is *Turing invariant*: $r \equiv_T s \Rightarrow h(r) = h(t)$. For example, we could take $h \colon r \mapsto \omega_1^r$.

Now fix, for each real r, a real $\hat{r}$ such that $\hat{r} \geq_T s$ for every $s \in \delta_{\mathcal{U}_\eta}(deg(r))$ with $\eta < h(r)$. Using 2 we can construct an ultrafilter $\mathcal{V}$ such that $\hat{r} \in \mathcal{V}(deg(r))$ for every real r. This $\mathcal{V}$ dominates each $\mathcal{U}_\eta$ on a cone, so we have $\mathcal{U}_\eta < \mathcal{V}$ for every $\eta \in \omega_1$. □

Note that this argument cannot be easily extended to give upper bounds of larger sets of ultrafilters. Indeed, it is consistent that there are exactly ω_2-many ultrafilters on ω, in which case ω_1-directedness is the most we could hope for.

Question 8. *What can be said about* $|\mathcal{D}_{ult}|$? *(Note that we have* $2^{\aleph_0} < |\mathcal{D}_{ult}| \leq 2^{2^{\aleph_0}}$; *the second inequality is trivial, and the first follows from an argument similar to that of Proposition 7.5.1. Moreover, it is consistent — and follows from* **GCH**— *that* $|\mathcal{D}_{ult}| = 2^{2^{\aleph_0}}$.)

Additionally, the proof of Proposition 7.5.1 says nothing about the *optimality* of the upper bound constructed.

Question 9. *What sets of ultrafilters have least upper bounds in* $\mathcal{D}_{ult}$?

Note that it is not even clear whether *finite* sets of ultrafilters have least upper bounds.

We now show that the set of ultrafilters carries a natural semigroup structure which is compatible with the degree structure $\mathcal{D}_{ult}$:

Definition 42. *For* $\mathcal{U}, \mathcal{V}$ *ultrafilters, let*

$$\mathcal{U} * \mathcal{V} = \{X : \{b : \{a : \langle a, b \rangle \in X\} \in \mathcal{V}\} \in \mathcal{U}\}.$$

It is clear that $\mathcal{U} * \mathcal{V}$ is again an ultrafilter, and that the operation $*$ is associative. The crucial property of $*$ is the following:

Proposition 7.5.2. *For all* $\mathcal{U}, \mathcal{V}$ *we have* $\delta_{\mathcal{U}} \circ \delta_{\mathcal{V}} = \delta_{\mathcal{U}*\mathcal{V}}$.

Proof. For a set X, let
$$X^\sharp = \{\langle \langle i, j \rangle, k \rangle : \langle i, \langle j, k \rangle \rangle \in X\}.$$
We claim that $\lim_{\mathcal{U}*\mathcal{V}}(X^\sharp) = \lim_{\mathcal{U}}(\lim_{\mathcal{V}}(X))$, as follows:

$$x \in \lim_{\mathcal{U}}(\lim_{\mathcal{V}}(X)) \iff \{j : \langle j, x \rangle \in \lim_{\mathcal{V}}(X)\} \in \mathcal{U} \iff \{j : \{k : \langle k, \langle j, x \rangle \rangle \in X\} \in \mathcal{V}\} \in \mathcal{U}$$

$$\iff \{j : \{k : \langle \langle k, j \rangle, x \rangle \in X^\sharp\} \in \mathcal{V}\} \in \mathcal{U} \iff \{\langle k, j \rangle : \langle \langle k, j \rangle, x \rangle \in X^\sharp\} \in \mathcal{U}*\mathcal{V} \iff x \in \lim_{\mathcal{U}*\mathcal{V}}(X^\sharp).$$

Since the operation $\sharp$ is invertible and preserves Turing degree, we have shown that $\delta_{\mathcal{U}} \circ \delta_{\mathcal{V}}(\mathbf{a}) = \delta_{\mathcal{U}*\mathcal{V}}(\mathbf{a})$ for every degree $\mathbf{a}$. □

Remark 7.5.3. *Note that Proposition 7.5.2 only holds on the level of* degrees: *in general, given ultrafilters $\mathcal{U}$ and $\mathcal{V}$ and a set X there need be no ultrafilter $\mathcal{W}$ with $\lim_{\mathcal{W}}(X) = \lim_{\mathcal{U}}(\lim_{\mathcal{V}}(X))$. For example, take $X = (X_i)_{i \in \omega}$ with $X_0 = \omega$ and $X_i = \emptyset$ for $i > 0$. Then $\lim_{\mathcal{W}}(X) = \{0\}$ and $\lim_{\mathcal{U}}(\lim_{\mathcal{V}}(X)) = \emptyset$ regardless of the choice of $\mathcal{U}, \mathcal{V}, \mathcal{W}$.*

Proposition 7.5.2 immediately yields:

Corollary 7.5.4. *The operation $*$ is compatible with $\mathcal{D}_{ult}$: if $\mathcal{U}_0 \leq \mathcal{U}_1$ and $\mathcal{V}_0 \leq \mathcal{V}_1$, then $\mathcal{U}_0 * \mathcal{V}_0 \leq \mathcal{U}_1 * \mathcal{V}_1$. Moreover, $*$ is well-defined on elements of $\mathcal{D}_{ult}$.*

Proposition 7.5.2 also provides us with a "jump" operator on $\mathcal{D}_{ult}$:

Definition 43. *For $\mathcal{U}$ an ultrafilter, let $\mathcal{U}' = \mathcal{U} * \mathcal{U}$.*

Proposition 7.5.5. *For every $\mathcal{U}$ we have $\mathcal{U} < \mathcal{U}'$.*

Proof. This is a refinement of Corollary 7.4.2. That $\mathcal{U} \leq \mathcal{U}'$ is immediate. To show that this is strict, fix a sufficiently large degree $\mathbf{a}$ and suppose $\mathcal{U}' \leq \mathcal{U}$. Then we have (using Lemma 7.2.1(1) for the first equality)

$$\delta_{\mathcal{U}}(\mathbf{a}') \subseteq \delta_{\mathcal{U}'}(\mathbf{a}) \subseteq \delta_{\mathcal{U}}(\mathbf{a}),$$

contradicting the relativized version of Proposition 7.4.1. $\qquad \square$

This natural algebraic structure, compatible with the preorder, suggests that $\mathcal{D}_{ult}$ may be an interesting degree structure in its own right. We end by providing further evidence for this: a connection between $\mathcal{D}_{ult}$ and a more classical ordering of ultrafilters, the *Rudin-Keisler ordering*:

Definition 44. *For $\mathcal{U}, \mathcal{V}$ ultrafilters, $\mathcal{U}$ is* Rudin-Keisler reducible *to $\mathcal{V}$ — and we write $\mathcal{U} \leq_{RK} \mathcal{V}$ — if for some $f : \omega \to \omega$ we have*

$$\mathcal{U} = f^{-1}(\mathcal{V}), \text{ that is, } X \in \mathcal{V} \iff f^{-1}(X) \in \mathcal{U}.$$

We write $\mathcal{U} \leq^f_{RK} \mathcal{V}$ if f witnesses $\mathcal{U} \leq_{RK} \mathcal{V}$.

The connection between Rudin-Keisler reducibility and our $\mathcal{D}_{ult}$ is provided by the following:

Theorem 7.5.6. *Suppose $\mathcal{U} \leq^f_{RK} \mathcal{V}$. Then if $f \leq_T \mathbf{a}$, we have $\delta_{\mathcal{U}}(\mathbf{a}) \subseteq \delta_{\mathcal{V}}(\mathbf{a})$.*

Proof. Given $X = (X_i)_{i \in \omega} \in \mathbf{a}$, define $Y_i = \{n : n \in X_{f(i)}\}$, $Y = (Y_i)_{i \in \omega}$. Now by our assumption on f we have

$$n \in \lim_{\mathcal{V}}(Y) \iff \{i : n \in X_{f(i)}\} \in \mathcal{V} \iff \{i : n \in X_i\} \in \mathcal{U} \iff n \in \lim_{\mathcal{U}}(X).$$

But this means $\delta_{\mathcal{U}}(\mathbf{a}) \subseteq \delta_{\mathcal{V}}(\mathbf{a})$, so we are done. $\qquad \square$

Corollary 7.5.7. *If $\mathcal{U} \leq_{RK} \mathcal{V}$, then $\mathcal{U} \leq \mathcal{V}$.*

Given this connection between $\mathcal{D}_{ult}$ and the Rudin-Keisler ordering, it is natural to ask:

Question 10. *Is there a characterization of $\leq$ in terms of combinatorial properties of ultra-filters? In particular, does $\leq^*$ coincide with $\leq_{RK}$?*

7.6 Further directions

We end by presenting two directions for further research.

Filter jumps

We have investigated maps $\delta_{\mathcal{U}}$ for $\mathcal{U}$ an ultrafilter. However, this construction applies equally well to filters:

Definition 7.6.1. *A filter is a collection of sets $\mathcal{F} \subseteq \mathcal{P}(\omega)$ satisfying conditions (1)-(3) of definition 7.1.1. For $\mathcal{F}$ a filter and $A = (A_i)_{i \in \omega}$ a sequence of sets, set $\lim_{\mathcal{F}}(A) = \{i : A_i \in \mathcal{F}\}$; then for $\mathbf{a}$ a Turing ideal, define*

$$\delta_{\mathcal{F}}(\mathbf{a}) = \{\lim_{\mathcal{F}}(A) : A \leq_T \mathbf{a}\}.$$

To preserve the analogy with limit computability, we will restrict our attention to filters containing $\mathcal{F}$.

Intuitively, this is a more "biased" notion of limit computability, since it is in general easier to have $X \notin \mathcal{F}$ than to have $X \in \mathcal{F}$. This is reflected in the fact that, in general, the resulting "filter jumps" $\delta_{\mathcal{F}}$ — while they may correspond to natural computability-theoretic operations — do not always yield Turing ideals. For example, $\delta_{\mathcal{F}_{cof}}(\mathbf{a}) = \Sigma_2^0(\mathbf{a})$, which is not closed under Turing reduction. On the positive side, note that $\delta_{\mathcal{F}}(\mathbf{a})$ is always closed under $\oplus$, and the limit lemma immediately implies that $\delta_{\mathcal{F}}(\mathbf{a}) \supseteq \Delta_2^0(\mathbf{a})$. Beyond this, however, it seems difficult to establish how these more general operations behave, and so the question of characterizing the possible images of filter jumps, in analogy with Theorem 7.3.1, is open:

Question 11. *Fix a Turing degree $\mathbf{a}$. For what classes $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is there some filter $\mathcal{F}$ with $\delta_{\mathcal{F}}(\mathbf{a}) = \mathcal{I}$?*

In particular, ensuring closure under Turing reducibility appears difficult.

Question 12. *What filters $\mathcal{F}$ have the property that $\delta_{\mathcal{F}}(\mathbf{a})$ is a Turing ideal for all $\mathbf{a}$?*

Moving on to trying to control the action of $\delta_{\mathcal{F}}$, we can (as in section 7.4) define a degree $\mathbf{a}$ to be *f-low* if there is a filter $\mathcal{F}$ such that $\delta_{\mathcal{F}}(\mathbf{a}) = \delta_{\mathcal{F}}(\mathbf{0})$. Clearly f-lowness is implied by u-lowness, and DNR_2 degrees are not f-low.

Question 13. *Which degrees are f-low?*

Ultrafilter jumps of Turing ideals

Our definition of $\delta_{\mathcal{U}}$ makes sense, not just for degrees, but for Turing ideals:

Definition 7.6.2. *For $\mathfrak{I}$ a countable Turing ideal, let $\delta_{\mathcal{U}}(\mathfrak{I}) = \{\lim_{\mathcal{U}}(A) : A \in \mathfrak{I}\}$.*

Then we can develop the theory of ultrafilter jumps in this broader context. By and large, the resulting picture is the same. Most importantly, by essentially the same proof as Theorem 7.3.1, we obtain:

Corollary 7.6.3. *Suppose $\mathfrak{I}$ is a countable Turing ideal and $\mathfrak{K} \supseteq \mathfrak{I}$. Then the following are equivalent:*

- *$\mathfrak{K}$ is a countable Scott set containing $\mathbf{a}'$ for every $\mathbf{a} \in \mathfrak{I}$.*

- *There is an ultrafilter $\mathcal{U}$ with $\delta_{\mathcal{U}}(\mathfrak{I}) = \mathfrak{K}$.*

There are, however, slight differences. For example, note that when generalized to ideals, Question 7 has a simple negative answer:

Proposition 7.6.4. *Let $\mathcal{U}, \mathcal{V}$ be ultrafilters, and fix a Turing ideal $\mathfrak{I}$. Then there is an ideal $\mathfrak{K} \supseteq \mathfrak{I}$ such that $\delta_{\mathcal{U}}(\mathfrak{K}) = \mathfrak{K} = \delta_{\mathcal{V}}(\mathfrak{K})$.*

Proof. We alternately apply $\delta_{\mathcal{U}}$ and $\delta_{\mathcal{V}}$ infinitely many times to $\mathfrak{I}$. Let $\mathfrak{I}_0 = \mathfrak{I}$, $\mathfrak{I}_{n+1} = \delta_{\mathcal{U}*\mathcal{V}}(\mathfrak{I}_n)$, and let

$$\mathfrak{K} = \bigcup_{i \in \omega} \mathfrak{I}_n.$$

It is easy to check that $\mathfrak{K}$ satisfies the desired properties. □

Having already generalized to countable Turing ideals, we can further consider the question of characterizing $\delta_{\mathcal{U}}(\mathfrak{I})$ for *uncountable* Turing ideals $\mathfrak{I}$. To a large extent, the possible behavior of ultrafilter jumps on uncountable ideals is already determined by their possible behavior on countable ideals, and even on individual degrees. However, the proof of Theorem 7.3.1 relied on enumerating the Turing ideals in question, and so breaks down as soon as we pass to uncountable ideals. This raises the question of whether our characterization still holds for uncountable ideals, and furthermore, to what extent the answer to this question depends on the axioms of set theory.

In general, this is unknown. However, at least a certain amount of set-theoretic independence does occur. By Theorem 7.3.1, if $\mathfrak{I}$ is countable and arithmetically closed then there is some ultrafilter $\mathcal{U}$ with $\delta_{\mathcal{U}}(\mathfrak{I}) = \mathfrak{I}$. This can fail to generalize to uncountable ideals in a strong way. The following theorem was stated without proof in [1]:

Theorem 7.6.5 (S)**.** *Suppose $V \models$ZFC+PD. Then there is a forcing extension $V[G]$ and a Turing ideal $M_G \subseteq \mathbb{R}$ such that*

- $(\omega, M_G; +, \times, \in)$ *is an elementary substructure of true second-order arithmetic*

$$(\omega, \mathcal{P}(\omega); +, \times, \in);$$

  but

- M_G *is not fixed by any nonprincipal ultrafilter — that is, $\delta_{\mathcal{U}}(M_G) \supsetneq M_G$ for every nonprincipal ultrafilter $\mathcal{U} \in V[G]$.*

Note that if $V = L$, then there is a projectively definable ultrafilter on $\mathbb{R}$, and so every projectively closed set of reals *is* fixed by an ultrafilter. Also, note that M_G will be closed under projective definability; in fact, assuming PD, this is equivalent to the first bullet point.

Proof. Suppose $V \models$ ZFC+PD, and consider the following notion of forcing $\mathbb{P}$. Elements of $\mathbb{P}$ are ordered pairs $(\mathcal{M}, \mathbb{A})$ with $\mathcal{M}$ a countable family of sets of natural numbers forming (the second-order part of) a countable elementary submodel of true second-order arithmetic, and $\mathbb{A}$ a countable family of sets of natural numbers such that $\mathbb{A} \cap \mathcal{M} = \emptyset$. $\mathbb{P}$ is ordered by coordinatewise reverse inclusion. Note that $\mathbb{P}$ is countably closed, since the union of a chain of elementary extensions is an elementary extension. Let G be $\mathbb{P}$-generic over V, and let

$$M_G = \bigcup_{(\mathcal{M},\emptyset)\in G} \mathcal{M};$$

we will prove that M_G is not fixed by any nonprincipal ultrafilter.

Since $\mathbb{P}$ is countably closed, no new reals are added, so the ultrafilters in the generic extension have only elements from the ground model. Let ν be a name for an ultrafilter, and p a condition; we will show that it is dense below p to force that M_G is not fixed by $\nu[G]$.

First, note that there is some $q := (\mathcal{M}, \mathbb{A}) \leq p$ such that for every $Y \in \mathcal{M}$, either $q \Vdash Y \in \nu$ or $q \Vdash Y \notin \nu$; this is because $\mathbb{P}$ is countably closed, and left and right coordinates of conditions in $\mathbb{P}$ are countable. Without loss of generality, assume p itself has this property.

Now let $p = (\mathcal{M}, \mathbb{A})$. Let $\mathbb{L}$ be the poset consisting of finite partial functions $\omega \to \mathcal{M}$ ordered by reverse inclusion. $\mathbb{L}$ and the forcing relation for $\mathbb{L}$ are definable in $\mathcal{M}$ (modulo appropriate coding of functions as sets). Finally, say that a *listing of $\mathcal{M}$* is a set $X \subseteq \omega$ such that for all $Y \in \mathcal{M}$, there is some $i \in \omega$ with $X_i = Y$.

Claim: If X is $\mathbb{L}$-generic over $\mathcal{M}$, then X is a listing of $\mathcal{M}$, and neither $\delta_\nu(X)$ (that is, $\{j : p \Vdash \delta_\nu(X^j) = 1\}$) nor any element of $\mathbb{A}$ is projectively definable from X.

Proof of claim. That X is a listing of $\mathcal{M}$ is immediate.

That elements of $\mathbb{A}$ are not projectively definable from X is just the standard argument that any set definable from every sufficiently generic filter extending some condition, is already definable in the ground model; since $\mathcal{M}$ yields an elementary submodel of true second-order arithmetic, $\mathcal{M}$ is closed under projective definability, and hence no element of $\mathbb{A}$ is projectively definable in $\mathcal{M}$.

The proof that $\delta_\nu(X)$ is not projectively definable from X proceeds as follows. Suppose X is a counterexample to the claim. Then there is some condition $c \in \mathbb{L}$ which is compatible with X such that $c = \langle A_0, A_1, ..., A_n \rangle \Vdash_{\mathbb{L}} \delta_\nu(X) = \varphi(X)$ for some second-order formula φ. Considering the possible ways to extend c, we must have that for all $B \in \mathcal{M}$,

$$\exists d \leq_{\mathbb{L}} \langle A_0, A_1, ..., A_n, B \rangle (d \Vdash_{\mathbb{L}} \varphi(X) = 1) \iff B \in \nu.$$

But this yields a projective definition of $\nu \cap \mathcal{M}$ in $\mathcal{M}$; since $\mathcal{M} \prec 2^\omega$, by PD there are no projectively definable ultrafilters on $\mathcal{M}$, so this is a contradiction.

$\square$

So let X be a sufficiently generic listing of $\mathcal{M}$. Let $\mathcal{N}$ be the projective closure of $\mathcal{M} \cup \{X\}$; by PD, $\mathcal{N}$ is an elementary substructure of 2^ω. By the claim, $\delta_\nu(X) \notin \mathcal{N}$, so $(\mathcal{N}, \mathbb{A} \cup \{\delta_\nu(X)\})$ is a condition extending p. But this condition forces $\delta_\nu(M_G) \neq M_G$.

$\square$

Chapter 8

Computability theoretic aspects of ordinals

The work in this section remains in progress.

8.1 Introduction

In this chapter we explore computability-theoretic aspects of ordinals. First, we examine the difficulty of *listing* certain countable sets of countable ordinals. We show that there is a class of ordinals whose associated copy-and-diagonalize game (a concept introduced by Montalban [56]) is undetermined. As a corollary, we get (assuming PD) that there is a countable collection of ordinals which is "difficult to list" in a precise sense.

We then turn to the question of comparing different ordinals as individual structures. In terms of Muchnik reducibility, the situation is completely understood: $\alpha \leq_w \beta$ iff α is less than the first admissible above β. However, once we add *uniformity* to the picture, things become much more complicated:

Definition 45. *For* countable *structures $\mathcal{A}, \mathcal{B}$, we write $\mathcal{A} \leq_s \mathcal{B}$ if every copy of $\mathcal{B}$ computes a copy of $\mathcal{A}$, uniformly:*

$$\exists e \forall \hat{B} \cong \mathcal{B}[\Phi_e^{\hat{B}} \cong \mathcal{A}].$$

Here the $\hat{B}$ range over copies of $\mathcal{B}$ *with domain ω.*

We study the coarse structure of the Medvedev degrees of countable ordinals. In particular, we give a strong positive answer a question of Hamkins and Li [26]: we show that there is a club of countable ordinals which are pairwise Medvedev incomparable.

One important tool we use is the generic analogue of Medvedev reducibility, similar to $\leq_w^*$:

Definition 46. *For* possibly uncountable *structures* $\mathcal{A}, \mathcal{B}$, *we write* $\mathcal{A} \leq_s^* \mathcal{B}$ *if — in every forcing extension in which* $|\mathcal{A}| + |\mathcal{B}| \leq \aleph_0$ *— every copy of* $\mathcal{B}$ *computes a copy of* $\mathcal{A}$, *uniformly:*

$$\text{For every forcing } \mathbb{P}, \text{ if } \Vdash_{\mathbb{P}} |\mathcal{A}|, |\mathcal{B}| \leq \aleph_0, \text{ then } \Vdash_{\mathbb{P}} \text{``}\exists e \forall \hat{B} \cong \mathcal{B}[\Phi_e^{\hat{B}} \cong \mathcal{A}].\text{''}$$

Again, the $\hat{B}$ range over copies of $\mathcal{B}$ with domain ω (which, regardless of the true cardinality of $\mathcal{B}$, will exist after forcing with $\mathbb{P}$). As with generic Muchnik reducibility, by Shoenfield absoluteness we could replace "every forcing extension" with "some forcing extension," and $\leq_s$ agrees with $\leq_s^*$ on countable structures.

We leave a number of interesting questions open; this part of the chapter is work in progress, and we hope to resolve some of these questions soon.

8.2 Copying ordinals

Throughout, $\mathbb{K}$ will always be a class of countable ordinals, and $G_{\mathbb{K}}$ will be the ineffective array copying game — player 1 (Diagonalize) builds a single structure $\mathcal{M}_R$, player 2 (Copy) builds an array of structures $\mathcal{M}_{C,i}$ $(i \in \omega)$, and player 2 wins iff every $\mathcal{M}_{C,i}$ is in $\mathbb{K}$ and $\mathcal{M}_R \in \mathbb{K} \Rightarrow \exists i (\mathcal{M}_R \cong \mathcal{M}_{C,i})$. Our first result is — assuming the continuum hypothesis — the construction of a class of ordinals $\mathbb{K}$ such that $G_{\mathbb{K}}$ is undetermined.

Lemma 8.2.1. *If* $\mathbb{K}$ *is unbounded in* ω_1, *then* $G_{\mathbb{K}}$ *is not a win for II.*

Proof. Fix a strategy Π for 2; for each strategy Σ for 1, Π produces a collection of orderings $(\Sigma \otimes \Pi)_i$ $(i \in \omega)$; let $\mathcal{L}_\Sigma = \bigoplus (\Sigma \otimes \Pi)_i$. If for some Σ the order $\mathcal{L}_\Sigma$ is ill-founded, then Π is not winning for II; so $\mathcal{L}_\Sigma$ is always an ordinal. But then the set

$$\{\alpha : \exists \Sigma [\alpha \cong \mathcal{L}_\Sigma]\}$$

is Σ_1^1 and hence bounded below ω_1, so I can defeat Π by simply playing an ordinal greater than that bound which is in $\mathbb{K}$ - and since $\mathbb{K}$ is unbounded in ω_1, this is always possible. $\square$

Lemma 8.2.2. *Let* $\mathcal{K} \subset \omega_1$ *be bounded, and* $r \in \mathbb{R}$. *Then there is an ordinal* α *and some set* $X = (X_i)_{i \in \omega}$ *such that*

- *each* X_i *is a copy of some* $\beta \in \mathcal{K}$,

- *each* $\beta \in \mathcal{K}$ *is isomorphic to some* X_i,

- *no copy of* α *is computable from* $r \oplus X$, *and*

- $\alpha > \beta$ *for every* $\beta \in \mathcal{K}$.

Proof. Fix any such X, and take $\omega_1^{r \oplus X}$ (note that the fourth condition is trivially implied by the third). $\square$

Corollary 8.2.3 (ZFC+CH)**.** *There is a class $\mathbb{K}$ of ordinals for which the game $G_{\mathbb{K}}$ is undetermined.*

Proof. By CH, fix a listing $(r_\eta)_{\eta \in \omega_1}$ of $\mathbb{R}$; using 8.2.2, we will define $\mathbb{K}$ in stages; $\mathbb{K}$ will have a "local listability" property which prevents 1 from winning.

Construction. At stage η, suppose we have defined $\mathcal{K}_\eta$; let α_η be the ordinal guaranteed to exist by 8.2.2 applied to $\mathcal{K}_\eta$ and r_η, that is, there is some listing X_η of $\mathcal{K}_\eta$ such that $X_\eta \oplus r_\eta$ does not compute a copy of α_η. We then let $\mathcal{K}_{\eta+1} = \mathcal{K}_\eta \cup \{\alpha_\eta\}$. At limit stages we take unions, and we let $\mathbb{K} = \bigcup_{\eta \in \omega_1} \mathcal{K}_\eta$.

Analysis. Since $\mathbb{K}$ is unbounded in ω_1, by 8.2.1 $G_{\mathbb{K}}$ is not a win for 2; so it is enough to show how to defeat a given strategy Σ for player 1.

Fix η such that $\Sigma \leq_T r_\eta$, and consider the strategy for 2 which simply plays the listing X_η of $\mathcal{K}_\eta$ as in two paragraphs previous. By construction of $\mathbb{K}$, $r_\eta \oplus X_\eta$ cannot compute any element of $\mathbb{K}$ other than those in $\mathcal{K}_\eta$; so if Σ builds an element of $\mathbb{K}$, it must be isomorphic to one of the orderings played by this strategy for 2. So Σ is not winning. $\square$

Question 10. *Can we remove CH from the argument above?*

Assuming projective determinacy (PD), we now turn to two corollaries of the argument above, on the complexity of listing certain countable sets of ordinals:

Corollary 8.2.4 (ZFC+PD)**.** *For all reals r on a cone, there is no r-computable listing of the admissibles below ω_1^r. Specifically, there is no uniformly r-computable sequence of reals x_i such that each x_i (codes a structure which) is isomorphic to an admissible ordinal $\alpha_i < \omega_1^r$, and every such admissible ordinal is isomorphic to some x_i.*

Note that this is immediate of ω_1^r is a limit of admissibles; the interesting case is when ω_1^r is a successor admissible.

Proof. Let $\mathbb{K}$ be the class of admissibles. This class is Π_1^1, and so the associated copy-diagonalize game $G_{\mathbb{K}}$ is projective and hence determined. By 8.2.1, $G_{\mathbb{K}}$ is not a win for II. So it is enough to show that, supposing the corollary is false, the game $G_{\mathbb{K}}$ is not a win for I either.

To see this, fix a strategy Σ for I. Now, if the corollary does not hold on a cone, then (by PD) it fails on a cone, so we can suppose it fails for some real $r \geq_T \Sigma$. But then the strategy Π for II which simply plays an r-computable listing of the admissibles below ω_1^r must beat Σ. $\square$

The second corollary is the existence of a countable set of ordinals which is hard to list — any listing of it provides nontrivial information about arbitrarily large countable ordinals:

Corollary 8.2.5 (ZFC+PD)**.** *There is a countable set $B \subset \omega_1$ of ordinals such that for every countable ordinal α, there is some real r such that*

- *r does not compute a copy of α (unless α is recursive), but*

- *for every listing X of B, $X \oplus r$ computes a copy of α.*

Proof. Otherwise, we can projectively define an unbounded class $\mathbb{K}$ of countable ordinals, such that for every $\alpha \in \mathbb{K}$ and every real r, if r does not compute a copy of α then there is some listing X of $\mathcal{K}_\alpha = \mathbb{K} \cap \alpha$ such that $r \oplus X$ does not compute a copy of α. But then this lets us defeat a given strategy for player I in the game $G_\mathbb{K}$, in the same manner as in 8.2.3: let Σ strategy for player I, and $\alpha \in \mathbb{K}$ be such that Σ does not compute a copy of α. Then let X be a listing of $\mathbb{K} \cap \alpha$ such that $X \oplus \Sigma$ does not compute a copy of α; the strategy Π for player II which just plays the list of structures X, regardless of what player I does, will then beat Σ. So player I does not have a winning strategy in $G_\mathbb{K}$. By lemma 8.2.1, player II does not have a winning strategy either. Since $\mathbb{K}$, and hence the game $G_\mathbb{K}$, is projective, PD implies that this cannot happen. $\square$

8.3 Medvedev degrees of ordinals

In [26], Hamkins and Li initiated the study of the Medvedev degrees of ordinals. They proved a number of basic results, but the global structure of the Medvedev degrees of ordinals remained largely mysterious. In particular, they asked:

Question 11. *[Hamkins, Li] Are there Medvedev incomparable ordinals?*

We give a strong positive answer to this question. The key definition is a set of indices associated to any structure, and the natural pre-wellordering on that set:

Definition 47. *For $\mathcal{A}$ a countable structure, we let*

- *$MIO(\mathcal{A})$ (the set of Medvedev indices of ordinals relative to $\mathcal{A}$) is the set*

$$MIO(\mathcal{A}) = \{e : \text{ for any } \omega\text{-copies } \hat{A}, \hat{B} \text{ of } \mathcal{A}, \ \Phi_e^A \cong \Phi_e^B \in ON\}.$$

- *$R(\mathcal{A})$ is the natural ordering on $MIO(\mathcal{A})$:*

$$R(\mathcal{A}) = \{(e_0, e_1) \in MIO(\mathcal{A})^2 : \Phi_{e_0}^\mathcal{A} \preccurlyeq \Phi_{e_1}^\mathcal{A}\}.$$

  We can also confuse $R(\mathcal{A})$ with the corresponding order on ω^2, where elements of $\omega \setminus MIO(\mathcal{A})$ are put at the bottom.

- *$Var(\mathcal{A})$ is the Mostowski collapse of the relation $R(\mathcal{A})$. (The variety of ordinals Medvedev below $\mathcal{A}$.) Alternately, $Var(\mathcal{A})$ is the ordertype of the ordinals Medvedev below $\mathcal{A}$.*

For uncountable $\mathcal{A}$ we define $MIO(\mathcal{A}), R(\mathcal{A}), Var(\mathcal{A})$ identically but with "$\leq_s^*$" in place of "$\leq_s$".

Before answer Question 11, we give an upper bound to the complexity of this index set:

Proposition 8.3.1. $MIO(\mathcal{A})$ *is* Π_1^1 *relative to any copy of* $\mathcal{A}$*; that is,* $MIO(\mathcal{A})$ *is relatively intrinsically* Π_1^1 *in* $\mathcal{A}$*.*

Proof. $e \in MIO(\mathcal{A})$ iff Φ_e^B is a well-ordering for every ω-copy B of $\mathcal{A}$, and $\Phi_e^{B_0} \cong \Phi_e^{B_1}$ for all copies B_0, B_1 of $\mathcal{A}$.

The first conjunct — "Φ_e^B is a well-ordering for every ω-copy B of $\mathcal{A}$" — is equivalent to saying that the following tree T is well-founded: a node on T is a pair (π, l) where π is a finite partial permutation of ω with domain and range at least $\frac{|\pi|}{4}$, and l is a finite descending sequence in the (finite) structure $\Phi_e^{\pi(A)}$. Extension is defined on both coordinates simultaneously: $(\pi, l) > (\pi', l')$ if $\pi \supsetneq \pi'$ and $l \supsetneq l'$. Clearly if T has no paths, then Φ_e^B is well-ordered for every ω-copy B of $\mathcal{A}$. Conversely, suppose T has a path F; let f and g be the left and right projections of F. Then by the domain/range requirement, f is in fact a total permutation of ω, so yields a copy of $\mathcal{A}$ given by $f(B)$, and g is then a descending sequence through $\Phi_e^{f(B)}$.

The second conjunct — "$\Phi_e^{B_0} \cong \Phi_e^{B_1}$ for all copies B_0, B_1 of $\mathcal{A}$" — uses the fact that we can compare ordinals in a uniformly Π_1^1 way. Specifically, suppose α and β are ordinals. Then $\alpha \leq \beta$ iff there is no injection from $\beta + 1$ to α. So the second conjunct can be written as "For all copies B_0, B_1 of $\mathcal{A}$, there is no embedding of $\Phi_e^{B_0} + 1$ into $\Phi_e^{B_1}$." $\qquad\square$

Turning now back to Question 11, we give a strong positive answer:

Proposition 8.3.2. *There is a club of countable ordinals which forms a Medvedev antichain.*

Proof. We "clubbify and thin."

First, for each n there is a club C_n such that either $\forall \alpha \in C_n(n \in MIO(\alpha))$ or $\forall \alpha \in C_n(n \notin MIO(\alpha))$; this is because "$MIO(\alpha)$" is Π_1^1 relative to α by Proposition 8.3.1, and (provably in ZFC) Π_1^1 sets of countable ordinals contain or are disjoint from a club. Let $C = \bigcap C_n$; then C is again a club, and $\forall \alpha, \beta \in C(MIO(\alpha) = MIO(\beta))$ (call this set "$MIO(C)$").

Next, for each pair $(e_0, e_1) \in MIO(C)^2$, there is a club $D_{(e_0, e_1)}$ such that either $\forall \alpha \in D_{(e_0,e_1)}((e_0, e_1) \in R(\alpha))$ or $\forall \alpha \in D_{(e_0,e_1)}((e_0, e_1) \notin R(\alpha))$, since "$(e_0, e_1) \in R(\alpha)$" is projective. Let $D = \bigcap_{(e_0,e_1) \in MIO(C)} D_{(e_0,e_1)}$; then D is again a club.

Now note that for $\alpha, \beta \in D$, we must have $Var(\alpha) = Var(\beta)$ since $MIO(\alpha) = MIO(\beta)$ and $R(\alpha) = R(\beta)$. Finally, we thin D as follows: let

$$E = \{\alpha \in D : \forall \beta \in D, \beta < \alpha \Rightarrow \omega_1^{CK}(\beta) < \alpha\}.$$

(Here "$\omega_1^{CK}(\beta)$" is the least ordinal which is not *Muchnik* below β; equivalently, the least admissible above β.) It's not hard to show that E is a club (clearly E is closed, and note that E contains the set of elements of D which are the least element of D above γ for some limit of admissibles γ, which is a club).

We claim E is a Medvedev antichain. Suppose $\alpha, \beta \in E$ with $\alpha < \beta$. Then clearly $\alpha \not\leq_s \beta$. Assume $\beta \leq_s \alpha$. Then $Var(\beta) \geq Var(\alpha) + 1$ (everything Medvedev below β is Medvedev below α, and then there's α itself). But then $Var(\beta) \neq Var(\alpha)$, a contradiction. $\qquad\square$

This argument yields a number of immediate corollaries:

Corollary 8.3.3. *There is an ordinal θ_{club} such that for club-many ordinals α, we have*

$$Var(\alpha) = \theta_{club}.$$

Similarly, there is a set M_{club} and a relation R_{club} on M_{club} corresponding to MIO and R.

Corollary 8.3.4. *For club-many countable ordinals α, there is some β with $\alpha < \beta < \alpha + \alpha$ such that $\beta \not\leq_s \alpha$.*

Proof. Let E be the club antichain produced in Proposition 8.3.2. Take any $\alpha \in E$ with $\alpha > \theta_{club}$. Since $\alpha \in E$, we have $Var(\alpha) = \theta_{club} < \alpha$. But the interval $(\alpha, \alpha + \alpha)$ has order-type α, so there must be some $\beta \in (\alpha, \alpha + \alpha)$ which is not $\leq_s \alpha$. $\square$

Using generic Medvedev reducibility, we can strengthen some of the results above — provided we have appropriate generic absoluteness theorems. For the remainder of this section, we assume PD (although in fact Σ_3^1-forcing absoluteness is sufficient), which among other things implies that projective sentences are absolute for set forcing.

Question 12. *Can the following results be proved in ZFC alone?*

We begin with a basic absoluteness result:

Definition 48. *For a possibly uncountable structure $\mathcal{A}$, we let $gMIO(\mathcal{A})$ be the set of indices for Medvedev reductions from $\mathcal{A}$ to ordinals, in some generic extension:*

$$gMIO(\mathcal{A}) = \{e : \exists \mathbb{P}(\Vdash_\mathbb{P} \mathcal{A} \text{ is countable and } e \in MIO(\mathcal{A}))\}.$$

Proposition 8.3.5. *For any (possibly uncountable) structure $\mathcal{A}$, the set $\{\beta : \beta \leq_s^* \mathcal{A}\}$ is countable, and if $V[G]$ is a forcing extension in which $\mathcal{A}$ is countable, this set is exactly the set of ordinals which are Medvedev below $\mathcal{A}$. Moreover, $gMIO(\mathcal{A})$ is forcing absolute: if $V[H]$ is a forcing extension of V, then $gMIO(\mathcal{A})^V = gMIO(\mathcal{A})^{V[H]}$.*

Proof. This is a straightforward application of Shoenfield absoluteness. We prove the first claim; the latter two use the same type of argument.

Suppose $V[G]$ is a generic extension where $\mathcal{A}$ is countable, G $\mathbb{P}$-generic over V, and α is an ordinal such that $V[G] \models \alpha \leq_s \mathcal{A}$. Then there is a condition $p \in G$ and a number $e \in \omega$ such that $p \Vdash$ "$e \in MIO(\mathcal{A})$ and $\Phi_e^{\hat{A}} \cong \alpha$ for every ω-copy $\hat{A}$ of $\mathcal{A}$." Let $\mathbb{Q}$ be any other forcing making $\mathcal{A}$ countable, and let $H_0 \times H_1$ be $\mathbb{P} \times \mathbb{Q}$-generic with $p \in H_0$. Then by Shoenfield absoluteness from $V[H_0]$ to $V[H_0 \times H_1]$, we have $V[H_0 \times H_1] \models$ "$e \in MIO(\mathcal{A})$ and $\Phi_e^{\hat{A}} \cong \alpha$ for every ω-copy $\hat{A}$ of $\mathcal{A}$."

But suppose $\hat{B}$ were an ω-copy of $\mathcal{A}$ in $V[H_1]$. Then $\hat{B} \in V[H_0 \times H_1]$, so $V[H_0 \times H_1] \models \Phi_e^{\hat{B}} \cong \alpha$; by absoluteness, $V[H_1] \models \Phi_e^{\hat{B}} \cong \alpha$. So $V[H_1] \models$ "$e \in MIO(\mathcal{A})$ and $\Phi_e^{\hat{B}} \cong \alpha$ for every ω-copy $\hat{B}$ of $\mathcal{A}$."

This means that for every ordinal α with $\alpha \leq_s^* \mathcal{A}$, we may find some e such that every forcing making $\mathcal{A}$ countable forces that $\Phi_e^{\hat{A}} \cong \alpha$, whenever $\hat{A}$ is an ω-copy of $\mathcal{A}$. Let e_α be the least such e. Clearly $e_\alpha = e_\beta \iff \alpha = \beta$, since distinct ordinals are not isomorphic in any forcing extension; but this yields in V an injection from $\{\alpha \in ON : \alpha \leq_s^* \mathcal{A}\}$ to ω, so the former is countable. $\square$

Remark 8.3.6. *Note that there is no special role played by ordinals, here; if $\mathcal{A}$ is any structure, the same argument shows that there are only countably many structures in V which are $\leq_s^* \mathcal{A}$.*

Definition 49. *For α an ordinal, let $\omega_1^M(\alpha)$ be the least ordinal $\delta > \alpha$ such that $\delta \nleq_s \alpha$.*

Under appropriate absoluteness assumptions, such as follow from PD, we are able to extend 8.3.4:

Proposition 8.3.7. *For all but countably many α, $\omega_1^M(\alpha) < \alpha + \alpha$.*

Proof. Let ω_1^V be countable in $V[G]$. Then $V[G]$ satisfies the following sentence:

$$\exists \alpha < \omega_1 \forall \beta [\alpha < \beta < \omega_1 \Rightarrow \omega_1^M(\beta) < \beta + \beta].$$

(Just take $\alpha = \omega_1^V$ and apply Proposition 8.3.5). Since all quantifiers are (in $V[G]$) over countable ordinals, this sentence is projective, so by PD holds in V already. $\square$

Definition 50. *For α an uncountable ordinal, let $Var(\alpha)$ be the order type of the set of ordinals $\leq_s^* \alpha$.*

Proposition 8.3.8. *The set $\{Var(\alpha) : \alpha < \omega_1\}$ is bounded strictly below ω_1.*

Proof. The crucial observation is that "$\leq_s^*$" is absolute between forcing extensions: for ordinals α, β (more generally, for structures $\mathcal{A}, \mathcal{B} \in V$) and a generic extension $V[G]$ of V, we have $V \models \alpha \leq_s^* \beta$ iff $V[G] \models \alpha \leq_s^* \beta$. The proof of this is, again, just Shoenfield absoluteness: given α, β and a generic extension $V[G]$ of V, let $\mathbb{P} \in V$ be a forcing making α and β countable, and let H be $\mathbb{P}$-generic over $V[G]$. Then applying Shoenfield absoluteness between $V[G][H]$ and $V[H]$, we have that either $\alpha \leq_s \beta$ in each or $\alpha \nleq_s \beta$ in each.

Using this, we have that $Var(\alpha)^{V[G]} = Var(\alpha)^V$ for every forcing extension $V[G]$. So in particular, since $Var(\alpha)$ is countable in V, we have $V[G] \models Var(\alpha) < \omega_1^V$ for every generic extension $V[G]$. Let $V[G]$ be a generic extension in which ω_1^V is countable; then $V[G]$ satisfies "There is a countable γ such that every countable ordinal α has $Var(\alpha) < \gamma$" (namely $\gamma = \omega_1^V$), which is a projective sentence, so by PD so does V. $\square$

With a bit more work, we can extend this to all ordinals, countable or not:

Corollary 8.3.9. *The set $\{Var(\alpha) : \alpha \in ON\}$ is bounded strictly below ω_1. (In particular, it is a set, even though a priori it is merely a class.)*

Proof. First, we show that the relation "$\leq_s^*$" is absolute between V and L; then, we use the existence of L-indiscernibles.

    Claim : if $\alpha \leq_s^* \beta$, then $L \models \alpha \leq_s^* \beta$.

Proof of claim. Suppose $V \models \alpha \leq_s^* \beta$ via Φ_e. Let G be $Col(\beta + \alpha, \omega)$-generic over V; note that $Col(\beta + \alpha, \omega) \in L$, so G is generic over L as well. Now suppose B is an ω-copy of β, $B \in L[G]$. $B \in V[G]$, so since $\alpha \leq_s^* \beta$ we have $V[G] \models \Phi_e^B \cong \alpha$. Since $\alpha \in L$, the tree of partial isomorphisms between α and Φ_e^B is in $L[G]$. This tree has a path (in fact, a unique path) in $V[G]$, so by Mostowski absoluteness it also has a path in $L[G]$ — so $L[G] \models \Phi_e^B \cong \alpha$. Since this is true for any ω-copy B of β in $L[G]$, we have $L[G] \models \alpha \leq_s \beta$, so $L \models \alpha \leq_s^* \beta$.

    The converse — showing that $L \models \alpha \leq_s^* \beta$ implies $V \models \alpha \leq_s^* \beta$ — uses the same idea. If $V[G] \models \alpha \not\leq_s \beta$, then — in $V[G]$ — for each e there is an ω-copy B of β such that $\Phi_e^B \not\cong \alpha$. This is a Σ_2^1 sentence, so absolute between $L[G]$ and $V[G]$. □

    We now show how the corollary follows from this claim.

Definition 51. *Let* $\theta_{sup} = \sup\{Var(\alpha) : \alpha < \omega_1\}$.

    By Proposition 8.3.8, we have $\theta_{sup} < \omega_1$. Let C be the proper class of L-indiscernible ordinals guaranteed by $0^\#$ (whose existence follows from PD, which we are assuming here); it is known that $C \cap \omega_1$ is club in ω_1, so we may find $\alpha_0 < \alpha_1 \in C$ with $\theta_{sup} < \alpha_0 < \alpha_1 < \omega_1$. In V, the pair α_0, α_1 satisfies "If $\beta < \alpha_1$, then $Var(\beta) < \alpha_0$" — by the claim above, this sentence is true in L as well. So for any $\alpha_2 \in C$ with $\alpha_2 > \alpha_0$, we have $L \models \forall \gamma < \alpha_2(Var(\gamma) < \alpha_0)$, and again by the claim this statement lifts to V. Since C is unbounded in ON, this means that $\{Var(\alpha) : \alpha \in ON\}$ is bounded below ω_1 (specifically, below the least L-indiscernible). □

Remark 8.3.10. *If* α_0, α_1 *is as above, then* $\sup\{Var(\gamma) : \gamma < \alpha_0\} = \sup\{Var(\gamma) < \alpha_1\} = \theta_{sup}$ *— so by indiscernibility* $\sup\{Var(\gamma) : \gamma < \alpha\} = \theta_{sup}$ *for every* $\alpha \in C$. *This shows that* $\theta_{sup} = \sup\{Var(\gamma) : \gamma \in ON\}$.

    Interestingly, Corollary 8.3.9 holds also for counterexamples to Vaught's conjecture!

Proposition 8.3.11. *Suppose T is a counterexample to Vaught's conjecture. Then there is a $\theta_{sup}^T < \omega_1$ such that for all $\mathcal{A} \models T$, $Var(\mathcal{A}) \leq \theta_{sup}^T$.*

Proof. The key fact is that counterexamples to Vaught's conjecture cannot be "randomly" added by forcing: if $\mathbb{P}$ is a forcing notion, $T \in V$ is a counterexample to Vaught's conjecture, and $\mathcal{A} \in V[G]$ is a model of T when G is $\mathbb{P}$-generic over V, then there was a name $\nu \in V^{\mathbb{P}}$ such that $\nu[G] \cong \mathcal{A}$ and $\Vdash_{\mathbb{P} \times \mathbb{P}} \nu[G_0] \cong \nu[G_1]$; otherwise, we could add a perfect set of nonisomorphic models of T, contradicting the fact that T is a counterexample to Vaught's conjecture in V and that being a counterexample to Vaught's conjecture is absolute. (Note that this was also how we reproved Harrington's theorem in chapter 4.)

    But if $\Vdash_{\mathbb{P} \times \mathbb{P}} \nu[G_0] \cong \nu[G_1]$, we must have $p \Vdash$ "$\Phi_e^{\hat{A}} \cong \beta$ for every ω-copy $\hat{A}$ of $\mathcal{A}$" if and only if *every* condition forces this, for $e \in \omega$ and $\beta \in ON$. This means that the set

$\{\beta \in ON : V[G] \models \beta \leq_s^* \mathcal{A}\}$ can be computed in V, as well as the injection from this set to ω given by sending each β to its least "Medvedev index." So that set is countable in V, and hence $V[G] \models Var(\mathcal{A}) < \omega_1^V$.

So $V[G]$ satisfies "$\{Var(\mathcal{A}) : \mathcal{A} \models T\}$ is bounded below ω_1" (namely, by ω_1^V); this is a projective sentence, so true in V. $\square$

There are a number of open questions remaining; we close by mentioning one in particular.

Question 13. *Does $\theta_{club} = \theta_{sup}$?*

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