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2016

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UNIVERSITY OF CALIFORNIA
Los Angeles

Essays on Macroeconomics and Firm Dynamics

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Economics

by

Lei Zhang

2016

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ABSTRACT OF THE DISSERTATION

Essays on Macroeconomics and Firm Dynamics

by

Lei Zhang

Doctor of Philosophy in Economics

University of California, Los Angeles, 2016

Professor Andrew Granger Atkeson, Co-chair

Professor Andrea Lynn Eisfeldt, Co-chair

This dissertation contains three essays at the interaction between macroeconomics and the financial market, with an emphasis on macroeconomic implications of heterogeneous firms under financial frictions. My dissertation explores the relationships among financial market friction, firms' entry and exit behaviors, and job reallocation over the business cycle.

Chapter 1 examines the macroeconomic effects of financial leverage and firms' endogenous entry and exit on job reallocation over the business cycle. Financial leverage and the extensive margin are the keys to explain job reallocation at both the firm-level and the aggregate level. I build a general equilibrium industry dynamics model with endogenous entry and exit, a frictional labor market, and borrowing constraints. The model provides a novel theory that financially constrained firms adjust employment more often. I characterize an analytical solution to the wage bargaining problem between a leveraged firm and workers. Higher financial leverage allows constrained firms to bargain for lower wages, but also induces higher default risks. In the model, firms adopt (S,s) employment decision rules. Because the entry and exit firms are more likely to be borrowing constrained, a negative shock affects the inaction regions of the entry and exit firms more than that of the incumbents. In the simulated model, the extensive margin explains 36% of the job reallocation volatility, which is very close to the data and is quantitatively significant.

Chapter 2 investigates firms' financial behaviors and size distributions over the business cycle. We propose a general equilibrium industry dynamics model of firms' capital structure and entry

and exit behaviors. The financial market frictions capture both the age dependence and size dependence of firms' size distributions. When we add the aggregate shocks to the model, it can account for the business cycle patterns of firm dynamics: 1) entry is more procyclical than exit; 2) debt is procyclical, and equity issuance is countercyclical; and 3) the cyclicalities of debt and equity issuance are negatively correlated with firm size and age.

Chapter 3 studies the equilibrium pricing of complex securities in segmented markets by risk-averse expert investors who are subject to asset-specific risk. Investor expertise varies, and the investment technology of investors with more expertise is subject to less asset-specific risk. Expert demand lowers equilibrium required returns, reducing participation, and leading to endogenously segmented markets. Amongst participants, portfolio decisions and realized returns determine the joint distribution of financial expertise and financial wealth. This distribution, along with participation, then determines market-level risk bearing capacity. We show that more complex assets deliver higher equilibrium returns to expert participants. Moreover, we explain why complex assets can have lower overall participation despite higher market-level alphas and Sharpe ratios. Finally, we show how complexity affects the size distribution of complex asset investors in a way that is consistent with the size distribution of hedge funds.

The dissertation of Lei Zhang is approved.

Hugo Andres Hopenhayn

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2016

To my wife, Dongwei, for her immense love and endless support.

To my parents, for their constant support.

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ACKNOWLEDGMENTS

I am greatly indebted to my committee Andrea Eisfeldt, Andrew Atkeson, Hugo Hopenhayn, and Pierre-Olivier Weill for inspiration, encouragement and valuable comments. It has been a privilege to work with and learn from them. Without their guidance, none of my achievement would have been possible. My deepest appreciation is to my co-advisors, Prof. Andrea Eisfeldt and Prof. Andrew Atkeson. I strongly appreciate Andrea for her continual faith in me and insightful advice. In addition, I would like to express my sincere gratitude to Andy for his patience and discussions during my difficult times.

I also thank Saki Bigio, Francisco Buera, Ariel Burstein, Roger Farmer, Gary Hansen, Zhen Huo, Matthew Kahn, Lee Ohanian, Vincenzo Quadrini, and Xiaolan Zhang for their discussions.

Many friends have helped me stay sane through these difficult years. Their support and care helped me overcome setbacks and stay focused on my graduate study. I would especially like to thank my dearest friends, Chao Guo, Haoyuan Li, Ji Ma, and Dongyang Zhang. I would also like to thank Yang Cao, Yi Chen, Ting Ji, Yangbo Song, Miao Sun, Cong Xie, Hongxiang Xu, Menghan Xu, Yujing Xu, Ye Wang, Yulong Wang, Tianyu Wu, Mofei Zhao, and Xuhua Zhou who brought me great joy and much encouragement.

Most importantly, none of this would have been possible without the love and patience of my family. I would like to express my heartfelt gratitude to my wife, Dongwei. She has been extremely supportive throughout this entire process and has made countless sacrifices to help me get to this point. I cannot imagine a more special soul-mate.

Chapter 3 is from my joint work with Andrea Eisfeldt and Hanno Lustig, which is in preparation for submission to a journal for publication consideration. All remaining errors are mine.

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CHAPTER 1

Financial Leverage, Job Reallocation, and Firm Dynamics over the Business Cycle

1.1 Introduction

Financial frictions are widely understood to shape macroeconomic dynamics, especially those of the recent financial crisis.¹ Little, however, is known about how financial market frictions affect firm dynamics over the business cycle, for example, the sizable and persistent decline in new business formation observed during the Great Recession. A few studies note that entry and exit do not significantly alter aggregate dynamics.² I challenge this view using empirical evidence and a quantitative model.

This paper aims to understand how financial leverage affects firms' entry and exit decisions, namely the extensive margin, which further explains job reallocation across firms over the business cycle.³ I develop a general equilibrium industry dynamics model to study the determinants of job reallocation over the business cycle. I incorporate two main ingredients to the model of Hopenhayn (1992a), search friction and financial leverage. In the model, firms randomly meet with workers in a frictional labor market subject to both hiring and firing cost. And firms can issue one-period debt

¹See Jermann and Quadrini (2012); Kiyotaki and Moore (2008); Mendoza (2010); Brunnermeier and Sannikov (2014). Brunnermeier et al. (2012) and Quadrini (2011) provide detailed surveys on this topic.

²Clementi et al. (2014) shows that entry and exit amplify and propagate the effects of aggregate shocks, while Hawkins (2011), Coles and Kelishomi (2011) and Elsby and Michaels (2013) find little evidence that entry and exit shape aggregate job creation and destruction.

³The literature documents the importance of understanding how reallocation relates to the business cycle. See Hopenhayn and Rogerson (1993); Schuh and Triest (1998); Davis et al. (1998); Hall (1998); Mortensen and Pissarides (1999) for topics on labor reallocation. See Eisfeldt and Rampini (2006); Cooper and Schott (2013) for topics on capital reallocation.

subject to borrowing constraints. I show that the firms adopt (S,s) employment decision rules.⁴ I provide an analytical solution to the wage bargaining problem between a leveraged firm and multiple workers along the lines of Stole and Zwiebel (1996a,b).

The equilibrium outcomes adequately capture a few patterns in firm dynamics. First, entry is procyclical and exit is countercyclical. The prices of debt reflect the default probability of a firm. The model generates countercyclical external financing costs, which are important to match entry and exit dynamics (Lee and Mukoyama, 2012). Second, the extensive margin accounts for 36% of the job reallocation variance over the business cycle in the model and for 41% of that observed in the empirical data. The contribution of the extensive margin to the aggregate volatility is surprisingly large given a small fraction of entry and exit firms in the whole economy. I argue that the model cannot match both first and second moments of job reallocation without the financial leverage and decentralized bargained wage. When we increase the firing cost, both the level and the volatility of job reallocation are higher because of smaller inaction regions and higher exit rate. The financial leverage mitigates the reallocation at the larger and financially unconstrained firms because they can adjust leverage rather than employment. The financial leverage balances the contributions of the intensive and extensive margins to job reallocation. I prove that small firms and financially constrained firms have smaller inaction regions. Because the entry and exit firms are more likely to be borrowing constrained, a negative shock affects the inaction regions of the entry and exit firms more than that of the incumbents. Third, financially constrained firms adjust employment more frequently (Benmelech et al., 2011). This follows directly from wage bargaining outcome. Even though higher financial leverage allows constrained firms to bargain for lower wages, small and financially constrained firms still pay a higher wage than unconstrained firms. As a result, small and financially constrained are more likely to adjust employment in the presence of a productivity shock.

The model economy is populated a continuum of firms, a representative risk-neutral household with a continuum of workers, and a continuum of risk-neutral financial intermediaries. Firms are owned by the household. The firms receives an aggregate shock and an idiosyncratic shock

⁴This is realistic as shown in empirical data that many firms have zero employment growth rates.

each period. The production function exhibits decreasing returns to scale with inputs of labor. The firm pays a fixed production cost in each period. If the continuation value, conditional on the realization of aggregate and idiosyncratic shocks, is negative, the firm endogenously exits the market. A fixed number of potential entrants randomly draw an idiosyncratic productivity from a known distribution function. If they draw a good productivity, they pay a fixed entry cost and enter the market.

The firm raises funds through one-period debt. Borrowing is constrained. The debt is a standard contract with agency problems. The firm borrows money and repays the borrowed funds plus interest in the next period. The firm defaults and exits the market if it cannot fully repay its debt after the realization of shocks. The firm cannot take on excessive debt, as default is costly. Once default occurs, lenders recover only a fraction of the firm's profits. The firm refinances its debt as its productivity varies, but its cost of doing so changes endogenously over the business cycle. The firm faces a higher default probability if it is relatively small and if the economy is in recession. Thus, the prices of debt are lower for small firms as well as during recessions. Small firms have higher external financing costs, which make it more difficult for potential entrants to enter the market during recessions.

To hire workers, firms post vacancies in a frictional labor market. Firms randomly meet with unemployed workers and offer bargained wages. Stole and Zwiebel (1996a,b) develop the standard solution to the bargaining problem between a firm and multiple workers.⁵ They propose a unique subgame perfect equilibrium in which the wage profile coincides with the Shapley values (Shapley, 1953).⁶ I show that the bargained wage is negatively correlated with financial leverage. Borrowing constraints limit the amount of debt that firms can carry over time and directly affect wage bargaining outcomes. In the model, hiring is risky because the expected future profits may

⁵This bargaining solution has been widely applied in many recent papers examining firm dynamics and labor search. Examples of applications include business cycle and firm dynamics (Elsby and Michaels, 2013; Fujita and Nakajima, 2014), international trade and labor markets (Helpman et al., 2010; Felbermayr et al., 2011; Itskhoki and Helpman, 2014), product market regulation and employment (Ebell and Haefke, 2009; Felbermayr and Prat, 2011), and wage and employment dynamics (Cahuc et al., 2008; Acemoglu and Hawkins, 2014).

⁶Brugemann et al. (2015) note that the Stole and Zwiebel game does not support the Shapley values as a subgame perfect equilibrium. Instead, they propose the Rolodex Game and present the same solution. The application of this game to the model is not affected.

not fully repay debt. Hiring an additional worker decreases a firm's borrowing limit and increases a firm's likelihood of default. Because both hiring and firing are costly, employment decisions take the form of (S,s) rules. Each firm has an inaction region. Together with financial leverage, this model illustrates interesting interactions between firm dynamics and inaction regions. Small firms, financially constrained firms and potential entrants have smaller inaction regions. Both job creation and job destruction are more volatile in these firms. Moreover, the inaction regions are larger in an economy with a higher borrowing capacity or higher financial leverage.

I then calibrate the model in the stationary equilibrium. The model adequately captures some of the empirical regularities of firm dynamics. Furthermore, I consider a model without the external financing. The firms finance the projects with internal profits. In this case, the inaction regions are smaller, resulting in a less persistent employment process. The job reallocation rate is more volatile. However, it is more costly to enter the market with only internal profits. The entry and job reallocation rates decrease.

Finally, I simulate the model with an aggregate productivity shock. Firms have to predict the entire firm size distribution to compute labor market tightness when they post vacancies. I solve the model for this environment by applying standard tools from Krusell and Smith (1998). In the simulated model, the extensive margin explains 36% of the job reallocation variance over the business cycle, while the data show that the extensive margin accounts for 41% of the variance. Further, I compare the simulated economy using different values for borrowing capacity. A higher value implies a higher borrowing limit and larger inaction regions for small firms. The extensive margin contributes to a higher percentage of job reallocation variance with a higher borrowing capacity. Last, when the firms have no access to the external financing, the explanatory power of the extensive margin declines. However, the endogenous entry and exit resulting from the fixed operating cost still explains 20% of the job reallocation variance.

I organize the remainder of this paper as follows. The next section of the introduction provides the literature review. Section 2 presents the empirical patterns of cyclical job reallocation at both the extensive and intensive margins. Section 3 describes the model setup. Section 4 characterizes the bargaining solution and the employment decisions. Section 5 presents the calibration and characterization of the stationary distribution. Section 6 simulates the model with aggregate

shocks and shows how financial leverage interacts with entry and exit decisions as well as with job reallocation. Section 7 concludes.

Related Literature

The main contribution of this paper is to develop a model that is theoretically tractable and empirically realistic to analyze the determinants of job reallocation at both the firm-level and the aggregate level, with an emphasis on the extensive margin and financial leverage. The paper relates to several branches of the literature.

First, this paper builds on a vast literature in industry organization on firm dynamics, size, age and growth (Lucas and Prescott, 1971; Lucas, 1978; Hopenhayn, 1992a; Hopenhayn and Rogerson, 1993). In these models, they characterize the stationary firm size distribution. Cooley and Quadrini (2001), Albuquerque and Hopenhayn (2004) and Cabral and Mata (2003) study the relationships among firm size, age, and financial borrowing constraints. Veracierto (2002) is the first paper to consider both aggregate shocks and idiosyncratic shocks at the firm level. Lee and Mukoyama (2012) develop a model with exogenous countercyclical entry costs to match the stylized procyclical entry rate and stable exit rate over the business cycle. In my model, I add the financial leverage to Hopenhayn (1992a) model with aggregate shocks and search frictions. Debt prices are determined in equilibrium. The default cost acts as countercyclical entry and operating costs, which are the keys to explain the cyclical behavior of entry and exit.

Second, this paper is closely related to the recent literature on the relationship between firm dynamics and labor search. Kaas and Kircher (2015) and Schaal (2015) adopt directed search with Nash bargaining whereby growth firms not only are posting more vacancies but also are more likely to fill the job with a higher wage. I use the random search framework. Stole and Zwiebel (1996a,b) and Brügemann et al. (2015) provide theoretic foundations for a wage bargaining problem between a firm and multiple workers. The results have been applied in a vast and growing literature on firm dynamics with a frictional labor market. Acemoglu and Hawkins (2014) characterize a steady-state equilibrium of bargained wage and firm size distribution. Elsby and Michaels (2013) and Fujita and Nakajima (2014) analyze wage and firm size distributions, focusing on cross-sectional

dispersion and aggregate dynamics. My model introduces endogenous entry and exit and financial constraints. I solve the optimal wage contract with financial leverage between a firm and multiple workers. The solution provides a cross-sectional implication between wage dynamics and financial constraints. Financially constrained firms are more likely to adjust employment in the presence of a productivity shock, Benmelech et al. (2011). I argue that the model without both elements is not able to simultaneously match both first and second moment of job reallocation, and is not able to explain the contribution of the extensive margin to job reallocation.

Third, this paper relates to the body of literature on the role of entry and exit in shaping aggregate dynamics. Clementi and Palazzo (2013) and Clementi et al. (2014) examine the cyclical implications of endogenous firm entry and exit decisions in response to an aggregate productivity shock. Clementi and Palazzo (2013) find that the entry and exit margins account for 20% of output growth over ten years. The extensive margin also generates persistence in aggregate time series. Clementi et al. (2014) computes standard business cycle moments and impulse response functions of shocks to entry and exit rates. They find that a shock that more directly affects young firms sharpens the missing generation effect, accounting for a significant reduction in employment. Siemer (2014) finds that the financial constraint affects employment growth in small firms to a greater extent than in large firms. A large financial shock in the model generates a long-lasting recession because of reduced and persistently low entry rates. However, Hawkins (2011), Coles and Kelishomi (2011) and Elsby and Michaels (2013) find little evidence that entry and exit rates alter the behaviors of aggregate job creation and job destruction. My paper contributes to the literature with a plausible mechanism that entry and exit shapes labor market dynamics. I document new evidence that the extensive margin is important to aggregate job reallocation. To explain this, I argue that both decentralized wage bargaining and the financial leverage, as amplification mechanisms, are necessary ingredients to generate the stylized facts.

1.2 Job Reallocation at the Extensive Margin: Empirical Evidence

In this section, I document some facts about firm dynamics over the business cycle, including job reallocation and entry and exit rates. First, entry is procyclical, and exit is countercyclical.

Second, job reallocation is very volatile over the business cycle, almost three times as the volatility of output. Third, I provide evidence that the extensive margin accounts for 41% job reallocation variance. Last, I demonstrate an important feature of firm behavior during the Great Recession. The entry rate decreased more than 25% and has remained at a low level. The job creation rate from entrants is highly correlated with entry rates and job reallocation.

1.2.1 Data

I use the Business Dynamics Statistics (BDS) as the primary data source. This is a reliable source of data for the job reallocation of firms and establishments in non-agricultural sectors in the US. Unlike the data in the Current Population Survey and Establishment Survey, the BDS data include cyclical movements in the aggregate employment. These data can be used to quantify different types of firms' and margins' contributions to fluctuations in aggregate job reallocation.

I define the intensive and extensive margins to partition all the firms and establishments included in BDS to characterize regularities in the business cycle. The intensive margin is the job creation and job destruction from incumbents. The extensive margin refers to job creation and destruction purely from entries and exits. I use this definition to demonstrate the empirical findings in the following section.

1.2.2 Firm Entry and Exit

I start by examining firm entry and exit dynamics over the business cycle. Figure 1.2 presents the entry and exit rates of firms since 1980. The rates vary significantly over the business cycle. On average, the entry rate is higher during booms and lower during recessions. The exit rate is higher during recessions and lower during booms. While the entry rate has been fairly stable since 1990, it decreased dramatically in 2007 and has remained at a low level. Moreover, there is no clear trend that business entry has recovered since 2010.

Figure 1.3 shows the cyclical component of the number of firms and establishments during the period from 1980 to 2012. The HP filter is subject to some measurement error in the early 1980s. The gray bars show NBER-dated recession periods. We observe that the number of firms declines

Table 1.1: Job creation and job destruction

	Volatility	Relative volatility	Correlation with output
Job creation	0.050	2.78	0.50
Job destruction	0.074	4.11	-0.53
Job reallocation	0.053	2.94	-0.18

Notes: Series are HP filtered with parameter $\lambda = 6.25$. 1980-2012 Source: Own calculations. Business Dynamics Statistics (BDS).

during recessions, especially during the Great Recession. The total number of firms decreased by more than 25 percent.

1.2.3 Job Reallocation

We define job reallocation as a sum of job destruction and job creation. Figure 1.4 depicts the job creation and job destruction rates since 1980. Figure 1.5 plots cyclical component of job creation and job destruction. Table 1.1 reports volatility and correlation of job reallocation to output. Job destruction is more volatile than job creation. Job reallocation rate is very volatile over the business cycle, which is 2.94 times of the volatility of aggregate output. The gross job reallocation is countercyclical. The job creation rate is procyclical, and the job destruction is countercyclical. I compute the average changes of job destruction and job creation in the recession relative to the change of job destruction and job creation not in the recession. There is about 2.7 times more job destruction in the recession, and 2.7 times less job creation.

However, if I only examine the job creation and destruction from entry and exit, I find different patterns. Figure 1.6 plots cyclical components of job creation from entry, job destruction from exit firms, entry and exit rates. The job destruction rate from exits is nearly acyclical, while the job creation rate from entry is countercyclical before 2007. This pattern occurs because of the cleansing effects. During recessions, both labor and capital are cheaper and labor is reallocated from low productivity firms to high productivity firms as well as from old firms to young firms. However, the job creation rate among new firms decreased during the Great Recession and has remained at a low level, even after the end of the recession. I further plot the relationship between job reallocation from the extensive margin and the entry and exit rates in figure 1.7. I find strong positive correlations among job reallocation from entry firms, entry and exit rates, and gross job

reallocation. This pattern suggests that the job reallocation at the extensive margin, from entry and exit, is important for explaining reallocation dynamics.

1.2.4 Decomposition of Job Reallocation

Now, I decompose job reallocation into the intensive margin and the extensive margin. Denote LR_t as gross job reallocation at time t ,

$$\begin{aligned}
LR_t &= \frac{\Delta L_{inc,t}^+ + \Delta L_{inc,t}^-}{L_t} + \frac{\Delta L_{entry,t}^+ + \Delta L_{exit,t}^-}{L_t} \\
&= \underbrace{\frac{\Delta L_{inc,t}^+ + \Delta L_{inc,t}^-}{L_t}}_{\text{intensive margin}} \\
&\quad + \underbrace{\frac{\frac{\Delta L_{exit,t}^-}{N_t^{exit}} \frac{N_t^{exit}}{L_t} + \frac{\Delta L_{entry,t}^+}{N_t^{entry}} \frac{N_t^{exit}}{L_t} + \frac{\Delta L_{entry,t}^+}{N_t^{entry}} \left(\frac{N_t^{entry}}{N_t} - \frac{N_t^{exit}}{N_t} \right)}{\frac{L_t}{N_t}}}_{\text{extensive margin from entry and exit}}
\end{aligned}$$

where $\Delta L_{inc,t}^+$ is job creation from incumbents, $\Delta L_{inc,t}^-$ is job destruction from incumbents, $\Delta L_{entry,t}^+$ is job creation from entrants, $\Delta L_{exit,t}^-$ is job destructions from exit, L_t is total labor supply, N_t is the number of firms in the market, N_t^{exit} is the number of firms exiting the market, and N_t^{entry} is the number of firms entering the market.

$\frac{\Delta L_{inc,t}^+ + \Delta L_{inc,t}^-}{L_t}$ represents the intensive margin of job reallocation between incumbents, where labor moves from one existing firm to another. The second term, total reallocation less the intensive margin, is defined as the extensive margin of job reallocation from entry and exit. It consists of three terms. $\frac{\frac{\Delta L_{exit,t}^-}{N_t^{exit}} \frac{N_t^{exit}}{L_t}}{\frac{L_t}{N_t}}$ is the relative size of exiter compared to incumbent times the exit rate,

representing total job destruction from exits. $\frac{\frac{\Delta L_{entry,t}^+}{N_t^{entry}} \frac{N_t^{exit}}{L_t}}{\frac{L_t}{N_t}}$ is the relative size of exit firms to the size of incumbent times the exit rate. This is the total number of jobs created if the number of entrants exactly replaces the number of firms exiting the market. Thus, in this case, the total number of

firms does not change. The last term, $\frac{\frac{\Delta L_{entry,t}^+}{N_t^{entry}} \left(\frac{N_t^{entry}}{N_t} - \frac{N_t^{exit}}{N_t} \right)}{\frac{L_t}{N_t}}$, is the job creation from the changing

Table 1.2: Decomposition of job reallocation

	Percentage of mean	Percentage of variance
Extensive margin	36.34%	40.89%
Intensive margin	63.66%	59.11%

Notes: Series are HP filtered with parameter $\lambda = 6.25$. 1980-2012 Source: Own calculations. Business Dynamics Statistics (BDS).

number of firms.

Table 1.2 computes the variance decomposition of job reallocation between the extensive margin and the intensive margin. The data show that the extensive accounts for 36% of gross job reallocation. The extensive margin also explains 41% of job reallocation variance.

1.3 Model

This section presents a general equilibrium industry dynamics model. The model is based on Hopenhayn (1992a) and Elsby and Michaels (2013). This model differs from Elsby and Michaels (2013) in a few respects. First, I consider the interaction between financial leverage and labor bargaining problems. A firm can raise funds via one-period debt but faces a borrowing constraint. Second, my model incorporates endogenous entry and exit, which enables us to examine both the intensive and extensive margins of job reallocation. Third, I introduce a firing cost.

1.3.1 Preferences and Technology

The economy is populated by a continuum of firms, a representative risk-neutral household with a continuum of workers, and a continuum of risk-neutral financial intermediaries. Firms are owned by the household. All agents discount future payoffs at rate $\beta < 1$.

There are two types of firms, incumbents and potential entrants. Firms observe both aggregate and idiosyncratic productivity shocks. Firms have a decreasing returns to scale production function with only labor input. In order to hire a worker, a firm must post a vacancy at a flow cost c_v . A firm fires a worker at a cost c_f . Firms can borrowing money from the financial intermediaries, subjected to a collateral constraint. Financial intermediaries are competitive with zero profits in

the equilibrium.

A fixed number of potential entrants draw an idiosyncratic productivity from a common distribution function. Given the draw, they make optimal entry decisions. The firms can borrow money from a financial intermediary, subjected to a collateral constraint. The price of debt depends on firms' characteristics and is determined endogenously in equilibrium.

Workers can either be employed or unemployed. An unemployed worker receives flow utility u from non-market activity ("leisure") and searches for a job. An unemployed worker meets with a firm randomly subjected to the labor market friction. An employed worker obtains a bargained wage but cannot search for a new job. Wage bargaining occurs between a firm and multiple workers as in Stole and Zwiebel (1996a,b). I will discuss wage determination shortly.

Labor market frictions limit the rate at which jobs can be filled. The matching function is $M = M(U, V)$, where U and V represent the total number of vacancies and unemployed workers, respectively. I assume that the matching function exhibits constant returns to scale. The vacancies posted by firms are filled with probability

$$q = \frac{M}{V} = M\left(\frac{U}{V}, 1\right).$$

Unemployed workers find a job at a rate

$$p = \frac{M}{U} = M\left(1, \frac{V}{U}\right).$$

Define labor market tightness

$$\theta = \frac{V}{U}.$$

With a probability of $q < 1$, posted vacancy will be filled. With a probability of $p < 1$, a worker finds a job. The market tightness is determined endogenously in equilibrium.

Timing

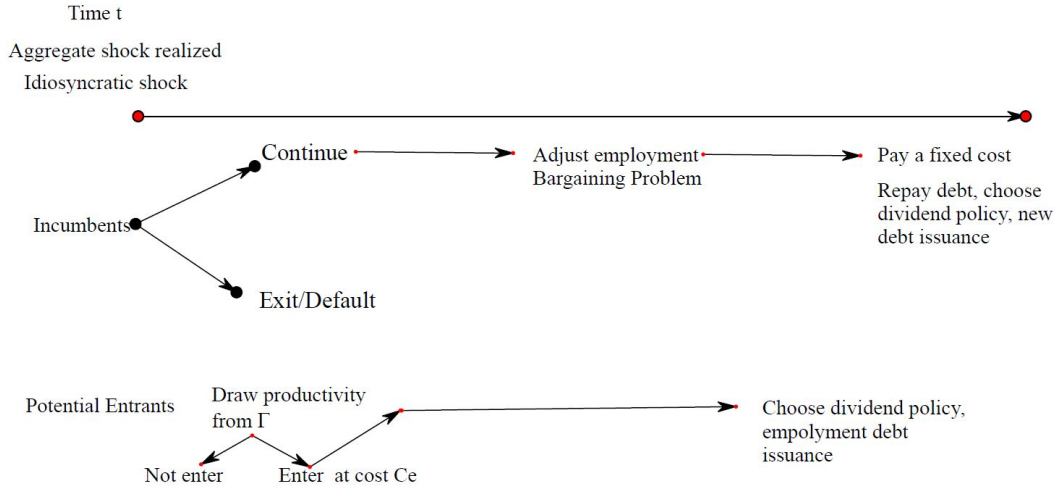


Figure 1.1: Timing of the model

1.3.2 Firms

1.3.2.1 Incumbents

Figure 2.1 summarizes the timeline of the model. At the beginning of period, there is a continuum of incumbent firms. The aggregate state z is known. Every period is divided into three subperiods: morning, midday and afternoon. In the morning, each incumbent sequentially observes an aggregate productivity shock z and an idiosyncratic productivity shock s . Given the realization of idiosyncratic productivity and aggregate shocks, the firm decides to remain in or exit the market. If a firm stays in the market, it incurs a fixed cost of production c_f in the afternoon. The fixed cost in each period introduces the endogenous exit of a firm. There are different ways to specify endogenous exits. We can either assume that the firm has outside options (Jovanovic, 1982; Lee and Mukoyama, 2012) or that there is a fixed cost of production (Hopenhayn, 1992a).

At midday, the firm adjusts its employment n . A firm can fire a worker at a cost of κ . Once the worker is fired, he cannot return to the market immediately. To hire a worker, firms must post

vacancies. Each vacancy costs c_v . Given the job finding rate, firms choose their optimal level of employment. Conditional on the employment level n , firms bargain with workers over the labor wage $w(z, s, n, b)$. The wage depends on aggregate productivity, idiosyncratic productivity, number of employees, and debt level.

In the afternoon, the firm produces consumption goods, makes investment decisions and chooses new level of debt. The production function exhibits decreasing returns to scale

$$F(z, s, n) = zsf(n),$$

where z is the aggregate shock, and n is the labor input. I make the following assumptions about the production function: 1) The function $f : R_+^1 \rightarrow R_+^1$ is strictly increasing, strictly concave, and continuously differentiable; and 2) both the aggregate shock z and idiosyncratic shock s are AR(1) processes.

$$\ln s' = \rho_s \ln(s) + \varepsilon_s$$

$$\ln z' = \rho_z \ln(z) + \varepsilon_z$$

where $\varepsilon_s \sim N(0, \sigma_s^2)$ and $\varepsilon_z \sim N(0, \sigma_z^2)$. Denote $h^a(z'|z)$ and $h^i(s'|s)$ as the conditional probability density functions of idiosyncratic productivity and aggregate shocks, respectively.

A firm finances its investment by issuing one-period debt. The firm begins the period with intertemporal liabilities b . Hiring a worker is risky such that default occurs in equilibrium. The firm first needs to repay the debt carried over from the last period. If it cannot fully repay this debt, bankruptcy occurs and lenders can recover only a fraction ζ of the firm's cash flow⁷

$$F(z, s, n_{-1}) - w_{-1}n_{-1},$$

where n_{-1} and w_{-1} ⁸ represent yesterday's employment and wage, respectively. I assume that the

⁷I abstract from the renegotiation process here, although renegotiation is more beneficial to the financial intermediary than liquidation. See Cooley and Quadrini (2001).

⁸I denote lagged values using subscript $_{-1}$ and forward values with prime $'$.

financial intermediary takes over the firms for an extra period of production if the firms default. There is no bargaining between workers and the financial intermediary. The cost of production to the intermediary is the total compensation to workers in the period before default, $w_{-1}n_{-1}$. Upon default, workers only receive a flow payoff u . The difference, $w_{-1} - u$, is forgone as part of default costs.

After firm fully repays its debt, it chooses its dividend payout d , and new intertemporal debt b' . The firm maximizes its dividend payouts $d \geq 0$. The expected present value of the firm $V^i(z, s, n_{-1}, b)$ is

$$V^i(z, s, n_{-1}, b) = \max_{d, b', n} d + \beta \mathbb{E} \max \{ V^i(z', s', n, b'), 0 \}, \quad (1.1)$$

subjected to a budget constraint

$$b + w(z, s, n, b)n + d + c_f + \frac{c_v}{q(z)}(n - n_{-1})1_{n > n_{-1}} + \kappa(n_{-1} - n)1_{n < n_{-1}} \leq F(z, s, n) + q_b(z, s, n, b')b',$$

where $q_b(z, s, n, b')$ is the price of a one-period debt. The new level of debt is constrained by

$$b' \leq \chi \{ \mathbb{E}[F(z', s', n)] - w(z, s, n, b)n \},$$

where χ represents the efficiency of the financial market, $\mathbb{E}[F(z', s', n)] - w(z, s, n, b)n$ is the financial intermediary's expected operating profits in the event of default. By varying the value of χ , we can trace out all degrees of efficiency of the financial market; $\chi = 0$ corresponds to the case in which the firms have no access to external debt, and $\chi = \infty$ corresponds to a perfect financial market.

The sequential timing of decisions for an incumbent firm does not matter.⁹ We can also assume that all of all these firm-level decisions occur simultaneously which does not change dynamics of firm behaviors. For the economic environment described here, it is impossible to know the aggregate state-contingent prices without knowing the distribution of productivity, employment and debt across firms. In particular, because the wage is now partly determined by the labor market

⁹See also Monacelli et al. (2011).

equilibrium, firms have to know the labor market tightness θ to predict future wages. The labor market demand curve is the aggregate labor demand across all types of firms. The distribution of firm size across all states determines the location of the labor market demand curve. Thus, firms have to incorporate such information into their decision.

1.3.2.2 Entrants

In the morning, a total measure of potential entrants N randomly draw an idiosyncratic productivity s from the distribution function $\Gamma(s)$. After drawing the s , a potential entrant decides whether to enter the market. If it enters, it pays the entry cost c_e and does not carry any initial labor and debt. A firm entering the market could finance its investment through one-period debt. The new entrant maximizes the expected value of the firm $V^i(z, s, 0, 0)$, where both the initial employment and the level of debt are zero. They choose the labor input, dividend payout and debt,

$$\begin{aligned}
 V^i(z, s, 0, 0) &= \max_{d, b', n} d + \beta \mathbb{E} \max \{V^i(z', s', n, b'), 0\} \\
 \text{s.t. } d + w(z, s, n)n + \frac{n}{q(z)}c_v + c_f &= F(z, s, n) + q_b(z, s, n, b')b' & (1.2) \\
 b' &\leq \chi \{ \mathbb{E}[F(z', s', n)] - w(z, s, n, b)n \} \\
 d &\geq 0.
 \end{aligned}$$

Tomorrow morning, entrants will face exactly the same problem as the incumbent firms. Thus, firms enter the market if and only if the discounted value of entering is higher than the entry cost,

$$V^i(z, s, 0, 0) \geq c_e.$$

This equation determines a threshold $s_e^*(z)$ at which only firms with idiosyncratic productivity shock $s \geq s_e^*(z)$ enter the market.

1.3.3 Workers

A worker is either employed or unemployed. An employed worker receives a flow payoff equal to the bargained wage, $w(z, s, n, b)$. He loses his job with a probability of $\lambda(z', s', n, b)$ in the next period. He obtains a value of unemployment $V^u(z')$. If he continues to work at the current firm, he receives a value of $V^e(z', s', n, b)$. Thus, the value of employment in a firm of productivity s , debt level b , and size n , $V^e(z, s, n, b)$, is defined as

$$V^e(z, s, n, b) = w(z, s, n, b) + \beta \mathbb{E} [\lambda(z', s', n, b) V^u(z') + (1 - \lambda(z', s', n, b)) V^e(z', s', n, b)]. \quad (1.3)$$

An unemployed worker receives a flow payoff u , which represents either the unemployment benefits or the value of leisure. She finds a job with probability p . If a worker finds a job, she receives the value of employment $V^e(z', s', n, b)$; otherwise, she remains unemployed and obtains the value of unemployment $V^u(z')$. The value of unemployment can be written as

$$V^u(z) = u + \beta \mathbb{E} [(1 - p(z')) V^u(z') + p(z') V^e(z', s', n, b)]. \quad (1.4)$$

1.3.4 Wage and Bargaining Problem

Wage bargaining occurs between a firm and multiple workers. The theoretical analysis of this bargaining problem is by Stole and Zwiebel (1996a,b). Stole and Zwiebel propose an extensive-form game with a unique subgame perfect equilibrium, where wages and profits coincide with the Shapley values.¹⁰

Firms and workers bargain over the marginal surplus generated by their employment relation-

¹⁰Brugemann et al. (2015) note that the Stole and Zwiebel game does not support the Shapley values as a subgame perfect equilibrium. Instead, they propose a Rolodex Game. Under some mild restrictions, this game has a unique subgame perfect equilibrium that satisfies the axiomatic solution offered by Shapley. The Stole and Zwiebel game and the Rolodex Game both show that each worker and firm captures a fraction $1/(n+1)$ of the total surplus in equilibrium.

ship. The marginal surplus, which I denote $J(z, s, n, b)$, is equal to

$$J(z, s, n, b) = zsf_n(n) - w(z, s, n, b) - w_n(z, s, n, b)n + \beta \mathbb{E}V_n(z', s', n, b') - \mu \chi \mathbb{E} \frac{\partial [F(z', s', n) - w(z, s, n, b)n]}{\partial n},$$

where $V(z, s, n, b)$ is the value of a firm, and μ is the Lagrangian multiplier of the borrowing constraint. The first term is the worker's marginal product of labor; the second term is the worker's wage compensation; the third term represents the worker's marginal impact on all existing workers; the fourth term is the worker's contribution to the firm's future value; and the last term, which is novel in this paper, shows the worker's marginal effect on borrowing capacity. In the case in which the firm is not borrowing constrained, $\mu = 0$, or in which the firms have no access to external debt, $\chi = 0$, the marginal surplus is identical to that in Elsby and Michaels (2013).

Wages are the outcome of a Nash bargaining game between a firm and multiple workers over the marginal surplus

$$(1 - \eta) [V^e(z, s, n, b) - V^u(z)] = \eta [J(z, s, n, b) + \kappa], \quad (1.5)$$

where η is the bargaining power of the worker. The firing cost κ is added to the marginal value of the firm. When the firms bargain with the workers, the workers are able to use firing cost as a credible threat to the firms. Firms have to pay firing costs if the worker-firm relationship breaks down during bargaining.

1.3.5 Exit and Default Decisions

Every morning, the firm makes its exit and default decisions. The firm exits the market if the continuation value, conditional on its productivity and aggregate productivity, is negative. A firm defaults on its debt if it cannot fully repay its debt with its current profits. If a firm defaults, I assume that there is no renegotiation and that the firm must exit the market. The firm's debt holders can recover only part of cash flow $\zeta [F(z', s', n) - w(z, s, n, b)n]$. The default decision is important in my model, as it generates the endogenous one-period debt price for each type of firm.

It also limits the growth rate of small firms. The price of debt varies with firm size and age.

The firm exits the market if and only if the firm's value is below 0. That is

$$V^i(z, s, n-1, b) \leq 0.$$

Therefore, the exit decision involves a reservation rule

$$v^e(z, s, n-1, b) = \begin{cases} 0 & \text{if } s \geq s_{exit}^*(z, n-1, b) \\ 1 & \text{o.w.} \end{cases},$$

where

$$s_{exit}^*(z, n-1, b) = \inf \{s \in S : V^i(z, s, n-1, b) \geq 0\}.$$

Default occurs if and only if the current period profits are smaller than the debt repayment. Equivalently, a firm will default if its debt repayment is too large or if the realized productivity shock is too small. The default will occur if and only if the firm's idiosyncratic shock s is smaller than a cutoff value

$$v^d(z, s, n-1, b) = \begin{cases} 0 & \text{if } s \geq s_d^*(z, n-1, b) \\ 1 & \text{o.w.} \end{cases},$$

where

$$s_d^*(z, n-1, b) = \sup \{s \in S : F(z, s, n-1) - w(z_{-1}, s_{-1}, n-1, b_{-1})n_{-1} - c_f \leq b\}.$$

1.3.6 Financial Intermediary

Financial intermediary lends money to firms. The market is perfect competitive in which all intermediaries earn zero profits. The price of debt depends on firms's characteristics. The firm begins the period with intertemporal liabilities b . The firm first needs to repay the debt carried over from the last period. If the financial intermediary cannot receive full payment of the debt, bankruptcy occurs and they recover only a fraction ζ of the firm's cash flow $F(z, s, n-1) - w_{-1}n_{-1}$.

I make a few more assumptions about the financial intermediary if the firm defaults. First,

the intermediary takes over the firms for an extra period of production with existing workers. Second, there is no wage bargaining between workers and the financial intermediary. The cost of production to the intermediary is the total compensation to workers in the period before default, $w_{-1}n_{-1}$. Third, workers only receive a flow payoff u . The difference is forgone as part of default costs. Therefore, in the equilibrium, the price of debt for each type of firm, $q_b(z, s, n, b')$, is pinned down by the following equation

$$\begin{aligned}
& q_b(z, s, n, b')b' \\
= & \underbrace{b' \int \int \mathbb{1}(z', s', n, b') h^a(z'|z) h^i(s'|s) ds dz}_{\text{probability of continuation}} \\
& + \zeta \int \int \underbrace{[1 - \mathbb{1}(z', s', n, b')] \max\{b, F(z', s', n) - w(z, s, n, b)n\}}_{\substack{\text{indicator of default or exit} & \text{liquidation value if default or exit}}} h^a(z'|z) h^i(s'|s) ds dz,
\end{aligned} \tag{1.6}$$

where $\mathbb{1}(z', s', n, b')$ is an indicator function that firm continue in the market in the following period

$$\mathbb{1}(z', s', n, b') = [1 - v^e(z', s', n, b')] [1 - v^d(z', s', n, b')].$$

The left-hand side of the equation (1.6) is the total value of debt b' that financial intermediary lends to the firm at the price of $q_b(z, s, n, b')$. On the right-hand side, the first term is the level of debt multiplied by the expected probability that the firm stays in the market and makes a full payment. The second term represents the expected liquidation value of the firm if it exits or defaults.

1.3.7 Recursive Equilibrium

I consider a recursive equilibrium. A key element here is the law of the motion of aggregate state of the economy. $(z, \Omega(z, s, n_{-1}, b))$ are the aggregate state variables in my model. z is exogenously given. But $\Omega(z, s, n_{-1}, b)$, which denotes the density of firm size distributions over aggregate productivity, idiosyncratic productivity shocks, employment, and debt, is endogenously computed from the model. To hire a worker, the firms have to know the expected labor market tightness to post the exact number of vacancies. The labor market tightness depends on total labor demand of all firms. Given the exogenous process z , the only objective of the firm is to know the updated

Ω , that is, firms need to predict $\Omega(z', s', n, b') = I(z', z, \Omega(z, s, n_{-1}, b))$. For convenience, I denote $\Omega = \Omega(z, s, n_{-1}, b)$ and $\Omega' = \Omega(z', s', n, b')$ from now on.

Given the entry, exit, and the default decisions and the policy functions of the firms, the evolution of the state of the industry $\Omega(z', s', n, b')$ satisfies

$$\begin{aligned}
\Omega' &= \Omega(z', s', n, b') = I(z', z, \Omega) \\
&= h^a(z'|z) \int_s \int_{n_{-1}} \int_b \underbrace{\Omega(z, s, n_{-1}, b) \mathbb{1}_{(n, b'|z, s, n_{-1}, b, \Omega)} h^i(s'|s)}_{\text{measure of incumbents with } z, s', n, b'} \underbrace{\mathbb{1}(z', s', n, b')}_{\text{indicator for continuation}} ds dn_{-1} db \\
&\quad + h^a(z'|z) N \int_{s \geq s^*(z, \Omega)} \underbrace{\mathbb{1}_{(n, b'|z, \Omega)} h^i(s'|s)}_{\text{measure of new entrants with } z, s', n, b'} d\Gamma(s), \tag{1.7}
\end{aligned}$$

where $\mathbb{1}_{(n, b'|n_{-1}, b, s, z, \Omega)}$ and $\mathbb{1}_{(n, b'|z, \Omega)}$ are the indicator functions that firms choose employment n and debt b' given incumbents and entrants' policy functions, respectively,

$$\begin{aligned}
\mathbb{1}_{(n, b'|n_{-1}, b, s, z, \Omega)} &= \mathbb{1}_{g_n^i(s, b, n_{-1}; z, \Omega) = n, g_b^i(s, b, n_{-1}; z, \Omega) = b'}, \\
\mathbb{1}_{(n, b')} &= \mathbb{1}_{g_n^e(s; z, \Omega) = n, g_b^e(s; z, \Omega) = b'},
\end{aligned}$$

where g_n^i, g_b^i are the policy functions of the incumbents, and g_n^e, g_b^e are the policy functions of the entrants.

The Equation (2.8) is the evolution of probability density of firm size distribution. On the left-hand side, $\Omega(z', s', n, b')$ is the updated distribution of firms. On the right-hand side, $h^a(z'|z)$ represents the transition probability of aggregate state z . The first part measures the transition probability of firms that continue in the market with future characteristics s', n, b' , given all possible realizations from today's states. The second part is the total measure of new entrants with employment n , debt b' , and idiosyncratic productivity s' .

A recursive competitive equilibrium consists of a market tightness $\theta(z, \Omega)$, a forecast rule $I(z', z, \Omega)$, a value function $V^i(z, s, n_{-1}, b)$, the default decision $v^d(z, s, n_{-1}, b)$, the exit decision $v^e(z, s, n_{-1}, b)$, the entry decision $s^*(z)$, the price of debt $q_b(z, s, n, b')$, the optimal decision rules $\{g_n^i(s, b, n_{-1}; z, \Omega), g_b^i(s, b, n_{-1}; z, \Omega), g_n^e(s; z, \Omega), g_b^e(s; z, \Omega)\}$, and the number of potential entrants N such that

1. Incumbent optimization: The value function $V^i(z, s, n_{-1}, b)$ solves the Bellman Equation (1.1). $v^e(z, s, n_{-1}, b)$ and $v^d(z, s, n_{-1}, b)$ are the exit and default rules, respectively, associated with $V^i(z, s, n_{-1}, b)$. $g_n^i(s, b, n_{-1}; z, \Omega)$ and $g_b^i(s, b, n_{-1}; z, \Omega)$ are associated policy functions to $V^i(z, s, n_{-1}, b)$.
2. Entrant optimization: The value function $V^i(z, s, 0, 0)$ solves the Bellman Equation (2.5). $g_n^e(s; z, \Omega)$ and $g_b^e(s; z, \Omega)$ are the associated policy functions, and firms enter the market if and only if $s \geq s^*(z)$.
3. Given the forecasting function, wage function and the firms' optimal decision rules, $V^e(z, s, n, b)$ and $V^u(z)$ solves the Bellman Equations 1.3 and 1.4.
4. The labor market clears. $w(z, s, n, b)$ solve the wage bargaining problem 1.5.
5. Consistency: The forecast function $I(z', z, \Omega)$ is consistent with the actual law of motion, Equation (2.8), implied by the optimal decision rules.

1.4 Bargaining Outcome and Job Reallocation

In this section, I analyze the firms' optimal employment dynamics resulting from bargaining with workers for any given aggregate states.

Proposition 1 *The bargained wage $w(z, s, n, b)$ solves the differential equation*

$$\begin{aligned}
& w(z, s, n, b) \\
= & \frac{\eta}{1 - \eta\mu\chi} \left[(zs - \mu\chi\mathbb{E}[zs]) f_n(n) - (1 - \mu\chi) w_n(z, s, n, b) n + \beta p(z) \frac{c_v}{q(z)} + \kappa(1 - \beta(1 - p(z))) \right] \\
& + \frac{1 - \eta}{1 - \eta\mu\chi} u. \tag{1.8}
\end{aligned}$$

Proof. See the Appendix. ■

The solution extends the results of Elsby and Michaels (2013). If the firm is not borrowing constrained or if the firm has no access to external debt, which means $\mu\chi = 0$, the solution is identical

to the results of Elsby and Michaels (2013). First, the firing cost term is included. Second, the level of debt affects the bargained wage if the firm is borrowing constrained, i.e., $\mu > 0$, and the firms have access to external debt, i.e., $\chi > 0$. The intuition for the above solution is straightforward. The wage is a weighted average of two terms, discounted by tightness of borrowing constraints $1 - \eta\mu\chi$. The first term is the worker's share of contribution to firm's value. The second term is the worker's outside option. As shown in Stole and Zwiebel (1996a,b) and Elsby and Michaels (2013), the worker's contribution includes the his impact on the wages of other employees of the firm. The first part, $(zs - \mu\chi\mathbb{E}[zs])f_n(n)$, is the marginal product of labor. The second part, $(1 - \mu\chi)w_n(z, s, n, b)n$, is the worker's impact on the wages of other employees if he quits. The last part, $\beta p(z)\frac{c_v}{q(z)} + \kappa(1 - \beta(1 - p(z)))$, represents the marginal cost of hiring and firing the worker. The additional part is the worker's marginal impact on the borrowing constraint, which is novel in the model. It represents the fact that the firm can strategically use debt as a bargaining tool to reduce the bargained wage.

As in a standard search model, the wage is increasing in the worker's bargaining power η , the marginal product of labor, the worker's job finding rate $p(z)$, the marginal cost of posting a vacancy for the hiring firm $\frac{c_f}{q(z)}$, the worker's flow value of unemployment insurance u , the firm's firing cost κ , and the worker's marginal impact on other workers $w_n(z, s, n, b)$. A higher value of each parameter stands for a higher opportunity cost between the worker and the firm, resulting in a higher bargained wage. However, if a firm carries a high level of debt, it has a higher default probability. A constrained firm offers a lower bargained wage to decrease its default probability, which is beneficial to both the firm and the worker avoiding separation. By reducing bargained wage, smaller and constrained firms can carry more debt and grow faster. The benefit to the worker is the future value of the employment within the firm once it is no longer borrowing constrained.

Proposition 2 *If the production function is of the Cobb-Douglas form*

$$f(n) = n^\alpha,$$

the bargained wage $w(z, s, n, b)$, from the differential Equation 1.8, has the following solution

$$w(z, s, n, b) = A + B\alpha n^{\alpha-1},$$

where

$$A = \frac{\eta}{1 - \eta\mu\chi} \left\{ \beta p(z) \frac{c_v}{q(z)} + \kappa [1 - \beta (1 - p(z))] \right\} + \frac{1 - \eta}{1 - \eta\mu\chi} u,$$

$$B = \frac{\eta}{1 - \eta(1 - \alpha) - \eta\mu\chi\alpha} [zs - \mu\chi\mathbb{E}(zs)].$$

Proof. See the Appendix. ■

A firm's optimal employment decision $n(z, s, n_{-1}, b)$ is characterized by taking the first-order condition with respect to n . We obtain

$$F_n(z, s, n) - w(z, s, n, b) - w_n(z, s, n, b)n - \frac{c_v}{q(z)} \mathbf{1}_{n > n_{-1}} + \kappa \mathbf{1}_{n < n_{-1}} + \beta \mathbb{E} \max \{ V_n^i(z', s', n, b'), 0 \} - \mu\chi \mathbb{E} [F_n(z', s', n) - w_n(z, s, n, b)n - w(z, s, n, b)] = 0.$$

The indicator function reflects the asymmetric marginal costs of hiring and firing workers. No cost is incurred if the firm does not change its employment level. Asymmetric marginal costs lead to an inaction region, in which the firm freezes employment such that $n = n_{-1}$.

Proposition 3 *The employment decision rule is (S, s) , where $[\underline{n}, \bar{n}]$ is an inaction region. That is*

$$n(z, s, n_{-1}, b) = \begin{cases} \bar{n}(z, s, n_{-1}, b) & \text{if } s > \bar{s}(z, n_{-1}, b) \\ n_{-1} & \text{if } s \in [\underline{s}(z, n_{-1}, b), \bar{s}(z, n_{-1}, b)] \\ \underline{n}(z, s, n_{-1}, b) & \text{if } s < \underline{s}(z, n_{-1}, b) \end{cases},$$

where $\bar{n}(z, s, n_{-1}, b)$ and $\underline{n}(z, s, n_{-1}, b)$ solve the following equations

$$\begin{aligned} F_n(z, s, n) - w(z, s, n, b) - w_n(z, s, n, b)n - \frac{c_v}{q(z)} + \beta \mathbb{E} \max \{V_n^i(z', s', n, b'), 0\} \\ - \mu \chi \mathbb{E} [F_n(z', s', n) - w_n(z, s, n, b)n - w(z, s, n, b)] &= 0, \\ F_n(z, s, n) - w(z, s, n, b) - w_n(z, s, n, b)n + \kappa + \beta \mathbb{E} \max \{V_n^i(z', s', n, b'), 0\} \\ - \mu \chi \mathbb{E} [F_n(z', s', n) - w_n(z, s, n, b)n - w(z, s, n, b)] &= 0. \end{aligned}$$

The optimal employment decision rule takes the form of an $[s, S]$ rule, with $[\underline{s}(z, n_{-1}, b), \bar{s}(z, n_{-1}, b)]$ as inaction regions. Specifically, if it receives an unfavorable idiosyncratic productivity shock, the firm will shed workers until it reaches the optimal level of employment in a separation regime. If it receives a favorable idiosyncratic productivity shock, the firm will post vacancies and hire workers until it reaches the optimal level of employment in a hiring regime.

Proposition 4 *The job creation and job destruction rates for firm type (s, n_{-1}, b) and aggregate state z are*

$$JC(z, s, n_{-1}, b) = \begin{cases} \frac{\bar{n}(z, s, n_{-1}, b)}{n_{-1}} - 1 & \text{if } s > \bar{s}(z, n_{-1}, b) \\ 0 & \text{if } s \in [\underline{s}(z, n_{-1}, b), \bar{s}(z, n_{-1}, b)] \\ \frac{\underline{n}(z, s, n_{-1}, b)}{n_{-1}} - 1 & \text{if } s < \underline{s}(z, n_{-1}, b) \end{cases},$$

where $JC > 0$ indicates job creation; $JC < 0$, indicates job destruction.

Proposition 5 *If $\rho_s = 0$ or ρ_s is close to 1, I can show that the range of the inaction region has the following property*

$$\begin{aligned} \frac{\bar{s}(z, n_{-1}, b)}{\underline{s}(z, n_{-1}, b)} &\simeq 1 + \frac{\frac{c_v}{q(z)} + \kappa}{(1 - \mu \chi) [w(z, \underline{s}, n_{-1}, b) + w_n(z, \underline{s}, n_{-1}, b)n_{-1}] - \kappa} \\ &\simeq 1 + \frac{\frac{c_v}{q(z)} + \kappa}{(1 - \mu \chi) [A + B\alpha^2 n_{-1}^{\alpha-1}] - \kappa}. \end{aligned}$$

Further,

$$\frac{\partial \frac{\bar{s}(z, n_{-1}, b)}{\underline{s}(z, n_{-1}, b)}}{\partial n_{-1}} < 0, \quad \frac{\partial \frac{\bar{s}(z, n_{-1}, b)}{\underline{s}(z, n_{-1}, b)}}{\partial \chi} > 0.$$

Proof. See the Appendix. ■

This proposition shows that the relationship between the range of inaction regions and firms' characteristics. A smaller inaction region, where firms more frequently adjust employment following a productivity shock, implies more volatile job reallocation.

First, consider a frictionless economy with competitive labor market and no financial leverage, that is $\chi = 0$. If firms pay competitive wages to the worker, which means the wage $w(z, s, n_{-1}, b)$ is constant across firms and the worker's marginal impact on other employee $w_n(z, \bar{s}, n_{-1}, b)$ is zero, then the range of inaction regions are constant for all firms.¹¹

Second, consider an economy with competitive labor market and financial leverage, that is $\chi > 0$. In this economy, financially constrained firms are less likely to adjust labor than financially unconstrained firms when hit by a productivity shock. First, financially constrained firms are more likely to be small firms with a higher marginal product than the market clear wage. In the model, firing is less costly to firms because they can always find a worker at the competitive wage. Since the wage in the model affects collateral constraints, when a negative productivity shock hits the economy, the large firms find it is optimal to fire workers to reduce cost of default, while smaller firms find it is optimal to maintain the employment to get maximum borrowing. Financially constrained firms have a larger inaction region, which is contrary to the empirical evidence in Benmelech et al. (2011), where they find a negative and statistically significant relationship between financially constrained firms and the total change in the firm employment.

Third, consider an economy with decentralized bargained wage and no financial leverage. In my model with the decentralized wage, the inaction regions are smaller for small firms. The intuition behind this is the small firms have a higher marginal product of labor, thus they pay a higher wage to the worker. Because of the fixed marginal adjustment cost, small firms are more likely to hire or fire the worker in the presence of a shock.

Fourth, consider an economy with both decentralized bargained wage and financial leverage. The range of inaction region is larger for the firms that are financially unconstrained, $\mu\chi = 0$. This is because, with the borrowing constraints, the financially unconstrained firms are more likely to

¹¹This is also a standard result in the model with adjustment cost and competitive wage.

adjust their optimal level of debt to save the cost of firing workers and searching for new workers. Their inaction regions are larger. This is also consistent with the empirical evidence in Benmelech et al. (2011).

Last, compare two economies with different level of financial borrowing capacity, more firm are financially constrained in the economy with a lower borrowing capacity. The resulting inaction regions are larger with a higher borrowing capacity than that with a lower borrowing capacity.

1.5 Calibration

In this section, I characterize firms' financial behaviors and industry dynamics. By linking firms' financial decisions to industry dynamics and the aggregate economy, the model setup allows me to examine important empirical and theoretical patterns of industry dynamics. In particular, the model without aggregate shocks, i.e., $z = 1$, provides the benchmark. I describe the invariant distribution of firms and their financial structures. I parameterize the model assuming that one period is one quarter. The discount factor is 0.985 with an implied annual risk-free interest rate of 6.09%. All parameter values are summarized in Table 2.1.

The production function exhibits decreasing returns to scale, zsn^α . I set α to a standard value of 0.60, with an implied labor share 0.72. The idiosyncratic productivity follows an AR(1) process,

$$\ln s' = \rho_s \ln(s) + \varepsilon_s,$$

where $\varepsilon_s \sim N(0, \sigma_s^2)$. The values of the persistence are obtained from Lee and Mukoyama (2012). The variance is calibrated to match the empirical patterns of employment growth volatility. The AR(1) process is approximated by a Markov process with ten states (Tauchen, 1986). The matching function is assumed to be Cobb-Douglas $M = \xi U^\phi V^{1-\phi}$, with matching elasticity $\phi = 0.5$ (Petrongolo and Pissarides, 2001) and matching efficiency $\xi = 0.129$ (Pissarides, 2007). The workers' bargaining power is 0.443. The unemployment benefit u is 0.387 (Elsby and Michaels, 2013).

The liquidation value of firm ζ in the event of default is 0.7 (Gourio, 2013). I assume that

potential entrants draw their productivity from the distribution $\Gamma(s) \sim B \exp(-s)$, where B is a scale parameter to make the distribution sum to one. The entry cost c_e is endogenously computed such that only the firms with productivity above the mean productivity enter the market in the stationary equilibrium.

There are still five parameters, $\{c_f, \chi, \sigma_s, c_v, \kappa\}$, that must be specified. Those parameters are selected to obtain the following targets: 1) a total exit probability, including the default rate, of 5.4% (Lee and Mukoyama, 2012); 2) a mean leverage ratio of 0.81 (Chen et al., 2009); 3) variance of the employment growth rate of 0.14; 4) a hiring cost of 14% of the quarterly wage; 5) persistence in the employment process of 0.97. N is solved such that the total measure of firm sums to 1 in the stationary equilibrium. The size of the labor force L is chosen to match a mean unemployment rate of 6.5%.

1.5.1 The Economy without Aggregate Shocks

The economy is characterized by a certain distribution of firms Ω over all state variables, n, b and s . In this section, I focus on the invariant distributions of firms. The existence and the uniqueness of the invariant distribution therefore depend on the properties of the transition matrix generated by the optimal policy functions, labor and debt, of both incumbents and entrants, which is characterized by equation (2.8) without aggregate shocks z . The invariant distribution is the fixed point of this contraction mapping.

The Appendix describes the details of the computation of the stationary distribution. I solve the value function first and then simulate the model with 10,000 firms over 5,000 periods. I exclude the first 10% of simulations and calculate the summary statistics from the remaining simulations. To maintain the stationary distribution from endogenous exit, I replace all exiting firms with the new entrants. The total number of firms, as well as the labor supply, remains unchanged. Table 2.2 compares the results of the model to the data targets. The simulated model approximates the targets. The job reallocation rate also approaches the values observed in the data.

It is critical to examine how the financial leverage affects the stationary distribution of firms in the model before I proceed to solve the model with aggregate shocks. I compare the calibrated

model to the model without financial leverage. I set $\chi = 0$. If firms have no access to the external financing, they are more likely to decrease either their employment or wage profile after experiencing a negative shock. If the firms have access to the external financing, constrained firms are also more likely to hire more workers in the presence of a positive shock. However, as I have shown in the Proposition 5, the ratio of the increased productivity for hiring workers and the decreased productivity for firing workers is larger in the economy with positive χ . This is because all firms are essentially borrowing constrained in the economy with $\chi = 0$. The inaction region is smaller, and employment growth is less persistent. Without the financial leverage, fewer firms enter with internal financing and more firms exit if their continuation values are negative. Finally, fewer jobs are created and destroyed. The intuition is straightforward. If firms have no access to the external financing, they have less incentive to hire too many workers, even they have adequate productivity. Productive firms can only finance their project and hire workers using their current operating profits, which limits their growth opportunities.

Figure 1.8 illustrates the bargained wage conditional on firm size. I take the number of employees as a proxy for firm size. I plot the relationship in log values. The pink line indicates the marginal product of labor. The red dotted line indicates the wage profile if firms have no debt, which is the optimal wage in the model without the financial market. The yellow line is the wage profile with an optimal level of debt. The difference between the wage with debt and the wage without debt is plotted in blue. The optimal wage contract with a Stole-Zwiebel bargaining protocol have three features. First, small firms pay higher wages because of their higher marginal product of labor and greater growth options. Second, workers are paid less than the marginal product of labor in small firms, while they are paid more than the marginal product of labor in the large firms. The workers in the small firms have to compensate their marginal negative impacts on other workers. The marginal impacts decrease with firm size. Therefore, the small firms are more likely to offer a lower wage contract than the marginal product of labor to offset the negative effect. As a firm becomes larger, the workers gain bargaining power in the wage determination process, because the marginal worker has a negligible impact. Third, the optimal wage contract with financial leverage, where small firms are constrained, is smaller than the wage contract without financial leverage. Hiring an additional worker is risky because of the default risk of small firms. Firms gain a slightly

better bargaining position to offer a lower wage today and to sustain growth. Since the workers are not allowed for the on-the-job search, in this case, it is also beneficial for workers to remain in the firm with a lower bargained wage in exchange for the continuation value of employment. For larger firms, borrowing constraints are not binding. The difference between the wage with debt and the wage without debt decreases as firm size increases.

Figure 1.9 plots financial leverage and labor share conditional on firm size and productivity. Financial leverage is defined as the total amount of debt divided by operating profits, output less total wage payout. The labor share is defined as the total wage payout over total output. The blue and red lines plot firms' financial leverage conditional on firm size. The pink and yellow lines represent the total labor share of an individual firm. The figure shows that 1) small firms are constrained and have higher financial leverage and that 2) more productive firms have lower financial leverage. This pattern is very intuitive. Small and low productive firms have smaller internal profits, and hence they are willing to borrow more debt to finance their projects. As firms become larger, they can finance their projects primarily using internal cash, profits generated less dividend payouts. Larger firms pay a lower wage to each worker, but their total wage payments are higher than those of small firms.

Figure 1.10 plots the size, age, exit and default distributions of firms. The top panels show the firm size and age distributions. The distribution exhibits a degree of skewness toward small and young firms. All these patterns confirm the empirical regularity of the data. The bottom panels report the exit and default densities of the model as a function of firm size and age. I have shown that the default and exit probabilities are higher for small firms. The density reported is the actual number of exits and default as a fraction of total exits and defaults within each size and age group. Small and younger firms face higher default and exit probabilities, and they are more likely to default on debt and exit the market.

Figure 1.11 plots the dynamics of job reallocation and the growth rate conditional on firm size and age. The top panels indicate the relationships between firm dynamics and job reallocation. Job destruction is defined as the sum of employment losses from contracting firms divided by total employment. Job creation is defined as the sum of employment added from expanding firms divided by total employment. Job creation is hump-shaped. For small and young firms, the job de-

struction rate is also very small. This is because small and young firms demand less labor, and this effect dominates the contractionary effects of small and young firms' higher probabilities of exit and default. As firms grow larger, their growth options dominate the default and exit probabilities, and hence the gross job reallocation rate increases. Job destruction is negatively correlated with firm size and age. However, the bottom panel of Figure 1.11 provides empirical evidence of firm dynamics: small and young firms grow faster and experience higher growth volatility. The growth rate and standard deviation of growth are decreasing functions of firm size and age.¹²

To summarize, this model with frictional labor and financial markets can capture both the “age dependence” and the “size dependence” of firm dynamics (Cooley and Quadrini, 2001). The model is also able to account for most of the stylized facts of firm behaviors. I replicate and extend many of the findings in Cooley and Quadrini (2001). This step is important before examining the model properties when aggregate shocks are included.

1.6 Aggregate Dynamics

This section presents the cyclical behaviors of firm dynamics by adding an aggregate shock. I assume that the aggregate shock follows an AR(1) process with persistence 0.9 and standard deviation 0.09. I discretize the AR(1) process into a Markov process over the state space $[0.99, 1, 1.01]$. The challenge of including both aggregate and idiosyncratic shocks is the computational complexity of the general equilibrium. My solution is based on Krusell and Smith (1998). I solve the model by iterating between an inner loop and an outer loop until the forecast rule for the aggregate state variables is consistent with the equilibrium outcome. I approximate the state variable, the distribution of firms Ω , using the first moments of market tightness θ and productivity z . The forecast rule is

$$\log(\theta') = a_0 + a_1 \log(\theta) + a_2 \log(z').$$

Given their forecast of the level of market tightness, incumbents and entrants post vacancies to fill jobs. The computation details are summarized in the Appendix. The results of the model with

¹²Cooley and Quadrini (2001) shows that the standard deviation of growth is a decreasing function of firm size, except for very small firms.

aggregate shocks are presented in Table 2.3, 1.6, and 1.7.

Table 2.3 shows the cyclical behavior of entry and exit. We observe a strong pattern of procyclical behavior for entries, with a rate of 4.42% during the recession versus 6.68% during the boom. The total exit rate is countercyclical: 5.85% during the recession and 5.20% during the boom. We observe both procyclical entry and countercyclical exit in the data. The model also captures countercyclical default rates of 0.58% during the recession and 0.35% during the boom. Entry is procyclical because of bargained wages and external financial costs. Countercyclical external financing costs make entry more difficult during the recession than during the boom. The average size of entrants is larger during the recession than during the boom. The average size of entrants during the boom is 92% of their average size during the recession.

Table 1.6 is the main result from the simulated model on the relationship between job reallocation and the extensive margin. The table presents the decomposition of job reallocation variance between the intensive and extensive margins. The extensive margin accounts for 41% of job reallocation over the business cycle in the data, while the simulated model explains 36% of job reallocation. This result is not surprising. The model features endogenous entry and exit. In response to a negative TFP shock, the option value of waiting is higher because of a higher default risk. Potential entrants delay entering the market. Incumbents fire workers and delay hiring workers. The inaction region becomes smaller. Thus, the procyclical entry rate and countercyclical exit rate adequately capture the contribution of the extensive margin to the job reallocation variance.

Table 1.7 summarizes the correlation between job reallocation and total output. I also compute the correlations for different size groups. The simulated time series replicate most of their counterparts in the data. Job destruction is countercyclical, with a value of -0.43 in the data versus -0.42 in the model. While job creation is nearly acyclical for the entire sample, it is procyclical for both small and large firms. Net job reallocation, defined as job creation minus job destruction, is procyclical. Gross job reallocation, defined as job creation plus job destruction, is countercyclical. The countercyclical character of job destruction largely drives the cyclicity of both net and gross job reallocation.

To examine the effectiveness of the model, I compare the simulated model results with the

model in which I vary the level of borrowing capacity. Table 1.8 presents the model results with varying levels of borrowing capacity. It enables me to investigate the role of the financial market and the extensive margin in shaping aggregate dynamics. The extensive margin accounts for more job reallocation if we relax the borrowing constraint. The intuition is straightforward: With a higher borrowing capacity, firms face a higher default risk and reduced costs for entrants characterized by higher productivity. With a more developed financial system and a higher borrowing limit, the market is more fluid with a higher job reallocation rate.¹³ However, the economy is also more vulnerable to financial shocks, e.g., a negative shock to the borrowing limit. A permanent shock to borrowing capacity will reduce reallocation from the extensive margin. This may explain the sizable and persistent decrease in job reallocation both during and after the Great Recession. A negative financial shock increases both default probabilities and exits. At the same time, firms are less likely to enter the market. If entrants are small, their initial impacts on aggregate dynamics are limited, but these young and survivor firms grow faster and create more jobs during and after recessions. This extensive margin amplifies the effects of aggregate dynamics.

1.7 Conclusion

In this paper, I propose a general equilibrium industry dynamics model with endogenous entry and exit and frictional labor and financial markets. I characterize the optimal bargained wage as a function of firm characteristics. Specifically, I prove that financial leverage negatively affects the bargained wage. I numerically solve the model, which can adequately explain several stylized facts concerning both the stationary distribution (size and age dependence) and the business cycle (cyclical behaviors of entry, exit and job reallocation). The model yields a procyclical entry rate and a countercyclical exit rate. The extensive margin resulting from endogenous cyclical entry and exit accounts for 36% of job reallocation variance over the business cycle. The model also indicates cross-sectional implications of financial leverage and the wage contract. The bargained wage is negatively correlated with firm size and financial leverage. This is an interesting empirical

¹³If reallocation helps correct misallocation among firms, a more fluid labor market may explain why misallocation is more severe in developing countries.

question left for future research.

Appendix

Proof of Proposition 1

Proof. First, we have firm's first order condition w.r.t. employment

$$F_n(z, s, n) - w(z, s, n, b) - w_n(z, s, n, b)n - \frac{c_v}{q(z)}1_{n>n_{-1}} + \kappa 1_{n<n_{-1}} + \beta \mathbb{E} \max \{V_n^i(z', s', n, b'), 0\} - \mu \chi \mathbb{E} [F_n(z', s', n) - w_n(z, s, n, b)n - w(z, s, n, b)] = 0$$

The decision rule is (S, s) , with $[\underline{n}, \bar{n}]$ as inaction region. That is

$$n(z, s, n_{-1}, b) = \begin{cases} \bar{n}(z, s, n_{-1}, b) & \text{if } s > \bar{s}(z, n_{-1}, b) \\ n_{-1} & \text{if } s \in [\underline{s}(z, n_{-1}, b), \bar{s}(z, n_{-1}, b)] \\ \underline{n}(z, s, n_{-1}, b) & \text{if } s < \underline{s}(z, n_{-1}, b) \end{cases}$$

Therefore, we can rewrite the marginal surplus to a firm as

$$\begin{aligned} & J(z, s, n, b) \\ &= zsf_n(n) - w(z, s, n, b) - w_n(z, s, n, b)n + \beta \mathbb{E} V_n(z', s', n, b') \\ &\quad - \mu \chi \mathbb{E} [F_n(z', s', n) - w_n(z, s, n, b)n - w(z, s, n, b)] \\ &= zsf_n(n) - w(z, s, n, b) - w_n(z, s, n, b)n - \mu \chi \mathbb{E} [F_n(z', s', n) - w_n(z, s, n, b)n - w(z, s, n, b)] \\ &\quad + \beta \int_s \int_z \int_{\bar{n}(z, s, n_{-1}, b)}^{\infty} \frac{c_v}{q(z)} dH^a(z'|z) dH^i(s'|s) - \beta \int_s \int_z \int_0^{\underline{n}(z, s, n_{-1}, b)} \kappa dH^a(z'|z) dH^i(s'|s) \\ &\quad + \beta \int_s \int_z \int_{\bar{n}(z, s, n_{-1}, b)}^{\bar{n}(z, s, n_{-1}, b)} J(z', s', n, b') dH^a(z'|z) dH^i(s'|s), \end{aligned}$$

where $H^a(z'|z)$ and $H^i(s'|s)$ are the conditional cumulative distribution functions of idiosyncratic productivity and aggregate shocks, respectively. In addition, we can write the value of unemployment

$$V^u(z) = b + \beta \mathbb{E} \left[(1 - p(z)) V^u(z') + p(z) \int_z \int_s \int_b \int_{\underline{n}(z, s, n_{-1}, b)}^{\infty} \frac{V^e(z', s', n, b')}{1 - G(\bar{s}(z, n_{-1}, b))} d\Omega(z, s, n_{-1}, b) \right].$$

Once unemployed worker find a job, the new job must be in a firm which is posting vacancies. And $\Omega(z, s, n_{-1}, b)$ is the firm size distribution. Given the Nash bargaining protocol in an expanding firm, we have

$$V^e(z, s, n, b) - V^u(z) = \frac{\eta}{1-\eta} [J(z, s, n, b) + \kappa] = \frac{\eta}{1-\eta} \left[\frac{c_v}{q(z)} + \kappa \right]. \quad (1.9)$$

we have

$$V^u(z) = b + \beta V^u(z') + \beta p(z) \frac{\eta}{1-\eta} \frac{c_v}{q(z)}.$$

The value of employment can be written as

$$\begin{aligned} & V^e(z, s, n, b) \\ &= w(z, s, n, b) + \beta \mathbb{E} [\lambda(z', s', n, b) V^u(z') + (1 - \lambda(z', s', n, b)) V^e(z', s', n', b')] \\ &= w(z, n, s, b) \\ &+ \beta \int_z \int_0^{\bar{s}(z, n_{-1}, b)} [\lambda V^u(z') + (1 - \lambda) V^e(z', s', n', b')] dH^a(z'|z) dH^i(s'|s) \\ &+ \beta \int_z \int_{\underline{s}(z, n_{-1}, b)}^{\bar{s}(z, n_{-1}, b)} V^e(z', s', n', b') dH^a(z'|z) dH^i(s'|s) \\ &+ \beta \int_z \int_{\bar{s}(z, n_{-1}, b)}^{\infty} V^e(z', s', n', b') dH^a(z'|z) dH^i(s'|s), \end{aligned}$$

where

$$\lambda = 1 - \frac{n(z, s, n_{-1}, b)}{n_{-1}}.$$

If the worker is fired, he transits into unemployment and receives a payoff $V^u(z')$; otherwise, worker continues to be employed in the firm. By firm order condition, in the firm firing workers, we have

$$J(z, s, n, b) + \kappa = 0.$$

Thus, we have

$$\begin{aligned}
& V^e(z, s, n, b) \\
= & w(z, s, n, b) + \beta V^u(z') \\
& + \beta \frac{\eta}{1-\eta} \int_z \int_{\bar{s}(z, n-1, b)}^{\infty} \frac{c_v}{q(z)} dH^a(z'|z) dH^i(s'|s) \\
& + \beta \frac{\eta}{1-\eta} \int_z \int_{\underline{s}(z, n-1, b)}^{\bar{s}(z, n-1, b)} J(z', s', n', b') dH^a(z'|z) dH^i(s'|s) \\
& + \beta \frac{\eta}{1-\eta} \int_z \int_0^{\underline{s}(z, n-1, b)} J(z', s', n', b') dH^a(z'|z) dH^i(s'|s)
\end{aligned}$$

This gives

$$\begin{aligned}
& V^e(z, s, n, b) - V^u(z) \\
= & w(z, s, n, b) - u \\
& + \beta \frac{\eta}{1-\eta} \int_z \int_{\bar{s}(z, n-1, b)}^{\infty} \left[\frac{c_v}{q(z)} + \kappa \right] dH^a(z'|z) dH^i(s'|s) \\
& + \beta \frac{\eta}{1-\eta} \int_z \int_{\underline{s}(z, n-1, b)}^{\bar{s}(z, n-1, b)} [J(z', s', n', b') + \kappa] dH^a(z'|z) dH^i(s'|s) \\
& + \beta \frac{\eta}{1-\eta} \int_z \int_0^{\underline{s}(z, n-1, b)} [J(n, s', b') + \kappa] dH^a(z'|z) dH^i(s'|s) \\
& - \beta p(z) \frac{\eta}{1-\eta} \left(\frac{c_v}{q(z)} + \kappa \right) \\
= & w(n, s, b) - u - \beta p \frac{\eta}{1-\eta} \left(\frac{c_v}{q(z)} + \kappa \right) \\
& + \beta \frac{\eta}{1-\eta} \int_z \int_{\bar{s}(z, n-1, b)}^{\infty} \left[\frac{c_v}{q(z)} + \kappa \right] dH^a(z'|z) dH^i(s'|s) \\
& + \beta \frac{\eta}{1-\eta} \int_z \int_{\underline{s}(z, n-1, b)}^{\bar{s}(z, n-1, b)} [J(z', s', n', b') + \kappa] dH^a(z'|z) dH^i(s'|s) \\
= & \frac{\eta}{1-\eta} [J(z, s, n, b) + \kappa] \tag{1.10}
\end{aligned}$$

Therefore, from Equation 1.9 and Equation 1.10, we have

$$\begin{aligned}
& w(z, s, n, b) - u - \beta p \frac{\eta}{1 - \eta} \left(\frac{c_v}{q(z)} + \kappa \right) \\
&= \frac{\eta}{1 - \eta} \left\{ \begin{array}{l} z s f_n(n) - w(z, s, n, b) - w_n(z, s, n, b) n \\ - \mu \chi [\mathbb{E}[z s] f_n(n) - w(n, s, b) - w_n(n, s, b) n] + \kappa(1 - \beta) \end{array} \right\} \\
&= \frac{\eta}{1 - \eta} \{ (z s - \mu \chi \mathbb{E}[z s]) f_n(n) - (w(z, s, n, b) - w(z, s, n, b) n) (1 - \mu \chi) + \kappa(1 - \beta) \},
\end{aligned}$$

Rearrange the equation, wage is determined by

$$\begin{aligned}
& w(z, s, n, b) \\
&= \frac{\eta}{1 - \eta \mu \chi} \left[(z s - \mu \chi \mathbb{E}[z s]) f_n(n) - (1 - \mu \chi) w_n(z, s, n, b) n + \beta p(z) \frac{c_v}{q(z)} + \kappa(1 - \beta(1 - p(z))) \right] \\
&+ \frac{1 - \eta}{1 - \eta \mu \chi} u.
\end{aligned}$$

■

Proof of Proposition 2

Proof. (Guess and verify) Assume

$$w(z, s, n, b) = A + B \alpha n^{\alpha-1}.$$

We have

$$\begin{aligned}
& A + B \alpha n^{\alpha-1} \\
&= \frac{\eta}{1 - \eta \mu \chi} \left[(z s - \mu \chi \mathbb{E}[z s]) \alpha n^{\alpha-1} - (1 - \mu \chi) B \alpha (\alpha - 1) n^{\alpha-1} + \beta p(z) \frac{c_v}{q(z)} + \kappa(1 - \beta(1 - p(z))) \right] \\
&+ \frac{1 - \eta}{1 - \eta \mu \chi} u.
\end{aligned}$$

Therefore

$$B = \frac{\eta}{1 - \eta\mu\chi} [(zs - \mu\chi\mathbb{E}[zs]) - (1 - \mu\chi)B(\alpha - 1)],$$

$$A = \frac{\eta}{1 - \eta\mu\chi} \left[\beta p(z) \frac{c_v}{q(z)} + \kappa(1 - \beta(1 - p(z))) \right] + \frac{1 - \eta}{1 - \eta\mu\chi} u.$$

Solve the above two equations, I get

$$A = \frac{\eta}{1 - \eta\mu\chi} \left\{ \beta p \frac{c_v}{q(z)} + \kappa[1 - \beta(1 - p(z))] \right\} + \frac{1 - \eta}{1 - \eta\mu\chi} u,$$

$$B = \frac{\eta}{1 - \eta(1 - \alpha) - \eta\mu\chi\alpha} [zs - \mu\chi\mathbb{E}(zs)].$$

■

Proof of Proposition 5

Assume that $\mathbb{E}(sz) = s\rho_s\mathbb{E}(z)$, I have

$$(1 - \mu\chi\rho_s)z\bar{s}n_{-1}^\alpha - (1 - \mu\chi) [w(z, \underline{s}, n_{-1}, b) + w_n(z, \bar{s}, n_{-1}, b)n_{-1}]$$

$$- \frac{c_v}{q(z)} + \beta\mathbb{E}\max\{V_n^i(z', s', n_{-1}, b'), 0\} = 0$$

$$(1 - \mu\chi\rho_s)z\underline{s}n_{-1}^\alpha - (1 - \mu\chi) [w(z, \underline{s}, n_{-1}, b) + w_n(z, s, n_{-1}, b)n_{-1}] + \kappa$$

$$+ \beta\mathbb{E}\max\{V_n^i(z', s', n_{-1}, b'), 0\} = 0$$

Assume

$$\mathbb{E}V_n^i(z', s', n_{-1}, b') \sim s\rho_s\mathbb{E}(z', s, n_{-1}, b')$$

and ρ_s is close to 1, I have

$$\begin{aligned}
\frac{\bar{s}(z, n_{-1}, b)}{\underline{s}(z, n_{-1}, b)} &\simeq \frac{(1 - \mu\chi) [w(z, \bar{s}, n_{-1}, b) + w_n(z, \bar{s}, n_{-1}, b) n_{-1}] + \frac{c_v}{q(z)}}{(1 - \mu\chi) [w(z, \underline{s}, n_{-1}, b) + w_n(z, \underline{s}, n_{-1}, b) n_{-1}] - \kappa} \\
&= 1 + \frac{\frac{c_v}{q(z)} + \kappa}{(1 - \mu\chi) [w(z, \underline{s}, n_{-1}, b) + w_n(z, \underline{s}, n_{-1}, b) n_{-1}] - \kappa} \\
&= 1 + \frac{\frac{c_v}{q(z)} + \kappa}{(1 - \mu\chi) [A + B\alpha^2 n_{-1}^{\alpha-1}] - \kappa}.
\end{aligned}$$

Take partial derivatives, it is straightforward to show that

$$\frac{\partial \frac{\bar{s}(z, n_{-1}, b)}{\underline{s}(z, n_{-1}, b)}}{\partial n_{-1}} < 0.$$

Next, I have

$$\begin{aligned}
&(1 - \mu\chi) [A + B\alpha^2 n_{-1}^{\alpha-1}] \\
&= \frac{1 - \mu\chi}{1 - \eta\mu\chi} \left\{ \eta \left\{ \beta p \frac{c_v}{q(z)} + \kappa [1 - \beta (1 - p(z))] \right\} + (1 - \eta) u \right\} \\
&\quad + \frac{1 - \mu\chi}{1 - \eta(1 - \alpha) - \eta\mu\chi\alpha} \eta [zs - \mu\chi \mathbb{E}(zs)] \alpha^2 n_{-1}^{\alpha-1},
\end{aligned}$$

which is an increasing function of $\mu\chi$. Therefore

$$\frac{\partial \frac{\bar{s}(z, n_{-1}, b)}{\underline{s}(z, n_{-1}, b)}}{\partial \chi} > 0.$$

Computation of the Stationary Distribution

This section presents the computational details of firm dynamics in the model with aggregate shocks. I omit the notation on z since it is constant here. The computational procedure is based on value function iteration and simulations.

1. Guess initial labor market tightness $\theta = \frac{U}{V}$. This gives us job finding rate, p , and job filling rate, q .
2. Set grids on n, b and μ . Discretize the AR(1) process of idiosyncratic shocks with ten states

Markovian process.

3. Solve the value functions and optimal policy rules.

- (a) Set any initial value for $V_0^i(s, n, b)$ and $V_0^e(s, n, b)$
- (b) Compute the bargained wage conditional on μ

$$w(s, n, b; \mu) = A + B\alpha n^{\alpha-1},$$

where

$$A = \frac{\eta}{1 - \eta\mu\chi} \left[\beta p \frac{c_v}{q} + \kappa(1 - \beta(1 - p)) \right] + \frac{1 - \eta}{1 - \eta\mu\chi} u,$$

$$B = \frac{\eta}{1 - \eta(1 - \alpha) - \eta\mu\chi\alpha} (s - \mu\chi\mathbb{E}[s]).$$

- (c) Calculate $V^i(s, n, b; \mu)$ by

$$V^i(s, n-1, b; \mu) = \max_{d, b', n} d + \beta \mathbb{E} \max \{ V^i(s', n, b'), 0 \} - \mu \{ \chi [\mathbb{E}F(s', n) - w(s, n, b; \mu)n] - b' \},$$

where

$$b' < \chi [\mathbb{E}F(s', n) - w(s, n, b; \mu)n] \text{ if } \mu = 0,$$

$$b' = \chi [\mathbb{E}F(s', n) - w(s, n, b; \mu)n] \text{ if } \mu > 0.$$

Therefore, I update $V^i(s, n, b)$ with

$$V^i(s, n, b) = \max_{\mu} V^i(s, n-1, b; \mu)$$

Then obtain the optimal policy functions $g_n^i(s, n, b)$ and $g_b^i(s, n, b)$.

(d) Solve the exit decisions $v^e(s, k, b)$

$$v^e(z, s, n_{-1}, b) = \begin{cases} 0 & \text{if } s \geq \inf \{s \in S : V^i(z, s, n_{-1}, b) \geq 0\} \\ 1 & \text{o.w.} \end{cases},$$

and default decisions, $v^d(s, n_{-1}, b)$

$$v^d(s, n_{-1}, b) = \begin{cases} 0 & \text{if } \omega \geq \omega^*(s, n_{-1}, b) \\ 1 & \text{o.w.} \end{cases},$$

where

$$\omega^*(s, n_{-1}, b) = \sup \{s \in S : F(s, n_{-1}) - w(s_{-1}, n_{-1}, b_{-1})n_{-1} - c_f \leq b\}.$$

(e) Update until

$$\|V^s - V_0^s\| \leq \varepsilon$$

and let $V^{s*} = V^s$.

4. Similarly, solve the entrants' Bellman equation

$$V^e(s) = \max_{d, b', n} d + \beta \mathbb{E} [\max \{V^i(s', n, b'), 0\}]$$

and get the optimal policy functions $g_n^e(s)$ and $g_b^e(s)$. The entry cost c_e is set such that only the entrants above the mean productivity can entry the market,

$$s^* = \text{median}(s), \quad c_e = V^e(s^*).$$

5. Simulate the stationary distribution of firms with 20,000 firms and 3,000 periods. Initially, all firms are new entrants. Generate the markovian chain for each firm. If the firm defaults or exits the market, it is replaced by a new entrant with a new sequence of productivity shocks.

6. Drop the first 10% simulations and calculate the invariant distributions. Compute the total

labor force

$$L = \frac{\int n(s, n, b) \Omega(ds, dn, db)}{1 - U}.$$

where U is unemployment rate and is fixed at the targeted level. The number of separations, S , equals to number of hires, M ,

$$\begin{aligned} S &= \frac{1}{L} \left\{ \int_{s^*(b, n_{-1})}^{s(n_{-1}, b)} [n_{-1} - n(s, b)] d\Omega(s, n, b) + \int_{s < s^*(b, n_{-1})} [n_{-1}(s, n, b)] dH^i(s'|s) \right\} \\ M &= \frac{1}{L} \left\{ \int_{s \geq \bar{s}(n_{-1}, b)} [n(s, b) - n_{-1}] d\Omega(s, n, b) + \int_{s \geq s_e^*} [n(s)] dH^i(s'|s) \right\} \end{aligned}$$

7. Update the labor market tightness

$$\theta' = \frac{U}{V} = q \frac{U}{M}.$$

Stop until

$$\|\theta' - \theta\| \leq \varepsilon.$$

Computation of the Model with Aggregate Shocks

The section presents the computational details of firm dynamics in the model with aggregate shocks. The computational procedure is based on value function iteration and simulations.

1. Set grids on n , b , s and z . Discretize the AR(1) process of idiosyncratic and aggregate shocks with ten-states Markovian process.
2. Guess market tightness θ a function of z and θ_{-1} .

$$\log(\theta') = a_0 + a_1 \log(\theta) + a_2 \log(z').$$

3. Solve the value functions and optimal policy rules. This is similar to calculate stationary distribution, but with one more state variable.
4. Simulate the model with 20,000 firms and 3,000 periods. Initially, all firms are new entrants.

Generate the aggregate shocks and markovian chain for each firm.

5. Drop the first 10% simulations and calculate the invariant distributions. Compute the total labor demand

$$L(z) = \int n(z, s, n, b) \Omega(dz, ds, dn, db).$$

The number of separations, S , and the new of new hires, M ,

$$\begin{aligned} S(z) &= \frac{1}{L} \int_{s^*(z, b, n_{-1})}^{s(z, n_{-1}, b)} [n_{-1} - n(z, s, b)] d\Omega(z, s, n, b) \\ &\quad + \frac{1}{L} \int_{s < s^*(z, b, n_{-1})} [n_{-1}(z, s, n, b)] dH^i(s'|s) \\ M(z) &= \frac{1}{L} \int_{s \geq \bar{s}(z, n_{-1}, b)} [n(z, s, b) - n_{-1}] d\Omega(z, s, n, b) \\ &\quad + \frac{1}{L} \int_{s \geq s_e^*(z)} [n(z, s)] dH^i(s'|s) \end{aligned}$$

We get dynamics of unemployment

$$U(z') = \frac{U(z) + S(z) - M(z)}{L},$$

and market tightness

$$\theta(z) = \frac{U(z)}{V(z)} = q(z) \frac{U(z)}{M(z)} = q(z) \frac{U(z)}{M(z)}$$

6. Run regressions on aggregate law of motion,

$$\log(\theta') = a_0 + a_1 \log(\theta) + a_2 \log(z') + \varepsilon,$$

7. update a_0 , a_1 , and a_2 until converge; otherwise go back to step 2.

Parameter	Symbol	Value	Source or Target
Discount factor	β	0.985	Annual risk-free rate
Curvature of production function	α	0.60	Elsby and Michaels (2013)
Liquidation value	ζ	0.70	Gourio (2013)
Matching elasticity	ϕ	0.5	Petrongolo and Pissarides (2001)
Matching efficiency	ξ	0.129	Pissarides (2007)
Worker's bargaining power	η	0.443	Elsby and Michaels (2013)
Value of unemployment	u	0.387	Elsby and Michaels (2013)
Aggregate productivity shock persistence	ρ_z	0.9	Standard in the literature
Aggregate productivity shock volatility	σ_z	0.09	Standard in the literature
Idiosyncratic productivity shock persistence	ρ_s	0.97	Lee and Mukoyama (2012)
Idiosyncratic productivity shock volatility	σ_s	0.1045	Variance of employment growth
Tightness of borrowing constraint	χ	1.5	Leverage ratio of 0.81
Vacancy posting cost	v_c	0.13	Resource cost associated with hiring
Firing cost	κ	0.16	Persistent of employment
Fixed production cost	c_f	1.7	Quarterly exit rate
Entry cost	c_e	2.8	Entry cutoff
Measure of potential entrants	N	0.2	Measure of all firms

Table 1.3: Parameterizations

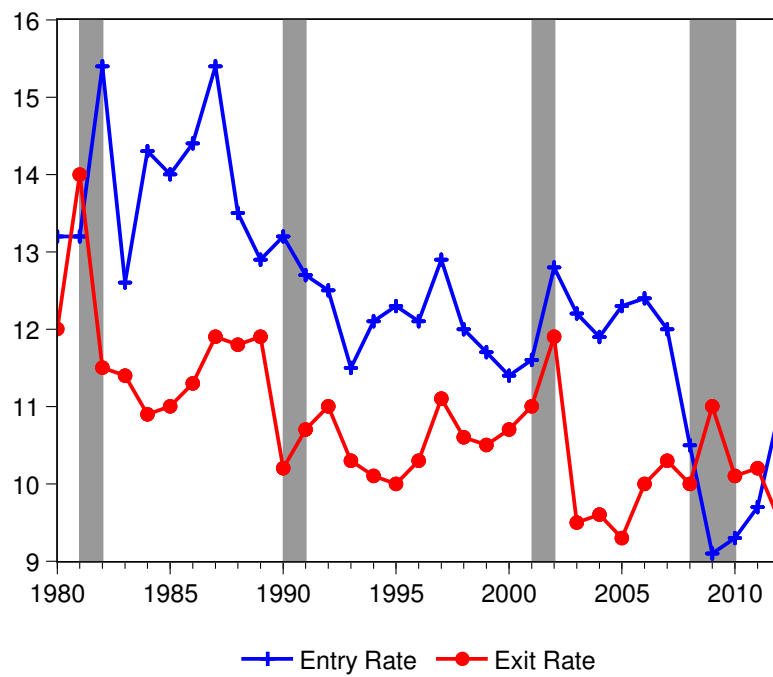
	Data	Model $\chi = 1.5$	Model $\chi = 0$
Corporate default rate	0.4%	0.5%	0
Total Exit rate	5.4%	5.4%	4.6%
Leverage ratio	0.81	0.85	0
Persistence of employment process	0.97	0.97	0.96
Variance of employment growth	0.14	0.15	0.18
Job reallocation rate	19.4%	18.22%	14.6%

Table 1.4: Data and model statistics in the stationary equilibrium

	Bad	Good
Corporate default rate	0.58%	0.35%
Total exit rate	5.85%	5.20%
Entry rate	4.42%	6.68%

Table 1.5: Entry and exit with aggregate shocks

Figure 1.2: Entry and exit rate



Source: Own calculations. Business Dynamics Statistics (BDS) 1980-2012. The gray bars are NBER-dated recession periods.

Table 1.6: Decomposition of job reallocation variance between groups

	Data	Model
Extensive margin	41%	36%
Intensive margin	59%	64%
Relative volatility of job reallocation	2.94	3.03

Notes: Series are HP filtered with parameter $\lambda = 6.25$. 1980-2012 Source: Own calculations. Business Dynamics Statistics (BDS).

Table 1.7: Correlation of job reallocation to output

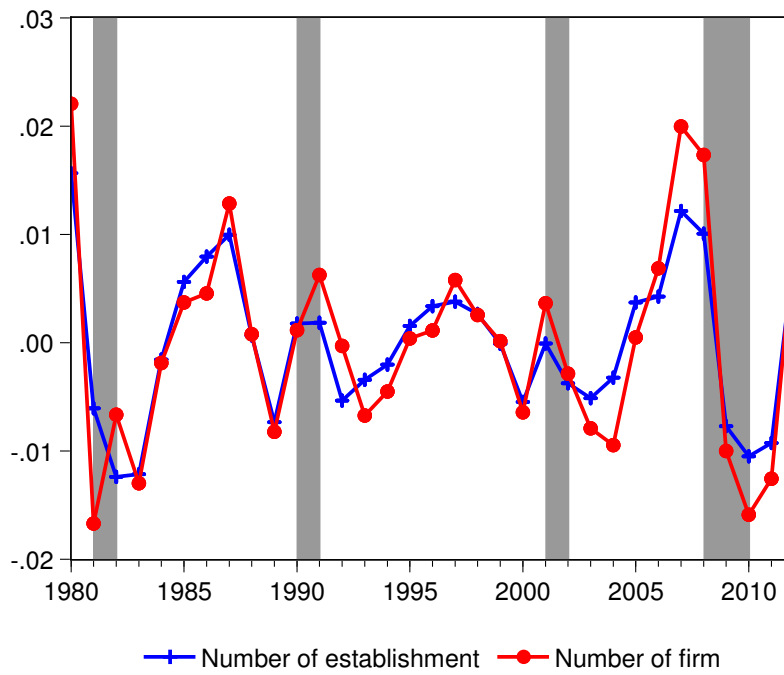
	Data	Model
Job reallocation	-0.18	-0.22
Job creation	0.50	0.40
Job destruction	-0.53	-0.42

Notes: Series are logged and HP filtered with parameter $\lambda = 6.25$. 1980-2012 Source: Own calculations. Business Dynamics Statistics (BDS).

Table 1.8: Job reallocation with different borrowing capacity

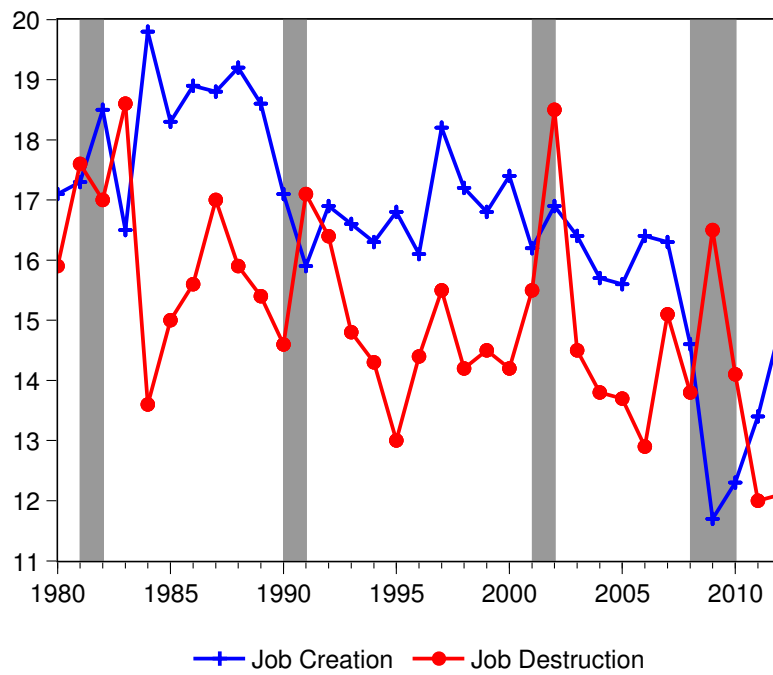
	$\chi = 5$	$\chi = 1.5$	$\chi = 0.5$	$\chi = 0$	Data
Extensive margin	0.44	0.36	0.26	0.20	0.41
Intensive margin	0.56	0.64	0.74	0.80	0.59
Relative volatility of job reallocation	1.66	3.03	3.25	4.08	2.94

Figure 1.3: Cyclical component of number of firms



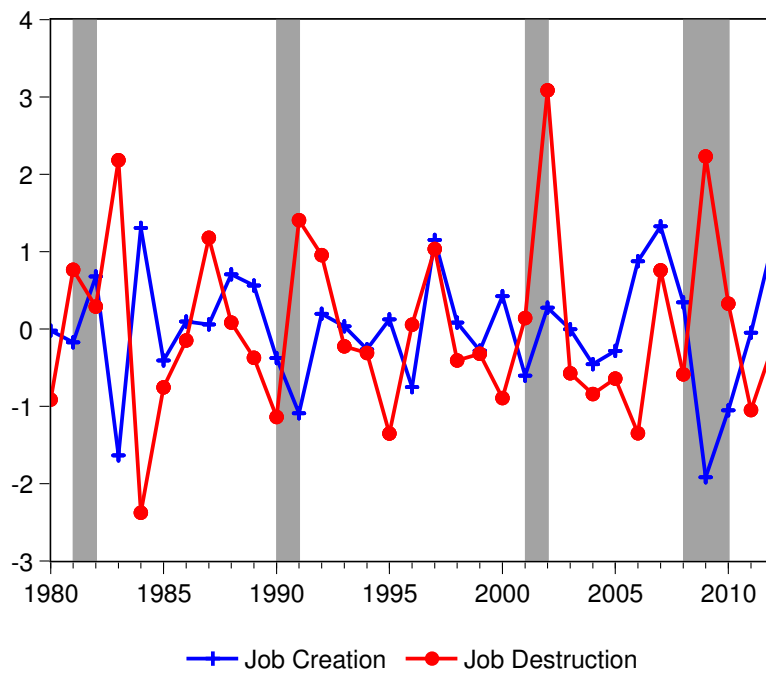
Notes: Series are logged and HP filtered with parameter $\lambda = 6.25$. 1980-2012 Source: Own calculations. Business Dynamics Statistics (BDS). The gray bars are NBER-dated recession periods.

Figure 1.4: Job creation and job destruction



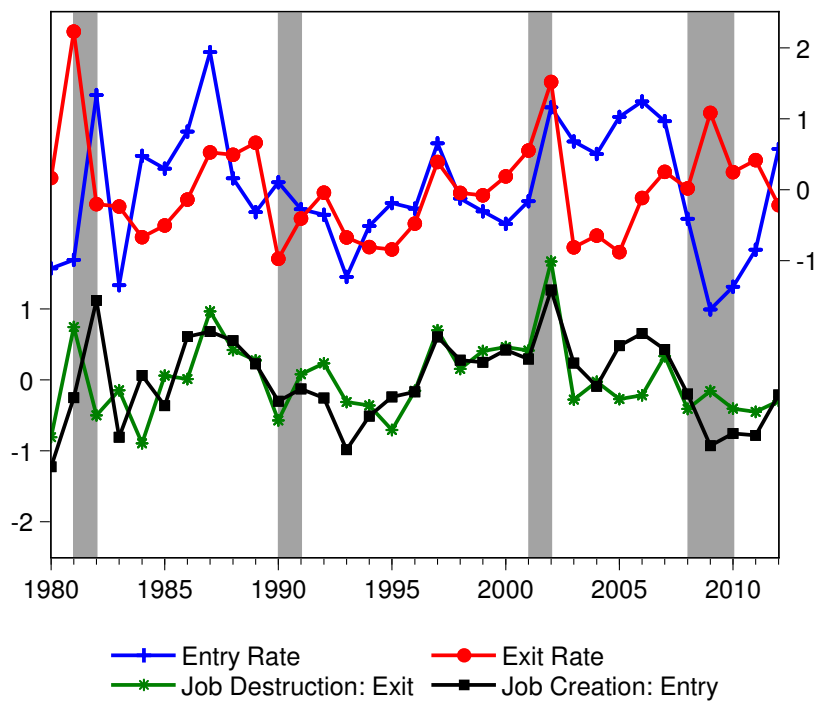
Notes: Series are logged and HP filtered with parameter $\lambda = 6.25$. 1980-2012 Source: Own calculations. Business Dynamics Statistics (BDS). The gray bars are NBER-dated recession periods.

Figure 1.5: Cyclical component of job creation and job destruction



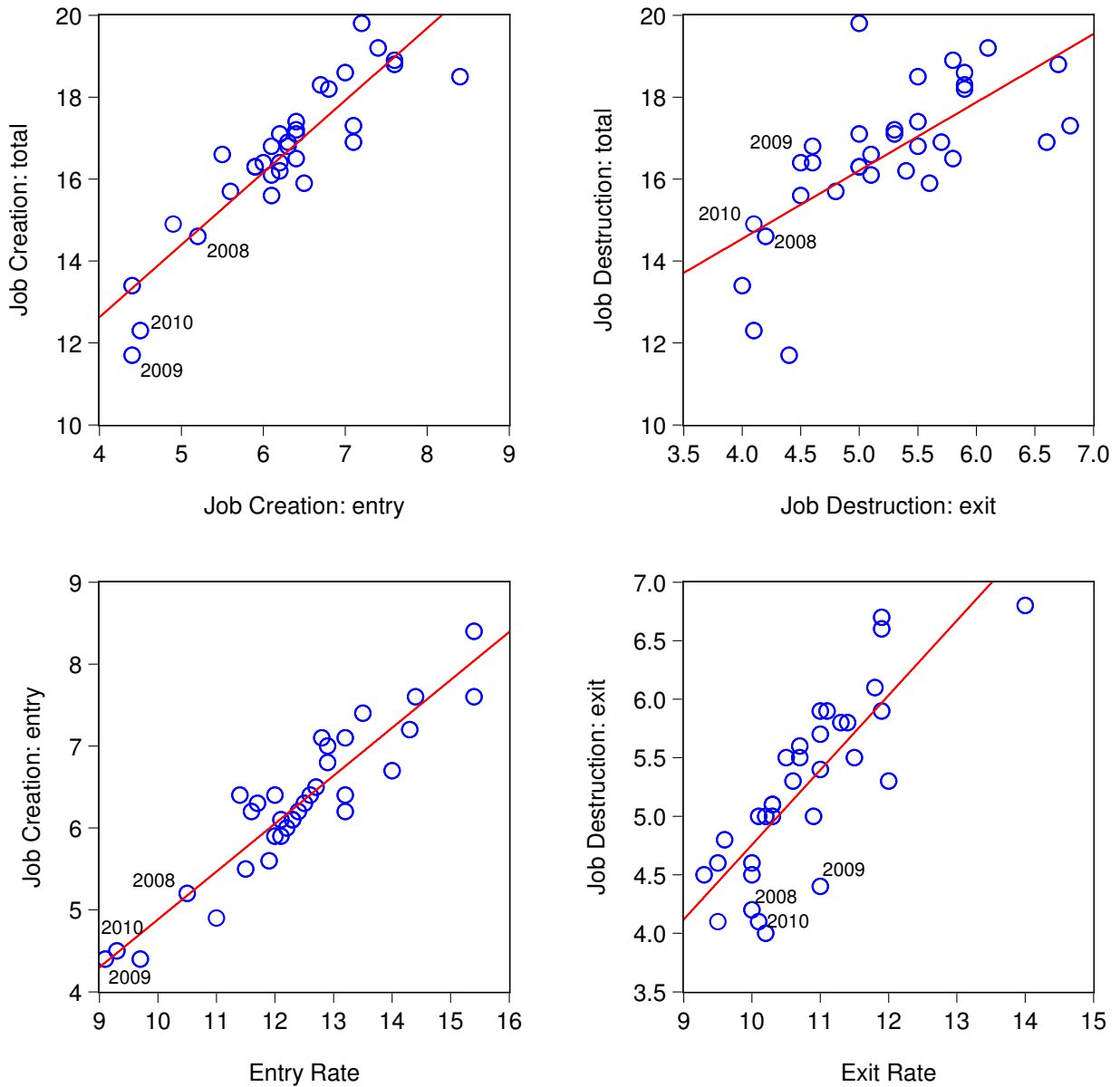
Notes: Series are logged and HP filtered with parameter $\lambda = 6.25$. 1980-2012 Source: Own calculations. Business Dynamics Statistics (BDS). The gray bars are NBER-dated recession periods.

Figure 1.6: Cyclical component of job reallocation: the extensive margin, entry and exit rates



Source: Own calculations. Business Dynamics Statistics (BDS) 1980-2012. The gray bars are NBER-dated recession periods.

Figure 1.7: Job reallocation, entry and exit rate



Source: Own calculations. Business Dynamics Statistics (BDS) 1980-2012. The gray bars are NBER-dated recession periods. The red lines are fitted value from OLS regressions.

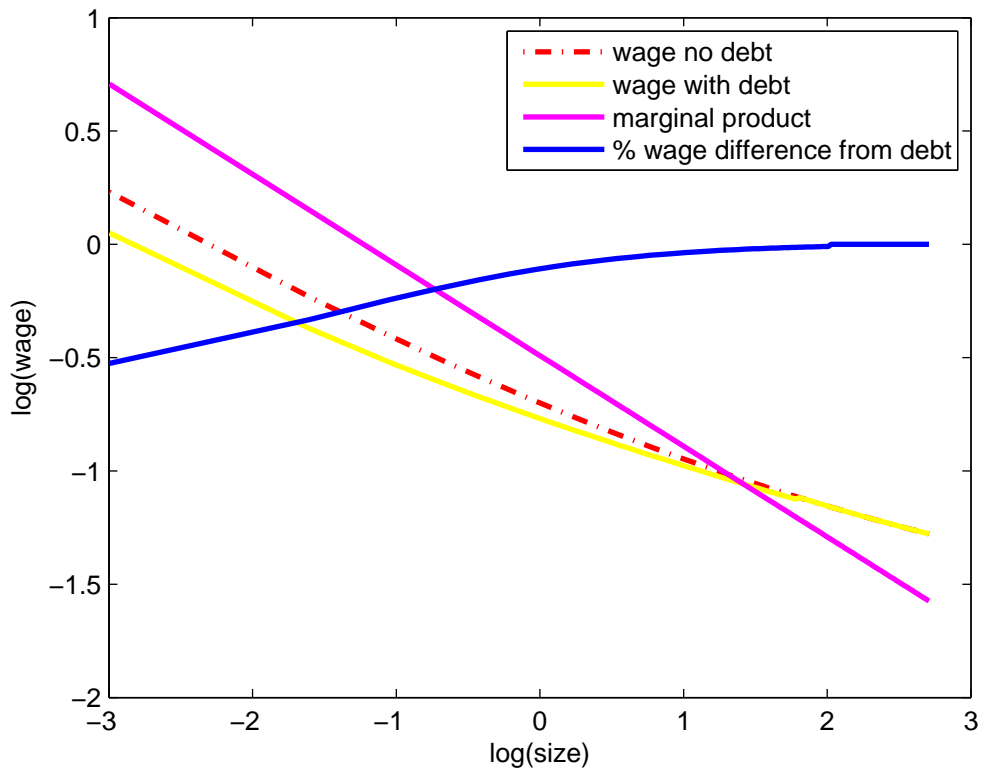


Figure 1.8: Bargained wage conditional on firm size and debt

Figure 1.9: Financial leverage and labor share conditional on firm size and productivity

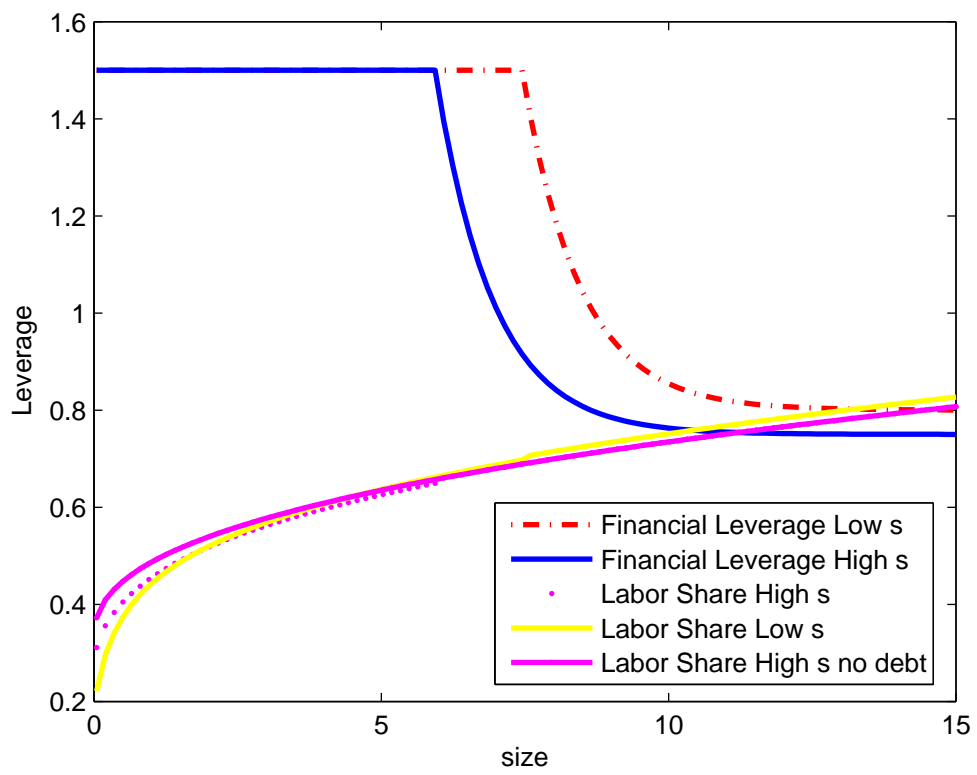


Figure 1.10: Firm size and age distributions

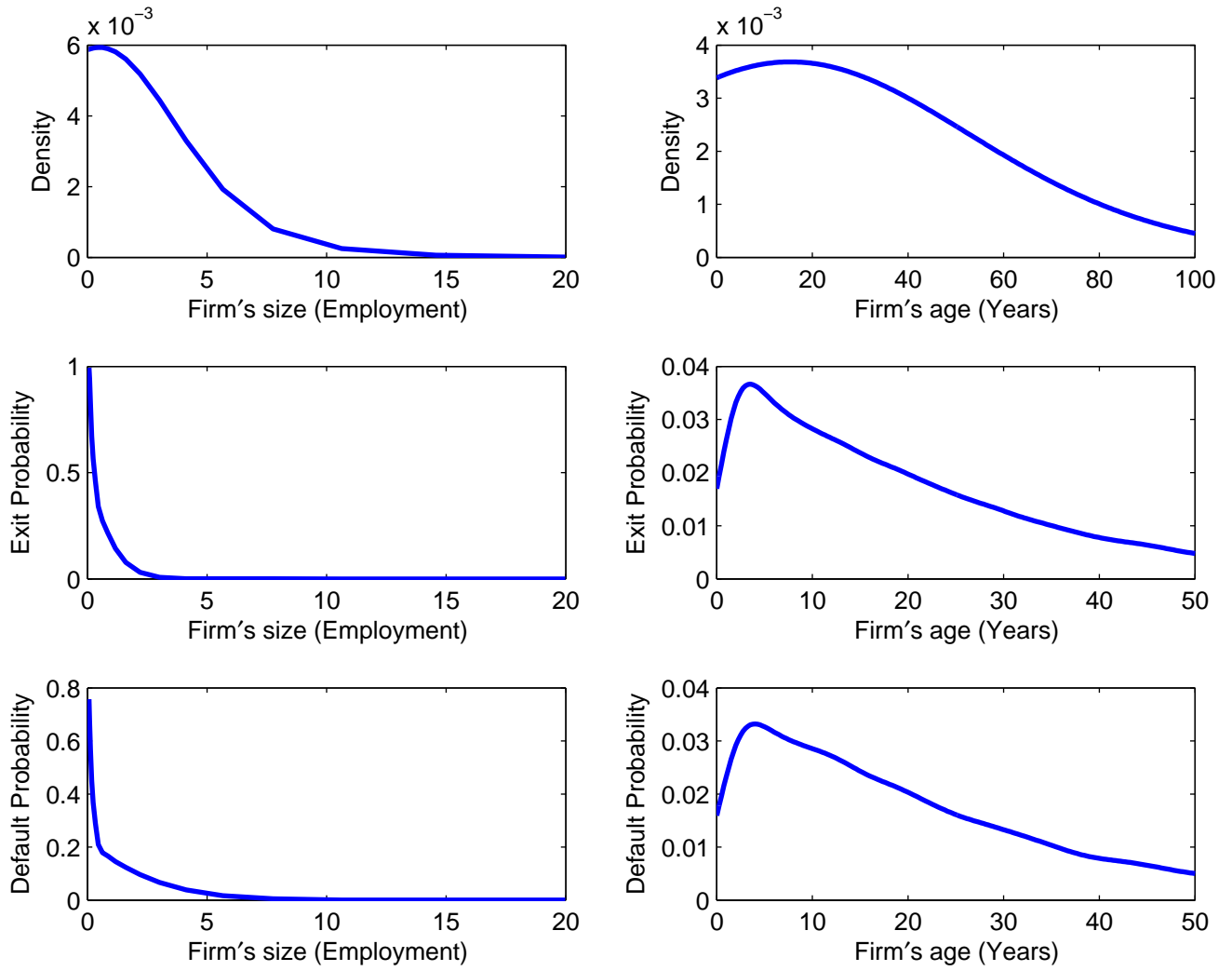
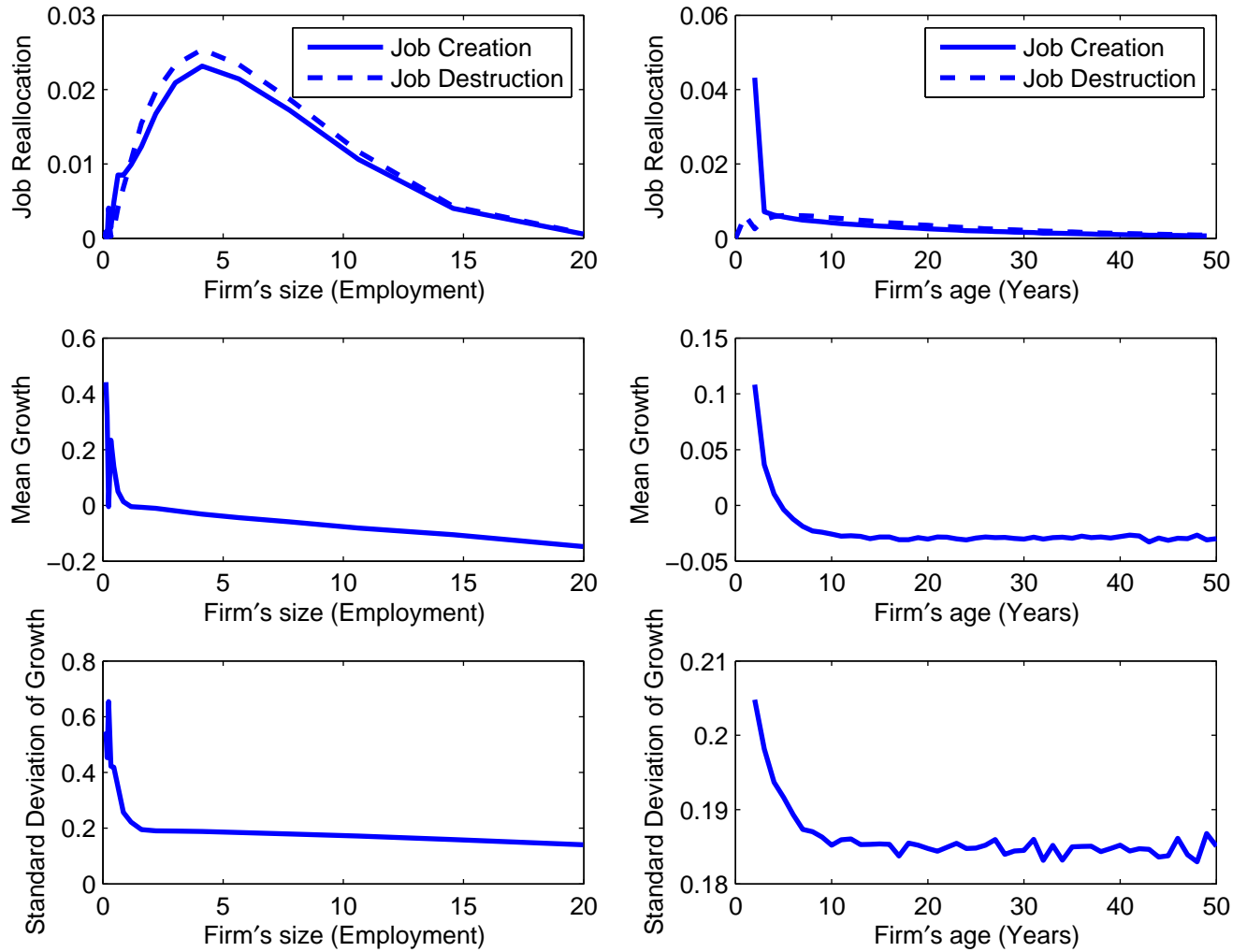


Figure 1.11: Job reallocation and firm growth distribution



CHAPTER 2

Entry, Exit, and Capital Structure over the Business Cycle

2.1 Introduction

The aim of this paper is to investigate dynamic capital structures and firms' entry and exit behaviors over the business cycle using a general equilibrium model. Two broad questions are how a firm's financial decisions and entry and exit decisions are correlated and how capital structure and financial friction affect firms' size distributions and the aggregate price dynamics.

Most efforts by macro-finance economists have focused on explaining the cyclical behaviors of asset prices and their comovements with macroeconomic aggregates using a general equilibrium model, where firms are *ex ante* identical and only differ *ex post* in the realization of shocks and where equilibrium can be characterized by a single representative agent. By contrast, most IO economists model the dynamic behaviors of firms' entry and exit decisions in a complete market, generally ignoring the agency problem in firms' financing decisions. They assume that the market is complete because we live in the Modigliani and Miller world, where capital structure does not matter.

We believe that each of these studies' parallel but separate approaches offers an incomplete picture of the importance of the relationship between firms' capital structure and their entry and exit decisions over the business cycle in macroeconomic models. In this paper, we deviate slightly from these research strategies. Here, we focus on integrating some important ideas from each of these approaches in a standard general equilibrium industry dynamics model of firms' financial structures of debt and equity. Firms are heterogeneous and face both idiosyncratic shocks and aggregate shocks. Because of endogenous entry and exit, firms can no longer generally be replaced by a representative firm, and the distribution of firm sizes plays a role in aggregate price dynamics.

Essentially, we are interested in explaining two stylized facts using a unified model: 1) the entry rate is procyclical, while the exit rates is fairly stable (Lee and Mukoyama, 2012); 2) debt issuance is procyclical, while the equity issuance of most firms is countercyclical (Covas and Den Haan, 2011a).

In our model, a firm receives an aggregate shock and an idiosyncratic shock every period. The production function exhibits decreasing returns to scale with capital and labor inputs. The firm pays a fixed production cost each period. If the continuation value, which is conditional on the realization of the aggregate and idiosyncratic shocks, is negative, the firm exits the market. If it continues, the firm receives an i.i.d. capital quality shock, which affects its capital stock¹. The capital quality shock is received before the actual production occurs. The firm defaults and exits the market if it cannot fully repay its debt after the realization of the capital shock, which determines the firm's profitability. Therefore, we can distinguish between the exit decisions and default decisions of each firm.

The firm raises funds via debt and equity. The financial market is not frictionless, and as a result, the firm size and age distributions depend on debt and equity. Equity and debt are not perfect substitutes. There is a flotation cost of issuing new shares and a default cost of debt. Default is costly, and no renegotiation is allowed. The firm refinances its debt and equity as productivity varies. However, the refinancing cost endogenously changes over the business cycle. In a recession, the firm is less profitable, and its default probability increases such that the price of debt declines. The firm must compare the marginal cost of issuing new shares with the marginal default cost of taking on more debt. A new firm enters the market with an initial equity and debt that evolve over time as the firm refinances its capital structure. Small and young firms encounter higher costs of external financing.

The debt contract is a standard one-period contract with agency problems. The firm borrows money from households and repays the borrowed funds plus interest in the next period. There is a tax advantage of debt. For each dollar of debt raised, the firm receives tax benefits from the government. The firm defaults and exits the market if it cannot fully repay its debt after the

¹We may also interpret this as the quality of capital (Bigio, 2012). However, we do not encounter adverse selection problems because it is unknown to both the producer (borrower) and the consumer (lender).

realization of the shock. Debt is preferred because of its associated tax advantage, but the firm cannot take on too much debt because default is costly. Once default occurs, the household only recovers a fraction of the firm's assets (liquidation value). Small and young firms take on more debt because default is less costly to them. The firm faces a higher default probability if it is relatively small or young and during a recession. Thus, the price of debt is lower for small and young firms and during recessions. Those firms have higher external financing costs.

The cyclical behaviors of entry and exit can be explained by the average size of the entrants and procyclical wages. Procyclical wages suggest that the employment of incumbents increases less than productivity shock during a boom. However, the labor supply increases because of higher wages. The countercyclical external finance cost makes entry more difficult during a recession than during a boom. The average size, measured in terms of employment, of the entrants is larger during a recession than during a boom in both the model and the data. During booms, first, a larger gap exists between the labor supply and the labor demand from incumbents. Second, the entrants are relatively small in size. These two effects imply that more entries are required during booms to fill the gap between the labor supply and the labor demand from incumbents. Because the wage effects are somehow offset by the aggregate productivity and wages are less volatile, the second effect is stronger than the first in our model. Additionally, financial market frictions generate the implicit countercyclical nature of external financing costs. The flat exit rates over the business cycle can be explained by age effects. The wage effects of incumbents are offset by aggregate productivity. Furthermore, because of age effects, older firms are larger and are more likely to exhibit their optimal production levels and capital structures; therefore, the countercyclical nature of external finance costs does not affect the entrants to a greater extent than the incumbents. These effects maintain the flat exit rates of the model. Without financial market frictions, it would be difficult to explain the procyclical nature of entry because wage effects are almost offset by the aggregate productivity effects, as in many IO models.

The cyclical behaviors of debt and equity issuance are explained as follows. Because default is more costly and the bond price is lower during a recession, the firm issues fewer bonds during recessions and more bonds during booms. Firms earn higher profits during booms, and therefore, they distribute more dividends during booms than during recessions. The procyclical nature of div-

idend payouts implies countercyclical equity issuance. By sorting firms into portfolios according to their sizes, our model describes the cyclical behaviors of debt and equity issuance of different firm groups. Debt is more procyclical for small firms, which also face higher default and exit probabilities. The marginal cost of an additional dollar raised by bonds is higher for these firms. Thus, the debt issuances of small firms are more procyclical than those of large firms. The cyclicity of equity issuance is driven by productivity. Small firms generally issue equity, while large firms pay out dividends. During a recession, for small firms, investments decrease, and as a result, they issue less equity. Large firms pay fewer dividends during a recession because of contracting profits. Therefore, equity issuance is procyclical for small firms and countercyclical for large firms.

The rest of the paper is organized as follows. The remainder of the introduction is a literature review. Section 2 presents the model and defines a recursive equilibrium. Section 3 describes the calibration and characterization of the stationary distribution of the model without aggregate shocks. Section 4 simulates the model with aggregate shocks and matches the cyclical behaviors of entry, exit and capital structure. Section 5 concludes.

Related Literature

This paper is relevant to several branches in the literature.

First, it builds on a vast body of literature concerning industry organization relating to firm dynamics, size, age and growth (Lucas and Prescott, 1971; Lucas, 1978; Hopenhayn, 1992a; Hopenhayn and Rogerson, 1993). In those models, there are no aggregate shocks and no financial market frictions. Researchers are interested in characterizing the stationary equilibrium. Cooley and Quadrini (2001), Albuquerque and Hopenhayn (2004) and Cabral and Mata (2003) characterize the relationships among firms' sizes, ages and financial borrowing constraints. Veracierto (2002) is the very first paper to consider both an aggregate shock and an idiosyncratic shock at the firm level. Lee and Mukoyama (2012) build a model with exogenous, time-varying entry costs to match the stylized procyclical entry rate, entry TFP and stable exit rate over the business cycle. They find strong evidence suggesting that the aggregate shock is important for firms' dynamic behaviors.

Second, this paper is highly relevant to the recent macro-finance literature concerning the relationship between a firm's financial decisions and aggregate dynamics. Jermann and Quadrini (2012) study the cyclicity of equity and debt over the business cycle using a model in which financial frictions affect firms' borrowing constraints. Eisfeldt and Muir (2012) focus on the cross-sectional correlation between external finance and liquidity accumulation. They find that firms tend to raise substantial external financing and accumulate liquidity. Covas and Den Haan (2011b) is similar to our paper, focusing on the procyclical nature of debt and equity issuance based on firms' sizes. Their model explains the size-dependent procyclical behavior of equity issuance observed for most listed firms.

Third, the paper is relevant to the macroeconomics literature concerning general equilibrium business cycle models with financial constraints. Bernanke and Gertler (1989) and Kiyotaki and Moore (1997) are the classical macroeconomics references on the subject of financial frictions. They focus on how a small shock could generate long-lasting, persistent effects. Recent developments based on their ideas include Kiyotaki and Moore (2008), Mendoza (2010), Brunnermeier and Sannikov (2014), and Jermann and Quadrini (2012).² In these studies, a significant role of the financial market frictions in understanding the dynamic macroeconomic features in the general equilibrium model is reported.

Fourth, the paper also draws from the finance literature on dynamic capital structure and macroeconomic risk (Hackbarth et al., 2006; Hennessy and Whited, 2007; Bhamra et al., 2010; Chen, 2010; Eisfeldt and Muir, 2012). These works include exogenous cash flow and no investment and are not solved using a general equilibrium model. Although they could measure the impacts of macroeconomic risk on asset pricing and capital structure, they could not capture the feedback between the financial frictions and firms' dynamics.

²Brunnermeier et al. (2012) and Quadrini (2011) present detailed surveys of this topic.

Timing

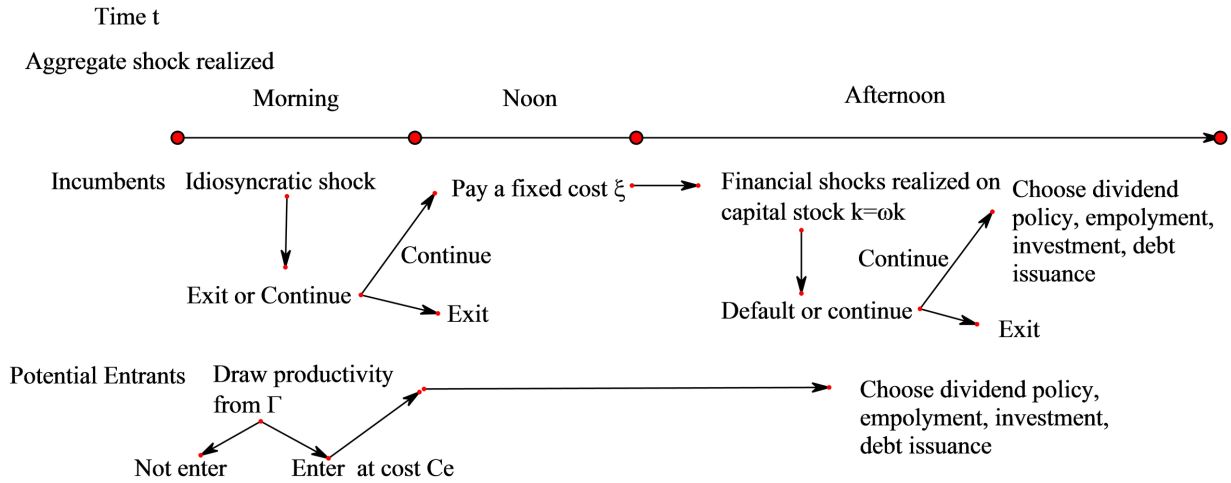


Figure 2.1: Timing of the model

2.2 Model

This section presents a baseline dynamic stochastic general equilibrium model including entry and exit. The model is based on Hopenhayn (1992a) and is similar to Cooley and Quadrini (2001) in some aspects. The model has both aggregate productivity shocks and idiosyncratic productivity shocks at the firm level. In addition to the productivity shocks, the firm faces an i.i.d. shock to its capital quality every period. The firm can raise funds via one-period debt and equity. The financial market is not perfect. There is a flotation cost of equity issuance and a default cost of debt.

2.2.1 Firms

Figure 2.1 summarizes the timeline of the model. At the beginning of period t , there is a continuum of incumbent firms. The aggregate state, z_t , is known. In the morning, each incumbent sequentially observes an idiosyncratic productivity shock, s_t , and a random shock, ω_t , to its capital stock. Without loss of generality, we set the timing of the model such that the firm makes an exit decision

before the realization of the capital quality shock.³ The capital stock, adjusted by its quality, evolves

$$k_t = \omega_t \tilde{k}_t,$$

where k_t is the capital input that could be used for production in the afternoon, and \tilde{k}_t is the capital stock in the morning. We assume that the shock, ω , is independently and identically distributed (i.i.d.) in the set of positive real numbers, $\omega \in [\underline{\omega}, \bar{\omega}]$, with $E\omega = 1$. The density function of ω is continuous and differentiable. This shock is important for a few reasons. First, we interpret the shock as the quality of capital (Bigio, 2012), but it is unknown to all firms. Second, the shock is important with regard to debt contraction, as in Bernanke and Gertler (1989) and Cooley and Quadrini (2001), where a firm that receives a bad shock will default on its debt. The capital quality shock identifies debt contraction and the default decision in our model, while the idiosyncratic productivity shock controls the exit decision.

In the afternoon, if the firm stays in the market, it incurs a fixed cost of production of ξ . The fixed cost of each period is used to introduce the firm's endogenous exit. There are different ways to specify an endogenous exit. We can either assume that the firm has some outside options (Jovanovic, 1982; Lee and Mukoyama, 2012) or that there is a fixed cost of production (Hopenhayn, 1992a). The firm produces consumption goods, makes investment decisions and chooses new debt and equity levels. The production function exhibits decreasing returns to scale

$$F(z_t, s_t, k_t, n_t) = z_t s_t f(k_t, n_t),$$

where z_t is the aggregate shock, n_t is the labor input, and k_t is the realized capital stock as the input for production. We make the following assumptions regarding the production function: 1) The function, $f : R_+^2 \rightarrow R_+^2$, is strictly increasing, strictly concave, and continuously differentiable. 2) Both the aggregate shock, z , and the idiosyncratic shock, s , follow AR(1) processes.

Instead of assuming firing costs (Hopenhayn and Rogerson, 1993; Lee and Mukoyama, 2012), we assume that there is a quadratic adjustment cost of capital. The adjustment cost is important

³We can establish the model to allow the simultaneous observation of productivity and capital quality shocks. All results hold.

because a firm cannot reach an efficient level of production immediately after receiving a good or bad shock to its productivity. Second, adding a small amount of adjustment cost could better match the job-reallocation rate. Third, the adjustment cost of capital controls the size of new entrants relative to the size of continuing firms. The law of motion of the capital stock takes the form

$$\tilde{k}_{t+1} = (1 - \delta)k_t + i_t,$$

where i_t is the total investment, and we assume that there is an adjustment cost of capital

$$g_t(i_t) = \frac{\rho_1}{1 - \kappa} \left(\frac{i_t}{k_t} \right)^{1-\kappa} k_t$$

where ρ_1 and κ are the coefficients that control the slope and curvature of the adjustment cost.

The firm finances its investments by raising external equity and one-period bonds. The choice between equity and debt is driven by the trade-off between bankruptcy costs and the tax advantage associated with debt. The firm starts the period with intertemporal liabilities b_t and capital stock k_t . Investment is risky such that default occurs at equilibrium. First, the firm must repay its debt carried from the last period; if it cannot fully repay the debt, bankruptcy occurs, and the bondholders can only recover a fraction, θ , of the capital value⁴. After the firm fully repays its debt, it makes labor, n_t , investment, i_t , equity payout, d_t , and new intertemporal debt, b_{t+1} , choices. A positive d_t means that dividends are distributed, and a negative d_t means that equity will be issued. Given the tax advantage associated with debt, a firm that issues debt at price q receives $(1 + \chi)q$, where $\chi > 1$; that is, for each dollar that the firm raises via debt, the government provides a subsidy of χ dollars. Thus, the firm's budget constraint is

$$b_t + w_t n_t + i_t + \varphi(d_t) + g_t(i_t) = F(z_t, s_t, k_t, n_t) + (1 + \chi)q_t b_{t+1},$$

where $\chi > 0$ reflects the tax advantage of debt, and q_t is the price of one-period bonds.

To formalize the rigidity of dividend payout and reflect the costs of internal finance and external

⁴We abstract from the renegotiation process here, although renegotiation is more desirable for the financial intermediary than liquidation. See Cooley and Quadrini (2001).

equity finance, we assume a flotation cost of equity issuance

$$\varphi(d_t) = (1 - \gamma 1_{d_t < 0}) d_t,$$

where $\gamma 1_{d_t < 0}$ is the flotation cost of equity issuance.

For the economic environment described here, it is impossible to know the aggregate state-contingent prices without knowing the distributions of productivity, capital stock and debts across firms. In particular, because the wage is now determined by the labor market equilibrium, firms must know the labor market demand curve to predict future wages. The labor market demand curve is the aggregation of labor demand across all types of firms. The distribution of firm size across all states determines the location of the labor market demand curve. Thus, firms must incorporate this information into their decisions.

2.2.1.1 Exit and Default Decisions

Every morning, the firm makes exit and default decisions. The firm exits the market if the continuation value, conditional on its own productivity and aggregate productivity, is negative. The firm defaults on its debt if it cannot fully repay the debt with its current profits. If the firm defaults, we assume that there is no renegotiation, and it must exit the market. The debtholders can only recover the liquidation value of the firm's capital, θk . The default decision is important in our model. It generates the endogenous one-period bond price of each type of firm. The price varies with firm size and age. The default mechanism differs from the exit mechanism. We separate the realizations of the productivity shock and the capital quality shock into three regions. If the firm receives a low-productivity shock, it exits the market; if the firm receives a bad capital quality shock and a high-productivity shock, it defaults on its capital because it cannot fully repay its debt with its current profits; and for all other realizations of the productivity shock and the capital quality shock, the firm remains in the market.

The value function, after the realization of the productivity shock but before the realization of

the capital quality shock, at the beginning of each period is defined as

$$V_t^b(s_t, \tilde{k}_t, b_t; z_t, \Omega_t) = \max\{E[V_t^s(s_t, k_t, b_t; z_t, \Omega_t)], 0\} \quad (2.1)$$

$$s.t. k_t = \omega_t \tilde{k}_t, \quad (2.2)$$

where Ω_t is the information set used by the firms to predict the aggregate prices, and V_t^s is called the firm's intra-period value function. The exit decision involves a reservation rule

$$v_t^e(\tilde{k}_t, b_t; z_t, \Omega_t) = \begin{cases} 0 & \text{if } s \geq s_i^*(b_t, k_t; z_t, \Omega_t) \\ 1 & \text{o.w.} \end{cases},$$

where

$$s_i^*(b_t, k_t; z_t, \Omega_t) = \inf\{s \in S : E[V_t^s(s_t, \omega \tilde{k}_t, b_t; z_t, \Omega_t)] \geq 0\}.$$

Default occurs if and only if the current period profit plus the liquidation value of the undepreciated capital is less than the debt repayment. Equivalently, the firm will default if its debt repayment is too high or if its realized capital stock is too low. Default will occur if and only if the firm's idiosyncratic shock ω_i is smaller than a cutoff value

$$v_t^e(\tilde{k}_t, b_t; z_t, \Omega_t) = \begin{cases} 0 & \text{if } \omega \geq \omega^*(s, b, \tilde{k}; z, \Omega) \\ 1 & \text{o.w.} \end{cases},$$

where

$$\omega^*(s, b, \tilde{k}; z, \Omega) = \inf\{\omega \in W : F(z_t, s_t, \omega \tilde{k}_t, n_t) - w_t n_t \leq b_t - \theta \omega \tilde{k}_t\}.$$

We assume that if the firm defaults on its debt, the exit value of the firm is negative: $-\zeta$. The negative exit value ensures that the firm will not gamble in equilibrium. If ζ was zero, the firm could always stay in the market and try its luck at receiving a positive quality shock on its capital realization, k_t . Thus, the intraperiod value function, after the realization of the capital quality

shock, is defined as

$$V_t^s(s_t, k_t, b_t; z_t, \Omega_t) = V_t^{sc}(s_t, k_t, b_t; z_t, \Omega_t) 1_{F(z_t, s_t, k_t, n_t) - w_t n_t \geq b_t - \theta k_t} - \zeta 1_{F(z_t, s_t, k_t, n_t) - w_t n_t < b_t - \theta k_t}, \quad (2.3)$$

where V_t^{sc} is the continuation value of the firm at the end of the day.

In the afternoon, if the firm stays in the market, conditional on realizing the capital quality shock and idiosyncratic productivity shock, it chooses its investment policy, dividend payout policy, and new level of debt to maximize the present discounted value of equity

$$\begin{aligned} V_t^{sc}(s_t, k_t, b_t; z_t, \Omega_t) &= \max_{d_t, i_t, b_{t+1}, n_t} d_t - \xi + E \left[m_{t+1} V_{t+1}^b(s_{t+1}, \tilde{k}_{t+1}, b_{t+1}; z_{t+1}, \Omega_{t+1}) | z_t, s_t \right], \\ s.t. \quad w_t n_t + b_t + \varphi(d_t) + i_t &= F(z_t, s_t, k_t, n_t) + (1 + \chi) q_t b_{t+1} - g_t(i_t) \\ \tilde{k}_{t+1} &= (1 - \delta) k_t + i_t, \end{aligned} \quad (2.4)$$

where m_{t+1} is the stochastic discount factor of a representative household, and ξ is the fixed cost of production.

2.2.1.2 Entry

In the morning, a total mass of N potential entrants randomly draws an idiosyncratic productivity, s_t , from the distribution function $\Gamma(s_t)$. After drawing the s_t , a potential entrant decides whether to enter the market. If it enters, it pays the entry cost c_e . A firm entering at period t can finance its investment via debt and equity. The new entrant maximizes the expected value of the firm, $V^e(z_t, s_t)$, by choosing labor and capital inputs and equity and debt to fund investment

$$\begin{aligned} V^e(s_t; z_t, \Omega_t) &= \max_{d_t, b_{t+1}, k_t, n_t} d_t + E \left[M_{t+1} V_{t+1}^b(s_{t+1}, \tilde{k}_{t+1}, b_{t+1}; z_{t+1}, \Omega_{t+1}) | z_t, s_t \right] \\ s.t. \quad w_t n_t + \varphi(d_t) + \frac{\rho_1}{1 - \kappa} k_t^{1 - \kappa} &= F(z_t, s_t, k_t, n_t) + (1 + \chi) q_t b_{t+1} + k_t \\ \tilde{k}_{t+1} &= (1 - \delta) k_t. \end{aligned} \quad (2.5)$$

At time $t + 1$, an entrant will face exactly the same problem as the incumbent. Thus, the firm enters the market if and only if the discounted value of entering exceeds the entry cost

$$V^e(s_t; z_t, \Omega_t) \geq c_e.$$

This equation determines the threshold, $s_e^*(z_t, \Omega_t)$, beyond which only firms with idiosyncratic productivity shocks, $s_t \geq s_e^*(z_t, \Omega_t)$, enter the market.

2.2.2 Household

The household is passive in the model. There is a continuum of homogeneous households maximizing the expected lifetime utility

$$\max E_0 \sum_{t=0}^{\infty} \beta^t U(c_t, L_t), \quad (2.6)$$

where β is the discount factor, c_t is consumption, and L_t is total labor supply.

The household supplies labor in a competitive market and trades the stocks and bonds issued by the firms. The budget constraint is

$$\begin{aligned} & c_t + \int q_{it} b_{it+1} \Omega_t(ds_{it}, dk_{it}, db_{it}) + \int x_{it+1} p_{it} \Omega_t(ds_{it}, dk_{it}, db_{it}) \\ & \leq w_t L_t + \int \rho_{it} b_{it} \Omega_{t-1}(ds_{it-1}, dk_{it-1}, db_{it-1}) + \int x_{it} (p_{it} + d_{it}) \Omega_{t-1}(ds_{it-1}, dk_{it-1}, db_{it-1}) - T_t, \end{aligned} \quad (2.7)$$

where q_{it} is the bond price of firm i ; b_{it+1} is the newly issued bonds of each firm; x_{it} represents the equity shares; p_{it} is the equity price of individual firms; d_{it} is the dividend payout received from owning shares; w_t is the equilibrium wage; ρ_{it} is the redemption value of the bond issued in the last period, which reflects the default and liquidation value; and T_t is the lump-sum taxes that finance the tax benefit associated with the firm's debt. The first-order conditions with respect to L_t , b_{it+1} ,

and x_{it+1} are

$$\begin{aligned} w_t U_c(c_t, L_t) + U_L(c_t, L_t) &= 0, \\ U_c(c_t, L_t) q_t - \beta E \rho_{it+1} U_c(c_{t+1}, L_{t+1}) &= 0, \\ U_c(c_t, L_t) p_{it} - \beta E (d_{it+1} + p_{it+1}) U_c(c_{t+1}, L_{t+1}) &= 0. \end{aligned}$$

The first two equations determine the supply of labor and the interest rate. The last equation determines the price of shares.

2.2.3 Recursive Equilibrium

We consider a recursive equilibrium definition. A key element here is the law of the motion of aggregate states of the economy. (z_t, Ω_t) are the aggregate state variables in our model. We call Ω_t the aggregate state of the industry, which depends on the current measurements of firms encountering capital stocks, idiosyncratic productivity shocks and debt. Given the exogenous process of z_t , the only objective is to know the updated values of Ω_t ; in other words, we need to predict $\Omega' = I(z', z, \Omega)$.

Given firms' entry, exit, and default decisions and policy functions, the evolution of the state of the industry Ω satisfies

$$\begin{aligned} &\Omega(s', \tilde{k}', b'; z') \tag{2.8} \\ = &\int \int \left\{ (1 - v^e(s', \tilde{k}', b'; z', \Omega')) h^a(z'|z) h^i(s'|s) \right. \\ &\left. \int 1_{(\tilde{k}', b' | \omega \tilde{k}, b, s, z, \Omega)} (1 - v^d(s, \omega \tilde{k}, b; z, \Omega)) \Omega(s, \tilde{k}, b; z) \right\} H^\omega(d\omega) ds dz \\ &+ N \int_{s \geq s^*(z, \Omega)} 1_{(\tilde{k}', b' | z, \Omega)} h^i(s'|s) \Gamma(ds), \end{aligned}$$

where h^i and h^a are the conditional probability density functions of idiosyncratic productivity and aggregate shocks, respectively; H^ω is the cumulative distribution function of capital shocks; and $1_{(\tilde{k}', b' | \omega \tilde{k}, b, s, z, \Omega)}$ and $1_{(\tilde{k}', b' | z, \Omega)}$ are the indicator functions given incumbents' and entrants' policy

functions

$$\begin{aligned} 1_{(\tilde{k}', b' | \omega \tilde{k}, b, s, z)} &= 1_{g_k^i(\omega \tilde{k}, b, s; z, \Omega) = k', g_b^i(\omega \tilde{k}, b, s; z, \Omega) = b'}, \\ 1_{(\tilde{k}', b')} &= 1_{g_k^e(s; z, \Omega) = k', g_b^e(s; z, \Omega) = b'}, \end{aligned}$$

where g_k^i, g_b^i are the policy functions of incumbents, and g_k^e, g_b^e are the policy functions of entrants.

A recursive competitive equilibrium consists of the pricing function $w(z, \Omega)$, the forecast rule $I(z', z, \Omega)$, the value function $\{V^s(s, \tilde{k}, b; z, \Omega), V^{sc}(s, \tilde{k}, b; z, \Omega), V^b(s, \tilde{k}, b; z, \Omega), V^e(s; z, \Omega)\}$, the default decision $v^d(s, \tilde{k}, b; z, \Omega)$, the exit decision $v^e(\tilde{k}, b; z, \Omega)$, the entry decision $s^*(z_t, \Omega_t)$, the optimal decision rules $\{g_k^i(s, k, b; z, \Omega), g_b^i(s, k, b; z, \Omega), g_k^e(s; z, \Omega), g_b^e(s; z, \Omega)\}$, and the measure of entrants N_t such that

1. Incumbents' optimization: value function $\{V^s(s, \tilde{k}, b; z, \Omega), V^{sc}(s, \tilde{k}, b; z, \Omega), V^b(s, \tilde{k}, b; z, \Omega)\}$ solves Bellman equations (2.1) to (2.3). $v^e(\tilde{k}, b; z, \Omega)$ and $v^d(s, \tilde{k}, b; z, \Omega)$ are associated exit and default rules for $V^b(s, \tilde{k}, b; z, \Omega)$ and $V^s(s, \tilde{k}, b; z, \Omega)$. $g_k^i(k, b, s; z, \Omega)$ and $g_b^i(k, b, s; z, \Omega)$ are associated policy functions for $V^{sc}(s, \tilde{k}, b; z, \Omega)$.
2. Entrants' optimization: value function $V^e(s; z, \Omega)$ solves Bellman equation (2.5). $g_k^e(s; z, \Omega)$ and $g_b^e(s; z, \Omega)$ are associated policy functions, and firms enter the market if and only if $s \geq s^*(z_t, \Omega_t)$.
3. The household maximizes its utility function, equation (2.6), subject to the budget constraint equation (2.7).
4. Market clearance: labor market; equity market; bond market.
5. Consistency: The forecast function $I(z', z, \Omega)$ is consistent with the actual law of motion, equation (2.8), that is implied by the optimal decision rules.

2.2.4 Comparative Analysis

Given the value functions, the firm exits the market if and only if its realized productivity shock is lower than a certain value $s^*(b, \tilde{k}; z, \Omega)$. The firm defaults if and only if its realized financial shock

is lower than a certain value $\omega^*(s', b', \tilde{k}'; z', \Omega')$

$$(1+r)qb' = (1-p) \left\{ b' \int_{\omega^*(s', b', \tilde{k}'; z', \Omega')}^{\bar{\omega}} H^\omega(d\omega) + \int_{\underline{\omega}}^{\omega^*(s', b', \tilde{k}'; z', \Omega')} \theta \tilde{k}' \omega H^\omega(d\omega) \right\} + p \min(b', \theta \tilde{k}'),$$

where p is the probability of exiting the market in the next period

$$p = E^x \int_{z'} H^i(s_i^*(b, \tilde{k}; z, \Omega, x) | s) H^a(dz' | z),$$

where $H^a(\cdot | z)$ and $H^i(\cdot | s)$ are the conditional cumulative distribution functions of aggregate and idiosyncratic productivity shocks, respectively.

Proposition 6 *The default threshold $\omega^*(s, b, \tilde{k}; z, \Omega)$ is a decreasing function of z, s and \tilde{k} and an increasing function of b_{-1} .*

Proof. See Appendix. ■

Proposition 7 *The exit point $s_i^*(b, \tilde{k}; z, \Omega)$ is a decreasing function of z and \tilde{k} and an increasing function of b .*

Proof. See Appendix. ■

Proposition 8 *The cost of external financing, q , is a decreasing function of z, s and \tilde{k} and an increasing function of b .*

Proof. It is straightforward from Proposition 1 and 2. Because $\omega^*(s', b', \tilde{k}'; z', \Omega')$ and $s_i^*(b, \tilde{k}; z, \Omega)$ are decreasing in s, z and k and increasing in b, q is a decreasing function of z and k and an increasing function of b . ■

For the same amount of investment, a small firm (with lower productivity and capital stock) faces a higher cost of external finance. The cost of financing is lower during booms than during recessions. Debt is preferred by firms because of the associated tax advantages, but higher bankruptcy costs exist if the firm has higher leverage.

2.3 Calibration

In this section, we characterize firms' financial behaviors and industry dynamics. By linking firms' financial decisions to industry dynamics and the aggregate economy, our model allows us to examine a number of important empirical and theoretical patterns of industry dynamics. In particular, we start with the model without aggregate shocks, $z = 1$, as the benchmark. We describe the invariant distribution of firms and their financial structures. We parameterize the model assuming that one period is equal to one year and normalize the wage rate, w , to 1 in the benchmark. The discount factor is 0.94 with an implied risk-free interest rate of 6.38%. All parameter values are summarized in Table 2.1.

The production function exhibits decreasing returns to scale, $z_s (k^\alpha n^{1-\alpha})^\nu$. α is the capital share, and we set a standard value of 0.36. The parameter ν determines the degree of returns to scale. We take the value $\nu = 0.85$ from Atkeson and Kehoe (2005). The idiosyncratic productivity follows an AR(1) process, $\ln s' = \alpha_s + \rho_s \ln(s) + \varepsilon_s$, where $\varepsilon_s \sim N(0, \sigma_s^2)$. The values of the drift, persistence and variance are calibrated to match the empirical patterns of employment processes (Lee and Mukoyama, 2012). The AR(1) process is approximated by a Markov process with ten states (Tauchen, 1986).

We include a quadratic adjustment cost in the model. The depreciation rate is set to 0.08. In the financial market, the flotation cost of new shares is set to 0.3 (Cooley and Quadrini, 2001; Hennessy and Whited, 2005). We assume a lognormal distribution for the capital quality shock, $\omega \sim \ln N\left(-\frac{\sigma_\omega^2}{2}, \sigma_\omega^2\right)$, which is discretized on the interval $[0, 5]$. The liquidation value of capital, θ , in the event of default is 0.7 (Gourio, 2013). The exit cost upon default, ζ , is equivalent to the production cost, ξ . The entry cost, c_e , is set such that only the firms whose productivity exceeds the mean productivity enter the market. With a corporate income tax of 35%, the risk-free interest rate of 6.38% implies a tax subsidy of 2.23%.

The consumer is passive in our model. The utility function is linear in consumption and can be divided between leisure and consumption,

$$U(c, L) = c_t - A \frac{L^{1+1/\eta}}{1+1/\eta}.$$

We assume that utility is linear in consumption such that firms' stochastic discount factors do not depend on the aggregate states. γ is the elasticity of labor supply. We set η equal to 1.761. The elasticity of labor supply is calibrated to match the volatility of the wage relative to productivity. A is solved such that it is consistent with the equilibrium wage. The measure of potential entrants, N , is calculated from the equilibrium labor supply using a 60% total employment rate.

There are still six parameters, $\{\alpha_s, \rho_s, \sigma_s, \sigma_\omega, \rho_1, \xi\}$, that must be set. These parameters are chosen to achieve the following targets: 1) the annual average corporate default rate is equal to 0.4% (Chen et al., 2009; Giesecke et al., 2011); 2) the mean leverage ratio (D/E) is equal to 0.81 (Chen et al., 2009); 3) the total exit probability, including the default rate, is equal to 5.4% (Lee and Mukoyama, 2012); 4) the average size of new entrants is 57% of the average size of incumbents; 5) the persistence of the employment process is 0.97; and 6) the variance of the employment growth rate is 0.14.

2.3.1 Stationary Equilibrium

The economy is characterized by a certain distribution of firms, μ , over all state variables, k, b and s . In the analysis described in this section, we focus on the invariant distributions of firms. The existence and uniqueness of the invariant distribution therefore depends on the properties of the transition matrix generated by the optimal policy functions, capital and debt, of both incumbents and entrants, which are characterized by the equation (2.8) without an aggregate shock z . The invariant distribution is the fixed point in this contraction mapping.

The convergence of the stationary equilibrium allows us to numerically solve the model. The computational details of the stationary distribution are described in the Appendix. We solve the value function first and then simulate the model with 10,000 firms over 5,000 periods. We drop the first 10% of the simulations and obtain the summary statistics from the remaining simulations. Table 2 compares the results of the model to the data targets. The simulated model is fairly close to the targets. The job reallocation rate, defined as the total jobs created relative to the percentage of total employment, is also similar to the value in the data. In the simulation, we replace all exiting firms with new entrants, and the total measure of firms and the labor supply remain unchanged.

In addition, the labor market clearing condition ensures that the entrants hire all workers who lost their jobs via exits. The average ratio of exits to entrants is close to one in our model. This is larger than the value reported in Lee and Mukoyama (2012): 0.7

Figure 2.2 plots the size, age, exit and default distributions of firms. We use the number of employees⁵ as a proxy for firm size. The top panels show the firms' size and age distributions: 1) the shape of the distribution presents a degree of skewness toward small and young firms; 2) conditional on size, larger firms tend to be older⁶; and 3) conditional on age, older firms tend to be larger. All of these patterns confirm the empirical regularity of the data. The bottom panels report the exit and default densities of the model as a function of firm size and age. We showed that the default and exit probabilities are higher for small firms in Proposition 6 and 7. The density we reported is the actual number of exit and default firms as a fraction of the total exits and defaults within each size and age group. Small and young firms face higher default and exit probabilities. They are more likely to default on their debt and exit the market.

It is critical to examine how the financial market affects the stationary distribution of firms in our model before we progress to the model with aggregate shocks. Figure 2.3 plots the leverage ratio, bond price and Tobin's Q against firm size and age. First, small and young firms take on more debt. They have a higher leverage ratio than large and old firms. Second, because small and young firms face higher exit and default probabilities, their bond price, q , is lower. Third, Tobin's Q is a decreasing function of firm age. Tobin's Q is calculated based on the value of firms in the afternoon (after the realization of capital quality shocks) over the capital input in production. However, Tobin's Q does not decrease as firm size increases (Cooley and Quadrini, 2001). In Cooley and Quadrini (2001), only one level of productivity is used. Additionally, because the production function causes decreasing returns to scale and the value of the firm is proportional to its production function, larger firms should have lower Tobin's Q values. Our model utilizes a different mechanism: production is specified by both an idiosyncratic productivity shock and firm size, and the size effect is dominated by the productivity effects. Therefore, larger firms have

⁵The result is similar when we use total assets, k , as a proxy for firm size.

⁶We exclude the new entrants from the sample simulation here. When the adjustment cost is very low, there are many firms entering the market with large labor demands, which significantly lowers the average age of the firms.

higher productivities and, thus, higher Tobin’s Q values.

To summarize, the model with financial frictions can capture both the “age dependence” and “size dependence” of firm dynamics (Cooley and Quadrini, 2001). In addition, the model is able to account for most of the stylized facts regarding firms’ dynamic behaviors and financial structures. We replicate and extend many of the findings in Cooley and Quadrini (2001) using a more general and flexible model setup. This is an important prerequisite for examining the properties of the model with aggregate shocks.

2.4 Adding Aggregate Shocks

This section presents the cyclical behaviors of firm dynamics by adding aggregate shocks. We assume that aggregate shocks follow an AR(1) process with a persistence of 0.654 and a standard deviation of 0.7%. We discretize the AR(1) process into a Markov process over the state space $[0.99, 1, 1.01]$. The challenge of including both aggregate and idiosyncratic shocks is the computational complexity of the general equilibrium. First, we simplify the consumer’s behaviors by assuming that the utility function is linear in consumption. The stochastic discount factor in the firm problems is constant, and it coincides with the consumer’s discount factor, β . Second, because the utility function can be separated into consumption and leisure, the first-order condition simplifies the labor market clearing condition, $w = AL^{1/\eta}$. Third, we assume that the total mass of potential entrants, N_t , varies over time. The new entrants fill the gaps between the labor supply and the labor demand from incumbents on the extensive margin, the number of firms entering, and the intensive margin, $s_t(z_t)$.

Following Krusell and Smith (1998), we approximate the state variable distribution of firms, Ω , with the first moments of labor supply, L , and productivity, z . The forecast rule is

$$\log(L') = a_0 + a_1 \log(L) + a_2 \log(z').$$

By forecasting the first moment of labor supply, the equilibrium wage is implied by the labor market clearing condition. Given the wage, incumbents make exit and default decisions and adjust

their labor demands. Entrants, N_t , are solved such that the gaps between the aggregate labor supply and the labor demand from incumbents are filled by the entrants' labor demands. The results of the model with aggregate shocks are summarized in Table 2.3 and Table 2.4.

Table 2.3 presents the entry and exit results of the simulated model. We can observe a strong pattern of procyclical behaviors of entry: 3.82% during recessions versus 7.23% during booms. The total exit rate is procyclical (almost acyclical): 5.85% during recessions and 5.20% during booms. In the data, we observe both procyclical entry and procyclical exit (almost acyclical). The model also captures the countercyclical default rate: 0.58% during recessions and 0.35% during booms.

The cyclical behavior of entry results from cyclical wages and the size of the entrants. The model predicts a higher wage during booms. All incumbents will cut off employment. Higher wages also imply higher labor supplies. The countercyclical external finance costs increase the difficulty of entry during recessions compared with during booms. The average size, measured in terms of employment, of entrants is larger during recessions than during booms. The entry threshold, measured in terms of productivity, is 1.74 during booms and 1.84 during recessions. The average size of the entrants during booms is 92% of that during recessions. Thus, the gap between the labor supply and the labor demand from incumbents is relatively large during booms, and the size of entrants is relatively small during booms. Those two effects ensure procyclical entry in the model. Because the wage effect is somehow offset by the aggregate productivity, the second effect is stronger than the first in the model. The wage volatility is less than half the productivity volatility in our model simulations.⁷

The flat exit rates over the business cycle can be explained by age effects. The wage effects of incumbents are offset by the aggregate productivity. Additionally, because of age effects, i.e., older firms are larger and are more likely to exhibit their optimal production levels and capital structures, the countercyclical nature of external finance costs affects the entrants to a lesser extent than incumbents. These effects maintain the flat exit rates in the model.

Compared to the result in Lee and Mukoyama (2012), our model has lower wage volatility

⁷Lee and Mukoyama (2012) observe the cyclical nature of entry when the wage volatility is as large as the productivity volatility.

and lower volatility of productivity. If we increase the volatility of aggregate productivity, the model can have an even stronger entry cyclical. Without financial market frictions, it is difficult to explain procyclical entry because wage effects are almost offset by the aggregate productivity effects, as in many IO models.⁸ Additionally, the financial market frictions generate the implicit countercyclical nature of external financing costs and relative entry size. These effects guarantee procyclical entry and flat exit rates.

Table 2.4 shows the aggregate dynamics of debt issuance and equity issuance. Debt issuance is procyclical, and equity issuance is countercyclical. The standard deviations of debt and equity issuance over aggregate GDP are similar to the values reported in Jermann and Quadrini (2012). Figure 2.4 further plots the cyclical behaviors of debt issuance and equity issuance in terms of the firm size distribution. We discard the bottom 5% of firms because many of them result from very severe capital quality shocks, and their cyclical behaviors of debt and equity issuances are inconsistent. We sort the firms into the size-based portfolios [5%,25%], [25%,50%], [50%,75%], [75%,90%], and [90%,100%] and the age-based portfolios [1,5], [5,10], [10,30], [30,100], and [100,200]. The cyclicalities of debt issuance and equity issuance are negatively correlated with firm size and age. The procyclicality is stronger for smaller and younger firms. Debt issuance is procyclical for most size groups but is acyclical for the top 10% of firms. Equity issuance is procyclical for small firms but is countercyclical for large firms. The aggregate cyclicity of equity issuance is substantially affected by the cyclicity of the top firms because of the large amounts of funds raised by those firms. These results are similar to the empirical cyclicity patterns documented in Covas and Den Haan (2011a).

2.4.1 Model Comparisons

To examine the effectiveness of our model, we compare the baseline model result with the following: 1) the model in which we eliminate the intensive margin of labor demand and 2) the model in which we eliminate external financing. If we assume that each firm can only have one worker, the total demand for workers will be equal to the number of firms. There is no intensive margin when a

⁸See (Lee and Mukoyama, 2012) Table 25.

severe shock hits. In this case, the entry rate is less volatile than the baseline model. Additionally, all dynamics result from entries and exits. Entry thus plays an important role over the business cycle. If we assume that no external financing is available by setting the bond price to zero and the flotation cost of equity issuance to infinity, the model exhibits a different pattern. The entry rate is 1% smaller than that in the baseline model. Thus, external financing explains 25% of the entry rate difference between good and bad times.

2.5 Conclusion

In this paper, we propose an industry dynamics model with financial market frictions in a general equilibrium setting. We numerically solve the model and find that it is able to explain several stylized facts in both the stationary distribution (size and age dependence) and the business cycle (cyclical behaviors of entry, exit and financial structure). The model reveals procyclical entry and almost acyclical exit. It also exhibits procyclical debt issuance and countercyclical equity issuance at both the aggregation level and the firm level. We demonstrate that the cyclicity of debt and equity depends on firm size and age.

Appendix

Proof of Proposition 1

Given the production function, labor demand is determined by

$$w = F_n(z, s, k, n).$$

Thus, the default decision is given by

$$v^d(s, \tilde{k}, b; z, \Omega) = \inf \{ \omega \in W : F(z, s, \omega \tilde{k}, n) - F_n(z, s, \omega \tilde{k}, n) n \leq b - \theta \omega \tilde{k} \}.$$

For any $\tilde{k}_1 \geq \tilde{k}_2$, it suffices to prove that $\omega_1 = v^d(s, \tilde{k}_1, b; z, \Omega) \leq v^d(s, \tilde{k}_2, b; z, \Omega) = \omega_2$. If all functional forms are continuous, we have

$$F(z, s, \omega_2 \tilde{k}_2, n) - F_n(z, s, \omega_2 \tilde{k}_2, n) n = b - \theta \omega_2 \tilde{k}_2.$$

Let $\omega'_1 = \frac{\omega_2 \tilde{k}_2}{\tilde{k}_1}$,. Then, we have

$$F(z, s, \omega'_1 \tilde{k}_1, n) - F_n(z, s, \omega'_1 \tilde{k}_1, n) n = b - \theta \omega'_1 \tilde{k}_1.$$

Therefore,

$$v^d(s, \tilde{k}_1, b; z, \Omega) = \inf \{ \omega \in W : F(z, s, \omega \tilde{k}_1, n) - F_n(z, s, \omega \tilde{k}_1, n) n \leq b - \theta \omega \tilde{k}_1 \} \leq \omega'_1 \leq \omega_2.$$

For any $b_1 \geq b_2$, if the production function exhibits decreasing returns to scale, let

$$g(z, s, \omega \tilde{k}, n) = F(z, s, \omega \tilde{k}, n) - F_n(z, s, \omega \tilde{k}, n) n + \theta \omega \tilde{k}.$$

We have

$$g_k(z, s, \omega \tilde{k}, n) = F_k(z, s, \omega \tilde{k}, n) - F_{kn}(z, s, \omega \tilde{k}, n) n + \theta > 0.$$

Thus, g is an increasing function of k . Because $\omega_1 = v^d(s, \tilde{k}, b_1; z, \Omega)$ and $\omega_2 = v^d(s, \tilde{k}, b_2; z, \Omega)$, we have

$$g(z, s, \omega_1 \tilde{k}, n) = b_1 \geq b_2 = g(z, s, \omega_2 \tilde{k}, n),$$

that is, $\omega_1 \geq \omega_2$.

A similar process can be used to prove that if $g(z, s, \omega \tilde{k}, n)$ is an increasing function of s and z ,, $v^d(s, \tilde{k}, b; z, \Omega)$ is a decreasing function of s and z .

Proof of Proposition 2

It suffices to show that $V_t^s(s_t, \tilde{k}_t, b_t; z_t, \Omega_t)$ is an increasing function of s, z and \tilde{k} and a decreasing function of b . This is a standard Bellman Equation problem, where

$$\begin{aligned} & V_t^s(s_t, \tilde{k}_t, b_t; z_t, \Omega_t) \\ = & \max_{d_t, i_t, b_{t+1}, n_t} d_t - \xi \\ & + E \left[m_{t+1} \max \{ EV_{t+1}^s(s_{t+1}, k_{t+1}, b_{t+1}; z_{t+1}, \Omega_{t+1}), 0 \} \mid z_t, s_t \right] 1_{F(z_t, s_t, \omega \tilde{k}_t, n_t) - w_t n_t \geq b_t - \theta \tilde{k}_t} \\ s.t. & \quad w_t n_t + \varphi(d_t) = F(z_t, s_t, \omega_t \tilde{k}_t, n_t) + \chi q_t b_{t+1} + \omega_t \tilde{k}_t \\ & \quad \tilde{k}_{t+1} = (1 - \delta) \omega_t \tilde{k}_t. \end{aligned}$$

Theorem 9.7 in Stokey and Lucas (1989) guarantees the existence and monotonicity of the value function $V_t^s(s_t, \tilde{k}_t, b_t; z_t, \Omega_t)$.

Computation of the Stationary Distribution

This section presents the computational details of the stationary distribution of firm dynamics in the model without aggregate shocks. We omit the notation of z because it is constant here. The computational procedure is based on value function iteration.

1. Set grids on k and b . Discretize the AR(1) process of idiosyncratic shocks using a ten-state Markovian process.

2. Solve the value functions and optimal policy rules.

(a) Set any initial value for $V_0^s(s, k, b)$ and $V_0^b(s, k, b)$.

(b) Calculate $V^{sc}(s, k, b)$ and $V^s(s, k, b)$ by

$$\begin{aligned} V^{sc}(s, k, b) &= \max d + E \left[\beta V_0^b(s', \tilde{k}', b') \mid s \right], \\ V^s(s, k, b) &= V^{sc}(s, k, b) 1_{F(s, k, n) - wn \geq b - \theta k} - \xi, \end{aligned}$$

obtain the optimal policy functions, $g_k^i(s, k, b)$ and $g_b^i(s, k, b)$.

(c) Solve the exit decisions, $v^e(s, k, b)$,

$$\begin{aligned} v^e(s, k, b) &= \inf \{s \in S : E[V^s(s, \omega k, b)] \geq 0\}, \text{ or} \\ v^e(s, k, b) &= 1 \text{ if this set is empty} \end{aligned}$$

and default decisions, $v^d(s, k, b)$,

$$\begin{aligned} v^d(s, k, b) &= \inf \{\omega \in W : F(s, \omega \tilde{k}, n) - wn \leq b - \theta \omega \tilde{k}\} \text{ or} \\ v^d(s, k, b) &= 1 \text{ if this set is empty.} \end{aligned}$$

(d) Calculate the value of the firm, $V^b(s, k, b)$, at the beginning of each day

$$V^b(s, \tilde{k}, b) = \max \{E[V^s(s, \omega \tilde{k}, b)], 0\}.$$

(e) Update until

$$\|V^s - V_0^s\| \leq \varepsilon, \text{ and } \|V^b - V_0^b\| \leq \varepsilon,$$

and let $V^{s*} = V^s, V^{b*} = V^b$.

3. Solve the entrants' Bellman equation

$$V^e(s) = \max d + E \left[\beta V^b(s', \tilde{k}', b') \mid s \right],$$

and obtain the optimal policy functions, $g_k^e(s)$ and $g_b^e(s)$. The entry cost, c_e , is set such that only the entrants whose productivity exceeds the mean value can enter the market

$$s^* = \text{median}(s), c_e = V^e(s^*).$$

4. Simulate the stationary distribution of firms with 10,000 firms and 5,000 periods. Initially, all firms are created by entry. Generate the capital quality shock and Markovian chain for each firm. If the firm defaults or exits the market, it is replaced by a new entrant with a new sequence of productivity shocks.
5. Drop the first 10% of simulations and calculate the invariant distributions. The total measure of firms, μ , is calculated based on the labor market clearing condition

$$L = \mu \int n(s, k, b) \Omega(ds, dk, db).$$

Computation of the Model with Aggregate Shocks

This section presents the computational details of firm dynamics in the model with aggregate shocks. The computational procedure is based on value function iteration and simulations.

1. Set grids on k , b , s and z . Discretize the AR(1) process of idiosyncratic and aggregate shocks using a ten-state Markovian process.
2. Guess L as a function of z and L_{-1} .
3. Solve the value functions and optimal policy rules. This process is similar to that used to calculate the stationary distribution but with one additional state variable.
4. Simulate the model with 10,000 firms and 5,000 periods. Initially, all firms are created by entry. Generate the aggregate shocks, the capital quality shock and Markovian chain for each firm. If the firm defaults or exits the market, it is replaced by a new entrant with a new sequence of productivity shocks.

5. Drop the first 10% of simulations and calculate the invariant distributions. The total measure of firms, μ , is calculated based on the labor market clearing condition
6. Run regressions on the aggregate law of motion

$$\log(L') = a_0 + a_1 \log(L) + a_2 \log(z') + \varepsilon.$$

7. Update a_0 , a_1 , and a_2 until convergence; otherwise, go back to step 2.

Parameter	Symbol	Value
Discount factor	β	0.94
Returns to scale	ν	0.85
Capital share	α	0.36
Elasticity of labor supply	η	1.70
Liquidation value	θ	0.70
Flotation cost of new shares	γ	0.30
Tax subsidy	τ	2.23%
Depreciation rate	δ	0.08
Aggregate productivity shock persistence	ρ_z	0.65
Aggregate productivity shock volatility	σ_z	0.66%
Fixed production cost	ξ	11.7
Adjustment cost	ρ_1	0.0022
Standard deviation of capital quality shock	σ_ω	0.22
Idiosyncratic productivity shock drift	α_s	0.0441
Idiosyncratic productivity shock persistence	ρ_s	0.9312
Idiosyncratic productivity shock volatility	σ_s	0.1045
Share of leisure in utility function	A	1.35

Table 2.1: Parameterizations

	Data	Model
Corporate default rate	0.4%	0.5%
Total exit rate	5.4%	5.4%
Leverage ratio (D/E)	0.81	0.75
Average size of new entrants relative to incumbents	0.60	0.53
Persistence of employment process	0.97	0.98
Variance of employment growth	0.14	0.22
Average size of exits relative to entrants	0.70	0.45
Tobin's Q	1.11	1.29
Job reallocation rate	19.4%	17.22%

Table 2.2: Data and model statistics in the stationary distribution

	Bad	Good
Wage	0.9952	1.0040
Corporate default rate	0.58%	0.35%
Total exit rate	5.85%	5.20%
Entry rate	3.82%	7.23%

Table 2.3: Entry and exit with aggregate shocks

	Std(Variable)	Corr(Variable,GDP)
Debt issuance	0.42%	0.23
Equity issuance	0.65%	-0.17

Table 2.4: Cyclical debt and equity issuance behaviors

	Bad	Good
Total exit rate	5.80%	5.06%
Entry rate	4.54%	6.32%

Table 2.5: Model with one unit of labor demand for each firm

	Bad	Good
Total exit rate	5.54%	5.15%
Entry rate	4.89%	5.80%

Table 2.6: Model with no external financing

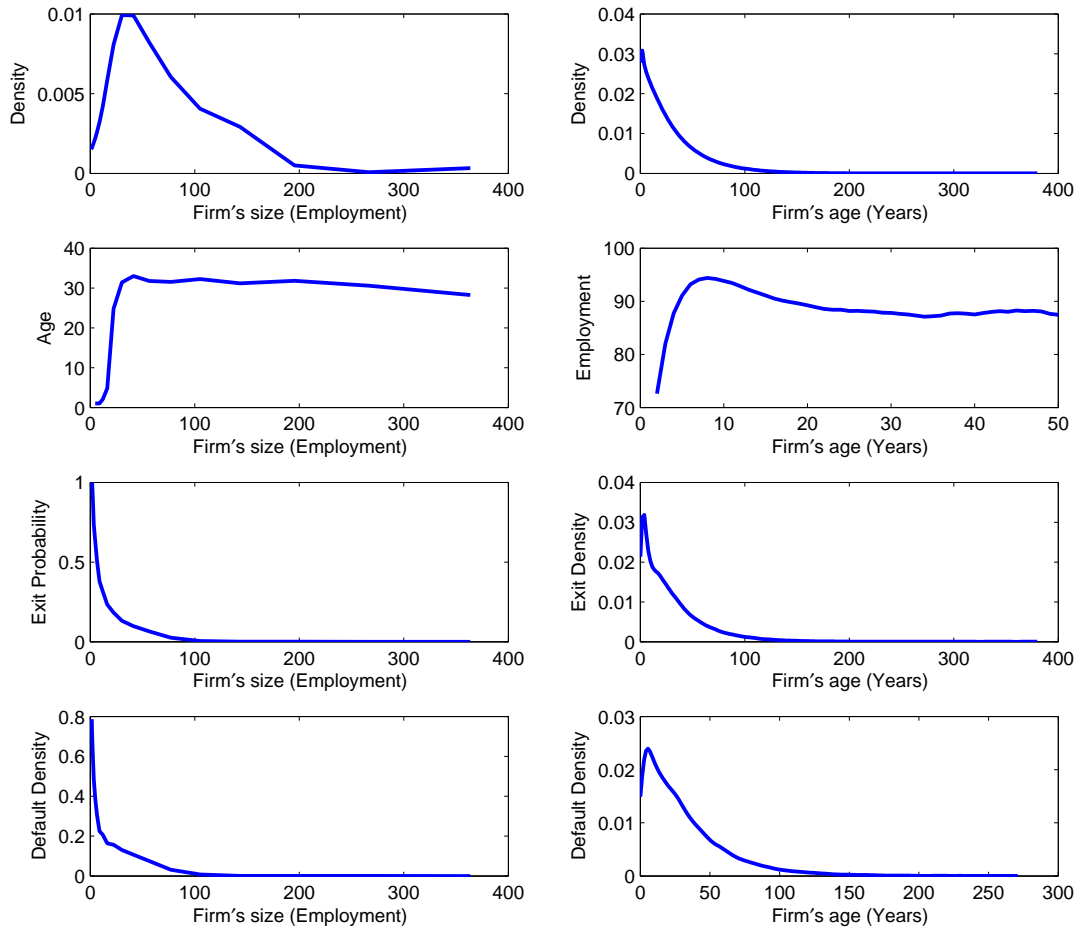


Figure 2.2: Firm size and age distributions

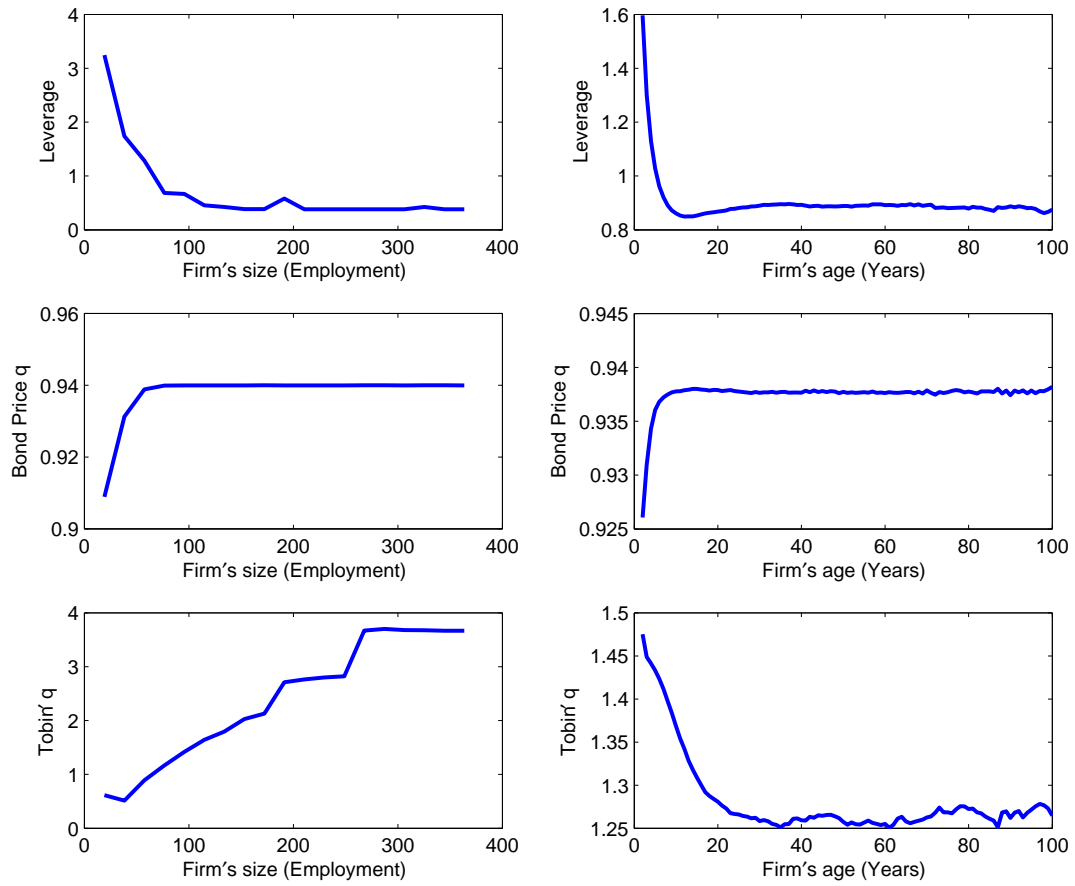


Figure 2.3: Financial structure and bond price

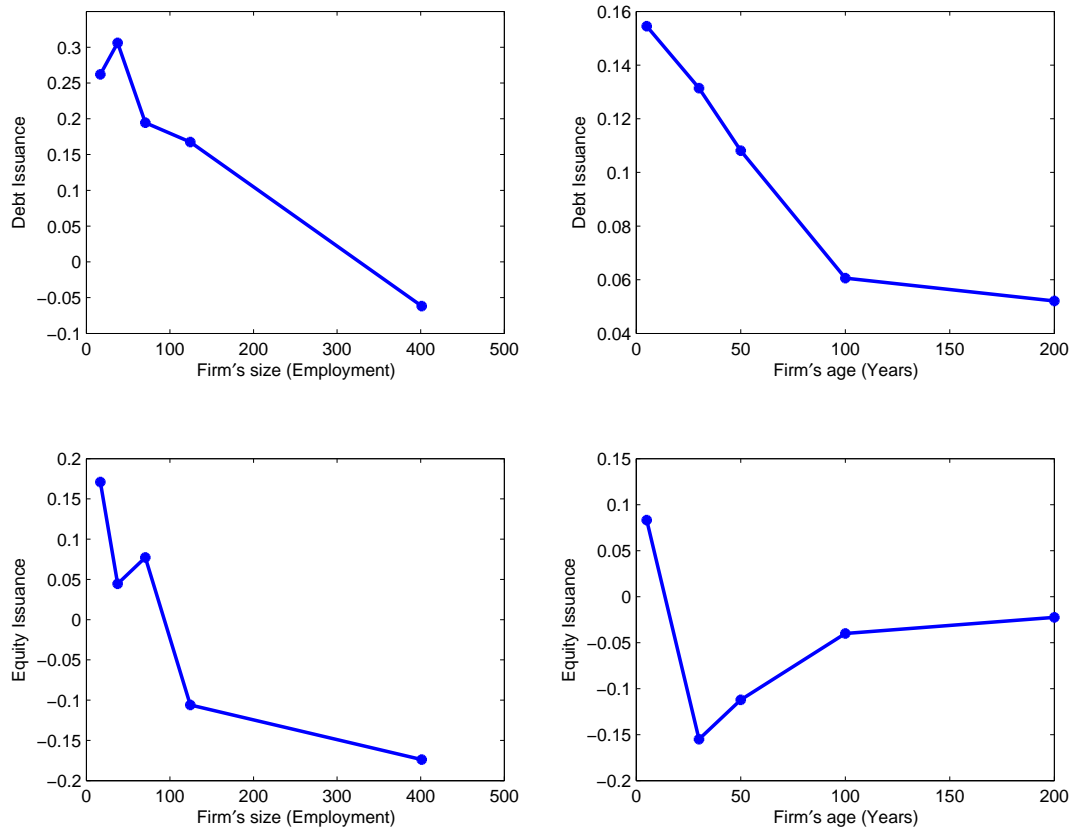


Figure 2.4: Cyclical debt and equity issuance conditional on firm size distribution

CHAPTER 3

Risk and Return in Segmented Markets with Expertise

3.1 Introduction

Complex investment strategies, such as those employed by hedge funds and other sophisticated investors, appear to generate persistent alphas, high Sharpe ratios¹, and to feature limited participation, despite free entry. We develop an industry equilibrium model of the complex asset management industry which explains these facts, and generates additional testable predictions about the industrial organization of complex asset markets which we show are consistent with data on hedge funds. Investing in a complex asset requires an investment not only in the asset itself, but also in a technology with which to manage the asset. We argue that this joint investment in complex asset strategies exposes investors to asset-specific, idiosyncratic risk, and that variation in expertise across investors leads to variation in the asset-specific risk investors face. Thus, we define a complex asset as one that imposes idiosyncratic risk on investors, and argue that more complex assets impose more asset-specific risk. We use our model to characterize how the equilibrium pricing of complex assets is determined by the endogenous joint distribution of expertise and financial wealth. In equilibrium, this joint distribution is in turn determined by the deep parameters which describe preferences, endowments, and technologies in our model economies, and which proxy for asset complexity.

Our model economy is populated by a continuum of agents who choose to be either non-experts who can invest only in the risk free asset, or experts who can invest in both the risk free and risky assets. Investors who choose to be experts make an initial investment in expertise, which represents the investor's personnel, data, hedging and risk management technologies, back office operations

¹See Sharpe (1966).

and trade clearing processes, relationships with dealers, and relationships with clients.

The acquisition and management of complex assets require a joint investment in the asset (or strategy) and in an implementation technology which requires financial expertise. All expert investors in the market earn a common equilibrium return that clears the market. However, their returns are subject to asset-specific (or strategy-specific) shocks. Expertise improves an investor's specific implementation technology and shrinks the asset-specific volatility of the returns to the risky asset, implying that more expert investors face a higher Sharpe ratio. Thus, expertise may be interpreted as the ability to implement complex strategies better either by developing a superior model or information technology, hiring better employees, or by gathering superior information.

By definition, true “alpha” must be due to idiosyncratic, not systematic, risk. In our stationary model, all risk is asset-specific and idiosyncratic. This is, of course, a useful assumption technically. However, we argue that emphasizing the role of idiosyncratic risk in asset pricing is also realistic, as argued in Merton (1987). There is a growing literature that documents the importance of idiosyncratic risk in complex asset strategies. Pontiff (2006) investigates the role of idiosyncratic risk faced by arbitrageurs in a review of the literature and argues that “The literature demonstrates that idiosyncratic risk is the single largest cost faced by arbitrageurs”. Greenwood (2011) states that “Arbitrageurs are specialized and must be compensated for idiosyncratic risk,” and lists this first as the key friction investors in complex strategies face. To paraphrase Emanuel Derman, if you are using a model, you are short volatility, since you will lose money when your model is wrong.²

Idiosyncratic risk is likely to be particularly important in markets for complex assets. Complex assets expose their owners to idiosyncratic risk through several channels. First, any investment in a complex asset requires a joint investment in the front and back office infrastructure necessary to implement the strategy. Second, their constituents tend to be significantly heterogeneous, so that no two investors hold exactly the same asset. Third, the risk management of complex assets typically requires a hedging strategy that will be subject to the individual technological constraints of the investor. Fourth, firms which manage complex assets may be exposed to key person risk due to the importance of specialized traders, risk managers, and marketers. Finally, complex assets may

²Derman (2016)

introduce or amplify idiosyncratic risk on the liability side of the balance sheet, through the fact that they are difficult for outside investors to understand, but tend to be funded with external finance.³ We provide one specific micro-foundation for the idiosyncratic risk complex assets impose on investors in strategies involving a long position in an underlying asset, and a short position in an imperfectly correlated, investor specific, tracking portfolio. Thus, an additional contribution of our paper is to provide a precise explanation for the idiosyncratic risk that the prior literature has argued is important for understanding complex asset returns.

We assume that funds cannot be reallocated across individual risk-averse investors. Clearly, since the risk in our economy is idiosyncratic, pooling this risk would eliminate the risk premium that experts require to hold it. Complex assets tend to be held in managed accounts. For incentive reasons, these managers cannot hedge their own exposure to their particular portfolio. In fact, Panageas and Westerfield (2009) and Drechsler (2014) provide important results for the portfolio choice of hedge fund managers who earn fees based on assets under management and portfolio performance. In particular, they show that these managers behave like constant relative risk aversion investors. This motivates why we endow expert investors in our model with CRRA preferences.

In our model, expertise varies in the cross-section but is fixed for each agent over time. This allows us to solve our model analytically, including the joint stationary wealth and expertise distribution, in closed form, up to the equilibrium fixed point for expected returns.⁴ The deep preference and technology parameters determine the joint distribution of wealth and expertise and the resulting equilibrium alpha and Sharpe ratio. More complex assets, characterized by a higher required expertise to achieve a given lower level of risk (or, equivalently, a larger amount of fundamental risk) have higher alphas, and under natural conditions on the distribution of expertise, lower participation and higher Sharpe ratios. The complex asset market is endogenously segmented, since expert demand lowers required returns. Although alphas and Sharpe ratios of participants may look attractive, they are not representative of what investors with less expertise can achieve. As a

³Broadly interpreted, these risks may come either from the asset side, or from the liability side, since funding stability likely varies with expertise. However, we abstract from the microfoundations of risks from the liability side of funds' balance sheets, and model risk on the asset side.

⁴We use a numerical algorithm to solve the market clearing fixed point problem. However, the solution is straightforward given our analytical solution for policy functions and distribution over individual states.

result, participation is naturally limited, and elevated excess returns with modest average market-level risk persist. We focus on differences in complexity arising from variation in the amount of idiosyncratic risk the asset class imposes on investors, for example because more complex assets impose more model-specific risk. However, we also show that other proxies for complexity display similar comparative statics. In particular, assets which have higher costs of maintaining expertise, or require expertise which is more scarce, also have higher alphas.

The equilibrium stationary wealth distribution of participants is Pareto conditional on each expertise level. The decay parameter depends on investors' portfolio choice and exposure to the risky complex asset. In particular, because investors with higher expertise choose a higher exposure to the risky asset, both the drift and the volatility of their wealth will be greater, leading to a fatter tailed distribution at higher expertise levels. Our model predicts, under natural conditions on the distribution of expertise (for example using a log-normal distribution), that more complex assets will have less concentrated wealth distributions, a fact that is consistent with data on strategy-level hedge fund data. This result is driven by the fact that more complex assets will have a higher threshold for expertise required for participation. Although there are likely to be many agents with lower levels of expertise, the distribution across expertise at higher levels is flatter, leading to a less concentrated wealth distribution for more complex assets. We provide evidence for this ancillary prediction using data from Hedge Fund Research on size distributions across different strategies.

The paper proceeds as follows. In Section 3.2 we review the related literature. Section 3.3 contains the construction and analysis of our dynamic model, and finally Section 3.5 concludes. Most proofs appear in the Appendix. In separate work (Eisfeldt et al. (2015)), we study a discrete time dynamic model with stochastic expertise, which we use to study the impact of unanticipated aggregate shocks and to develop quantitative results. In particular, using intuition developed in this paper, we show that expertise can act as an excess capacity-like barrier to entry, leading to interesting dynamics for market excess returns and volatility following shocks to investor wealth and to fundamental asset volatility.

3.2 Literature

Our paper contributes to a large and growing literature on segmented markets and asset pricing. Relative to the existing literature, we provide a model with endogenous entry, a continuous distribution of heterogeneous expertise, and a rich distribution of expert wealth that is determined in stationary equilibrium. Thus, we have segmented markets, but allow for a participation choice. Our market has limited risk bearing capacity, determined in part by expert wealth, but in addition to the amount of wealth, the efficiency of the wealth distribution also matters for asset pricing.

We group the existing literature into three main categories, namely financial constraints and limits to arbitrage, intermediary asset pricing, and segmented market models with alternative microfoundations to agency theory. Although our model is not one of arbitrage per se, our study shares the goal of understanding the returns to complex assets and strategies. Our model also shares the features of segmented markets and trading frictions with the limits to arbitrage literature. Gromb and Vayanos (2010b) provide a recent survey of the theoretical literature on limits to arbitrage, starting with the early work by Brennan and Schwartz (1990) and Shleifer and Vishny (1997).⁵ Shleifer and Vishny (1997) emphasize that arbitrage is conducted by a fraction of investors with specialized knowledge, but similar to He and Krishnamurthy (2012), they focus on the effects of the agency frictions between arbitrageurs and their capital providers. Although we do not explicitly model risks to the liability side of investors' balance sheets, one can interpret the shocks agents in our model face to include idiosyncratic redemptions.⁶

Recently, the broader asset pricing impact of financially constrained intermediaries has been studied in the literature on intermediary asset pricing following He and Krishnamurthy (2012) and He and Krishnamurthy (2013).⁷ This literature applies results from the literature on asset

⁵See also Aiyagari and Gertler (1999), Froot and O'Connell (1999), Basak and Croitoru (2000), Xiong (2001), Gromb and Vayanos (2002), Yuan (2005), Gabaix, Krishnamurthy and Vigneron (2007), Mitchell, Pedersen and Pulvino (2007), Acharya, Shin and Yorulmazer (2009), Kondor (2009), Duffie (2010), Gromb and Vayanos (2010a), Hombert and Thesmar (2011), Edmond and Weill (2012), Mitchell and Pulvino (2012), Pasquariello (2013), and Kondor and Vayanos (2014).

⁶For other models of risks stemming from redemptions and fund outflows and the resulting asset pricing implications, see Berk and Green (2004), and Liu and Mello (2011).

⁷See also, for example, Adrian and Boyarchenko (2013). For empirical applications, see for example, Adrian, Etula and Muir (Forthcoming) and Muir (2014).

pricing with heterogeneous agents, following Dumas (1989), to segmented markets with financial constraints.⁸ In doing so, the intermediary asset pricing literature both connects to empirical applications, and to the asset price dynamics which are the focus of the limits to arbitrage literature. Finally, several papers develop alternative microfoundations to agency theory for segmented markets. Allen and Gale (2005) provide an overview of their theory of asset pricing based on “cash-in-the-market”. Plantin (2009) develops a model of learning by holding. Duffie and Strulovici (2012) develop a theory of capital mobility and asset pricing using search foundations. Glode, Green and Lowery (2012) study asset price dynamics in a model of financial expertise as an arms race in the presence of adverse selection. Kurlat (2013) studies an economy with adverse selection in which buyers vary in their ability to evaluate the quality of assets on the market, and, like us, emphasizes the distribution of expertise on the equilibrium price of the asset. Garleanu, Panageas and Yu (2014) derive market segmentation endogenously from differences in participation costs. Kacperczyk, Nosal and Stevens (2014) construct a model of consumer wealth inequality from differences in investor sophistication.

Our model is an example of an “industry equilibrium” model in the spirit of Hopenhayn (1992a) and Hopenhayn (1992b). These models study the important effects of firm dynamics, entry and exit in the heterogeneous agent framework developed in Bewley (1986). This literature focuses in large part on explaining firm growth, and moments describing the firm size distribution. Recent progress in the firm dynamics literature using continuous time techniques to solve for policy functions and stationary distributions include Miao (2005), Luttmer (2007), Gourio and Roys (2014), Moll (Forthcoming), and Achdou, Han, Lasry, Lions and Moll (2014). We draw on results in these papers as well as discrete time models of firm dynamics, as in recent work by Clementi and Palazzo (2013), which emphasizes the role of selection in explaining the observed relationships between firm age, size, and productivity. We also draw on work in the city size literature in Gabaix (1999) and the literature on the consumer wealth distribution with idiosyncratic risk and fiscal policy in Benhabib et al. (2014).

We use the the hedge fund industry, and in particular the asset backed fixed income (ABFI)

⁸For closely related work on asset pricing with heterogeneous risk aversion and segmented markets, see also Basak and Cuoco (1998), Kogan and Uppal (2001), Chien, Cole and Lustig (2011), and Chien, Cole and Lustig (forthcoming).

segment, for some motivating empirical moments describing size and performance. As such, we draw from the literature on hedge funds performance and compensation.⁹ In particular, we use ABFI funds as one example of a complex strategy using the evidence in Duarte, Longstaff and Yu (2006). They provide evidence that MBS strategies are relatively complex and earn higher returns even in comparison to other sophisticated fixed income arbitrage strategies. Several papers provide evidence for the importance of idiosyncratic risk in the hedge fund returns, following the idea in Merton (1987) that idiosyncratic risk will be priced when there are costs associated with learning about or hedging a specific asset.¹⁰ Relatedly, Fung and Hsieh (1997) find that hedge fund returns have low and sometimes negative correlation with asset class returns. Our model features investors with constant relative risk aversion (CRRA) preferences. While we do this for tractability and parsimony to retain our focus on the effects of the joint wealth and expertise distribution, Panageas and Westerfield (2009) show that hedge fund compensation contracts with long horizons lead to portfolio choice which aligns perfectly with that of a CRRA investor. Drechsler (2014) extends these results to include variation in managers' outside options and shows the CRRA result holds as long as such reservation values are neither too high nor too low. These results extend the analysis of the impact of high-water marks in Goetzmann et al. (2003).

3.3 Model

3.3.1 Preferences, Endowments, & Technologies

We study a model with a continuum of investors of measure one, with CRRA utility functions over consumption:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}.$$

⁹Fung, Hsieh, Naik and Ramadorai (2008) is a well known paper describing performance. Jagannathan, Malakhov and Novikov (2010) carefully correct for selection bias and smoothed returns in a study of hedge fund performance persistence. Carlson and Steinman (2007) consider the relationship between hedge fund survival and market conditions. In a related spirit to our work, Getmansky (2012) empirically studies the effects of size and competition on hedge fund returns.

¹⁰See Titman and Tiu (2011) and Lee and Kim (2014). Jurek and Stafford (Forthcoming) emphasize that scarce and specialized knowledge may drive both hedge fund returns and put pricing.

Investment Technology Investors are endowed with a level of expertise which varies in the cross section, but is fixed for each agent over time. Each individual investor is born with a fixed expertise level, x , drawn from a distribution with pdf $\lambda(x)$, and cdf $\Lambda(x)$.

Investors can choose to be experts, and have access to the complex risky asset, or non-experts, who can only invest in the risk free asset. Each investor's complex risky asset delivers a stochastic return which follows a geometric Brownian motion:

$$\frac{dP(t,s)}{P(t,s)} = [r_f + \alpha(s)] ds + \sigma(x) dB(t,s) \quad (3.1)$$

where $\alpha(s)$ is the common excess return on the risky asset and $B(t,s)$ is a standard Brownian motion which is investor specific and i.i.d. in the cross section. For parsimony, we suppress the dependence of the Brownian shock on investor i in our notation. The volatility of the risky technology $\sigma(x)$ decreases in the investor's level of expertise x , i.e. $\frac{\partial \sigma(x)}{\partial x} < 0$. For now, we focus on describing the equilibrium for a single asset, and we suppress the positive dependence of $\sigma(x)$ on the fundamental volatility of the asset class σ_v . Below, we describe comparative statics across assets with varying complexity, with more complex assets characterized by a higher σ_v , or "fundamental volatility". We refer to $\sigma(x)$ as "effective volatility", meaning the remaining fundamental volatility the investor faces after expertise has been applied. We require that $\lim_{x \rightarrow \infty} \sigma(x) = \underline{\sigma} > 0$. The lower bound on volatility, $\underline{\sigma}$, represents complex asset risk that cannot be eliminated even by the agents with the greatest expertise, and it guarantees that the growth rate of wealth is finite.

One interpretation of the return process in Equation (3.1) is that in order to invest in the risky asset and to earn the common market clearing return, an investor must also jointly invest in a technology with a zero mean return and an idiosyncratic shock. This technology represents each investor's specific hedging and financing technologies, as well as the unique features of their particular asset. According to its general definition, α cannot be generated by bearing systematic risk. However, capturing α is risky because it requires a model and execution, and each investor's model and execution technology is unique. For example, hedging portfolios tend to vary substantially across different investors in the same asset class.¹¹ We present an example of an explicit micro-

¹¹For example, for MBS, there is no agreed upon method to hedge mortgage duration risk, though most all active

foundation for equation (3.1) based on a long position in a fundamental asset and a short position in a tracking portfolio in the Appendix.

To be an expert, an investor must pay the entry cost F_{nx} to set up their specific technology for investing in the complex risky asset. Experts must also pay a maintenance cost, F_{xx} to maintain continued access to the risky technology. We specify that both the entry and maintenance costs are proportional to wealth:

$$\begin{aligned} F_{nx} &= f_{nx}w, \\ F_{xx} &= f_{xx}w, \end{aligned}$$

which yields value functions which are homogeneous in wealth.

Optimization We first describe the Bellman equations for non-experts and experts respectively, and characterize their value functions, as well as the associated optimal policy functions. With the value functions of experts and non-experts in hand, we then characterize the entry decision.

We begin with non-experts, who can only invest in the risk free asset. Let $w(t, s)$ denote the wealth of investors at time s with initial wealth W_t at time t . The riskless asset delivers a fixed return of r_f . All investors choose consumption, and an optimal stopping, or entry time according to the Bellman Equation:

$$V^n(w(t, s), x) = \max_{c^n(t, s), \tau} E \left[\int_t^\tau e^{-\rho(s-t)} u(c^n(t, s)) ds + e^{-\rho(\tau-t)} V^x(w(t, s) - F_{nx}, x) \right] \quad (3.2)$$

$$\text{s.t. } dw(t, s) = (r_f w(t, s) - c^n(t, s)) ds \quad (3.3)$$

where ρ is their subjective discount factor, $c(t, s)$ is consumption at time s , F_{nx} is the entry cost, and τ is the optimal entry date.

Under the assumptions of linear entry and maintenance costs, and expertise which is fixed over time, the optimal entry date in a stationary equilibrium will be either immediately, or never. Thus,

investors do so. Some hedge according to empirical durations, using various estimation periods and rebalancing periods. Others hedge according to the sensitivity of MBS prices yield curve shifts using their own (widely varying) proprietary model of MBS prepayments and prices.

assuming an initial stationary equilibrium, investors who choose an infinite stopping time are then non-experts, and investors who choose a stopping time $\tau = t$ are experts.¹²

Experts allocate their wealth between current consumption, a risky asset, and a riskless asset. They also choose an optimal stopping time to exit the market.

$$V^x(w(t,s),x) = \max_{c^x(t,s),T,\theta(x,t,s)} \mathbb{E} \left[\int_t^T e^{-\rho(s-t)} u(c^x(t,s)) ds + e^{-\rho(T-t)} V^n(w(t,s),x) \right] \quad (3.4)$$

$$\begin{aligned} \text{s.t. } dw(t,s) &= [w(t,s)(r_f + \theta(x,t,s)\alpha(t,s)) - c^x(t,s) - F_{xx}] ds \\ &+ w(t,s)\theta(x,t,s)\sigma(x)dB(t,s), \end{aligned} \quad (3.5)$$

where $\alpha(s)$ is the equilibrium excess return on the risky asset, $\theta(x,t,s)$ is the portfolio allocation to the risky asset by investors with expertise level x at time s , $c(t,s)$ is consumption, F_{xx} is the maintenance cost, and T is the optimal exit date. We include exit for completeness. However, exit will not occur in this homogeneous model with fixed expertise.

The following proposition states the analytical solutions for the value and policy functions in our model. We prove this Proposition by guess and verify in the Appendix.

Proposition 9 Value and Policy Functions: *The value functions are given by*

$$V^x(w(t,s),x) = y^x(x,t,s) \frac{w(t,s)^{1-\gamma}}{1-\gamma} \quad (3.6)$$

$$V^n(w(t,s),x) = y^n(x,t,s) \frac{w(t,s)^{1-\gamma}}{1-\gamma} \quad (3.7)$$

where $y^x(x)$ and $y^n(x)$ are given by:

$$y^x(x) = \left[\frac{(\gamma-1)(r_f - f_{xx}) + \rho}{\gamma} + \frac{(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)} \right]^{-\gamma} \text{ and} \quad (3.8)$$

$$y^n(x) = \left[\frac{(\gamma-1)r_f + \rho}{\gamma} \right]^{-\gamma}. \quad (3.9)$$

¹²Outside of a stationary equilibrium, because α is not constant, both entry and exit are possible.

The optimal policy functions $c^x(x, t, s)$, $c^n(t, s)$, and $\theta(x)$ are given by:

$$c^x(x, t, s) = [y^x(x)]^{-\frac{1}{\gamma}} w(t, s), \quad (3.10)$$

$$c^n(t, s) = [y^n(x)]^{-\frac{1}{\gamma}} w(t, s) \text{ and} \quad (3.11)$$

$$\theta(x, t, s) = \frac{\alpha(t, s)}{\gamma \sigma^2(x)}. \quad (3.12)$$

Furthermore, the wealth of experts evolves according to the law of motion:

$$\frac{dw(t, s)}{w(t, s)} = \left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma + 1) \alpha^2(t, s)}{2\gamma^2 \sigma^2(x)} \right) dt + \frac{\alpha(t, s)}{\gamma \sigma(x)} dB(t, s) \quad (3.13)$$

Finally, investors choose to be experts if the excess return earned per unit of wealth exceeds the maintenance cost per unit of wealth:

$$\frac{\alpha^2(t, s)}{2\sigma^2(x)\gamma} \geq f_{xx}. \quad (3.14)$$

We define \underline{x} as the lowest level of expertise amongst participating investors, for which Equation (3.14) holds with equality. Note that the law of motion for wealth is a sort of weighted average of the return to the risky and riskless assets, as determined by portfolio choice, net of consumption. The drift and volatility of investors' wealth are increasing in the allocation to the risky asset. This mechanism has important implications for the wealth distribution in the stationary equilibrium of our model.

3.3.2 The Distribution(s) of Expert Wealth

The total amount of wealth allocated to the complex risky asset, as well as the distribution of expert wealth across expertise levels, are key aggregate state variables for the the first and second moments of the equilibrium returns to the complex risky asset. Once the entry decision has been made, given that we do not clear the market for the riskless asset, the wealth of non-experts is irrelevant for the returns to the complex risky asset. We solve for the cross-sectional distribution of expert wealth in a stationary equilibrium of our model. Given that expertise is fixed over time

for each investor, constructing the wealth distribution at each expertise level is sufficient to obtain the cross-sectional joint distribution of wealth and expertise.

First, we note that in order to construct a stationary equilibrium given that experts' wealth on average grows over time, it is convenient to study the ratio $z(t, s)$ of individual wealth to the mean wealth of agents with highest expertise, $\mathbb{E}[w|_{\bar{x}}(t, s)]$.

$$z(t, s) \equiv \frac{w(t, s)}{\mathbb{E}[w|_{\bar{x}}(t, s)]}.$$

Next, note that the law of motion for the mean wealth of agents with a given level of expertise x is given by

$$d\mathbb{E}[w|_x(t, s)] \equiv [g(x)] dt.$$

where $g(x)$ will be determined in equilibrium. Define the average growth rate amongst agents with the “highest” level of expertise as $g(\bar{x}) \equiv \sup_x g(x)$. Then, the ratio $z(t, s)$ follows a geometric Brownian motion given by

$$\frac{dz(t, s)}{z(t, s)} = \left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1)\alpha^2(t, s)}{2\gamma^2\sigma^2(x)} - g(\bar{x}) \right) dt + \frac{\alpha(t, s)}{\gamma\sigma(x)} dB(t, s), \quad (3.15)$$

where $\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1)\alpha^2(t)}{2\gamma^2\sigma^2(x)} - g(\bar{x}) < 0$ represents the negative drift, or growth rate.

Let the cross-sectional p.d.f. of expert investors' wealth and expertise at time t be denoted by $\phi^x(z, x, t)$. Without additional assumptions, the relative wealth of lower expertise agents will shrink to zero. Two methods are commonly used to generate a stationary distribution. The first, for example used in Benhabib et al. (2014), is to employ a life cycle model, or Poisson elimination of agents. The second, employed by Gabaix (1999), is to introduce a reflecting barrier at a minimum wealth share, z_{\min} .¹³ We adopt the assumption of a minimum wealth share because it leads to a more elegant expression for the wealth distribution. Moreover, for asset pricing, only the higher ends of the wealth distribution are quantitatively relevant, so this elegance comes at a low cost.

¹³Gabaix (1999) constructs a model of the city size distribution, and thus his share variable represents relative population shares. See also the Appendix of that paper for a related method of constructing a stationary distribution using a Kesten (1973) process, which introduces a random shock with a positive mean to normalized city size.

We will show that the stationary distribution of wealth at each expertise level will be a Pareto distribution.¹⁴ Note that the reflecting barrier at z_{\min} implies that the growth rate of any individual agent, even those with the highest level of expertise, will grow more slowly than the mean wealth of the highest expertise agents.

Since the reflecting boundary mainly affects low wealth investors, decisions near the boundary matter little for equilibrium pricing. However, we adopt an interpretation of exit and entry at z_{\min} which ensures that policies are not distorted there. Then, since both time and state variables are continuous in our model, if policies are not distorted at z_{\min} , then they will not be distorted elsewhere. The strategy we employ is to ensure that the value at z_{\min} from adopting the optimal policy functions under non-reflecting wealth share dynamics is equal to the value of adopting those policies given that with some probability the investor will be punished by being forced to exit, and with some probability the investor will be rewarded by being able to infuse funds themselves, or by receiving new external funds. In the case of exit, we assume the investor is replaced by a new entrant with wealth share z_{\min} and the same level of expertise x as the exiting agent.¹⁵

We derive the Kolmogorov forward equations describing the evolution of the wealth distribution, taking $\alpha(t)$ as given, as follows:¹⁶

$$\begin{aligned} \partial_t \phi^x(z, x, t) &= -\partial_z \left[\left((r_f + \theta(x, t) \alpha(t, s)) - [y^x(x)]^{-\frac{1}{\gamma}} - f_{xx} - g(\bar{x}) \right) z \phi^x(z, x, t) \right] \\ &\quad + \frac{1}{2} \partial_{zz} \left([z \theta(x, t) \sigma(x)]^2 \phi^x(z, x, t) \right) \\ &= -\partial_z \left[\left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma + 1) \alpha^2(t, s)}{2\gamma^2 \sigma^2(x)} - g(\bar{x}) \right) z \phi^x(z, x, t) \right] \\ &\quad + \frac{1}{2} \partial_{zz} \left[\left(z \frac{\alpha(t, s)}{\gamma \sigma(x)} \right)^2 \phi^x(z, x, t) \right]. \end{aligned} \quad (3.16)$$

We then study the stationary distribution of wealth shares, in which $\partial_t \phi^x(z, x, t) = 0$. We take

¹⁴Adopting the assumption of Poisson death with a fixed initial wealth, for example, would instead lead to a double Pareto distribution, with a cutoff at the initial value of wealth. For example, see Benhabib et al. (2014) for the wealth distribution under the alternative assumption of Poisson elimination in a closely related model. This is also the assumption we adopt in our quantitative study in Eisfeldt et al. (2015). The alternative, initializing agents according to the stationary distribution involves solving a challenging fixed point problem.

¹⁵We discuss the interpretation we adopt in detail in the Appendix.

¹⁶See Dixit and Pindyck (1994) for a heuristic derivation, or Karlin and Taylor (1981) for more detail.

as given, for now, that $\alpha(t,s)$ will be constant, as in the stationary equilibrium we define in the following section. This will be true given a stationary distribution over investors' individual state variables. A stationary distribution of wealth shares $\phi^x(z,x)$ satisfies the following equation:

$$0 = -\partial_z \left[\left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1)\alpha^2}{2\gamma^2\sigma^2(x)} - g(\bar{x}) \right) z\phi^x(z,x) \right] + \frac{1}{2}\partial_{zz} \left[\left(z \frac{\alpha}{\gamma\sigma(x)} \right)^2 \phi^x(z,x) \right]. \quad (3.17)$$

We use guess and verify to show that the stationary distribution of wealth shares at each level of expertise is given by a Pareto distribution with an expertise specific tail parameter. This tail parameter, which we denote by β , is determined by the drift and volatility of the expertise specific law of motion for wealth shares. Intuitively, the larger the drift and volatility of the expertise specific wealth process, the fatter the tail of the wealth distribution at that level of expertise will be.

Proposition 10 *The stationary distribution of wealth shares $\phi^x(z,x)$ has the following form:*

$$\phi(z,x) \propto C(x)z^{-\beta(x)-1},$$

where

$$\begin{aligned} \beta(x) &= C_1 \frac{\sigma^2(x)}{\alpha^2} - \gamma \geq 1, \\ C_1 &= 2\gamma(f_{xx} + \rho - r_f + \gamma g(\bar{x})), \\ C(x) &= \frac{1}{\int z^{-\beta} dz} = \frac{C_1 \frac{\sigma^2(x)}{\alpha^2} - \gamma}{-C_1 \frac{\sigma^2(x)}{\alpha^2} + \gamma} \Big|_{z_{\min}}. \end{aligned}$$

See the Appendix for the Proof, where we also show that, in the stationary distribution, $\beta > 1$, which ensures a finite integral, and confirms that the distribution satisfies stationarity. The following Corollary, which we also prove in the Appendix, gives the tail parameters for the highest expertise agents, as well as all other investors.

Corollary 11 *For the highest expertise agents, we have*

$$\beta(\bar{x}) = \frac{1}{1 - z_{\min}/\bar{z}} = C_1 \frac{\sigma^2(\bar{x})}{\alpha^2} - \gamma$$

where \bar{z} is mean of normalized wealth of experts with highest expertise,

$$\bar{z} = \int_{z_{\min}}^{\infty} z\phi(z, \bar{x})dz = z_{\min} \left[1 + \frac{1}{\beta(\bar{x}) - 1} \right]$$

and

$$g(\bar{x}) = \frac{r_f - f_{xx} - \rho}{\gamma} + \frac{\alpha^2}{2\gamma\sigma^2(\bar{x})} + \frac{\alpha^2}{2\gamma^2\sigma^2(\bar{x})} \frac{1}{1 - z_{\min}/\bar{z}}$$

For all other expertise levels, we have

$$\beta(x) = \left(\gamma + \frac{z_{\min}/\bar{z}}{1 - z_{\min}/\bar{z}} \right) \frac{\sigma^2(x)}{\sigma^2(\bar{x})} - \gamma > 1. \quad (3.18)$$

The parameter β controls the tail of each expertise specific wealth distribution. The smaller is β , the more slowly the distribution decays, and the fatter is the upper tail. Clearly, β is an increasing function of risk aversion, γ , and an increasing function of expertise level volatility, $\sigma(x)$. The dependence of the tail parameter on expertise is given by $\frac{\sigma^2(x)}{\sigma^2(\bar{x})}$. Since expertise-specific effective volatility $\sigma(x)$ is decreasing in x , the wealth distribution of experts with a higher level of fixed expertise has a fatter tail. Investors with higher expertise allocate more wealth to the risky asset, which increases the mean and volatility of their wealth growth rate. Both a higher drift, and a wider distribution of shocks, lead to a fatter upper tail for wealth. Moreover, equation (3.18) shows that if the relation between expertise and effective volatility is steeper, then the difference in the size of the right tails of the wealth distribution across expertise levels increases. In equilibrium, variation in effective volatilities in complex asset markets will be driven both by the functional form for effective volatility, and by participation decisions which determine how different effective volatilities of participating agents will be. We can also measure the degree of wealth inequality within each expertise level as $\frac{1}{\beta(x)}$. High expertise levels exhibit greater size “inequality”, and again, if the relation between expertise and effective volatility is steeper, indicating a more complex asset, then the difference in size inequality within expertise levels increases.

It is intuitive that investing more in the risky asset leads to a fatter tailed wealth distribution. However, perhaps surprisingly, as Lemma 12 illustrates, not every parameter which increases difference in the fraction of wealth allocated to the risky asset leads to an increase in the degree of fat tails of the expertise specific wealth distributions. We show in Lemma 12 that, while differences in portfolio choice driven by differences in effective volatilities lead to greater differences in decay parameters, this is not true for variation in portfolio choice driven by higher excess returns or lower risk aversion. This result offers a unique prediction for our model of complexity as differences in risk vs. risk aversion. See the Appendix for the proof.

Lemma 12 Relation Between $\theta(x)$ and $\beta(x)$

Consider two level of expertise, x_{\min} and x_{\max} , we have

$$\theta(x_{\max}) - \theta(x_{\min}) = \frac{\alpha \sigma^2(x_{\min}) - \sigma^2(x_{\max})}{\gamma \sigma^2(x_{\max}) \sigma^2(x_{\min})},$$

and

$$\beta(x_{\max}) - \beta(x_{\min}) = 2\gamma^2 (f_{xx} + r - r_f + \gamma g(\bar{x})) \frac{\sigma^2(x_{\max}) \sigma^2(x_{\min})}{\alpha^3} [\theta(x_{\min}) - \theta(x_{\max})].$$

If a larger difference in portfolio choice is due to either a higher excess return or a lower risk aversion, the dispersion in β is smaller. If it is due to an increase in the difference in effective volatilities, then the difference in β 's is larger.

3.3.3 Aggregation and Stationary Equilibrium

In this section, we define a stationary equilibrium, and state the condition which determines the market clearing α in a stationary equilibrium. Slightly abusing notation by suppressing dependence on the distribution of wealth and expertise, or equivalently on α , we define aggregate investment in the complex risky asset to be I , given each sum of expertise level investment $I(x) \forall x$, where:

$$I = \int \lambda(x) I(x) dx. \tag{3.19}$$

We first define a stationary equilibrium. In order to ensure that the supply of the complex risky asset does not become negligible as investor wealth grows, we assume that the supply grows proportionally to the mean wealth of the highest expertise investors. That is, we assume:

$$S(t) = Sg(\bar{x})t.$$

For convenience, we assume that the support of expertise is bounded above by \bar{x} , although most of our results only require that $\sigma(x)$ satisfies $\lim_{x \rightarrow \infty} \sigma(x) = \underline{\sigma} > 0$.

Definition 13 *A stationary equilibrium consists of a market clearing α , policy functions for all investors, and a stationary distribution over investor types $i \in \{x, n\}$, expertise levels x , and wealth shares z , $\phi(i, z, x, t)$, such that given an initial wealth distribution, an expertise distribution $\lambda(x)$, and parameters $\{\gamma, \rho, S, r_f, f_{nx}, f_{xx}, \sigma_v\}$ the economy satisfies:*

1. *Investor optimality: Investors choose participation in the complex risky asset market according to Equation (3.14), as well as optimal consumption and portfolio choices $\{c^n(t), c(x, t), \theta(x, t)\}_{t=0}^{\infty}$ according to Equations (3.10)-(3.23), such that their utilities are maximized.*
2. *Market clearing: The equilibrium market clearing α is determined by equating supply and demand:*

$$S(t) = \int \lambda(x, t) \theta(x, t) W(x, t) dx.$$

In a stationary equilibrium, we have:

$$I \equiv \int \lambda(x) I(x) dx = S, \tag{3.20}$$

Define $Z(x)$ to be the total expertise level wealth share,

$$Z(x) = z_{\min} \left(1 + \frac{1}{\beta(x) - 1} \right).$$

Then, define $I(x)$ to be the detrended total expertise level investment in the complex risky

asset, namely,

$$I(x) = \frac{\alpha}{\gamma\sigma^2(x)}Z(x). \quad (3.21)$$

3. The distribution over all individual state variables is stationary, i.e. $\partial_t\phi(i, z, x, t) = 0$.

3.4 Results

3.4.1 Analytical Asset Pricing Results

With policy functions, stationary distributions, and the equilibrium definition in hand, we turn to our asset pricing results. We focus on the definition of a more complex asset as one that introduces more idiosyncratic risk. Comparing across assets, we use σ_v to denote the fundamental volatility of the asset before expertise is applied, so that the risk in each investor's asset is $\sigma(\sigma_v, x)$, and is increasing in the first argument, and decreasing in the second. We provide a specific example below, but begin with any general function satisfying two these properties. Importantly, we describe natural conditions under which more complex assets, or assets which introduce more idiosyncratic risk, have lower participation despite higher α 's and higher Sharpe ratios.

We begin by studying comparative statics over the equilibrium market clearing α . Although we focus on comparative statics over fundamental volatility, we also provide results for the market clearing α for changes other parameters which might proxy for asset complexity, such as the cost of maintaining expertise, or investor risk aversion. Next, we analyze individual Sharpe ratios. We emphasize heterogeneity across investors with different levels of expertise in changes in the risk return tradeoff as fundamental volatility changes. Because the other parameters which proxy for complexity do not change investor specific volatility, the results for individual Sharpe ratios are the same as those for α . Finally, we study market level Sharpe ratios, with a focus on the effects of the intensive and extensive margins of participation by investors with heterogeneous expertise.

Investor Demand, Aggregate Demand, and Equilibrium α We first describe the comparative statics for demand conditional on investors' expertise levels in Lemma 14.

Lemma 14 *Using Equation (3.21) for investor demand conditional on expertise, x , we have following comparative statics, $\forall x$:*

1. $\frac{\partial I(x)}{\partial \sigma^2(x)} < 0$
2. $\frac{\partial I(x)}{\partial \sigma_v} < 0$
3. $\frac{\partial I(x)}{\partial \alpha} > 0$
4. $\frac{\partial I(x)}{\partial \gamma} < 0$
5. $\frac{\partial I(x)}{\partial f_{xx}} < 0$

Demand for the risky asset at each level of expertise is increasing in the squared investor specific Sharpe ratio, and it is increasing in α . Demand is decreasing in effective variance, fundamental volatility, risk aversion, and the maintenance cost.

With expertise level total demands in hand, we can construct comparative statics for aggregate demand. We cannot express the equilibrium excess return in closed form. However, the following Proposition shows that the equilibrium excess return, α , and aggregate demand, I , form a bijection. This uniqueness result in turn ensures that α can be numerically solved for as the unique fixed point to Equation (3.20).

Proposition 15 *Aggregate market demand for the complex risky asset is an increasing function of the excess return, α , and α and I form a bijection. Mathematically,*

$$\frac{\partial I}{\partial \alpha} > 0.$$

Proposition 16 provides comparative statics over the aggregate demand for the complex risky asset, I . Using the result in Proposition 15, these comparative statics also hold for α .

Proposition 16 *Using the market clearing condition, we have that the following comparative statics hold:*

1. $\frac{\partial I}{\partial \sigma_v} < 0$, thus α is an increasing function of fundamental risk
2. $\frac{\partial I}{\partial \gamma} < 0$, thus α is an increasing function of risk aversion
3. $\frac{\partial I}{\partial f_{xx}} < 0$, thus α is an increasing function of the maintenance cost.

Demand for the risky asset is decreasing in fundamental volatility, risk aversion, and the maintenance cost. As a result, α is increasing in fundamental volatility, risk aversion, and the maintenance cost. We argue that an increase in these parameters proxies for greater asset complexity, and thus that our model predicts that α will be higher in more complex asset markets.

We now turn to the effect of the efficiency of the joint distribution of wealth and expertise on equilibrium pricing. In particular, we demonstrate that the equilibrium required excess return on the complex risky asset is decreasing in the amount of wealth commanded by agents with higher levels of expertise. The wealth distribution at each expertise level is a Pareto distribution with an expertise specific tail parameter. By shifting the distribution of expertise rightward, leading to a new distribution with a relatively larger fraction of higher expertise investors, relatively more wealth will reside with agents with higher expertise. Thus, with any rightward shift, the joint distribution of wealth and expertise becomes more efficient. Moreover, because the wealth distribution at higher expertise levels exhibits fatter right tails, there is an additional direct effect on overall wealth from a rightward shift in the distribution of expertise. Accordingly, Proposition 17 shows that if the density of experts shifts to the right, then demand for the complex risky asset will increase, and the required equilibrium excess return will decrease. The equilibrium excess return is decreasing in the amount of wealth which resides in the hands of agents with higher expertise. Note that this result can also be interpreted to state that in asset markets in which higher levels of expertise are more widespread, or less rare, equilibrium required returns will be lower. We argue that the scarcity of relevant expertise is increasing with asset complexity, again implying a higher α in more complex markets. The proof appears in the Appendix.

Proposition 17 *If $\frac{\partial \sigma(x)}{\partial x} < 0$, and Λ_1 exhibits first-order stochastic dominance over Λ_2 , $I(\Lambda_1) \geq I(\Lambda_2)$. As a result $\alpha(\Lambda_1) < \alpha(\Lambda_2)$.*

Investor Specific Sharpe ratios, Investor Participation, and Market Level Sharpe ratios With the analysis of equilibrium excess returns in hand, we now turn to the equilibrium risk-return trade-off at the investor and market level as described by the investor-specific, and market level Sharpe ratios. We emphasize the variation across individual Sharpe ratios as a function of expertise; all investors face a common market clearing α , but their effective risk varies. For the market level Sharpe ratio, two effects are present. First, there is the effect of any changes on parameters on the individual Sharpe ratios of participants. Second, there is a selection effect, or the effect on participation. We provide a natural condition under which participation declines as the asset becomes more complex. We focus on the equally weighted market-level equilibrium Sharpe ratio in our analysis. In addition to offering cleaner comparative statics because it does not depend on investor portfolio choices and market shares, the equally weighted Sharpe ratio represents the expected Sharpe ratio that an investor who could pay a cost to draw from the expertise distribution above the entry cutoff would earn. In that sense, it is the “expected Sharpe ratio”. Note that the Sharpe ratio for non-experts is not defined.

Investor-specific Sharpe ratios: We define the investor-specific Sharpe Ratio as:

$$SR(x) = \frac{\alpha}{\sigma(x)}.$$

We provide results for how investor-specific Sharpe ratios change as fundamental volatility changes under the three possible cases for the elasticity of investor specific risk with respect to fundamental volatility in Proposition 18. The sign of this elasticity is a key determinant of our Sharpe ratio results.

Proposition 18 *The comparative statics for the investor-specific Sharpe ratios with respect to fundamental volatility depend on which of the three possible cases for the elasticity of investor-specific risk with respect to fundamental volatility applies:*

1. *Case 1, Constant Elasticity: If $\frac{\partial \log \sigma(x)}{\partial \log \sigma_v}$ is a constant, that is*

$$\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}}{\partial x} = 0,$$

we must have that $SR(x)$ is either an increasing or a decreasing function of fundamental risk for all expertise levels.

2. *Case 2, Increasing Elasticity: If $\frac{\partial \log \sigma(x)}{\partial \log \sigma_v}$ is an increasing function of expertise, that is*

$$\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}}{\partial x} > 0,$$

there is a cutoff level x^ , such that for all $x < x^*$, we have $\frac{\partial SR(x)}{\partial \sigma_v} > 0$; and for all $x > x^*$, we have $\frac{\partial SR(x)}{\partial \sigma_v} < 0$. Further, x^* exists if for any small $\varepsilon < 10^{-6}$*

$$(0, \varepsilon) \subseteq \left\{ \frac{\partial \log \sigma(x)}{\partial \log \sigma_v} \mid \text{for all } x \right\} \subseteq [0, \infty).$$

3. *Case 3 Decreasing Elasticity: If $\frac{\partial \log \sigma(x)}{\partial \log \sigma_v}$ is a decreasing function of expertise, that is*

$$\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}}{\partial x} < 0,$$

then there is a cutoff level x^ , such that for all $x < x^*$, we have $\frac{\partial SR(x)}{\partial \sigma_v} < 0$; and for all $x > x^*$, we have $\frac{\partial SR(x)}{\partial \sigma_v} > 0$.*

Proposition 18 demonstrates that the effect of an increase in fundamental volatility on individual Sharpe ratios varies in the cross section, except in Case 1. The intuition is that the change in investors' Sharpe ratios depends on the percentage change in α relative to the percentage change in effective volatility. The change in α is aggregate, the same for all investors. So, the changes in individual Sharpe ratios with respect to changes in fundamental volatility depend on the percentage changes in effective volatility relative to the percentage change in fundamental volatility. If this elasticity is the same for all investors (Case 1), then the percentage change in α relative to the

percentage change in effective volatility is the same for all investors. On the other hand, if the elasticity of effective volatility with respect to fundamental volatility is increasing in expertise (Case 2), then Sharpe ratios increase below a cutoff level of expertise and decrease above as fundamental volatility increases. This case is interesting if one interprets the increase in fundamental volatility as coming from a change in the asset which hurts incumbent higher expertise agents worse than potential new entrants. Finally, if this elasticity is declining in expertise, so that higher expertise investors can weather an increase in fundamental volatility better (Case 3), then Sharpe ratios increase above a cutoff level of expertise and decrease below. We focus our analysis on this case, because it is the only case which leads to the empirically plausible implication that more complex assets, with higher fundamental volatilities, have *lower* participation despite having persistently elevated excess returns. Thus, we argue that the decreasing elasticity case is the most relevant for describing a long-run, stationary equilibrium in a complex asset market.

Investor Participation: Before turning to the market-level Sharpe ratio, we describe investor participation. There are two key inputs into the market level risk return tradeoff. First, incumbents' individual Sharpe ratios change. Second, as equilibrium α changes, participation also changes. This selection effect plays a key role in determining comparative static results in general equilibrium. We show in the Appendix that participation increases with fundamental volatility in Cases 1 and 2 of Proposition 18. This is intuitive because α must increase with fundamental volatility σ_v in order to clear the market. If all elasticities of $\sigma(x)$ with respect to σ_v are the same, or if they are lower for lower expertise investors, then participation will increase with fundamental volatility. Thus, we focus on Case 3, and provide a natural condition under which participation declines as the asset becomes more complex and fundamental volatility increases.

Proposition 19 *Define the entry cutoff \underline{x} as in Equation (3.14). We have*

$$\frac{\partial \underline{x}}{\partial \sigma_v} > 0$$

if the following conditions hold:

1. $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}}{\partial x} < 0$, (Case 3 of Proposition 18) and

$$2. l_{\text{sup}}^{\sigma_v} > \left(1 + \frac{1}{1 + \frac{2}{\beta(x)} \frac{\beta(x)+\gamma}{\beta(x)-1}} \right) E \left[\frac{\partial \log \sigma(x)}{\partial \log \sigma_v} \mid x \geq \underline{x} \right],$$

where $l_{\text{sup}}^{\sigma_v}$ is defined to be the highest elasticity of all participating investors' effective volatility with respect to fundamental volatility.

The first condition, namely that the elasticity of effective volatility with respect to fundamental volatility is decreasing in expertise, is necessary for participation to decline as complexity, and fundamental volatility, increase. The second condition gives a sufficient condition which states that the elasticity of the lowest expertise agent who participates, i.e. the agent with the highest sensitivity of effective volatility to fundamental volatility, must be sufficiently different from the average. Intuitively, what is necessary for participation to decline as fundamental volatility increases is that there is enough variation in the effect of the change in fundamental volatility across agents with high and low expertise so that α does not need to increase enough to satisfy the marginal investor or entice lower expertise investors to participate. We argue that it is natural for more complex assets, in addition to exposing investors to more risk overall, to pose a larger difference in risk across investors with different levels of expertise. Under the conditions in Proposition 19 our model generates higher persistent α 's and lower participation, despite free entry, as fundamental volatility and asset complexity increase.

Equilibrium Market-Level Sharpe Ratio We define the equally weighted market equilibrium Sharpe ratio as:

$$SR^{ew} = E \left[\frac{\alpha}{\sigma(x)} \mid \frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx} \right].$$

We focus on comparative statics for the equally weighted market equilibrium Sharpe ratio for simplicity.¹⁷

Proposition 20 *The equally weighted market Sharpe Ratio is increasing with fundamental risk in general equilibrium, i.e.,*

$$\frac{\partial SR^{ew}}{\partial \sigma_v} > 0,$$

¹⁷See the Appendix for the definition of the value-weighted market equilibrium Sharpe ratio.

if:

1. Participation increases, $\frac{\partial x}{\partial \sigma_v} < 0$ or,
2. Participation decreases, $\frac{\partial x}{\partial \sigma_v} > 0$ and $l_{\text{sup}}^{\sigma_v} > \frac{d\Lambda(x)}{E\left[1 - \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v/\sigma_v} | x > \underline{x}\right]}$, where we restrict the the average elasticity of participants to be less than 1, so that the denominator is positive.

Condition 1 of Proposition 20 shows that the equally weighted market Sharpe ratio increases with fundamental volatility if participation increases. However, we argue that the more relevant case is in Condition 2, which covers the case when participation is lower when assets have higher fundamental volatility and are more complex. Note that the restriction that when fundamental risk is increased by 1%, the average effective vol is increased less by 1% is easily satisfied, as expertise reduces fundamental volatility. Thus, for the equally weighted market Sharpe ratio to increase with fundamental volatility while participation declines, the model first requires that agents with more expertise are less sensitive to increases in volatility (a necessary condition for participation to decline with fundamental volatility). The second condition is a sufficient condition that if there are many investors around the entry threshold, these investors do not have such low Sharpe ratios that the market Sharpe ratio is overwhelmed by their participation.¹⁸ Note also the similarity between the second conditions in Propositions 19 and 20. Both require the elasticity of the lowest participating investor to be sufficiently different from the average. Thus, another intuitive statement of the requirement in Condition 2 in Proposition 20 is that the average elasticity will be very different from that of the threshold investors if there are relatively few investors at the threshold.

We argue that the declining elasticity case of Proposition 18 is the most natural in a stationary equilibrium for complex assets with limited participation. Moreover, it seems reasonable to assume a distribution for expertise which does not put too much weight on investors near the threshold. For example, we show below that a log-normal distribution easily delivers the relevant result. Under the conditions in Proposition 20, our model delivers a rational explanation for why more complex assets have a higher α , a higher equally-weighted equilibrium market Sharpe ratio, but low participation, despite free entry. Intuitively, as in a standard industrial organization model, the

¹⁸We also note that the value weighted Sharpe ratio puts less weight on agents at the threshold, as they have a lower share of wealth and a smaller share of their wealth allocated to the risky asset.

superior volatility reduction technologies of more expert investors provide them with an excess of (risk-bearing) capacity, which serves to reduce the entry incentives of newcomers despite attractive conditions for incumbents.

3.4.2 Numerical Examples

This section presents complementary numerical results and comparative statics for Case 3 from Proposition 18, in which the elasticity of effective volatility with respect to fundamental volatility declines with expertise. Results for the other cases are available upon request. The model generates closed form policy functions and wealth distributions conditional on expertise levels. To provide intuition for the effects of equilibrium pricing, we provide the comparative statics in both partial equilibrium and general equilibrium. In partial equilibrium, the excess return is given exogenously, and held fixed, while aggregate demand (and hence implicitly supply) varies. In general equilibrium, the excess return is computed endogenously given a fixed supply of the risky asset. Because α and I form a bijection (Proposition 15 provides conditions for which they are one to one and onto), for any given supply of the complex risky asset, we can solve for the market equilibrium α in the following steps:

1. Choose an upper and a lower bound for α , namely α_1 and α_2 , ($\alpha_1 > \alpha_2$).
2. Let $\alpha = \frac{\alpha_1 + \alpha_2}{2}$, and compute the total demand for the risky asset

$$\int \lambda(x) I(x) dx$$

3. If $S - \int \lambda(x) I(x) dx < -10^{-4}$, let $\alpha_1 = \alpha$ and back to step 1; if $S - \int \lambda(x) I(x) dx > 10^{-4}$, let $\alpha_2 = \alpha$ and back to step 1; otherwise, STOP.

We provide results under specific parametric assumptions. Specifically, we specify that:

$$\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}}{\partial x} < 0, \sigma(x) = a + x^{-b} \sigma_v^2. \quad 19$$

¹⁹ x^{-b} can be replaced by any function $f(x)$ as long as $\frac{\partial f(x)}{\partial x} < 0$.

Our baseline parameters are summarized in Table 3.1. The time interval is one quarter. The risk-free rate is 1%. The discount factor is 1%. The maintenance cost is also 1%. The coefficient of relative risk-aversion is 5. The log-normal distribution of expertise has a mean of 0 and volatility of 5. The minimum wealth share is set to 0.05. The fundamental standard deviation of the risky asset return is 20%. We set $a = 0.0112$ and $b = 1$. This implies that the highest expertise investors can eliminate 47% of fundamental risk, and face an effective standard deviation of 10.6%.

Figure 3.1 studies the effects of changes in fundamental volatility, with more complex assets characterized by higher fundamental volatility. Starting in the top row, as fundamental volatility increases, demand for the risky asset in partial equilibrium decreases, implying a higher α in general equilibrium. The left hand side of the second row displays the entry cut-off, which is increasing in fundamental volatility, consistent with our result in Proposition 19. Accordingly, participation, graphed on the right hand side of the second row, declines. We note that participation declines by less in general equilibrium, due to the positive effect of fundamental volatility on α , but still the decline is nearly as large as in partial equilibrium given our parametric assumptions. Finally, the third row plots the equally weighted standard deviation of the risky asset returns, which are increasing in both partial and general equilibrium. The effect is magnified in general equilibrium because participation declines by more, and hence there is more positive selection to higher expertise investors, since α is held constant in partial equilibrium. Finally, the bottom right panel of Figure 3.1 shows that despite the fact that the equally weighted standard deviation is increasing, the larger, positive effect of the increase in α in general equilibrium implies that the equally weighted Sharpe ratio increases, consistent with Proposition 20. Thus, the numerical example confirms the model's ability to generate persistently higher α 's and larger Sharpe ratios, but lower participation despite free entry, for more complex assets characterized by higher fundamental volatility.

Finally, we present results on the size distribution of funds in our model, and in the data, across asset classes which are more and less complex. Although in the model, it is easy to define a complex asset as one with a higher fundamental volatility, fundamental volatility (before expertise is applied) is unobservable in the data. Thus, we use the implication of our model that Sharpe ratios are higher in more complex asset classes. We use the subset of the Hedge Fund Research (HFR) data which describes Relative Value fund performance, as these strategies are likely to involve

long-short positions as in the micro-foundation for our return process. We compute “pseudo” Sharpe ratios as the ratio of the average industry level return to the time series average of the cross section standard deviation of returns. We then rank strategies from most to least complex by these pseudo Sharpe ratios. This ranking is essentially unchanged if we instead use the cross sectional average of time series standard deviations of returns by fund in the denominator. We note also that the time series average of the cross section standard deviation of returns and the cross sectional average of time series standard deviations of returns by fund are very similar supports the structure of our stationary model.

The top panel of Figure 3.2 displays the relative concentration of wealth across strategies in the HFR Relative Value data by plotting the cumulative wealth shares by wealth decile. Although the relationship is not quite monotonic, on average the more complex, higher Sharpe ratio strategies display lower concentration. The bottom panel of Figure 3.2 plots the relative concentration of wealth in the model across strategies with varying levels of complexity, given by the level of fundamental volatility. The model generates the pattern seen in the data; more complex strategies have less wealth concentration. This might seem surprising given that in our model high expertise agents have fatter tailed wealth distributions, and have Sharpe ratios (and hence portfolio allocations to the risky asset) which increase with fundamental volatility. The reason more complex assets have less concentrated wealth distributions in the model are twofold. First, participation is limited, so agents who are in the market cannot have too different of individual Sharpe ratios. Second, our specification for effective volatility has a positive second derivative, therefore there are essentially decreasing returns to expertise. As a result, agents who participate in the most complex asset classes are not as different from each other as those in less complex asset markets, which results in a less concentrated wealth distribution.²⁰

²⁰Our model does a better job matching the upper end of the wealth distribution than the lower end. This is a well-known problem in models of the wealth distribution featuring a Pareto distribution. See Castaneda et al. (2003) for a review of the literature, and specifically Table 1 for the errors in six prominent models for the low end of the wealth distribution.

3.5 Conclusion

We study the equilibrium returns to complex risky assets in segmented markets with expertise. We show that required returns increase with asset complexity, as proxied for by higher fundamental volatility, higher costs of maintaining expertise, and by expertise being scarce in the population. We emphasize heterogeneity in the risk-return tradeoff faced by investors with different levels of expertise. Accordingly, we show that in our model, under reasonable conditions, improvements in market level Sharpe ratios can be accompanied by lower market participation, consistent with empirical observations. Finally, we describe the implications of our model for the industrial organization of markets for complex risky assets. Markets for more complex assets have a less concentrated size distribution, which we show is consistent with data on relative value hedge fund strategies.

Appendix

Table 3.1: Parameter values: numerical example

Parameter	Symbol	Value	Target
Discount factor	ρ	0.01	Annual interest rate
Risk-free rate	r_f	0.01	Annual interest rate
Risk aversion	γ	5	Data/mean portfolio choice
Entry cost	fnx	0.03	
Maintenance cost	fxx	0.01	
Risky asset supply	S	0.52	$\alpha = 5.5\%$
Volatility of risky asset return	σ_v	20%	
Mean of expertise process	μ_x	0	
Volatility of expertise process	σ_x	5	
Constant in σ_x^2	a	0.0112	
Slope of σ_x^2	b	1	
Minimum wealth share	z_{min}	0.05	

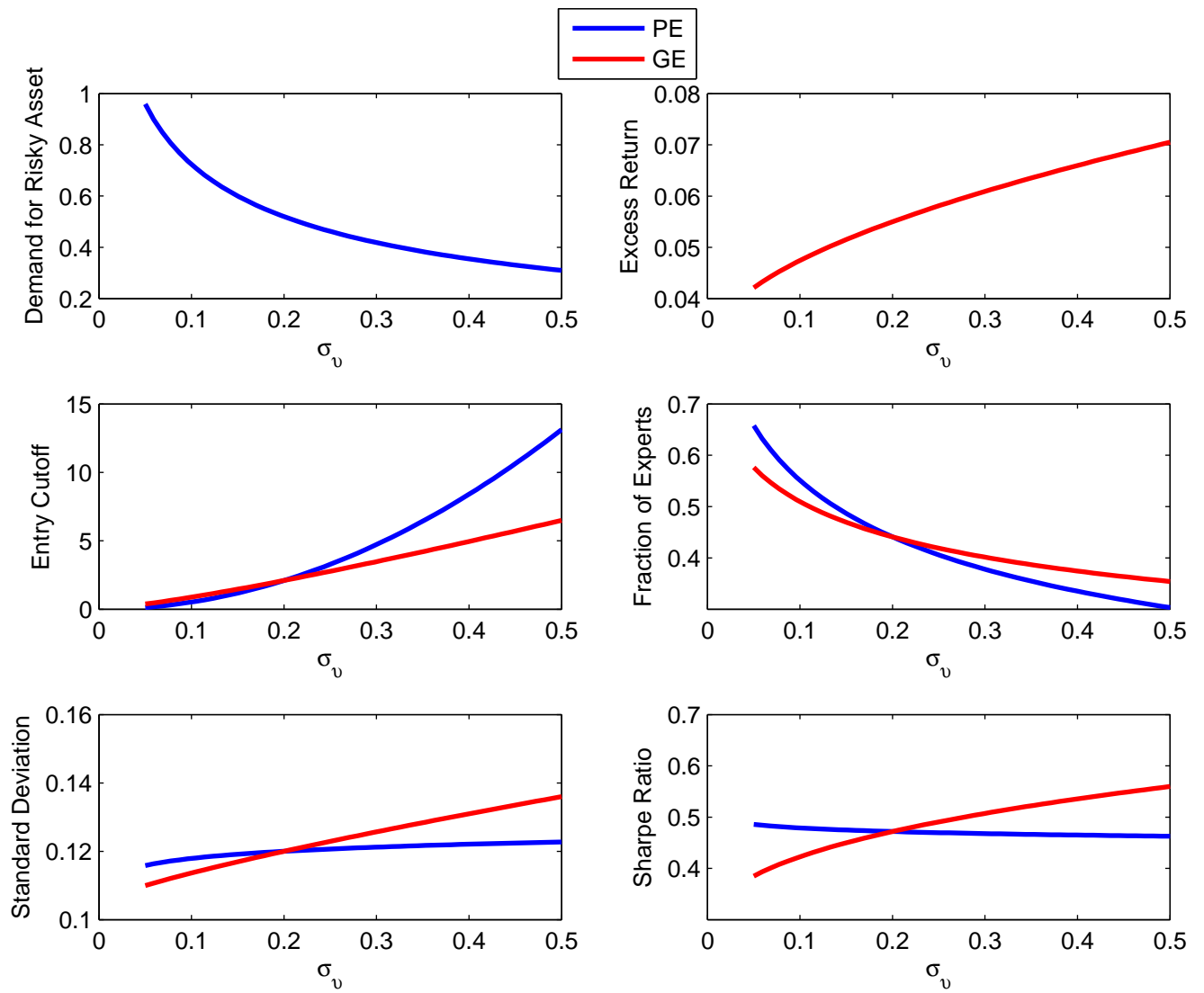


Figure 3.1: Case 3 Model comparative statics: fundamental risk

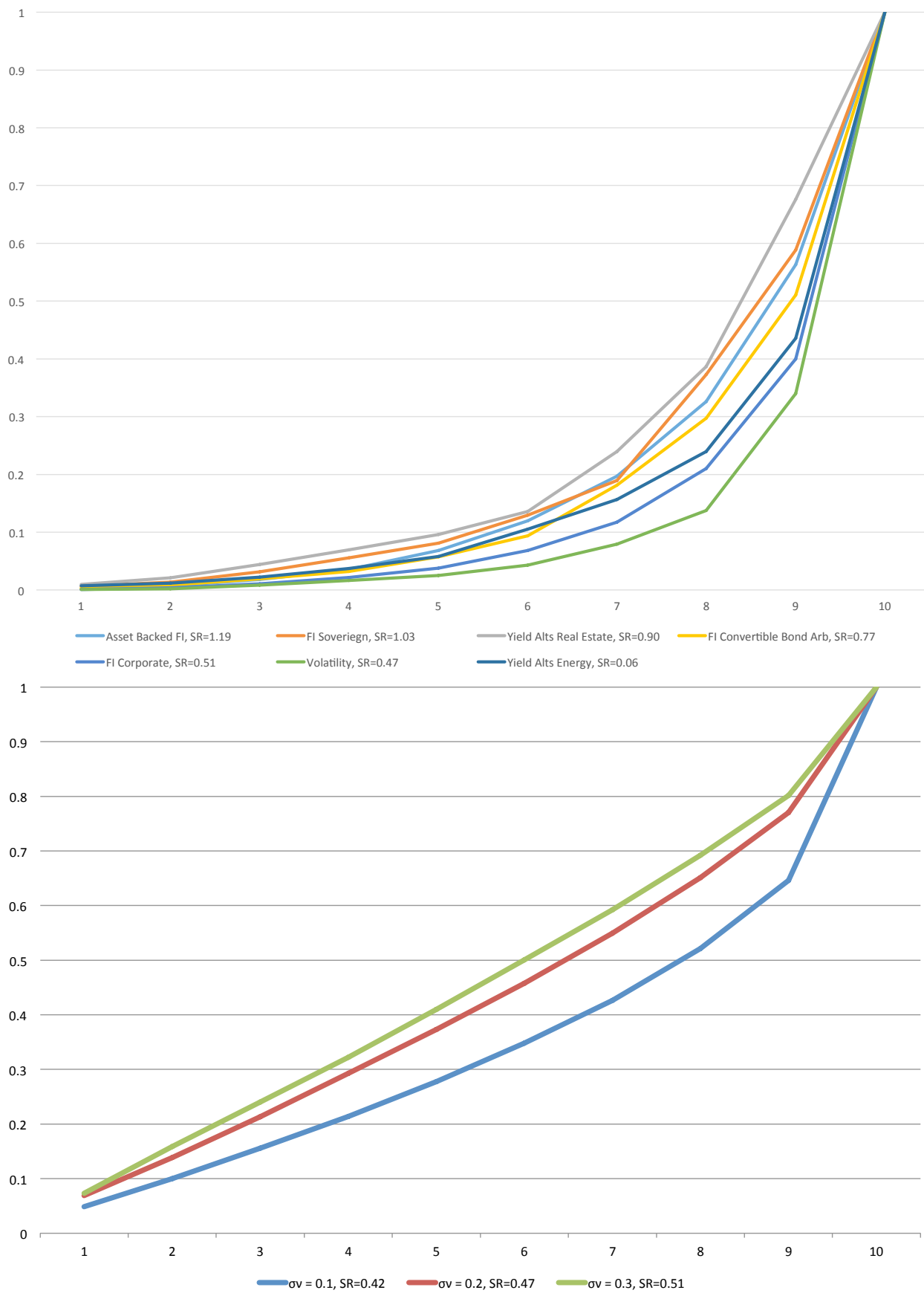


Figure 3.2: Cumulative wealth shares in the data (top) and model (bottom) across asset classes. Complex assets have higher Sharpe ratios, and (on average) lower concentration. FI = Fixed Income. Data is from HFR Relative Value strategies, excluding multi-strategy.

Complex Asset Return Process: Hedging with Tracking Portfolios. We construct an example motivation for the return process in Equation (3.1) based on executing an arbitrage opportunity via a long position in an underlying asset and a short position in a hedging or tracking portfolio. We interpret the α as the “mispricing” of the complex asset, and it is equal to the equilibrium return it earns because investors must bear idiosyncratic model risk to invest in the long-short strategy. There is an underlying complex asset, such as an MBS or convertible bond, which returns:

$$\frac{dU(t,s)}{U(t,s)} = [r_f + \alpha(s) + a(s)] dt + \sigma^U dB^U(t,s).$$

Investors have heterogeneous access to, or knowledge of, tracking or hedging portfolios. The value of each agent’s “best” tracking portfolio per unit of the underlying asset evolves according to $\frac{dT_i(t,s)}{T_i(t,s)}$. Thus, each agent takes a unit short position in their tracking portfolio for each unit long position they hold in the underlying asset $U(t,s)$.²¹ Tracking portfolio returns evolve according to:²²

$$\frac{dT_i(t,s)}{T_i(t,s)} = a(s)dt + \vartheta \sigma^U dB^U(t,s) - \sigma(x, \vartheta) dB_i^*(t,s),$$

where

$$\sigma^2(x, \vartheta) = \left(\frac{\vartheta}{h(x)} \right)^2 - (\vartheta \sigma^U)^2.$$

Note that, by definition, if the tracking portfolio returns are not perfectly correlated with the underlying asset returns (in which case there would exist a risk-less arbitrage opportunity), then the tracking portfolio will introduce independent risk. We assume this risk is uncorrelated across investors. Because each investor has their own model and strategy implementation, tracking portfolios introduce investor-specific shocks. We then use the fact that any Brownian shock which is partially correlated with the underlying Brownian shock $dB^U(t,s)$ can be decomposed into a linear combination of a correlated shock and an independent shock. We denote this independent, investor-specific shock $dB_i^*(t,s)$. Our assumption for the amount of idiosyncratic risk the tracking

²¹We do not clear the market for tracking portfolios. We instead argue that it is realistic to assume that demand for the tracking portfolio from hedging the complex asset is “small” relative to total demand.

²²Note we leave r_f out of the tracking portfolio return for parsimonious (and familiar) expressions for expert portfolio returns but this is without loss of generality. The equilibrium excess return will simply increase by r_f if the net asset’s drift is decreased by r_f .

portfolio introduces implies that this risk is larger the lower is ϑ , the loading on the underlying asset's Brownian shock, which is intuitive. Here, the effect of expertise on risk is captured by $h(x)$, with $h'(x) > 0$. Within an asset class, investors with higher expertise have superior models and tracking portfolios, hence they face lower risk. Across asset classes, more complex assets are characterized by more imperfect models and tracking portfolios, and hence more complex assets impose more risk on investors. For example, one can interpret a more complex asset as one for which $h(x)$ is lower for all agents.

Given these returns to the underlying asset and tracking portfolio, we have

$$\begin{aligned} \text{corr}(dB^F(t,s), dB_i^*(t,s)) &= 0, \\ \text{corr}\left(\frac{dU(t,s)}{U(t,s)}, \frac{dT_i(t,s)}{T_i(t,s)}\right) &= h(x)\sigma^U. \end{aligned}$$

Thus, investors with more expertise have tracking portfolios with a higher correlation with the underlying asset, as is intuitive. The linear form ensures that the idiosyncratic risk introduced by the tracking portfolio will remain even if there is no underlying risk, which is also intuitive, and consistent with our assumptions.

Returns for the net asset evolve according to

$$\frac{dU(t,s)}{U(t,s)} - \frac{dT_i(t,s)}{T_i(t,s)} = [r_f + \alpha(s)] dt + (1 - \vartheta)\sigma^U dB^U(t,s) + \sigma(x, \vartheta) dB_i^*(t,s)$$

We have for the net asset, then:

$$\begin{aligned} E\left(\frac{dU(t,s)}{U(t,s)} - \frac{dT_i(t,s)}{T_i(t,s)}\right) &= r_f + \alpha(s) \\ \text{Var}\left(\frac{dU(t,s)}{U(t,s)} - \frac{dT_i(t,s)}{T_i(t,s)}\right) &= [1 - \vartheta]^2 (\sigma^U)^2 + \left(\frac{\vartheta}{h(x)}\right)^2 - (\vartheta\sigma^U)^2 \\ &= \left(\frac{\vartheta}{h(x)}\right)^2 + (1 - 2\vartheta) (\sigma^U)^2 \end{aligned}$$

Since we abstract from aggregate risk, we study the case in which ϑ goes to one, which implies that, given our assumptions, the underlying Brownian risk drops out as follows. Taking $\vartheta \rightarrow 1$, we

have:

$$\frac{dF(t,s)}{F(t,s)} - \frac{dT_i(t,s)}{T_i(t,s)} = [r_f + \alpha(s)] dt + \sigma(x, \vartheta) dB_i^*(t,s).$$

We thus micro-found the return process in Equation (3.1) with the volatility of the independent shock given by:

$$\sigma(x, \vartheta) = \left(\frac{1}{h(x)} \right)^2 - (\sigma^U)^2,$$

where we note that $dB^U(t,s)$ drops out, leaving only the fixed parameter σ^U and a term which is decreasing in expertise.²³

Note that we can generate the example functional form from Section 3.4.2 by assuming the following for $h(x)$:

$$\sigma^2(x) = a + x^{-b} \sigma_v^2, \text{ implies } \left(\frac{1}{h(x)} \right)^2 = a + x^{-b} \sigma_v^2 + (\sigma^U)^2.$$

Proof. Proposition 9. We prove this Proposition by guess and verify. First, we write the HJB equations of our model

$$\begin{aligned} \max_{c^x(t,s), \theta(x,t,s)} 0 &= u(c^x(t,s)) + V_w^x [w(t,s) (r_f + \theta(x,t,s) \alpha(t,s)) - c^x(t,s) - f_{xx} w(t,s)] \\ &\quad + \frac{\theta^2(x) \sigma^2(x) w(t,s)^2}{2} V_{ww}^x - \rho V^x \\ \max_{c^n(t,s)} 0 &= u(c^n(t,s)) + V_w^n (r_f w(t,s) - c^n(t,s)) - \rho V^n \end{aligned}$$

The first order conditions are

$$u'(c^x(t,s)) = V_w^x,$$

$$u'(c^n(t,s)) = V_w^n,$$

$$V_w^x \alpha(t,s) + \theta(x,t,s) \sigma^2(x) w(t,s) V_{ww}^x = 0.$$

²³We note that if one instead takes $\sigma^F \rightarrow 0$, we have $\frac{dF(t,s)}{F(t,s)} - \frac{dT_i(t,s)}{T_i(t,s)} = [r_f + \alpha(s)] dt + \sigma(\vartheta) dB_i^*(t,s)$, where $\sigma^2(x, \vartheta) = \left(\frac{\vartheta}{h(x)} \right)^2$, which is also yields a micro-foundation consistent with our assumptions, and no aggregate risk.

Next, we guess that

$$\begin{aligned} V^x(w(t,s),x) &= y^x(x,t,s) \frac{w(t,s)^{1-\gamma}}{1-\gamma}, \\ V^n(w(t,s),x) &= y^n(x,t,s) \frac{w(t,s)^{1-\gamma}}{1-\gamma}. \end{aligned}$$

Thus

$$\begin{aligned} c^x &= [y^x(x,t,s)]^{-\frac{1}{\gamma}} w(t,s), \\ c^n &= [y^n(x,t,s)]^{-\frac{1}{\gamma}} w(t,s), \end{aligned}$$

and portfolio choice is given by

$$\theta(x,t,s) = \frac{\alpha(t,s)}{\gamma\sigma^2(x)}.$$

Plugging these choices into the HJB equations, we get

$$\begin{aligned} 0 &= [y^x(x,t,s)]^{-\frac{1-\gamma}{\gamma}} + y^x(x,t,s) \left(r_f + \frac{\alpha^2(t,s)}{\gamma\sigma^2(x)} - [y^x(x,t,s)]^{-\frac{1}{\gamma}} - f_{xx} \right) (1-\gamma) \\ &\quad - \frac{\alpha^2(t,s)}{2\gamma\sigma^2(x)} y^x(x,t,s) (1-\gamma) - \rho y^x(x,t,s) \\ &= \gamma [y^x(x,t,s)]^{-\frac{1-\gamma}{\gamma}} + y^x(x,t,s) \left(r_f + \frac{\alpha^2(t,s)}{2\gamma\sigma^2(x)} - f_{xx} \right) (1-\gamma) - \rho y^x(x,t,s), \\ 0 &= \gamma [y^n(x,t,s)]^{-\frac{1-\gamma}{\gamma}} + y^n(x,t,s) (1-\gamma) r_f - \rho y^n(x,t,s). \end{aligned}$$

Rearranging the equations, we solve for $y^x(x,t,s)$ and $y^n(x,t,s)$,

$$\begin{aligned} y^x(x,t,s) &= \left[\frac{(\gamma-1)(r_f - f_{xx}) + \rho}{\gamma} + \frac{(\gamma-1)\alpha^2(t,s)}{2\gamma^2\sigma^2(x)} \right]^{-\gamma}, \\ y^n(x,t,s) &= \left[\frac{(\gamma-1)r_f + \rho}{\gamma} \right]^{-\gamma}. \end{aligned}$$

Given all policy functions, we get the experts' wealth growth rates:

$$\frac{dw(t,s)}{w(t,s)} = \left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1)\alpha^2(t,s)}{2\gamma^2\sigma^2(x)} \right) dt + \frac{\alpha(t,s)}{\gamma\sigma(x)} dB(t,s)$$

Finally, given homogeneity of the value functions in wealth, the participation cutoff is constructed by direct comparison between $y^x(x, t, s)$ and $y^m(x, t, s)$. ■

Proof of equivalence of policy functions under the reflecting barrier z_{\min}

Interpretation of z_{\min} : We assume that one of two things can happen to an investor at z_{\min} . With probability q , the investor is eliminated from the market, and replaced with a new agent with wealth share z_{\min} and the same expertise as the exiting agent. Note that elimination in isolation would cause the incumbent agent to be conservative, to avoid z_{\min} . With probability $1 - q$, the agent is rewarded by being able to infuse funds themselves, or by receiving new external funds, and the wealth share reflects. Note that this reward in isolation would cause the agent to risk shift, to take advantage of limited liability at z_{\min} . We require that $E[V^x(z, x)_{true}] = qE[V^x(z, x)_{die}] + (1 - q)E[V^x(z, x)_{reflect}]$, conditional on the optimal policies under the true wealth share dynamics. Since the value under the true, non-reflecting, dynamics lies between the punishment value of dying and the reward value of reflection, we conjecture that there exists some probability, conditional on parameters, that this is the case. For simplicity, we assume that $V^x(z, x)_{die} = 0$. It seems quite realistic that investors face uncertainty about what will happen to them as their assets fall below a threshold level. Will they be liquidated, or rescued? Note that our proof offers a technical contribution, since in Gabaix (1999) cities do not choose size, unlike the case for our investors, who choose their savings and portfolio allocations.

We show that the optimal policies in the model with reflecting barrier z_{\min} are equivalent to those in the original model under our assumptions of a zero value at death, which is traded off with the positive value of reflection. Our proof assumes an optimal exit date. This is without loss of generality in a stationary equilibrium with no entry or exit.

Model 1:

$$\begin{aligned}
V^x(w(t,s),x) &= \max_{c^x(t,s),T,\theta(x,t,s)} \mathbb{E} \left[\int_t^T e^{-\rho(s-t)} u(c^x(t,s)) ds + e^{-\rho(T-t)} V^n(w(t,s),x) \right] \\
\text{s.t. } dw(t,s) &= [w(t,s)(r_f + \theta(x,t,s)\alpha(t,s)) - c^x(t,s) - F_{xx}] ds \\
&\quad + w(t,s)\theta(x,t,s)\sigma(x)dB(t,s),
\end{aligned}$$

Model 2:

$$\begin{aligned}
V^y(w(t,s),y) &= \max_{c^y(t,s),T,\theta(y,t,s)} \max \left\{ V^x(w(t,s),y), \mathbb{E} \left[\begin{aligned} &\int_t^{s'} e^{-\rho(s-t)} u(c^y(t,s)) ds \\ &+ (1-q)e^{-\rho(s'-t)} V^y(w_{\min},y) \end{aligned} \right] \right\} \\
\text{s.t. } dw(t,s) &= [w(t,s)(r_f + \theta(y,t,s)\alpha(t,s)) - c^y(t,s) - F_{yy}] ds \\
&\quad + w(t,s)\theta(y,t,s)\sigma(y)dB(t,s)
\end{aligned}$$

Assume $F_{xx} = F_{yy}$. They are both linear in wealth. By definition, we have

$$V^y(w(t,s),x) = (1-q)V^y(w_{\min},x), \text{ for } w(t,s) \leq w_{\min}.$$

Define

$$q(w(t,s),w_{\min}) = 1 - \left[\frac{w(t,s)}{w_{\min}} \right]^{1-\gamma}, \text{ for } w(t,s) \leq w_{\min}.$$

Therefore, we have

$$V^x(w(t,s),x) = (1-q)V^x(w_{\min},x), \text{ for } w(t,s) \leq w_{\min}.$$

It suffices to show that

$$V^y(w(t,s),x) = V^x(w(t,s),x), \text{ for all } x \text{ and } w(t,s),$$

when agent's wealth hits w_{\min} before he/she exits the market. That is

$$\begin{aligned}
V^y(w(t,s),y) &= \max_{c^y(t,s),\theta(y,t,s)} \mathbb{E} \left[\int_t^{s'} e^{-\rho(s-t)} u(c^y(t,s)) ds + (1-q) e^{-\rho(s'-t)} V^y(w_{\min},y) \right] \\
\text{s.t. } dw(t,s) &= [w(t,s)(r_f + \theta(y,t,s)\alpha(t,s)) - c^y(t,s) - F_{yy}] ds \\
&\quad + w(t,s)\theta(y,t,s)\sigma(y)dB(t,s)
\end{aligned}$$

First,

$$\begin{aligned}
V^y(w_{\min},x) &= \mathbb{E} \left[\int_t^{s'} e^{-\rho(s-t)} u(c^y(t,s)) ds + (1-q) e^{-\rho(s'-t)} V^y(w_{\min},x) \right] \\
&\geq \mathbb{E} \left[\int_t^{s'} e^{-\rho(s-t)} u(c^x(t,s)) ds + (1-q) e^{-\rho(s'-t)} V^y(w_{\min},x) \right],
\end{aligned}$$

that is,

$$\begin{aligned}
&E \int_t^{s'} e^{-\rho(s-t)} u(c^x(t,s)) ds \\
&\leq E \int_t^{s'} e^{-\rho(s-t)} u(c^y(t,s)) ds \\
&= \frac{1}{1 - \mathbb{E} [(1-q) e^{-\rho(s'-t)}]} V^y(w_{\min},x).
\end{aligned}$$

Second,

$$\begin{aligned}
V^x(w_{\min},x) &= \mathbb{E} \left[\int_t^{s'} e^{-\rho(s-t)} u(c^x(t,s)) ds + e^{-\rho(s'-t)} V^x(w(t,s'),x) \right] \\
&= \mathbb{E} \left[\int_t^{s'} e^{-\rho(s-t)} u(c^x(t,s)) ds + (1-q) e^{-\rho(s'-t)} V^x(w_{\min},x) \right] \\
&\geq \mathbb{E} \left[\int_t^{s'} e^{-\rho(s-t)} u(c^y(t,s)) ds + (1-q) e^{-\rho(s'-t)} V^x(w_{\min},x) \right],
\end{aligned}$$

that is,

$$\begin{aligned}
& E \int_t^{s'} e^{-\rho(s-t)} u(c^y(t, s)) ds \\
& \leq E \int_t^{s'} e^{-\rho(s-t)} u(c^x(t, s)) ds \\
& = \frac{1}{1 - \mathbb{E}[(1-q)e^{-\rho(s'-t)}]} V^x(w_{\min}, x).
\end{aligned}$$

Therefore, we must have

$$E \int_t^{s'} e^{-\rho(s-t)} u(c^y(t, s)) ds = E \int_t^{s'} e^{-\rho(s-t)} u(c^x(t, s)) ds,$$

and

$$V^y(w_{\min}, x) = V^x(w_{\min}, x).$$

Next,

$$\begin{aligned}
V^y(w(t, s), x) &= \mathbb{E} \left[\int_t^{s'} e^{-\rho(s-t)} u(c^y(t, s)) ds + (1-q)e^{-\rho(s'-t)} V^y(w_{\min}, x) \right] \\
&\geq \mathbb{E} \left[\int_t^{s'} e^{-\rho(s-t)} u(c^x(t, s)) ds + (1-q)e^{-\rho(s'-t)} V^x(w_{\min}, x) \right] \\
&= \mathbb{E} \left[\int_t^{s'} e^{-\rho(s-t)} u(c^x(t, s)) ds + e^{-\rho(s'-t)} V^x(w(t, s'), x) \right] \\
&= V^x(w(t, s), x), \text{ for all } w(t, s)
\end{aligned}$$

with equality iff $c^x(t, s) = c^y(t, s)$ and $\theta^x(x, t, s) = \theta^y(x, t, s)$.

Lastly,

$$\begin{aligned}
V^x(w(t, s), x) &= \mathbb{E} \left[\int_t^{s'} e^{-\rho(s-t)} u(c^x(t, s)) ds + e^{-\rho(s'-t)} V^x(w(t, s'), x) \right] \\
&\geq \mathbb{E} \left[\int_t^{s'} e^{-\rho(s-t)} u(c^y(t, s)) ds + (1-q)e^{-\rho(s'-t)} V^y(w_{\min}, x) \right] \\
&= V^y(w(t, s), x), \text{ for all } w(t, s)
\end{aligned}$$

with equality iff $c^x(t, s) = c^y(t, s)$ and $\theta^x(x, t, s) = \theta^y(x, t, s)$.

Therefore,

$$\begin{aligned} V^y(w(t, s), x) &= V^x(w(t, s), x), \text{ for all } x \text{ and } w(t, s). \\ c^x(t, s) &= c^y(t, s), \\ \theta^x(x, t, s) &= \theta^y(x, t, s). \end{aligned}$$

Proof. Proposition 10 We prove this Proposition by guess and verify. We guess that:

$$\phi(z, x) = C(x)z^{-\beta(x)-1},$$

Then, we have

$$\begin{aligned} 0 &= -\partial_z \left(z^{-\beta(x)} \left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1)\alpha^2}{2\gamma^2\sigma^2(x)} - g(\bar{x}) \right) \right) \\ &\quad + \frac{1}{2} \partial_{zz} \left(z^{1-\beta(x)} \frac{\alpha^2}{\gamma^2\sigma^2(x)} \right) \\ &= \beta(x) \left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1)\alpha^2}{2\gamma^2\sigma^2(x)} - g(\bar{x}) \right) \\ &\quad - \frac{1}{2} \beta(x)(1-\beta(x)) \left[\frac{\alpha}{\gamma\sigma(x)} \right]^2 \\ &= \beta(x) \left[\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{\alpha^2(\gamma+\beta(x))}{2\gamma^2\sigma^2(x)} - g(\bar{x}) \right] \end{aligned}$$

Thus

$$\begin{aligned} \beta(x) &= C_1 \frac{\sigma^2(x)}{\alpha^2} - \gamma \geq 1, \\ C_1 &= 2\gamma(f_{xx} + \rho - r_f + \gamma g(\bar{x})), \\ C(x) &= \frac{1}{\int z^{-\beta-1} dz} = \frac{C_1 \frac{\sigma^2(x)}{\alpha^2} - \gamma}{-C_1 \frac{\sigma^2(x)}{\alpha^2} + \gamma} \Big|_{z_{\min}}. \end{aligned}$$

Note there are two roots of equation

$$0 = \beta(x) \left[\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{\alpha^2(\gamma+\beta(x))}{2\gamma^2\sigma^2(x)} - g(\bar{x}) \right].$$

We only take the root that is larger than 1 to ensure the mean wealth has a finite mean. ■

Proof. Corollary 11. For the highest expertise agents, we have

$$\bar{z} = \int_{z_{\min}}^{\infty} z \phi(z, \bar{x}) dz = \int_{z_{\min}}^{\infty} C z^{-\beta(\bar{x})} dz = z_{\min} \left[1 + \frac{1}{\beta(\bar{x}) - 1} \right].$$

This gives us another expression of $\beta(\bar{x})$,

$$\beta(\bar{x}) = \frac{1}{1 - z_{\min}/\bar{z}}.$$

Also, we know

$$\beta(\bar{x}) = 2\gamma(f_{xx} + \rho - r_f + \gamma g(\bar{x})) \frac{\sigma^2(\bar{x})}{\alpha^2} - \gamma$$

Therefore, we have

$$2\gamma(f_{xx} + \rho - r_f + \gamma g(\bar{x})) \frac{\sigma^2(\bar{x})}{\alpha^2} - \gamma = \frac{1}{1 - z_{\min}/\bar{z}},$$

Rearrange the above equation, we get

$$g(\bar{x}) = \frac{r_f - f_{xx} - \rho}{\gamma} + \frac{\alpha^2}{2\gamma\sigma^2(\bar{x})} + \frac{\alpha^2}{2\gamma^2\sigma^2(\bar{x})} \frac{1}{1 - z_{\min}/\bar{z}}.$$

Plug $g(\bar{x})$ into $\beta(x)$, we derive

$$\beta(x) = \left(\gamma + \frac{z_{\min}/\bar{z}}{1 - z_{\min}/\bar{z}} \right) \frac{\sigma^2(x)}{\sigma^2(\bar{x})} - \gamma.$$

■

Proof. Lemma 12

$$\begin{aligned}\text{Recall that: } \theta(x) &= \frac{\alpha}{\gamma \sigma^2(x)} \\ \beta(x) &= 2\gamma(f_{xx} + r - r_f + \gamma g(\bar{x})) \frac{\sigma^2(x)}{\alpha^2} - \gamma\end{aligned}$$

Consider two levels of expertise, x_{\min} and x_{\max} , we have

$$\begin{aligned}\theta(x_{\max}) - \theta(x_{\min}) &= \frac{\alpha}{\gamma} \left[\frac{1}{\sigma^2(x_{\max})} - \frac{1}{\sigma^2(x_{\min})} \right] \\ &= \frac{\alpha}{\gamma} \frac{\sigma^2(x_{\min}) - \sigma^2(x_{\max})}{\sigma^2(x_{\max}) \sigma^2(x_{\min})},\end{aligned}$$

and

$$\begin{aligned}\beta(x_{\max}) - \beta(x_{\min}) &= 2\gamma(f_{xx} + r - r_f + \gamma g(\bar{x})) \frac{1}{\alpha^2} [\sigma^2(x_{\max}) - \sigma^2(x_{\min})] \\ &= 2\gamma^2(f_{xx} + r - r_f + \gamma g(\bar{x})) \frac{\sigma^2(x_{\max}) \sigma^2(x_{\min})}{\alpha^3} [\theta(x_{\min}) - \theta(x_{\max})].\end{aligned}$$

If a larger dispersion of portfolio choice is due to either a higher excess return or a lower risk aversion, the dispersion in β is smaller, since:

$$\begin{aligned}\frac{\partial [\beta(x_{\max}) - \beta(x_{\min})]}{\partial \alpha} &< 0, \text{ and } \frac{\partial [\theta(x_{\min}) - \theta(x_{\max})]}{\partial \alpha} > 0 \\ \frac{\partial [\beta(x_{\max}) - \beta(x_{\min})]}{\partial \gamma} &> 0, \text{ and } \frac{\partial [\theta(x_{\min}) - \theta(x_{\max})]}{\partial \gamma} < 0\end{aligned}$$

Consider the case where $\sigma^2(x_{\max}) \sigma^2(x_{\min})$ is a constant, then

$$\frac{\partial [\beta(x_{\max}) - \beta(x_{\min})]}{\partial [\theta(x_{\min}) - \theta(x_{\max})]} = 2\gamma^2(f_{xx} + r - r_f + \gamma g(\bar{x})) \frac{\sigma^2(x_{\max}) \sigma^2(x_{\min})}{\alpha^3} > 0.$$

A larger dispersion in portfolio choice, resulting from a larger difference between effective volatility, implies a larger dispersion of tail distribution. The condition on the product of the effective variances is not necessary, however, as can be seen by simple algebra. ■

Proof. Proof of Lemma 14 Direct calculation. We use 1 to denote a positive sign.

First,

$$\begin{aligned}\log I(x) &= \log \frac{\alpha}{\gamma \sigma^2(x)} + \log Z(x) \\ &= \log \alpha - \log \gamma - \log \sigma^2(x) + \log Z(x).\end{aligned}$$

We have

$$\begin{aligned}\text{sign} \left(\frac{\partial I(x)}{\partial \sigma^2(x)} \right) &= \text{sign} \left(\frac{\partial \log I(x)}{\partial \sigma^2(x)} \right) \\ &= \text{sign} \left(-1 - \frac{1}{Z(x)} \frac{z_{\min}}{(\beta(x) - 1)^2} C_1 \frac{1}{\alpha^2} \right) \\ &= -1\end{aligned}$$

Second, for each level of expertise, we have

$$\begin{aligned}&\text{sign} \left(\frac{\partial I(x)}{\partial \sigma_v} \right) \\ &= \text{sign} \left(\frac{\partial I(x)}{\partial \sigma^2(x)} \frac{\partial \sigma^2(x)}{\partial \sigma_v} \right) \\ &= \text{sign} \left(\frac{\partial I(x)}{\partial \sigma^2(x)} \right) \text{sign} \left(\frac{\partial \sigma^2(x)}{\partial \sigma_v} \right) \\ &= -1.\end{aligned}$$

Third, for each level of expertise, we have

$$\begin{aligned}\text{sign} \left(\frac{\partial I(x)}{\partial \alpha} \right) &= \text{sign} \left(\frac{\partial \log I(x)}{\partial \alpha} \right) \\ &= \text{sign} \left(1 + \frac{2}{Z(x)} \frac{z_{\min}}{(\beta(x) - 1)^2} C_1 \frac{\sigma^2(x)}{\alpha^3} \right) \\ &= 1\end{aligned}$$

Fourth, for each level of expertise:

$$\begin{aligned}
\text{sign}\left(\frac{\partial I(x)}{\partial \gamma}\right) &= \text{sign}\left(\frac{\partial \log I(x)}{\partial \gamma}\right) \\
&= \text{sign}\left(-1 - \frac{1}{Z(x)} \frac{z_{\min}}{(\beta(x) - 1)^2} \left(\frac{\sigma^2(x)}{\alpha^2} \left(\frac{C_1}{\gamma} + 2\gamma g(\bar{x})\right) - 1\right)\right) \\
&\leq \text{sign}\left(-1 - \frac{1}{Z(x)} \frac{z_{\min}}{(\beta(x) - 1)^2} \left(\frac{\sigma^2(x)}{\alpha^2} \frac{C_1}{\gamma} - 1\right)\right) \\
&= -1
\end{aligned}$$

Lastly, for each level of expertise:

$$\begin{aligned}
\text{sign}\left(\frac{\partial I(x)}{\partial f_{xx}}\right) &= \text{sign}\left(\frac{\partial \log I(x)}{\partial f_{xx}}\right) \\
&= \text{sign}\left(-\frac{1}{Z(x)} \frac{z_{\min}}{(\beta(x) - 1)^2} \frac{\sigma^2(x)}{\alpha^2} 2\gamma\right) \\
&= -1
\end{aligned}$$

■

Proof. Proof of Proposition 15 For each level of expertise, we have

$$\text{sign}\left(\frac{\partial I(x)}{\partial \alpha}\right) = 1, \text{ for all } x \text{ such that } \frac{\alpha^2}{2\sigma^2(x)\gamma} \geq f_{xx}$$

And when α is higher, more experts enter. Thus

$$\frac{\partial I}{\partial \alpha} > 0.$$

■

Proof. Proof of Proposition 16 Direct calculation. We use 1 to denote a positive sign.

$$\begin{aligned} & \text{sign} \left(\frac{\partial I(x)}{\partial \sigma_v} \right) \\ &= \text{sign} \left(\frac{\partial I(x)}{\partial \sigma^2(x)} \frac{\partial \sigma^2(x)}{\partial \sigma_v} \right) \\ &= \text{sign} \left(\frac{\partial I(x)}{\partial \sigma^2(x)} \right) \text{sign} \left(\frac{\partial \sigma^2(x)}{\partial \sigma_v} \right). \end{aligned}$$

We also have

$$\text{sign} \left(\frac{\partial I(x)}{\partial \sigma^2(x)} \right) = -1$$

Thus for each level of expertise, when fundamental risk is higher, the demand for the complex risky asset is smaller. And when σ_v is higher, fewer experts enter the complex risky asset market.

Thus

$$\frac{\partial I}{\partial \sigma_v} < 0.$$

Next, for each level of expertise:

$$\text{sign} \left(\frac{\partial I(x)}{\partial \gamma} \right) = -1,$$

Lastly, for each level of expertise:

$$\text{sign} \left(\frac{\partial I(x)}{\partial f_{xx}} \right) = -1,$$

Therefore:

$$\frac{\partial I}{\partial \gamma} < 0 \text{ and } \frac{\partial I}{\partial f_{xx}} < 0$$

■

Proof. Proof of Proposition 17 We have

$$\text{sign} \left(\frac{\partial I(x)}{\partial x} \right) = \text{sign} \left(\frac{\partial I(x)}{\partial \sigma(x)} \frac{\partial \sigma(x)}{\partial x} \right) = 1$$

And

$$\begin{aligned}
I(\Lambda_1) - I(\Lambda_2) &= \int [\lambda_1(x) - \lambda_2(x)] I(x) dx \\
&= -I(x) [\Lambda_1(x) - \Lambda_2(x)] - \int \frac{\partial I(x)}{\partial x} [\Lambda_1(x) - \Lambda_2(x)] dx \\
&> 0
\end{aligned}$$

■

Proof. Proof of Proposition 18. Given

$$\frac{\partial SR(x)}{\partial \sigma_v} = \frac{\frac{\partial \alpha}{\partial \sigma_v} \sigma(x) - \alpha \frac{\partial \sigma(x)}{\partial \sigma_v}}{\sigma^2(x)}$$

we have

$$\frac{\partial SR(x)}{\partial \sigma_v} > 0 \text{ iff } \frac{\partial \log \sigma(x)}{\partial \log \sigma_v} < \frac{\partial \log \alpha}{\partial \log \sigma_v}.$$

If $\frac{\partial \log \sigma(x)}{\partial \log \sigma_v}$ is a constant, we must have either $\frac{\partial \log \alpha}{\partial \log \sigma_v} > \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}$ for all x or $\frac{\partial \log \alpha}{\partial \log \sigma_v} < \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}$ for all x .

If $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}}{\partial x} < 0$, and assume there is a cutoff x^* such that

$$\frac{\partial \log \sigma(x^*)}{\partial \log \sigma_v} = \frac{\partial \log \alpha}{\partial \log \sigma_v},$$

then for all $x < x^*$, we have $\frac{\partial SR(x)}{\partial \sigma_v} < 0$; and for all $x > x^*$, we have $\frac{\partial SR(x)}{\partial \sigma_v} > 0$.

If $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}}{\partial x} > 0$, and assume there is a cutoff x^* such that

$$\frac{\partial \log \sigma(x^*)}{\partial \log \sigma_v} = \frac{\partial \log \alpha}{\partial \log \sigma_v},$$

then for all $x < x^*$, we have $\frac{\partial SR(x)}{\partial \sigma_v} > 0$; and for all $x > x^*$, we have $\frac{\partial SR(x)}{\partial \sigma_v} < 0$. ■

Value Weighted Equilibrium Sharpe ratio The market value weighted Sharpe ratio can be written as

$$\begin{aligned}
SR^{vw} &= E \left[\frac{\theta Z(x)}{I} \frac{\alpha}{\sigma(x)} \middle| \frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx} \right] \\
&= E \left[\frac{\theta Z(x)}{I} \frac{\alpha}{\sigma(x)} \middle| \frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx} \right] \\
&= \frac{\alpha}{\gamma I} E \left[\frac{Z(x)}{\sigma^3(x)} \middle| \frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx} \right]
\end{aligned}$$

Participation: Intermediate results and proofs We begin by describing results for bounds on the elasticity of α with respect to changes in fundamental volatility, and the implications of these bounds for participation. First, we show that the percentage change in α has to be large enough to at least satisfy the investors whose risk-return tradeoff deteriorates the least as fundamental volatility increases.

Lemma 21 *In the equilibrium, we have*

$$\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} > l_{\text{inf}}^{\sigma_v},$$

where $l_{\text{inf}}^{\sigma_v}$ is the lowest elasticity of all participating investors' effective volatility with respect to fundamental volatility

$$l_{\text{inf}}^{\sigma_v} \equiv \inf \left\{ \frac{\partial \log \sigma(x)}{\partial \log \sigma_v} \middle| \frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx} \right\}.$$

Proof. Proof of Lemma 21 Proof by contradiction. Suppose σ_v is increased by 1%, but the equilibrium α is increased by less than $l_{\text{inf}}^{\sigma_v}\%$, that is

$$\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} \leq l_{\text{inf}}^{\sigma_v}$$

We have

1. Less participation: because $\frac{\alpha^2}{2\sigma^2(x)\gamma} = f_{xx}$ and $\frac{\partial\alpha/\alpha}{\partial\sigma_v/\sigma_v} < l_{\text{inf}}^{\sigma_v}$, \underline{x} is higher.
2. Less investment in the complex risky asset:

$$\begin{aligned}
& \frac{\partial \log I(x)}{\partial \sigma_v} \\
&= -\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v} + \frac{1}{\sigma_v} \left[1 + \frac{z_{\min}}{Z(x)} \frac{2(\beta(x) + \gamma)}{(\beta(x) - 1)^2} \right] \left[-\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v/\sigma_v} + \frac{\partial \alpha/\alpha}{\partial \sigma_v/\sigma_v} \right] \\
&= -\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v} + \frac{1}{\sigma_v} \left[1 + \frac{1}{\beta(x)} \frac{2(\beta(x) + \gamma)}{\beta(x) - 1} \right] \left[-\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v/\sigma_v} + \frac{\partial \alpha/\alpha}{\partial \sigma_v/\sigma_v} \right] \\
&< 0, \text{ for all } x.
\end{aligned}$$

Therefore, in the new equilibrium, the total demand for risky asset is less than the total supply.

Contradiction. It must be that

$$\frac{\partial \alpha/\alpha}{\partial \sigma_v/\sigma_v} > \inf \left\{ \frac{\partial \log \sigma(x)}{\partial \log \sigma_v} \mid \frac{\alpha^2}{2\sigma^2(x)\gamma} \geq f_{xx} \right\}.$$

■

We can also put an upper bound on the percentage change in α relative to the percentage change in fundamental volatility. The change will not be greater than twice the elasticity of the agent with the highest elasticity, which we prove by contradiction.

Lemma 22 *In the equilibrium, we have*

$$\frac{\partial \alpha/\alpha}{\partial \sigma_v/\sigma_v} < 2l_{\text{sup}}^{\sigma_v},$$

where $l_{\text{sup}}^{\sigma_v}$ is the highest elasticity of all participating investors' effective volatility with respect to fundamental volatility,

$$l_{\text{sup}}^{\sigma_v} \equiv \sup \left\{ \frac{\partial \log \sigma(x)}{\partial \log \sigma_v} \mid \frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx} \right\}.$$

Proof. Proof of Lemma 22 Proof by contradiction. Suppose σ_v is increased by 1%, but the equilibrium α is increased by more than $2l_{\text{sup}}^{\sigma_v}$ %, that is

$$\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} \geq 2l_{\text{sup}}^{\sigma_v}$$

We have

1. More participation: because $\frac{\alpha^2}{2\sigma^2(x)\gamma} = f_{xx}$ and $\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} \geq 2l_{\text{sup}}^{\sigma_v} > l_{\text{sup}}^{\sigma_v}$, \underline{x} is lower.
2. More investment in the complex risky asset:

$$\begin{aligned} & \frac{\partial \log I(x)}{\partial \sigma_v} \\ &= \left\{ -\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v} + \frac{1}{\sigma_v} \left[1 + \frac{z_{\min} 2(\beta(x) + \gamma)}{Z(x) (\beta(x) - 1)^2} \right] \left[-\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v / \sigma_v} + \frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} \right] \right\} \\ &= \frac{1}{\sigma_v} \left\{ -\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v / \sigma_v} + \left[1 + \frac{1}{\beta(x)} \frac{2(\beta(x) + \gamma)}{\beta(x) - 1} \right] \left[-\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v / \sigma_v} + \frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} \right] \right\} \\ &= \frac{1}{\sigma_v} \left\{ \left[1 + \frac{1}{\beta(x)} \frac{2(\beta(x) + \gamma)}{\beta(x) - 1} \right] \frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} - \left[2 + \frac{1}{\beta(x)} \frac{2(\beta(x) + \gamma)}{\beta(x) - 1} \right] \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v / \sigma_v} \right\} \\ &= \frac{1 + \frac{1}{\beta(x)} \frac{2(\beta(x) + \gamma)}{\beta(x) - 1}}{\sigma_v} \left\{ \frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} - \frac{2 + \frac{1}{\beta(x)} \frac{2(\beta(x) + \gamma)}{\beta(x) - 1}}{1 + \frac{1}{\beta(x)} \frac{2(\beta(x) + \gamma)}{\beta(x) - 1}} \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v / \sigma_v} \right\} \\ &\geq \frac{1 + \frac{1}{\beta(x)} \frac{2(\beta(x) + \gamma)}{\beta(x) - 1}}{\sigma_v} \left\{ 2 - \frac{2 + \frac{1}{\beta(x)} \frac{2(\beta(x) + \gamma)}{\beta(x) - 1}}{1 + \frac{1}{\beta(x)} \frac{2(\beta(x) + \gamma)}{\beta(x) - 1}} \right\} \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v / \sigma_v} \\ &= \frac{1 + \frac{1}{\beta(x)} \frac{2(\beta(x) + \gamma)}{\beta(x) - 1}}{\sigma_v} \left\{ 1 - \frac{1}{1 + \frac{1}{\beta(x)} \frac{2(\beta(x) + \gamma)}{\beta(x) - 1}} \right\} \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v / \sigma_v} \\ &> 0, \end{aligned}$$

Therefore

$$\frac{\partial \log I(x)}{\partial \sigma_v} > 0$$

Therefore, in the new equilibrium, the total demand for risky asset is more than the total supply.

Contradiction. It must be that

$$\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} < 2l_{\text{sup}}^{\sigma_v}.$$

■

The following lemma describes bounds on the percentage change in α for a given percentage change in fundamental volatility for the case of decreasing elasticities of effective volatility with respect to fundamental volatility (Case 3 of Proposition 18). We show that the percentage change in α for a given percentage change in fundamental volatility will be greater than the highest elasticity of effective volatility with respect to fundamental volatility (displayed by the participating investor with the lowest expertise) if that highest elasticity is less than a constant times the average elasticity over participating investors. The constant will be near one if β is close to one, which it will be as it is the tail parameter from a Pareto distribution. Note we derive a sufficient condition which is based on the wealth distribution of the highest expertise agents, as using the entire distribution, a mixture of Pareto distributions, is more complicated but would yield similar intuition. We also show the converse: The percentage change in α for a given percentage change in fundamental volatility will be less than the highest elasticity of effective volatility with respect to fundamental volatility (displayed by the participating investor with the lowest expertise) if that highest elasticity is less than a constant near one times the average elasticity over participating investors. Case 3 of Proposition 18 is the only case which yields a decline in participation as fundamental volatility increases. It does so under natural conditions, related to these bounds. We show below that participation increases if Condition 1 of Lemma 23 holds, but decreases if Condition 2 holds. Intuitively, participation will increase if the change in α is large enough to satisfy lower expertise investors in Case 3, but will decrease otherwise. Lemma 23 provides bounds on the percentage change in α for a given percentage change in fundamental volatility for Case 3. We provide a sufficient condition for participation to decline as fundamental volatility increases below.

Lemma 23 When $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}}{\partial x} \leq 0$, in the equilibrium, we have,

1.

$$\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} > l_{\text{sup}}^{\sigma_v} \text{ if } l_{\text{sup}}^{\sigma_v} < \left(1 + \frac{1}{1 + \frac{2}{\beta(\bar{x})} \frac{\beta(\bar{x}) + \gamma}{\beta(\bar{x}) - 1}} \right) E \left[\frac{\partial \log \sigma(x)}{\partial \log \sigma_v} \mid \frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx} \right].$$

and

2.

$$\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} < l_{\text{sup}}^{\sigma_v} \text{ if } l_{\text{sup}}^{\sigma_v} > \left(1 + \frac{1}{1 + \frac{2}{\beta(x)} \frac{\beta(x) + \gamma}{\beta(x) - 1}} \right) E \left[\frac{\partial \log \sigma(x)}{\partial \log \sigma_v} \mid \frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx} \right],$$

Proof. Proof of Lemma 23 In case 3, we have $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}}{\partial x} < 0$.

First, we show that $\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} > l_{\text{sup}}^{\sigma_v}$ if

$$l_{\text{sup}}^{\sigma_v} < \left(1 + \frac{1}{1 + \frac{2}{\beta(\bar{x})} \frac{\beta(\bar{x}) + \gamma}{\beta(\bar{x}) - 1}} \right) E \left[\frac{\partial \log \sigma(x)}{\partial \log \sigma_v} \mid \frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx} \right].$$

Proof by contradiction. Assume $\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} \leq l_{\text{sup}}^{\sigma_v} < \frac{2 + \frac{2}{\beta(\bar{x})} \frac{\beta(\bar{x}) + \gamma}{\beta(\bar{x}) - 1}}{1 + \frac{2}{\beta(\bar{x})} \frac{\beta(\bar{x}) + \gamma}{\beta(\bar{x}) - 1}} E \left[\frac{\partial \log \sigma(x)}{\partial \log \sigma_v} \mid x \geq \underline{x} \right]$, We have

- Less participation: because $\frac{\alpha^2}{2\sigma^2(x)\gamma} = f_{xx}$ and $\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} < l_{\text{sup}}^{\sigma_v}$, \underline{x} is higher.
- Less investment in the complex risky asset:

$$\begin{aligned} & \frac{\partial \log I(x)}{\partial \sigma_v} \\ &= -\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v} + \frac{1}{\sigma_v} \left[1 + \frac{z_{\min}}{Z(x)} \frac{2(\beta(x) + \gamma)}{(\beta(x) - 1)^2} \right] \left[-\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v / \sigma_v} + \frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} \right] \\ &= -\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v} + \frac{1}{\sigma_v} \left[1 + \frac{1}{\beta(x)} \frac{2(\beta(x) + \gamma)}{\beta(x) - 1} \right] \left[-\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v / \sigma_v} + \frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} \right] \end{aligned}$$

Thus,

$$\begin{aligned} \frac{\partial I}{\partial \sigma_v} &= \int_{\underline{x}}^{\infty} \frac{\partial I(x)}{\partial \sigma_v} d\Lambda(x) - I(x) d\Lambda(x) \Big|_{\sigma^2(x) = \frac{\alpha^2}{2\gamma f_{xx}}} \frac{\partial \underline{x}}{\partial \sigma_v} \\ &< E \left\{ -\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v} + \frac{1}{\sigma_v} \left[1 + \frac{1}{\beta(x)} \frac{2(\beta(x) + \gamma)}{\beta(x) - 1} \right] \left[-\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v / \sigma_v} + \frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} \right] \right\} \\ &< E \left\{ -\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v} + \frac{1}{\sigma_v} \left[1 + \frac{1}{\beta(\bar{x})} \frac{2(\beta(\bar{x}) + \gamma)}{\beta(\bar{x}) - 1} \right] \left[-\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v / \sigma_v} + l_{\text{sup}}^{\sigma_v} \right] \right\} \\ &= \frac{1}{\sigma_v} \frac{1}{\beta(\bar{x})} \frac{2(\beta(\bar{x}) + \gamma)}{\beta(\bar{x}) - 1} l_{\text{sup}}^{\sigma_v} - \frac{2}{\sigma_v} \frac{1}{\beta(\bar{x})} \frac{2(\beta(\bar{x}) + \gamma)}{\beta(\bar{x}) - 1} E \left\{ \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v} \right\} \\ &< 0 \end{aligned}$$

Therefore, in the new equilibrium, the total demand for the complex risky asset is less than the total supply. Contradiction. Therefore, it must be that

$$\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} > l_{\text{sup}}^{\sigma_v} = \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v / \sigma_v}.$$

Second, we show that $\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} < l_{\text{sup}}^{\sigma_v}$ if

$$l_{\text{sup}}^{\sigma_v} > \left(1 + \frac{1}{1 + \frac{2}{\beta(\underline{x})} \frac{\beta(\underline{x}) + \gamma}{\beta(\underline{x}) - 1}} \right) E \left[\frac{\partial \log \sigma(x)}{\partial \log \sigma_v} \mid \frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx} \right].$$

Proof by contradiction. Assume $\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} \geq l_{\text{sup}}^{\sigma_v} > \frac{2 + \frac{2}{\beta(\bar{x})} \frac{\beta(\bar{x}) + \gamma}{\beta(\bar{x}) - 1}}{1 + \frac{2}{\beta(\bar{x})} \frac{\beta(\bar{x}) + \gamma}{\beta(\bar{x}) - 1}} E \left[\frac{\partial \log \sigma(x)}{\partial \log \sigma_v} \mid x \geq \underline{x} \right]$, We have

- More participation: because $\frac{\alpha^2}{2\sigma^2(\underline{x})\gamma} = f_{xx}$ and $\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} > l_{\text{sup}}^{\sigma_v}$, \underline{x} is lower.
- More investment in the complex risky asset:

$$\begin{aligned} & \frac{\partial \log I(x)}{\partial \sigma_v} \\ &= -\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v} + \frac{1}{\sigma_v} \left[1 + \frac{1}{\beta(\underline{x})} \frac{2(\beta(\underline{x}) + \gamma)}{\beta(\underline{x}) - 1} \right] \left[-\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v / \sigma_v} + \frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} \right] \\ &> -\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v} + \frac{1}{\sigma_v} \left[1 + \frac{2}{\beta(\underline{x})} \frac{\beta(\underline{x}) + \gamma}{\beta(\underline{x}) - 1} \right] \left[-\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v / \sigma_v} + l_{\text{sup}}^{\sigma_v} \right] \end{aligned}$$

Next

$$\begin{aligned} \frac{\partial I}{\partial \sigma_v} &= \int_{\underline{x}}^{\infty} \frac{\partial I(x)}{\partial \sigma_v} d\Lambda(x) - I(x) d\Lambda(x) \mid_{\sigma^2(x) = \frac{\alpha^2}{2\gamma f_{xx}}} \frac{\partial \underline{x}}{\partial \sigma_v} \\ &> E \left\{ -\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v} + \frac{1}{\sigma_v} \left[1 + \frac{1}{\beta(\underline{x})} \frac{2(\beta(\underline{x}) + \gamma)}{\beta(\underline{x}) - 1} \right] \left[-\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v / \sigma_v} + l_{\text{sup}}^{\sigma_v} \right] \right\} \\ &= \frac{1 + \frac{1}{\beta(\underline{x})} \frac{2(\beta(\underline{x}) + \gamma)}{\beta(\underline{x}) - 1}}{\sigma_v} l_{\text{sup}}^{\sigma_v} - \frac{2 + \frac{1}{\beta(\underline{x})} \frac{2(\beta(\underline{x}) + \gamma)}{\beta(\underline{x}) - 1}}{\sigma_v} E \left\{ \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v} \right\} \\ &> 0 \end{aligned}$$

Therefore, in the new equilibrium, the total demand for risky asset is more than total supply. Contradiction. Therefore, it must be that

$$\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} < l_{\text{sup}}^{\sigma_v} = \frac{\partial \sigma(\underline{x}) / \sigma(\underline{x})}{\partial \sigma_v / \sigma_v}.$$

■

We now show conditions under which participation increases, i.e. under which the cutoff level of expertise for participation \underline{x} declines, as fundamental volatility increases. In particular, we show that participation increases with fundamental volatility in Cases 1 and 2 of Proposition 18, but only under a tight restriction in Case 3. In Case 3, participation only increases if the elasticity of the effective volatility of the lowest expertise investor is not too different from that of the average participating investor. In other words, participation increases if there is very little difference across expertise levels in the effect of changes in fundamental volatility on effective volatility, so that elasticities are nearly constant, as in Case 1. Notice that the condition restricting the differences in elasticities across investors is the same as Condition 1 in Lemma 23 which bounds the change in α from below. Thus, participation will increase only if the change in α is large enough, which will be the case if all participating investors face similar changes to their effective volatility as fundamental volatility changes. We discuss the more empirically relevant case, when elasticities vary more across high expertise and low expertise agents, and participation thus declines, in the text.

Proposition 24 *Define the entry cutoff \underline{x} ,*

$$\underline{x} = \sigma^{-1} \left(\frac{\alpha}{\sqrt{2\gamma f_{xx}}} \right),$$

where $\sigma^{-1}(\cdot)$ is the inverse function of $\sigma(x)$. We have that participation increases with fundamental volatility,

$$\frac{\partial \underline{x}}{\partial \sigma_v} < 0$$

if the following conditions hold

1. $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}}{\partial x} \geq 0$, (Proposition 18 Cases 1 and 2) or
2. $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}}{\partial x} < 0$, (Proposition 18 Case 3) and $l_{\text{sup}}^{\sigma_v} < \frac{2 + \frac{2}{\beta(\bar{x})} \frac{\beta(\bar{x}) + \gamma}{\beta(\bar{x}) - 1}}{1 + \frac{2}{\beta(\bar{x})} \frac{\beta(\bar{x}) + \gamma}{\beta(\bar{x}) - 1}} E \left[\frac{\partial \log \sigma(x)}{\partial \log \sigma_v} | x \geq \underline{x} \right]$.

Proposition 24 shows that participation increases in Cases 1 and 2 as fundamental volatility increases. The reason is that demand for the complex asset by incumbent experts declines, and new wealth must be brought into the market to clear the fixed supply. However, in Case 3, it is possible that because higher expertise agents' risk-return tradeoff deteriorates by less as fundamental volatility increases, that participation declines. This can be seen in the condition for increased participation in Case 3, which requires a very small difference between the highest and lowest elasticities, since $\beta \approx 1$, and we confirm this formally in Proposition 19.

Proof. Proof of Proposition 24 First,

$$\frac{\partial \underline{x}}{\partial \sigma_v} < 0 \text{ iff } \frac{\partial \log \frac{\alpha^2}{\sigma^2(\underline{x})}}{\partial \log \sigma_v} > 0.$$

We have

$$\frac{\partial \log \frac{\alpha^2}{\sigma^2(\underline{x})}}{\partial \log \sigma_v} = 2 \left(\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} - \frac{\partial \sigma(\underline{x}) / \sigma(\underline{x})}{\partial \sigma_v / \sigma_v} \right)$$

Therefore

$$\frac{\partial \log \frac{\alpha^2}{\sigma^2(\underline{x})}}{\partial \log \sigma_v} > 0 \text{ iff } \frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} > \frac{\partial \sigma(\underline{x}) / \sigma(\underline{x})}{\partial \sigma_v / \sigma_v}$$

If $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}}{\partial x} \geq 0$, from Proposition 21 we have

$$\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} > l_{\text{inf}}^{\sigma_v} = \frac{\partial \sigma(\underline{x}) / \sigma(\underline{x})}{\partial \sigma_v / \sigma_v}.$$

If $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}}{\partial x} < 0$ and $l_{\text{sup}}^{\sigma_v} < \frac{2 + \frac{2}{\beta(\bar{x})} \frac{\beta(\bar{x}) + \gamma}{\beta(\bar{x}) - 1}}{1 + \frac{2}{\beta(\bar{x})} \frac{\beta(\bar{x}) + \gamma}{\beta(\bar{x}) - 1}} E [l^{\sigma_v} | x \geq \underline{x}]$, from Lemma 23, we know

$$\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} > l_{\text{sup}}^{\sigma_v} = \frac{\partial \sigma(\underline{x}) / \sigma(\underline{x})}{\partial \sigma_v / \sigma_v}.$$

■

Proof. Proof of Proposition 19 First,

$$\frac{\partial \underline{x}}{\partial \sigma_v} > 0 \text{ iff } \frac{\partial \log \frac{\alpha^2}{\sigma^2(\underline{x})}}{\partial \log \sigma_v} < 0.$$

We have

$$\frac{\partial \log \frac{\alpha^2}{\sigma^2(\underline{x})}}{\partial \log \sigma_v} = 2 \left(\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} - \frac{\partial \sigma(\underline{x}) / \sigma(\underline{x})}{\partial \sigma_v / \sigma_v} \right)$$

Therefore

$$\frac{\partial \log \frac{\alpha^2}{\sigma^2(\underline{x})}}{\partial \log \sigma_v} < 0 \text{ iff } \frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} < l_{\text{sup}}^{\sigma_v} = \frac{\partial \sigma(\underline{x}) / \sigma(\underline{x})}{\partial \sigma_v / \sigma_v}.$$

If $\frac{\partial \frac{\partial \log \sigma(\underline{x})}{\partial \log \sigma_v}}{\partial \underline{x}} < 0$ and $l_{\text{sup}}^{\sigma_v} > \frac{2 + \frac{2}{\beta(\underline{x})} \frac{\beta(\underline{x}) + \gamma}{\beta(\underline{x}) - 1}}{1 + \frac{2}{\beta(\underline{x})} \frac{\beta(\underline{x}) + \gamma}{\beta(\underline{x}) - 1}} E[l^{\sigma_v} | x \geq \underline{x}]$, from Lemma 23, we know

$$\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} < l_{\text{sup}}^{\sigma_v} = \frac{\partial \sigma(\underline{x}) / \sigma(\underline{x})}{\partial \sigma_v / \sigma_v}.$$

■

We note that the conditions in Proposition 24 and Proposition 19 are sufficient, but not necessary.

As discussed in the main text, we use the tail parameters for the highest and lowest expertise levels since the entire wealth distribution is a mixture of Pareto distributions (a complicated object).

These conditions are also not overlapping, because

$$\frac{2 + \frac{2}{\beta(\bar{x})} \frac{\beta(\bar{x}) + \gamma}{\beta(\bar{x}) - 1}}{1 + \frac{2}{\beta(\bar{x})} \frac{\beta(\bar{x}) + \gamma}{\beta(\bar{x}) - 1}} < \frac{2 + \frac{2}{\beta(\underline{x})} \frac{\beta(\underline{x}) + \gamma}{\beta(\underline{x}) - 1}}{1 + \frac{2}{\beta(\underline{x})} \frac{\beta(\underline{x}) + \gamma}{\beta(\underline{x}) - 1}}.$$

Proof. Proof of Proposition 20

We first consider the case in which participation increases. There are two subcases, with slightly different proof strategies:

1. $\frac{\partial \underline{x}}{\partial \sigma_v} < 0$ and $\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} \geq l_{\text{sup}}^{\sigma_v}$,
2. $\frac{\partial \underline{x}}{\partial \sigma_v} < 0$ and $\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} < l_{\text{sup}}^{\sigma_v}$,

First, we show that, for Case 1,

$$\frac{\partial SR^{ew}}{\partial \sigma_v} > 0 \text{ if } \frac{\partial \underline{x}}{\partial \sigma_v} < 0 \text{ and } \frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} \geq l_{\text{sup}}^{\sigma_v}.$$

Suppose

$$\frac{\partial SR^{ew}}{\partial \sigma_v} < 0.$$

We have

$$\begin{aligned} & \frac{\partial SR^{ew}}{\partial \sigma_v} \\ = & E \left[\frac{1}{\sigma(x)} \frac{\partial \alpha}{\partial \sigma_v} - \frac{\alpha}{\sigma^2(x)} \frac{\partial \sigma(x)}{\partial \sigma_v} \Big| x \geq \underline{x} \right] - \frac{\alpha}{\sigma(x)} d\Lambda(x) \Big|_{\sigma^2(x) = \frac{\alpha^2}{2\gamma f_{xx}}} \frac{\partial \underline{x}}{\partial \sigma_v} \\ = & \frac{\alpha}{\sigma_v} E \left[\frac{1}{\sigma(x)} \left(\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} - \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v / \sigma_v} \right) \Big| x \geq \underline{x} \right] - \frac{\alpha}{\sigma(x)} d\Lambda(x) \Big|_{\sigma^2(x) = \frac{\alpha^2}{2\gamma f_{xx}}} \frac{\partial \underline{x}}{\partial \sigma_v} \\ < & 0, \end{aligned}$$

Therefore

$$\frac{\sigma(\underline{x})}{\sigma_v} E \left[\frac{1}{\sigma(x)} \left(\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} - \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v / \sigma_v} \right) \Big| x \geq \underline{x} \right] < d\Lambda(x) \Big|_{\sigma^2(x) = \frac{\alpha^2}{2\gamma f_{xx}}} \frac{\partial \underline{x}}{\partial \sigma_v} < 0.$$

But

$$E \left[\frac{1}{\sigma(x)} \left(\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} - \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v / \sigma_v} \right) \Big| x \geq \underline{x} \right] \geq 0 \text{ because } \frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} \geq l_{\text{sup}}^{\sigma_v}.$$

Second, we show that, for Case 2,

$$\frac{\partial SR^{ew}}{\partial \sigma_v} > 0 \text{ if } \frac{\partial \underline{x}}{\partial \sigma_v} < 0 \text{ and } \frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} < l_{\text{sup}}^{\sigma_v}.$$

Suppose

$$\frac{\partial SR^{ew}}{\partial \sigma_v} < 0.$$

We have

$$\begin{aligned}
& \frac{\partial SR^{ew}}{\partial \sigma_v} \\
&= E \left[\frac{1}{\sigma(x)} \frac{\partial \alpha}{\partial \sigma_v} - \frac{\alpha}{\sigma^2(x)} \frac{\partial \sigma(x)}{\partial \sigma_v} \middle| x \geq \underline{x} \right] - \frac{\alpha}{\sigma(x)} d\Lambda(x) \Big|_{\sigma^2(x)=\frac{\alpha^2}{2\gamma f_{xx}}} \frac{\partial \underline{x}}{\partial \sigma_v} \\
&= \frac{\alpha}{\sigma_v} E \left[\frac{1}{\sigma(x)} \left(\frac{\partial \alpha/\alpha}{\partial \sigma_v/\sigma_v} - \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v/\sigma_v} \right) \middle| x \geq \underline{x} \right] - \frac{\alpha}{\sigma(x)} d\Lambda(x) \Big|_{\sigma^2(x)=\frac{\alpha^2}{2\gamma f_{xx}}} \frac{\partial \underline{x}}{\partial \sigma_v} \\
&< 0.
\end{aligned}$$

Therefore, we must have

$$E \left[\frac{1}{\sigma(x)} \left(\frac{\partial \alpha/\alpha}{\partial \sigma_v/\sigma_v} - \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v/\sigma_v} \right) \middle| x \geq \underline{x} \right] < 0.$$

Next,

$$\begin{aligned}
& \frac{\partial \log I(x)}{\partial \sigma_v} \\
&= -\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v} + \frac{1}{\sigma_v} \left[1 + \frac{1}{\beta(x)} \frac{2(\beta(x) + \gamma)}{\beta(x) - 1} \right] \left[-\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v/\sigma_v} + \frac{\partial \alpha/\alpha}{\partial \sigma_v/\sigma_v} \right] \\
&< \frac{1}{\sigma_v} \left[1 + \frac{1}{\beta(x)} \frac{2(\beta(x) + \gamma)}{\beta(x) - 1} \right] \left[-\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v/\sigma_v} + \frac{\partial \alpha/\alpha}{\partial \sigma_v/\sigma_v} \right].
\end{aligned}$$

So,

$$\begin{aligned}
\frac{\partial I}{\partial \sigma_v} &= \int_{\underline{x}}^{\infty} \frac{\partial I(x)}{\partial \sigma_v} d\Lambda(x) - I(x) d\Lambda(x) \Big|_{\sigma^2(x)=\frac{\alpha^2}{2\gamma f_{xx}}} \frac{\partial \underline{x}}{\partial \sigma_v} \\
&< E \left\{ \frac{I(x) \sigma(x)}{\sigma_v} \left[1 + \frac{1}{\beta(x)} \frac{2(\beta(x) + \gamma)}{\beta(x) - 1} \right] \frac{1}{\sigma(x)} \left[-\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v/\sigma_v} + \frac{\partial \alpha/\alpha}{\partial \sigma_v/\sigma_v} \right] \right\}
\end{aligned}$$

Define

$$J(x) = \frac{I(x) \sigma(x)}{\sigma_v} \left[1 + \frac{1}{\beta(x)} \frac{2(\beta(x) + \gamma)}{\beta(x) - 1} \right].$$

It is straightforward to show that

$$J'(x) > 0.$$

In Case 2 we have $\frac{\partial \frac{\log \sigma(x)}{\log \sigma_v}}{\partial x} > 0$ and $E \left[\frac{1}{\sigma(x)} \left(\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} - \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v / \sigma_v} \right) | x \geq \underline{x} \right] < 0$. Therefore,

$$E \left[\frac{J(x)}{\sigma(x)} \left(\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} - \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v / \sigma_v} \right) | x \geq \underline{x} \right] < 0.$$

Therefore

$$\frac{\partial I}{\partial \sigma_v} < 0,$$

Contradiction. We must have

$$\frac{\partial SR^{ew}}{\partial \sigma_v} > 0.$$

Last, we show that if participation is increasing, SR^{ew} increases as long as a condition on the distribution of expertise holds. In particular, we require that, if there are many investors around the cutoff level of expertise, that their effective volatility does not increase by so much that it drives the market Sharpe ratio down.

$$\frac{\partial SR^{ew}}{\partial \sigma_v} > 0 \text{ if } \frac{\partial \underline{x}}{\partial \sigma_v} > 0 \text{ and } E \left[1 - \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v / \sigma_v} | x > \underline{x} \right] > d\Lambda(\underline{x}) \frac{1}{I_{\text{sup}}^{\sigma_v}}.$$

$$\begin{aligned} & \frac{\partial SR^{ew}}{\partial \sigma_v} \\ = & \frac{\alpha}{\sigma_v} E \left[\frac{1}{\sigma(x)} \left(\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} - \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v / \sigma_v} \right) \middle| \frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx} \right] - \frac{\alpha}{\sigma(x)} d\Lambda(x) \middle|_{\sigma^2(x) = \frac{\alpha^2}{2\gamma f_{xx}}} \frac{\partial \underline{x}}{\partial \sigma_v} \end{aligned}$$

Next

$$\begin{aligned} \frac{\partial I}{\partial \sigma_v} &= \int_{\underline{x}}^{\infty} \frac{\partial I(x)}{\partial \sigma_v} d\Lambda(x) - I(x) d\Lambda(x) \middle|_{\sigma^2(x) = \frac{\alpha^2}{2\gamma f_{xx}}} \frac{\partial \underline{x}}{\partial \sigma_v} \\ &= E \left\{ \frac{I(x)}{\sigma_v} \left(1 + \frac{1}{\beta(x)} \frac{2(\beta(x) + \gamma)}{\beta(x) - 1} \right) \left(\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} - \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v / \sigma_v} \right) \right\} \\ &\quad - E \left[\frac{I(x)}{\sigma_v} \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v / \sigma_v} \right] - I(\underline{x}) d\Lambda(\underline{x}) \frac{\partial \underline{x}}{\partial \sigma_v}, \end{aligned}$$

We also have $\sigma^2(\underline{x}) = \frac{\alpha^2}{2\gamma f_{xx}}$, thus

$$\frac{\partial \underline{x}}{\partial \sigma_v} = \frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} \frac{1}{l_{\text{sup}}^{\sigma_v}}.$$

Since $\frac{\partial I}{\partial \sigma_v} = 0$, we have

$$\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} = \frac{E \left[\left(2 + \frac{1}{\beta(x)} \frac{2(\beta(x)+\gamma)}{\beta(x)-1} \right) \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v / \sigma_v} \right]}{E \left(1 + \frac{1}{\beta(x)} \frac{2(\beta(x)+\gamma)}{\beta(x)-1} \right) - I(\underline{x}) d\Lambda(\underline{x}) \frac{1}{l_{\text{sup}}^{\sigma_v}}}.$$

Furthermore,

$$\begin{aligned} & \frac{\partial SR^{ew} I(\underline{x}) \sigma(\underline{x})}{\partial \sigma_v} \frac{1}{\alpha} \\ &= \frac{I(\underline{x}) \sigma(\underline{x})}{\sigma_v} E \left[\frac{1}{\sigma(x)} \left(\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} - \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v / \sigma_v} \right) \middle| \frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx} \right] - I(\underline{x}) d\Lambda(\underline{x}) \frac{\partial \underline{x}}{\partial \sigma_v} \\ &= \frac{I(\underline{x}) \sigma(\underline{x})}{\sigma_v} E \left[\frac{1}{\sigma(x)} \left(\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} - \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v / \sigma_v} \right) \middle| \frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx} \right] - \int_{\underline{x}}^{\infty} \frac{\partial I(x)}{\partial \sigma_v} d\Lambda(x) \\ &= E \left[\frac{1}{\sigma_v} \frac{I(\underline{x}) \sigma(\underline{x})}{\sigma(x)} \left(\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} - \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v / \sigma_v} \right) \right] + E \left[\frac{I(x)}{\sigma_v} \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v / \sigma_v} \right] \\ &\quad - E \left[\frac{I(x)}{\sigma_v} \left(1 + \frac{1}{\beta(x)} \frac{2(\beta(x)+\gamma)}{\beta(x)-1} \right) \left(\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} - \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v / \sigma_v} \right) \right] \\ &= \frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} E \left[\frac{1}{\sigma_v} \frac{I(\underline{x}) \sigma(\underline{x})}{\sigma(x)} - \frac{I(x)}{\sigma_v} \left(1 + \frac{1}{\beta(x)} \frac{2(\beta(x)+\gamma)}{\beta(x)-1} \right) \right] \\ &\quad + E \left\{ \left[\left(2 + \frac{1}{\beta(x)} \frac{2(\beta(x)+\gamma)}{\beta(x)-1} \right) \frac{I(x)}{\sigma_v} - \frac{1}{\sigma_v} \frac{I(\underline{x}) \sigma(\underline{x})}{\sigma(x)} \right] \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v / \sigma_v} \right\} \end{aligned}$$

Therefore, $\frac{\partial SR^{ew}}{\partial \sigma_v} > 0$ iff

$$\begin{aligned} \frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} &= \frac{E \left[\left(2 + \frac{1}{\beta(x)} \frac{2(\beta(x)+\gamma)}{\beta(x)-1} \right) \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v / \sigma_v} \right]}{E \left[1 + \frac{1}{\beta(x)} \frac{2(\beta(x)+\gamma)}{\beta(x)-1} \right] - I(\underline{x}) d\Lambda(\underline{x}) \frac{1}{l_{\text{sup}}^{\sigma_v}}} \\ &< \frac{E \left[\left(2 + \frac{1}{\beta(x)} \frac{2(\beta(x)+\gamma)}{\beta(x)-1} \right) \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v / \sigma_v} \right] - E \left[\frac{I(\underline{x}) \sigma(\underline{x})}{\sigma(x)} \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v / \sigma_v} \right]}{E \left[1 + \frac{1}{\beta(x)} \frac{2(\beta(x)+\gamma)}{\beta(x)-1} \right] - E \left[\frac{I(\underline{x}) \sigma(\underline{x})}{\sigma(x)} \right]}, \end{aligned}$$

It suffices to show that

$$E \left[\frac{\sigma(\underline{x})}{\sigma(x)} \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v/\sigma_v} \middle| x > \underline{x} \right] < E \left[\frac{\sigma(\underline{x})}{\sigma(x)} \right] - d\Lambda(\underline{x}) \frac{1}{l_{\text{sup}}^{\sigma_v}}.$$

This is true because

$$\begin{aligned} & E \left[\frac{\sigma(\underline{x})}{\sigma(x)} \left(1 - \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v/\sigma_v} \right) \middle| x > \underline{x} \right] \\ & > E \left[1 - \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v/\sigma_v} \middle| x > \underline{x} \right] \\ & > d\Lambda(\underline{x}) \frac{1}{l_{\text{sup}}^{\sigma_v}}. \end{aligned}$$

■

Static Model

We present a static model to build intuition about the interaction between the size and expertise distribution of investors and equilibrium returns.

Model and Results

Setup Investors have constant relative risk aversion preferences over date 1 consumption, with coefficient of relative risk aversion γ . At date 0, they are endowed with financial wealth W and expertise X . There is a riskless asset with gross return R_f , and a risky asset with gross returns R , which are distributed log normally. We use lower case letters to denote logs.

We assume that the log return on the risky asset for any given investor, which we denote by r , is distributed according to $r \sim N(\mu - \frac{1}{2} \frac{\sigma_v^2}{X}, \frac{\sigma_v^2}{X})$, given the distribution of W and X . We denote the variance of log returns on the fundamental asset, before expertise is applied, by σ_v^2 , and call this fundamental variance, and its square root fundamental volatility. The effective variance and

volatility of an investor's return on the risky asset then decreases as expertise X increases, while the innovation v itself is independent from W and X . We provide an example microfoundation for a closely related return process in the context of our dynamic model in the Appendix.

Investing in the complex asset implies a joint investment in a common market clearing return, as well as a specific risk from hedging or asset specificities. We assume the specified functional form for log return volatility for simplicity, as it allows for straightforward calculations of all expectations, and minimal parameters. It is straightforward to show that our main conclusions for the static model are robust to a family of functions $\frac{\sigma_v^2}{k_0+k_1X+k_2X^2+\dots}$, with all coefficients k_0, k_1, k_2, \dots being non-negative. In levels, expected returns μ are the same for all investors, regardless of their individual expertise.

Solving the Portfolio Choice Problem Using the approximation described in Campbell and Viceira (2002b), and the associated appendix Campbell and Viceira (2002a), which relates log individual-asset returns to log portfolio returns over short time intervals, the investor's optimization problem becomes:

$$\max_{\theta} \left\{ \theta (\mu - r_f) - \frac{\gamma}{2} \theta^2 \frac{\sigma_v^2}{X} \right\} \quad (3.22)$$

where r_f represents the log return on the riskless asset. In this section, for emphasis, we use bold notation to denote equilibrium returns.²⁴ The solution for the optimal fraction of wealth allocated to the risky asset is:

$$\theta^* = \frac{(\mu - r_f)}{\gamma \sigma_v^2} X. \quad (3.23)$$

Thus, portfolio choice in a lognormal model with power utility resembles that of a mean variance investor. The allocation to the risky asset is increasing in the equilibrium average excess return, decreasing in risk aversion, and decreasing in the fundamental shock variance. Moreover, the fraction of wealth that an investor allocates to the risky asset strictly increases with expertise. The

²⁴Because an individual investor's return volatility depends on their expertise, for the approximation to be good given our specification for log return volatility, we have to impose a technical restriction that the majority of distribution of expertise X is bounded away from zero. This assumption is unnecessary if one adopts the general functional form for volatility discussed in footnote 3.5.

relationship is linear under our functional form assumptions.²⁵

Equilibrium We now describe how the equilibrium excess return depends on the parameters for preferences, technology, and the joint distribution of wealth and expertise. We focus on comparative statics over the equilibrium average excess return, market level Sharpe ratio, and individual Sharpe ratios. We normalize the mass of investors to one, define the value of the supply of the risky asset to be S , determine the market clearing log expected return μ , and then back out the equilibrium expected level return and therefore α . We assume that W and X are jointly log-normally distributed. We denote the joint pdf of the log variables $f(w, x)$, with means and variances μ_w, μ_x, σ_w^2 , and σ_x^2 respectively, and covariance $\rho_{w,x}\sigma_w\sigma_x$. Thus, an economy ψ is described by $\psi \equiv \{r_f, \gamma, I, \sigma_v, \mu_w, \sigma_w, \mu_x, \sigma_x, \rho_{w,x}\}$. The equilibrium log expected return μ solves the market clearing condition:

$$\text{Supply} \equiv S = \text{Demand} = \int \int \exp(w)\theta^*(\exp(x))f(w, x)dw dx = \frac{\mu - r_f}{\gamma\sigma_v^2} \mathcal{X} \quad (3.24)$$

where $\theta^*(\exp(x))$ is the portfolio choice given in Equation (3.23) and \mathcal{X} is the wealth and population weighted average of expertise:

$$\int \int \exp(w+x)f(w, x)dw dx = \exp\left(\frac{1}{2}(\sigma_w^2 + \sigma_x^2 + 2\rho_{w,x}\sigma_w\sigma_x + 2\mu_w + 2\mu_x)\right) \quad (3.25)$$

utilizing the result for the expectation of log normally distributed variables.

Rearranging, we have:

$$\mu - r_f = \left(\frac{\sigma_v^2}{\mathcal{X}}\right) \gamma S. \quad (3.26)$$

The equilibrium log expected excess return is increasing in the amount of risk relative to the risk bearing capacity of investors. We decompose the inputs into two components. The first term is the effective risk in the market, namely the fundamental risk σ_v^2 , scaled down by the wealth and population weighted average of expertise. The second term is the risk aversion scaled supply of

²⁵Without restrictions on the distribution of X , θ can be larger than one, implying borrowing at the risk free rate.

the risky asset which must be cleared. The higher is investors' coefficient of relative risk aversion, and the larger is the supply of the asset, the higher is the required return. Conversely, the wealth and population weighted average of expertise, \mathcal{X} , scales μ down due to the positive impact of expertise on investors' allocation to the risky asset.

Using the equilibrium log expected return μ , we can rewrite agents' optimal portfolio allocations to the risky asset as:

$$\theta^* = \frac{X}{\mathcal{X}} S. \quad (3.27)$$

This expression captures the fact that, in equilibrium, the optimal portfolio allocations to the risky asset by an agent with expertise X turns out to be a fraction of total supply equal to their expertise relative to the wealth and population weighted average of expertise.

The equilibrium mean of the level of the gross risky return over the level of the gross risk free rate, α , is a monotonic transformation of μ . In particular, we show in the Appendix that the equilibrium α is then given by:

$$\alpha = \exp(\mu) - R_f \quad (3.28)$$

which gives a one to one mapping from μ to α conditional on parameters. Note also that writing θ^* (Equation 3.23) as a function of either μ or α will always yield identical equilibrium outcomes.

Lemma 25 *Using Equation (3.28) describing the equilibrium market clearing α , the following comparative statics can be directly calculated:*

1. $\frac{\partial \alpha}{\partial \sigma_v^2} = \exp(\mu) \frac{\gamma S}{\mathcal{X}} > 0$. α increases with fundamental risk.
2. $\frac{\partial \alpha}{\partial \gamma} = \exp(\mu) \frac{\sigma_v^2}{\mathcal{X}} S > 0$. α increases with the coefficient of relative risk aversion.
3. $\frac{\partial \alpha}{\partial S} = \exp(\mu) \frac{\sigma_v^2}{\mathcal{X}} \gamma > 0$. α increases with the risky asset supply investors must absorb.
4. $\frac{\partial \alpha}{\partial \mu_w} = -\exp(\mu) \frac{\sigma_v^2}{\mathcal{X}} \gamma S < 0$. α decreases as aggregate wealth increases.
5. $\frac{\partial \alpha}{\partial \mu_x} = -\exp(\mu) \frac{\sigma_v^2}{\mathcal{X}} \gamma S < 0$. α decreases as aggregate expertise increases.
6. $\frac{\partial \alpha}{\partial \rho_{w,x}} = -\exp(\mu) \frac{\sigma_v^2}{\mathcal{X}} \gamma S \sigma_w \sigma_x < 0$. As $\rho_{w,x}$ increases, there is a more efficient allocation of expertise and α decreases.

$$7. \frac{\partial \alpha}{\partial \sigma_w} = -\exp(\mu) \frac{\sigma_v^2}{\mathcal{L}} \gamma \mathcal{S}(\sigma_w + \rho_{w,x} \sigma_x)$$

- > 0 if $\rho_{w,x} < -\frac{\sigma_w}{\sigma_x}$, i.e. if wealth and expertise are strongly negatively correlated.
- < 0 if $\rho_{w,x} > -\frac{\sigma_w}{\sigma_x}$, i.e. if wealth and expertise are positively or only weakly negatively correlated.

$$8. \frac{\partial \alpha}{\partial \sigma_x} = -\exp(\mu) \frac{\sigma_v^2}{\mathcal{L}} \gamma \mathcal{S}(\sigma_x + \rho_{w,x} \sigma_w)$$

- > 0 if $\rho_{w,x} < -\frac{\sigma_x}{\sigma_w}$, i.e. if wealth and expertise are strongly negatively correlated.
- < 0 if $\rho_{w,x} > -\frac{\sigma_x}{\sigma_w}$, i.e. if wealth and expertise are positively or only weakly negatively correlated.

Proof. By direct calculation. ■

All comparative statics are intuitive. An increase in the correlation of wealth and expertise will reduce α , as investors with more expertise account for a larger share of the wealth distribution. The effect of an increase in $\rho_{w,x}$ on the market clearing α will be larger the larger is amount of fundamental risk, σ_v^2 , the larger is the coefficient of relative risk aversion, γ , the larger is the supply of the risky asset, S , the smaller is the mean of log wealth, μ_w , and the smaller is the mean of log expertise, μ_x .

We also derive results for the equilibrium market-level and investor-specific Sharpe ratios. The market level Sharpe ratio requires a definition appropriate for our environment. Here, we define the equilibrium market level Sharpe ratio to be the equal-weighted cross-sectional average of excess returns divided by the equal-weighted cross-sectional standard deviation. Thus this market Sharpe ratio can, for example, be interpreted as the expected Sharpe ratio for an investor “behind the veil” drawing from the distribution of possible levels of expertise, before the investment stage of the model. This Sharpe ratio would be relevant, for example, in a model with entry in which an investor must decide whether to enter before drawing an expertise level from the given distribution. We then refer to what is technically the equilibrium equally weighted market Sharpe ratio as the “Sharpe ratio” for exposition purpose:

$$SR = \frac{1 - R_f \exp(-\mu)}{\sqrt{\mathbb{E} \left[\exp \left(\frac{\sigma_v^2}{X} \right) \right] - 1}} = \frac{1 - R_f \exp(-\mu)}{\sqrt{\sum_{k=1}^{\infty} \frac{1}{k!} \sigma_v^{2k} \exp(-k\mu_x + \frac{1}{2}k^2\sigma_x^2)}} \approx \frac{1 - R_f \exp(-\mu)}{\sigma_v \exp(-\frac{1}{2}\mu_x + \frac{1}{4}\sigma_x^2)}, \quad (3.29)$$

where \mathbb{E} denotes the cross-sectional expectation. This ratio aggregates all investor decisions and measures the *market level* risk return tradeoff.²⁶ The market-level Sharpe ratio increases as the average log expertise μ_x in this economy increases, but it decreases as the cross-sectional standard deviation of log expertise σ_x^2 increases.

Lemma 26 *Using Equation (3.29) describing the equilibrium market clearing equally weighted Sharpe ratio, the following comparative statics can be directly calculated:*

1. Let η denote any parameter $\eta \in \{\gamma, S, \mu_w, \sigma_w, \rho_{w,x}\}$.

Then, $\text{Sign} \left(\frac{\partial(SR)}{\partial \eta} \right) = \text{Sign} \left(\frac{\partial \alpha}{\partial \eta} \right)$.

2. The signs for comparative statics with respect to parameters $\hat{\eta} \in \{\sigma_v^2, \mu_x, \sigma_x\}$, are indeterminate.

Proof. By direct calculation, see Appendix. ■

Expected returns rise proportionally relative to the volatility of the risky asset return in our static model, so that the Sharpe ratio improves with any parameter change that increases α . Thus, we confirm that, at the market level, parameter changes which lead to an increase in the equilibrium expected excess return in fact lead to better investment opportunities given the market risk in equilibrium.

In our model, each investor confronts a different risk-return trade-off. Since the volatility of log returns depends on individual investors' expertise, an observed increase in the market Sharpe ratio does not necessarily imply a higher Sharpe ratio for every investor in the market. Moreover,

²⁶See Appendix for derivation. We also compute and analyze the market value weighted Sharpe ratio in the Appendix.

even if the Sharpe ratio improves for each agent individually, the magnitude of the improvement an individual investor faces will not, in general, coincide with the market improvement. To see this, consider the investor-specific Sharpe ratio. For an investor with wealth W and expertise X , we show in the Appendix that this investor's Sharpe ratio is given by:

$$SR(X) = \frac{1 - R_f \exp(-\mu)}{\sqrt{\exp\left(\frac{\sigma_v^2}{X}\right) - 1}}. \quad (3.30)$$

Equation (3.30) clearly shows that the model can deliver considerable cross-sectional dispersion in investor-specific Sharpe ratios. Investors with very low effective risk, $\frac{\sigma_v^2}{X}$, face significantly higher Sharpe ratios than their counterparts with low expertise. We can determine the signs of the following comparative statics:

Lemma 27 *Using Equation (3.30) describing the investor-specific Sharpe ratio, and Equation (3.23) describing the portfolio allocation θ^* , the following comparative statics can be directly calculated. Let η denote any parameter $\eta \in \{\gamma, S, \mu_w, \sigma_w, \rho_{w,x}\}$.²⁷*

1. $\frac{\partial SR(X)}{\partial X} > 0$. *Higher expertise generates lower effective risk, and a correspondingly higher individual Sharpe ratio.*
2. $Sign\left(\frac{\partial SR(X)}{\partial \eta}\right) = Sign\left(\frac{\partial \alpha}{\partial \eta}\right) = Sign\left(\frac{\partial(SR)}{\partial \eta}\right)$. *All investor-specific Sharpe ratios co-move with the equilibrium excess return and the market level equilibrium Sharpe ratio.*
3. $Sign\left(\frac{\partial Var(SR(X))}{\partial \eta}\right) = Sign\left(\frac{\partial \alpha}{\partial \eta}\right) = Sign\left(\frac{\partial(SR)}{\partial \eta}\right)$. *Whenever a parameter change increases the market level equilibrium Sharpe ratio, it leads to a larger cross-sectional dispersion in the investor-specific Sharpe ratio.*
4. $Sign\left(\frac{\partial^2 SR(X)}{\partial \eta \partial X}\right) = Sign\left(\frac{\partial SR(X)}{\partial \eta}\right) = Sign\left(\frac{\partial \alpha}{\partial \eta}\right) = Sign\left(\frac{\partial(SR)}{\partial \eta}\right)$.

Whenever a parameter change increases the investor-specific Sharpe ratio, it leads to a larger increase for high expertise investors relative to low expertise investors.

²⁷Derivatives with respect to μ_x and σ_x follow the same formulas as those that support parts 2 to 5 of lemma 27. However, the changes are not comparable to the market Sharpe ratio, as we can't determine the signs in lemma 26, part 2. Derivatives with respect to σ_v^2 cannot be signed generally.

$$5. \text{Sign} \left(\frac{\partial^2 \theta^*(X)}{\partial \eta \partial X} \right) = \text{Sign} \left(\frac{\partial SR(X)}{\partial \eta} \right) = \text{Sign} \left(\frac{\partial \alpha}{\partial \eta} \right) = \text{Sign} \left(\frac{\partial (SR)}{\partial \eta} \right).^{28}$$

Whenever a parameter change increases the investor-specific portfolio allocation, it leads to a larger increase for high expertise investors relative to low expertise investors.

6. $\exists \bar{X} > \underline{X} > 0$ such that $\forall X > \bar{X}, \frac{\partial SR(X)}{\partial \sigma_v^2} > 0$, and $\forall X < \underline{X}, \frac{\partial SR(X)}{\partial \sigma_v^2} < 0$. An increase in the fundamental risk generates a higher Sharpe ratio for high expertise investors and a lower Sharpe ratio for low expertise investors.

Proof. By direct calculation. See Appendix. ■

Lemma (27) has rich implications. First, we emphasize the co-movement between cross sectional variation in investor-specific Sharpe ratios and the level of the market Sharpe ratio. Any increase in the market-level Sharpe ratio will also increase the cross-sectional dispersion in Sharpe ratios. Furthermore, because an increase in the market level Sharpe ratio improves investment opportunities for high expertise investors by more than for low expertise investors, such an increase accordingly increases their allocation to the risky asset θ^* by more. Thus, an improvement in the market-level risk return tradeoff in large part reflects the improved risk-return trade-off faced by high expertise investors, and not by their low expertise counterparts.

Any parameter change which increases the market level Sharpe ratio increases the investor specific Sharpe ratio for high expertise by more, and increases the influence of high expertise investors' Sharpe ratios on the market level risk return tradeoff. In our model, measured improvements in the aggregate Sharpe ratio are a misleading indicator of improvements in individual investors' risk-return tradeoff, and can indeed more accurately reflect changes in the Sharpe ratio of higher expertise investors. The converse is also true.

Furthermore, part 6 of Lemma (27) states that changes to fundamental risk can lead to changes in individual Sharpe ratios that vary in sign. For example, if σ_v^2 increases, all investors face the same increase in the equilibrium excess return, but investors with high expertise face a considerably smaller increase in risk. Thus, more complex assets with higher σ_v^2 can have higher market level

²⁸Except for γ , where $\frac{\partial^2 \theta^*(X)}{\partial \gamma \partial X} = 0$.

Sharpe ratios but lower demand from non-experts. We also emphasize that because an increase in fundamental risk improves the investor-specific Sharpe ratio for some investors but not others, in a dynamic model a shock to fundamental risk can lead potentially lead to variation in investors' participation decisions. In other words, an increase in risk which improves the market level equally weighted Sharpe ratio may still lead low expertise investors to exit, or not to enter.

Proofs for Static Model Results

This section contains proofs and additional results for the static model.

Optimal Portfolio Choice

This section describes how to solve the optimal portfolio allocation problem in the static model. We also use upper case letters for level variables, and lower case letters for log variables. Under the assumptions in the main text, the optimization problem for an investor with wealth W and expertise X , can be written as:

$$v(W, X) = \max_{\theta} \mathbb{E} \left[\frac{(WR_p)^{1-\gamma}}{1-\gamma} \right]$$

subject to

$$\begin{aligned} R_p &= \theta R + (1 - \theta) R_f, \\ r_p | (W, X) &\sim N \left(\mu - \frac{1}{2} \frac{\sigma_v^2}{X}, \frac{\sigma_v^2}{X} \right). \end{aligned}$$

Campbell and Viceira (2002b) and Campbell and Viceira (2002a) show that the log portfolio return r_p over a short time horizon with bounded variance, can be approximated by:

$$r_p \approx r_f + \theta (r - r_f) + \frac{1}{2} \theta (1 - \theta) \frac{\sigma_v^2}{X}.$$

As a result,

$$r_p | (W, X) \sim N \left(r_f + \theta (\mu - r_f) - \frac{1}{2} \theta^2 \frac{\sigma_v^2}{X}, \theta^2 \frac{\sigma_v^2}{X} \right).$$

Then the value function equals:

$$v(W, X) = \max_{\theta} \frac{W^{1-\gamma}}{1-\gamma} \exp \left((1-\gamma) r_f + (1-\gamma) \theta (\mu - r_f) - \frac{1}{2} \gamma (1-\gamma) \theta^2 \frac{\sigma_v^2}{X} \right).$$

Hence, The investor's optimization problem becomes:

$$\max_{\theta} \left\{ \theta (\mu - r_f) - \frac{\gamma}{2} \theta^2 \frac{\sigma_v^2}{X} \right\}.$$

Equilibrium Market Excess Return

This section describes how to derive the equilibrium market excess return, α , from the log expected return, μ , given all parameters. Because

$$r|(W, X) \sim N\left(\mu - \frac{1}{2} \frac{\sigma_v^2}{X}, \frac{\sigma_v^2}{X}\right),$$

Then

$$\mathbb{E}(R|W, X) = \exp(\mu).$$

In addition,

$$\mathbb{E}[R] = \mathbb{E}[\mathbb{E}(R|W, X)].$$

Hence,

$$\mathbb{E}[R] = \exp(\mu).$$

Finally,

$$\alpha = \exp(\mu) - R_f.$$

Equilibrium Equally Weighted Market Sharpe ratio

This section describes how to derive the equilibrium equally weighted market Sharpe ratio, SR , from the log expected return, μ , given all parameters. Because

$$r|(W, X) \sim N\left(\mu - \frac{1}{2} \frac{\sigma_v^2}{X}, \frac{\sigma_v^2}{X}\right),$$

Then

$$\text{Var}(R|W, X) = \exp(2\mu) \left(\exp\left(\frac{\sigma_v^2}{X}\right) - 1 \right).$$

In addition, we have proven that

$$\mathbb{E}(R|W, X) = \mathbb{E}[R] = \exp(\mu).$$

Therefore, the equally weighted variance of the risky asset, is given by:

$$\text{Var}[R] = \mathbb{E}(\text{Var}(R|W, X)) = \exp(2\mu) \left(\mathbb{E}\left[\exp\left(\frac{\sigma_v^2}{X}\right)\right] - 1 \right).$$

Hence, the equally weighted market Sharpe ratio, can be written as:

$$SR = \frac{1 - R_f \exp(-\mu)}{\sqrt{\mathbb{E}\left[\exp\left(\frac{\sigma_v^2}{X}\right)\right] - 1}},$$

where $\mathbb{E}\left[\exp\left(\frac{\sigma_v^2}{X}\right)\right] = \sum_{k=0}^{\infty} \frac{1}{k!} \sigma_v^{2k} \mathbb{E}[\exp(-kx)]$, using a Taylor expansion of $\exp(\sigma_v^2 X^{-1}) = 1 + \sigma_v^2 X^{-1} + \frac{1}{2!} \sigma_v^4 X^{-2} + \frac{1}{3!} \sigma_v^6 X^{-3} + \dots$, which is equivalent to:

$$\mathbb{E}\left[\exp\left(\frac{\sigma_v^2}{X}\right)\right] = \sum_{k=0}^{\infty} \frac{1}{k!} \sigma_v^{2k} [\exp(-k\mu_x + \frac{1}{2} k^2 \sigma_x^2)],$$

where we have used the moment-generating function of the normal distribution. Hence, the equally

weighted market Sharpe ratio, can be written as:

$$SR = \frac{1 - R_f \exp(-\mu)}{\sqrt{\sum_{k=0}^{\infty} \frac{1}{k!} \sigma_v^{2k} \exp(-k\mu_x + \frac{1}{2}k^2\sigma_x^2) - 1}},$$

Provided that σ_v^4 is small enough, this SR is approximately equal to the following expression:

$$SR \approx \frac{1 - R_f \exp(-\mu)}{\sigma_v \exp(-\frac{1}{2}\mu_x + \frac{1}{4}\sigma_x^2)}.$$

Equilibrium Investor-Specific Sharpe ratio

This section describes how to derive the equilibrium investor-specific Sharpe ratio, $SR(X)$, from log expected return, μ , given all parameters. For an investor with wealth W and expertise X , Because

$$r|(W,X) \sim N\left(\mu - \frac{1}{2} \frac{\sigma_v^2}{X}, \frac{\sigma_v^2}{X}\right),$$

Then

$$\mathbb{E}(R|W,X) = \exp(\mu),$$

And

$$\text{Var}(R|W,X) = \exp(2\mu) \left(\exp\left(\frac{\sigma_v^2}{X}\right) - 1 \right).$$

Hence, the investor-specific Sharpe ratio is given by:

$$SR(X) = \frac{1 - R_f \exp(-\mu)}{\sqrt{\exp\left(\frac{\sigma_v^2}{X}\right) - 1}}$$

$$E[SR(X)] = E\left[\frac{1 - R_f \exp(-\mu)}{\sqrt{\exp\left(\frac{\sigma_v^2}{X}\right) - 1}}\right]$$

Proof of Lemma 26 and 27

This section describes how to prove lemma 26 and 27. From equations (3.23), (3.28), (3.29) and (3.30), we can derive that, if η denotes any parameter $\eta \in \{\gamma, S, \mu_w, \sigma_w, \rho_{w,x}\}$:

1. $\frac{\partial \alpha}{\partial \eta} = \exp(\mu) \frac{\partial \mu}{\partial \eta}$;
2. $\frac{\partial(SR)}{\partial \eta} = \frac{R_f \exp(-\mu)}{\sqrt{\mathbb{E}\left(\exp\left(\frac{\sigma_v^2}{X}\right)\right) - 1}} \frac{\partial \mu}{\partial \eta}$;
3. $\frac{\partial SR(X)}{\partial \eta} = \frac{R_f \exp(-\mu)}{\sqrt{\exp\left(\frac{\sigma_v^2}{X}\right) - 1}} \frac{\partial \mu}{\partial \eta}$;
4. $\frac{\partial \text{Var}(SR(X))}{\partial \eta} = 2(1 - R_f \exp(-\mu)) R_f \exp(-\mu) \text{Var}\left(\frac{1}{\sqrt{\exp\left(\frac{\sigma_v^2}{X}\right) - 1}}\right) \frac{\partial \mu}{\partial \eta}$;
5. $\frac{\partial^2 SR(X)}{\partial \eta \partial X} = \frac{\partial\left(\frac{R_f \exp(-\mu)}{\sqrt{\exp\left(\frac{\sigma_v^2}{X}\right) - 1}}\right)}{\partial\left(\frac{\sigma_v^2}{X}\right)} \frac{\partial\left(\frac{\sigma_v^2}{X}\right)}{\partial X} \frac{\partial \mu}{\partial \eta}$;
6. $\frac{\partial^2 \theta^*(X)}{\partial \eta \partial X} = \frac{1}{\gamma \sigma_v^2} \frac{\partial \mu}{\partial \eta}, \forall \eta \neq \gamma, \text{ and } \frac{\partial \theta^*(X)}{\partial \gamma} = 0$
7. $\frac{\partial SR(X)}{\partial \sigma_v^2} = \frac{R_f \exp(-\mu) \frac{\mu - r_f}{\sigma_v^2} - \frac{1}{2}(1 - R_f \exp(-\mu)) \frac{\exp\left(\frac{\sigma_v^2}{X}\right) \frac{1}{X}}{\exp\left(\frac{\sigma_v^2}{X}\right) - 1}}{\sqrt{\exp\left(\frac{\sigma_v^2}{X}\right) - 1}}$.

$$\begin{aligned} \text{Hence, } \text{Sign}\left(\frac{\partial \mu}{\partial \eta}\right) &= \text{Sign}\left(\frac{\partial \alpha}{\partial \eta}\right) = \text{Sign}\left(\frac{\partial(SR)}{\partial \eta}\right) = \text{Sign}\left(\frac{\partial SR(X)}{\partial \eta}\right) \\ &= \text{Sign}\left(\frac{\partial \text{Var}(SR(X))}{\partial \eta}\right) = \text{Sign}\left(\frac{\partial^2 SR(X)}{\partial \eta \partial X}\right) = \text{Sign}\left(\frac{\partial^2 \theta^*(X)}{\partial \eta \partial X}\right) \end{aligned}$$

In addition, we have:

$$1. \text{ Because } \frac{\exp\left(\frac{\sigma_v^2}{X}\right)^{\frac{1}{X}}}{\exp\left(\frac{\sigma_v^2}{X}\right) - 1} > \frac{1}{X}, \forall X,$$

$$\text{then } \frac{\partial SR(X)}{\partial \sigma_v^2} < \frac{R_f \exp(-\mu) \frac{\mu - r_f}{\sigma_v^2} - \frac{1}{2} (1 - R_f \exp(-\mu))^{\frac{1}{X}}}{\sqrt{\exp\left(\frac{\sigma_v^2}{X}\right) - 1}} < 0, \forall X < \underline{X},$$

$$\text{where } \underline{X} = \frac{\frac{1}{2} (1 - R_f \exp(-\mu))}{R_f \exp(-\mu) \frac{\mu - r_f}{\sigma_v^2}} > 0;$$

$$2. 0 = \frac{1}{\sigma_v^2} \left(1 + \frac{\sigma_v^2}{X}\right) - \frac{1}{X} - \frac{1}{\sigma_v^2} < \frac{1}{\sigma_v^2} \exp\left(\frac{\sigma_v^2}{X}\right) - \frac{1}{X} - \frac{1}{\sigma_v^2} = \left(\exp\left(\frac{\sigma_v^2}{X}\right) - 1\right) \left(\frac{1}{X} + \frac{1}{\sigma_v^2}\right) - \exp\left(\frac{\sigma_v^2}{X}\right) \frac{1}{X},$$

$$\text{then } \frac{\exp\left(\frac{\sigma_v^2}{X}\right)^{\frac{1}{X}}}{\exp\left(\frac{\sigma_v^2}{X}\right) - 1} < \frac{1}{X} + \frac{1}{\sigma_v^2},$$

$$\text{and } \frac{\partial SR(X)}{\partial \sigma_v^2} > \frac{R_f \exp(-\mu) \frac{\mu - r_f}{\sigma_v^2} - \frac{1}{2} (1 - R_f \exp(-\mu)) \left(\frac{1}{X} + \frac{1}{\sigma_v^2}\right)}{\sqrt{\exp\left(\frac{\sigma_v^2}{X}\right) - 1}}, \forall X.$$

$$\text{Hence, if } \bar{X} = \frac{1}{R_f \exp(-\mu) \frac{\mu - r_f}{\sigma_v^2} - \frac{1}{2} (1 - R_f \exp(-\mu)) \left(\frac{1}{\bar{X}} + \frac{1}{\sigma_v^2}\right)} > 0, \text{ then } \forall X > \bar{X}, \frac{\partial SR(X)}{\partial \sigma_v^2} > 0.$$

$$\bar{X} > 0 \text{ if and only if } F(\mu - r_f) \equiv \exp(-(\mu - r_f)) (\mu - r_f + \frac{1}{2}) - \frac{1}{2} > 0.$$

We can prove $F(\mu - r_f) > 0$ if and only if $0 < \mu - r_f < 1.2564$, but $\mu - r_f = 1.2564$ corresponds to an α around 250%. Then we can conclude $\bar{X} > 0$ for all reasonable parameters.

3. we can prove by direct computation that $\bar{X} > \underline{X}$ whenever $\bar{X} > 0$.

In sum, for all reasonable parameters, $\exists \bar{X} > \underline{X} > 0$ such that $\forall X > \bar{X}, \frac{\partial SR(X)}{\partial \sigma_v^2} > 0$, and $\forall X < \underline{X}, \frac{\partial SR(X)}{\partial \sigma_v^2} < 0$. The general functional form for effective risk yields similar results.

Equilibrium Value-Weighted Market Sharpe ratio

This section shows that our main conclusions still hold with respect to the value-weighted equilibrium market Sharpe ratio. Because

$$r|(W, X) \sim N\left(\mu - \frac{1}{2} \frac{\sigma_v^2}{X}, \frac{\sigma_v^2}{X}\right),$$

Then

$$\mathbb{E}(R|W, X) = \exp(\mu).$$

Then the value-weighted market expected return also equals to:

$$\exp(\mu),$$

In addition,

$$\text{Var}(R|W, X) = \exp(2\mu) \left(\exp\left(\frac{\sigma_v^2}{X}\right) - 1 \right).$$

Therefore, the value-weighted variance of the risky asset, is given by:

$$\int \int \text{Var}(R|W, X) \frac{\exp(w)\theta^*(\exp(x))f(w, x)}{\int \int \exp(w)\theta^*(\exp(x))f(w, x)dwde} dwde,$$

Which equals to

$$\exp(2\mu) \frac{\mathbb{E}\left[\left(\exp\left(\frac{\sigma_v^2}{X}\right) - 1\right)\exp(w+x)\right]}{\mathcal{X}}$$

Hence, the value-weighted market Sharpe ratio, can be written as:

$$\frac{1 - R_f e^{-\mu}}{\sqrt{\frac{\mathbb{E}\left[\left(\exp\left(\frac{\sigma_v^2}{X}\right) - 1\right)\exp(w+x)\right]}{\mathcal{X}}}}.$$

where $\mathbb{E}\left[\exp\left(\frac{\sigma_v^2}{X} + w + x\right)\right]$, using a Taylor expansion of $\exp(\sigma_v^2 X^{-1} + w + x) = 1 + \sigma_v^2 X^{-1} +$

$w + x + \frac{1}{2!}(\sigma_v^2 X^{-1} + w + x)^2 + \frac{1}{3!}(\sigma_v^2 X^{-1} + w + x)^3 + \dots$, which is equivalent to:

$$\mathbb{E} \left[\exp \left(\frac{\sigma_v^2}{X} + w + x \right) \right] = \sum_{k=0}^{\infty} \frac{1}{k!} (\sigma_v^2 X^{-1} + w + x)^k,$$

This will be approximately equal to

$$\begin{aligned} E[\exp(\sigma_v^2 X^{-1} + w + x)] &\approx 1 + E[\sigma_v^2 X^{-1}] + E[w] + E[x], \\ &+ \frac{1}{2} E[\sigma_v^4 X^{-2} + w^2 + x^2 + 2wx\sigma_v^4 X^{-2} + 2w^2 x \sigma_v^2 X^{-1} + 2wx^2 \sigma_v^2 X^{-1}] \end{aligned}$$

The moment-generating function is given by:

$$M(t_1, t_2) = E[\exp(t_1 w) \exp(t_2 x)] = \exp(t_1 \mu_x + t_2 \mu_w + (1/2)(t_1 \sigma_x^2 + t_2 \sigma_w^2 + 2t_1 t_2 w x \rho_{w,x} \sigma_x \sigma_w))$$

$$\partial M(t_1, t_2)$$

Then, if η denotes any parameter $\eta \in \{\gamma, S\}$,

$$\frac{\delta(SR)}{\delta \eta} = \frac{R_f e^{-\mu}}{\sqrt{\frac{\mathbb{E} \left[\left(\exp \left(\frac{\sigma_v^2}{X} \right) - 1 \right) \exp(w+x) \right]}{\mathcal{X}}}} \frac{\delta \mu}{\delta \eta},$$

However, unlike in the case of the equally weighted market equilibrium Sharpe ratio, for the value weighted Sharpe ratio, the derivatives needed to sign the comparative statics in lemma 26 and 27 for $\eta \in \{\mu_w, \sigma_w, \rho_{w,x}\}$ are indeterminate.

Wealth effect of Expertise

This section shows that while savings rates can theoretically be slightly decreasing in expertise, due to the wealth effect from higher expertise and the associated larger present value of investment opportunities, this effect tends to be dominated by the portfolio choice effect.

The static model with a consumption savings decision can be written as:

$$v(W, X) = \max_{(I, \theta)} \frac{(W - I)^{1-\gamma}}{1-\gamma} + \beta I^{1-\gamma} \mathbb{E} \left[\frac{R_p^{1-\gamma}}{1-\gamma} \right]$$

subject to:

$$\begin{aligned} R_p &= \theta R + (1 - \theta) R_f, \\ r | (W, X) &\sim N \left(\mu - \frac{1}{2} \frac{\sigma_v^2}{X}, \frac{\sigma_v^2}{X} \right). \end{aligned}$$

Clearly, the portfolio choice problem is independent from the consumption savings decision, and the solution to the portfolio choice problem coincides with that of the static model without the consumption saving decision. For any choice of investment I , the optimal portfolio allocation always solves the same problem, maximizing the expected utility derived from the chosen investment level, given the return process for the riskless and risky assets. Therefore, we can plug the optimal portfolio choice back into the value function, and then derive the optimal investment. Finally we get:

$$I^* = W \frac{\left(\beta E [R_p^{1-\gamma}] \right)^{\frac{1}{\gamma}}}{1 + \left(\beta E [R_p^{1-\gamma}] \right)^{\frac{1}{\gamma}}}$$

where

$$E [R_p^{1-\gamma}] = \exp \left((1 - \gamma) r_f + \frac{1}{2} \frac{(1 - \gamma)}{\gamma} \frac{(\mu - r_f)^2}{\frac{\sigma_v^2}{X}} \right).$$

Then, we can show that:

$$\frac{\partial I^*}{\partial X} = W \frac{\left(\beta E \left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}{\left(1 + \left(\beta E \left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}\right)^2} \frac{1}{2} \frac{(\gamma-1) (\mu - r_f)^2}{\gamma^2} \frac{\partial \left(\frac{\sigma_v^2}{X}\right)}{\partial X}.$$

Observe that the saving rate decreases with the expertise if and only if $\gamma > 1$.

However, for the investment in the risky asset, $I^* \theta^*$, we have:

$$I^* \theta^* = W \frac{\left(\beta E \left[R_p^{1-\gamma} \right]\right)^{\frac{1}{\gamma}} (\mu - r_f)}{1 + \left(\beta E \left[R_p^{1-\gamma} \right]\right)^{\frac{1}{\gamma}} \gamma \frac{\sigma_v^2}{\bar{X}}}$$

Then,

$$\frac{\partial (I^* \theta^*)}{\partial X} = W \frac{\left(\beta E \left[R_p^{1-\gamma} \right]\right)^{\frac{1}{\gamma}} (\mu - r_f)}{1 + \left(\beta E \left[R_p^{1-\gamma} \right]\right)^{\frac{1}{\gamma}} \gamma \left(\frac{\sigma_v^2}{\bar{X}}\right)^2} \left(\frac{1}{2} \frac{(\gamma-1)}{\gamma^2} \frac{(\mu - r_f)^2}{\frac{\sigma_v^2}{\bar{X}}} \frac{1}{1 + \left(\beta E \left[R_p^{1-\gamma} \right]\right)^{\frac{1}{\gamma}}} - 1 \right) \frac{\partial \left(\frac{\sigma_v^2}{\bar{X}}\right)}{\partial X}$$

There are two cases, depending on the coefficient of relative risk aversion:

1. If $\gamma < 1$, the saving rate does not fall with the expertise, neither does the investment in the risky asset.

We have $\frac{1}{2} \frac{(\gamma-1)}{\gamma^2} \frac{(\mu - r_f)^2}{\frac{\sigma_v^2}{\bar{X}}} \frac{1}{1 + \left(\beta E \left[R_p^{1-\gamma} \right]\right)^{\frac{1}{\gamma}}} - 1 < 0$.

Therefore, $\frac{\partial (I^* \theta^*)}{\partial X} > 0, \forall X$.

2. If $\gamma > 1$, the saving rate falls with the expertise, while the investment in the risky asset doesn't, as long as the expertise level is not too high.

We have $\frac{1}{2} \frac{(\gamma-1)}{\gamma^2} \frac{(\mu - r_f)^2}{\frac{\sigma_v^2}{\bar{X}}} \frac{1}{1 + \left(\beta E \left[R_p^{1-\gamma} \right]\right)^{\frac{1}{\gamma}}} - 1 < \frac{1}{2} \frac{(\gamma-1)}{\gamma^2} \frac{(\mu - r_f)^2}{\frac{\sigma_v^2}{\bar{X}}} - 1, \forall X$.

Then $\frac{1}{2} \frac{(\gamma-1)}{\gamma^2} \frac{(\mu - r_f)^2}{\frac{\sigma_v^2}{\bar{X}}} \frac{1}{1 + \left(\beta E \left[R_p^{1-\gamma} \right]\right)^{\frac{1}{\gamma}}} - 1 < 0, \forall X < \bar{X}$, where $\bar{X} = \frac{2\gamma^2}{(\gamma-1)} \frac{\sigma_v^2}{(\mu - r_f)^2}$.

Therefore, $\frac{\partial (I^* \theta^*)}{\partial X} > 0, \forall X < \bar{X}$. The signs for comparative statics for $\forall X > \bar{X}$ are indeterminate.

In sum, investment in the risky asset increases with expertise, as long as the expertise level is not too high. The general functional form for effective risk yields similar results.

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