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ABSTRACT

The assumption of a vanishing neutrino mass leads to a group of transformations of the neutrino field which transform the beta decay interaction into equivalent interactions. By physical observations one cannot distinguish between equivalent interactions. The results of observations can be expressed in terms of nuclear matrix elements and combinations of the coupling constants which are invariant under the group. These invariants have recently been put forward by Pursey and on a more general basis by Pauli. They are explored further in this paper. Their mathematics is studied and relations between them are established. The conditions for invariance with respect to reflections in space, charge conjugation, and time reversal are expressed in terms of these invariants. Interactions which conserve lepton charge and/or couple to only two components of the neutrino field are characterized by relations between the invariants. (For a reader who does not want to follow the detailed arguments the main results are summarized in the last section.) In the Appendix possible experiments on beta decay are expressed in terms of the invariants.

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ON THE PURSEY-PAULI INVARIANTS

IN THE THEORY OF BETA DECAY

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1. As recently shown by Pauli[1] the assumption that the mass of the neutrino is exactly equal to zero leads to a group of linear transformations of the neutrino field operators which leave both commutation relations and free field Hamiltonian invariant. This group is generated by the following two commuting subgroups

$$\psi' = a\,\psi + b\,\gamma_5\, C^{-1}\,\bar{\psi}\,; \qquad \bar{\psi}' = a^*\,\bar{\psi} + b^*\,\psi\, C\,\gamma_5 \tag{1}$$

with

$$|a|^2 + |b|^2 = 1 \tag{2}$$

(transformation (I) in LC) and[2]

$$\psi' = e^{i\alpha\gamma_5}\,\psi\,; \qquad \bar{\psi}' = \bar{\psi}\,e^{i\alpha\gamma_5} \tag{3}$$

with α real (transformation (II) in LC).[3] The symbol C in Eq. (1) denotes the 4 x 4 charge conjugation matrix.[4] Throughout this paper the

[1] W. Pauli, On the Conservation of the Lepton Charge, Il Nuovo Cimento (to be published). We shall use the symbol LC when referring to this paper.

[2] Transformation (II) was often used in the literature. It forms the basis of a discussion by D. L. Pursey of Invariance Properties of Fermi Interactions (Il Nuovo Cimento, to be published).

[3] In the following we shall refer to these two transformations by (I) and (II) as done in LC.

[4] We use the definition given in LC. The present author used a slightly different definition in a recent paper (Annals of Physics 2, 1 (1957)). The relation between these two definitions is $C_{Pauli} = -C^{-1}_{Lüders}$.

conventional four-component theory of the neutrino shall be used; the two-component theory can be treated as a special case in which only one half of the degrees of freedom of the neutrino appears in the interaction Hamiltonian (c.f. Section 6). The existence of especially transformation (I) shows that the concept of particles versus antiparticles is not well defined for free neutrinos. To each momentum and spin parallel or antiparallel to the momentum, there are two linearly independent states of the neutrino; but it is not clear which particular linear combination of these states has to be used to define the neutrino (in the conventional sense) or the antineutrino. Therefore the concept of antineutrino shall be avoided in this paper and the word neutrino shall be used for the whole physical entity described by the four-component spinor.

Pauli's discussion of beta decay is based upon the interaction Hamiltonian

$$H_{int} = \sum_{i=1}^{5} (\bar{\psi}_N O_i \psi_P) \Big[g_{I,i}(\bar{\psi}_\nu O_i \psi_e) - f_{I,i}(\psi_\nu \gamma_5 O_i \psi_e) + g_{II,i}(\psi_\nu C O_i \psi_e) + f_{II,i}(\psi_\nu C \gamma_5 O_i \psi_e) \Big] + \text{herm. conj.} \tag{4}$$

(LC, Eq. (1)) where local interaction is assumed but neither parity conservation nor conservation of lepton charge in the conventional sense. An application of the transformations of the neutrino field does not leave the interaction Hamiltonian invariant but rather can be expressed as a linear transformation of the four coupling constants carrying the same subscript i , i.e., referring to the same type of coupling. Pauli indicates that physical results are not affected by such a transformation (cf. also Sec. 2) and, therefore, can depend only upon invariant combinations of the coupling constants. A complete list of these invariants is given by

$$K_{ij} = K^*_{ji} = g^*_{Ii} g_{Ij} + f^*_{Ii} f_{Ij} + g^*_{IIi} g_{IIj} + f^*_{IIi} f_{IIj} , \tag{5}$$

$$L_{ij} = L^*_{ji} = g^*_{Ii} f_{Ij} + f^*_{Ii} g_{Ij} - g^*_{IIi} f_{IIj} - f^*_{IIi} f_{IIj} , \tag{6}$$

$$I_{ij} = I_{ji} = g_{Ii} g_{IIj} + g_{IIi} g_{Ij} + f_{Ii} f_{IIj} + f_{IIi} f_{Ij} , \tag{7}$$

$$F_{ij} = -F_{ji} = g_{Ii} f_{IIj} - f_{IIi} g_{Ij} + f_{Ii} g_{IIj} - g_{IIi} f_{Ij} . \tag{8}$$

We mention that K_{ii} and L_{ii} are both real and that

$$K_{ii} \geqslant 0 \qquad ; \qquad -K_{ii} \leqslant L_{ii} \leqslant +K_{ii} . \tag{9}$$

The sign of equality in the first equation only holds if that particular type of interaction does not appear at all in the Hamiltonian.

The general structure of these invariants is better understood when one looks at three special transformations contained in the group:

(a) Phase transformation of the neutrino field (transformation (I) with $b = 0$). This transformation amounts to multiplying coupling constants with subscript I by some phase factor and multiplying those with subscript II by the complex conjugate phase factor. Consequently there appear in the invariants either products of coupling constants and complex conjugate coupling constants carrying the same Roman subscript (K_{ij} and L_{ij}) or products of two coupling constants carrying different Roman indices (I_{ij} and J_{ij}).

(b) Charge conjugation of the neutrino field (apply first (I) with $a = 0$, $b = -i$ and subsequently (II) with $\alpha = \pi/2$). This transformation essentially (i.e., apart from signs) leads to an exchange of coupling constants with subscripts I and II. So these coupling have to appear in an essentially symmetrical manner.

(c) Multiplication of ψ_ν by γ_5. This case has been discussed already in LC. As a consequence the constants f and g enter in an essentially symmetrical way.[5]

The invariants K_{ij} and L_{ij} on the one hand and I_{ij} and J_{ij} on the other hand are not quite on the same footing since the neutron, proton, and electron fields can be multiplied by an arbitrary phase factor without changing any physical results. If this transformation again is expressed as a transformation of the coupling constants, K_{ij} and L_{ij} still stay invariant whereas I_{ij} and J_{ij} take up the same phase factor. So physical results must be expressable in terms of K_{ij}, L_{ij} and the combinations $I_{ij} I^*_{\ell m}$, $I_{ij} J^*_{\ell m}$, $J_{ij} J^*_{\ell m}$.

In LC also relative invariants are given, i.e., expressions which remain invariant under (I) but take up a phase factor under (II). If one, however, forms strictly invariant combinations, e.g., $N_{I,ij} N_{II,\ell m}$,[6] one

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5 These considerations could in fact be used for a systematic construction of the invariants starting from any product of coupling constants.— C. P. Enz (Fermi Interaction with Non-Conservation of Lepton Charge and of Parity, to be published in Il Nuovo Cimento) treated the case of double processes with nonvanishing neutrino mass. The bilinear combinations of coupling constants which in the results appear multiplied by m_ν can be constructed in a similar manner if one observes that a neutrino with nonvanishing rest mass still admits the group generated by phase transformations, charge conjugation, and γ_5 multiplication combined with the substitution $m_\nu \rightarrow -m_\nu$. The latter transformation was used in a different context by D. C. Peaslee, Phys. Rev. 91, 1447 (1953).

6 For the notation see LC, Eqs. (18a) and (18b).

sees that they can be expressed in terms of bilinear combinations of the original invariants. This is a consequence of the completeness of the bilinear invariants (c.f. Sec. 3).

The whole physics of beta decay including all double processes can be expressed in terms of the invariants (5) through (8). Particularly, ordinary beta decay depends only on the invariants K_{ij} and L_{ij} (and, of course, the nuclear matrix elements). Double beta decay can be expressed in terms of I_{ij} and J_{ij} (or rather their fully invariant combinations). The chain beta decay-inverse beta decay depends either on I_{ij} and J_{ij} or on K_{ij} and L_{ij} depending on whether the charge of the electron emitted in the two processes is the same or opposite. These statements are true in the lowest nonvanishing order. Practically uninteresting higher order terms might depend on all four types of invariants. We believe that these invariants are not only of theoretical interest but that they also might represent an effective tool for the analysis of experimental data.[7] Therefore we explored these invariants beyond the analysis given by Pursey and Pauli in their papers.

2. It has already been shown Pauli that states which contain neutrons, protons, and electrons but no neutrinos are not affected by the group of transformations. But the same holds with slight modifications for final states which do contain neutrinos. If the neutrinos are not absorbed, e.g., in some subsequent inverse beta decay, they escape essentially unobserved. The most one can hope to measure is their linear momentum (in the case of only one neutrino from momentum conservation) and perhaps the component of the spin parallel to the momentum. Therefore in all statements of physical significance

[7] Cf. the Appendix to this paper.

one has to sum over internal degrees of freedom of the neutrino. Such a situation can be expressed by formulating the final situation in terms, not of a state vector, but of a projection operator (density matrix).

For this purpose the neutrino operator shall be decomposed in the usual way in terms of plane waves

$$\psi_\sigma(\underline{r}) = \frac{1}{\sqrt{V}} \sum_{\underline{p},\lambda} \left(a_{\underline{p}\lambda}\, u_{\underline{p}\sigma}^{\lambda}\, e^{i\underline{p}\cdot\underline{r}} + b^*_{\underline{p}\lambda}\, v_{-\underline{p}\sigma}^{\lambda}\, e^{-i\underline{p}\cdot\underline{r}} \right) \tag{10}$$

$u_{\underline{p}\sigma}^{\lambda}$ ($\lambda = 1,2$) are, for fixed $\underline{p}$, a pair or orthonormal four-spinors[8,9] obeying

$$(\underline{\gamma}\cdot\underline{e} + i\gamma_4)_{\rho\sigma}\; u_{\underline{p}\sigma}^{\lambda} = 0 \tag{11}$$

where $\underline{e}$ is the three dimensional unit vector in the direction of motion. In the same way one has[10]

$$(\underline{\gamma}\cdot\underline{e} + i\gamma_4)_{\rho\sigma}\; v_{-\underline{p}\sigma}^{\lambda} = 0 \;. \tag{12}$$

On the one hand $a_{\underline{p}\lambda}$, $b_{\underline{p}\lambda}$ and $a^*_{\underline{p}\lambda}$, $b^*_{\underline{p}\lambda}$ on the other hand are the well known annihilation and creation operators for neutrinos of momentum $\underline{p}$.

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8 The normalization is $a^*_{p\lambda}\, a_{p\mu} = \delta_{\lambda\mu}$ etc. A covariant normalization (using $\bar{a}_{p\lambda}\, a_{p\mu}$) is not possible for mass zero.

9 The spinors $u_{\underline{p}}^{\lambda}$ and $v_{-\underline{p}}^{\lambda}$ actually do not depend upon the magnitude of the momentum vector $\underline{p}$ but only on its direction, the unit vector $\underline{e}$.

10 Notice that because of the absence of a mass term there is no real difference between spinors of positive and negative energy.

For the present discussion it is advisable to relate the spinors $u_{\underline{p}}^{\lambda}$ and $v_{-\underline{p}}^{\lambda}$ in the following manner,

$$v_{-\underline{p}}^{\lambda} = \gamma_5 \gamma_4 \; C^{-1} (u_{\underline{p}}^{\lambda})^* \tag{13}$$

This correspondence is compatible with the Dirac equations (11) and (12) and conserves orthonormality. If the spinors are eigenstates of the spin component in the direction of motion, Eq. (12) relates spinors of equal eigenvalue.[11] Once this correspondence has been established the creation operators transform under (I) in the following way

$$a'^*_{\underline{p}\lambda} = a^* \; a^*_{\underline{p}\lambda} + b^* \; b^*_{\underline{p}\lambda} \; ; \qquad b'^*_{\underline{p}\lambda} = -b \; a^*_{\underline{p}\lambda} + a \; b^*_{\underline{p}\lambda} \; . \tag{14}$$

Transformation (II) on the other hand leads to a multiplication of each creation operator separately by a phase factor if the spinors have been chosen as eigenstates of the spin component in the direction of motion.

Now we can take up the discussion of projection operators. For the sake of simplicity this discussion shall be limited to final states containing only one neutrino. Let $| \; >$ be a state of the system which is different from this final state only by the absence of the neutrino and $a^*_{\underline{p}\lambda} | \; >$ as well as $b^*_{\underline{p}\lambda} | \; >$ be the neutrino containing states which actually appear as final states. Let

$$P = | \; > < \; | \tag{15}$$

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[11] This follows without calculation from the observation that relation (13) does not depend upon any space direction apart from $\underline{e}$.

be the projection operator corresponding to $| >$. The projection operator for the states with neutrino, summed over the internal degrees of freedom of the neutrino, is then given by

$$P_{\underline{p}\lambda} = a^*_{\underline{p}\lambda} \, P \, a_{\underline{p}\lambda} + b^*_{\underline{p}\lambda} \, P \, b_{\underline{p}\lambda} \quad . \tag{16}$$

Here we assume that the neutrino states correspond to a particular spin component in the direction of the momentum. In most cases one will have to sum over λ (= 1, 2). Now it is easily seen that this projection operator is indeed invariant under the transformation (14) (which corresponds to transformation (I)) and of course also under a multiplication of the creation operators by phase factors (corresponding to transformation (II)).[12]

Neutrinos in initial states cannot be treated in an analogous way. One rather has to include their production mechanism in the physical process.

<u>These considerations show that interaction Hamiltonians which can be transformed into each other by a combination of the transformations (I) and (II) lead to the same physical conclusions</u>. Within the framework of beta decay there is no possibility of distinguishing between them. Such Hamiltonians shall be called equivalent. The invariants (5) through (8) are the same for equivalent Hamiltonians. In the following section we shall show that if two interaction Hamiltonians lead to the same values for the invariants then there is one and generally only one transformation of the group which transforms these Hamiltonians into each other. So equality of

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12 In the second case a simpler proof can be given in terms of the Casimir projection operator in spinor space. One only has to check that

$$e^{i\alpha\gamma_5}(\underline{\gamma}\cdot\underline{e} + i\gamma_4)e^{i\alpha\gamma_5} = (\underline{\gamma}\cdot\underline{e} + i\gamma_4) \quad .$$

the invariants is a necessary and sufficient condition for the equivalence of interaction Hamiltonians. We shall also show that these invariants form a complete set but are not entirely independent.

3. For discussions of a more mathematical character[13] the coupling constants g_I, g_{II}, f_I, f_{II} are less practical than the following linear combinations[14]

$$F_1 = g_I - f_I \ , \qquad G_1 = g_I + f_I \ ,$$
$$F_2 = g_{II} + f_{II} \ , \qquad G_2 = -g_{II} + f_{II} \ ; \tag{17}$$

here the subscript i referring to the type of coupling has been omitted. If one further introduces

$$H_1 = G^*_2 \ , \qquad H_2 = -G^*_1 \tag{18}$$

the transformation of the coupling constants is simply given by[15]

$$(F'_1, F'_2) = (F_1, F_2)T \ , \qquad (H'_1, H'_2) = (H_1, H_2)T \tag{19}$$

with

$$T = e^{-i\alpha} \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} . \tag{20}$$

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[13] The reader not interested in mathematical rigor might very well skip this section. He should however take notice of the inequalities (25), (26), and (27) between the invariants and of the existence of rather complicated identities between them.

[14] LC Eq. (5).

[15] LC Eq. (10).

So the pairs F_1, F_2 and H_1, H_2 transform under the same irreducible representation of the group. If one regards $\underline{F} = (F_1, F_2)$ and $\underline{H} = (H_1, H_2)$ as vectors in a two-dimensonal complex vector space[16] one recognizes that the group generated by transformations (I) and (II) induces a unitary transformation in this space.[17]

The invariants under this group of unitary transformations are given by the scalar products

$$\underline{A} \cdot \underline{B} = A^*_1 B_1 + A^*_2 B_2 \tag{21}$$

of pairs of vectors or, more explicitly, by

$$\underline{F}_i \cdot \underline{F}_j , \qquad \underline{H}_i \cdot \underline{H}_j , \qquad \underline{H}_i \cdot \underline{F}_j \tag{22}$$

where now the subscript referring to types of coupling have been written down explicitly. The relation between these invariants and the invariants (5) through (8) is given by

$$\underline{F}_i \cdot \underline{F}_j = K_{ij} - L_{ij}, \qquad \underline{H}_i \cdot \underline{H}_j = K_{ji} + L_{ji} , \qquad \underline{H}_i \cdot \underline{F}_j = -I_{ij} - J_{ij} = -I_{ji} + J_{ji} . \tag{23}$$

The invariants (22), i.e. (5) through (8), are obviously characteristic for the unitary group in two dimensions and consequently for the group of transformations of the neutrino field.[18] From the theory of invariants of the unitary group it also follows that all invariant combinations of the

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[16] To avoid misunderstanding we should like to mention that the ordinary complex plane in the sense of this terminology is not a two-dimensional but a one-dimensional complex space.

[17] This incidentally shows that the group of transformations of the neutrino field is isomorphic with the unitary group in two dimensions.

[18] This has already been shown in LC. The author realizes that it is more difficult to establish a mathematical fact than to simplify a proof already given.

coupling constants can be expressed in terms of the basic invariants (22);[19] these invariants form a complete set.

At present we are interested in a different question. Provided there are two sets of vectors $\underline{F}_i$, $\underline{H}_i$ and $\underline{F}'_i$, $\underline{H}'_i$ which lead to the same invariants (22) (or (5) through (8)). Does there always exist a transformation of the group which transforms one set into the other? In other words, is equality of all invariants not only a necessary but also a sufficient condition for the equivalence of the two interactions in the sense explained in Section 2 (identical values for all physically meaningful quantities)? From simple geometrical considerations, or from the theory of the unitary group, it follows that the answer is indeed in the affirmative. From the equality of the scalar products between corresponding dashed and undashed vectors, one concludes that lengths and relative orientations of the two sets of vectors are the same; therefore they can be transformed into each other by a unitary transformation. If there are at least two linearly independent vectors the transformation is determined uniquely.

The scalar products (or invariants) are not all independent. First there are inequalities between them which are a consequence of the Cauchy Schwarz inequality

$$(\underline{A}\cdot\underline{A})(\underline{B}\cdot\underline{B}) \geqslant |\underline{A}\cdot\underline{B}|^2 \quad . \tag{24}$$

In terms of the invariants (5) through (8) one finds

$$(K_{ii} \pm L_{ii})(K_{jj} \pm L_{jj}) \geqslant |K_{ij} \pm L_{ij}|^2 \tag{25}$$

$$(K_{ii} + L_{ii})(K_{jj} - L_{jj}) \geqslant |I_{ij} + J_{ij}|^2 \quad . \tag{26}$$

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[19] This is especially true for invariant products of the relative invariants (LC, Eqs. (18) - (18b)).

Inequality (25) is valid both for upper and lower sign. Putting $i = j$ in the second inequality one obtains

$$(K_{ii})^2 \geqslant (L_{ii})^2 + |I_{ii}|^2 \qquad (27)$$

which is a stronger relation than Eq. (9).[20]

There are also identities between the invariants which are obtained in the following way. Since the number of dimensions of the complex vector space is equal to two, any three vectors $\underline{A}$, $\underline{B}$, $\underline{C}$ are linearly dependent, i.e. there are numbers λ, μ, ν not all equal to zero so that

$$\lambda\underline{A} + \mu\underline{B} + \nu\underline{C} = 0. \qquad (28)$$

Forming scalar products of this relation with three vectors $\underline{D}$, $\underline{E}$, $\underline{G}$ one obtains three equations which can be regarded as linear equations for λ, μ, ν. The condition for a non-trivial solution is

$$\begin{vmatrix} \underline{D}\cdot\underline{A} & \underline{D}\cdot\underline{B} & \underline{D}\cdot\underline{C} \\ \underline{E}\cdot\underline{A} & \underline{E}\cdot\underline{B} & \underline{E}\cdot\underline{C} \\ \underline{G}\cdot\underline{A} & \underline{G}\cdot\underline{B} & \underline{G}\cdot\underline{C} \end{vmatrix} = 0 . \qquad (29)$$

From this general expression one can derive relations between the invariants (5) through (8) but at present it seems not worthwhile to do so. An experimental test of such relations might eventually mean a test of the general ansatz (4) for the interaction.

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20 It should be noticed that Eq. (9) is <u>not</u> a consequence of Eq. (24) but rather follows immediately from Eq. (23) since a scalar product of a vector with itself is a real non-negative quantity.

4. Since there is no physical distinction between equivalent Hamiltonians the concept of invariance with respect to symmetry operations (especially reflections in space, charge conjugation, and time reversal) is to be modified. One might very well use the usual definitions of these operations.[21] But it would be unphysical to require that the interaction Hamiltonian is unchanged by such a transformation. One rather has to postulate that the transformed Hamiltonian is equivalent to the original one in the sense of this paper. This means that one only has to study the action of a symmetry operation on the invariants (5) through (8). The conditions that a particular theory is invariant with respect to a symmetry operations have to be expressed entirely in terms of these invariants.

As an example we treat reflections in space in some detail. If one applies the customary parity operation one obtains the following transformation of the coupling constants

$$g'_{Ii} = g_{Ii}\, e^{i\eta}, \qquad g'_{IIi} = -g_{IIi}\, e^{i\zeta},$$
$$f'_{Ii} = -f_{Ii}\, e^{i\eta}, \qquad f'_{IIi} = f_{IIi}\, e^{i\zeta} \tag{30}$$

where $\exp(i\eta)$ and $\exp(i\zeta)$ are arbitrary phase factors. This leads to the following transformation of the invariants

$$K'_{ij} = K_{ij}, \qquad L'_{ij} = -L_{ij}, \qquad I'_{ij} = -I_{ij}\, e^{i(\eta+\zeta)}, \qquad J'_{ij} = J_{ij} e^{i(\eta+\zeta)}. \tag{31}$$

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[21] Cf. e.g., G. Lüders, Annals of Physics 2, 1 (1947). Before application of charge conjugation notice, however, footnote 3 of the present paper.

The conditions for invariance with respect to reflections in space (or conservation of parity) are therefore given by[22]

$$L_{ij} = 0 , \qquad I_{ij} J^{*}_{k\ell} = 0 . \qquad (32)$$

The last condition of course means that either all I_{ij} or all J_{ij} have to vanish. In ordinary beta decay all effects from which a nonconservation of parity can be recognized depend upon the invariants L_{ij}; parity violating effects in double beta decay can be expressed in terms of $I_{ij} J^{*}_{k\ell}$.

If one treats the operations of charge conjugation and time reversal in a similar manner one obtains the following conditions[23]: invariance with respect to charge conjugation:

$$\mathrm{Im}\, K_{ij} = \mathrm{Re}\, L_{ij} = \mathrm{Re}\, I_{ij} J^{*}_{k\ell} = 0 \qquad (33)$$

invariance with respect to time reversal:

$$\mathrm{Im}\, K_{ij} = \mathrm{Im}\, L_{ij} = \mathrm{Im}\, I_{ij} J^{*}_{k\ell} = 0 . \qquad (34)$$

Incidentally, since the quantities L_{ii} are real, one sees that invariance with respect to charge conjugation requires the vanishing of all L_{ii}. One result of the TCP theorem is immediately recognized from the conditions (32)

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22 For the more special case $I_{ij} = J_{ij} = 0$ these conditions and the others presented in this section were already given by Pursey, l.c..

23 Re = real part, Im = imaginary part.

through (34): from the invariance with respect to any two of these symmetry operations, invariance with respect to all three can be inferred. The condition for invariance under all three operations is indeed given by

$$\text{Im}\, K_{ij} = L_{ij} = I_{ij} J^*_{k\ell} = 0 \quad . \tag{35}$$

5. The question whether a particular interaction Hamiltonian conserves lepton charge is to be handled in a similar way. One has to analyze whether there is an equivalent Hamiltonian in which all coupling constants with subscript II vanish. One condition on the invariants is easily recognized

$$I_{ij} = J_{ij} = 0 \quad . \tag{36}$$

Since double beta decay depends only on I_{ij} and J_{ij} this condition physically means that there is no double beta decay (and no effect in the chain beta decay-inverse beta decay with the emission of equally charged electrons in both processes).

The other conditions are obtained if one puts all $f_{II,i}$ and $g_{II,i}$ equal to zero in the invariants K_{ij} and L_{ij}.[24] One gets

$$(K_{ij} \pm L_{ij})(K_{k\ell} \pm L_{k\ell}) = (K_{i\ell} \pm L_{i\ell})(K_{kj} \pm L_{kj}) \quad ; \tag{37}$$

the equations are to be postulated both with plus signs and with minus signs.

[24] The derivation of these equations is more easy if one works with the first two invariants (22) putting $F_{2i} = G_{2i} = 0$; cf. also Eq. (38).— The equations are equivalent to the conditions $N_{I,ij} = N_{II,ij} = 0$ in LC.

We shall show presently that Eqs. (36) and (37) are not only necessary but also sufficient conditions for the conservation of lepton charge. It is remarkable that Eq. (36) alone constitutes almost a sufficient condition.[25] If

$$K_{ij} \pm L_{ij} = 0 \tag{38}$$

(with upper or lower sign) does not hold for all combinations of indices then Eq. (37) can be inferred from (36) through the identities (29). If, however, (38) holds one of the conditions (37) is evidently satisfied but the other has to be postulated.[26] It should also be mentioned that once the conditions for the conservation of lepton charge are satisfied, all inequalities (24) and identities (29) between the invariants are automatically fulfilled; so no further information can be obtained from them.[27]

We now want to show that conditions (36) and (37) together are sufficient for conservation of lepton charge in the sense that there exists an equivalent interaction Hamiltonian with $f_{II,i} = g_{II,i}' = 0$. For this purpose we assume that there is a set of coupling constants $g'_{I,i}$, $g'_{II,i}$, $f'_{I,i}$, $f'_{II,i}$ which leads to invariants fulfulling these

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[25] The reverse, however, is not true; Eq. (36) cannot be concluded from Eq. (37). Equation (37) also holds if there is no conservation of lepton charge but only two components of the neutrino field are coupled to the other fields (cf. Sec. 6).

[26] Enz, l.c., got hold of such an exceptional case with $K_{SS} + L_{SS} = K_{ST} + L_{ST} = K_{TT} + L_{TT} = 0$.

[27] Inequality (25) is obviously satisfied with the sign of equality; the second inequality is fulfilled with vanishing right hand side since both terms on the left hand side are positive (Eq. (9)). In the identity (29) all possible choices of the vectors have to be discussed separately.

relations. Then we ask whether it is possible to choose another set of coupling constants $g_{I,i}$, $f_{I,i}$, to put $g_{II,i} = f_{II,i} = 0$, and still to obtain the same invariants. If Eq. (37) is expressed in terms of the invariants (22) one has

$$(\underline{F}_i \cdot \underline{F}_j)(\underline{F}_k \cdot \underline{F}_\ell) = (\underline{F}_i \cdot \underline{F}_\ell)(\underline{F}_k \cdot \underline{F}_j); \qquad (\underline{H}_j \cdot \underline{H}_i)(\underline{H}_\ell \cdot \underline{H}_k) = (\underline{H}_\ell \cdot \underline{H}_i)(\underline{H}_j \cdot \underline{H}_k) \tag{39}$$

and the question is whether it is possible to choose constants $F_{1,i}$ and $H_{2,i}$ so that

$$\underline{F}_i \cdot \underline{F}_j = F^*_{1i} F_{1j} , \qquad \underline{H}_i \cdot \underline{H}_j = H^*_{2i} H_{2j} . \tag{40}$$

For the general argument is is only necessary to analyze in more detail the invariants $\underline{F}_i \cdot \underline{F}_j$. First we put

$$F_{1j} = e^{i\alpha_j} \sqrt{\underline{F}_j \cdot \underline{F}_j} \tag{41}$$

for all j where the phase factors $\exp(i\alpha_j)$ remain to be determined. We can fix one of these phase factors, e.g., $\exp(i\alpha_1)$, ambiguously[28] and determine the others from putting

$$F^*_{11} F_{1j} = e^{i(\alpha_j - \alpha_1)} \sqrt{(\underline{F}_1 \cdot \underline{F}_1)(\underline{F}_j \cdot \underline{F}_j)} = \underline{F}_1 \cdot \underline{F}_j \tag{42}$$

for all $j \neq 1$. That this is indeed possible follows from (39) with a special choice of indices

$$|\underline{F}_1 \cdot \underline{F}_j|^2 = (\underline{F}_1 \cdot \underline{F}_1)(\underline{F}_j \cdot \underline{F}_j) . \tag{43}$$

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[28] It is assumed that $\underline{F}_1 \cdot \underline{F}_1 \neq 0$.

The F_{1i} determined in this way also give the correct value for $\underline{F}_i \cdot \underline{F}_j$ $(i \neq 1)$[29] since

$$\underline{F}_i \cdot \underline{F}_j = \frac{(\underline{F}_i \cdot \underline{F}_1)(\underline{F}_1 \cdot \underline{F}_j)}{\underline{F}_1 \cdot \underline{F}_1} \quad . \tag{44}$$

6. In contrast to Pauli in LC we do not want to treat the two-component theory of the neutrino as a different case for which new invariants have to be formulated. We rather want to work with the full four-component formulation throughout and to treat the two-component theory as a specialization. This means that we always work with the same invariants (5) through (8) and express experimental information in terms of these invariants only. If one has a two-component neutrino there are identities between the various four-component invariants which one has to test on the experimental data.

The ordinary two-component theory of the neutrino can be written either in the Weyl formulation or in the Majorana formulation. If one works with the Weyl formulation and translates it into the four-component theory it means that one has the following conditions on the coupling constants

$$g_{Ii} = \pm f_{Ii} \quad ; \qquad g_{IIi} = \pm f_{IIi} \tag{45}$$

with either the upper or the lower signs throughout. The Majorana formulation is in four-component language given by

$$g_{Ii} = \mp g_{IIi} \quad ; \qquad f_{Ii} = \pm f_{IIi} \quad . \tag{46}$$

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29 This argument shows that not all conditions (37), (39), respectively, are independent.

The equivalence between the two formulations[30] can be expressed as equivalence in the sense of this paper by recognizing that Eq. (45) is transformed into (46) under transformation (I)[31] with $a = -b = \sqrt{\frac{1}{2}}$.

The statement that we have a two-component theory does not mean that the interaction Hamiltonian really has to fulfill the conditions (45),(46), respectively. It rather means that the particular interaction Hamiltonian is equivalent to a two-component theory fulfilling either of these conditions. Therefore the two-component character can be expressed in terms of identities between the invariants (5) through (8). Inserting either (45) or (46) into these invariants one sees that the following relation is a necessary condition for a two-component theory in the above sense[32]

$$(K_{ij} + L_{ij})(K_{k\ell} - L_{k\ell}) = (I^*_{ik} + J^*_{ik})(I_{j\ell} + J_{j\ell}). \tag{47}$$

Further Eq. (37) has to be satisfied[33] so that from ordinary beta decay alone

[30] Serpe, Physica 18, 295 (1952) and more recent papers by other authors.

[31] This follows most easily from LC, Eq. (6) if it is noticed that Eq. (45) (for upper sign) is equivalent to $F_{1i} = G_{2i} = 0$ and Eq. (46) equivalent to $F_{1i} + F_{2i} = G_{1i} - G_{2i} = 0$. A slightly more general condition on a and b is $a + b^* = 0$. Both the Weyl and the Majorana formulation still admit transformation (II) (cf., LC, Eq. (7)) and charge conjugation; the latter operation changes upper into lower signs in the conditions (45) and (46).

[32] This equation is equivalent to $(\underline{H}_j \cdot \underline{H}_i)(\underline{F}_k \cdot \underline{F}_\ell) = (\underline{H}_i \cdot \underline{F}_k)^*(\underline{H}_j \cdot \underline{F}_\ell)$. The necessity of this condition and of Eq. (37) in the form of Eq. (39) is most conveniently derived in the Weyl formulation with $F_{1i} = G_{2i} = 0$, c.f. footnote 31. This representation is also suitable for proving the sufficiency of the conditions.

[33] One also derives $(I_{ij} + J_{ij})(I_{k\ell} + J_{k\ell}) = (I_{i\ell} + J_{i\ell})(I_{kj} + J_{kj})$ or $(\underline{H}_i \cdot \underline{F}_j)(\underline{H}_k \cdot \underline{F}_\ell) = (\underline{H}_i \cdot \underline{F}_\ell)(\underline{H}_k \cdot \underline{F}_j)$ which, however, is not independent of Eqs. (37) and (47).

(i.e., K_{ij} and L_{ij}) one cannot decide between a four-component interaction which conserves lepton charge and a two-component interaction for which conservation of lepton charge has not been postulated. All inequalities (25) and (26) and identities (29) are again satisfied; in fact all two-rowed subdeterminants of (29) vanish. That the conditions (37) and (47) are also sufficient is shown in a similar manner as in Section 5 for the conservation of lepton charge.

Now the particular case of a two-component theory which conserves lepton charge[34] shall be treated. It follows from Eqs. (36) and (47) that

$$(K_{ij} + L_{ij})(K_{k\ell} - L_{k\ell}) = 0 \tag{48}$$

or[35]

$$K_{ij} \pm L_{ij} = 0 \tag{49}$$

with the same sign for all indices. Consequently Eq. (37) reduces to

$$K_{ij} K_{k\ell} = K_{i\ell} K_{kj} \quad . \tag{50}$$

Eqs. (49) and (50) together form necessary and sufficient conditions for a two-component interaction with conservation of lepton charge.[36] For all i with $K_{ii} \neq 0$ [37] one finds L_{ii} $(=\pm K_{ii}) \neq 0$. Consequently one necessarily has violation of both parity and charge conjugation (cf. Eqs. (32) and (33)) in a two-component theory which conserves lepton charge.[38]

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[34] In most of the current literature such a theory is simply called a two-component theory.

[35] The relations for $i \neq j$ can be inferred from those for $i = j$ by means of inequality (25).

[36] Notice that inequality (26) leads to Eq. (36) as a consequence of Eq. (49).

[37] Cf. our remarks in connection with Eq. (9).

[38] This has been recognized recently by several physicists on the basis of less general formulations of beta decay theory.

Since the second inequality (9) imposes a limitation on the possible values of L_{ii} one so gets maximum violation of parity and charge conjugation.

Both the Weyl equation and the Majorana equation lead to a neutrino the physical state of which is entirely specified by momentum and component of the spin in the direction of the momentum. In the four-component theory these quantum numbers do not specify the state of a neutrino completely; one rather has an additional two-fold degeneracy. The fact that the group generated by transformations (I) and (II) acts on this additional degree of freedom but does not change physical results means that this degree of freedom is physically redundant. This could be regarded as an argument in favor of the realization of the two-component neutrino in nature. We think, however, that one should be most reluctant with arguments of this kind.

7. All experimental information in the field of beta decay can, under the assumption of vanishing neutrino rest mass and local interactions (Eq. (4)), be expressed in terms of the bilinear combinations (invariants) (5) through (8) of the coupling constants and of nuclear matrix elements. Especially ordinary beta decay depends only on the quantities K_{ij} and L_{ij}; without additional assumptions or conventions more detailed information about the coupling constants themselves cannot be obtained from experiments. Invariance with respect to reflections in space (i.e. conservation of parity), charge conjugation, and time reversal are expressed by the conditions (32), (33), (34), respectively. Conservation of lepton charge is fulfilled if the conditions (36) and (37) are satisfied; these conditions do not only forbid double beta decay but they also put limitations on the quantities entering into ordinary beta decay. If beta decay is adequately described by a two-component neutrino, Eqs. (37) and (47) have to be fulfilled. A two-component thery which conserves lepton charge is characterized by Eqs. (49) and (50).

Unfortunately Eq. (37) connecting quantities which can be derived from ordinary beta decay alone is a necessary condition for both conservation of lepton charge and two-component interaction (with no requirements as to conservation of lepton charge). So from single beta decay data one cannot decide between the two cases. The stronger requirement of a two component neutrino interaction which simultaneously conserves lepton charge can, however, be tested on information from beta decay alone.

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Appendix

Allowed Transitions

For more detailed formulae one has to specify the Dirac matrices O_i in Eq. (4). This can for instance be done by postulating that the square of each of them is equal to one which leads to the following list[39]

$$1, \quad \gamma_\mu , \quad i\,\gamma_{\mu\nu} , \quad i\,\gamma_\mu \gamma_5 , \quad \gamma_5 .$$

Pauli's notation for the coupling constants is not identical with the one used in current beta decay theory. The relation between Pauli's coupling constants g_{Ii}, f_{Ii} and C_i, C'_i [40] is given by

$$g_{Ii} = C^*_i , \qquad f_{Ii} = -C'^*_i .$$

Here, it is understood that in the term for tensor interaction it is to be summed only once over each pair of tensor indices (or that a factor of $\frac{1}{2}$ is to be added if free summation is permitted).

In the following table[41,42] many observable quantities in allowed beta decay are expressed in terms of the invariants K_{ij} and L_{ij}. To obtain such expressions one only has to make use of calculational results for an interaction containing both g_{Ii} and f_{Ii} (or C_i and C'_i); from the general arguments given in LC and in this paper it then follows that

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39 This distribution of imaginary units does, however, not give tensors which are bilinear in Dirac fields and have simple Hermiticity properties.

40 T. D. Lee and C. N. Yang, Phys. Rev. 104, 254 (1956).

41 The table has been compiled mainly by Dr. T. Kotani. It is based on recent papers (cf., footnotes 40 and 42) and on unpublished work by himself; cf., also his University of California Radiation Laboratory Report No. 3798. The present author is very grateful to Dr. Kotani for his permission to publish the table, and for many discussions of its content.

42 J. D. Jackson, S. B. Treiman, and H. W. Wyld, Phys. Rev. 106, 517 (1957); M. E. Ebel and G. Feldman, Phys. Rev. (to be published); M. Morita and R. S. Morita, Phys. Rev. 107, 139 (1957), and Phys. Rev. (to be published).

the coupling constants g_{IIi} and f_{IIi} can only occur in such a way as to complete the invariants K_{ij} and L_{ij}.[43] The table is mainly presented to show explicitly in what observable effects the various invariants enter. Since complete formulae for these effects are not presented here the reader should in any particular case use formulae already given in the literature and then generalize them in the same way as has been done for the construction of the table.

In the table, essentially the factors are given which in the various observable quantities appear multiplied by the squared Fermi matrix element, the squared Gamow-Teller matrix element, or products between these two matrix elements. First order Coulomb corrections (terms proportional to $(\alpha Z)^1$) are presented besides the main terms (no dependence upon αZ). Experiments 1, 2, 7, and 12 do not show any violation effects in the Coulomb independent part; the results depend only upon Re K_{ij}. Indications for invariance under charge conjugation and under time reversal can, however, be obtained from the Coulomb term in Experiment 2. Experiments 3, 4, 5, 6, and 13 are typical experiments for testing the violation of parity; in some of the cases conservation or violation of time reversal can be read from the terms proportional to αZ. Experiments 10 and 11 in principle also test parity violation; the main effect vanishes, however, in these cases if time reversal is not violated. Experiments 8 and 9 (depending upon Im K_{ij} in the main term) check invariance with respect to charge conjugation and time reversal whereas the second condition for invariance under time reversal

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[43] For nonviolation effects it is even sufficient to use calculations for parity conserving interactions; cf., also T. D. Lee and C. N. Yang, Phys. Rev. 104, 254 (1956), especially Eqs. (A.3) through (A.5). For Eq. (A.4) cf. Errata in Phys. Rev. 106, 1371 (1957).

($\mathrm{Im}\, L_{ij} = 0$) is tested in Experiments 10, 11, and 14. Experiments 12, 13, and 14 are β-γ correlation experiments. The symbol m denotes the mass of the electron and E its total energy. Where two signs ($\pm$ or $\mp$) are given the upper sign refers to emission of positive electrons, the lower to emission of negative electrons.

In the following we give short formal expressions for the various observable quantities. Notations: $\underline{J}$ = spin of oriented nucleus, $\underline{p}$ = electron momentum, $\underline{\sigma}$ = electron spin, $\underline{q}$ = neutrino momentum, $\underline{k}$ = γ momentum, τ(= $\pm$ 1) symbol for circular polarization of γ quantum.

2:	$\underline{p}\cdot\underline{q}$	8:	$\underline{J}\cdot\underline{p} \times \underline{q}$
3:	$\underline{p}\cdot\underline{J}$	9:	$\underline{\sigma}\cdot\underline{p} \times \underline{q}$
4:	$\underline{p}\cdot\underline{\sigma}$	10:	$\underline{\sigma}\cdot\underline{J} \times \underline{p}$
5:	$\underline{q}\cdot\underline{J}$	11:	$\underline{\sigma}\cdot\underline{J} \times \underline{q}$
6:	$\underline{q}\cdot\underline{\sigma}$	12:	$\underline{k}\cdot\underline{J}$
7a:	$\underline{J}\cdot\underline{\sigma}$	13:	$\tau(p\cdot k)$
7b:	$(\underline{J}\cdot\underline{p})(\underline{p}\cdot\underline{\sigma})$	14:	$(\underline{J}\cdot\underline{p} \times \underline{k})(\underline{J}\cdot\underline{k})^n$, $(n = 1, 3)$ and $\tau\,(\underline{J}\cdot\underline{p} \times \underline{k})$.

	Type of Experiment		Fermi		Gamow-Teller		Cross Terms	
			$(\alpha Z)^{o}$	$(\alpha Z)^{1}$	$(\alpha Z)^{o}$	$(\alpha Z)^{1}$	$(\alpha Z)^{o}$	$(\alpha Z)^{1}$
1.	Spectrum and Lifetime	Non-Cross Term	$K_{SS} + K_{VV}$		$K_{TT} + K_{AA}$		--	
		Fierz Term	$\pm \frac{2}{E}$ Re K_{SV}		$\pm \frac{2}{E}$ Re K_{TA}		--	
2.	β-ν Angular Correlation		$-K_{SS}+K_{VV}$	$\pm K_{SV}$	$K_{TT}-K_{AA}$	$\mp K_{TA}$	--	--
3.	β Distribution from Oriented Nuclei		--	--	$\mp(L_{TT}-L_{AA})$	Im L_{TA}	$-\text{Re}(L_{ST}-L_{VA})$	$\pm \text{Im}(L_{SA}-L_{VT})$
4.	Longitudinal Polarization of β		$\mp(L_{SS}-L_{VV})$	Im L_{SV}	$\mp(L_{TT}-L_{AA})$	Im L_{TA}	--	--
5.	ν Distribution from Oriented Nuclei		--	--	$\mp(L_{TT}+L_{AA})$ $-\frac{2m}{E}$ Re L_{TA}	--	$\text{Re}(L_{SA}+L_{VT})$ $\pm \frac{m}{E} \text{Re}(L_{ST}+L_{VA})$	--
6.	β Polarization along Direction of ν		2Re L_{SV} $\pm \frac{m}{E}(L_{SS}+L_{VV})$	--	-2Re L_{TA} $\mp \frac{m}{E}(L_{TT}+L_{AA})$	--	--	--

Table continued

Type of Experiment	Fermi		Gamow-Teller		Cross Terms	
	$(\alpha Z)^0$	$(\alpha Z)^1$	$(\alpha Z)^0$	$(\alpha Z)^1$	$(\alpha Z)^0$	$(\alpha Z)^1$
7a. β Polarization from Oriented Nuclei*	--	--	$\pm 2\mathrm{Re}\, K_{TA} + \frac{m}{2E}(K_{TT}+K_{AA})$	--	$\mathrm{Re}(K_{SA}+K_{VT}) \pm \frac{m}{E}\mathrm{Re}(K_{ST}+K_{VA})$	--
7b.	--	--	$K_{TT}+K_{AA} \mp 2\,\mathrm{Re}\,K_{TA}$		$-\mathrm{Re}(K_{SA}+K_{VT}) \pm \mathrm{Re}(K_{ST}+K_{VA})$	
8. β-ν Angular Correlation from Oriented Nuclei	--	--	--	--	$\mathrm{Im}(K_{VA}-K_{ST})$	$\mp \mathrm{Re}(K_{SA}-K_{VT})$
9. Correlation of β and ν Directions with β Polarization	$-\mathrm{Im}\,K_{SV}$	$\mp(K_{SS}-K_{VV})$	$\mathrm{Im}\,K_{TA}$	$\pm(K_{TT}-K_{AA})$	--	--
10. Polarization of β Perpendicular to Plane of Decay, for Oriented Nuclei**	--	--	$\pm \mathrm{Im}\,L_{TA}$	$L_{TT}-L_{AA}$	$\mathrm{Im}(L_{SA}-L_{VT})$	$\pm \mathrm{Re}(L_{ST}-L_{VA})$

* There are two distinct types of such experiments, cf., summary on page 26.

** The plane is here defined by the direction of polarization of the decaying nucleus (but not its recoil) and by the direction of emission of the electron.

Table continued

	Type of Experiment	Fermi		Gamow-Teller		Cross Terms	
		$(\alpha Z)^0$	$(\alpha Z)^1$	$(\alpha Z)^0$	$(\alpha Z)^1$	$(\alpha Z)^0$	$(\alpha Z)^1$
11.	β Polarization Correlated with Direction of ν, for Aligned Nuclei	--	--	--	--	$\pm\frac{m}{E}\mathrm{Im}(L_{ST}+L_{VA})$ $+\mathrm{Im}\,(L_{SA}+L_{VT})$	--
12.	γ Distribution from Oriented Nuclei	$K_{SS}+K_{VV}$ $\pm\frac{2m}{E}\,\mathrm{Re}\,K_{SV}$	--	$K_{TT}+K_{AA}$ $\pm\frac{2m}{E}\,\mathrm{Re}\,K_{TA}$	--	--	--
13.	Correlation between β and Polarized γ	--	--	$\pm(L_{TT}-L_{AA})$	$\mathrm{Im}L_{TA}$	$\mathrm{Re}(L_{ST}-L_{VA})$	$\mp\mathrm{Im}(L_{SA}-L_{VT})$
14.	β-γ Correlation from Oriented Nuclei	--	--	--	--	$\mathrm{Im}(L_{ST}-L_{VA})$	$\pm\mathrm{Re}(L_{SA}-L_{VT})$