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ABSTRACT

The solution of the ABFST multiperipheral integral equation with a narrow-resonance kernel is investigated. First, an approximation scheme that leads to a tractable analytic approximate solution is presented for both the forward and nonforward equations. Next, the exact numerical solutions are displayed for the relevant values of the input parameters: these results serve as a measure of the accuracy of various analytic approximate solutions. The approximate solution presented here, which is found to be good to within about 10% in the region of interest, should be useful both in the general study of the output of the multiperipheral model and in the Pomeranchukon perturbation theory.

I. INTRODUCTION

In this paper we shall investigate in two complementary ways the solution of the ABFST multiperipheral integral equation,[1] in the modern version formulated by Chew, Rogers, and Snider,[2] and by Abarbanel, Chew, Goldberger, and Saunders.[3] We shall study in detail the solution of the equation with the simplest kernel consisting of a single sharp resonance, and discuss only briefly the straightforward generalization to the case of a kernel with many resonances. This solution, in the language of Ref. 3, corresponds to the "unperturbed solution," since we neglect the small high-subenergy diffractive scattering part in the input kernel.

In order to gain insight into the nature of the output, we first obtain an analytic approximate solution by replacing the original kernel by a factorizable kernel. This replacement is guided in some sense by "peripheralism," that is, the factorizable kernel should behave like the original kernel in the peripheral region, where the contribution to any convergent integral involved is expected to be important. We shall demonstrate that the solution so obtained reproduces itself under the action of the original kernel in the most peripheral region.

On the other hand, we have also solved the equation numerically for certain values of the input parameters. This solution provides a measure of the accuracy of various analytic approximate solutions.

Our analytic approximate method is presented in Sec. II (for forward scattering) and Sec. III (for nonforward scattering). There is no pretense of rigor; rather, in a practical way we shall develop a tractable explicit form that is simple enough and yet has reasonable accuracy. The latter point is justified by comparing with the exact numerical solution which is presented in Sec. IV. Some generalizations and the question of the uniqueness of our approximation scheme are presented at the end of the paper.

II. THE FORWARD EQUATION AND THE APPROXIMATE SOLUTION

We shall first illustrate the properties of our approximation in the case of forward scattering ($q = 0$ in Fig. 1). Let us here ignore the problem of internal symmetry; this can easily be incorporated into the model by introducing crossing matrices as described in Ref. 3. The absorptive part A of the elastic amplitude T of pseudoscalar-meson—pseudoscalar-meson scattering is normalized in such a way that

$$A(s,\mu^2,\mu^2) = \Delta^{\frac{1}{2}}(s,\mu^2,\mu^2)\, \sigma^{tot}(s,\mu^2,\mu^2) , \qquad \text{(II-1)}$$

where μ^2 is the meson mass squared, $\Delta(x,y,z) = x^2 + y^2 + z^2 - 2(xy + yz + zx)$, and σ^{tot} is the total meson-meson (μ-μ) cross section. The elastic μ-μ cross section σ^{el} enters in the input potential of the equation in the (on-shell) form

$$V(s,\mu^2,\mu^2) = \Delta^{\frac{1}{2}}(s,\mu^2,\mu^2)\, \sigma^{el}(s,\mu^2,\mu^2) . \qquad \text{(II-2)}$$

The $O(1,3)$ partial wave of A is defined as

$$A_\lambda(\tau_1,\tau_2) = \int_{4\mu^2}^{\infty} ds\, e^{-(\lambda+1)\theta(s,\tau_1,\tau_2)} A(s,\tau_1,\tau_2) ; \qquad \text{(II-3)}$$

the inverse transform is

$$A(s,\tau_1,\tau_2) = \int_{c-i\infty}^{c+i\infty} \frac{d\lambda}{2\pi i} \frac{e^{+(\lambda+1)\theta(s,\tau_1,\tau_2)}}{2(-\tau_1)^{\frac{1}{2}}(-\tau_2)^{\frac{1}{2}} \sinh \theta(s,\tau_1,\tau_2)} A_\lambda(\tau_1,\tau_2) , \qquad \text{(II-4)}$$

where the contour is taken to the right of any singularity of A_λ in the λ plane. In terms of A_λ, the ABFST equation is

$$A_\lambda(\tau_1,\tau_2) = V_\lambda(\tau_1,\tau_2) + \frac{1}{16\pi^3(\lambda+1)} \int_{-\infty}^{0} \frac{d\tau'}{(\mu^2-\tau')^2}$$

$$\times\ V_\lambda(\tau_1,\tau')\, A_\lambda(\tau',\tau_2)\ . \qquad \text{(II-5)}$$

The essence of our method is to approximate

$$e^{-\Theta(s,\tau_1,\tau_2)} = \frac{2(-\tau_1)^{\frac{1}{2}}(-\tau_2)^{\frac{1}{2}}}{(s-\tau_1-\tau_2) + [(s-\tau_1-\tau_2)^2 - 4(-\tau_1)(-\tau_2)]^{\frac{1}{2}}} \qquad \text{(II-6)}$$

by a factorizable expression[4]

$$\xi(s,\tau_1,\tau_2) \equiv \frac{(-s\tau_1)^{\frac{1}{2}}(-s\tau_2)^{\frac{1}{2}}}{(s-\tau_1)(s-\tau_2)}\ . \qquad \text{(II-7)}$$

The function ξ is actually a lower bound to $e^{-\Theta}$. Notice that

$$e^{-\Theta(s,\tau_1,\tau_2)} \rightarrow \xi(s,\tau_1,\tau_2)\left[1 + \frac{2(-\tau_1)(-\tau_2)}{s^2}\right] \qquad \text{(II-8)}$$

when either τ_1 or τ_2, or both, approach zero. When either τ_1 or τ_2, or both, approach (minus) infinity, the two expressions are different. One hopes that, in any convergent integral involved in the calculation, the contributions from these "nonperipheral" regions do not matter very much. Notice also that ξ is a small quantity for all values of τ_1 and τ_2. For a given s, it has an absolute maximum

$$\xi(s,\tau_1,\tau_2)\Big|^{\text{abs.max.}}_{\tau_1=\tau_2=-s} = \frac{1}{4}\ , \qquad \text{(II-9)}$$

whereas

$$e^{-\Theta(s,\tau_1,\tau_2)}\Big|^{\text{max.}}_{\text{fixed } \tau_2,\, -\tau_1=s-\tau_2} = \frac{(s-\tau_2)^{\frac{1}{2}} - (s)^{\frac{1}{2}}}{(-\tau_2)^{\frac{1}{2}}} \leq 1,$$

approaching the absolute maximum value of 1 for $-\tau_1 = -\tau_2 \gg s$.

With this approximation, Eq. (II-5) is immediately soluble. Here we consider the solution for the kernel with a single (sharp) resonance. A kernel with many resonances will be discussed in Sec. V. Thus we put for the (on-shell) potential

$$V(s,\mu^2,\mu^2) = \Delta^{\frac{1}{2}}(s,\mu^2,\mu^2)\pi m x\Gamma\ \sigma^{el}_{max}\delta(s-m^2) \equiv m^2R(0)\delta(s-m^2)\ , \qquad \text{(II-10)}$$

where m^2, x, and Γ are the squared mass, elasticity, and width of the μ-μ resonance. We shall assume $\mu^2 \ll m^2$. The solution to Eq. (II-5) is then

$$A_\lambda(\tau_1,\tau_2) = \frac{m^2R(0)\left[\dfrac{(-m^2\tau_1)^{\frac{1}{2}}(-m^2\tau_2)^{\frac{1}{2}}}{(m^2-\tau_1)(m^2-\tau_2)}\right]^{\lambda+1}}{1-\mathrm{Tr}K_\lambda}\ , \qquad \text{(II-11)}$$

where

$$\mathrm{Tr}K_\lambda = \frac{m^2R(0)}{16\pi^3(\lambda+1)}\int_{-\infty}^{0}\frac{d\tau}{(\mu^2-\tau)^2}\left[\frac{-m^2\tau}{(m^2-\tau)^2}\right]^{\lambda+1}\ , \qquad \text{(II-12)}$$

$$= \frac{R(0)}{16\pi^3}\, B(\lambda+2,\lambda+2)\ F\!\left(2,\lambda+2,\ 2\lambda+4,\ 1-\frac{\mu^2}{m^2}\right), \qquad \text{(II-13)}$$

$$= \frac{R(0)}{16\pi^3}\,\frac{B(\lambda,\lambda)}{2(2\lambda+1)}\ , \qquad \text{for } \mu^2 = 0. \qquad \text{(II-14)}$$

In Eq. (II-13) and Eq. (II-14), B is the Euler beta function and F is the hypergeometric function. The eigenvalue condition is given by the vanishing of the Fredholm determinant

$$D(\lambda) = 1 - \mathrm{Tr}K_\lambda = 0 . \qquad \text{(II-15)}$$

A special property of this approximate solution is that, under the action of the original kernel, it "reproduces itself" for $-\tau_1$, $-\tau_2$, or both, small (in comparison with m^2). This can best be illustrated by going back to the s plane. From Eq. (II-4) and Eq. (II-11), we get for the leading behavior of the full amplitude

$$A(s,\tau_1,\tau_2) \underset{s\to\infty}{\sim} 16\pi^3 \beta_\alpha \left(\frac{m^2}{m^2-\tau_1}\right)^{\alpha+1} \left(\frac{m^2}{m^2-\tau_2}\right)^{\alpha+1} \left(\frac{s}{m^2}\right)^{\alpha}, \qquad \text{(II-16)}$$

where α is the largest value of λ satisfying Eq. (II-15), and

$$\beta_\alpha = -\left[\frac{\partial}{\partial\lambda} \frac{16\pi^3}{R(0)} \mathrm{Tr}K_\lambda\right]^{-1}_{\lambda=\alpha} . \qquad \text{(II-17)}$$

In the interest of simplicity and clarity, let us put $\mu^2 = 0$ for the moment; then the amplitude at the physical (and most peripheral) point is

$$A(s,0,0) \underset{s\to\infty}{\sim} 16\pi^3 \beta_\alpha \left(\frac{s}{m^2}\right)^{\alpha} . \qquad \text{(II-18)}$$

On the other hand, in this asymptotic region of the s plane, the full amplitude, when written in the form

$$A(s,\tau_1,\tau_2) \underset{s\to\infty}{\sim} \phi_\alpha(\tau_1,\tau_2) s^{\alpha} , \qquad \text{(II-19)}$$

satisfies an equation corresponding to Eq. (II-5)

$$[(-\tau_1)^{\frac{1}{2}}(-\tau_2)^{\frac{1}{2}}]^{\alpha+1} \phi_\alpha(\tau_1,\tau_2)$$

$$\sim \frac{1}{16\pi^3(\alpha+1)} \int ds\, V(s,\tau_1,\tau') \int_{-\infty}^{0} \frac{d\tau'}{(\mu^2-\tau')^2}\, e^{-(\alpha+1)\theta(s,\tau_1,\tau')}$$

$$\times [(-\tau')^{\frac{1}{2}}(-\tau_2)^{\frac{1}{2}}]^{\alpha+1} \phi_\alpha(\tau',\tau_2) . \qquad \text{(II-20)}$$

If we put Eq. (II-16) as a trial function into the right-hand side of Eq. (II-20) with the original kernel, the <u>output</u> physical amplitude is

$$A(s,0,0) \sim \lim_{\tau_1,\tau_2\to 0} \phi_\alpha(\tau_1,\tau_2) s^{\alpha} \qquad \text{(II-21)}$$

$$\sim \frac{m^2 R(0)}{16\pi^3(\alpha+1)} \int_{-\infty}^{0} \frac{d\tau'}{\tau'^2} \left(\frac{-\tau'}{m^2-\tau'}\right)^{\alpha+1} 16\pi^3 \beta_\alpha \left(\frac{m^2}{m^2-\tau'}\right)^{\alpha+1} \left(\frac{s}{m^2}\right)^{\alpha} \qquad \text{(II-22)}$$

which is just Eq. (II-18) by virtue of Eq. (II-12) and Eq. (II-15). (Actually the condition $\tau_2 = 0$ is not necessary in this part of the argument; τ_2 can take any value.) The corresponding property can of course be demonstrated in the λ plane. It should be noted that some previously proposed approximate solutions[5-8] do not possess this property. Comparisons of the solution proposed here and other approximate solutions with the exact numerical solution will be given in Sec. IV.

III. THE NONFORWARD EQUATION AND ITS APPROXIMATE SOLUTION

Away from $t = 0$, the on-shell potential is given by

$$V(s,t) = \frac{1}{16\pi^2 \Delta^{\frac{1}{2}}(s,\mu^2,\mu^2)} \int \frac{dt_1 dt_2 \theta\left(-\Delta(t,t_1,t_2) + \frac{4tt_1t_2}{s-4\mu^2}\right)}{\left[-\Delta(t,t_1,t_2) + \frac{4tt_1t_2}{s-4\mu^2}\right]^{\frac{1}{2}}} T^*(s,t_1)T(s,t_2), \quad \text{(III-1)}$$

where T is the complete elastic amplitude; $\operatorname{Im} T(s,t) = A(s,t)$.

A single (sharp) resonance contributes a potential

$$V(s,t) = 2 \times \frac{16\pi s}{\Delta^{\frac{1}{2}}(s,\mu^2,\mu^2)} (2L+1)P_L(z_s)\pi m x \Gamma \delta(s - m^2)$$

$$\equiv m^2 R(t)\delta(s - m^2), \quad \text{(III-2)}$$

where

$$z_s = 1 + \frac{2s}{\Delta(s,\mu^2,\mu^2)} t$$

and L is the spin of the resonance.

The appropriate $O(1,2)$ partial-wave amplitude is

$$A_\ell(\tau_1,z_1,\tau_2,z_2;\, t) = \int_{4\mu^2}^{\infty} ds\, Q_\ell(\cosh\psi)\, A(\psi,\tau_1,z_1,\tau_2,z_2;\, t), \quad \text{(III-3)}$$

and the inverse transform is

$$A(\psi,\tau_1,z_1,\tau_2,z_2;\, t) = \int_{c-i\infty}^{c+i\infty} \frac{d\ell}{2\pi i} \frac{(2\ell+1)P_\ell(\cosh\psi)}{2[(-\tau_1)(1-z_1^2)(-\tau_2)(1-z_2^2)]^{\frac{1}{2}}} A_\ell(\tau_1,z_1,\tau_2,z_2;t), \quad \text{(III-4)}$$

where the contour is taken to the right of any singularity of A_ℓ in the ℓ plane.

In terms of A_ℓ, the nonforward ABFST equation is

$$A_\ell(\tau_1,z_1,\tau_2,z_2;\, t) = V_\ell(\tau_1,z_1,\tau_2,z_2;\, t) + \frac{1}{16\pi^4} \int_{-\infty}^{0} d\tau' \int_{-1}^{+1} \frac{dz'(1-z'^2)^{-\frac{1}{2}}}{[(\mu^2 - \tau' - \frac{t}{4})^2 - \tau' t z'^2]} V_\ell(\tau_1,z_1,\tau',z';\, t)$$

$$\times A_\ell(\tau',z',\tau_2,z_2;\, t). \quad \text{(III-5)}$$

In order to make an approximation similar to that discussed in Sec. II, we note that the function $Q_\ell(\cosh\psi)$ can be expanded as[9]

$$Q_\ell(\cosh\psi) = \sum_{n=0}^{\infty} \frac{\Gamma^2(\ell+1)2^{2\ell+1} n!}{\Gamma(2\ell+2+n)} [(1-z_1^2)^{\frac{\ell+1}{2}} C_n^{\ell+1}(z_1)]$$

$$\times [(1-z_2^2)^{\frac{\ell+1}{2}} C_n^{\ell+1}(z_2)]\, e^{-(\ell+1+n)\theta(s,\tau_1,\tau_2)}, \quad \text{(III-6)}$$

where $C_n^{\ell+1}$ is a Gegenbauer polynomial. Now, as before, we shall replace $e^{-\theta(s,\tau_1,\tau_2)}$ by $\xi(s,\tau_1,\tau_2)$ of Eq. (II-7). With the input potential Eq. (III-2), the kernel in Eq. (III-5) is then a sum of factorized terms. We shall discuss this case in Sec. V. As a first approximation here, we take only the first term of the sum in

Eq. (III-6).[10] This is not unreasonable since, as we have realized above [Eq. (II-9)], $\xi(s,\tau_1,\tau_2)$ is a small quantity throughout the range of integration. Thus

$$\mathrm{TrK}_\ell = \frac{m^2 R(t)\, B(\ell+1,\frac{1}{2})}{16\pi^4}\int_{-\infty}^{0} d\tau \int_{-1}^{+1} \frac{dz(1-z^2)^{\ell+\frac{1}{2}}}{[(\mu^2-\tau-\frac{t}{4})^2-\tau t z^2]}$$

$$\times \left[\frac{-m^2\tau}{(m^2-\tau)^2}\right]^{\ell+1} , \qquad \text{(III-7)}$$

$$= \frac{(m^2)^{\ell+2}R(t)}{16\pi^3(\ell+1)}\int_0^\infty du\, \frac{u^{\ell+1}}{(m^2+u)^{2\ell+2}}\, \frac{1}{(\mu^2+u+\zeta)^2}$$

$$\times\ F\left(1,\frac{1}{2},\ell+2,\frac{4u\zeta}{(\mu^2+u+\zeta)^2}\right) , \qquad \text{(III-8)}$$

where we have used the notation $u = -\tau$ and $\zeta = -\frac{t}{4}$ for convenience. Now observe that, for a given $\zeta > 0$, the expression $(4u\zeta)/(\mu^2+u+\zeta)^2$ is always less than or equal to 1 throughout the range of integration

$$\left.\frac{4u\zeta}{(\mu^2+u+\zeta)^2}\right|^{\max.}_{u=\mu^2+\zeta} = \frac{\zeta}{\mu^2+\zeta} \begin{array}{ll} < 1 & \text{for } \mu^2 \neq 0, \\ = 1 & \text{for } \mu^2 = 0 . \end{array} \qquad \text{(III-9)}$$

Thus, the expansion of F as a hypergeometric series in powers of $(4u\zeta)/(\mu^2+u+\zeta)^2$ always stays within the radius of convergence of the series for Re $\ell > -2$. After this expansion has been made, the series can be integrated term by term, each term being expressed as a hypergeometric function. Thus we get a series of hypergeometric functions with coefficients $(\zeta/m^2)^n$. The first two terms are as follows:

$$\mathrm{TrK}_\ell = \frac{R(t)}{16\pi^3(\ell+1)}\left\{B(\ell+2,\ \ell+2)\ F\left(2,\ell+2,2\ell+4,\ 1-\frac{\mu^2+\zeta}{m^2}\right)\right.$$

$$+\ \frac{4}{2(\ell+2)}\left(\frac{\zeta}{m^2}\right) B(\ell+1,\ell+3)\ F\left(4,\ell+1,2\ell+4,1-\frac{\mu^2+\zeta}{m^2}\right)$$

$$\left. +\ O\left(\left(\frac{\zeta}{m^2}\right)^2 F\right)\right\} . \qquad \text{(III-10)}$$

The next step is to transform[11] $F(a,b,c,1-x)$ into $F(a',b',c',x)$ and then to express $F(a',b',c',x)$ as a hypergeometric series in powers of $x \equiv (\mu^2+\zeta)/(m^2)$ [since we shall be interested only in the small t region where $(\mu^2+\zeta)/(m^2) \le 1$]. After this manipulation, we obtain the eigenvalue condition

$$1 = \mathrm{TrK}_\ell$$

$$= \frac{R(t)}{16\pi^3}\left\{\frac{B(\ell,\ell)}{2(2\ell+1)}\left(1+\frac{2(\ell+2)}{-\ell+1}\left(\frac{\mu^2+\zeta}{m^2}\right)\right.\right.$$

$$\left. +\ \frac{2(\ell+2)}{-\ell+1}\,\frac{3(\ell+3)}{-\ell+2}\,\frac{1}{2}\left(\frac{\mu^2+\zeta}{m^2}\right)^2+\cdots\right)$$

$$-\ \frac{\pi}{\sin\pi\ell}\left(\frac{\mu^2+\zeta}{m^2}\right)^\ell$$

$$\times\left(1+\frac{(2\ell+2)(\ell+2)}{\ell+1}\left(\frac{\mu^2+\zeta}{m^2}\right)+\frac{(2\ell+2)(\ell+2)}{\ell+1}\,\frac{(2\ell+3)(\ell+3)}{\ell+2}\right.$$

$$\left.\times\ \frac{1}{2}\left(\frac{\mu^2+\zeta}{m^2}\right)^2+\cdots\right)$$

Equation (III-11) Continued

Equation (III-11) Continued

$$+ \frac{4}{2(\ell + 2)}\left(\frac{\zeta}{m^2}\right)\left[-B(\ell,\ell)\ \frac{\ell}{-\ell + 1}\left(\frac{1}{\ell + 1} + \frac{4}{-\ell + 2}\left(\frac{\mu^2 + \zeta}{m^2}\right) + \cdots\right)\right.$$

$$\left. + \frac{\ell(\ell + 2)}{6}\ \frac{\pi}{\sin \pi\ell}\left(\frac{\mu^2 + \zeta}{m^2}\right)^{\ell-1}\left(1 + \frac{2\ell(\ell + 3)}{\ell}\left(\frac{\mu^2 + \zeta}{m^2}\right) + \cdots\right)\right]$$

$$+ 0\left(\left(\frac{\zeta}{m^2}\right)^2 F\right)\Bigg\} \quad . \qquad \text{(III-11)}$$

Notice that in Eq. (III-11), the radius of convergence of the series is controlled by m^2. Therefore, for $t \neq 0$, even if $\mu^2 \to 0$, the solution $\ell(t)$ of Eq. (III-11) remains finite. If, instead of the procedure following Eq. (III-8), a direct expansion of the nonforward propagator were made in the form

$$\frac{1}{[(\mu^2 - \tau - \frac{t}{4})^2 - \tau t z^2]} = \frac{1}{(\mu^2 - \tau)^2}\left[1 + \frac{t}{2}\ \frac{1}{(\mu^2 - \tau)} - \frac{t\tau z^2}{(\mu^2 - \tau)^2} + \cdots\right], \qquad \text{(III-12)}$$

one would obtain a series representation of TrK_ℓ with a radius of convergence that is essentially controlled by μ^2. Thus such a procedure would suggest that $\mathrm{TrK}_\ell \to \infty$ for $\mu^2 \to 0$ and $\ell < 1$, even though the corresponding integral representation of TrK_ℓ is actually finite in this limit.[6,8]

The slopes of the trajectories can easily be computed from Eq. (III-11) by the formula

$$\frac{d\ell}{dt} = -\frac{\partial \mathrm{TrK}_\ell}{\partial t}\Bigg/\frac{\partial \mathrm{TrK}_\ell}{\partial \ell}\ .$$

Thus

$$\left.\frac{d\alpha}{dt}\right|_{t=0} = \frac{1}{Y(0)}\left[\frac{1}{4m^2}\,X(0) - \frac{16\pi^3}{R(0)}\cdot\left(\frac{2m^2}{\Delta(m^2,\mu^2,\mu^2)}\right)P_L'(1)\right] \quad ,\text{(III-13)}$$

in which α satisfies Eq. (III-11) and

$$X(0) = \left[\frac{\partial}{\partial\zeta}\ \frac{16\pi^3}{R(t)}\ \mathrm{TrK}_\ell\right]_{\ell=\alpha(0),\,\zeta=0} \qquad \text{(III-14)}$$

$$= \frac{B(\alpha,\alpha)(\alpha + 2)}{(-\alpha + 1)(2\alpha + 1)}\left[1 + \frac{3(\alpha + 3)}{(-\alpha + 2)}\left(\frac{\mu^2}{m^2}\right) + \cdots\right]$$

$$- \frac{\pi}{\sin \pi\alpha}\left(\frac{\mu^2}{m^2}\right)^{\alpha-1}\left[\alpha + 2(\alpha + 1)(\alpha + 2)\left(\frac{\mu^2}{m^2}\right) + \cdots\right]$$

$$- \frac{2\alpha\ B(\alpha,\alpha)}{(-\alpha + 1)(\alpha + 1)(\alpha + 2)}\left[1 + \frac{4(\alpha + 1)}{(-\alpha + 2)}\left(\frac{\mu^2}{m^2}\right) + \cdots\right]$$

$$+ \frac{\pi}{\sin \pi\alpha}\ \frac{\alpha}{3}\left(\frac{\mu^2}{m^2}\right)^{\alpha-1}\left[1 + 2(\alpha + 3)\left(\frac{\mu^2}{m^2}\right) + \cdots\right] , \qquad \text{(III-15)}$$

$$\underset{\alpha=1}{=}\left(\frac{43}{18} + \frac{2}{3}\log\left(\frac{\mu^2}{m^2}\right)\right) + \left(\frac{169}{9} + \frac{28}{3}\log\left(\frac{\mu^2}{m^2}\right)\right)\left(\frac{\mu^2}{m^2}\right) + \cdots , \qquad \text{(III-16)}$$

$$Y(0) = \left[\frac{\partial}{\partial \ell} \frac{16\pi^3}{R(t)} \mathrm{Tr}K_\ell\right]_{\ell=\alpha(0),\zeta=0} \qquad \text{(III-17)}$$

$$= \left(\frac{\dot{B}(\alpha,\alpha)}{2(2\alpha+1)} - \frac{B(\alpha,\alpha)}{(2\alpha+1)^2}\right)$$

$$+ \left(\frac{\dot{B}(\alpha,\alpha)(\alpha+2)}{(-\alpha+1)(2\alpha+1)} - \frac{3B(\alpha,\alpha)}{(-\alpha+1)(2\alpha+1)^2}\right.$$

$$\left. + \frac{B(\alpha,\alpha)(\alpha+2)}{(-\alpha+1)^2(2\alpha+1)}\right)\left(\frac{\mu^2}{m^2}\right) + \cdots$$

$$+ \left[\frac{-\pi}{\sin \pi\alpha}\left(\frac{\mu^2}{m^2}\right)^{\alpha} \log\left(\frac{\mu^2}{m^2}\right) + \left(\frac{\pi}{\sin \pi\alpha}\right)^2 (\cos \pi\alpha)\left(\frac{\mu^2}{m^2}\right)^{\alpha}\right]$$

$$\times \left(1 + 2(\alpha+2)\left(\frac{\mu^2}{m^2}\right) + \cdots\right) - 2\frac{\pi}{\sin \pi\alpha}\left(\frac{\mu^2}{m^2}\right)^{\alpha+1} + \cdots , \qquad \text{(III-18)}$$

$$\underset{\alpha=1}{=} -\frac{4}{9} - \frac{1}{2}\left(\frac{67}{9} - \frac{\pi^2}{3} - \log^2\left(\frac{\mu^2}{m^2}\right)\right)\left(\frac{\mu^2}{m^2}\right) - \cdots . \qquad \text{(III-19)}$$

In Eq. (III-18), $\dot{B}(\alpha,\alpha) = 2B(\alpha,\alpha)[\psi(\alpha) - \psi(2\alpha)]$. From these relations one sees immediately that owing to the presence of the factor $[(\mu^2)/(m^2)]^{\alpha-1}$ in $X(0)$, $(d\alpha)/(dt)|_{t=0} \to \infty$ when $\mu^2 \to 0$ and $\alpha \leq 1$. Alternatively one can see this from the derivative of the integral representation of $\mathrm{Tr}K_\ell$ in Eq. (III-8): The integral $[\partial/(\partial\zeta)]\mathrm{Tr}K_\ell$ diverges at the lower end of integration when $\mu^2 = 0$ and $\zeta = 0$. By taking $\mu^2 \to 0$ we have moved the threshold from $t = 4\mu^2 > 0$ to $t = 0$.

We also have

$$A_\ell(\tau_1,z_1,\tau_2,z_2;\ t)$$

$$= \frac{B(\ell+1,\frac{1}{2})[(1-z_1^2)(1-z_2^2)]^{\frac{\ell+1}{2}} R(t)}{1 - \mathrm{Tr}K_\ell}\left[\frac{(-m^2\tau_1)^{\frac{1}{2}}(-m^2\tau_2)^{\frac{1}{2}}}{(m^2-\tau_1)(m^2-\tau_2)}\right]^{\ell+1} , \qquad \text{(III-20)}$$

$$A(s,\tau_1,\tau_2;\ t) \underset{s\to\infty}{\sim} 16\pi^3\, \beta_{\alpha(t)}(t)$$

$$\times \left(\frac{m^2}{m^2-\tau_1}\right)^{\alpha(t)+1}\left(\frac{m^2}{m^2-\tau_2}\right)^{\alpha(t)+1}\left(\frac{s}{m^2}\right)^{\alpha(t)} , \qquad \text{(III-21)}$$

where, as before, $\beta_{\alpha(t)}(t) = -[Y(t)]^{-1}$.

Notice that when t (i.e., ζ) goes to zero, Eq. (III-21) coincides with Eq. (II-16). That is, in the forward limit, the leading member of the family of Regge poles ($\ell = \alpha - n$, $n = 0,1,2,\cdots$) and the corresponding Toller pole ($\lambda = \alpha$) are the same, as far as the high-energy behavior of the full amplitude is concerned. This result is true in general and does not depend on the approximation we have made. On the other hand, from Eq. (III-20) and Eq. (II-11) we see that $A_\ell \xrightarrow[t\to 0]{} A_\lambda$ apart from the function $B(\ell+1,\frac{1}{2})$ and a factor with dependence on z_1 and z_2; however, this result follows only from the fact that we have discarded all the daughters ($\ell = \alpha - 1, \alpha - 2, \cdots$) in A_ℓ,[9] owing to the approximation made after Eq. (III-6).

IV. EXACT NUMERICAL SOLUTION AND COMPARISON WITH APPROXIMATE SOLUTIONS

We have also solved Eq. (II-5) and Eq. (III-5) numerically, using the method described by Wyld[12] to find the leading pole and its residue. We have considered the two cases $\mu^2 = 0$ and $\mu^2 = m_\pi^2$ (i.e., $\mu^2/m^2 = 1/30$) for a kernel consisting of a sharp resonance of mass squared $m^2 = m_\rho^2 = 0.585$ GeV^2. The quantity R is treated as a variable parameter.

In Fig. 2 we show the numerical solution for the intercept of the leading pole when $\mu^2 = 0$. It should be noted that the method of numerical solution is not precise in this zero μ limit: the error in α might be as high as ± 0.1. The value of $\alpha(0)$ calculated from Eq. (III-11) with $\mu^2 = 0$ and $\zeta = 0$ is also plotted; it differs from the numerical solution by about 6% when $\alpha = 1$. For the sake of comparison we have also plotted the values of $\alpha(0)$ calculated in the trace approximation[2]

$$1 = \frac{R(0)}{16\pi^3} \frac{2}{\alpha(\alpha + 1)(\alpha + 2)} \qquad \text{(IV-1)}$$

(dashed line), and the approximate solution of Ref. 5

$$1 = \frac{R(0)}{16\pi^3} \frac{1}{\alpha(\alpha + 1)} \qquad \text{(IV-2)}$$

(dotted line), which is also the expression obtained in Ref. 6.

Figure 3 shows the numerical solution for $\alpha(0)$ when $(\mu^2)/(m^2) = 1/30$, together with the value of $\alpha(0)$ calculated from Eq. (III-11) to first order in $(\mu^2)/(m^2)$.

Figure 4 shows the numerical solution for the residue of the leading pole defined in Eq. (II-19) when $\mu^2 = 0$. In this figure we have also plotted our approximate solution from Eq. (II-18)

$$\phi_\alpha(0,0) \sim 16\pi^3 \beta_\alpha \frac{1}{(m^2)^\alpha} , \qquad \text{(IV-3)}$$

where $\beta_\alpha(0) = -[Y(0)]^{-1}$ with $\mu^2 = 0$ in Eq. (III-17). Again we also show the value of $\phi_\alpha(0,0)$ calculated from Ref. 7 (dashed line) and from the eigenvalue condition [our Eq. (IV-2)] of Refs. 5 and 6 (in dotted line). For $[R(0)]/(16\pi^3) = 4$, say, Eq. (IV-3) misses the exact solution by about 10%, whereas the other approximate solutions miss by more than 100%.

In Fig. 5 we show the numerical solution for $\phi_\alpha(0,0)$ when $(\mu^2)/(m^2) = 1/30$. The approximate solution given by Eq. (IV-3) with $\beta_\alpha = -[Y(0)]^{-1}$ is also shown to first order in $(\mu^2)/(m^2)$.

Figure 6 shows the off-shell dependence of the residue of the leading pole when $\mu^2 = 0$. From Eq. (II-16), we have

$$\phi_\alpha(\tau_1,0) = \phi_\alpha(0,0)\left(\frac{m^2}{m^2 - \tau_1}\right)^{\alpha+1} . \qquad \text{(IV-4)}$$

It is seen that the approximate solutions are "more peripheral" than the exact ones. This is not surprising since we have replaced the original kernel by the one which is more peripheral [c.f., Eq. (II-6) and Eq. (II-7)].

Finally, in Fig. 7 we show the slope of the leading pole at $t = 0$, when the μ-μ resonance is (i) a scalar and (ii) a vector. The numerical solution for $t = 0$ lies somewhere within the shaded area. We have also computed the values of the slope at $t = 0$

according to Eq. (III-13) up to order $[(\mu^2)/(m^2)]^2$ at three different values of $R(0)$ corresponding to $\alpha(0) = 0.5$, 0.8, and 1.0.[13] They are shown as X's in the figure.

V. GENERALIZATIONS

Let us consider briefly the case of a kernel consisting of a finite sum of factorized (and symmetric) terms. The sum may arise either from the input of many resonances in Eq. (II-10) or from taking more terms in the series of Eq. (III-6). For example, if we put $V(s,\mu^2,\mu^2) = \sum_{i=1}^{n} m_i^2 R_i \, \delta(s - m_i^2)$, then we have

$$V_\lambda(\tau_1,\tau_2) = \sum_i \left[m_i R_i^{\frac{1}{2}} \left(\frac{m_i(-\tau_1)^{\frac{1}{2}}}{m_i^2 - \tau_1} \right)^{\lambda+1} \right] \left[m_i R_i^{\frac{1}{2}} \left(\frac{m_i(-\tau_2)^{\frac{1}{2}}}{m_i^2 - \tau_2} \right)^{\lambda+1} \right], \qquad \text{(V-1)}$$

which is no longer factorizable in τ_1 and τ_2. The resulting equation can be solved by an algebraic method. That is, for an integral equation of the type

$$f(\tau_1,\tau_2) = V(\tau_1,\tau_2) + \int V(\tau_1,\tau') \, S(\tau') \, f(\tau',\tau_2) d\tau' \qquad \text{(V-2)}$$

with $V(\tau_1,\tau_2) = \sum_{i=1}^{n} v_i(\tau_1) v_i(\tau_2)$, the solution is just

$$f(\tau_1,\tau_2) = \sum_{i=1}^{n} v_i(\tau_1) v_i(\tau_2) + \sum_{i,j,k=1}^{n} v_i(\tau_1)(1 - T)^{-1}_{ij} \, T_{jk} \, v_k(\tau_2), \qquad \text{(V-3)}$$

where

$$T_{ij} = T_{ji} = \int v_i(\tau) \, S(\tau) \, v_j(\tau) \, d\tau \, . \qquad \text{(V-4)}$$

For the case $n = 2$, for example, the solution is

$$f(\tau_1,\tau_2)$$

$$= [(1 - T_{22})v_1(\tau_1)v_1(\tau_2) + (1 - T_{11})v_2(\tau_1)v_2(\tau_2) + T_{12}v_1(\tau_1)v_2(\tau_2)$$

$$+ T_{21}v_2(\tau_1)v_1(\tau_2)][(1 - T_{11})(1 - T_{22}) - T_{12}T_{21}]^{-1}, \quad \text{(V-5)}$$

which is similar to the solution to a coupled-channel problem.

VI. DISCUSSION AND CONCLUSION

We have seen above from the numerical solution that the leading pole position $\alpha(0)$ and the residue ϕ_α display monotonic behavior as a function of the kernel strength, which is characterized by $R(0)/16\pi^3$.

As for the explicit approximate solutions, we see from Eq. (II-16), Eq. (III-11), and Eq. (III-13-21) the following characteristics:

(1) The leading behaviors of the trajectory and residue do not depend on the external mass μ^2; in fact, the full expressions for them remain finite in the limit $\mu^2 \to 0$ (but not for the slope at $t = 0$ for $\alpha \leq 1$).

(2) The mass m^2 plays the role of the "scale parameter" in the factor $[(s)/(m^2)]^\alpha$ as well as in the slope formula Eq. (III-13); this scale parameter is usually asserted to be about 1 GeV^2 in Regge phenomenology. And, apart from the masses μ^2 and m^2, the residue is completely determined by the location of the pole $\alpha(t)$. In the Veneziano model, the situation is similar to that mentioned in (2) above, in that it is the reciprocal of the slope of the trajectories which serves as the scale parameter. But in that model there is an overall factor in the residues (the constant usually denoted by β) which is not determined by the theory itself.

Let us now turn to the question of the uniqueness of our approximation scheme. There exist, of course, a lot of factorized forms similar to the particular one given in Eq. (II-7). Even the conditions that we imposed in Eq. (II-8) and Eq. (II-22) do not seem to determine $\zeta(s,\tau_1,\tau_2)$ uniquely. However, if we multiply Eq. (II-7) by a factor $\gamma(s,\tau_1)\gamma(s,\tau_2)$ in which we require

$\gamma(m^2,\tau_1)\gamma(m^2,\tau_2) = 1$ when either τ_1 or τ_2 goes to zero,[i.e., that $\xi(m^2,\tau_1,\tau_2)\gamma(m^2,\tau_1)\gamma(m^2,\tau_2)$ matches $e^{-\theta(s,\tau_1,\tau_2)}$ along the line $\tau_1 = 0$ $(\tau_2 = 0)$ for all values of τ_2 (τ_1) in the $\tau_1 - \tau_2$ plane], then it must be true that $\gamma(m^2,\tau) \equiv 1$ (apart from a sign). On the other hand, if we only require $\gamma(m^2,0) = 1$, then the condition in Eq. (II-22) imposes

$$\int_{-\infty}^{0} \frac{d\tau}{\tau^2} \left[\frac{-m^2\tau}{(m^2 - \tau)^2}\right]^{\lambda+1} [\gamma^2(m^2,\tau) - \gamma(m^2,\tau)] = 0 . \qquad \text{(VI-1)}$$

Of course $\gamma(m^2,\tau) \equiv 1$ (the original proposal) fulfills this requirement. There do exist, however, non-null functions which are orthogonal to $1/\tau^2[(-m^2\tau)/(m^2 - \tau)^2]^{\lambda+1}$; and thus one may be able to choose a suitable $\gamma(m^2,\tau)$ to match the high τ behavior of $e^{-\theta(s,\tau_1,\tau_2)}$. We shall not investigate this possibility further here. It suffices to say that Eq. (II-7) seems to be the simplest choice and produces a solution with reasonable analytic properties and in fairly good numerical agreement with the exact solution. We believe that such a solution will be useful in the semi-quantitative study of the general physical output of the multiperipheral model, as well as in the Pomeranchukon perturbation theory.[3]

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FOOTNOTES AND REFERENCES

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1. L. Bertocchi, S. Fubini, and M. Tonin, Nuovo Cimento 25, 626 (1962); D. Amati, A. Stanghellini, and S. Fubini, ibid, 26, 6 (1962).

2. G. F. Chew, T. Rogers, and D. Snider, Phys. Rev. D2, 765 (1970).

3. H. D. I. Abarbanel, G. F. Chew, M. L. Goldberger, and L. M. Saunders, Princeton Preprint (March 1971) and references therein.

4. When this work was almost completed, one of us (B.R.W.) received from Professor M. L. Goldberger a portion of his Varenna Lecture Notes in which a similar approximation has also been made for the kernel.

5. In the second paper of Ref. 1.

6. D. Silverman and P. D Ting, University of California, San Diego preprint UCSD-10P10-82 (1971).

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8. J. Dash, Cambridge preprints DAMTP 71/12 (April 1971), DAMTP 71/19 (May 1971).

9. L. M. Saunders, O. H. N. Saxton, and C. I. Tan, Phys. Rev. D3, 1005 (1971).

10. It can be shown, from Eq. (V-3) below and the properties of Gegenbauer polynomials, that higher terms in Eq. (III-6) affect the trajectories determined by Eq. (III-7) only to the order t^2 and higher. Similar conclusion was obtained by V. Chung and D. Snider [Phys. Rev. 162, 1639 (1967)], and M. L. Goldberger, (private communication to G. F. Chew, June 1971).

11. See, for example: W. Magnus, F. Oberhettinger, and R. P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, Die Grundlehren der Mathematischen Wisenschaften in Einzeldarstellungen, Band 52 (Springer-Verlag, New York, Inc., 1966). The formula we needed is

$$F(a,b,c,1-x) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)} F(a,b,a+b-c+1,x)$$

$$+ x^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} F(c-a,\ c-b,\ c-a-b+1,\ x),$$

for $|\arg x| < \pi$. When $c - a - b = \pm 0, \pm 1, \pm 2, \cdots$, this expression is still valid but we must pass to the limit with care, e.g.,

$$F(a,b,a+b+k,\ 1-x)$$

$$= \frac{\Gamma(k)\Gamma(a+b+k)}{\Gamma(a+k)\Gamma(b+k)} \sum_{n=0}^{k-1} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(1-k)}{\Gamma(1-k+n)} \frac{x^n}{n!}$$

$$- \frac{\Gamma(a+b+k)}{\Gamma(a)\Gamma(b)} x^k \sum_{n=0}^{\infty} \frac{\Gamma(a+k+n)\Gamma(b+k+n)}{\Gamma(a+k)\Gamma(b+k)} \frac{x^n}{n!(k+n)!}$$

$$\times [\log x - \psi(n+1) + \psi(a+k+n) + \psi(b+k+n) - \psi(1+k+n)]$$

for $|\arg x| < x$, $|x| < 1$, $k = 1,2,3,\cdots$. This gives, for example, Eq. (III-11) below when $\zeta = 0$ and $\ell \to 1$,

$$1 = \frac{R(0)}{16\pi^3} \left\{ \frac{1}{6} + \left(\log \frac{\mu^2}{m^2} + \frac{7}{3}\right)\left(\frac{\mu^2}{m^2}\right) + 6\left(\log \frac{\mu^2}{m^2} + \frac{17}{12}\right)\left(\frac{\mu^2}{m^2}\right)^2 + \cdots \right\} .$$

12. H. W. Wyld, Jr., Phys. Rev. D3, 3090 (1971).

13. We computed the slope to order $[(\mu^2)/(m^2)]^2$ because the series X in Eq. (III-15) and Y in Eq. (III-18) converge less rapidly than that in Eq. (III-11). Although the $O\{[(\mu^2)/(m^2)]^2\}$ terms are in magnitude only about 15% of the $O(1) + O[(\mu^2)/(m^2)]$ terms, they contribute with opposite signs to X and Y, making the quotient X/Y change by about 30%.

Table I. The variables.

$$s = (p_1 - p_2)^2 \qquad \cosh\theta = \frac{-p_1 \cdot p_2}{(-\tau_1)^{\frac{1}{2}}(-\tau_2)^{\frac{1}{2}}} = \frac{s - \tau_1 - \tau_2}{2(-\tau_1)^{\frac{1}{2}}(-\tau_2)^{\frac{1}{2}}}$$

$$t = q^2$$

$$\tau_1 = p_1^{\,2} \qquad z_1 = \frac{p_1 \cdot q}{(-\tau_1)^{\frac{1}{2}}(-t)^{\frac{1}{2}}}$$

$$\tau_2 = p_2^{\,2} \qquad z_2 = \frac{p_2 \cdot q}{(-\tau_2)^{\frac{1}{2}}(-t)^{\frac{1}{2}}}$$

$$\tau' = p'^{\,2} \qquad z' = \frac{p' \cdot q}{(-\tau')^{\frac{1}{2}}(-t)^{\frac{1}{2}}}$$

$$-1 \leq z \leq 1 \quad \text{for all } z\text{'s}$$

$$\cosh\psi = \frac{\cosh\theta - z_1 z_2}{(1 - z_1^{\,2})^{\frac{1}{2}}(1 - z_2^{\,2})^{\frac{1}{2}}} .$$

FIGURE CAPTIONS

Fig. 1. The kinematic structure of the multiperipheral integral equation.

Fig. 2. Solutions for the intercept of the leading pole when $\mu^2 = 0$.

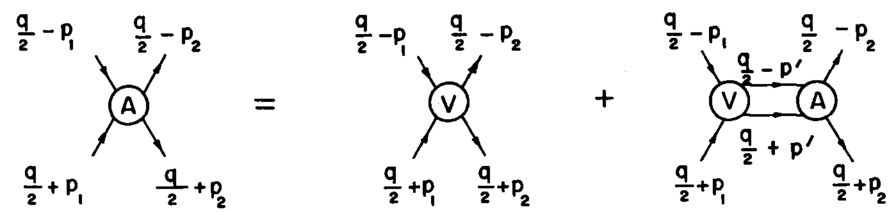
Fig. 3. Solutions for the intercept of the leading pole when $\mu^2 = m_\pi^2$.

Fig. 4. Solutions for the residue of the leading pole when $\mu^2 = 0$.

Fig. 5. Solutions for the residue of the leading pole when $\mu^2 = m_\pi^2$.

Fig. 6. Off-shell dependence of the residue of the leading pole when $\mu^2 = 0$. The exact numerical solutions are shown in heavy lines, whereas $\phi_\alpha(\tau_1,0)$ calculated from Eq. (IV-4) are shown in light lines. Curve I: $\alpha(0) = 1$, $\frac{R(0)}{16\pi^3} = 4.95$. Curve II: $\alpha(0) = 1$, $\frac{R(0)}{16\pi^3} = 6$. Curve III: $\alpha(0) = 0.94$, $\frac{R(0)}{16\pi^3} = 4.95$. Curves 1 and 2: $\alpha(0) = 0.7$, $\frac{R(0)}{16\pi^3} \approx 2.5$.

Fig. 7. Slope of the leading pole at $t = 0$.



XBL719-4376

Fig. 1

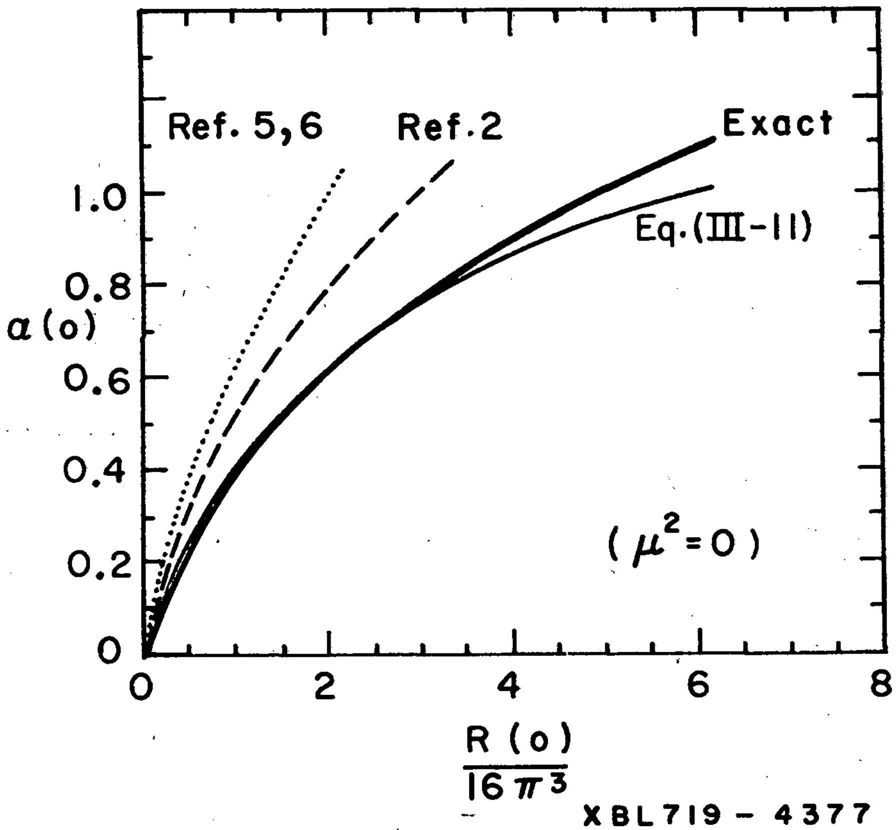

Ref. 5,6
Ref. 2
Exact
Eq. (III-11)
α(o)
1.0
0.8
0.6
0.4
0.2
0
0
2
4
6
8
(μ²=0)
R (o) / 16π³
XBL719 - 4377

Fig. 2

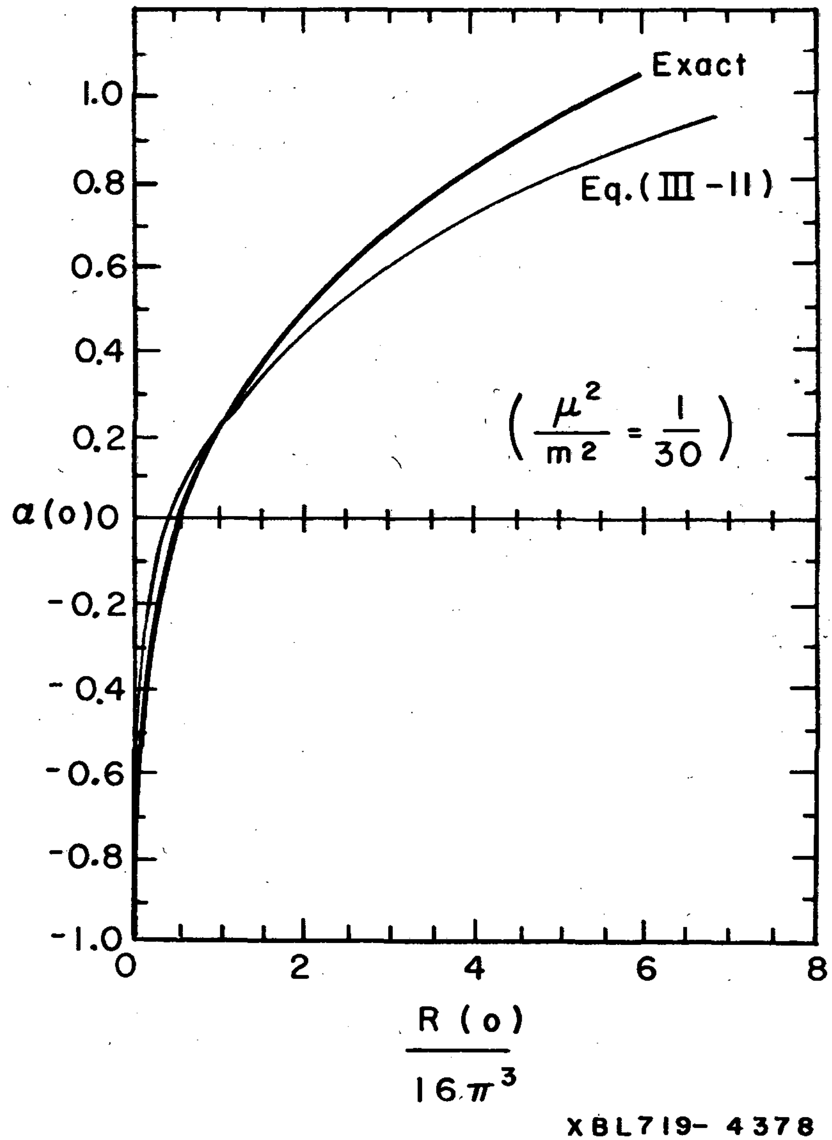

Exact
Eq. (III-11)
1.0
0.8
0.6
0.4
0.2
α(o) 0
-0.2
-0.4
-0.6
-0.8
-1.0
0
2
4
6
8
(μ²/m² = 1/30)
R (o) / 16π³
XBL719- 4378

Fig. 3

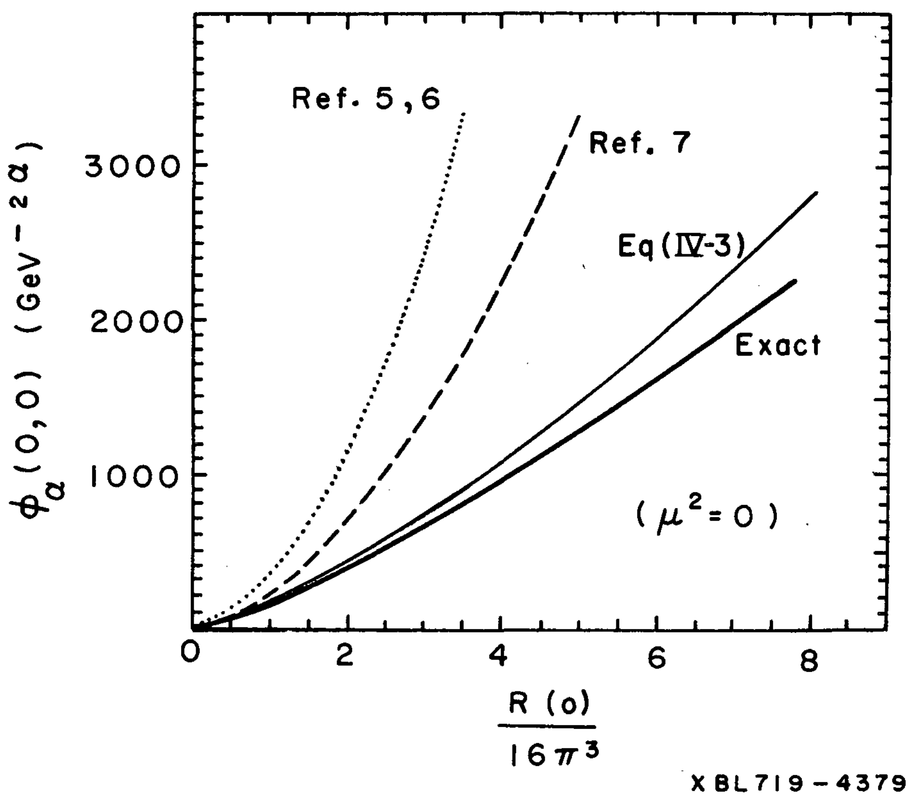

Ref. 5,6
Ref. 7
Eq(IV-3)
Exact
(μ^2=0)
3000
2000
1000
0
2
4
6
8
ϕ_α(0,0) (GeV$^{-2\alpha}$)
$\frac{R(o)}{16\pi^3}$
XBL719-4379

Fig. 4

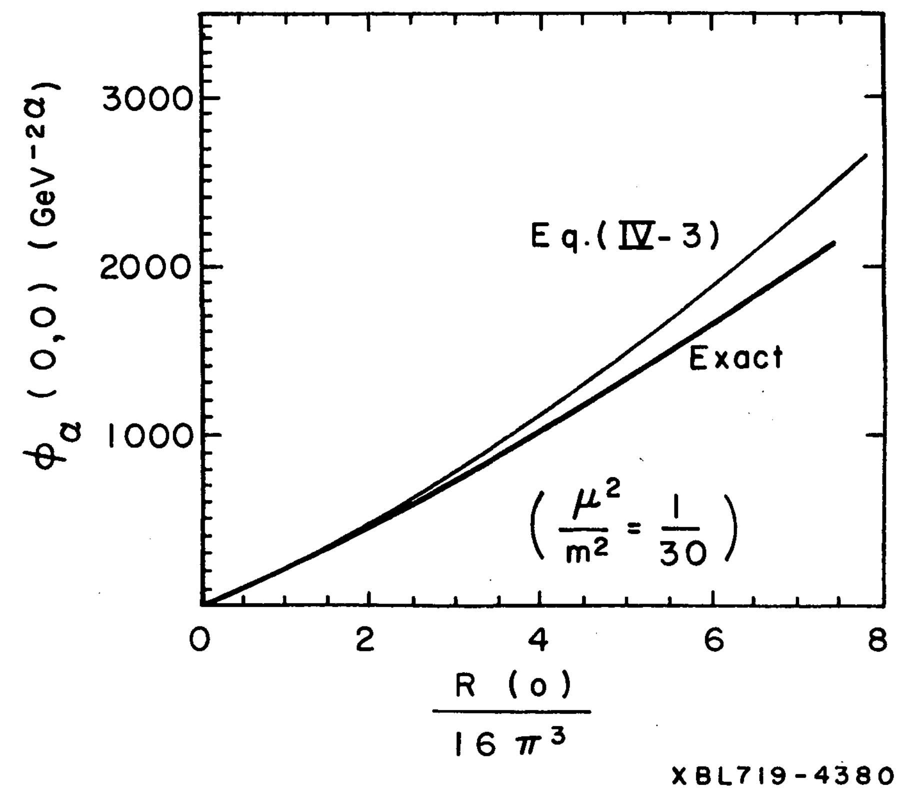

Eq.(IV-3)
Exact
$\left(\frac{\mu^2}{m^2}=\frac{1}{30}\right)$
3000
2000
1000
0
2
4
6
8
ϕ_α(0,0) (GeV$^{-2\alpha}$)
$\frac{R(o)}{16\pi^3}$
XBL719-4380

Fig. 5

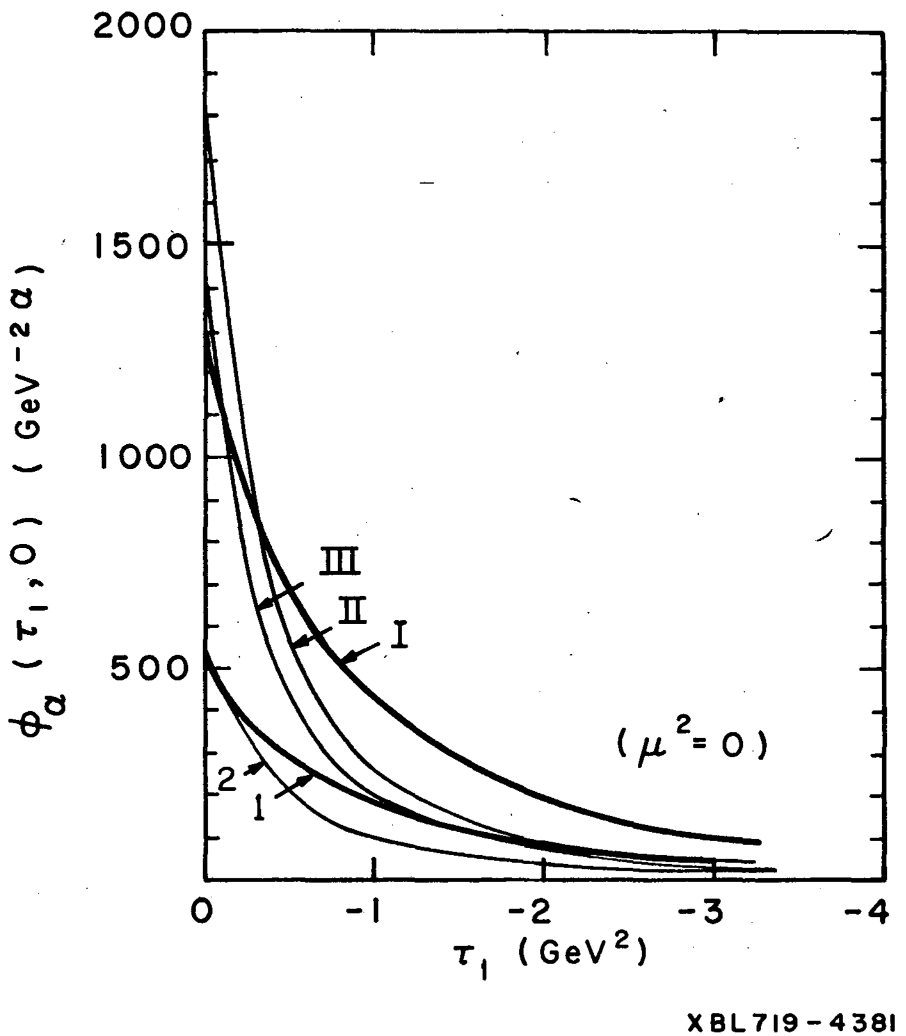

2000
1500
1000
500
$\phi_a(\tau_1, 0)$ (GeV^{-2a})
III
II
I
2
1
$(\mu^2 = 0)$
0
-1
-2
-3
-4
τ_1 (GeV^2)
XBL719-4381

Fig. 6

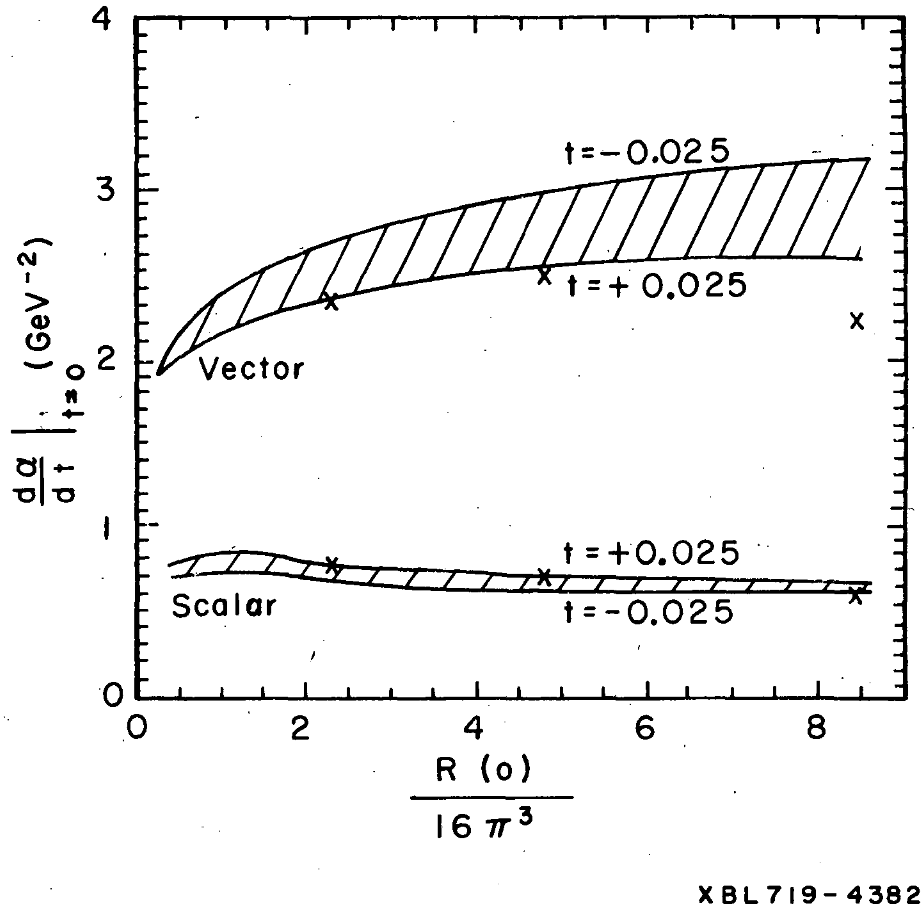

4
3
2
1
0
$\frac{d\sigma}{dt}\Big|_{t=0}$ (GeV^{-2})
t=-0.025
t=+0.025
Vector
t=+0.025
Scalar
t=-0.025
0
2
4
6
8
$\frac{R(0)}{16\pi^3}$
XBL719-4382

Fig. 7

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