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Husserlian Philosophy of Mathematical Practice: An Empathy-First Approach

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Philosophy

by

Stella Moon

Dissertation Committee:
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2023

DEDICATION

To

My cousin, Kyeong-man Kim,
may he rest in peace.

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ABSTRACT OF THE DISSERTATION

Husserlian Philosophy of Mathematical Practice: An Empathy-First Approach

By

Stella Moon

Doctor of Philosophy in Philosophy

University of California, Irvine, 2023

Distinguished Professor Emeritus David Woodruff Smith, Chair

In the history of mathematics and philosophy, the connections and interactions between the two disciplines were clearer than they are today. For example, Descartes is today a well-respected figure in both philosophy and mathematics. Edmund Husserl was also a philosopher and a mathematician: he is best known to be a founder of phenomenology, but he earned his Ph.D. in Mathematics and worked as an assistant to Karl Weierstrass, known as the 'father of modern analysis'. In my dissertation, I develop Husserlian phenomenological methods for studying contemporary philosophy of mathematics.

Recent literature in philosophy of mathematics often advocates the use of non-philosophical methods, turning towards the methods of empirical or social sciences (e.g. Maddy, 2000). I suggest, challenging this view, that phenomenology offers philosophical methods for studying mathematical practice. Phenomenology can be described as an 'explicatory science', whose methods should be adopted, along with cognitive science, as the means to study human cognition and understanding, especially when it comes to mathematics. Importantly, this explicatory science takes the first-person perspective seriously when clarifying our mathematical cognition. Thus, I call this an '*empathy-first* approach'. An empathy-first approach in mathematics is important, especially when we consider that philosophy of mathematics should take a 'mathematics-first' as opposed to a 'philosophy-first' approach. That latter strategy often begins with certain philosophical first principles and applies those to mathematical issues. The former kind of approach aims to begin from mathematical practice and to consider philosophical questions arising from that practice. The empathy-first approach to philosophy of

mathematics not only would begin from mathematical practice, but would also focus on understanding the mathematics as experienced by the mathematicians. In doing so, we are able to evaluate philosophical problems based on how important or relevant they are to mathematical practice.

To demonstrate the empathy-first approach, I begin applying the method to our ordinary perspective on numbers, rather than the mathematicians' perspective. By looking at the number sequence, expressed by the sequence of numerals '1, 2, 3, ...', I describe different acts of accessing the numbers expressed in '...'. In our ordinary experience, there are ways, other than counting, of accessing a larger number. The numbers accessed by such acts could be considered non-arithmetical numbers, as opposed to those that can be accessed in principle only by counting, the *arithmetical* numbers. The demarcation of non-arithmetical and arithmetical numbers by the empathy-first approach suggests a way of demarcating between mathematical concepts and non-mathematical concepts in other areas. But not only that, this demarcation can be supported by other empirical evidence (e.g. Relaford-Doyle & Núñez, 2017, 2018, 2021). This shows how phenomenology, as an explicatory science, can work with cognitive science, and be developed into an interdisciplinary research programme.

I also show that Husserlian methods offer a way of studying group knowledge, which I consider mathematical knowledge to be. Beyond the general method of phenomenological analysis, Husserl also offers a method for studying group knowledge by analysing scientific practice as teleological. This method, known as *Besinnung*, involves standing in the 'community of empathy' with scientists and clarifying the aims and goals that drive their discipline. I argue that Husserl's notion of community is different from other existing notions of groups/communities in that it is defined from a first-person perspective, and that it is defined based on certain properties/experiences shared between an individual and others in relation to a teleological group subjectivity. When the method of *Besinnung* is applied to mathematical practice, it can help philosophers to evaluate whether a philosophical question is genuinely important to practising mathematicians. Once mathematicians' goals and aims are clarified, we can then consider whether a given philosophical question needs a philosophical answer with respect to the goals and aims. This is called 'radical *Besinnung*'. This feature makes the method superior to the methods found in other disciplines, which do not offer this meta-analysis.

I demonstrate this by applying *Besinnung* to a contemporary foundational theory in mathematics, called Homotopy Type Theory (HoTT). Philosophers of HoTT (e.g. Ladyman and Presnell (2015); P. Walsh (2017)) have argued that the definition of identity in HoTT (also known as path induction) needs a philosophical justification. Once we have clarified what path induction is from the empathised perspective of the mathematicians, the definition can be internally justified, without appealing to external philosophical assumptions. In this clarification, we can further identify the goals of the homotopy type theorists, as *rigour* and *homotopical autonomy.* These goals are to be found within the community of empathy, rather than presupposed when looking at the mathematical theory.

INTRODUCTION

The typical stance in philosophy of mathematical practice is *mathematics-first*, instead of *philosophy-first*. This means looking at mathematical practice and theories and asking philosophical questions concerning them, rather than asking philosophical questions independently of the practice of mathematics, usually treating mathematics as a canonical example for abstract objects or *a priori* knowledge. In contemporary philosophy of mathematical practice, it is common to turn to empirical or social-scientific methods for investigating philosophical questions about mathematics. On the one hand, this seems appropriate given scientific advances and the fact that cognitive scientific investigation appears the most appropriate way to develop an understanding of mathematical cognition. On the other hand, what makes this *philosophy* of mathematical practice? Is it simply the kind of questions asked? That is, are philosophers to ask metaphysical and epistemological questions about mathematical practice, and scientists to ask different kinds of questions? Or, in an extreme case, are philosophers to stop doing philosophy completely and simply turn to other disciplines? This dissertation responds to these worries by offering a *philosophical* method for studying mathematical practice.

In this dissertation, I aim to develop a Husserlian method for the philosophy of mathematical practice. Despite the fact that Edmund Husserl was a mathematician and a philosopher in the late nineteenth/early twentieth centuries, philosophy of mathematics in the analytic tradition has often ignored Husserl's contributions to the field. On the one hand, there have been a few discussions on Husserl and mathematics recently, including work by Mark van Atten (e.g. 2007, 2015, 2017), Richard Tieszen (e.g. 1984, 1998, 2002, 2005, 2010, 2011, 2012, 2017), and Dagfinn Føllesdal (e.g. 1995); but none of these authors considers the methodological programme that Husserl offered for studying mathematics and other sciences. Mirja Hartimo, on the other hand, has explicitly engaged with Husserl's methodological programme and highlighted his philosophical views with regard to studying mathematics. As Hartimo shows (2019a, 2020a, 2020b, 2021a, 2021b), Husserl viewed mathematics and other sciences as teleological disciplines – differentiating them from other social practices, and he offered the method of *Besinnung* (i.e. Reflection) as a way to study mathematical and scientific practices of his time.

I build on Hartimo's historical analysis and offer my own interpretation of Husserl's methods. I describe the methods outlined as *Husserlian* (rather than *Husserl's*), as my concern is how Husserl's methodology *ought* to be understood in light of contemporary mathematical practice. I propose to describe the Husserlian approach as an *empathy-first approach* to philosophy (generally) and to philosophy of mathematical practice in particular.

According to Zahavi (2014), Husserl's notion of empathy [*Einfühlung*] was influenced by (but distanced from) that of the psychologist, Theodor Lipps (1905, 1907a, 1907b, 1909). Zahavi describes that, for Lipps,

> [to] feel empathy is to experience a part of one's own psychological life as belonging to or in an external object, it is to penetrate and suffuse that object with one's own life. (2014, p. 130)

Roughly, given that one separates one's own psychological life (i.e. a psychological sense of self) from one's own physical body, which is external to the psychological sense of self, one *feels empathy* if one can treat the physical body as part of or fused with one's own psychological self. Thus, the German term *Einfühlung* can be understood as 'feeling oneself into', as one feels one's psychological self into one's physical body. But generally, it is not clear whether Lipps had a single account of empathy that can be easily summarised, although it is largely agreed that Lipps's notion is the origin of the phenomenological notion (see Zahavi, 2014, and Burns, 2021).

For Husserl (and Edith Stein, 1964), the term *empathy* refers to an intentional experience from one's own ego leading into a foreign ego (Zahavi, 2014, p. 134). This contrasts with Lipps's account, since the focus in Husserl's and Stein's is on the experience of the *foreign ego*, i.e. an *other* ego that is distinct from one's own ego. Furthermore, empathy is a form of *understanding* another person's experience (Stein, 1964, see also D. W. Smith, 1989, p. 117). As I empathise with you, I understand what it is like to be you, to have your experiences, given your other experiences and background. It is distinct from sympathy, which refers to my *own* experience of (e.g.) sadness, starting from *your* experience of (e.g.) sadness, i.e. 'feeling *with* the other what she or he feels' (D. W. Smith, 1989, p. 117). Thus, empathy is a form of understanding of the other's experience, from the other's perspective.

For Husserl and Stein, empathy is a fundamental experience that grounds our understanding of other minds, social interactions and social relationships, including scientific practice. By 'scientific practice', I mean the social human practice of various academic disciplines, in so far as the practice has certain methods or aims in common. For example, history is an academic discipline, as researchers accept and adopt certain methods with the shared aim of uncovering truths about our historical past. Historical practice, then, refers to the way historians carry out their research – their methods, use of concepts, etc. My main interest is in *mathematical practice*, understood in the Husserlian sense as a practice grounded in empathy. That means, the methods or concepts used in mathematics can be understood via our human understanding of the mathematicians' experience. This does not mean to *reduce* mathematics and mathematical practice in line with some variant of social constructivism, but instead to *understand* mathematics in terms of the mathematicians' social activity. In this sense, my approach here is an *empathy-first* approach in philosophy of mathematical practice.

In general, Husserlian methods aim at clarifying various concepts, methods, and/or goals that are found within mathematical practice. I present that the Husserlian phenomenological methods have applications in *contemporary* mathematical practices, and, by using them, we can have a clarified understanding of mathematical practice, as opposed to (e.g.) ordinary usage of mathematical concepts. Furthermore, I suggest that the Husserlian method of *Besinnung*, a method of reflection that aims to clarify the goals of scientific practice, can be used to answer or settle certain philosophical questions about mathematical practice.

0.1 The Goals of the Dissertation

One of the general aims of the dissertation is to demonstrate the importance of *empathy* in philosophy of mathematical practice. Briefly, I characterise Husserlian empathy in mathematics as an act of engaging with mathematics from the perspective of the mathematician(s) and, at the same time, understanding the shared motivation behind their mathematical practice. In my view, *empathy* is what makes phenomenology a philosophical method for studying conscious experience from the *first-person perspective*. It does not necessarily mean focusing on an individual's experience and phenomenological content, but it does require *empathising* with others, understanding their experience from their perspective and analysing their experience. Eventually, through empathy, their types of experiences could become mine: if I empathise with mathematicians and their practice, I could eventually learn to practise mathematics myself. Thus, empathy is an approach to doing philosophy that is most appropriate for pursuing philosophy of mathematical practice.

Beyond applying the empathic approach glossed above, my dissertation has two further main goals. The first is to show that phenomenology and empirical/social sciences can collaborate on interdisciplinary research projects. In Chapter 1, I introduce Husserl's phenomenological methods. Very briefly, phenomenological methods consist in first describing one's experiences from the first-person standpoint and then reflecting on those experiences in order to describe and clarify the meaning or content found in them. Clarification of the meaning or the content of experiences is the general aim of phenomenological methods. In Chapter 2, I apply the phenomenological methods to describe the concept of the number sequence '$1, 2, 3, \ldots$' as experienced from an ordinary, common-folk standpoint, and contrast it with the mathematical concept of the sequence of natural numbers. To test the result of my phenomenological analysis, I present and evaluate it in the light of some empirical evidence. In doing so, I show that phenomenological methods primarily *describe* and *clarify* our experienced concepts, while social/empirical scientific methods aim to *justify* or *verify* the accuracy of such concepts. In that sense, phenomenological methods ought to be an important part of interdisciplinary projects that aim to understand human cognition and concepts.

The second goal is to develop a Husserlian/phenomenological method of philosophy of mathematical practice, extending Mirja Hartimo's contribution in highlighting the importance of Husserl's *Besinnung* as a phenomenological method. A well-developed Husserlian method of *Besinnung* could offer a *philosophical method* for studying mathematical practice. In contemporary philosophy of mathematical practice, there has been a move to adopt scientific methodologies, often at the cost of philosophy. Although this trend could be seen as an anti-philosophical movement, I aim to bring a philosophical method – i.e. one from phenomenology – into the picture and demonstrate the fruitfulness of its approach. Furthermore, Husserl's method of *Besinnung* offers a unique account of group knowledge, offering a unique alternative perspective to those of social epistemology and social ontology.

Before giving an overview of the dissertation, I want to emphasise what I am *not* aiming to do here. My dissertation is not focused on what Husserl's philosophy of mathematics was historically (although I do offer a philosophical interpretation of Husserl as a philosopher of mathematics and mathematical practice). The methodological programme I develop here is *Husserlian*, not necessarily *Husserl's*.

My starting point is the ongoing discussion on the philosophy of mathematical practice. I aim to show that Husserl's phenomenology can offer a fruitful methodology for the philosophy of mathematical practice. For those interested in Husserl's historical views on mathematics, see (e.g.) Tieszen (2010; 2017), Hartimo (2006, 2019, 2021a, 2021b).

0.2 An Overview of the Dissertation

This dissertation contains four chapters. The first chapter serves both as an introduction to philosophy of mathematical practice and as an introduction to phenomenology. The second chapter then demonstrates how the general phenomenological method (which focuses on the first-person perspective) can be used to understand mathematical cognition and contribute to interdisciplinary research. In particular, I focus on the phenomenological understanding of number-cognition and the cognitive scientific work that supports the phenomenological analysis. The third chapter focuses on the particular methodological approach (called *Besinnung*), developed by Husserl, for studying scientific practice, providing an interpretation of that method, in preparation for the fourth chapter, in which the method is applied to contemporary problems in philosophy of mathematical practice.

I shall now provide a more detailed overview of the chapters.

The first chapter argues that Husserlian phenomenology[1] satisfies the contemporary understanding of appropriate philosophy of mathematical practice – as philosophy that takes a *mathematics-first* approach. The chapter further shows that Husserlian phenomenology satisfies the conditions proposed by Gray and Ferreirós (2006) for what philosophy of mathematical practice ought to focus on: i.e. that it should prioritise conceptual and methodological questions that are interesting to practising mathematicians, and engage with mathematical practice in all aspects. Roughly, this would include engaging with informal discussion with mathematicians (about mathematics), not only looking at published work. In Chapter 1, I also show that Husserl viewed mathematics (as he viewed each of the sciences) as a teleological programme, and that the teleological structure of mathematical practice is similar to the structure of intentionality in an individual subject's experience.

The second chapter presents how (sophisticated) mathematical practice is distinct from ordinary mathematical practice. In particular, I give a phenomenological analysis of our ordinary experience of number concepts. I argue that in the ordinary experience of the number sequence, we allow experiences other than counting as ways to grasp large numbers. Hence, the number sequence, understood from ordinary practice, is distinct from the sequence of natural numbers found in mathematical practice. I also show that the conclusion I draw from the phenomenological analysis can also be verified by empirical experiments. In doing so, I demonstrate that phenomenology and empirical/social sciences often are concerned with similar questions, and both approaches should be used together for a better and more accurate understanding of human cognition.

The third chapter is devoted to developing the Husserlian method of *Besinnung* in relation to contemporary mathematical practice. By focusing on *Formal and Transcendental Logic, Crisis* and *Cartesian Meditations*, I give an interpretation of the Husserlian method of *Besinnung*. Hartimo (e.g.

[1] I use 'Husserlian Phenomenology' to refer to phenomenology primarily influenced by Husserl, while 'Husserl's Phenomenology' will be used for the historical views Husserl held for his phenomenology.

2020a, 2020b) characterises Husserl's method of *Besinnung* so as to involve 'standing [*stehend*] in the community of empathy' in order to 'clarify the final goal' of the community. I explicate what Husserl could have meant by the 'community of empathy' [*Einfühlungsgemeinschaft*]. In particular, I argue that the 'community of empathy' (per *FTL*) is a structural notion of social group (in the sense of Ritchie 2020), defined by the empathic relation between its members. This means that, in a community of empathy, each member has an understanding of other members' experiences, where these experiences are appropriate for the practice of their discipline. I also show that Husserl's notion of community is a unique notion of a social community or group. Unlike contemporary accounts, such as those offered by Margaret Gilbert (2009, 2020) and Kate Ritchie (2013, 2015, 2016, 2020), Husserl's account involves a *first-person* notion of community. The emphasis on community and empathy displays the importance of the social aspects of mathematics, which has only recently gained attention in philosophical approaches to understanding mathematics (see, e.g., Ferreirós, 2016).

On the basis of the relevant empathic experiences, the community has shared goals that are implicit in its practice. I explain the way in which *Besinnung* aims to clarify such shared goals (which I shall call 'motivational goals'), which have a historical motivation for the community. Chapter 3 is devoted to characterising what steps are necessary to carry out *Besinnung* successfully; that is, in such a way that the motivational goals of mathematics, as a discipline practised by the relevant communities of empathy, can be understood. Simply put, *Besinnung* initially involves some historical investigation, to provide an understanding of the historical motivation behind certain assumptions made for the practice of a mathematical theory. Once the assumptions have been clarified, we can enter the community of empathy and identify what concepts, methods, goals, etc., could be further clarified from the empathised perspective. Such clarification will lead us to identifying the motivational goals of the practice. These goals are understood as what the community is aiming for, but also what guides the community's practice. For example, rigour in mathematics is a motivational goal: mathematical definitions or mathematical proofs can be described as rigorous or precise, and this rigour is a motivation behind the discipline of mathematics. Furthermore, mathematicians aim to continue this rigorous practice as the discipline develops and changes. In this sense, rigour is a motivational goal found within mathematical practice.

The fourth chapter applies the method of *Besinnung* to a contemporary mathematical theory. In particular, I consider Homotopy Type Theory (HoTT) and I clarify the concept of identity in HoTT. Understood from the perspective of the homotopy type theorists – i.e. from the empathised perspective – identity is understood homotopically, in particular as following from the path lifting property in topology. I clarify this homotopical understanding and thus offer a novel argument for justifying path induction. The justification I offer is an *internal* justification, in the sense that it is an argument offered from the perspective of the community, rather than from a philosophical perspective that goes beyond the mathematical practice. By carrying out *Besinnung*, we are able to settle whether identity in HoTT needs a further philosophical justification or not, and this puts it at a strong advantage compared with other social/empirical scientific methods. This is a fruitful consequence of carrying out *Besinnung*. By employing it we can focus on the philosophical questions that are the most relevant to mathematical *practice*, instead of giving equal weighting to all of the philosophical questions that can arise concerning mathematical practice. In this chapter I also show that two kinds of motivational goals can be found in HoTT: rigour (in the logical syntax and computational implementation) and homotopical autonomy. The motivational goal of homotopical autonomy suggests a new sense of foundation based on the mathematical practice of HoTT.

The chapters show that phenomenological methods can be fruitful for mathematical practice. Not only do they offer a demarcation between mathematical practice strictly speaking and ordinary, commonsense 'mathematical' practice, in terms of the different acts or experiences involved in them (Chapter 2), they help us to raise and answer philosophical questions that are interesting or important for mathematical practice (Chapters 3 and 4). While some questions might seem interesting for mathematicians, not all questions will be important for its practice. The phenomenological method of *Besinnung* involves first identifying which concepts, methods, etc., must be clarified before going on to clarify them, all from the perspective of the mathematicians. Furthermore, *Besinnung* aims to bring out the motivational goals shared in the practice, enabling the phenomenologist to suggest revisions to the practice to help the mathematicians in achieving their goals.[2] Thus, the Husserlian *empathy-first* approach, and in particular the Husserlian method of *Besinnung*, should be considered seriously as providing a method for studying philosophy of mathematics.

[2]Husserl calls this approach 'radical *Besinnung*'.

My dissertation contributes further to areas of philosophy beyond mathematical practice. For social epistemology/ontology, I offer a Husserlian account of scientific practice as a social practice, in terms of community of empathy (Chapter 3). I also demonstrate that phenomenology should be adopted as a descriptive/clarificatory science in relation to human understanding (Chapters 1 and 2). By doing so, I confirm the importance of *philosophical* thinking and methodologies, which are often neglected in other disciplines – and by some within philosophy itself!

With all this said, it must be admitted that phenomenology, like any methodological programme, has its limitations. It is not one of its aims to offer evidential justifications, but rather clarifications. (So, to critically evaluate phenomenology on the basis of its provision of justifications would be to misunderstand its values.[3]) My aim here is *not* to argue that phenomenology should replace other methods (methods that are better suited to providing evidential justifications), but to emphasise and highlight what is unique about the phenomenological methods outlined. Understanding the values of the phenomenological approach (and allied methodologies) and using them appropriately for what they do best, will advance human knowledge.

0.3 Methodological Choices

In his lifetime, Husserl wrote a vast amount in published works and lecture notes, and he delivered many lectures in different locations. His philosophical works are frequently seen as falling into three phases. First come his habilitation, *On the Concept of Number* (1887), and the later published *Philosophy of Arithmetic* (1891), which aim to provide logical and psychological analyses of the concept of numbers. The second phase comes with the *Logical Investigations*, in which he renounces the 'psychological analysis' of his previous works and introduces phenomenology. And the third phase is captured in the *Ideas*, *Crisis*, *Cartesian Meditations* and *Formal and Transcendental Logic* – often described as articulating 'Transcendental Phenomenology' and hence said to mark Husserl's 'transcendental phase'. Although there are some good reasons for understanding Husserl's

[3]Although I offer a justification for path induction in Chapter 4, it is a *clarificatory* justification, in that it is an argument that clarifies how path induction is understood from the perspective of homotopy type theorists. It is not an evidential justification in the sense of offering empirical evidence for how mathematicians cognitively process path induction.

philosophical thinking in terms of these three distinct phases, I adopt a more unifying approach in reading Husserl.

Throughout the dissertation, I treat Husserl's changing ideas to be continuous with his previous thoughts. On this reading, his later work can sometimes be clarified by his earlier writings. Reflecting back on Husserl's prior concepts gives us an insight into Husserl's original motivation concerning certain concepts. A good example of this can be seen in relation to Husserl's explicit remarks when discussing the notion of *Zwecksinn* in *Formal and Transcendental Logic*. He identifies *Zwecksinn*, or goal-sense, with the notion of fulfilled sense, found in the *Logical Investigations*. Thus, in Chapter 3, I explicate Husserl's *Zwecksinn* by looking at the *Logical Investigations*.

The two key methods I use in the dissertation can be broadly characterised as textual analysis and phenomenological analysis. In textual analysis, I compare the contemporary published literature on both philosophy of mathematical practice (Chapter 4) and phenomenology (Chapter 3) with Husserl's views on scientific practice (Chapter 1). Furthermore, I engage with the published work in cognitive science (Chapter 2) in order to argue that empirical evidence can support the findings of my phenomenological analysis presented earlier in that same chapter. The *phenomenological* analysis I use, in Chapter 2, involves reflecting on the empathised experience of the number sequence and clarifying the meaning found in those experiences of the number sequence. In general, by 'phenomenological analysis', I mean carrying out some self-reflective act to clarify the content found in one's own or another subject's experience. Phenomenology, however, is not just one method, but rather a collection of methods. In particular, in Chapter 1, I explain the phenomenological method of *Besinnung* as an extension of the general phenomenological method: i.e. first-person perspectival reflection aimed at clarifying goals.

As noted, phenomenological methods involve the empathised first-person perspective, and, on some occasions, additional methods based on the perspective will be adopted. Here, I briefly describe how the textual analysis and phenomenological analysis pursued in Chapters 3 and 4 can, in combination, lead us to adopting the formal mathematical method of proof. In Chapter 3, I give a textual analysis of Husserl's *Formal and Transcendental Logic*, *Cartesian Meditations*, *Crisis of European Science* and *Ideas I*, in order to clarify Husserl's method of *Besinnung*. Having developed this particular phenomenological method, I provide, in Chapter 4, a phenomenological analysis of the practice of homotopy type theorists on the basis of textual evidence found in the book Univalent Foundations Program (2013) and in various online forums in which HOTT originated. This textual analysis is supplemented by my engagement and interaction with the practitioners of HoTT (e.g.) at conferences – this kind of engagement/interaction being a necessary part of *Besinnung*. In doing so, I naturally adopt the method preferred by the mathematical community (i.e. the mathematical method of proof) to clarify the meaning of identity in HoTT.

In general, I see Husserl's work as being in continuous development from his earliest philosophical work, *On the Concept of Number* (1887). With this framework in mind, I support my interpretation of Husserl based on the general or broad picture of his views that can be found in his writings across his lifetime. This means I do not distinguish between his earlier work – e.g. *Logical Investigations* – and his later work – e.g. *Crisis* – as embodying distinct philosophical programmes, divided by a 'transcendental turn'.[4] Although I hold by a unified understanding of Husserl's work overall, I do focus on particular primary texts in individual discussions throughout my dissertation. I shall identify these texts below, after giving a brief description of Husserl's educational background.

0.4 Biography of and Works by/about Husserl

Edmund Husserl was born in Prossnitz (Moravia) on 8th April 1859 to Jewish parents. After studying mathematics, physics, and philosophy in Leipzig and Berlin, he went to Vienna to complete his Ph.D. in Mathematics on Calculus of Variations (1883). He then returned to Berlin and worked as Weierstrass's assistant until Weierstrass fell ill. In 1884, he moved back to Vienna to study philosophy with Franz Brentano.

[4]This view is consistent with that articulated by David Woodruff Smith (1989, 2013), Zahavi (2003), and Hartimo (2021). Here, Husserl's work is seen as continuous, with a 'transcendental turn' marking a less-than-drastic shift. Please note, however, that my interpretation of transcendental phenomenology is different from Hartimo's (for example), in that I take it that the 'transcendental' is what we, as phenomenologists, should be aiming for generally, rather than a particular stance in relation to which we evaluate only certain practices. This will be clarified in Chapters 1 and 3.

Since Brentano had given up his professorship in 1880 to marry, he could not advise Husserl on his Habilitation thesis. So, on Brentano's advice, Husserl completed his Habilitation dissertation in philosophy, *On the Concept of Number* (1887), under Carl Stumpf in Halle.

Husserl's first book, *Philosophy of Arithmetic: Psychological and Logical Investigations* (1891), was based on his habilitation thesis. Then he went on to publish many other books – e.g. *Logical Investigations* (1900–1901) in two volumes: Volume 1, *Prologomena to Pure Logic*, and Volume 2, *Investigations in Phenomenology and Knowledge* – and frequently gave lectures. Below, I list some of Husserl's major publications (in order of original publication), some of which were published posthumously. On the left of each title, I label them with the abbreviation to be used throughout the dissertation. The dates that follow are in the following order (unless specified otherwise): original publication, Husserliana volumes, and English Translation.

PA *Philosophy of Arithmetic: Psychological and Logical Investigations* (1891), Hua vol.12 (1970), English Translation (2003)

LI I/II *Logical Investigations* (1900–1), Hua vol.18–19 (1984), Hua vol.20 (2002–05), English Translations (2000)

Ideas I *Ideas Pertaining to a Pure Phenomenology and to Phenomenological Philosophy – First Book: General Introduction to Pure Phenomenology* (1913), Hua vol.3 (1976), English Translation (1989)

FTL *Formal and Transcendental Logic* (1929), Hua vol.17 (1974), English Translation (1969)

CM *Cartesian Meditations*, French publication (1931; Lectured in Paris 1929), German publication and Hua vol.1 (1950), English Translation (1960)

Crisis *The Crisis of European Sciences and Transcendental Phenomenology* (Partial publication 1936), Hua vol.6 (1954), Hua vol.29 (1993), English Translation (1970)

Henceforth, I shall reference the texts by their abbreviation, unless a reminder of the full title might be appropriate, and the Husserliana Volume (when appropriate). The pagination is from the English translations, unless stated otherwise. A variety of English translations were used, where available, in addition to the original German, and in some cases I offer my own translations.

With regard to secondary sources, I focus on the following texts Hartimo (2006, 2008, 2010, 2019a, 2019b, 2020a, 2020b, 2021b, 2022a, 2022b), David W. Smith (1989, 2013, 2018), Zahavi (2003, 2004), Cohen and Moran (2015), Hopp (2020). The point of focusing on these particular secondary texts is to give a more succinct and coherent interpretation of Husserl. The works of Edmund Husserl have been studied and analysed by various scholars across the globe but, since my aim here is to develop the phenomenological methods for studying philosophy of mathematical practice, it is important for me to provide a coherent unifying account of Husserl's thought that is most relevant to my project, and I have therefore opted to be selective with regard to secondary sources in developing my interpretation of Husserl.

We shall now turn to the first chapter of the dissertation. In the first chapter, I provide introductions to both philosophy of mathematical practice and phenomenology, before going on to show that there could be a phenomenology of mathematical practice. The discussion has been framed with the intention that the reader, regardless of their level of familiarity with philosophy of mathematical practice or phenomenology, will be able to grasp the ideas and arguments. I also offer an original interpretation of Husserl's account of the teleological structure of science/scientific practice as an application of his account of the structure of the intentionality of subjective experience.

Chapter 1

Husserlian Phenomenology of Mathematics

In this chapter, I argue that Husserlian philosophy of mathematics is a philosophy of mathematical practice. In section 1.1, I give a brief overview of philosophy of mathematical practice. Recent advocates for philosophy of mathematical practice have been critical of the systematic methods and approaches in philosophy regarding mathematics. I argue that Husserlian phenomenology, while it offers a systematic philosophical methodology, does not fall victim to the criticisms levelled at other philosophical methods and approaches. In sections 1.2 and 1.3, I clarify the nature of Husserlian phenomenology, and show that Husserlian phenomenology is an *empathy-first approach* to philosophy. I further highlight the importance of Husserl's view of science as a teleological practice and compare its structure with the structure of intentionality.

The goals of this chapter are threefold. The first goal is to introduce philosophy of mathematical practice. The second goal is to explain what Husserlian phenomenology is. The third goal is to show that Husserlian phenomenology offers methods for the philosophy of mathematical practice.

1.1 Philosophy of Mathematical Practice

In this section, I introduce *philosophy of mathematical practice* as a response to concerns over traditional philosophy of mathematics.

As Shapiro (1997) says

> For some time, philosophers and mathematicians held that ontology and other philosophical matters determine the proper *practice* of mathematics. (Shapiro, 1997, p. 6)

According to this perspective, in order to practice mathematics, we ought to 'first figure out what we are talking about, by describing or discovering the metaphysical nature of mathematical entities' (Shapiro, 1997, p. 6). Shapiro calls this 'the *philosophy-first principle*'. The perspective placed philosophy of mathematics *prior* to mathematical practice: it was thought necessary to secure a foundation for mathematics in order for mathematical practice to be viable. An alternative to this view is the *philosophy-last-if-at-all principle* (Shapiro, 1997, p. 7). The extreme of this view would claim that there should be no *philosophy* in mathematics, since 'it is mathematicians, after all, who practice and articulate their field' (Shapiro, 1997, p. 7). If one were to follow this extreme view, there ought not to be a discipline called *philosophy of mathematics* at all – which, as Shapiro says, would be 'unhealthy for mathematics' (Shapiro, 1997, p. 7).

A search for a happy middle-ground between the extremes of the two principles leads us to philosophy of mathematical practice. Owing to there being a wide range of views among the advocates for philosophy of mathematical practice, we cannot characterise it simply in terms of necessary and sufficient conditions. For instance, some have advocated their position as going *against* the foundational questions (e.g. Kitcher, Tymoczko, Corfield), while others have suggested that we should go *beyond* them (see Gray & Ferreirós, 2006, p. 5). What brings the philosophers of mathematical practice together is perhaps that they focus on expanding or extending discussion around the existing epistemological and ontological problems about mathematics to engage with issues found in mathematical practice:

> the epistemology of mathematics needs to be extended well beyond its present confines to address epistemological issues having to do with fruitfulness, evidence, visualization, diagrammatic reasoning, understanding, explanation, and other aspects of mathematical epistemology[. The] ontology of mathematics could also benefit from a closer look at how interesting ontological issues emerge both in connection to some of the epistemological problems mentioned above [. . .] and from mathematical practice itself. (Mancosu, 2008, pp. 1–2)

The general suggestion is that we ought to consider the philosophical questions about mathematics by engaging with mathematical practice, and focus on those questions that are interesting to practising mathematicians, not only to philosophers.

A natural question is, then, what kind of philosophical questions would interest practising mathematicians? In this chapter, I answer this question by focusing on the views of Gray and Ferreirós (2006), as these offer more general desiderata for philosophy of mathematical practice.

Other discussions on philosophy of mathematical practice can be found in Mancosu (1996, 2008), Lakatos (2015), Corfield (2003), Ernest (1994, 1998), and Maddy (2007). Note here that I am not claiming that Gray and Ferreirós's view on philosophy of mathematical practice is *the* only view, nor that it is the *best* view. I simply consider it to be *one* of many views of philosophy of mathematical practice that describe the general desiderata for what philosophy of mathematical practice ought to focus on. It is, I hold, a view that Husserlian philosophy of mathematics/mathematical practice can satisfy. So, let us turn to Gray and Ferreirós.

Here, I characterise two desiderata of philosophy of mathematical practice suggested by Gray and Ferreirós (2006). The first one states that philosophers ought to engage with conceptual or methodological problems found in mathematics. They write that philosophical reflections on conceptual or methodological problems, which are 'practised by mathematicians themselves', tend to be 'perceived as relevant and interesting by mathematicians and historians' (2006, p. 10). The view is that these conceptual or methodological problems are already what mathematicians are engaged in, so philosophical perspectives on these would be useful for mathematicians. From this we can define the first desideratum of philosophy of mathematical practice (PMP):

PMP1: PMP should engage with conceptual or methodological problems.

While one could argue that the foundational problems are already effectively conceptual or methodological, what Gray and Ferreirós propose is to focus *primarily* on these rather than ontological or epistemological problems. If the foundational problems about mathematics are *only* about ontological or epistemological problems, these ought *not* to be the problems in philosophy of *mathematical practice*, but in philosophy of *mathematics*. So, what makes PMP *philosophy* is its engagement with the conceptual and methodological problems, which are philosophical problems, *over* the more traditional ontological and epistemological problems, such as the epistemic accessibility of abstract mathematical objects (see, e.g., Mancosu, 2008, pp. 1–2).

Gray and Ferreirós also ask that philosophers 'engage in a real exchange with the complexities of mathematical practice, past or present, and to be open to the questions that worry practising mathematicians about the nature of their subject, especially in times of change' (2006, p. 11). The desire to engage in a real exchange has some methodological implications. It is not a matter of simply engaging with published works in mathematics but, in addition, of engaging with other, informal aspects of the practice. This is where 'real exchange' between mathematicians occurs. By engaging with the 'real exchange', philosophers can learn more about the complexities of conceptual and methodological developments within the practice. This suggests a more open-minded approach to studying mathematics, including the approaches of empirical or social sciences. Thus, this desideratum presses for an interdisciplinary methodology. For instance, informal correspondences between mathematicians have historical value in showing how certain ideas came to be: we have a better understanding of Descartes's mathematics because of his correspondence with Mersenne and other mathematicians of his time. So, we can characterise this as the second desideratum of PMP:

PMP2: PMP ought to engage in a real exchange with the complexities of mathematical practice.

The two desiderata together suggest what kinds of philosophical questions we ought to focus on (PMP1) and what kinds of methodologies we might consider in engaging with the 'complexities of mathematical practice' (PMP2). Thus, Jessica Carter suggests that PMP is a sub-discipline of philosophy of mathematics such that

> (i) mathematics is taken to be mathematics in every shade and not idealisations of mathematics, and (ii) an extension of methods [i.e. interdisciplinarity] is allowed, i.e., the possibility of bringing in results and tools from other disciplines. (2019, p. 2)

One well-known approach to the 'extension of methods' is to follow a 'naturalist' method, in particular in the methods of social and empirical sciences (see, e.g., Maddy, 1997, 2007). However, one ought to be careful that we do not neglect the desideratum PMP1 in pursuit of the interdisciplinary methodology motivated by PMP2. If we focus exclusively on the interdisciplinary method, this might lead us to ignore the *philosophical* problems and investigate the non-philosophical – sociological or anthropological – issues. In the next subsection, I briefly explain how such a mis-step might arise from the desiderata, but clarify what the desiderata mean.[1]

1.1.1 Interdisciplinarity or Anti-Philosophy?

As we saw, engaging in the real complexities of mathematical practice, interdisciplinary approaches might be adopted. For example, one could focus on the natural language of mathematics as it appears in journal publications in terms of speech acts (see, e.g., Ruffino, San Mauro, and Venturi, 2021), perhaps deploying social scientific methods to analyse the samples. Another example is the use of interviews and surveys (i.e. the methods used in the social sciences) to provide philosophical insight on the set theoretic practice of today (e.g. Dzamonja and Kant, 2019). Along this line, Ferreirós (2016) claims:

> [An] open-minded approach to the study of mathematical practice can only act for the good. The study of mathematical knowledge and how it is produced is an important topic, and it is certainly desirable that the work of philosophers be of interest to mathematicians, mathematical educationalists, and scientists who are mathematics users. (Ferreiorós, 2016, Foreword)

[1] To demonstrate fully that engaging with the complexities of mathematics requires an interdisciplinary methodology, I would need to provide a philosophical argument. This is not the main focus of this chapter, thus I omit this argument here.

At the same time, Ferreirós warns his readers to be 'careful' of simply *applying* 'established' theories from other disciplines to mathematical practice (2016, p. 3). Importantly, he is not interested in *reducing* the practice to the terms of a particular theory or position, but rather in looking at its 'meaning' in terms of its 'use' (Ferreirós, 2016, p. 5).

Another advocate for interdisciplinary methods is Penelope Maddy, whose methodological position is known as *Second Philosophy* (1997; 2007; 2014). The general attitude of a Second Philosopher is that philosophy is contiguous with empirical or social sciences, and some questions traditionally considered philosophical are better answered by other scientific methods. In order to explain what Second Philosophy is, Maddy introduces an idealised simple enquirer 'who sets out to discover what the world is like, the range of what there is and its various properties and behaviors' (2011, p. 39). This character is 'equally at home in all the various empirical investigations, from physics, chemistry, and astronomy to botany, psychology, and anthropology' (2011, p. 39). Hence, Maddy's philosophy of mathematical practice ought to accept the methods from all these disciplines.

Despite the description that they are 'open-minded', these interdisciplinary approaches are sometimes considered to be *anti-philosophical*.[2] For instance, Gray and Ferreirós (2006) are critical of philosophy's systematic approach which attempts to fix the basic fundamental principles of mathematics, or 'lay out the ontology of mathematics [...] to show that there are no objects' (2006, pp. 8–9). This sort of view is further clarified by Aspray and Kitcher (1988): the traditional philosophy of mathematics, they say,

> appears to be a microcosm for the most general and central issues in philosophy – issues in epistemology, metaphysics, and philosophy of language – and the study of those parts of mathematics to which philosophers most often attend (logic, set theory, arithmetic) seems designed to test the merits of large philosophical views about the existence of abstract entities or the tenability of a certain picture of human knowledge. (Aspray & Kitcher, 1988, p. 77)[3]

[2]As we shall see later on, this charge is not entirely accurate. The sense in which they are 'anti-philosophical' is actually a matter of resistance to particular philosophical approaches.

[3]A target of this criticism might be Benacerraf (1973). In his famous paper, Benacerraf aims to argue for a general account of truth that could also apply to mathematics.

Although the criticisms offered by Gray and Ferreirós (2006) and Aspray and Kitcher (1988) are more generally aimed at a systematic philosophical approach, Maddy also offers a particularist criticism of the traditional philosophical approaches. Maddy claims in *Naturalism in Mathematics* (1997) that

> on the basis of historical analysis, [...] certain types of typically philosophical considerations have turned out to be irrelevant in the past. [...I propose,] guided by this fact, a naturalized model of the underlying justificatory structure of the practice that can then be tested empirically (1997, p. 200).

Not only that, but in *Second Philosophy* (Maddy, 2007), she criticises various traditional philosophical methods (and/or their consequences) attributed to Descartes, Kant, Carnap, and Quine, by considering the method introduced by each philosopher and arguing that it is not suitable for the Second Philosopher's investigation.

Although Maddy does not engage with Husserlian phenomenology, Mirja Hartimo has highlighted some similarities between Second Philosophy and Husserlian phenomenology in a number of works. I shall briefly discuss this in the next section, after defending Husserlian phenomenology from the generalist criticism.

1.2 Husserlian Phenomenology as Pertaining to a Philosophy of Mathematical Practice

In this section, I argue that Husserlian phenomenology is an instance of philosophy of mathematical practice satisfying the general desiderata proposed by Gray and Ferreirós for PMP. To show this, I briefly discuss (in section 1.2.1) the *principle of freedom from presuppositions*. Then in sections 1.2.2 and 1.2.3, I describe Husserl's method of *Besinnung* for studying mathematical practice and compare it with Maddy's metaphilosophy, '*Second Philosophy*'.

1.2.1 Presuppositionlessness as 'First' Philosophy

The criticisms offered by Gray and Ferreirós and by Aspray and Kitcher target the systematic approaches of fixing and reducing mathematics. An example of this is provided by Benacerraf's paper on 'Mathematical Truth' (Benacerraf, 1973). One of the problems raised in the paper is whether there is a general account of truth (provided by knowledge) such that it is also an appropriate account of mathematical knowledge (with respect to truth).[4] The paper raises a dilemma, by arguing that a general account of truth and/or knowledge that is also an appropriate account of mathematical truth/knowledge is not possible. The starting point of the enquiry is the general philosophical issues (concerning truth and knowledge) rather than a mathematical problem.

Instead of starting with a particular philosophical perspective, Husserlian phenomenology begins with the *natural attitude* of the practitioners – i.e. the naïve standpoint of the practitioners prior to any extra theorising about their practice. (Being in the natural attitude is to presuppose only what the practitioners already presuppose.) From this beginning, Husserlian phenomology goes on to attempt to describe and clarify mathematics and its aims from the *practitioners'* perspective. In this sense, phenomenological methods start from an ontologically and epistemically neutral position – as long as the practitioners typically have neutral positions on the matter of ontology – and, consequentially, describe and clarify the ontological and epistemic positions found in the *natural attitude*[5] of practising mathematicians. Husserl writes:

> Phenomenology must lay claim [. . .] to being 'first' philosophy and providing the means for every rational critique [*Vernunftkritik*] that needs to be carried out. Thus, it requires the uttermost *presuppositionlessness* and an absolute, reflective insight into itself. (My emphasis, *Ideas I*, §63; Dahlstrom, translation, p. 117)

Husserl calls this requirement the '*principle of freedom from presuppositions*'. This principle suggests that one should not assume a philosophical stance beyond what is found in the natural attitude of the practitioners, before pursuing an investigation:

[4] In Benacerraf's case, he assumes a correspondence theory of truth.

[5] I will clarify this in section 1.3.1.

> An epistemological [phenomenological] investigation that can seriously claim to be scientific must [...] satisfy the *principle of freedom from presuppositions*. This principle [...] only seeks to express the strict exclusion of all statements not permitting of a comprehensive *phenomenological* realization. (*LI II*, Part I, Introduction §7; 2000, p. 189)

At base, the principle asserts that the statements that cannot be phenomenologically experienced or understood should be excluded from phenomenological investigation. Correlatively, in the context of mathematical practice, whatever is part of mathematical practice would be up for phenomenological investigation.

In order to engage phenomenologically with mathematics, and be in the mathematicians' natural attitude, we must begin by *empathising* with mathematicians. '*Empathy*', in the context of Husserlian phenomenology, roughly refers to understanding another person's experience (e.g. of mathematics) from that other's perspective (i.e., in this case, the mathematician's perspective).[6] Once this is achieved, we can reflect on the relevant concepts and methods from the mathematicians' perspective. For this reason, I would like to describe phenomenology as an '*empathy-first approach*' to philosophy, and hightlight pre-suppositionlessness as what is necessary to empathise with the relevant subject(s). In this sense, *I* could phenomenologically investigate what mathematical *experiences* (such as proving a theorem) are like or how mathematical things or facts appear to the mathematicians in these experiences.

But phenomenology is not only restricted to investigation of the experiences of individual mathematician. When we focus on the experiences of a community of mathematicians, rather than individual mathematicians, we first need to understand what kind of experiences are shared among the members of the given community. Then we aim to clarify what the community's goals are. Husserl calls this method '*Besinnung*'. In the next subsection, I shall briefly explain what *Besinnung* is and how this method can be used to study mathematical practice. A more detailed account of *Besinnung* will be presented in Chapter 3.

[6] I shall clarify what this comes to in more detail in Chapter 3.1.

1.2.2 Husserl's Method of *Besinnung*: Empathising with the Community

Besinnung is explicitly described in *Formal and Transcendental Logic* (1929, 1969, 1974), but is also mentioned in *Cartesian Meditations* (1929/31, 1960) and *Crisis* (1936, 1970). This method aims to clarify the *Zwecksinn* (i.e. the goal-sense or final-sense) of a practice, which broadly refers to the aims and goals of the practice. It involves 'standing in the community of empathy' (*FTL*, introduction), which roughly means to engage in the real practice of the community and understand its motivation.[7]

In the context of mathematics, to stand in the community of empathy, phenomenologists should be interacting with the mathematicians as they practise their discipline. This means understanding what kinds of experiences are involved in their practice – e.g. proving. To achieve this, one might attend lectures, solve mathematical problems, attend conferences, or converse with working mathematicians to engage in the real exchange with 'complexities of mathematical practice' (satisfying PMP2). Of course, not all mathematicians do all of these things, but these are *some* of the activities involved in the complexities. In relation to historical practice in mathematics, the phenomenologists can read primary mathematical sources engaging with the mathematics – not merely as a reference but as a source – in the way historians of mathematics would. Importantly, it is the *community of empathy* (in particular, *mathematical empathy*) that phenomenologists are aiming to stand within: phenomenologists ought to empathise with the mathematicians and understand what it is like to be a working mathematician, not simply as an individual subject but as a member of the mathematical community. This does not simply mean empathising with a person who happens to be a mathematician, but engaging with the mathematical content with which the mathematicians engage, and understanding the mathematician's perspective as they engage with that content.

In conversing and engaging with working mathematicians, phenomenologists cannot avoid the methodological/conceptual questions and worries of the mathematicians. If phenomenologists were to reject these worries, then they would simply be failing to empathise with the working mathematicians. Thus, being 'open to questions that worry mathematicians' (Gray and Ferreirós) is

[7] I shall expand on what Husserl means by the 'community of empathy' in Chapter 3.

a feature of standing in the community of empathy, and hence, and contributes to the satisfaction of PMP1.

Thus far, I have shown that Husserlian phenomenology of mathematics is a systematic philosophical approach starting with presuppositionlessness, that it is an *empathy-first* approach, and that the method of *Besinnung* satisfies PMP1 and PMP2. I shall now briefly show how Husserlian phenomenology compares with an existing methodological programme, Maddy's *Second Philosophy*. As Hartimo has pointed out (see, e.g., her 2020a and 2021b), there are similiarities between Maddy's Second Philosophy and the Husserlian phenomenological method of *Besinnung*. The next section aims to highlight some differences between the approaches and suggest some advantages of the Husserlian method. In particular, I argue that an understanding of Husserlian phenomenology allows us to see how it might be pursued effectively in practice, while Maddy's *Second Philosophy* remains, in contrast, an idealised methodological programme.

1.2.3 Phenomenology and Second Philosophy

As Mirja Hartimo has pointed out, there are many similarities between Second Philosophy and Husserlian phenomenology, especially when it comes to the method of *Besinnung* (see, e.g., Hartimo, 2020a, 2020b, 2021a, 2021b). This might come as a surprise to phenomenologists because of Husserl's anti-naturalist positions. Hartimo, (2021b, p. 18) writes that

> Husserl [argued] in 'Philosophy as Rigorous Science' (1911) that phenomenology should overcome naturalism. By 'naturalism' he means a reductive philosophical attitude according to which [. . .] human consciousness, ideas, ideals, and norms are reduced to physics, which results in relativism and skepticism.

Since Maddy's Second Philosophy is sometimes seen as a kind of Naturalism (see, e.g., Maddy, 1997, 2022), the knee-jerk reaction of phenomenologists might be 'Second Philosophy is bad!' However, the 'Naturalism' Husserl opposes is one intent on metaphysical reduction, in particular to physics, rather than the methodological one advocated by Second Philosophers.

One similarity between the two methods concerns how they view mathematics/mathematical practice. Second Philosophers and phenomenologists alike '[examine] developments in mathematics as they are motivated by particular goals and values' (Hartimo, 2021b, p. 19). In *Formal and Transcendental Logic*, Husserl writes that the mathematical community has a 'goal sense [*Zwecksinn*] toward which [mathematicians] have been continually aiming' (*FTL*, p. 9, quoted, with my minor modification to the translation, in 2021b, p. 20).[8] Similarly, Maddy identifies foundational goals (e.g. in Maddy, 2019) toward which set theorists and homotopy type theorists have been continually aiming. Thus, Second Philosophers and phenomenologists agree in their teleological view of mathematical practice, and in that share the aim to 'identify the goals and evaluate the methods by their relations to those goals' (1997, p. 194, quoted from 2021b, pp. 19–20).

However, they would disagree on what qualify as the goals of mathematical practice are. For phenomenologists, *Zwecksinne* include minimal requirements, 'without [which] one would think that there is something wrong with the theory, or that the theory is not a theory at all' (2021b, p. 189). For example, Hartimo (2021b) argues that the law of non-contradiction is a *Zwecksinn* of mathematical practice. These requirements are then one of the most trivial necessary conditions found in mathematics, and they ought to be understood when empathising with the community. However, such a general law would simply fall under *logic* for the Second Philosophers, and thus the law will be further reduced to the combination of the logical structure of the empirical world as well as our cognitive mechanisms (Maddy, 2007, p. 225ff.). So the law of non-contradiction is, for Second Philosophers, 'true of the world' and human beings' 'most primitive cognitive mechanisms allow them to detect [the law]' (Maddy, 2007, p. 233). In that sense, *Zwecksinne* are more broadly conceived than the *goals* of Second Philosophers. The law of non-contradiction is a *Zwecksinn* (as a minimal requirement for mathematical practice) while, for Second Philosophers, it is simply a law of logic which the practitioners can empirically verify.

There is a more important difference between the Second Philosopher and the phenomenologist. While the Second Philosopher is an idealised enquirer who is 'equally at home in all the various empirical investigations', the phenomenologist is not. Because the Second Philosopher is an idealisation, it is hard to find any practical guidance by which one can carry out the method of Second Philosophy. A

[8]See Chapter 3 for more on 'goal sense' (*Zwecksinn*).

phenomenologist is a real person with varying levels of skill in different disciplines, and someone who is curious to learn about different disciplines. A *human being* could be a phenomenologist by learning and carrying out phenomenological methods, while the Second Philosopher seems to be an *unattainable* ideal whose methods human beings will never be able to carry out.

Furthermore, any phenomenological method would require the enquirer to look at mathematics – to take one example – from a mathematician's perspective, via empathy – i.e. looking at the mathematics from a first-person perspective – unlike a Second Philosopher, who might pursue a third-person investigation, looking at a practice from an outsider's perspective. To conduct this phenomenological enquiry then, one must pay attention to the mathematics with which the mathematicians are engaging, rather than simply conducting a sociological or anthropological investigation concerning mathematicians (which might be considered satisfactory by a Second Philosopher).

Husserlian phenomenology qualifies as a Philosophy of Mathematical Practice because of its *principle of freedom from presuppositions* and its method relying on empathy: both of these elements put mathematical practice *first*, prior to any philosophical commitments. *Besinnung* is an example of a particular phenomenological method which focuses on the community's practice. Its aims resemble the aims of Maddy's Second Philosophy, but the two approaches differ, both in what they count as such goals, and in that the phenomenological method *must* be carried out from a first-person perspective, paying attention to the mathematics that the mathematicians are looking at.

1.3 What is Phenomenology?: Intentionality and Teleology

The previous section introduced Husserlian phenomenology as a candidate for a philosophy of mathematical practice. In this section, I provide a detailed account of the nature of Husserlian phenomenology. In particular, I argue that the basic structure of intentionality, considered to be fundamental to phenomenology, is paralleled by the teleological structure that Husserl sees in scientific practice, including mathematics.[9]

[9] This section also aims to clarify some terminology that I shall use throughout the dissertation.

1.3.1 Phenomenology as the Study of Consciousness: The Basic Structure of Intentionality

Husserl's phenomenology begins with the *Logical Investigations* (1900/1) in which he articulates the structure of intentionality. The term 'intentionality' comes from Husserl's teacher, Franz Brentano, whose work *Psychology from an Empirical Standpoint* (1874) had been a great influence on Husserl. Brentano claimed that, in a mental phenomenon, we have two distinct objects – the physical object and the *intentionally inexisting* object. The physical object is independent of our mental phenomenon, but is that at which our experience is directed. However, the *intentionally inexisting* object exists *within* our mental phenomenon. What Brentano sought was a new *science* of mental phenomena, focusing on acts of consciousness. But Husserl takes Brentano's descriptive psychology and builds his own theory, focusing on the idea that conscious experience has the intentional structure that Brentano described, but diverges from Brentano's views in some respects.[10] For instance, Husserl does not use the notion of intentional inexistence, but instead discusses the *meaning* (which he calls, 'sense' [*Sinn*]) in an intentional experience. On the Brentanian and Husserlian accounts, intentionality is a description of the structure of our conscious experience.

For example, our ordinary sense-perceptual experiences can be described as *intentional*, since they are *of* or *about* something. This structure can be found when we simply reflect on our own experiences. When we are seeing, hearing, touching, smelling, etc., there is always *something* that we see, hear, touch, smell, etc. This description is also true of hallucinations. If I were hallucinating a pink elephant, the *something* (regardless of whether it physically exists or not) is the pink elephant. Hence, 'intentionality' is a description of a subjective experience.[11]

In our experiences, the *something* is not restricted to particular physical objects like a tree. It can be of an event or a situation. Suppose that I see a cat falling out of an apple tree. In this experience, I am not merely seeing the cat, but I also see the cat falling from the tree onto the ground. Perhaps I even see the cat land safely on four legs. These experiences of seeing are about more than just the cat: they are about the cat falling from a tree, the cat landing on the ground, and so on. The

[10] Brentano, (1874/2012) claims that the term 'intentionality' [*Intentionalität*] comes from the medievals, but it is not clear whether the term was ever used by any medieval philosopher.

[11] I follow David Woodruff Smith (2013) in my characterisation of the general structure of intentionality.

structure of intentionality can also be found in non-sense-perceptual experiences such as thinking, remembering, imagining, etc. – for example, when I think, I am thinking *of* or *about* something: I am thinking of the cat outside; or I am remembering what happened yesterday; or I am imagining a dragon flying away from a castle, and so on. Phenomenology focuses on these kinds of experiences, which can be described as intentional.[12]

Given these features of intentional experiences, the *structure of intentionality* [*Intentionalität*] or *of intentional experience* can be clarified in more detail as follows. In an intentional experience, a *subject* experiences the (intended) object. This subject is usually referred to as 'I'. As I described earlier, the *something* of an intentional experience does not need to be a particular thing, but it could be an event or even an abstract entity. Regardless of the nature of *what* the experience is of or about, we refer to it as an *object* [*Gegenstand*, *Objekt*] or an *intended object*.[13] The question about the nature of the intended object is not always of concern to a phenomenologist, in contrast with the situation in (e.g.) metaphysics.

In an intentional experience, the intended object is said to be presented [*Vorgestellt*] in a certain way, and the *way* the object is presented is called the *content* [*Inhalt*, or *Gehalt*]. We use the terms 'sense [*Sinn*]', 'meaning [*Meinung*]', 'meaning content', or 'concept' to refer to the content of the experience, and throughout the dissertation, I shall use these words interchangeably. In particular in Chapter 2, I use the word 'concept' to refer to the *content* of intersubjective experiences.[14] Even though I am seeing the apple tree in my garden, if I do not know what an *apple* tree looks like, the content of the experience could be of a *tree* rather than an *apple tree*. Even though the intended object, apple tree, can be presented as a <tree>, as well as an <apple tree>, the content of *my* experience (i.e. this particular experience of my seeing) of the tree is the <tree>, not <apple tree>. That is, I see the thing in front of me *as* a <tree>. In this sense, the content is the *meaning as*

[12] I am not opposed to the idea that there could be experiences which are not intentional. In that case, we can simply say that phenomenologists are interested only in *intentional* experiences.

[13] We can understand the 'intended object' to refer to the object (or the objective event or situation) at which our subjective intentional experience is directed.

[14] As Husserl develops his phenomenology as a systematic science and sharpens his structure of intentionality, he replaces the term 'content' with the term 'sense [*Sinn*]' : 'Each noema has a "*content*", that is to say its "sense", and is related through it to "its" *object*' (Ideas I, §129), Sometimes he refers to the 'noematic *Sinn*'. In this dissertation, I shall simply drop the adjective 'noematic'.

presented by the intended object in the experience. Hence, phenomenology can also be seen as the study of *meaning* in our conscious experience.

Henceforth, I shall use angle brackets '<', '>' to indicate reference to the content of the experience or the presentation of the relevant object, in order to distinguish it from the object itself. By making this distinction, I do not mean to assert an ontological difference between the two, but rather to indicate that we can treat them differently, since the content can be described only within the experience itself, while it is possible that the intended object is described independently of the subject's experience.[15]

Occasionally, we might say the object is presented to me *as* a <tree> to emphasise that the <tree> is the content of my experience of seeing the apple tree, that is, the tree is *constituted* as a <tree> for me.

As we shall see shortly, the term '*Sinn*' is important for the project of this dissertation, since Husserl's phenomenological methods for studying mathematical practice, which he calls *Besinnung*, aims to clarify the *Zwecksinn* – i.e. the goal-sense. I take Husserl's terminological usage of '*Sinn*' very seriously for understanding *Besinnung*. Although '*Besinnung*' can be translated as 'reflection' with some spiritual or religious connotation, I am looking at *Besinnung* as a reflection on the *Sinn* of a given practice.[16]

From this basic structure of intentionality, D. W. Smith and R. McIntyre (1982) and David Woodruff Smith (2013) further clarify and sharpen different features of our consciousness. Some of these features will become important later in the dissertation, but for now, I shall focus only on the simple structure of intentionality described here.

[15] There is a debate among phenomenologists about the nature of the intentional content (see, e.g., Zahavi (2004); Hirvonen (2022)). In so-called 'West-Coast phenomenology', the content is distinct from the object. In the case of a tree, the object tree is in nature, but the content <tree> is not. However, the 'East-Coast phenomenology' reading interprets the tree in nature as the same tree as the content of the experience. In this reading, it is the subject's attitude that changes with respect to the experience of the tree – the subject brackets certain metaphysical questions about the tree and turns towards their phenomenological attitude, focusing on the meaning of the experience, or how the tree is presented. The distinction is an important one philosophically, but it is not significant for my project. Thus, I take an ontologically neutral stance on this issue and allow that we can talk about the content as a separate thing from the object by referring to them via certain noun phrases. On the other hand, we can also talk about the content in terms of adverbial ways in which we experience the object – e.g. the thing appears to me visually in a tree-like manner. For further discussion of these issues, see Zahavi, 2003, pp. 58–60; Zahavi, 2004; Follesdal, 1969; Follesdal, 1990; Crowell 2006, pp. 18–19; and D. W. Smith, 2013.

[16] For this reason, Dorian Cairns translates '*Besinnung*' as 'sense-investigation' in Husserl (1969).

With the basic structure of intentionality clarified, let us turn towards phenomenology as a method/collection of methods.

1.3.2 Phenomenology as a Methodological Approach

Another view of phenomenology is that it is a method, or that it consists of a collection of methods (see, e.g., D. W. Smith, 2013, p. 226ff.). The most fundamental phenomenological methods are bracketing, and phenomenological reductions (*epoché*). In order to explain what these methods involve, we must first explain what a natural attitude is.

As we go about our everyday business, we do not (always) stop to ask philosophical questions. We naturally assume that the world is out there, and that we live in this world. I do not doubt whether I have two hands, nor do I doubt whether $2 + 2 = 4$. This naïve attitude in which we have certain assumptions is called the '*natural attitude*' or '*natural standpoint*' ['*natüraliche Einstellung*'].[17]

The general methods of phenomenology aim to describe and clarify the meaning, or *Sinn*, found in our experience. To do so, we turn away from our natural attitude by *bracketing* (i.e. ignoring or suspending) judgements, questions or naïve assumptions; and by *phenomenological reduction* (or *epoché*), which is a kind of reflection, we turn our attention towards, for example, the phenomenological/intentional content – i.e. our perception of the objects. For example, when seeing an apple tree, we can reflect on this seeing experience and focus on the content <apple tree>. Once we focus our attention to the experiences, we can clarify what it *means* for something to be an *apple tree* from a subjective perspective. This is what phenomenological reduction (*epoché*) is, in the simplest sense.[18]

[17]There are different translations of '*Einstellung*' in Husserl scholarship. For example, Hartimo generally translates it as 'attitude' while David W. Smith translates it to 'standpoint'. For the sake of consistency, I shall typically use 'attitude'. The important point in my interpretation, however, is that the *Einstellung* refers to the perspectives of different subjects, in addition to the different contexts or situations in which they are placed.

[18]The notion of *epoché* goes back to Sextus Empiricus, a Pyrrhonian sceptic, who describes it as a 'mental rest stasis' [*stasis dianoias*] from which we neither deny nor affirm anything (Sextus, 1933, p. 10). Literally in Greek, '*epoché*' means *stopping*, but it is often translated as 'to suspend judgement' or 'to withhold assent'. I would like to thank Charles Leitz for pointing this out to me.

When we suspend our judgement from the intended object of our experience and focus on the content of our experience, we are said to be in the '*phenomenological attitude* [*phänomenologische Einstellung*]'.

Although in the example above, phenomenological reduction aimed to clarify the intentional content, it can also clarify the meaning/*Sinn* of other parts of intentionality. Carrying out phenomenological reduction begins in our natural attitude, bracketing the naïve assumptions, and then, by reflection, we clarify the *meaning* found from the experiences we have. This is not restricted to clarifying only the intentional content; it can also involve clarifying the kinds of experiences we have or the structure of our experience. For example, in my experience of seeing an apple tree, I can focus on *seeing* and clarify what it *means* to *see* from the subjective perspective.

Along these lines, Husserl characterises phenomenological reduction as a generalisation of Descartes's method of sceptical doubt (see, e.g., *Cartesian Meditations*). In his *Meditations*, Descartes begins to doubt the existence of the things he experiences and concludes that the *I* who is capable of doubting exists. The existence of the *I* is characterised as the *Cogito*, and it is considered to be a first principle that grounds all knowledge. Similarly, Husserl begins in his natural attitude and (instead of doubting) brackets all these naïve assumptions he has. Eventually, he comes to the realisation that the meaning or the *Sinn* (sense) of the *I* is the one who can think, experience, or has consciousness. Hence, the *I* is the precondition that makes his thinking experience possible. Going even further, for an arbitrary individual subject, they can also phenomenologically reduce to the existence of themselves, so this *I* can be understood as a universal idealised subject. Thus, Husserl refers to the universal, ideal *I*, as the *transcendental Ego*. While phenomenological reduction could be applied to understand the meaning within a particular intentional experience, with a *transcendental aim*, the method can result in a universal or general condition that makes intentional experience or certain experiences possible. Husserl's transcendental ego is an example of that: while considering various kinds of experiences, the transcendental aim leads Husserl to clarify the meaning of the *I*.

The term 'transcendental' has a variety of interpretations, but based on a loosely Kantian notion of the 'transcendental', it is broadly characterised as the pre-conditions that make cognition possible.[19] Like the Kantian term, Husserl's term 'transcendental' also has various interpretations. Instead of providing a review of other interpretations, I focus on my interpretation of Husserl's 'transcendental'.

Husserl's aim in transcendental phenomenology is, in my view, to clarify what is universal or general to our cognition or experience, such that every particular experience is an instance of the universal. For example, a phenomenologist might start with the ordinary experiences of *seeing*, and turn towards what is universally common in all the *seeing* experiences. Once this universal commonality is clarified, every experience of *seeing* is understood as an instance of the universal account of *seeing*. Then, the universality can be described as the condition that makes *seeing* possible. In this sense, phenomenology is an explicatory or clarificatory science: it is a science because it offers a systematic method, and it aims to clarify the universal conditions found from our experiences.

This is clearer in Husserl's characterisation of Descartes's *Cogito*. Descartes's *Cogito* shows that there must be a subject who is having these experiences. Any conscious subject must be able to conclude that the thinker/experiencer exists, and so the *Cogito* can be viewed as a transcendental condition – i.e. a universal condition – for cognition. This transcendental *Cogito*, or transcendental ego [*transzendentale Ich*](*CM*, §9ff., §11) as Husserl refers to it, is not the ego of a particular subject, but rather the general form or structure that is universal to any ego, and it is understood in virtue of carrying out the phenomenological method:

> By phenomenological *epoché* I reduce my natural human Ego and my psychic life – the realm of my *psychological self-experience* – to my transcendental-phenomenological Ego. (*CM*, §11)

Once we are able to clarify the transcendental condition, we are said to be in a *transcendental attitude* – a phenomenological attitude in which the content of the experience is the most *pure*. (See *Crisis*, Part III, A; 1970, p. 179.)

[19]For Kant, transcendental philosophy is what *explains* the *a priori* conditions for the possibility of knowledge/cognition. I would like to thank Jeremy Heis for clarifying this distinction for me. For a review of different interpretations of Kant's 'transcendental', see Stang, 2022.

In general, the transcendental conditions can be understood as *universal* features of our conscious experience, which are accessible by following the phenomenological methods. In that sense, we can understand 'Transcendental Phenomenology' as phenomenology that *aims* to clarify the universal conditions of our cognition. Although many scholars (e.g. Cobb-Stevens, 1990, p. 165) claim Husserl makes the 'transcendental turn' in his *Ideas I*, I do not consider Husserl's work to be divided as such. I follow the reading of Husserl that he is continuously developing his philosophical ideas (see, e.g., Hartimo, 2021b, D. W. Smith 2013), starting from the *Philosophy of Arithmetic* (Husserl, 1891/2003). On my account, the phenomenological methods *aim* to arrive at the transcendental (or universal) conditions, and *transcendental attitude* is a kind of phenomenological attitude where we have successfully clarified the universal conditions.[20] So, let us return to the more simple kind of phenomenological reduction and describe how the structure of intentionality can be understood by phenomenological reduction.

In phenomenological reduction applied to a particular experience, we turn our attention to the various parts of the experience. The structure of intentionality can then be found by phenomenological reduction. In any conscious experience, there is always a subject, a something that the subject is experiencing, and a meaning that the subject attributes to the thing experienced (or the object *as* experienced). Thus, we can understand the notion of *Sinn* (i.e. the meaning content) to be a transcendental condition of cognition. Furthermore, the structure of intentionality is also a transcendental condition for (intentional) consciousness, since it is necessary for conscious experience.

The ultimate aim of Husserlian phenomenology is to identify and clarify what conditions make our conscious experience possible. And we can apply the phenomenological methods to specific types of conscious experiences in an attempt to clarify what universal conditions there are for those types of experiences. For instance, if we apply it to the experiences of *knowing*, we ask what are the conditions that make knowledge possible? We reflect on our experiences of knowing, and discover what is universally common in those experiences and characterise them as transcendental conditions

[20]This would contrast with Hartimo's and Cobb-Stevens's treatments of transcendental reduction as a separate methodology from *epoché* or phenomenological reduction. And yet I agree with Cobb-Stevens that the transcendental attitude (if one must distinguish it) is a 'philosophical attitude that permits appropriate thinking about the relationship between knower and known' (1990, p. 162). On my account, in contrast with that of Cobbs-Stevens, 'transcendental' phenomenology describes an aim that extends the method of *epoché*. I shall return to this in Chapter 3.

for knowing. We can narrow the focus still further, to mathematical knowledge, for example, and ask what conditions make mathematical knowledge possible.

When Husserl claims that scientific practice is teleological, he is not simply assuming that, but stating what he has found through the application of phenomenological methods. What all of the sciences (including philosophy) have in common is teleological structure and continuity from their own individual histories. In the next subsection, I provide textual evidence to support Husserl's teleological view of the sciences. Then, I compare the structure of intentionality with the teleological structure.

1.3.3 The Teleological Practice of Science

According to Husserl, both subjective experience and scientific practice have the structure of intentionality. In subjective experience, our experience is directed at or aimed at the intended object via the intentional content. Through phenomenological reduction, however, we clarify the meaning of the object that is universally present in all our experience with the object. This clarified meaning is what Husserl calls, in *Logical Investigations*, '*fulfilled sense*'. It refers to the objective or ideal meaning or the complete meaning presented by the intended object, without which all our experiences of the said object would not be possible. In empirical sciences, the practice is concerned with the empirical world. The scientists study the empirical world around us in order to understand it better. Similarly, in mathematical sciences, we can describe ourselves to be concerned with the 'mathematical world'. Whatever groups, topological spaces, real numbers, sets, functions, etc., are, mathematical practice is concerned with them, whether or not they occupy some mathematical world. However, Husserl claims that the practice is directed at or aims at the *goal sense* [*Zwecksinn*] (see, e.g., Hartimo, 2020b, pp. 70–1). Husserl claims that *Zwecksinn* is similar to 'fulfilled sense' in *LI*. I shall show later, in Chapter 3, that *Zwecksinn* refers to the motivational goals of the practice, which guide or motivate the practice to be such, but also they are what the practitioners aim for. This suggests that *Zwecksinn* is a universal condition for the practice in consideration. With this view in mind, we can visualise the teleological structure of Husserl's as in Figure 1.1.

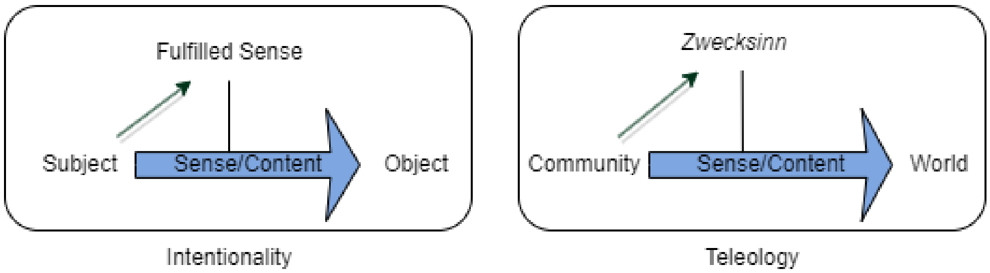


Figure 1.1: Intentionality and Teleology Comparison

Since our experience and our scientific practice share the structure of intentionality, we can apply the same general method to clarify the meaning (i.e. sense) of our experience and practice – in the former, this is the fulfilled intentional sense; and in the latter it is the goal sense [*Zwecksinn*]. Here, I provide textual evidence from Husserl's writings to show that he held this teleological view of science throughout his career.

For Husserl, science is an intersubjective or spiritual [*geistes*] creation by which the scientists jointly aim to achieve a particular goal:

> Sciences are creations of the spirit [*Geist*] which are directed to a certain end, and which are for that reason to be judged in accordance with that end. (*LI*, Prolegomena, §11; Findlay translation, p. 25)

This sentiment is similar to what we find in Dedekind's view (in *Was sind und was sollen die Zahlen?*; Dedekind, 1888) of the numbers as the 'free creation of the human spirit' [freie Schöpfungen des menschlichen Geistes,] (Dedekind, 1995), as well as in the Berlin school of mathematics according to Poincaré: 'The mathematical continuum from [the point of the Berlin school] would be a pure creation of the spirit [une pure création de l'esprit] in which experiment would have no part' (Poincaré, 2018, p. 19; 1903; modified with my translation). By 'spirit' (*Geistes* in German or *esprit* in French), Husserl, Dedekind, and Poincaré seem to be referring to the *intersubjectively* shared culture or practice (see, e.g., Reck & Keller, 2021, footnote 15). In other words, scientific and other mathematical objects are creations of the intersubjective spirit, the intersubjective practice, motivated by and directed to 'certain end[s]' (*LI*, §11).

For Husserl, then, *philosophy* of science (or of scientific practice) is the study based on these teleological structures, which cannot be completed by the natural sciences themselves:

> Furthermore, the systematic investigation of all teleologies that are to be found in the empirical world itself is not completed simply by the natural-scientific explanations of all such formations, on the basis of given factual circumstances and according to natural laws. (*Ideas I*, §58; Dahlstrom translation, p. 106)

Furthermore, the study of such structures involves the search for and the uncovering of the goal-idea or final idea [*Zweckidee*]:

> the interest in uncovering the teleological structures [*teleologischen Strukturen*] immanent in the final idea [*Zweckidee*] of a theory of science; the interest in developing, in orginary evidence, the other ideas included in the intentional sense [*intentionalen Sinn*] of that idea – ideas of component logical disciplines – and the essentially united set of problems peculiar to each of them. (*FTL*, p. 66, §23; Janssen, p. 79; Cairns translation, p. 75)

In fact, it is not only in the natural sciences that we find this teleological structure, but also in philosophy (*Crisis*, §15). And to understand the teleology of philosophy – and, as I consider, of other sciences including mathematics – we must understand how the discipline was formed historically, and also understand what our current final goals and ideas are:

> Our task is to make comprehensible the teleology in the historical becoming of philosophy, especially modern philosophy, and at the same time to achieve clarity about ourselves, who are the bearers of this teleology, who take part in carrying it out through our personal intentions. (*Crisis* §15; Carr translation, p. 70)

So how is this teleological structure of practice described as intentional? Recall that the structure of intentionality consisted broadly of a subject, intentional content, and the intended object. In the teleological picture of the practice, instead of a subject, we have a community of subjects – that is, the practitioners of a given discipline. The intended object can be replaced by anything that the scientists are concerned about. Since teleological practice is focused on the aims or goals, in place of intentional content, we have *Zwecksinn*, the goal sense. As Husserl describes, *Zwecksinn* is similar to the 'fulfilled sense' in the structure of intentionality. A sense, in a conscious experience, is said to be *fulfilled* if the subject can instantaneously and accurately grasp the object via the fulfilled sense. Thus, the object as experienced is the object as itself. In other words, we have a complete picture of the intended object, which cannot be faulted. Hence, *Zwecksinn* is the complete meaning or purpose of the practice, that which the practitioners are aiming for, and the practice cannot be understood completely without grasping this meaning.

When we consider mathematics as a science, and see it as a *teleological* practice, then Husserl's approach would suggest what philosophy of mathematical practice ought to be. Husserlian philosophy of mathematics, or the phenomenology of mathematics, ought to focus on understanding the historical/cultural becoming of the discipline, and aim to clarify the finals goals and ideas of the practice.

1.4 Conclusion

In this chapter, I have argued that Edmund Husserl, better known within philosophy as the founder of phenomenology, was a philosopher of mathematical *practice*. Phenomenology takes an empathy-first approach to philosophy of mathematics. This means taking the perspective of the mathematicians first in understanding mathematics without any external presuppositions (section 1.2.1). Furthermore, Husserl's phenomenology (in the context of mathematical practice) is a methodological programme that deals with the philosophical questions (i.e. conceptual and methodological questions) and aims to clarify the concepts and methods found within the community of mathematicians (section 1.3.2). Achieving this aim involves accepting that mathematical practice is an intersubjective practice with a teleological structure, similar to the structure of intentionality (see section 1.3.3).

Phenomenology neither asks nor answers ontological or epistemological questions independently of mathematics, the mathematical community, or mathematical practice; rather, Husserl's methods suggest philosophising *within* the community's practice (i.e. in empathic experience with the community – see section 1.2.2). The important methodological feature of phenomenology is the first-person perspectival analysis (and, hence, the empathy-first approach) and phenomenology aims at *clarification* from such a perspective. Any explication of our experience or practice that does not start from the first-person perspective is not one that can be considered phenomenological.

In the following chapters, I shall expand on these features of Husserlian phenomenology and clarify further how an empathy-first approach can be a fruitful approach to philosophy of mathematics and mathematical practice. The next chapter, for instance, focuses on this empathy-first approach in the broader context of epistemology of mathematics. Instead of taking the mathematicians' perspective, I take the perspective of the ordinary folk as the first-person perspective to understand human number cognition. By doing so, I aim to demonstrate how this empathy-first approach can be fruitful to epistemology generally.

The subsequent chapters will focus on developing the Husserlian method of *Besinnung* with the aim of applying it to contemporary discussions in philosophy of mathematical practice. With the empathy-first approach articulated in Chapter 2, I explain how empathy is important in understanding Husserl's notion of *community of empathy* in Chapter 3. Then in Chapter 4, I characterise what must be involved in empathising with the community of mathematicians who practice homotopy type theory.

Chapter 2

A Phenomenological Analysis of the Number Sequence

The general aim of phenomenology is to clarify the meaning or sense that we find in our experience or in our practice. A common criticism of phenomenology from the perspective of other empirical sciences is that it does not offer any empirical evidence for the accuracy of the first-person analysis of our conscious experience. But this criticism is misguided, since phenomenology is *not* an explanatory science. Instead, it aims to explicate and clarify what might be *intersubjectively* valid conditions for human conscious experiences, while paying close attention to the community and culture of which we are a part. Whether these descriptions are *objectively* valid is not a question for the phenomenologists, but perhaps for anthropologists or empirical psychologists. Hence, I describe phenomenology as an '*explicatory science*'.

This, in fact, is how I view phenomenology as a methodological programme. It provides descriptions and clarifications of our experience and/or scientific practice. Whether we can provide a causal explanation of shared human experience should be left to empirical or social scientists, while the phenomenologists focus on explicating such experience. In this chapter, I demonstrate how phenomenology and empirical sciences can interact and collaborate.

I shall take a step back from scientific practice at this point, and focus instead on our ordinary experience. Since phenomenology is an empathy-first approach, its methods can be applied to ordinary experiences, not just scientific practice. Here, I shall spend some time focusing on our experience of the number sequence '$1, 2, 3, \ldots$' and work to clarify the meaning of the sequence as experienced by an ordinary subject.

This chapter has two main aims. One is to show that phenomenological methods (i.e. the methods of the empathy-first approach) can be used to demarcate sophisticated mathematical practice from the ordinary use of mathematical concepts. A second aim is to outline a way in which phenomenology and cognitive science might be pursued collaboratively, as an explicatory science and an explanatory science, respectively.

To achieve these aims, I focus on our experience of the natural or counting numbers from the subjective perspective. The general approach is as follows. I first attempt to understand and describe our ordinary common-folk experience of the natural or counting numbers. This means, rather than assuming a mathematical definition or an account of the natural numbers, I simply begin with the ordinary description of the sequence of numbers as '$1, 2, 3, \ldots$'. Thus, the phenomenological method involves simply describing certain kinds of experiences *about* the sequence '$1, 2, 3, \ldots$' from the subjects' standpoint. Supposing that the mathematicians' experience of the description '$1, 2, 3, \ldots$' refers to the structure of the natural numbers $\mathbb{N}$, I show that the ordinary experience of $1, 2, 3, \ldots$ is not concerned with $\mathbb{N}$.

My approach here is similar to that taken by Husserl in his *Philosophy of Arithmetic* (1891, 2003). In *PA*, Husserl focused on the *origins* of the concept of numbers, but I shall focus on the experience of the sequence of numbers as '$1, 2, 3, \ldots$'. I then argue that the ordinary notion of numbers is distinct from the mathematical notion of natural numbers. Thus, I distinguish ordinary mathematics from mathematical practice. In fact, Husserl often demarcated between different disciplines as he studied philosophy: in *FTL*, Husserl demarcated logic and mathematics, and in *Crisis*, physics and mathematics (see Hartimo, 2020b, for more details).

2.1 Background: Understanding the Natural Numbers

In this section, I focus on the question, *How do we come to understand the natural numbers?* Various philosophers have offered their solution, which focused on the notion associated with '...' in the expression '$1, 2, 3, \ldots$'. I shall present their views in section 2.1.1.

2.1.1 Understanding the Natural Numbers?

Given that mathematicians characterise the sequence of natural numbers as '$1, 2, 3, \ldots$' (see, e.g., Davenport, 2008), how do we actually come to understand the natural numbers? There are different ways to answer this question. Shapiro (1997, p. 177ff.) argues that we can have epistemic access to the natural number structure, despite its abstractness, by using ellipses (...). Using ellipses, one can generalise from finite sequences of collections of strokes to a potentially infinite sequence. For instance, a child is given the following sequence of collections of strokes:

$$|, ||, |||, ||||.$$

The child understands that starting with the first collection (|), the next collection of strokes in the given sequence (||) is generated by drawing a stroke | next to the first collection. The third collection of strokes (|||) in the sequence is generated by the same operation on the previous one (||). This pattern of generating the next collection of strokes can be generalised, so the child can *project* the next collections to be ||||| and then ||||||:

> Reflecting on these finite patterns, the subject realizes that the sequence of patterns goes well beyond those she has seen instances of. [...] Our subject thus gets the idea of a sequence of 9,422 strokes, and she gets the idea of the 9,422 pattern. Soon she grasps the quadrillion pattern. (Shapiro, 1997, p. 119)

Then the subject can *abstract* from this pattern the notion of a *finite sequence* (Shapiro, 1997, p. 118) and express it as

$$`|, ||, |||, ||||, \ldots'.$$

Here the '...' expresses the potentially infinite nature of the sequence of collections. Once our subject has successfully understood what is expressed by '...', Shapiro adds:

> [we] can coherently discuss the infinite pattern and can teach it to others. [Then we] have grasped (an instance of) the natural-number structure. (Shapiro, 1997, p. 119)

Similar to Shapiro, Maddy (2014; 2018) emphasises the notion of '...' in understanding the natural number sequence, and she argues that the '...' arises from a *language-learning device* (as well as other human cognitive features) (Maddy 2018): this language-learning device refers to the 'recursivity' of spoken or written natural languages.[1] Maddy claims that

> Insofar as arithmetic [whose subject matter is the standard model or the omega-sequence] is 'about' anything, it's about an intuitive picture of a recursive sequence of potentially infinite extent, an intuitive picture we humans share thanks to the evolved linguistic faculty common to our species (Maddy, 2018, p. 26).[2]

Both Shapiro and Maddy are concerned with human subjects and their acquisition of the notion[3] of the structure of the natural numbers denoted by 'ℕ'. But in Shapiro's case, he is responding to an anti-realist worry: if ℕ does not exist, knowledge of ℕ might not be possible. Thus, he aims to show how we can come to have knowledge of abstract (perhaps non-existing) entities by highlighting that if we are able to talk about abstract entities, we can have knowledge of them. Maddy, in contrast, is attempting to answer a question regarding how we humans *actually* come to know or understand the structure of the natural numbers ℕ. Thus, she appeals to some empirical evidence that highlights the different features and stages of learning numerical notions from cognitive sciences. The question I am interested in is closer to Maddy's than it is to Shapiro's. I am interested how we come to understand the structure of the natural numbers ℕ, rather than the problem of epistemic access to abstract structures.

[1]The recursivity of natural languages should not be confused with the mathematical notion of recursivity. Rather, natural language is recursive in the sense that we can concatenate words to generate a phrase or a sentence, or even a new word.

[2]I would like to highlight that Maddy is no longer committed to this view in light of new empirical results which will be discussed in section 2.3. The point is explicitly stated in (Maddy and Väänänen, 2022, pp. 42f.): 'believing there's no largest number, appreciating the endlessness of the sequence, $1, 2, 3, \ldots$, isn't always enough to equip one with the notion of an orthodox omega sequence.' In footnote 74 of Maddy and Väänänen, (2022), they further point out that Maddy had made an error concerning this in her earlier work.

[3]I use the term 'notion' here since the term 'concept' is philosophically overloaded.

As we see, Maddy focuses on the *sequence* of the natural numbers (sometimes called the ω-sequence), which is often denoted by '$1, 2, 3, \ldots$'. On Maddy's account, what is meant by the sequence of numerals '$1, 2, 3, \ldots$' is the natural number sequence, or the $\mathbb{N}$-sequence. In more phenomenological terminology, we say that the $\mathbb{N}$-sequence is *given* to a subject by '$1, 2, 3, \ldots$' – meaning that, for this subject, what is meant by the expression '$1, 2, 3, \ldots$' is the $\mathbb{N}$-sequence. Since Maddy's claim is that understanding the recursivity of the '$\ldots$' will give us (the humans) an intuitive picture of the $\mathbb{N}$-sequence, we can interpret Maddy to be claiming that what is *given* (to an arbitrary subject) by '$1, 2, 3, \ldots$' is the $\mathbb{N}$-sequence.

Let us call this the '*Dots-*$\mathbb{N}$ *thesis*':

Dots-$\mathbb{N}$ **thesis**: the $\mathbb{N}$-sequence is *given* by '$1, 2, 3, \ldots$'.

Although Maddy (2014) uses empirical evidence for the Dots-$\mathbb{N}$ thesis, a recent empirical finding suggests that the thesis fails. If the above thesis were true, then anyone who understands what is meant by '$1, 2, 3, \ldots$' would not deny any instance of mathematical induction. However, across three papers, Relaford-Doyle and Núñez, (2017; 2018; 2021) argue that (at least some) university-educated students do not have the natural number concept, because they do not understand mathematical induction.

I think the problem is a little more complex than a simple denial of induction. Instead, I offer another perspective, in which the common-folk understanding of '$1, 2, 3, \ldots$' involves some non-arithmetical acts by which we access the larger numbers that occur in the '$\ldots$.' Hence, we can conceptually distinguish 'arithmetical numbers' from 'non-arithmetical numbers' based on the acts involved with accessing the larger numbers in ordinary understanding of the number sequence.

The main argument in this chapter is as follows:

- Premise 0: In order to understand the structure of the natural numbers, every number in the number sequence must be accessed by counting alone;[4]

[4]Note that I have already advocated for premise 0 in this introduction.

- Premise 1: Numbers in the sequence '$1, 2, 3, \ldots$' can be accessed by non-arithmetical acts, as well as arithmetical acts (e.g. counting), in our ordinary experiences;
- Premise 2: A conceptual boundary between 'arithmetical numbers' and 'non-arithmetical numbers' can be made based on how they are accessed;
- Premise 3: Empirical evidence shows that the above conceptual boundary can be found even in adult understanding of the numbers;
- Conclusion: Even cardinal principle knowers[5] do not understand $\mathbb{N}$. That is, the Dots-$\mathbb{N}$ thesis is false.

I argue for premise 1 by analysing what kinds of acts might be involved in accessing numbers that appear in the number sequence. While we could simply say *counting* or *adding one* is how we access all the numbers in principle, I argue that, for the common-folk, there are other ways of accessing the numbers. In section 2.2.1, I shall identify certain acts that can be interpreted as an arithmetical operation (such as successor operation, addition, or multiplication) and those which cannot be. Then, in section 2.2.2, I shall introduce the notion of feasibility from Dean (2018) in order to demarcate the numbers that are accessed by arithmetical acts from those that are not (Premise 2). I refer to these as 'arithmetical' and 'non-arithmetical' numbers, respectively. Informally, a number is *arithmetical* if we can *count* to that number starting from $1, 2$, and 3, while it is *non-arithmetical* if we cannot in a given context owing to some human or physical limitations. In the end, I argue that only arithmetical numbers are the natural numbers.

In section 2.3, I present empirical evidence that suggests that even adults who have some understanding of what is meant by '$1, 2, 3, \ldots$' allow non-arithmetical acts to access numbers (Premise 3). Then we can make a distinction between the arithmetical and the non-arithmetical numbers in their understanding of numbers. Hence, I conclude that they do not have an understanding of the structure of the natural numbers. And this is not simply because they deny an instance of induction as suggested by Relaford-Doyle and Núñez (2018).

[5] By 'cardinal principle knower', I mean those who have an in-principle understanding of counting indefinitely. This will be clarified shortly.

For the purposes of this investigation, I shall use the following operational definitions. By the (structure of the) *natural numbers*, I mean the collection of things (which I refer to as 'numbers')[6] often denoted by $\mathbb{N}$ with the appropriate arithmetical operations/relations defined on the elements. Note that, although the term 'natural numbers' is grammatically plural, by 'natural numbers' I mean the collection (singular). To minimise confusion, while being consistent with natural language usage, I shall be explicit whether I am referring to $\mathbb{N}$ or a particular element of $\mathbb{N}$. I shall simply refer to each element of $\mathbb{N}$ as a '*natural number*'. In general, by '*numbers*', I mean *finite quantities*, rather than natural numbers that are elements of the structure $\mathbb{N}$. Generally, the numbers can be expressed by *numerals* (e.g. '1' and '2') or by *number words* (e.g. 'one' and 'two'). I shall also use the numerals or number words to denote them. If I *mention* number words instead of *using* them I put quotation marks around the expressions, and if I am *using* and *mentioning* them at the same time, I *italicise* them in addition to using quotation marks.[7] Whether $\mathbb{N}$ contains 0 or not is a disciplinary difference within mathematics/computer science, and since this distinction is less important in my research, I shall arbitrarily include 0 as a natural number (unless specified otherwise). By including 0 as a natural number, we can define 0 as an additive identity, and 1 as a multiplicative identity.

I say that a subject has the understanding *of* the natural numbers $\mathbb{N}$ just in case the subject knows that the structure of natural numbers is unique and has the properties of the standard model of arithmetic. The *standard model* is mathematically characterised as the second-order structure $(N, \mathbf{s}, \mathbf{0})$ where N is the domain containing a constant $\mathbf{0}$, and $\mathbf{s}$ is the unary successor function on N[8]. To *know* the structure of natural numbers $\mathbb{N}$ does not mean that one needs to know of *all* truths in the standard model. But if one knows the natural numbers, or if one has understood the concept[9] of

[6] I am not interested here whether each natural number is an object or they are simply positions in a structure. For the ontological debate about this, see Benacerraf, 1965, and Shapiro, 1991.

[7] I will also italicise when I am emphasising certain expressions.

[8] The distinction between first-order and second-order structures is usually important when considering the corresponding deductive system. When we are concerned about the theories of arithmetic, a second-order theory will be much more expressive than a first-order theory, as the background logic is more expressive. For that reason, the first-order theory has to include further axioms for additional symbols such as $+$ and $\times$. The second-order structure does not have $+$ or $\times$ in its language, but they can be defined in second-order logic with the unary predicate N, unary successor function $\mathbf{s}$, and a first-order constant $\mathbf{0}$.

[9] Throughout the dissertation, I will use the term 'concept' as we use the term in English. Philosophically speaking, by 'concept' I mean the intentional content the experience of a subject or group of subjects. The intentional content is understood to be intensional in the sense that it is characterised by certain properties it has rather than by the elements that fall under the concept.

the natural numbers, it would seem unusual to deny a well-known fact about the standard model.[10] For instance, if one denies an instance of mathematical induction or more generally an axiom of first-order Peano arithmetic, one does not have the knowledge of the structure of natural numbers. Let us call this the **ℕ-test**:

> **ℕ-test**: if a subject has the knowledge of ℕ, then the subject accepts the axioms of first-order PA and some of their elementary consequences.[11]

The ℕ-test has (at least) some limitations as an operational test. First, the ℕ-test informs us only when a subject does *not* have the knowledge of ℕ rather than when they *have* the knowledge of ℕ. That is, if the subject denies an axiom of first-order PA, this shows that the subject does not have the knowledge of ℕ. Another limit is that some philosophers or mathematicians might have an understanding of the structure of natural numbers but deny that the elements of ℕ are what *numbers* really are . Given these cases, it seems that what I am pursuing here is how we come to *understand* the structure of natural numbers, rather than how we come to know them. Such a distinction, although important, can be made only once one has already understood what the structure of natural numbers is. Thus, for the present purposes, I shall not make the distinction.

In the next section, I give a short review of how contemporary empirical studies explain the stages of learning about the numbers for humans. Since this chapter concerns how *human* mathematicians come to understand the structure of the natural numbers ℕ, empirical research might provide useful insights (see, e.g., Maddy, 2014). In appealing to some empirical research, I do not mean to say that philosophy of mind or even epistemology should be replaced by psychology and cognitive science. But, in so far as I am engaging with how human mathematicians come to understand ℕ, it is important to engage with what might be the precursors to understanding ℕ.

[10] Of course, not all truths or facts of the standard model are known to us. For example, what is the status of the Goldbach conjecture?

[11] The test ought to include not only the axioms of first-order PA but also some elementary consequences of them. But for succinctness, I will omit the clause concerning 'elementary consequences' when we describe the ℕ-test again.

2.1.2 From Proto-Numbers to Natural Numbers: Empirical Studies

I begin by looking at some developmental psychology to guide us through the stages of learning about the *proto-numbers*. By 'proto-numbers', I mean the finite quantities that we understand prior to our acquisition of more sophisticated mathematical notions. I shall begin by discussing some biologically fundamental abilities found in pre-linguistic human infants, and explain what kind of things are considered as proto-numbers.

The Biological Origin of the Proto-Numbers

Cognitive scientists and developmental psychologists have identified several human abilities as precursors to the acquisition of numbers. For a long time, *subitising* was considered a biological numerical ability (see, e.g., Starkey and Cooper, 1980) by which a subject instantaneously recognises the quantity (up to three or four) of objects (usually) visually presented. This ability is now considered to be a more fundamental ability of *object tracking* (see, e.g., Watson, Maylor and Bruce, 2007, and Carey, 2009). In more contemporary literature, the cognitive system that allows pre-linguistic humans to estimate exact quantities is called the *approximate number system* (see Dehaene, 2011), which can be found across different non-human animals as well.

Regardless of whether there genuinely is a cognitive system that functions over finite exact quantities or *numbers*, it is plausible that the numbers *one, two*, and *three* are common across known human cultures as evidenced by their use of number words (for, e.g., Pirahã in Amazonia – see Gordon, 2004, Pica et al., 2004). Núñez (2017) proposes that we should make a distinction between the *quantical* abilities, which are biologically evolved abilities to discriminate different quantities generally, from the *numerical* abilities, which involve discriminating particular finite exact quantities. The quantical abilities are found across different cultures and species, since it is likely that they are biologically evolved abilities. But numerical abilities are found only in industrialised cultures (in our time). For instance, the exact quantity *three* is not commonly found in the Pirahã culture (Gordon, 2004), although they still make a distinction (through their language) between the quantities *one* and *two*. Following Núñez's distinction of abilities, I shall call *one, two*, and *three* the *proto-numbers*.

In contemporary industrialised societies (by which I mean most of the world), we find exact quantities greater than three.[12] Focusing on these societies, developmental psychologists characterised the stages of learning the number concepts in terms of *knower-levels* (see Sarnecka and Lee, 2009, for more details). The knower-levels characterise different stages in which

> children learn the exact cardinal meanings of the first three or four number words in order. That is, children begin by learning the meaning of 'one' first, then 'two', then 'three', and then (for some children) 'four', at which point they make an inductive leap, and infer the meaning of the rest of the words in their counting list. (Sarnecka and Lee, 2009, p. 52)

Such proficiency can be captured by empirical experiments such as the 'Give n' tasks (Wynn 1990; 1992). In a *Give* n task, a child is asked to bring n objects. If they can successfully bring n objects, they are considered an n-knower.

For proto-numbers one, two, and three (and occasionally four), children tend to become n-knowers in gradual stages. First, they begin not knowing any finite number concepts.[13] In this case, when a child is asked to bring '*one* toy', they might bring one, two, or three toys – clearly failing to understand what is meant by the number word 'one'. But once the child learns what is meant by 'one' – that is, they can successfully bring one thing when asked – the child has become a *1-knower*.

At this stage, the child can be a 1-knower but not yet a 2-knower. In this case, they might successfully bring one object when asked for 'one', but fail to bring two objects when asked for 'two'. Up to three or four years of age, children tend to become an $(n+1)$-knower only after being an n-knower, so the knower-levels can be correlated to certain age groups.[14] In developmental psychology, the children who understand the word 'ten' correctly, for example, are called *subset-knowers*[15] – they can correctly bring some finite number of objects (e.g. ten), although they do not know the number words beyond the word for that number . It is believed that human beings are eventually able

[12] By 'industrialised' here, I mean the mode of subsistence for a human society that obtains just in case its social structure integrates industrial processes of production. This contrasts with agricultural or horticultural societies, whose mode of subsistence involves only agricultural or horticultural production. Since the empirical evidence I consider focuses on contemporary societies, I shall restrict the discussion to the contemporary ones.

[13] By 'knowing' here, I simply refer to whether they are an n-knower or not.

[14] At what age a child becomes an n-knower may vary depending on the child's linguistic background. See Sarnecka, Kamenskaya, et al. (2007)

[15] Note that this notion of 'subset' is not the same as the mathematical notion of subset.

to understand the notion of any arbitrarily large number n in so far as there is a corresponding symbolic/numerical or linguistic expression for it delivered by the process of counting.[16] In this they become *cardinal-principle knowers*.[17]

One might think that cardinal-principle knowers, in virtue of being able to count to arbitrarily large numbers, understand the notion of potential infinity (...) in the sequence '$1, 2, 3, \ldots$'. Hence, according to Maddy, they understand (or 'have an intuitive picture of') the $\mathbb{N}$-sequence. In fact, this is also a view among cognitive scientists such as Rips and his co-authors (2007; 2008a; 2008b; 2015). They claim that children *do* grasp the natural number concept, which is characterised by the *Dedekind-Peano Axioms*:

> What information must children include in their math schema in order to possess the concept of natural number? As we mentioned earlier, it is hard to escape the conclusion that they need to understand that there is a unique initial number (0 or 1); that each number has a unique successor; that each number (but the first) has a unique predecessor; and that nothing else (nothing other than the initial number and its successors) can be a natural number. These are the ideas that the Dedekind-Peano axioms for the natural numbers codify ... (Rips, Bloomfield, and Asmuth, 2008, p. 638)

The Dedekind-Peano Axiom system is a mathematical theory which is usually characterised as a set of second-order[18] axioms defined on the *language of second-order arithmetic* $\mathcal{L}_2 := \{\mathbf{s}, \mathbf{0}\}$.[19] But for our purposes, and for what Rips and his co-authors appear to be alluding to, the theory of second-order Peano arithmetic PA_2 is enough:

Definition 2.1 (Second-Order Peano Arithmetic (see, e.g., Button and Walsh, 2018, p. 29)**.** *Let* $\mathbf{s}$ *and* $\mathbf{0}$ *be the language of second-order arithmetic, where* $\mathbf{s}$ *stands for a unary total successor function and* $\mathbf{0}$ *is a first-order constant for zero. Then, the pair* $(\mathbf{s}, \mathbf{0})$ *satisfies the Dedekind-Peano axioms just in case all the following conditions are satisfied:*

[16] Generally, we can take 'counting' to mean the process of reciting the number words or the numeral list – i.e. 'one, two, three, ... ', or '$1, 2, 3, \ldots$'.

[17] Some have argued that to go from a subset-knower to a cardinal-principle-knower, one must acquire the *successor principle* (see, e.g., Sarnecka and Carey, 2008): given a numeral n, the 'next number' is the numeral $n + 1$.

[18] That is, a set of formulas stated in the language of second-order logic. Second-order logic extended first-order logic with quantifiers for predicates or classes. Class variables are usually denoted as capital Roman letters $X, Y, Z, \ldots$ and the class constants are denoted as $A, B, C, \ldots$.

[19] To be clear, the Dedekind-Peano axiom system can be characterised in many ways which are bi-interpretable in second-order deductive system. For more details see Reck (2003; 2008).

(P1) $\forall x, y(\mathbf{s}(x) = \mathbf{s}(y) \to x = y)$,
i.e. 'the successor function is injective';

(P2) $\forall x \; \mathbf{s}(x) \neq \mathbf{0}$,
i.e. 'zero is not a successor of a number';

(P-IND) $\forall X \; ([(\mathbf{0} \in X \wedge \forall y \; (y \in X \to \mathbf{s}(y) \in X)] \to \forall y \; (y \in X))$,
i.e. 'for any predicate X such that $\mathbf{0}$ is X, and whenever a number y is X, its successor is X, then every number is X'.

In this definition, '$\mathbf{0}$' stands for the initial element zero, and '$\mathbf{s}$' stands for an injective function that takes any number $\mathbf{n}$ and maps it to its successor '$\mathbf{s}(\mathbf{n})$'.[20] And the induction axiom (*P-IND*) states very roughly that if X is a predicate that is closed under the successor $\mathbf{s}$ then every number is X. This mathematical definition characterises a unique abstract structure often known as the *standard model* of arithmetic, denoted by $\mathbb{N} := (N, \mathbf{s}, \mathbf{0})$. That is, for any structure $\mathcal{M} = (M, \mathbf{s}, \mathbf{0})$ satisfying the second-order axioms of PA_2, there is an isomorphism between $\mathcal{M}$ and $\mathbb{N}$.[21] Henceforth, I shall use the bold-face font (e.g.) $\mathbf{s}$ for the symbols that are internal to the logical language, and a plain (non-bold) font (e.g.) s for the meta-theoretic successor function.[22]

There are a few other ways to characterise the standard model $\mathbb{N}$. Informally, we often characterise the domain N of the standard model as '$\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots\}$' and call them the 'counting numbers' or the 'natural numbers'. In some disciplines, the counting numbers and the natural numbers refer to different collections. For instance, in mathematics, the series of the latter starts from '0', while the series of the former starts from '1'. In this dissertation, I shall not make the distinction and simply refer to the *natural number(s)* as the elements of the sequence which is informally expressible as '$0, 1, 2, 3, \ldots$' or '$1, 2, 3, \ldots$' (depending on the choice of initial element) and formally equivalent to the standard model.[23]

If Rips and his co-authors are correct, and one accepts Maddy's explanation, then what the cardinal-principle knowers mean by '$1, 2, 3, \ldots$' – i.e. their concept of number sequence – should

[20]When I am *mentioning* the symbols such as '$\mathbf{0}$', and '$\mathbf{n}$', I shall use single quotation marks; no quotation marks will be used when *using* the terms.

[21]This is known as *Dedekind's Categoricity Theorem*: for any two structures satisfying the axioms of second-order Peano arithmetic, there is an isomorphism between them.

[22]This is necessary only for distinguishing the meta-theoretic arithmetical notions from the notions defined within an arithmetical theory. For that reason, when I am considering the common-folk natural language understanding of certain numerical properties, I do not invoke§ the distinction.

[23]In set theory, we might define the domain N of the standard model $\mathbb{N}$ as ω in which each *number* is defined as a particular set (e.g. $0 := \emptyset$), and the function s is defined taking a set x to a set $x \cup \{x\}$, i.e. $s : x \mapsto x \cup \{x\}$.

be mathematically modelled as the sequence of natural numbers. On their accounts, the cardinal-principle knowers consider *adding one* or *counting* as the *only* ways to access more numbers.[24] This is precisely what I deny. The successor operation (i.e. adding one or counting) is not the only way we access numbers for ordinary common-folk – as I shall argue in the next section.

For the sake of rigour, we shall use the formal definition of mathematical induction as defined in first-order arithmetic. *First-order Peano Arithmetic* (PA_1) is a theory in the language of first-order arithmetic $\mathcal{L}_1 := \{\mathbf{s}, +, \times, <, \mathbf{0}, \mathbf{1}\}$ such that the unary successor function $\mathbf{s}$, the binary functions addition $+$ and multiplication $\times$, and the binary relation $<$ are defined recursively with the additive identity $\mathbf{0}$ and the multiplicative identity $\mathbf{1}$. Then, any first-order formula $\varphi(x)$ defined in the language of first-order arithmetic $\mathcal{L}_1$ is called an *arithmetical formula.* In addition to the recursive definitions of the language, PA_1 contains the following axiom schema:

Definition 2.2 (Mathematical Induction)**.** *For any arithmetical formula* $\varphi(x)$*,*

$$\varphi(\mathbf{0}) \land \forall x \ (\varphi(x) \to \varphi(\mathbf{s}(x))) \to \forall x \ \varphi(x). \qquad \text{(PA-Ind}(\varphi))$$

The definition states that whatever condition is expressed by the arithmetical formula φ, if it holds for $\mathbf{0}$ and whenever it holds for an arbitrary number, it also holds for the next number, then φ holds for all numbers. This formal definition shows that induction cannot be applied to any first-order formula but only the arithmetical ones. But if we are able to extend the language $\mathcal{L}_1$ with other predicates $\mathbf{P}$ then whether induction should also apply to the formulas in the language $\mathcal{L}_1 \cup \{\mathbf{P}\}$ becomes important.

Over a series of papers, Núñez and Relaford-Doyle (2017; 2018; 2021) have argued that (at least some) university educated adults do not have an understanding of the notion of natural numbers. That is, despite being *cardinal-principle knowers*, their number concept does not coincide with the natural number concept as Rips and co-authors, and Maddy claim.

[24]In general, this seems to be a common problematic assumption among those who rely either on purely neuropsychological accounts (e.g. Dehaene, 2001; Gallistel and Gelman, 2000) – which explain our acquisition of numbers purely from biological evidence – or cognitive accounts (e.g. Burge, 2007, 2010; Giaquinto, 2007) – which explain acquisition by looking at developmental psychology and identify what is logically or conceptually necessary. They both assume that common-folk will ultimately get to the sophisticated mathematical concept. See Marshall (2018) for a detailed argument against such accounts.

In the next section I offer a phenomenological descriptive analysis to show that in our ordinary experience, counting is not the *only* experience by which we access the unending sequence of numbers. Thus, what is meant by the sequence of numerals '$1, 2, 3, \ldots$' cannot be the ℕ-sequence for the common-folk.

2.2 From '1, 2, 3,...' to the Arithmetical and Non-Arithmetical Numbers

In this section, I shall argue that what is meant by the sequence of numerals '$1, 2, 3, \ldots$' is not the ℕ-sequence for the common-folk. The '$\ldots$' in the expression suggests that we can access more numbers beyond 1, 2, and 3. In our ordinary linguistic experience, we can access numbers for the unending sequence expressed by '$1, 2, 3, \ldots$' by using the features of our natural language that are not essentially arithmetical/mathematical. (This despite of the fact that cardinal-principle knowers will also characterise the unending sequence of numbers as '$1, 2, 3, \ldots$'.) Thus, I argue that being a cardinal-principle knower is not enough to know or understand the structure of natural numbers ℕ.

2.2.1 What does '$1, 2, 3, \ldots$' Mean? The Common-Folk Notion of '$1, 2, 3, \ldots$'

In this subsection, I shall give a phenomenological analysis of what is meant by '$1, 2, 3, \ldots$' based on the way people talk about numbers in our natural language. Before doing so, there are two assumptions I must make explicit here. My first assumption (and here I am similar to other authors on this topic) is that '$1, 2, 3, \ldots$' is about the unending sequence of numbers (whatever one might mean by numbers). This condition is also assumed by Maddy, given her claim that '$1, 2, 3, \ldots$' gives an intuitive picture of the natural numbers. It is plausible that this assumption might turn out to be empirically false. Perhaps common-folk do not consider '$1, 2, 3, \ldots$' to be expressing the sequence of numbers, but numbers and also something else! However, such a problem is beyond the scope of this philosophical investigation, so I do not discuss this further here.

My second assumption is that common-folk have the understanding that the numerals '1', '2', and '3' denote exact quantities. Although it can be debated whether their being *n-knowers* really entails that the subjects *know* what the number n is, I shall not discuss this further in this dissertation. Since, ultimately, I am interested in the possibility of sophisticated mathematical knowledge, in particular, knowledge of $\mathbb{N}$, what is important in my dissertation is that there is a distinction between the mathematically sophisticated notion of numbers (i.e. the structure of natural numbers) and the common-folk notion of numbers.

Let us begin by stating a few obvious things. '$1, 2, 3, \ldots$' is a sequence of numerals, where each numeral '1', '2', '3' denotes an exact quantity – 1, 2, and 3. Then the '$\ldots$' expresses the notion of potential infinity, as Maddy claims. Below, I will present four possible ways of understanding '$\ldots$' from a subject's standpoint. First, note that, while '$\ldots$' is understood to describe the potential infinity of the number sequence, it seems that, in order for the subject to comprehend that, they should have committed at least a few acts of accessing the numbers which occur in the '$\ldots$'. The four forms such acts can take are: (1) *counting*, (2) *numeralising*, (3) *predicating*, and (4) *naming*. To explain what these acts are from a subject's standpoint, I refer to an arbitrary subject as '*I*' in italics. The four acts are described below as though *I* is carrying them out.

Counting simply refers to the act of 'adding one' to an existing number. In logic and mathematics, we can characterise this by the successor operation $\mathbf{s}$. So for any number n, the successor $\mathbf{s}(n)$ is also a number. Repeating this process of 'adding one', *I* can access infinitely many numbers appearing in the number sequence. This, as I show shortly, is the only *arithmetical act* by which *I* can understand the infinitude of the number sequence. The other acts I will term *non-arithmetical acts*, since they cannot be understood by the language of arithmetic in logic, and are, rather, features of ordinary language.

Numeralising refers to the act of generating numerals, and understanding them to refer to numbers. For instance, in our number sequence, '$1, 2, 3, \ldots$', *I* am given the numerals '1', '2', and '3'. From these numerals, *I* can make other numerals, (e.g.) '123', which stands for the number 123. Once *I* have other numerals, '4', '5', '6', '7', '8', '9', and '0', *I* can generate all kinds of numerals by putting them next to one another. Thus, *numeralising* is a distinct act from *counting* that allows a subject to access infinitely many numbers, and is based on the features of the numerical symbols *I* use to express numbers.

Predicating refers to the act of describing a finite number by using the predicate 'the number of'. For example, *I* can refer to a particular finite number as 'the number of trees on campus', despite not knowing exactly what that number might be. Perhaps *I* know that there are more than 3, so whatever the number might be, it must be occurring in '...'. Since *I* can refer to any large number, such as, the number of stars in the sky, by *predicating*, this act can be used to understand the potential infinity of the number sequence, depending on the concepts *I* can apply the predicate 'the number of'. Thus, this act is non-arithmetical since it relies on the subject's knowledge of concepts.

Naming is yet another non-arithmetical act, distinct from *numeralising* or *predicating*. In *naming*, *I* assign a name to an arbitrary number. *I* start from naming 3 as 'three', 4 as 'four', 5 as 'five', etc. But at some point, *I* start to give new names for every 10 numbers, 'twenty', 'thirty', 'forty', etc., then *I* name 'hundred', 'thousand', 'million', 'billion', etc.[25] This process of number-naming has no termination. *I* can (and do) decide to name a number 'gazillion'. Although *I* do not know exactly what number 'gazillion' refers to, *I* believe that by *naming* a number, this number must occur in the number sequence.

These four acts together allow a subject to access the infinity of the number sequence. It is not simply by *counting*, but in other acts that are 'non-arithmetical'. So why does Maddy think '$1, 2, 3, \ldots$' means the $\mathbb{N}$-sequence? That is, why does Maddy think that being a cardinal-principle knower means to be an $\mathbb{N}$-sequence knower? My best understanding is that Maddy is focusing only on the arithmetical act(s) on the numbers. For Maddy, what is meant by '...' is that the rest of the numbers can be generated by arithmetical act(s) and by arithmetical act(s) alone. On this view, the sequence of numerals express the sequence of natural numbers.

If, however, we are really interested in how *real humans* come to understand $\mathbb{N}$, then we should also consider the different ways that '...' can be understood by common-folk rather than simply assuming that '...' might be adequate for understanding $\mathbb{N}$. If my interpretation of a common-folk

[25]It used to be the case that, in UK English, 'one billion' referred to 10^{12}, while in US English, it refers to 10^9. The American usage became increasingly common in the UK, and is now the official usage in the UK parliament, and this shows that even these known number words do not necessarily refer to one particular number.

understanding of the number sequence is accurate, we ought to be able to empirical verify this. Although I am not familiar with any existing empirical studies focusing on these understandings, I argue (in section 2.3) that Relaford-Doyle and Núñez's findings are consistent with the separation between arithmetical and non-arithmetical acts. But before presenting the empirical findings, I offer a conceptual demarcation between the numbers generated by arithmetical act(s) and the numbers generated by non-arithmetical acts.

In the next subsection, I offer a conceptual distinction between the numbers generated by arithmetical acts and those generated by non-arithmetical acts.

2.2.2 Arithmetical and Non-Arithmetical Numbers

The conceptual boundary between arithmetical and non-arithmetical numbers I offer here is motivated by the distinction between 'feasibility' and 'infeasibility' from finitist mathematics. I give a quick overview of what 'feasible numbers' are. In his paper 'Strict Finitism, Feasibility, and the Sorites', Dean (2018) characterises 'feasible numbers' as found in Yessenin-Volpin (1961; 1970). He begins by describing an 'infeasible' number, as one that is 'not possible to count' (Yessenin-Volpin, 1970, p. 5). Although the infeasible numbers seem to be greater than all feasible numbers, infeasible numbers are still 'finite' numbers. For an example of an infeasible number, consider the number 'one million'. Although in principle we could begin from 0 or 1 and count up to $1,000,000$, there are bound to be some errors and physical difficulties in counting that many things.[26] In that sense, a million is an 'infeasible number'. Unlike the feasible/infeasible numbers distinction, the arithmetical/non-arithmetical number distinction is not primarily concerned with the notion of finiteness. The conceptual boundary I am drawing here is based on *how* we access these numbers. The arithmetical numbers are accessible simply by counting alone. But non-arithmetical numbers can be accessed only by non-arithmetical acts, which could be linguistic acts.

Despite some conceptual differences between Dean's feasible/infeasible number distinction and my arithmetical/non-arithmetical number distinction, I present a possible (but rather incomplete)

[26] If a million does not seem to be infeasible, consider the number 'googolplex', which can be characterised numerically as 10^{googol}, and a googol is expressed as 10^{100}.

definition of the arithmetical/non-arithmetical number distinction based on Dean's characterisation (2018, p. 12).

Definition 2.3 (Arithmetical/Non-Arithmetical Numbers)**.** *The following conditions give a meta-theoretic description of 'arithmetical/non-arithmetical numbers':*

- *(F1)* $\mathbf{0}$ *is an arithmetical number;*
- *(F2) for any* $\mathbf{x}$*, if* $\mathbf{x}$ *is an arithmetical number then* $\mathbf{s}(\mathbf{x})$ *(or the 'next number') is an arithmetical number;*
- *(F3) there is a non-arithmetical number* $\mathbf{i_F}$*.*

The first two conditions defining the arithmetical numbers resemble some (first-order variations) of the conditions of definition 2.1. Note that exactly what the non-arithmetical number $\mathbf{i_F}$ is will vary depending on what context we are in.[27] But by using this mathematical characterisation, we can avoid specifying a context.

Instead of defining the demarcation meta-theoretically, if we wish to define it in the language of arithmetic, we could extend our arithmetical theory with a predicate F. For example, we could have *Robinson's arithmetic* (Q), which does not include mathematical induction,[28] and in which associativity and commutativity of $+$ and $\times$ are not provable. We can then extend Q to the theory PA^- which includes associativity and commutativity of $+$ and $\times$, but in which the induction schema is not an axiom schema.[29] Naturally, both Q and PA^- can be extended with the predicate F to distinguish between arithmetical and non-arithmetical numbers.

However, I show here that we can also consistently accept PA_1 (including induction) and the meta-theoretic distinction between arithmetical/non-arithmetical numbers. This is to show that the failure of mathematical induction is not precise enough to show that the common-folk do not understand $\mathbb{N}$, but the matter is mathematically and philosophically more complex than it seems.

[27]Since feasible/infeasible numbers can be distinguished by considering the context in which certain numbers cannot be counted, non-arithmetical numbers could also depend on the context. See the discussion on Wang's Paradox (Dummett, 1975, p. 303) for the contextualist interpretation of the feasible/infeasible number distinction. In my work, I hope to characterise the common-folk account of numbers independently of contexts, yet let it be applicable in any context.

[28]Robinson's Arithmetic is a first-order theory that does not include the first-order induction schema but has addition and multiplication defined recursively.

[29]If we extend either Q or PA^- with the induction schema, then we simply obtain the theory PA_1.

The condition (F2) of the statement of the arithmetical/non-arithmetical number distinction states that if **n** is an arithmetical number, its successor is also an arithmetical number. So, we must have **1**, **2**, and **3** are all arithmetical numbers. So any number obtained by counting from 1, 2, and 3 are arithmetical numbers, as expected.

The definition does not tell us how a non-arithmetical number is defined, but only that there is at least one. Accepting the three conditions, (F1), (F2), and (F3), tells us that the sequence of numerals '$1, 2, 3, \ldots$' expresses the sequence of arithmetical and non-arithmetical numbers, where the former can be accessed by arithmetical acts (e.g. counting) alone, but the latter are accessible only by non-arithmetical acts. For example, the number expressed by 'gazillion' is not something I can access by counting or adding other numbers, but this number could still appear in the sequence $1, 2, 3, \ldots$. So we might call the value of gazillion a non-arithmetical number.

One upshot of using the meta-theoretic distinction is that we can offer a consistent mathematical model capturing it. Namely, by using *non-standard models of* PA_1. Non-standard models of PA_1 are models of PA_1 that are not isomorphic to the standard model. These models contain all the natural numbers, but in addition to the naturals, they also have *non-standard numbers*. For any non-standard number $m \in M$, where M is the domain of a non-standard model $\mathcal{M}$, $\mathcal{M} \models n < m$ where $n \in \mathbb{N}$. But since $\mathcal{M}$ cannot express whether a number is standard or not, it does not 'know' that n is a standard number. We can apply this meta-theoretic distinction between standard and non-standard numbers to the arithmetical and non-arithmetical numbers, and provide a consistent model for the common-folk understanding of numbers.[30]

Now, let us turn to some empirical evidence that supports the claim that the sequence '$1, 2, 3, \ldots$' contains both arithmetical and non-arithmetical numbers.

[30]Although I shall not discuss this further in this paper, this illustrates a way in which tools from mathematical logic can be used for research in number cognition. In particular, we can design empirical questions based on properties of non-standard models to see whether ordinary common-folk will consider these properties to be true of their understanding of the numbers.

2.3 '1, 2, 3, ...' does not mean the ℕ-sequence: Empirical Study

Before explaining the experiment in more detail, let me emphasise the limitations of the study. First, this is *one* study concerning cardinal-principle-knowers that suggests a surprising result. For this result to generalise to the common-folk in industrialised societies, or even just to university students, further empirical work needs to be done. Second, the study is designed to focus on a particular diagram. If further empirical studies on number concepts are conducted that are independent of using sophisticated diagrams, the evidence from these will strengthen support for the claim.

Despite the limitations of this study – and the empirical evidence – there are, to my knowledge, no other similar empirical studies to date. The conclusion of the experiment is interesting since it suggests that cardinal-principle-knowers do not have the concept of natural numbers, challenging the typical philosophical view that they are ℕ-knowers (e.g. Maddy, 2014; 2018; Shapiro, 1991; Burge, 2010), and it supports my view that their understanding of '$1, 2, 3, \ldots$' treats non-arithmetical acts of generating numbers as equally legitimate as arithmetical acts.

2.3.1 The Experiment

There were a total of 49 participants (university students) from two distinct groups. One group ($n = 24$) consisted of students who had never taken a course in mathematical proofs.[31] The other ($n = 25$) were recruited from the students who received at least a B- in the 'Mathematical Reasoning' course that included mathematical induction as a method of proof. Let us call the first group '*the Educated group*' and the second '*the Proof-Trained group*'.[32]

Each participant was presented with a diagram (figure 2.2) that depicts a 'visual proof'[33] of the mathematical claim, 'the sum of the first n odd numbers is equal to n^2.' (Brown 1997) Henceforth, I

[31] Although none of the educated students had taken the 'Mathematical Reasoning' course, four of these students claimed to be familiar with mathematical induction.

[32] Some might be worried that achieving a B- does not indicate that the students have successfully understood the course content. However, as we shall see, the Proof-Trained group has a different understanding of numbers from the Educated group.

[33] Whether such a diagram is a 'proof' is controversial. See Brown, 2010, and Giardino, 2010, for more on this. We can consider it at least as a 'generic proof by figurate number' per Kempen and Biehler (2019).

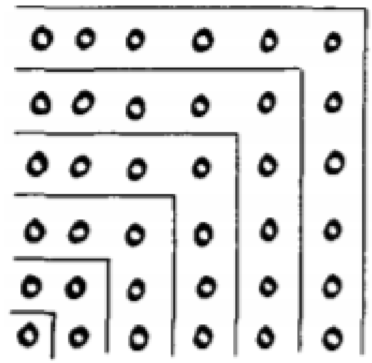


Figure 2.2: Picture Proof from Brown 1997

refer to figure 2.2 as 'the diagram' and the claim 'the sum of the first n odd numbers is equal to n^2' as 'the target claim'.

To begin, the participants were asked to explain how the diagram shows the target claim to be 'true'.[34] After the participants successfully completed their task, they were asked the following two questions by the researchers:

(Q1) Do you think the statement ['the sum of the first n odd numbers is n^2'] is true in all cases?;

(Q2) What would be the sum of the first 8 odd numbers?

If they answered 'yes' and '64', then they were asked an additional question:

(Q3) could there be a large number where the statement fails?

Note that a positive answer to (Q1) would logically imply that the answer to (Q3) is negative. That is, if the target statement is true in *all* cases, there cannot be a large number that makes the statement false. However, the researchers allow for a consistent interpretation here. The participants'

[34] More precisely, the participants were presented with only one of the following three conditions: (1) they were presented with the target claim and the diagram, and 'asked to explain how the picture shows that the statement is true'; or (2) they were presented with the diagram and the incomplete target claim ('The sum of the first n odd numbers is equal to _ ') and were to complete the target claim and 'to explain how the picture shows that the statement is true'; or (3) presented with the diagram and were 'told that a mathematician drew the picture while trying to prove a statement about the sum of odd numbers', and they were asked 'to guess what the mathematician was trying to prove and explain their answer' (Relaford-Doyle and Núñez, 2017, p. 1006).

use of ‘all’ is to be understood as a very general use, similar to ‘most’ or ‘many’ according to Relaford-Doyle and Núñez (2017; 2018; 2021). Later, in section 2.2.2, I shall give an alternative consistent interpretation that focuses on the properties on the numbers rather than the quantifier ‘all’.

Let us now turn to the results. If the participant had answered (Q1) and (Q2) with ‘yes’ and ‘64’ respectively, the researchers considered them to generalise the statement to ‘nearby cases’. Since the diagram given to them is of a 7 by 7 square, being able to apply the diagram to an 8 by 8 square would entail such answers to (Q1) and (Q2). From the experiment, almost all participants answered (Q1) and (Q2) as expected.[35]

In the interviews conducted, the researchers observed that some counterexample-resisting participants explicitly talked about mathematical induction in their reasoning (2018, p. 244), while others claimed that *counting* should prohibit a counterexample (2018, p. 245). This suggests that any cardinal-principle-knowers, who consider the numbers as the *counting* numbers, ought to have been able to make the same claims.

Furthermore, none of the Educated participants mentioned ‘counting’ in their explanation of their (Q3)-answers (2018, p. 245). Instead, they explain that there could be a counterexample because ‘in extremes things tend to not work as they do normally’ (2018, p. 247) or that it would fail for large numbers because ‘it’s too hard to draw a million dots’ (2018, p. 248). Thus, we can clearly see from this study that the Educated group makes a distinction between the ‘large’ counterexamples that are beyond their ability to count. So these ‘large’ numbers must be accessed (if they can be obtained or accessed) by acts that are beyond counting.

[35]Additionally, the participants’ answers to (Q3) were coded on a scale of 0 to 5 based on their ‘resistance’. The scores 0 to 3 meant ‘low resistance’ while 4 or 5 meant ‘high resistance’. The researchers compared the resistance scores between the two groups. Interestingly, the Educated group showed less resistance to the possibility of a counterexample than the Proof-Trained group. That is, the Educated group was more likely to think that there is a large number n such that the sum of the first n odd numbers is *not* n^2.

2.3.2 The Educated Group Distinguishes Arithmetical and Non-Arithmetical Numbers

Now I shall show that the Educated participants are making a distinction between arithmetical and non-arithmetical numbers in their answer to (Q3). I shall show that they do not make the distinction in their answers to (Q1) and (Q2), since those answers satisfy (F1) and (F2). But in their answer to (Q3), they bring some properties of the 'large' numbers which could be interpreted as 'non-arithmetical'.

Let us first consider how the conditions (F1) and (F2) should be interpreted with respect to the experiment. (F1) states that 'zero' or some other number serves as an initial case, and (F2) claims that for any number, if it is an arithmetical number then its successor is also an arithmetical number. Since the experiment concerned the statement 'the sum of the first n odd numbers is n^2,' we can consider this as a meta-theoretically defined property 'the sum of the first $\mathbf{n}$ odd numbers is $\mathbf{n^2}$' in place of 'arithmeticality'. Then we can re-state the conditions for the arithmetical numbers as (F1') and (F2') as follows:

(F1') the sum of the first $\mathbf{0}$ odd numbers is $\mathbf{0^2}$

(F2') if the sum of the first $\mathbf{n}$ odd numbers is $\mathbf{n^2}$ then the sum of the first $(\mathbf{n+1})$ odd numbers is $(\mathbf{n+1})^\mathbf{2}$.[36]

Since all the participants had to explain how the diagram shows the target claim to be true, we can safely accept that all the participants agree with (F1'). To see whether the participants accept (F2'), we must do some further interpretation.

Recall the questions (Q1) and (Q2): (Q1) *Do you think the statement is true in all cases?*; and (Q2) *What would be the sum of the first* 8 *odd numbers?*. Fifteen out of sixteen participants of the Educated group answered 'yes' and '64', respectively. Their answer to (Q1) can be interpreted as (A1) 'for any *arithmetical* number $\mathbf{n}$, the sum of $\mathbf{n}$ odd *arithmetical* numbers is $\mathbf{n^2}$.' Here I interpret the use of 'any' not as a generic generalisation (contrary to Relaford-Doyle and Núñez 2017; 2018; 2021) but rather a universal quantification over the 'arithmetical numbers'. Thus, we have,

[36]Expanding the brackets, we have the sum of the first $(\mathbf{n+1})$ odd numbers is $\mathbf{n^2+2n+1}$.

(A1) for any *arithmetical* number $\mathbf{n}$, the sum of $\mathbf{n}$ odd numbers is $\mathbf{n^2}$.

Now in first-order logic, we can show from that every $\mathbf{x}$ is arithmetical, it follows that for any arithmetical number $\mathbf{x}$, its successor is arithmetical. Thus (F2') follows from (A1).

In order to show that the Educated group distinguishes between arithmetical and non-arithmetical numbers, I must show the third condition (F3') is satisfied. (F3') can be stated as follows:

(F3') there is a number $\mathbf{n_F}$ such that the sum of the first $\mathbf{n_F}$ odd numbers is *not* $\mathbf{n_F^2}$.

To do so, I shall now look at the answers of three Educated-group participants, whose initials are DA, KL, and AT, to (Q3). The scripts below are taken directly from Relaford-Doyle and Núñez (2018).[37] Recall the question (Q3): *could there be a large number where the statement fails?* When this question was asked, DA answered as follows.

> DA: I guess that makes sense. Like the larger numbers could be, like, outliers, or something like that. (Relaford-Doyle and Núñez, 2018, p. 246)

DA's answer not only suggests that there is a 'large number' that fails the condition, but compares it to a 'statistical outlier'. The others provide more information about their answers to (Q3).

> KL: Based on my impression, just based on this observation, I think it would work, but when it gets to really high numbers, um, it's possible that, like (pauses). I can see maybe it gets kind of fuzzy. Because at extremes, things tend to not work as they do normally. (Relaford-Doyle and Núñez, 2018, p. 247)

Although KL shows more resistance than DA, they indicate that for 'really high numbers' the equation could fail to hold. They talk about how the 'really high numbers' can get 'fuzzy' and that 'at extremes things tend to not work as they do normally'.

[37] As mentioned earlier, the Educated group showed less resistance to a 'large number' counterexample than the Proof-Trained group. In the scripts below, I interpret their lack of resistance more strongly. For instances, 'I guess that makes sense' and 'I think it would work' are interpreted as positive answers to the question (Q3).

Although DA's and KL's answers show that ($F3'$) is satisfied, they do not explicitly refer to an example for this 'large number'. But AT's interview specifies that a 'large number' counterexample could be after '99'.

> AT: I guess this model proves to be true for, until, maybe like 99. I know it would be true. I don't know, I consider 99 a big number. I don't know how big this person . . . like maybe the model deconstructs at a thousand or a million, I don't know, but it's too hard to draw a million dots. (Relaford-Doyle and Núñez, 2018, p. 248)

Unlike DA, who refers to 'outliers', and KL, who adverts to 'fuzziness', in describing the possible 'large number', AT explains why there could be a counterexample. AT appeals to certain physical limitations of the 'model' (i.e. the diagram) and of a human-limitation of drawing the diagram for large numbers. At the same time, AT refers to 99 as a 'big number' which could be the 'non-arithmetical number' in (F3'). Hence, AT considers 'big numbers' to be restricted by human and physical limitations.

Thus far, I have shown that the Educated concept is of the arithmetical/non-arithmetical numbers. It might be argued that the participants did not understand the question, as they seemed to be applying the question to the diagram or to a statistical situation. This, however, is precisely the distinction between arithmetical and non-arithmetical numbers. The arithmetical numbers are those that can be accessed by the successor operation (or by arithmetical acts) while the non-arithmetical ones are accessed by non-arithmetical acts. So, thought about, arithmetical numbers are not troubled by concerns over their applicability or their instantiations in physical world. Human and physical limitations are only a consideration in relation to the non-arithmetical numbers.

The responses show that even when considering a mathematical statement, the Educated group considers it in relation to some physical or statistical context, unlike the Proof-Trained group. Given that the question (Q3) was not about the diagram but simply about the mathematical equation and the possibility of a 'large number', the Educated group's interpretation of the question shows an important difference in their understanding of the numbers from the Proof-Trained group. The Proof-Trained group treats the property 'large' not to really mean an arithmetical property for the numbers. So, it is irrelevant whether a number is large or small. But the Educated participants interpret ‘large’ to bring in the non-arithmetical properties in understanding the mathematical equation. In this sense, the Proof-Trained group shows strong resistance to a ‘large number’ counterexample, as their understanding of numbers is abstract and not necessarily connected to their applicability.

So what does this tell us? Let us suppose that the Educated participants are all cardinal-principle-knowers. If we can generalise from the empirical evidence to all cardinal-principle-knowers, it suggests that being a cardinal-knower – despite having an understanding of the recursivity (in the Maddy sense) of the ‘...’ – does not mean that they understand $\mathbb{N}$. The ‘...’ for the Educated group includes different non-arithmetical acts or methods of generating numbers which I have characterised as ‘non-arithmetical’. For those who are Proof-Trained, the ‘...’ simply refers to what can be generated by counting only.

So *why* are the number concepts between ordinary and mathematical practice different? One way to answer this question is to consider what the aims of using the concept might be. This can be understood as the goals of the practice, which ought to be clarified, as suggested by Husserl’s method of *Besinnung*. I shall analyse *Besinnung* in more detail in Chapter 3, and apply it to HoTT in Chapter 4, but, for now, here is a very simplified version of it applied to our conceptions of numbers.

One goal of ordinary practice of numbers is to measure and compare different *exact* quantities. A trivial example might simply be in comparing how much food you have. If I have ten apples, and you have nine apples, I have more than you. It doesn’t tell us who has bigger apples, but it tells us that I must have more apples than you! If I wanted to feed my family of ten, I better have ten apples rather than nine! But arithmetic or number theory for mathematical practice has fundamentally different goals. It aims to provide general properties of the finite numbers, which could be used as in the ordinary practices, but it is less about particular situations and more about generally applicable notions. Another goal is that arithmetic serves as a foundation to the rest of mathematics. Prior to learning any particular mathematical theory, arithmetic and geometry are taught. The history of the two disciplines is immense, and entangled with the ordinary practice, but they are considered to be *pure* sciences. These goals of mathematical practice are not goals of ordinary practice. Ordinary practice with numbers is not concerned about the general properties of numbers.

Adopting a Husserlian approach, we can not only answer the question of *why* there are two distinct number concepts (of ordinary or mathematical practices, respectively): Husserlian method also promises to answer *how* we can come to know the *mathematical* concept from the *ordinary* concept of numbers, by providing a clearer demarcation between the two.

2.4 From Mathematical Enculturation Towards *Besinnung*

Before diving into giving an account of the nature of *Besinnung*, let us entertain an alternative answer to the question of *how* we can come to *mathematical* knowledge from a starting point in *ordinary* knowledge. In his paper 'Objectivitiy in Mathematics, Without Mathematical Objects', Markus Pantsar (2021) argues that knowledge of natural numbers is acquired via 'enculturation':

> It is indeed the evolution on the *cultural* level that we need to include in order to explain how the proto-arithmetical abilities can develop into mathematical knowledge and skills. (Pantsar, 2021, p. 338; emphasis in the original)

> The enculturation framework can then provide the link between the proto-arithmetical abilities, which are the product of *biological evolution*, and the arithmetical ability that is the product of cumulative *cultural* evolution. (Pantsar, 2021, p. 341)

For Pantsar, *enculturation* refers to the 'transformative processes in which interactions with the surrounding culture determine the way cognitive practices are acquired and developed' (2021, p. 340).[38] Thus, if $\mathbb{N}$ is learnt by enculturation, our understanding of $\mathbb{N}$ is attributable to the process of interacting with the surrounding 'mathematical culture', which establishes that $\mathbb{N}$ is the correct notion of the set of finite numbers. This 'mathematical culture' goes beyond general university education, as suggested by the empirical study, but perhaps includes learning to do mathematical proofs.

[38]Pantsar (2021, p. 340) generally follows Richard Menary's account of enculturation:

> Enculturation rests in the acquisition of cultural practices that are cognitive in nature. The practices transform our existing biological capacities, allowing us to complete cognitive tasks, in ways that our un-enculturated brains and bodies will not allow. (Menary, 2015, p. 4)

The participants in the experiment discussed above had sufficient educational background to be cardinal-principle-knowers. Yet, they failed to understand the $\mathbb{N}$-sequence from the arithmetical/non-arithmetical sequence '$1, 2, 3, \ldots$'. This suggests that doing mathematical proofs is a kind of enculturation needed for the knowledge of $\mathbb{N}$. In that sense, mathematical knowledge is grounded in cultural norms and practices. However, this is perhaps not an interesting conclusion: of course, in order to learn A, where A is found in a specific community, one must learn the culture in which A is found![39]

In fact, the importance of enculturation to the acquisition of mathematical knowledge has already been highlighted by other empirical evidence concerning cultures outside industrialised societies. The proto-number concepts that we find in these cultures (e.g. Pirahã or Munduruku – see Gordon, 2004; Pica et al., 2004) are certainly different from the ones found in industrialised cultures to the point that Núñez argues that we should further make a distinction from what is common across cultures and call those notions *quantical* rather than *numerical* (Núñez, 2017). In some cultures, although they have the words referring to the proto-numbers, they do not commonly use words that pick out the exact quantity 5. For any quantity larger than 4, they will often use the word for 'many' instead of referring to a particular exact quantity, despite that the language acquisition device should allow them to express a term for four by (e.g.) concatenating 'one' and 'three'.[40]

Despite the (perhaps unsurprising) fact that the knowledge of $\mathbb{N}$ seems to be a result of enculturation, Pantsar claims that there is something 'objective' about mathematics and mathematical knowledge. In particular, why is mathematics so applicable to explaining natural world phenomena? Pantsar's own answers refer to Maddy (2014) and the way the world is structured, as well as our evolved cognitive ability of the language-learning device:

> Much as our primitive cognitive architecture, designed to detect [the logical structure of the world], produces our firm conviction in simple cases of rudimentary logic, our human language-learning device produces a comparably unwavering confidences in this potentially infinite pattern. (Maddy, 2014, p. 234; quoted from Pantsar, 2021, p. 343)

[39] By claiming the importance of enculturation, I do not mean to reduce mathematics to social constructivism (see, e.g., Ernest, 1998, for a social-constructivist account of mathematics).

[40] For example, in English, we say 'twenty-one' to refer to the exact quantity after the quantity 20, simply by concatenating the word 'twenty' and 'one'.

Based on his claim, one might misunderstand Pantsar's argument as running as follows. Our empirical world is structured in a certain way[41] from which we validate our logic. This logic of the empirical world is weaker than classical logic, but through our evolved human cognitive abilities, we idealise the logical structure of the world to the structure of classical logic where every proposition is either true or false. When it comes to the knowledge of the sequence of numbers, expressed as '$1, 2, 3, \ldots$', it is the human language-learning device (as well as other features of human cognition) that allows us to understand the infinitude of the sequence of numbers. And since this cognitive device is an evolved *human* ability, we can have objective (or intersubjective) knowledge of arithmetic, which is applicable to all exact quantities in the world with which we interact. However, if the language-learning device is what gets us to the objectivity of arithmetical knowledge, then the proto-numerical notions we find among the Pirahã or Munduruku peoples ought to be equivalent to ours in the industrialised societies. And, furthermore, we must allow for the linguistic acts such as numeralising, in so far as it allows us to generate numbers in a recursive manner – write more '0's next to the '1'. Given that this is not the case, it seems to follow that the language-learning device alone does not get us to mathematical knowledge, never mind its objectivity.

What Pantsar perhaps is claiming is that these cognitive features are necessary for having mathematical knowledge, but we also need enculturation: the non-industrialised societies also have all the cognitive features that are needed for understanding the natural numbers, and for whatever cultural reasons they have, they do not have $\mathbb{N}$. I fully agree with Pantsar that it is not simply the cognition that is enough for the human understanding of $\mathbb{N}$. And similarly for the participants of the Relaford-Doyle & Núñez experiement, they have the cognitive capacities, but they are not enculturated to the mathematical community.

Pantsar's enculturation is one answer (perhaps a sociological answer), but I think the Husserlian picture of standing in the community of empathy could provide a *philosophical* explanation. The Husserlian view also adds to the case that the mathematical community is a goal-oriented community. Then one reason why common-folk do not have the knowledge of $\mathbb{N}$ is because they do not share the goals of mathematical practice.

[41] Maddy calls the way in which the world is structured the 'KF-structure' after Kant and Frege.

2.5 Conclusion

In this chapter, I have argued that for the common folk, what is meant by '$1, 2, 3, \ldots$' is not the $\mathbb{N}$-sequence. This challenges Shapiro's and Maddy's view that once a subject understands what '$1, 2, 3, \ldots$' means, then they understand $\mathbb{N}$. Instead, I showed in section 2.2.2 that '$1, 2, 3, \ldots,$' could be modelled by the demarcation between the arithmetical and non-arithmetical numbers: the arithmetical numbers are those that we can access by counting or adding one (or by other arithmetical acts), while the non-arithmetical numbers are accessed only by non-arithmetical acts. Then I argued that arithmetical/non-arithmetical numbers were supported by the empirical evidence in section 2.3.

In section 2.4, I described *enculturation* as a sociological explanation for acquiring mathematical knowledge from ordinary knowledge, per Pantsar (2021). However, I propose to give a Husserlian account of this enculturation.

What this chapter has shown is how the Husserlian subjective analysis (or what I call an 'empathy-first approach') can be adopted in interdisciplinary research concerning mathematical cognition. In the particular analysis of '$1, 2, 3, \ldots$', the Husserlian approach aims to give a clear description of our ordinary experience of the number sequence. The empirical cognitive sciences then aim to verify the accuracy of the description. Of course, such Husserlian analysis is already being adopted in different areas of empirical work – for example, when interpreting the subjects' interviews from the Relaford-Doyle & Núñez experiment. What distinguishes my account of interdisciplinary work is the general aim of the methods. Phenomenology, or an empathy-first approach, aims to explicate our experience, while cognitive science aims to verify that explication.

As I mentioned in the introductory chapter, I believe the Husserlian view is that the communities have different aims, goals, or values (i.e. the *Zwecksinne*) that can shape the norms of their practices. Upon clarifying what these *Zwecksinne* are, we are able to explain *why* the mathematical community has opted in for the natural numbers $\mathbb{N}$ as opposed to (e.g.) some non-standard model. In order to uncover and explicate the *Zwecksinne*, we must carry out *Besinnung.* We are to stand within or enter the community of *empathy. Empathy* in phenomenology describes an experience of other conscious subjects and understanding the other's experience. Thus, the *community of empathy* can be understood as a community (formed by shared experiences) whose members understand each other's shared experiences. To stand within or enter this community means to learn to empathise with its members and/or the whole community. *Enculturation* on this Husserlian account would mean standing in or entering the community of empathy. As philosophers, we ought to enculturate ourselves with the practising community and clarify its *Zwecksinne*. Thereby, we can explain why (e.g.) the natural numbers are *given* to mathematicians by the sequence of numerals '$1, 2, 3, \ldots$'.

In the next chapter, I shall clarify the nature of *Besinnung*. This involves explaining what the phenomenological notion of empathy is, and in what sense the mathematical community is a community of empathy. Then, I shall explicate *Zwecksinn*.

Chapter 3

Husserl's Method of ***Besinnung*** for Mathematical Practice

Husserl describes his *Formal and Transcendental Logic* (1929) as his 'most mature work' (Schuhmann, 1977, pp. 484–485; translation quoted from Hartimo, 2021a). In it, he introduces and characterises his method of *Besinnung*, which he also uses in the *Cartesian Meditations* (1929/31) and the *Crisis* (1936). The method of *Besinnung* was highlighted only recently by Mirja Hartimo (e.g. in her 2018). Hartimo characterises *Besinnung* as a method that 'aims to make the goals of intentional activities explicit (Hartimo, 2021b, p. 4), and further suggests that phenomenological philosophy of mathematics in the twenty-first century 'would use radical *Besinnung* – that is, it would aim to capture the values and goals of present-day mathematicians' (Hartimo, 2021b, p. 189).

In this chapter, I clarify the nature of Husserlian method of *Besinnung*[1] – briefly introduced in Chapter 1. I argue that *Besinnung* is a method which aims to clarify the *motivational goals* of a scientific community, which Husserl calls '*Zwecksinne*' (i.e. goal-senses or final senses). In order to clarify the motivational goals of a community's practice, we first need to engage with that

[1]I prefer to use the German term '*Besinnung*' instead of choosing a particular translation into English. As I shall discuss shortly, there are several different English terms that might be used as translations, including 'reflection', 'sense-investigation', 'meditation', and 'clarification.'

community's practice, by *empathising* with the members of the community. Then we must identify those concepts, methods, goals, or questions that might need some further clarification. By clarifying these (or by determining answers the questions), we can then turn to the motivational goals that are implicit in the community's practice.

Husserl's own aim in pursuing *Besinnung* was to produce a Leibnizian *mathesis universalis* (Centrone & Da Silva, 2017, see also Hartimo, 2021b, §5.2.5): Husserl wanted to characterise a general theory of sciences, which offers a universal method for studying any science (see e.g. *FTL*, p. 16). By 'sciences [*Wissenshaften*]', Husserl would have included natural and social sciences and humanities disciplines (as is usual with the German term). Hence, Husserl appears to take the general scientific practice to apply to the community that he is interested in, instead of referring explicitly to a particular scientific community. However, I follow Husserl's own *presuppositionlessness* and do not assume that we can discover *mathesis universalis* through the method of *Besinnung*. Thus, my interpretation of *Besinnung* does not have the aim of *mathesis universalis*, but rather aims in each particular case to clarify the motivational goals of a particular scientific community. These motivational goals identify what a scientific community aims to achieve in their practice, but they also guide the community to practise their discipline in a certain way. Hence, I refer to the method as '*Husserlian Besinnung*', instead of '*Husserl's*', since it is motivated by Husserl's own work without being strictly what Husserl would have aimed for.

There are several reasons for my departing from Husserl. One is that academic disciplines today are not quite the same as in Husserl's time. Mathematics today is broadly divided into pure and applied mathematics. Even in these sub-disciplines, we find various mathematical theories (e.g. model theory) with which not all mathematicians would necessarily engage. Another reason is that, while I agree with Husserl's more historical approach to understanding each discipline, Husserl often refers back to Leibniz, Galileo, or Euclid, with whose actual work contemporary mathematicians do not engage. Instead of focusing on these particular figures, or thousands of years of history of the discipline as a whole, I believe that it is best to understand the local history of each mathematical theory. This will help us to understand the motivation behind the theory, and thus help us to clarify the motivational goals.

The goal of this chapter is to explicate the Husserlian method of *Besinnung* as a method for studying contemporary mathematical practice. In section 3.1, I give an overview of the nature of *Besinnung*, focusing on *CM*, *FTL*, and *Crisis*. Roughly, Husserl claims that, in order to carry out *Besinnung*, we must (1) 'enter the community of empathy with the scientists' and (2) clarify the 'intending sense' leading to the 'fulfilled sense' (*FTL*, p. 9). In order to understand what Husserl means by them, I explicate (in sections 3.2 and 3.3) Husserl's phrases 'community of empathy [*Einfühlungsgemeinschaft*]', and 'intending sense'/'goal-sense' ['*Zwecksinn*'], respectively. While doing so, I also compare these Husserlian accounts with contemporary notions of community or group intentions, as articulated by Ritchie (2020) and Gilbert (2009; 2013; 2020). Furthermore, I explain the community of empathy within the mathematical context in section 3.2.2. To conclude the chapter, I explain radical *Besinnung* and discuss how it can be applied to contemporary mathematical practice (before going on, in Chapter 4, to apply it to the community of homotopy type theorists). In particular, I suggest that we ought to start with particular mathematical theories and practices, and look at their histories.

My interpretation of Husserl's *community of empathy* offers an explanation for the teleological structure of science expounded in Chapter 1. To enter the community of empathy, one starts from the fundamental experience of empathy, i.e. an understanding of another's experience from that other's perspective (D.W. Smith, 1989, p. 112), and the goals of the community arise from the shared empathised experiences of the community of empathy. Naturally, then, *Besinnung* can be carried out to clarify the goals shared among the members of the community.

3.1 The Method of *Besinnung*

In this section, I describe Husserl's notion of *Besinnung* as found in *Formal and Transcendental Logic* (*FTL*, 1929, 1969), the *Cartesian Meditations* (1929, 1960), and the *Crisis* (1936, 1954, 1970). In each of these texts, different kinds of *Besinnung* are emphasised. In *FTL*, *Besinnung* aims at clarifying the values that motivate the sciences, by highlighting that there are two meanings of 'logic'. In the *Cartesian Meditations*, *Besinnung* focuses on our understanding of the *Cogito*, clarifying its structure. In the *Crisis*, Husserl focuses on the method of *historical Besinnung*, which aims to clarify the origins of the sciences. Rather than distinguishing between each of these methods, I characterise *Besinnung* as a single method that can be found in these different applications depending on the community that Husserl was engaged in.

Here, I briefly highlight some difficulties in understanding *Besinnung* that arise from various English translations of the *Crisis*, *Formal and Transcendental Logic*, and the *Cartesian Meditations.* The term '*Besinnung*' is translated variously into English as 'reflection' (*Crisis*, 1936, 1970), or 'sense-investigation' (*FTL*, 1929, 1969). Throughout the *Crisis*, Husserl explains and carries out *Besinnung* on the European community and their view on science, and it is easy here to mistake *Besinnung* as a naïve kind of reflection, without any philosophical or methodological significance. But, in the introduction of *FTL*, Husserl emphasises that *Besinnung* is a method with a particular aim and a method that can be carried out in a certain way.[2]

In fact, *FTL* was published in German in 1929, a few years before the partial publication of *Crisis* in 1936.[3] In this sense, *FTL* is an earlier work in which Husserl introduces the method of*Besinnung*, a method he goes on to develop more thoroughly and in more detail in his later work, *Crisis* With regard to the English translations, while the translation of *FTL* (by Dorian Cairns) was published in 1969, a year earlier than that of *Crisis*, the translator of *Crisis*, David Carr, did not use Cairns' carefully chosen phrase 'sense-investigation'. In fact, this might have been intentional on Carr's part, as he writes in his translator's introduction:[4]

> [For] some writers who are careful in their use of technical terms, precisely defined in advance, it may be possible to devise a sort of one-to-one correspondence in translation so that the reader will always know what German word is meant and so that a glossary of terms can be produced. Husserl was not such a writer[. . . . W]hen he does speak of terminology in general, it is usually to warn the reader that the phenomenologist's

[2] It is possible that the introduction was added after Husserl delivered his *Cartesian Meditations* lecture in Paris in 1929. The term '*Besinnung*' is not used frequently in the remainder of *FTL*, despite being emphasised in the introduction. This historical investigation is not the central theme of this chapter, so I won't discuss it further.

[3] The publication history of *Crisis* is rather complicated. In 1936, a selection of key parts were published based on lectures by Husserl. But the full publication, as we know it today, dates from 1954 and was subject to editorial selection.

[4] Carr refers to Husserl's remark from *Ideas I*, §66: 'The words used may stem from ordinary language, being ambiguous and vague in their changing meaning. [But] as soon as they "coincide" [*sic decken*] in the manner of immediate expression, with what is given intuitively, they take on a definite sense which is their immediate and clear sense *hic et nunc*' (quoted from Carr, 1970, footnote 10).

language must necessarily remain 'in flux' and that a demand for mathematical exactness of definitions is totally inapprorpiate in phenomenology. (xxi–xxii)

My reading of Husserl, contrary to Carr's, is that his work does contain exact terminology, terminology which Husserl had clarified by the method of *Besinnung*. For example, the term '*Besinnung*' is consistently used throughout Husserl's writings (at least since *FTL*) to advert to a particular phenomenological method.

The situation is further complicated by the English translation of the *Cartesian Meditations*. *CM* was first given as a lecture in Paris in 1929, then the French translation, by Michelle Pfeiffer and Emmanuel Levinas, was published in 1931. The German edition was published in 1950, and the English translation (by Dorian Cairns) in 1960. Cairns' translation was not based on the 1950 German publication, but on the typescript that Husserl gave Cairns in 1933 (Typescript C). Although the content of *Cartesian Meditations* and *FTL* can be believed to have developed around the same time, the English translations are separated by a nine-year gap. *FTL* was published in 1969, nine years *after* the English publication of the *Cartesian Meditations*. Although both translations are by Cairns, the translation of '*Besinnung*' in these volumes differs. In the *Cartesian Meditations*, '*Besinnung*' is often translated as 'reflection', 'meditation', or 'self-investigation', while in *FTL*, it is 'sense-investigation'.

Another issue we find in English translations is that the term 'reflection' is used as a translation for '*Reflexion*' and as a translation for '*Besinnung*' for both translators. This makes it necessary to look at the German text carefully to verify which term was used by Husserl. Thus, in my investigation, I pay closer attention to Husserl's use of '*Besinnung*' and related terms such as '*besinnen*' and '*besinnlich*'.

3.1.1 Husserl's Characterisation of *Besinnung*

Here, I give a characterisation of the method of *Besinnung* as follows. *Besinnung* aims to clarify the motivational goals of the scientific practice of a particular scientific community. Along these lines, Hartimo observes that '*Besinnung* means clarification of the sense of the activity by explication of the implicit goals that determine the activity' (Hartimo, 2018, p. 249). By 'motivational goals', I mean both the goals that are pursued by the scientific community under consideration,[5] and also the goals pursued by their historical predecessors. In order to carry out *Besinnung*, there are three broad steps one must follow. The first step is to 'enter the community of empathy with scientists' (*FTL*). This means to *empathise* with the scientists, i.e. to understand the science from their perspective and also to understand what motivates them. The second step is to identify and clarify the concepts, methods, goals, or questions within the practice that could be philosophically clarified. Husserl refers to these as 'intending senses'. Roughly, we can understand the intending senses to include (e.g.) the intended meaning of certain concepts, or (e.g.) the intended purpose of certain methods. The third step is to clarify the motivational goals of the community, based on the clarification of the intending senses.

Now, let us turn to Husserl's characterisations and uses of *Besinnung* in the full range of his relevant works. In order to understand *Besinnung*, we need to remind ourselves of Husserl's view that sciences have teleological structure similar to the structure of intentionality (Chapter 1). Underlying this structure is Husserl's focus on scientific practice as a *social* practice. Hence, in describing the method of *Besinnung*, Husserl first states that sciences are generally formed by the *practice* of the relevant *scientists*, and indeed successive *generations* of scientists:

> [Sciences] are formations produced indeed by the practice of the scientists and generations of scientists who have been building them. (*FTL*, p. 9)

For Husserl, when philosophically engaging with science, we must not forget that it is a practice of the *scientists* and that, as a result, sciences cannot be considered independently of the *scientists.* For example, if we are interested in studying philosophy of physics, we cannot simply look at physics as a science independently of physicists. For Husserl, what makes physics a science is that it is formed by the practice of physicists. Thus, each scientific discipline can be understood as formed by the practice of the discipline's practitioners.

[5]Who these scientists are would depend on which particular sciences or scientific theories are my concern. In this chapter, I do not narrow down to a particular community, but aim to clarify what Husserl's general view is.

Importantly, Husserl's view of scientists includes not only the contemporary practictioners, but also their predecessors. Husserl's interest in historically informed practice is clear: for him, sciences, as disciplines, are historically formed by those who practised them in the past as much as by those who are currently continuing the practice.

As mentioned earlier, for Husserl, 'science' includes not only natural sciences, such as physics and chemistry, but also humanities disciplines such as anthropology and history. Each discipline is to be understood in terms of the community of scientists who are continuing the practices, started by their predecessors, 'directed to a certain end, and which are for that reason to be judged in accordance with that end' (*LI*, Prologomena, §11). In *FTL*, Husserl calls such goals, the 'goal-senses' [*Zwecksinne*]:

> [as] produced, they [the sciences] have a goal-sense [*Zwecksinn*], toward which the scientists have been continually striving, at which they have been continually aiming. (*FTL*, p. 9)

In *Crisis*, Husserl also states that *Besinnung* aims at clarifying 'what was originally and always sought in [sciences]' (Crisis, §7), and 'what was continually sought by all the [scientists and sciences] that have communicated with one another historically' (Crisis, §7). Thus, the goal-sense is something that's shared by the scientists or within the *community* of scientists, continued through the historical practice. I shall clarify this sense of community later, in section 3.2. Given that the community shares a goal-sense, *Besinnung* aims at 'a final clarification [*letzten Besinnung*]' of 'the ultimate sense [*letzten Sinn*]' (*CM*:§61) or goal-sense of the sciences.

Following Hartimo (2020, 2021a, 2021b), the nature of goal-senses [*Zwecksinn*] are not pre-determined but can refer to aims, goals, or values of the scientists. This is because the goal-senses are to be discovered within the scientific practice of a given community, and each community might have different goal-sense. But the general structure of the goal-senses is clear: they are the goals at which scientists have *always* been aiming, according to their historically continued practice. Thus, I call them the '*motivational goals*' of the scientists. Similarly, Hartimo often refers to what is clarified by *Besinnung* as the 'ideals guiding the practice' (see, e.g., Hartimo, 2019b, p. 430; 2022b, p. 88). So, while these motivational goals are what the scientists (or the particular community of scientists) are aiming at, they are also what guides the scientists to use certain concepts, or adopt certain methods in their practice.

For example, consider the broad mathematical community. It is undeniable that *rigour* is something that any mathematical community is aiming for (see, e.g., Marfori, 2010; Tatton-Brown, 2021; Burgess & Toffoli, 2022). For instance, a proof ought to be rigorous so that it provides sufficient evidence for the conclusion. At the same time, *rigour* is also what motivated certain mathematical practice: e.g. category theory was motivated to provide a rigorous account of *structure* for solving 'problems involving different kinds of mathematical structure' (Awodey, 1996, p. 212). So, rigour is a motivational goal that can be found in various mathematical communities.

Although the general scientific community might share these motivational goals, how these goals appear in practice could differ among the particular scientific communities. For example, one could say that every science aims at truths, but what determines truth in each discipline would vary. In mathematics, deductive reasoning is privileged, but in empirical sciences inductive reasoning is adequate. Thus, while they can all aim for *truth*, how this aim can be achieved, or what methods are allowed to achieve such aim, can vary across disciplines. Furthermore, what exactly the nature of these goals is would vary depending on the scientific community. Some communities might be focused on a motivational goal that provides an ontology of their practice, e.g. set theoretic V or type theoretic universes $\mathcal{U}$. Others might simply be motivated by and aim for something epistemic. Despite each community's difference, each community has their own motivational goals: they motivate the community's practice but also are what the community is aiming for.

By clarifying the motivational goals of the scientists, *Besinnung* clarifies the historical foundation of the science:

> Every attempt of the historically developed sciences to attain a better grounding or a better understanding of their own sense and performance is a bit of self-investigation [*Selbstbesinnung*] on the part of the scientist. (*CM*, §64)

This suggests how Husserl considered *Besinnung* an important method that ought to be carried out by all scientific communities, so that they can have a better understanding of their own discipline.

While Husserl considered each community of scientists to have a motivational goal, he did not consider the community to have a good understanding of it. Thus, it is left to the phenomenologists to clarify them for a better understanding of the scientific disciplines.

Importantly, it is carried out from the perspective of the scientific community, as a 'self-investigation [*Selbstbesinnung*]' (*CM*, §64) i.e. it is a phenomenological – empathy-first – investigation. So how can we (as philosophers) carry out *Besinnung* from the perspective of the scientists? Husserl claims that we can do so by immersing ourselves in the intentions (i.e. the motivations) of the scientists and, by doing that, discovering the motivational goals [*Zweckidee*] of their practice:

> by immersing ourselves meditatively [besinnliche] in the general intentions of scientific endeavor, we discover fundamental parts of the final idea [*Zweckidee*], genuine science, which, though vague at first, governs that striving. (*CM*, §5)

Immersing ourselves does not simply mean practising the science in question, but seeking to understand the motivations for the community. These motivations are not always explicit in the practice, and they are often vague. For example, no mathematician explicitly discusses whether their proof is rigorous, yet rigour is a motivation that can be found within the practice. As philosophers, we ought to identify these vague motivations and carry out *Besinnung* in order to clarify them.

Husserl describes this process of immersing ourselves as entering the community empathy with the scientists:

> Standing in, or entering, a community of empathy [*Einfühlungsgemeinschaft*] with the scientists, we can follow and understand the goal sense [*Zwecksinn*] — and carry on 'sense-investigation' ['*Besinnung*']. (*FTL*, p. 9)

Thus far, we have looked at how *Besinnung* aims to clarify the motivational goals of the scientists, and have seen that, in order to carry out *Besinnung*, we need to enter the community of empathy. In the next section (section 3.2), I further clarify in what sense the scientific community is a community of empathy.

Before moving on to that section, however, I want to discuss briefly how Hartimo's interpretation of *Besinnung* differs from mine in relation to the term '*transcendental*'. I treat *Besinnung* as a method similar to phenomenological reduction/*epoché*, but a method applied to the teleological structure of the sciences, rather than to the structure of individual acts of intentionality. On Hartimo's account, *Besinnung* is not described as an application of *epoché* applied to the teleological structure. For instance, Hartimo (2022a) distinguishes the method of 'transcendental clarification' from that of 'sense-investigation' (i.e. *Besinnung*), and claims that 'Husserl's phenomenological method is a combination of methods: sense-investigation and transcendental clarification'. This difference is due both to our different interpretations of the term 'transcendental' in Husserl and to differences in our overall aims. Hartimo interprets 'transcendental clarification' as a separate method from *epoché* and *Besinnung*, while I treat it as an *aim* of transcendental phenomenology. Thus, phenomenology (for me) is an explicatory science that aims at (transcendental) clarification, i.e. clarification of the general or universal condition such that each experience or practice is an instance of that condition. The difference in interpretation here perhaps arises from our differing aims. I am focused on developing *Besinnung* as a method to apply to contemporary mathematics, while Hartimo aims at historical clarification of Husserl's method of *Besinnung*, with Husserl's own aim of *mathesis universalis* in mind. Despite the differences between our interpretations, Hartimo and I agree that *Besinnung* generally aims to clarify the goal-senses (i.e. the motivational goals) of the community of scientists.

3.2 Community of Empathy

In this section, I explicate the notion of 'community of empathy [*Einfühlungsgemeinschaft*]' (*FTL*) in terms of 'we-subjectivity [*Wir-Subjectivität*]' (*Crisis* §28, and *Die Lebenswelt*, Hua, vol. 39, 2008). Husserl claims that, in order to enter the 'community of empathy with the scientists', we 'let ourselves be guided by our empathic experience of the sciences' (*FTL*, p. 9). The scientists with whom we empathise here are individuals who practice the particular science that I am interested in. If I am interested in mathematics, then I would attempt to enter the community of empathy with mathematicians.

To be 'guided by our empathic experience of the sciences' (*FTL*, p. 9) means to empathise with their community by understanding what experiences are shared within that community, and to engage with the science from the community's perspective. Empathising with the community could initially begin by empathising with an individual scientist, who is a member of the community that I am interested in. But in order to fully empathise with the community, one must understand what experiences or assumptions are shared within it. Furthermore, engaging with the science from the community's perspective involves understanding the motivation behind the community's practice, since the goal-sense formed in the practice can be found in the 'living intention of' the scientists (*FTL*, p. 10). I shall clarify what it means to empathise with a scientist or with a community of scientists shortly.

Elsewhere, Husserl characterises the notion of 'we-subjectivity'. I argue that we can further explicate the notion of communities of empathy by understanding them as a we-subjectivities. In a we-subjectivity, instead of focusing on an individual subject *I*, the focus is on a group of subjects, a *we*. In this *we*, an *I* can be described as being at the centre, with the others in the *we* surrounding the *I* as actors:

> I, in the centre, the others around me — not as objects, but as actors. (*Die Lebenswelt*, Hua, vol. 39, p. 385; quotation from Caminada, 2015, p. 39)

Importantly, each actor is also a central subject that observes others as actors.

> In this way I have my others, but each of these others has me and its others around it, eccentrically, while each centre is there as a subject of interests, as a first person, while the others are second and self-mediating persons. (Hua, vol. 39, p. 385; quotation from Caminada, 2015, p. 39)

A community of empathy does not simply refer to an individual central subject and its actors, but rather to the totality of each central subject and its actors, understood from a particular central subject's perspective. For example, a group of philosophers to which I belong might be referred to as a community of empathy, in so far as I see them as sharing certain practices with me when practising philosophy. This could include a shared interest in Husserl, or mathematics, or logic, or other specific topics. From my perspective, each of these individuals also sees me as sharing the practice. While I might not know who all these individuals are, I can still have a sense of *we*, and I suppose that any *you* in my *we* would also have a sense of *we*.

The relationship that each central subject has with its actors is called 'empathy [*Einfühlung*]'. The German word '*Fühlung*' translates to the English word 'feeling'. So we can literally understand 'empathy' as 'feeling into', referring to an experience of another that involves positioning (or feeling) oneself in another's perspective. More broadly, we say that a subject *empathises* with (or has an *empathic experience* of) another (an other) just in case the subject has an intentional relation such that the intended object is a distinct conscious subject. Although this characterisation seems to be simplistic, the underlying understanding of empathy is much deeper. The Husserlian/Steinian account of empathy I am invoking here suggests that empathy is possible because the subject has a sense of self as a conscious being, *I*, who is capable of experience. *My* sense of self-hood is constantly there, regardless of the status of *my* physical body. But when *I* have physical experiences, *my* body moves in a certain way, which *I* can perceive. So when *I* see another physical body moving in a similar way, *I* can understand the other's experience in a similar way to *my* experience. So *I* understand the other's self-hood (as a conscious being), or the other *I*, in an empathic experience. See Stein (1964, Ch. 3) and D.W. Smith (1989, pp. 112ff.) for more details.

Hence, one of the necessary features of empathy or empathising is that the intended object is a *conscious* subject, i.e. another *I*. For example, consider yourself being at a wax museum. You see many wax figures, and you do not think of them as being conscious subjects. Suddenly, one of these figures starts to move and interact with others, and it turns out that this individual was not a wax figure after all, but a person who was pretending to be a wax figure. Let us call this individual 'the pretender'. You now have an empathic experience of the pretender (i.e. you empathise with this individual), an experience that you did not have before. Despite the fact the pretender was a conscious being, prior to your own recognition of them as such, your previous experience of them was not empathy. You did not recognise them as a conscious being (even though you now know that you were previously mistaken) and that is crucial in empathic experiences.

Note that to be a conscious subject means to be capable of intentional acts/experiences. In this sense, empathy is 'understanding another's experience from the other's point of view' (D. W. Smith, 1989, p. 112). It is not a matter of asserting my own perspective when it comes to the other's experience, but of understanding, or *empathising with*, the other person's perspective and their intentions. When you were seeing the pretender as a wax figure, you did not consider the pretender to have a perspective or a point of view. They were seen as a figure without consciousness, and thus without a perspective. However, once you empathised with the pretender, you were seeing the pretender as a conscious being with a perspective. From this broad account of empathy, we can also define particular empathic experiences, in which the other is perceived as having a particular intentional experience. Returning to the pretender example, consider if the pretender starts to cry in front of you. You will probably perceive the pretender as doing or experiencing many different kinds of things. The obvious one is that the pretender is crying, but you might also think that the pretender is crying because they are sad. Crying might be a physical experience that you can see, but *understanding* that they are sad is an empathic experience. These are particular empathic experiences a subject can have of another subject, and we shall describe them in the following sort of way: 'a subject empathically experiences/empathises with another subject *as* doing X', where X refers to a particular experience.

In general, I shall refer to the core empathising subject as the 'central subject', and while the empathised subject as the 'other subject'. In an empathic experience, we can also call the particular experience X the 'empathised experience'. Consider, for example, a subject looking at another subject who is eating. The central subject perceives (empathically) the other subject as to be *eating something*, i.e. having this particular kind of intentional experience. To the subject, the experience of eating is an *empathised experience.* These particular, empathised experiences could also be experiences of mathematical or scientific practice. A central subject might look at a mathematician proving a theorem. Then the empathised experience is *proving* of a theorem.[6]

Importantly, Husserl's account of empathy is *not* simply an act or an attempt to *feel* another person's emotions, but it is an act of *understanding.* Empathy is an understanding of the other's perspective,

[6]It is not the case that in all cases of empathy, the central subject might be fully aware of the empathised experience. Suppose I am speaking with someone who speaks in a foreign language that I am not familiar with. Then, while I see this person to be *talking about something*, I do not fully understand what they are saying.

the other's experience, rather than feeling the other emotions or making those experiences my own. In some places, this feeling another's emotion is characterised as 'sympathy'. Thus, in seeing the other cry, the central subject empathised with – i.e. understood – the other's feeling of sadness. See also Jardine (2014, p. 274) on this.

Husserl (in *Crisis*, §28; and see also Stein, 1964) further elaborates on empathy by distinguishing the living body [*Leib*][7] from the physical body [*Körper*] in order to distinguish the experience of the embodied self as opposed to the others. The 'living body' refers to the abstract notion of an embodied subject that is conscious. It is not materially located, but centrally located in our own consciousness. My living body, then, occupies my physical body that is materially located. My experience of another then can be analysed as an experience of their physical body, which I physically perceive, where I also empathically experience their living body, as a conscious being, which I empathically perceive through the physical perception of the physical body. Stein (1964) clarifies further that empathy is not a primordial experience, but a reproductive experience. This means that when the subject experiences (e.g.) the emotions of another, the experienced emotion is not the subject's own, but the other's. Another reproductive experience is memory, or remembering. When we remember what we had for dinner yesterday, we are not actually experiencing, or eating, the dinner. The experience of dinner is relived in our memory, so we say memory is a reproductive experience. See D. W. Smith (1989, pp. 115ff.) for more details.

Let me briefly digress and elaborate my account of empathy. Empathy is not only an experience of understanding another's perspective, but also a way in which we can learn certain things. If I am simply empathising with mathematicians, while I understand that they are (e.g.) proving something, I might not understand exactly every detail of the proof, nor reproduce the proof myself. But eventually, as I attempt to write a mathematical proof myself, I come to make the experience of proving my own. I shall not go into more details here, but this is generally how I consider empathy to be playing a role in our mathematical education, and also in our attempt to enter the community of empathy with mathematicians.[8]

[7]Some translations refer to *Leib* as the 'lived body'.

[8]Although there are some discussions of the importance of empathy in, e.g., art or history education (see Krieger, 2023, and Brooks, 2009), there is no current discussion about empathy in mathematical education in the literature.

In the next section, I build on from this account of empathy, as a form of understanding, and explain how it gives rise to the community of empathy and the teleological structure of scientific practice.

3.2.1 We-Subjectivity

How do we form a community of empathy? This can be better understood by looking at Husserl's description of community in the *Cartesian Meditations*. As Ronald McIntyre (2013) has shown, Husserl claims that a community can be formed by *empathic pairing*. *Pairing*, for Husserl, is a kind of *association* rather than *identification*:

> Pairing is a *primal form* of that *passive synthesis* which we designate as '*association*', in contrast to passive synthesis of 'identification'. (*CM*; Hua, vol. 1:142)

When we experience two distinct things that are alike in some ways, we start to associate them. Through this association, we can experience the two objects as one pair. In doing so, we are not identifying two objects to be the same, but making an association of the two into a single unity and experiencing the unity in one experience (McIntyre, 2013, p. 75). We find the experience of pairing things, when considering social kinds. For example, we might group a pair of tigers as 'two tigers' without considering each of them distinctly. When we have more than two things that are experienced in unity, this unity is called a *community*.

A *community of empathy* is a unity formed by subjects who empathise with one and other as having similar kinds of experiences. For example, suppose that I am a mathematician. I might describe the natural numbers as the sequence '$1, 2, 3, \ldots$', as we saw in chapter 2. When I engage with ordinary people and their understanding of '$1, 2, 3, \ldots$', I am surprised to see that they have certain beliefs about the numbers that I do not have. But when I meet another mathematician, I understand that this mathematician also thinks of the natural numbers as expressed by the sequence '$1, 2, 3, \ldots$'. I empathise that you, another mathematician, have the experience of the natural numbers when considering the sequence '$1, 2, 3, \ldots$'. Then you and I, along with others, form a *community of empathy* as you and I share the same experience.

One helpful way to understand Husserl's community of empathy is the *structuralist* account of a social group given by Ritchie (2020). Ritchie's structuralist account understands social groups in such a way that they can 'be represented as (although they are not identical to) graphs composed of nodes and edges' (Ritchie, 2020), where each node is a member of the group, and the members are connected to one another by the edges, or some relations. Husserl's community of empathy is similarly structured. There are individual members who are related to one another by their shared empathic experiences. What matters here is not the individual members and who they are, but rather that the members share certain kinds of experiences, and can empathise with each other as having these experiences.

However, Husserl's notion goes beyond this: the community is defined from the first-person perspective. Importantly, *I*, a member of the community, empathise with others in my community as *we* share in its history and culture. *I* might not know every single member of my community, as *I* have not interacted with all of them. Regardless, *I* still recognise that the members of *my* community of empathy share or understand certain experiences that *I* have. Then these shared experiences give rise to one or more shared goal (*Zwecksinne*).

These shared goals are not explicitly stated by the members of the community. Rather, they are formed by the shared intentions, or shared experiences, of the members. For example, mathematicians might share the experience of proving theorems, and be connected by their understanding of the proofs made by others, and so on. Or they might share the view that certain concepts are fundamental to mathematics – e.g. the concept of the natural numbers. But they need not explicitly discuss why certain ways of proving or certain concepts are important in mathematical practice. In adopting and accepting certain concepts and proof-methods, they form shared goals that are implicit in their practice. These goals are what need to be clarified by the method of *Besinnung*.

Given that Husserl's community of empathy gives rise to the shared goals of the community, one might consider it to be similar to Margaret Gilbert's notion of *joint commitment*. However, there are several important differences between Gilbert's *joint commitment* and Husserl's *community of empathy*, specifially, in relation to how they are formed. For the former, 'there are no special background understandings' (Gilbert, 2013, p. 65), or at least background understandings are not *required* in joint commitments. This contrasts with Husserl's *community*, which is a cultural formation 'by the practice of the scientists and generations of scientists who have been building them' (*FTL*, p. 9). The background understanding for the community of empathy is a shared culture and history, which the members of the community share by empathising with each other's experience. This shared (cultural or historical) experience is what forms a community of empathy.

Another difference is that Gilbert's *joint commitment* assumes each participant to *explicitly express* the shared mental state or commitment to their commitment (see Gilbert, 2009, p. 180, and 2013, p. 65). Whereas, for Husserl, the aims are *implicit* in the practice within the community, and they are to be clarified by carrying out the method of *Besinnung*.

Before going into further details about *Besinnung*, let me remark on what entering the community of empathy with mathematicians would look like.

3.2.2 Community of Empathy with Mathematicians

To enter the community of empathy with mathematicians, one must empathise with mathematicians. This does not mean empathising with a single random mathematician and their experience. Instead, there are particular experiences that are shared among the mathematicians, and we should attempt to understand what these experiences are. These experiences include taking certain concepts to be fundamental to their mathematical practice, or accepting certain proof-strategies. In the case of accepting certain fundamental mathematical concepts, this can narrow down our focus to a particular mathematical theory and its practitioners. For instance, taking groups as the fundamental concepts of mathematics narrows down our community to that of group theorists. Hence we enter the community of empathy: i.e. the community formed by their shared experience and understanding. The general mathematical community is perhaps held together by the empathised experience of proving a theorem, but a smaller community of mathematicians can be characterised by particular kinds of practices included within the sub-community. For instance, if one wishes to enter the community of empathy of topologists, one would have to understand certain diagrams to be referring to certain topological notions. Or if one is looking at the community of category theorists, one should understand what the commutative diagrams mean. These are just some of the examples of how one can empathise with mathematicians, who form a community of empathy. What would be difficult for me to provide is necessary and sufficient criteria for empathising with mathematicians, for each mathematical community might have different means of doing so. But the less controversial account for the general mathematical community is the activity or experience of proving a theorem is certainly a mathematical one.

Empathising with the mathematicians also involves understanding the mathematics from the mathematical community's perspective. Once understood, we could also provide a clarified characterisation of the mathematics used by the community. For example, what the natural numbers are for the mathematical community can be characterised by the second-order axioms of PA. However, characterising the set of natural numbers by second-order PA is not the business of the broader mathematical community, but for a particular sub-community of mathematicians, namely mathematical logicians. But this is not the only available characterisation of the set of natural numbers. Despite the different ways of characterising the set of natural numbers, the fact that the set of natural numbers can be characterised as the minimal set closed under the successor operation is something that is shared among the mathematicians. In virtue of this shared experience, mathematicians continue to prove theorems about the natural numbers, and it is (perhaps) assumed that facts about (the set of) the natural numbers are decidable in the broad mathematical practice – otherwise, why would mathematicians (or in particular number theorists) continue to try to prove currently unsettled problems about the natural numbers?

However, if we are not considering the set of numbers from the mathematical community's perspective, we can have a different account of numbers, holding beliefs about them which would be false in the standard model of arithmetic. As we saw in the previous chapter, common-folk beliefs about the numbers include facts that are not true of the minimal structure closed under the successor operation. Thus, empathising with the mathematicians ought to involve looking at mathematics from the perspective of mathematicians, whether it be the general mathematical community or a smaller community of mathematicians.

Another feature involved in entering the mathematical community, or empathising with the mathematicians, is to understand the shared motivation which emerges from their history. For contemporary mathematical practice, this might be more difficult. Mathematics has branched out into several different theories, and not every mathematician can be an expert in all the theories. However, given a particular mathematical theory (usually formed by shared assumptions or experience about certain concepts), it is important to understand the motivation of the relevant mathematicians for practising and developing such a theory. This might not be something that is explicitly stated by the mathematicians in their research work and discussions, but we might discover their motivations in their textbooks – either in the form of explicit statements or by 'reading between the lines' – or through informal discussions exploring motivations.

To summarise, entering the community of empathy with mathematicians, or empathising with mathematicians, involves (1) understanding the shared experiences of the mathematical community (e.g. the activity of proving a theorem), (2) understanding the mathematics from their perspective, and (3) understanding the motivation behind practising certain mathematical theories. Although these conditions might not be sufficient (but I would think they are necessary), we can then characterise empathising with the mathematicians as '*mathematical empathy*'. Thus, the first step of the method of *Besinnung* is to *mathematically empathise*.

3.3 Intending Sense, Fulfilling Sense, and *Zwecksinn*

In the previous section, I clarified Husserl's account of community of empathy, and thus how one can enter the community of empathy with mathematicians. Here, I show how we can clarify the motivational goals of the scientists – i.e. the third step of *Besinnung* characterised in 3.1. But before clarifying the motivational goals, Husserl explains that *Besinnung* involves clarifying the *Zwecksinne* by converting the intending sense, which is vaguely available to us in the practice, to the clear sense:

> *Besinnung* implies nothing other than the attempt at the genuine making of the Sense 'itself', which is presupposed in the bare aiming meaning [*Meinung gemeinter*]; or the attempt to convert the 'intending sense' [*intendierenden Sinn*] (as it was called in the

> Logical Investigations), [which is] in the unclearly aiming, 'vaguely floating before us', the unfulfilled sense [*erfüllten Sinn*] into the clear one, and hence it provides the evidence of its [i.e. the fulfilled one's] clear possibility. (*FTL*: Introduction; my own modified translation based on Cairns)

In this section, I clarify what Husserl means by the 'intending sense' in the *Logical Investigations* and explain that *Besinnung* involves identifying the concepts, methods, goals, or questions that are already found within the practice, and attempt to unify these by looking at the motivations of the practitioners.

3.3.1 Intending Senses in *Logical Investigations*

As Husserl described in *FTL*, *Besinnung* involves an attempt to clarify the intending sense in relation the fulfilled sense. He explicitly refers to the *Logical Investigations* to clarify what he means by the 'intending sense' [*intendierenden Sinn*]. In *Logical Investigations*, Husserl distinguishes between the intending sense and the fulfilling sense [*erfüellender Sinn*]– note that this is not the *fulfilled* sense, which is what a *Zwecksinn* (or a motivational goal) is. Here, I clarify Husserl's notions of intending sense and fulfilling sense, before turning towards the *fulfilled sense*.

In *Logical Invesitigations I, Prologomena*, §14, Husserl writes that content [*Inhalt*] can be distinguished into three kinds: intending sense [*intendierender Sinn*] or meaning [*Bedeutung*]; fulfilling sense [*erfüellender Sinn*]; and the object [*Gegenstand*]. On the one hand, the intending sense refers to the meaning that is out there with the object, without any reference to the intentional act or experience of a subject. This describes the *possible* meanings (or purposes) that an object could have, which could *possibly* be grasped by a subject. This meaning could be of a linguistic expression, concept, or even method, since Husserl's intended objects simply refer to the *something* that a subject has a relation to in an intentional experience (recall section 1.3.1). For example, consider a table. A table has an intending sense, i.e. the meaning or the purpose that the table is for. Regardless of whether there is a subject who is using the table, it has an intended purpose that (e.g.) it is an item of furniture that can be used to eat meals on, etc. Thus, intending sense can be understood as the intended purpose of an object or an intended meaning of a concept, etc. Just as Husserl's characterisation of intended object could be particulars or events, Husserl's characterisation of meaning is not pre-predicated with certain ontological features.

The fulfilling sense, on the other hand, is the meaning in virtue of a *fulfilling act*. Husserl provides perception, e.g. seeing, as an example of a fulfilling act. But, using the example of the table, simply *seeing* the table will not get us to fully grasp the intending sense (or intended meaning) of the table, although seeing another person using the table appropriately might get us closer to it. In general, a fulfilling act refers to an intentional act that aims to grasp the object (i.e. the table) via the intending sense (i.e. the intended meaning or purpose of the table) that is around us. When an appropriate fulfilling act is aimed at the object, the force of evidentness, or intuitive *fulfilment*, which brings further clarity to the intending sense itself. Thus, Husserl here is not referring to some reasoning involved in experiencing (e.g.) the table, but simply that we have such evident experience, that allows us to understand what the intending meaning of the table is, and thus can experience the table itself.[9]

In the context of *Besinnung*, Husserl describes the 'intending sense' as the 'meaning that is vaguely floating around us' (*FTL*, p. 9). Hence, the intending sense is the intended meaning that we can find in the scientific practice, i.e. in the concepts or methods available in the scientific practice. For example, when mathematicians practice number theory, the intended meaning of the set of *natural numbers* is the intending sense of the set of *natural numbers*. One could describe it as the minimal structure closed under the successor operation, or via mathematical induction, etc. But in order to fully grasp the set of natural numbers, we must perform a fulfilling act. This fulfilling act could be proving theorems about the natural numbers, e.g. by applying mathematical induction. In carrying out the proofs, we have the fulfilling sense of the natural numbers.

In some cases, the fulfilling sense might not coincide with the intending sense (*LI I, Prologomena*:§15), as in the case of *seeing* a table. How could one understand what a table is for just by staring at it? We can also find similar examples when considering mathematical concepts and some fulfilling experiences. For instance, suppose that a subject counts the numbers, so counting would be the

[9] In this particular section, Husserl further distinguishes between the *subjective content* and the *objective content* without further clarification. The three kinds of content are about the objective content, not the subjective content.

fulfilling act. The fulfilling sense of the numbers in the act of (intransitive[10]) counting might not be of the natural numbers (as I have argued in Chapter 2). While the subject continues to count, they might simply allow other *non-arithmetical acts* when describing certain *large numbers*. But in order for the intending sense and fulfilling sense to coincide, *unity of fulfillment* is required (*LI I, Prologomena*:§15). Only in the unity of fulfilment (or unity of coincidence) do the intending sense and fulfilling sense coincide and thereby the subject *fully* experiences the object. Thus, in proving a theorem about the natural numbers, the subject performs a fulfilling act in the unity of fulfilment.

Thus far, I have clarified Husserl's notions of intending sense and fulfilling sense. But to understand '*Zwecksinn*', we must understand the notion of *fulfilled sense* [*erfüllten Sinn*]. As the adjective 'fulfilled' suggests, it can be understood as the final result of the fulfilling sense. In particular, if the fulfilling act is performed in the unity of fulfilment, then the intentional content is the intending sense (i.e. the intended meaning), and through the unity of fulfilment, we can further clarify the *fulfilled* sense (i.e. the motivational goal). Thus, a fulfilled sense refers to the sense that arises as the intending and fulfilling senses coincide in an act, so that the intended object is intuitively grasped at a glance:

> The child who already sees physical things understands, let us say, for the first time the final sense [*Zwecksinn*] of scissors; and from now on he sees scissors at the first glance *as* scissors – but naturally not in an explicit reproducing, comparing, and inferring. (*CM*, §50)

However, in the context of *Besinnung*, *Zwecksinn* seems to carry more than simply the fulfilled meaning by which we can grasp the relevant object. Since Husserl's explanation in *Logical Investigations* concerns the structure of intentionality between a subject and an intended object, when we come to consider the teleological structure of sciences, the fulfilled sense carries a slightly different structure. In particular, I am referring to the *transcendental* aspect of the science, i.e. to universal or general features that make the science possible. The *Zwecksinn* is not simply the intended meaning, but more abstract motivational goals, which are implicit in practice. Once we have clarified these motivational goals via *Besinnung*, we can then look at the practice and understand certain features

[10]Following Benacerraf (1965), by 'transitive counting', I mean the act of counting a particular set of objects, while intransitive counting does not have a particular set of objects.

of the practice as the instances of the clarified motivational goals. For example, having clarified *rigour* as a motivational goal of mathematics, we understand (e.g.) axiomatic characterisations of definitions as instances of mathematical rigour. Hence the clarified motivational goals are more abstract, implicit aims whose instances can be intuitively or immediately seen in practice. In line with this, Mirja Hartimo (2021b, p. 189) writes that the *Zwecksinne* can also refer to values or virtues, which could be further divided into ideal desiderata or minimal requirements (see, e.g., Douglas, 2013). Thus, the *Zwecksinne* include values that motivate sciences to be a certain way, but they are also what sciences must aim to meet. The fulfilled senses, or *Zwecksinne*, are the motivational goals as described earlier.

To summarise, the intended sense refers to the intended meaning or purpose that is openly available within the practice of the scientific community in question. These meanings could be of concepts, goals, methods, or even questions. *Besinnung* involves carrying out an appropriate act such that we can clarify the intended meaning. This refers to clarifying the concepts, goals, or methods, or to answering the question based on the mathematical community's perspective. In such clarification, we can then find the motivational goals of the community. In the next section, we turn to some textual examples of how Husserl carries out *Besinnung*.

3.4 Husserl's Use of *Besinnung*

Having described what Husserl means by intending senses and *Zwecksinne*, I shall discuss how Husserl uses the method of *Besinnung*. For this discussion, I consider how Husserl himself carries out *Besinnung* in *FTL*, *CM*, and *Crisis*, and show that he begins with a general question within the community of empathy. This question then guides us to the particular concepts or other features within the practice that need to be further clarified. Clarifying those, we can turn to convert the intending senses to the *Zwecksinne* (i.e. motivational goals) of the practice. In describing Husserl's own approach, I also comment on how a more precise understanding of *Besinnung* can be provided for contemporary usage.

The aim of this section is to show that carrying out *Besinnung* successfully requires explicitly specifying the community of empathy in question in terms of their shared experience or practice, by looking at the historical motivation of the community. I find that while Husserl was not always so explicit about the community of empathy with which he is engaging, this is an important step in carrying out *Besinnung*.

In *FTL*, *CM*, and *Crisis*, Husserl's starting community of empathy is different in each case. In *Crisis*, he begins within the European community, rather than the scientific community. His question is *Is there, in view of their constant successes, really a crisis of the sciences?* (*Crisis*, §1), which can be answered negatively if he were starting from the scientific community's perspective, given the success of scientific methodologies, as it would be hard to consider that there is a threat to the sciences. However, he continues in §2 that there is another perspective, within the European community, where there *is* a crisis in the sciences, i.e. 'the crisis of our culture and the role [ascribed] to the sciences' (*Crisis*, §2).[11] As we see in the *Crisis*, he aims to clarify the question that has risen within the European community, and carries out 'a teleological-historical reflection upon the origins of our critical scientific and philosophical situation' (see Carr's footnote 1 on p.3 or *Crisis*, p.xiv). I shall clarify Husserl's *Besinnung* in the *Crisis* shortly, but let me briefly remark on his approaches in *CM* and *FTL* here.

In the *Cartesian Meditations*, he begins by situating himself as a (neo-)Cartesian mediatator (as far as is possible, without presupposing any sciences), presenting his phenomenology as a continuation of Descartes's *Meditations on First Philosophy*. Thus, Husserl's community of empathy is with the philosophers who see themselves as following Cartesian philosophy. By carrying out *Besinnung*, Husserl raises several questions leading to the claim that Descartes's meditative, sceptical method ought to be re-envisioned so that Descartes's original goal could be continued: 'to uncover [...] the genuine sense of the necessary regress to the ego, and consequently to overcome the hidden but already felt naïveté of earlier philosophising' (*CM*, §2). The question that Husserl is asking here is 'what is the genuine sense of the ego?' By answering the question, Husserl can be seen to clarify the renewed motivational goals of Descartes and the Cartesian meditators.

[11]Here, perhaps Husserl is referring to the Nazis coming to power in Europe.

In *Formal and Transcendental Logic*, his main questions are 'What is the genuine sense of science?' and 'What is the genuine sense of logic?' In his introduction, he describes the sense (i.e. meaning) of science as found in Plato, 'as a place for exploring the essential requirements of "genuine" "knowledge"'. The sense of logic in Plato, according to Husserl, 'arose from the reaction against the universal denial of science by sophistic skepticism'. Logic is then the 'theory of sciences', which makes 'genuine science possible' by '[guiding] its practice'. His worry is then the abandonment of this genuine sense of science in modern science. By answering these questions, carrying out *Besinnung*, Husserl aims to clarify the motivational goals of the modern scientists as a continuation of this historical genuine sense of science.

Although Husserl generally focuses on the sense of science of particular historical scientists, e.g. Plato and Descartes, I do not think that this is the interesting methodological feature of Husserl's *Besinnung*. His methodological claim is that we ought to enter the community of empathy with *practitioners*, and then we ought to clarify the motivational goals by looking at their questions, concepts, or goals (i.e. the intended objects that have the intended meaning in the practice). But the communities Husserl focuses on can sometimes be too broad. In order to productively carry out *Besinnung*, we ought to start with smaller communities, not (e.g.) European humanity, which consists of speakers of various languages who have complicated and diverse histories. It is also important to specify to which community we are referring, defined by what kind of shared experience, as a community of empathy. Then, within the community of empathy, we can follow Husserl's approach, starting with questions that arise in the practice or in the community. These questions then guide us to particular concepts, methods, etc., which the community would use, but would require further clarification in an attempt to clarify the motivational goals of the community.

In the next subsection, I shall argue that, in considering the shared experience of the community, we ought to look at the historical motivation behind the experience. To do so, I follow Husserl's use of *Besinnung* in *Crisis*.

Historical *Besinnung* in *Crisis*

The main goal of *Crisis* is to understand the meaning (or sense) of science, which began with the question, *Is there a crisis in the sciences?* The question arises from within humanity (or particularly, that European section of humanity of which he is a part), thus it demands an understanding of what the goals of European humanity are. He quickly narrows down to the European philosophical community, as it influenced European humanity in general. To clarify the goals of the community, Husserl carries out the method of *Besinnung* by reflecting [*besinnen*][12] on our philosophical history. He first identifies the historical motivation of European philosophy as 'in the possibility of universal knowledge' (*Crisis*, §7), and then questions 'how do we hold onto this belief?' (*Crisis*, §7).

The identifying of the historical motivation of the community is one of the important steps in empathising with the community. Such historical motivation is understood as the common experience by the members of the community. Thus, a historical reflection, which clarifies the motivation, is a necessary step in empathising with the community of European philosophers. He further clarifies that the contemporary European philosophical community is connected to the historical predecessors by the 'concepts, problems, and methods' used:

> Our first historical reflection [*historiche Besinnung*] has not only made clear to us the actual situation of the present and its distress as a sober fact; it has also reminded us that we as philosophers are heirs of the past in respect to the goals which the word "philosophy" indicates, in terms of concepts, problems, and methods. (*Crisis*, §7)

Since the philosophical community is a continuation of the practices of its predecessors, 'in terms of concepts, problems, and methods', starting with the historical motivation is then more important. Without a clarified understanding of the historical motivation of the community, it would be harder to find which concepts, problems, or methods ought to be clarified for the philosophical practice.

Returning to Husserl's main question of whether there is a crisis in the sciences, we then ought to consider what is meant by 'sciences' in the question by looking at its historical meaning according to

[12] Husserl writes: 'In dieser Not uns besinnend, wandert unser Blick zurück in die Geschichte unseres jetzigen Menschentums' (*Crisis*, §5)

the philosophical community. That is, what was meant by 'sciences' by the historical predecessors of philosophers contemporary with Husserl. This is why it is necessary here to 'reflect back [*Rückbesinnung*], in a thorough historical and critical fashion, in order to provide, *before all decisions*, for a radical self-understanding' (*Crisis*, §7).

What these passages show is that there are at least two steps for a successful *Besinnung*. One step is the historical reflection, which will point us to the historically shared motivation of the community that we are interested in. Another step involves starting with a general question (as found within the community) – e.g. *Is there a crisis in the sciences?* – to identify what concepts and methods we must focus on. Ultimately, *Besinnung* will lead us to 'what was originally and always sought in philosophy' (*Crisis*, §7), which is more general than the question concerning the crisis in the sciences we began with.

In other words, *Besinnung* involves clarifying the original motivations of the relevant community in question. Within this community, we are then led to certain questions that need to be answered. Answering these questions will guide us towards the concepts, methods, goals, etc., that need to be clarified. And the clarifications of these will take us to the clarification of the motivational goals of the community.

3.5 Radical *Besinnung* and Contemporary Mathematical Practice

In some places, Husserl describes *Besinnung* as 'radical', by which he means *Besinnung*, which aims to be critical in a way that will produce an original, clarified sense of the practice:

> Radical *Besinnung*, as such, is at the same time criticism for the sake of original clarification. Here original clarification means shaping the sense anew, not merely filling in a delineation that is already determinate and structurally articulated beforehand[. . . . O]riginal sense-investigation [*Besinnung*] signifies a combination of determining more precisely the vague predelineation, distinguishing the prejudices that derive from associational overlappings, and cancelling those prejudices that conflict with the clear sense-fulfilment – in a word, then: critical discrimination between the genuine and the spurious. (*FTL*, p. 10)

The result of radical *Besinnung* is that the resulting *Zwecksinne* are 'genuine', rather than 'spurious' (*FTL*, p. 10). Hartimo (2018, p. 265) observes that '[t]he outcome of radical *Besinnung* is criticism of existing practices.' This does not mean that radical *Besinnung* simply aims to criticise the practice, but rather that it re-directs the practice in the direction of its genuine *Zwecksinne*. This ensures the *Zwecksinne* to be actually possible, or achievable, for the practitioners. Husserl, in the *Cartesian Meditations*, also emphasises that radical *Besinnung* is to be 'phenomenological':

> Every attempt of the historically developed sciences to attain a better grounding or a better understanding of their own sense and performance is a bit of self-investigation [*Selbstbesinnung*] on the part of the scientist. But there is only one *radical* self-investigation, and it is phenomenological. (*CM*, §64)

Here, what makes *Besinnung* radical, or phenomenological, is the aspect of critical evaluation. Although scientists can attempt to understand their own practice and aims, so that this might be called *Besinnung*, the genuine *Besinnung* must be radical or phenomenological (*CM*, §11). This means to go beyond the clarification and offer a critical perspective that might not be easily available from within the community's perspective.

Hartimo (2018) describes this radical *Besinnung* as involving the 'transcendental method' or 'transcendental clarification'. By 'transcendental', Hartimo means a method of clarification that involves reflecting on the practice. Hence, for Hartimo, radical *Besinnung* involves two methods: sense-investigation (or simply *Besinnung*) and transcendental clarification. (See Hartimo, 2022a, for more details.) In my account, looking to mathematical practices, radical *Besinnung* is a particular kind of *Besinnung* with a critical *aim*. What makes *Besinnung* radical is the particular aim in consideration rather than an additional method. Once we have clarified the motivational goals, we can take a critical perspective and evaluate whether the goals are achievable or whether particular revisions might be necessary in the mathematicians' methods. Although Hartimo's account and my account of both *Besinnung* and radical *Besinnung* are more or less in agreement, where we diverge is on our interpretation of the term 'transcendental'. The 'transcendental', for me, does not describe a distinct method, but an *aim* of Husserl's *transcendental* phenomenology, i.e. to clarify universal structural conditions that guides the practice. Naturally, with this reading of Husserl's notion of *transcendental*, radical *Besinnung* involves critical evaluation of the motivational goals, and whether these goals are achievable for the practitioners, given their current methods (as Hartimo would agree). In order to successfully carry out radical *Besinnung*, one does not stop at clarifying the goals, but refers back to the practitioners and challenges their practice in light of the motivational goals. The practitioners might then reflect upon their clarified motivational goals and review or revise their practice.

An example of such revision could be the contemporary use of interactive proof checkers in mathematical practice. In light of concerns that there were problems arising in mathematical practice (see, e.g., Voevodsky, 2014) and given that having a rigorous method was a motivational goal of mathematical practice, interactive proof checkers have been used in algebraic topology to make the goal of rigorous method more achievable. Another example, along the same line, is the change in what concepts are considered *foundational* in mathematics. Naïve set theory is usually taken to be the foundational theory that (e.g.) provides the background language for mathematics. But for algebraic topology, it is widely claimed that type theory (or some form of naïve type theory) is better suited for the relevant practice. If mathematical practice needs a foundational theory that provides the background language for the practice (and this is a motivational goal), then the relevant and appropriate revision to the practice is necessary. In the long term, radical *Besinnung* is then an important part of mathematical practice. *Besinnung* clarifies the motivational goals found in practice. Then we refer back to the current state of practice to evaluate whether the motivational goals are achievable. It ensures that mathematical practice can continue to achieve the motivational goals of the practitioners, while continuing the practice in an open ended way. An example of such revision could be the contemporary use of interactive proof checkers in mathematical practice. In light of concerns that there were problems arising in mathematical practice (see, e.g., Voevodsky, 2014), given that having a rigorous method was a motivational goal of mathematical practice, interactive proof checkers have been used in algebraic topology to make the goal of rigorous method more achievable. Another example, along similar lines, is the change in what concepts are considered *foundational* in mathematics. Naïve set theory is usually taken to be the foundational theory that (e.g.) provides the background language for mathematics. But for algebraic topology, it is widely claimed that type theory (or some form of naïve type theory) is better suited for the relevant practice. If mathematical practice needs a foundational theory that provides the background language for the practice (and this is a motivational goal), then the relevant and appropriate revision to the practice is necessary.

In the long term, then, radical *Besinnung* is an important part of mathematical practice. *Besinnung* clarifies the motivational goals found in practice, and then we refer back to the current state of practice to evaluate whether the motivational goals are achievable. Radical *Besinnung* ensures that mathematical practice can continue to achieve the motivational goals of the practitioners, while continuing the practice in an open ended way.

In the next chapter, I apply the method of *Besinnung* to mathematical practice. In particular, I focus on HoTT that developed out of Voevodsky's worries about mathematical practice. We shall find that carrying out *Besinnung* offers a new philosophical approach to HoTT that contrasts with other contemporary approaches (e.g. that of Ladyman & Presnell, 2015). HoTT is a particularly interesting mathematical theory, with a relatively clear history. It is being developed by a group of mathematicians (and computer scientists and philosophers) who call themselves 'the Univalent Foundations Program'. Understanding their empathised experiences is especially valuable and important, since HoTT brings together various practitioners of mathematics, not necessarily all from mathematics departments. For example, there are mathematical logicians who have been interested in type theory and category theory, among those from computer science or philosophy departments. Among the mathematicians are those who work on homotopy theory or algebraic topology more broadly, and have no prior experience of type theory. Despite their differences in training and mathematical practice, they have come together to develop HoTT as a new mathematical theory with a shared goal. In the next chapter, I highlight the mathematical assumptions shared among the homotopy type theorists, which one must accept and understand in order to join the community of empathy. Then I clarify the concept of identity in HoTT, in an attempt to clarify what motivational goals can be understood from such clarification.

3.6 Conclusion

As I have shown, *Besinnung* (if applied to mathematical practice) aims to clarify the motivational goals of (the appropriate community of) mathematicians by (1) empathising with the community of mathematicians, understanding their historical motivation and engaging with the mathematics from within the community's perspective, (2) finding the concepts, methods, goals, or questions which are implicit in the practice, and (3) clarifying these. Once they are clarified, we can further clarify the motivational goals. Then, we can perform a critical evaluation of whether these goals are achievable – this is called *radical Besinnung*.

While Husserl's own application of radical *Besinnung* refers back to the Greek origin of European sciences, when we come to focus on contemporary mathematics, it is important to start with a particular theory and the community (or perhaps sub-communities) that practice that theory. Mathematics is an enormous discipline with many applications in other scientific disciplines. Thus, I suggest that radical Besinnung of contemporary practice should focus on certain mathematical theories, whose practitioners have a shared historical motivation, and carefully empathise with the relevant practitioners in an attempt to clarify their motivational goals.

To show more concretely how the method of *Besinnung* is used in mathematical practice, I apply *Besinnung* to HoTT in the next chapter. I shall show by demonstration that radical *Besinnung* is a successful philosophical method for studying mathematical practice in the special case of HoTT.

Chapter 4

Homotopy Type Theory

Homotopy type theory (HoTT) is a mathematical theory that was developed by a group of mathematicians and computer scientists supporting the so-called *Univalent Foundations Program* (UFP). This programme is often described as an 'alternative foundation for mathematics', going against set theory. The usual criticism of taking set theory as a foundation is that the language of set theory is much too expressive, making it possible to pose questions which are not relevant to mathematical practice: e.g. Is the number two $\{\emptyset, \{\emptyset\}\}$ or $\{\{\emptyset\}\}$? Some also argue that set theory is not a *foundation*, since it does not reflect mathematical practice (e.g. Leinster, 2014). HoTT, in contrast, is claimed to reflect the more *structural* aspects of mathematical practice, particularly in relation to topology and homotopy theory. In it, for instance, the natural numbers are defined as 'terms' in a 'natural number type' $\mathbb{N}$, so each number is simply defined by how it is related to other numbers in the type $\mathbb{N}$. Thus, as a foundation for mathematics that reflects the structural features of mathematical practice, HoTT can be seen as a more appropriate theory than set theory.

The aim of this chapter is to demonstrate how the method of *Besinnung* can be used to clarify the motivational goals of HoTT. In particular, I focus on clarifying what the definition of identity is in HoTT, by empathising with the homotopy type theorists. I identify three minimal requirements that we must accept in order to empathise with the homotopy type theorists. These are (a) the univalence axiom, (b) the logical syntax of HoTT, and (c) its intrinsic homotopical content. I explain each of them in section 4.1 while providing a motivation for each of them in the development of HoTT. Given these minimal requirements, we turn our attention to the definition of identity in HoTT, called 'path induction'. Philosophers (e.g. Ladyman & Presnell, 2015, and P. Walsh, 2017) have recently argued that path induction, the definition of identity in HoTT, needs to be justified. By looking at path induction from the perspective of the homotopy type theorists, in section 4.5, I offer a new *internal* justification for path induction. By an 'internal' justification, I mean a justification, i.e. an argument, which simply follows from what is already assumed within the mathematical theory. In this case, the given theory is HoTT and the presumed mathematics includes topology and homotopy theory, as stated by the UFP (2013, p. 3). Based on my justification of path induction, I then clarify the motivational goals of homotopy type theory in section 4.6.

There are two main arguments in this chapter. One argument aims to show that path induction is internally justified. This argument can be summarised in the following way.

- P1: In order to be internally justified, path induction must be shown to follow from the minimal requirements of empathy with the homotopy type theorists.
- P2: The minimal requirements of empathy with homotopy type theorists include accepting the univalence axiom, HoTT's logical syntax of Martin-Löf type theory (MLTT), and HoTT's intrinsic homotopical content. (Sections 4.1 and 4.2)
- P3: Path induction, written in MLTT, follows from the intrinsic homotopical content via the univalence axiom. (Sections 4.4 and 4.5)
- C1: Hence, path induction is internally justified.

The premise P3 can be demonstrated by the method of *Besinnung*. I do so by explicitly stating which topological/homotopical assumptions are needed to fully understand path induction, and showing that path induction actually follows from the path lifting property in topology. This argument particularly challenges the philosophical motivation of Ladyman and Presnell (2015). I shall discuss their argument in more detail in section 4.3, but briefly, they claim that if HoTT is a foundation, it must be understood from logical principles alone, independently from mathematical content. This, in my view, goes against the motivation of HoTT as a foundational theory that reflects the practice of topology and homotopy theory. Instead of starting with a particular sense of foundation prior to empathising with the homotopy type theorists, we should aim to clarify in what sense HoTT is a foundation.

The other main argument in the chapter aims to clarify the *Zwecksinne* (i.e. the motivational goals) of HoTT as homotopical autonomy and rigour. The general argument is as follows:

- P1: HoTT has three vaguely floating senses (*Sinne*) – the univalence axiom, the logical syntax of MLTT, and the intrinsic homotopical content (section 4.1) – understanding these is necessary for empathising with the mathematicians.
- P2: By *Besinnung*, the *Zwecksinne* can be clarified based on the three senses.
- C1: Hence, the logical syntax of MLTT aims to guarantee *rigour* in HoTT; and from the intrinsic homotopical perspective, HoTT provides an autonomous *homotopical* foundation. (section 4.6)

While the conclusion above suggests that the univalence axiom is not being further clarified as a motivational goal, this is not the case. The univalence axiom, as we shall see later, provides that we should treat the logical syntax and the intrinsic homotopical content as the same. Thus, through acceptance of the univalence axiom, HoTT has the motivational goals of rigour and homotopical autonomy.

By the two arguments noted above, I demonstrate that *Besinnung* is a fruitful philosophical method for studying mathematical practice. The first argument shows a way to engage with an existing philosophical question about mathematical practice. This satisfies the claim that philosophical questions in PMP ought to be interesting to practising mathematicians, and it offers a philosophical solution appropriate for the mathematicians. The second argument highlights the values or virtues (i.e. the motivational goals) of HoTT. More attention has been given to values and virtues of mathematics in the recent literature (e.g. Aberdein, Rittberg, and Tanswell, 2021), and *Besinnung*, as a method that aims to clarify the *Zwecksinne*, can be understood to clarify values and/or virtues found within the practice.

4.1 Empathising with Homotopy Type Theorists

In this section, I clarify the requirements for entering the community of empathy with the homotopy type theorists, as understanding (a) the univalence axiom, (b) HoTT's logical syntax, and (c) HoTT's intrinsic homotopical content. These can be understood as the minimal requirements for HoTT, or as what is involved in empathising with the homotopy type theorists. I provide some brief explanations and historical motivations for the requirements, before focusing on path induction and diving further into the mathematics behind it.

4.1.1 Univalence Axiom

The UFP was introduced in order to provide a foundation for algebraic topology that can use computer verification software (or interactive theorem provers/checkers) for mathematical practice. The term 'univalent' comes from the axiom that was stated by Vladimir Voevodsky, the founder of UFP. The axiom is presented as the following statement by Awodey (2018):

Definition 4.1 (Univalence Axiom (UA))**.** *Equivalence is equivalent to identity.*

$$(A \simeq B) \simeq (A = B).$$

The axiom states that certain mathematical concepts that are shown to be equivalent can be treated as identical, but the notion of equivalence can vary depending on the mathematical context. For example, within group theory, if two groups are isomorphic, we treat them as identical, despite them having different elements. This is similar in models of arithmetic, where we consider there to be the *unique* standard model of arithmetic, often with the tag 'up to isomorphism'. Unlike in group theory or model theory, in topology, the notion of equivalence is homeomorphism, not isomorphism. But it is not only the objects of a given theory that can be equivalent to one another. If we take a theory and consider a category of its objects (e.g. the category of topological spaces), then we can define certain notions of equivalence between this category and others (e.g. an equivalence between the category of topological spaces and the category of groups).

The univalence axiom can be seen as asserting an identity between structures/theories, when there is an equivalence between them. For example, since some equivalences can be found between Martin-Löf type theory, topology, and homotopy theory, certain notions used in these theories can be identified as the same. So, given a type theoretic judgement $p : A = B$, UA identifies (at least) the following concepts in HoTT (UFP, 2013, p. 5):

(1) a *logical proof* p of the *proposition* $A = B$,

(2) a *topological path* $p : [0,1] \to \mathcal{U}$[1] between the *points* A and B, i.e. $p(0) = A$ and $p(1) = B$, and

(3) *a homotopy equivalence* p between topological spaces A and B.

With these three distinct concepts identified as the same, we can freely use any one of the interpretations – logical, topological or homotopical – to understand the judgement $p : A = B$ (see Figure 4.1 for the pictorial versions of the topological and homotopical interpretations). For instance, consider the following statement: let A and B be topological spaces such that there is a proof p showing A and B are homotopy equivalent. In the following, I shall use '(HT)' to denote the homotopical interpretation, '(topology)' to denote the topological interpretation, and '(logic)' to denote the logical interpretation. The logical interpretation of types is obtained by *Curry-Howard isomorphism* – it is often used as an intutionistic semantics (also known as BHK – Brouwer-Heyting-Kolmogorov/Kreisel interpretation) for type theory (Howard, 1980; Troelstra, 2011). Briefly, the logical interpretation treats types as propositions, and terms as proofs of the proposition. So the following types, 0, 1, $A + B$, $A \times B$, $A \to B$, $\sum_{x:A} B(x)$, $\prod_{x:A} B(x)$, and $\mathsf{Id}_A(x, y)$, are logically interpreted as the propositions $\bot$, $\top$, $A \vee B$, $A \wedge B$, $A \to B$, $\exists x \in A, B(x)$, $\forall x \in A, B(x)$, and $x =_A y$.[2]

[1] $\mathcal{U}$ stands for the universe of types where all types occur.

[2] Explaining the topological and homotopical interpretations requires more work than the logical interpretation. I shall provide the mathematical details shortly, but the point to take away (for now) is that we are welcome to use any of these interpretations when reading the type theoretic statements. See Figure 4.1 for a pictorial understanding of these interpretations.

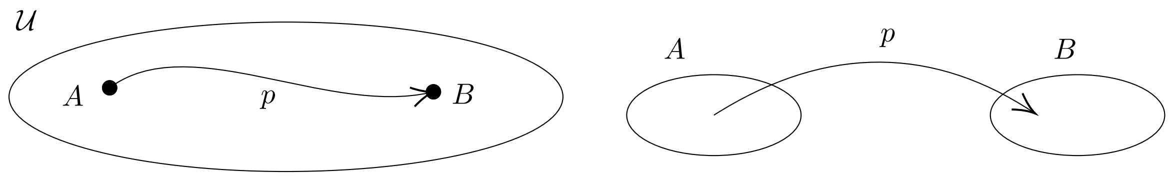


Figure 4.1: Interpretations via UA

Let us begin by numbering each sub-clause: (1) *let A and B be topological spaces*, (2) *there is a proof p*, and (3) *p shows that A is homotopy equivalent to B.* So, with the univalence axiom, (1) can be variously interpreted as any of the following three statements: (1a: HT) *let A and B be topological spaces*; (1b: topology) *let A and B be points in the topological space $\mathcal{U}$*; and (1c: logic) *let A and B be propositions.* Then (2) is interpreted as the following: (2a: HT) *there is a homotopy equivalence $p : A \to B$*; (2b: topology) *there is a path $p : [0,1] \to \mathcal{U}$*; and (2c: logic) *there is a proof p.* And finally, (3) is interpreted as the following: (3a: HT) *p shows that A is homotopy equivalent to B*; (3b: topology) *the starting point of p is $p(0) = A$ and the end of point of p is $p(1) = B$*; and (3c: logic) *p proves the proposition $A = B$.*

One challenge in following HoTT and standing in the community of empathy with the homotopy type theorists is in using these three interpretations concurrently. One must learn to choose whichever is most appropriate for the situation. This requires one to be at least comfortable, if not fluent, in all three interpretations, and to observe that there is a superior interpretation in certain instances.[3] Individually, each homotopy type theorist might have a preference for one interpretation over another, but insofar as the community is seen as a whole, it is only the situation that determines which interpretation is preferred.

4.1.2 Logical Syntax

Another requirement, independent of the univalence axiom, is that HoTT has a formal logical syntax for computer implementation. This allows the mathematical content from topology and homotopy theory to be expressed as judgements in type theory, and any proofs of the judgements be checked

[3] This is a difficult aspect of the practice of HoTT, an aspect that I believe one can master by trying out some problems.

by a computer. To understand why logical syntax is a minimal requirement for HoTT, we must look at the history of its development. In this section, I give a brief account of how HoTT was developed by Vladmir Voevodsky as a mathematical theory that could be computer implemented.

Voevodsky introduced the UFP with an aim to provide a proof-checking assistant for working mathematicians (Voevodsky, 2010). This was to resolve the situations where a mathematician's social reputation was influencing the verification of a mathematical argument. As an example, Voevodsky (2014) discusses a situation from his own experience. Despite the fact that Carlos Simpson had provided a counterexample in 1998 to a 'theorem' by Voevodsky and Kapranov, Simpson's counterexample was disregarded in the community because of Voevodsky and Kapranov's eminence.

To develop a 'foundation' with the goal of providing a computer assistant for proof verfication, Voevodsky introduces (what I call) the *foundational desiderata* in (Voevodsky, 2010). The following presentation is made precise based on Voevodsky, (2010; 2014). A theory T satisfies the desiderata for a univalent foundation just in case

(FD1) there is a formal deductive system $T' \subseteq T$ such that $L_{T'} = L_T$,

(FD2) for any T'-sentence φ, φ can be interpreted as some existing mathematical concepts in T, and

(FD3) T' can be implemented on a computer.

The first desideratum was put forward so that the theory relies on a deductive reasoning that is similar to the deductive reasoning found in mathematics. While one might question whether it is possible to formalise mathematical deductive reasoning, I do not wish to question this in this dissertation. I simply highlight it as a desideratum for achieving Voevodsky's goal of computer implementation of mathematical proofs. The second desideratum claims that the sentences of the formal system be interpreted by some existing mathematical concepts. This is so the working mathematicians who are less familiar with the formal system are able to work with the formal language. Otherwise, the theory presented would require all those who wish to work in the new theory to first learn the formal language. By allowing an interpretation of the sentences with the existing mathematical concepts, any mathematician already familiar with the concepts can immediately practice in this new theory. The third desideratum states that the formal system must be apt to be implemented in a programming language. This means that there is a programming language that corresponds with the relevant formal system such that a proof in the formal system can be written in the programming language, and be checked by the program. This is the essential feature that must be met by the theory, so that proofs in the theory can actually be verified by a computer. By interpreting the sentences in the formal system in terms of existing mathematical concepts, the computer program that runs the implemented programming language can inform the mathematician whether the proof is correct or not. Therefore, if all three desiderata are met by a theory T, then mathematicians who work with the concepts interpreting the sentences of T can verify whether their proofs are correct without relying on other expert opinion or the reputation of other mathematicians.

As a univalent foundation, HoTT was introduced by the UFP (2013). HoTT is a theory that uses the language of Martin-Löf type theory (MLTT), so a proof in HoTT can be written in MLTT – this satisfies the first desideratum. Then for any sentence in MLTT, which are called 'judgements', the univalence axiom provides the homotopical/topological interpretations of the judgements. For example, a judgement of the form $a : A$ is interpreted as a point a in a topological space A. And, since MLTT was already implemented in the programming language Coq (see, e.g., Coquand and Coquand, 1999; Bertot and Castéran, 2013), this satisfies the third desideratum. So, Voevodsky started working on a univalent foundation in Coq, by sharing the code on GitHub, which can be found in UniMath/Foundations (2014).

To realise Voevodsky's foundational desiderata, certain mathematical results played important roles. In 2009, Awodey and Warren (2009) published a proof showing that there is a homotopy theoretic model of identity types in MLTT. This means for homotopy theorists, they could simply follow their homotopical thinking to read the identity types of MLTT. If we can generally apply a homotopical model to MLTT, then Voevodsky's desiderata could be satisfied by HoTT: let T be HoTT and T' be MLTT. In HoTT, the judgements in MLTT are interpreted topologically and/or homotopically, and MLTT is implemented on a computer.

Naturally, the requirement of logical syntax is tied with the requirement of univalence axiom. If there is an equivalence between the logical interpretation of the language L_T and another mathematical interpretation, these are treated as the same, and thus, we are free to use the logical syntax and the mathematical interpretations.

Now let us turn to the third requirement for HoTT: intrinsic homotopical content.

4.1.3 Intrinsic Homotopical Content

Generalising from the homotopical models for identity types in MLTT, HoTT treats types in MLTT as homotopy types.[4] By doing so, HoTT contains *intrinsic homotopical content*. Other univalent foundations could involve other mathematical contents, as Voevodsky's desideratum (FD2) does not specify which existing mathematical concepts can be adopted.

To interpret all types, not just identity types, homotopically, we need something stronger than the results by Awodey and Warren (2009). One reason it is possible to interpret all types is the ∞-groupoid model for type theory by Hofmann and Streicher (1998).[5] The notion of ∞-groupoids can be found in higher-order category theory. Roughly, groupoids are obtained by weakening certain conditions on groups (e.g. replacing the binary operation with a partial function). ∞-groupoids are then understood as a generalisation of groupoids, which contains an infinite structure of groupoids nested one on top of another. (See nLab authors, Oct. 2022, for more details.) If there are only n many groupoids stacked together, we call this structure 'n-groupoid'. Without going into the details of the mathematics, we can simply understand that there is a reasonable interpretation between groupoids and homotopy types: an n-groupoid is interpreted as an n-homotopy type in homotopy type theory. Thus, an ∞-groupoid is interpreted as a general homotopy type. If types can be interpreted as ∞-groupoids, and ∞-groupoids can be interpreted as homotopy types, then types can also be interpreted as homotopy types. Hence, interpreting MLTT types as homotopy types would seem appropriate, and so, HoTT contains intrinsic homotopical content as its minimal requirement.

[4] I shall explain this briefly here, and offer more details in the next section.

[5] For a more detailed account of the mathematical history of homotopical interpretation of types, see Awodey, 2012.

Given the prior mathematical results, and Voevodsky's motivation, we can understand (1) univalence axiom, (2) logical syntax, and (3) intrinsic homotopical content as the minimal requirements for HoTT. It is due to its mathematical history that HoTT satisfies these conditions. Understanding and engaging with these minimal requirements is then essential for practising HoTT. By doing so, we can enter the community of empathy with the homotopy type theorists. With these requirements in mind, I give a short introduction to topology and homotopy theory, which are necessary for the topological and homotopical interpretations. We then focus on path induction, the definition of identity in HoTT.

We shall see that, since types in HoTT are homotopically interpreted as homotopy types, and propositional equalities as paths (UFP, 2013, pp. 6–10), path induction can be thought of as characterising a property on paths. In particular, once we have fully grasped the homotopical and topological interpretations, we can see that path induction follows from the path lifting property.

4.2 Mathematical Introduction: Introduction to HoTT and Path Induction

In this section, a short mathematical introduction to HoTT will be given, so the readers can begin to practise *mathematical empathy* with the homotopy type theorists. Unfortunately, the mathematics behind HoTT is complex and might require several reads. To help the reader understand the mathematics, I also offer a pictorial interpretation of HoTT, pertaining to the topological and homotopical interpretations in section 4.2.2. The pictorial interpretation, however, should not replace the homotopical and topological interpretations. Detailed homotopical and topological interpretations will be given in section 4.4 as we look at path induction in close detail.

4.2.1 The syntax of MLTT

Roughly,[6] MLTT is a two-sorted language, consisting of *types* and *terms*. In general, I will use the upper case letters, A, B, C, and so on, to denote the types, and the lower case letters, a, b, c, and so on, to denote the terms. There are three different ways to express *judgements* in MLTT. A judgement is of one of the following forms:

- A *Type*;
- $a : A$; and
- $a \equiv b : A$.

Judgements of the first form state that A *is a type*. In addition to types, there are also terms. So $a : A$ states that a is a *term* of type A. The colon : separates the term a from the type A. We can simply read this either as a sentence, 'a is a term of type A', or as a noun phrase, 'a term a of type A'. As suggested by the form of the judgements above, a term cannot be stated independently of a type: whenever a term is mentioned in a judgement, there *must* be a type that the term belongs with. In some cases, a type A might not have a term. In that case, we say that A is *uninhabited*. If A has a term, then A is *inhabited*.[7] The third form states that a and b are terms of type A and they are *judgementally equal*. All judgements in MLTT are in one of the three forms.

Before moving onto the logical interpretation, note that there are two kinds of identities or equalities in MLTT and HoTT. I shall use 'identity' and 'equality' interchangeably, as in the HoTT literature. We have already introduced the judgemental equality $\equiv$, but path induction concerns another identity/equality, known as the *propositional identity* $=$. The judgemental equality $\equiv$ is part of the primitive judgements in MLTT, while the propositional equality $=$ is introduced later as an identity type. In some occasions, propositional equality is expressed as $\mathsf{Id}_A(x, y)$, like a predicate on terms $x, y : A$. When we turn to consider path induction, we shall discuss this in more detail.

[6] Of course, if one thinks set theoretically, MLTT *is* a two-sorted language consisting of terms and types. But for the type theorists, this is the primitive language of type theoretic logic.

[7] When I discuss the logical interpretation of types, I shall explain that MLTT is the language of intuitionistic logic. However, by assuming that for any type A, we have either A is inhabited or its negation $\neg A$ is inhabited, the logic becomes classical.

Logical interpretation of MLTT

The logical interpretation of MLTT, mentioned briefly earlier, extends the *Curry-Howard isomorphism* from the implication portion of propositional logic to the full intuitionistic system (Howard, 1980). That is, Curry-Howard for propositional logic interprets the *function type* $A \to B$ in type theory as the proposition $A \to B$ (if A then B). Hence, every type is interpreted as a proposition, and each term of the type is interpreted as a proof of that proposition.[8] So, a term f of the function type $A \to B$ is interpreted as a *proof* f of the *proposition* $A \to B$. By extending this isomorphism to first-order intuitionistic logic, the symbols $\times, +, \prod$ and $\sum$ in MLTT are logically interpreted as conjunction, disjunction, universal quantifier, and existential quantifier ($\wedge, \vee, \forall$ and $\exists$) in first-order logic.

In general, the fact that A is uninhabited does not imply that $\neg A$ (which abbreviates $A \to 0$) is inhabited: the basic logic for MLTT is intuitionistic. However, particular types B, where the type $B + \neg B$ is inhabited, will exhibit classical behaviour.

While $\prod$ and $\sum$ are interpreted as the quantifiers of first-order logic, they are *bounded quantifiers*. That is, the quantifiers always apply to terms in a *given* type, which bounds the term. So to say, e.g., 'for any a of type A, $P(a)$', we write $\prod_{a:A} P(a)$, which is a type.

In the above example, P is interpreted as a predicate in first-order logic. This is only possible by the logical interpretation of types. In general, P is a *type family* or *family of types*, and is expressed as $P : A \to \mathcal{U}$ for a type A. The $\mathcal{U}$ denotes the *type universe* containing all of the types. Hence the judgement of the form A *Type* can be re-expressed as $A : \mathcal{U}$.[9] So the family $P : A \to \mathcal{U}$ takes a term a of type A to output $Pa : \mathcal{U}$. With the type universe notation $\mathcal{U}$, we can treat $Pa : \mathcal{U}$ be equivalent to the judgement that Pa *Type*. I shall often use the notation $P(a)$ for Pa to highlight that a is

[8] By interpreting the terms as proofs for the propositions, when the type theoretic judgements are computer implemented, we can treat the proofs as the execution of the the given program.

[9] Generally, the hierarchy of type universes are defined as follows:

- $\mathcal{U}_m : \mathcal{U}_n$ for all $m < n$;
- if $A : \mathcal{U}_m$ and $m \leq n$, then $A : \mathcal{U}$; and
- if $\vdash A : \mathcal{U}_n$ and x is a new variable, then $x : A$ is a judgement. The type universe is then hypothesised as the universe $\mathcal{U}$ such that for any n, $\mathcal{U}_n : \mathcal{U}$ See UFP (2013:§A.1.1) for more details.

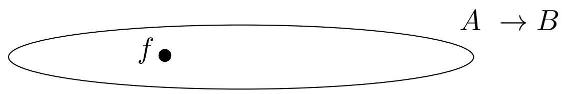
a term from a different type and P is taking the term a.[10] To explicate what a family of types is, consider the following example. Let $\mathsf{even} : \mathbb{N} \to \mathcal{U}$ be a family of types, representing the predicate 'is even' on natural numbers. In this, even takes a natural number $n : \mathbb{N}$ to a new type, $\mathsf{even}(n)$. Since a type is interpreted as a proposition by Curry-Howard, $\mathsf{even}(n)$ is interpreted as the proposition 'n is even'. Hence a family of types can be treated as a predicate for terms of a given type; and the truth of the proposition $\mathsf{even}(n)$ is determined by whether the type $\mathsf{even}(n)$ is inhabited.

4.2.2 Logical and Pictorial Interpretations

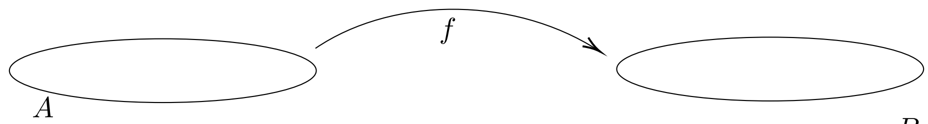
I shall provide an informal introduction to the homotopical interpretation here, which I shall call a 'pictorial interpretation' of HoTT. We can use the pictorial interpretation of HoTT, as it will carry the intrinsic homotopical content shared by the mathematicians. Later, in section 4.4, I give a more thorough homotopical interpretation of types, with a particular focus on path induction.

In general, types in HoTT are pictured as *(topological) spaces* with certain homotopical features. Throughout this discussion, I shall picture a space as an oval. A term of a given type is interpreted as a *point* in the given space. Points are depicted as dots within the oval figure. As the types become more complicated, the pictures of the spaces become more complicated. For example, given the function type $A \to B$ with the term that is a function $f : A \to B$, there are at least three different ways we can picture the judgement $f : A \to B$. One is by pictorially interpreting the given judgement directly as 'f is a point in space $A \to B$'. However, for $A \to B$ to be a type, we must have A and B be types. So, we can picture A and B as spaces, and f as a map between the spaces – depicted as an arrow. A third picture is obtained by treating the type universe $\mathcal{U}$ as a type and A and B are terms of the type $\mathcal{U}$. Thus we can picture A and B as points in the space $\mathcal{U}$. In this picture, the term f of type $A \to B$ is then pictured as a *path* from A to B within the space $\mathcal{U}$. We can simply interpret a path to be an arrow starting from one point and going to a (same or different) point within the given space.

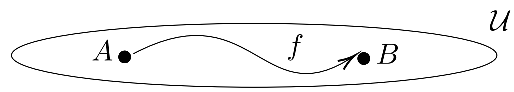
[10] In type theory, every term is defined within a type. Hence it is not necessary to use $(,)$ since the application of the types are always read left to right with respect to a given type. The notation $P(a)$ is used to make the notation easier for the reader to follow. But what is important is that the term is appropriate for the given type.



$A \to B$ as a space and f as a point



A and B as spaces and f as a map.



$\mathcal{U}$ as a space, A and B as points and f as a path between the points.

Figure 4.2: $f : A \to B$

The reason we can rely on the pictorial interpretation is that mathematical results show the types in MLTT can be interpreted as *homotopy types* (Awodey and Warren, 2009; Hofmann, 1998).[11] Since homotopy types are also spaces in homotopy theory, it is safe to picture types in HoTT as spaces.

The mathematical content of types is particularly interesting for identity =. Given the identity type $x =_A y$ where x and y are terms of a type A, any term $p : x =_A y$ is topologically interpreted as a *path* from x to y.[12] Homotopically, p is a homotopy equivalence from spaces x to y, so p is a continuous map from x to y with certain properties. With this topological/homotopical interpretations of identity =, we have the following pictures (Figure 4.3) available. While these pictures are similar to the pictures for the function $f : A \to B$, the underlying homotopical content between the term $p : x =_A y$ is different from the term $f : x \to y$. This is because the function term $f : x \to y$ is homotopically interpreted as a continuous map without the assumption of the property that it behaves like a homotopy equivalence (UFP, 2013, p. 6). We shall use pictures to help with our topological/homotopical reasoning, but it is important not to treat the pictures as a definitive description of our topological/homotopical reasoning. They do not replace the mathematical content and details, but rather supplement them.

[11] I shall provide the definition of homotopy types in section 4.4.

[12] We cannot immediately assume the usual properties of identity under the homotopical interpretation, because a term $p : x =_A y$ is interpreted as a path from x to y, while a term $q : y =_A x$ is interpreted as a path from y to x. However, it can be proven that identity = is an equivalence relation.

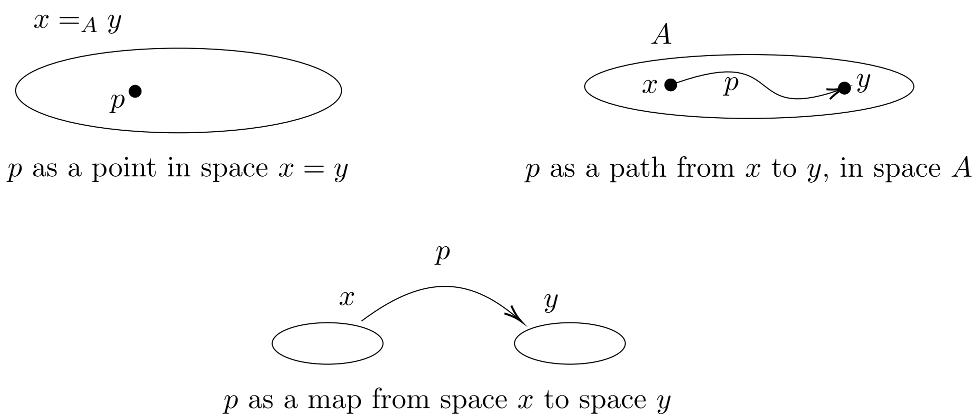


Figure 4.3: $p : x = y$

In section 4.4, I shall provide the mathematical definitions needed for understanding the topological/homotopical interpretations of types. For now, I shall rely on the logical and pictorial interpretations to explain path induction.

4.2.3 What is Path Induction?

Path induction is described as the elimination rule of identity types (UFP, 2013, p. 48). The introduction rule states that for any given type A, the identity type is characterised as $\prod_{x,y:A}(x =_A y)$, and for any term $a : A$, the identity type has the following term $\mathsf{refl}_a : a =_A a$.[13] I will occasionally drop the $\prod_{x,y:A}$, and simply refer to $(x =_A y)$ as the identity type. Note that, while $\prod_{x,y:A}(x =_A y)$ is the identity type for arbitrary terms $x, y : A$, when we define the term refl_x, we can focus on the particular identity type $x =_A x$. The index A on the propositional identity $a =_A a$ means the term a is from the type A. Here, we can logically interpret the $\prod$ as the universal quantifier $\forall$. Thus the type $\prod_{x,y:A}(x =_A y)$ is interpreted as the statement that for any $x, y : A$, $(x =_A y)$ is a type. Then path induction is the elimination rule that removes the propositional identity $=$ from the identity type $\prod_{x,y:A}(x =_A y)$. The term 'induction' in 'path *induction*' comes from the computational properties of MLTT, in which certain types are called 'inductive':

[13] The introduction rule must show how a type is introduced as well as its terms.

> Intuitively, we should understand an inductive type as being freely generated by its constructors. That is, the elements of an inductive type are exactly what can be obtained by starting from nothing and applying the constructors repeatedly. (UFP, 2013:§5.1)

Consider the natural number type, $\mathbb{N}$, which is defined by the term $0 : \mathbb{N}$ and a function $\mathsf{succ} : \mathbb{N} \to \mathbb{N}$. Starting with $0 : \mathbb{N}$, we can apply succ repeatedly to obtain all terms in the type $\mathbb{N}$. So the type $\mathbb{N}$ is inductive. Now take the identity type and note that its terms are inductively generated by $\mathsf{refl}_.$. This is how the introduction rule is given. The elimination rule is characterised by *induction principle*, among other rules (see UFP, 2013:§6.2, for discussion), and it generally informs us how we can *use* the terms of the type (UFP, 2013:§A.2.4). So, we can understand path induction as telling us how we can *use* the terms in the identity type.

I shall first give the formal characterisation of path induction, then turn to the logical and pictorial interpretations to explain it.

Path induction is stated as the following in the language of MLTT.

Definition 4.2 (Path induction)**.** *Let A be a type, and*

$$C : \prod_{x,y:A} (x =_A y) \to \mathcal{U}$$

be a family of types, and let a function

$$c : \prod_{x:A} C(x, x, \mathsf{refl}_x)$$

be such that $c(x) : C(x, x, \mathsf{refl}_x)$. Then there is a function

$$f : \prod_{x,y:A} \prod_{p:x=y} C(x, y, p),$$

such that

$$f(x, x, \mathsf{refl}_x) \equiv c(x).$$

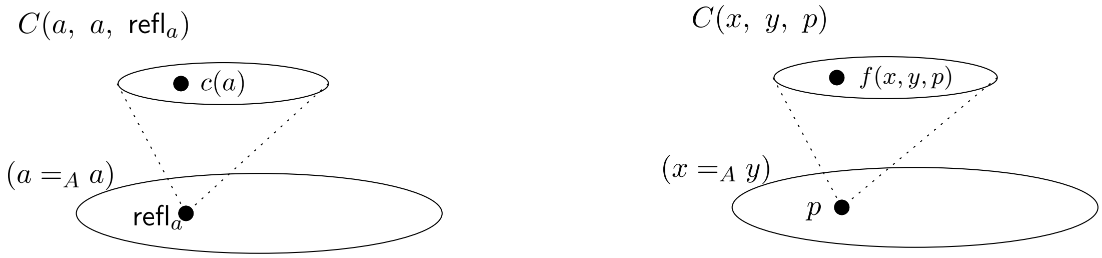


Figure 4.4: Type Family C in Path Induction

Explicating the statement with the logical interpretation, we can see that path induction claims the following. We are given a type A (or a set A, if one prefers), and a predicate

$$C : \prod_{x,y:A} (x = y) \to \mathcal{U},$$

which takes a triple (x, y, p) where p is a proof that $x = y$, for $x, y : A$. We are also given a dependent function

$$c : \prod_{x:A} C(x, x, \mathsf{refl}_x),$$

which takes an $x : A$ and outputs a proof $c(x)$ of the statement $C(x, x, \mathsf{refl}_x)$, where refl_x is a proof that $x =_A x$. From these given conditions, path induction informs us that there is a way to generalise the proof $c(x)$ of $C(x, x, \mathsf{refl}_x)$ to the proof $f(x, y, p)$ of $C(x, y, p)$, where p is a proof of $x =_A y$.[14]

Now we can further clarify path induction with the topological terminologies, such as 'points', 'paths', and 'spaces', and it can be pictured as in Figure 4.4.

To begin, we are given a type A and a family of types $C : \prod_{x,y:A}(x =_A y) \to \mathcal{U}$. So A can be topologically interpreted as a space. Since C is a family from $\prod_{x,y:A}(x =_A y)$ to the type universe $\mathcal{U}$, we can picture $C(x, y, p)$ as a space, as in the right diagram in Figure 4.4.

Importantly, $C(x, y, p)$ is a space that depends on p. For any arbitrary points $x, y : A$ and $p : x =_A y$ is pictured as a *path* in space A, but it can also be pictured as a point p in space $(x =_A y)$. This space $(x =_A y)$ is called a '*path space*'. Similarly $\prod_{x,y:A}(x =_A y)$ is topologically interpreted as a

[14] While $\prod_{x,y:A}(x =_A y)$ is a type, this does not mean that everything in A is identical. For this to be true, the identity type needs to be inhabited. Logically interpreted, this means there would have to be a proof of the proposition $\prod_{x,y:A}(x =_A y)$.

path space consisting of points, which are paths from an arbitrary point to an arbitrary point in space A. You can see, in Figure 4.4, the path space $(x =_A y)$ of arbitrary paths (on the right), and the path space $(a =_a)$ of paths from a to itself (on the left).

In path induction, we are given a function c that takes a term $x : A$ to $c(x) : C(x, x, \mathsf{refl}_x)$. Then we can picture $c(a)$ for an arbitrary point $a : A$ as a point in $C(a, a, \mathsf{refl}_a)$, as in the diagram on the left side of Figure 4.4. Since $C(a, a, \mathsf{refl}_a)$ depends on refl_a, which in turn depends on a, we also picture refl_a as a point in the path space $(a =_A a)$ in Figure 4.4.

We can summarise the given conditions for path induction with the logical and pictorial interpretations: we are given a space A, a predicate C on paths in A, and, for any point a in A, we have a proof $c(a)$ that shows C holds of the reflexive path refl_a. We can also depict $C(a, a, \mathsf{refl}_a)$ as a space with a point $c(a)$ (as in left diagram in Figure 4.4) via the pictorial interpretation.

Now, path induction states that from these given conditions, we have a function $f : \prod_{x,y:A} \prod_{p:x=y} C(x, y, p)$, which generalises from c. Recall that we can logically interpret the type as the following proposition: 'for any x and y of A, for any proof p showing that $x = y$, the predicate C holds for the triple (x, y, p)'. This means that $f(x, y, p)$ is a proof of $C(x, y, p)$. By further topologically/pictorially interpreting $C(x, y, p)$ as a space, we find that $f(x, y, p)$ is a point in the space $C(x, y, p)$ – (as in right diagram in Figure 4.4). So f is a function that maps any triple of points (x, y, p) to a point in the space $C(x, y, p)$.

Unsurprisingly, if we fix both x and y to be a in A, and fix p as the path refl_a, we have $f(a, a, \mathsf{refl}_a)$ is a point in $C(a, a, \mathsf{refl}_a)$. Recall that the given condition of path induction stated that $c(a)$ is a point in $C(a, a, \mathsf{refl}_a)$, and note that this function f is specified to be such that $f(a, a, \mathsf{refl}_a) \equiv c(a)$. As the diagrams in Figure 4.4 suggest, the function f generalises the function c to apply to any arbitrary path. But with a homotopical interpretation, we shall be given additional structures on the type families such that the role of function f becomes clearer.

Recall the minimal requirements of HoTT, which are (1) HoTT contains an intrinsic homotopical content, and (2) HoTT can be machine-implemented. The reason why HoTT is a theory written in MLTT is that it is a feature of type theory that the definitions can be 'executed' as computer programs – see UFP (2013, p. 6). While the logical interpretation of types offers an explanation for path induction, it does not in itself explain the homotopical content in path induction. Hence the complete explanation of path induction can be given only when the logical interpretation and the topological and homotopical interpretations are all provided.

In the remainder of this chapter, I shall use a pictorial interpretation of types that is similar to that used by topologists and homotopy theorists. This will give an informal introduction to the topological/homotopical ideas without the mathematical definitions. (For examples of diagram usage in HoTT, see UFP, 2013 pp. 69, 73; and, for algebraic topology, see Hatcher, 2002, pp. 2, 4, 5, 8, 9, ...) However, as I mentioned earlier, the pictorial interpretation should not replace the mathematical definitions. Thus, I also provide the mathematical definitions necessary for fully understanding the topological and homotopical interpretations of path induction.

Before going into the mathematical details behind path induction, let us look at the philosophical discussions concerning path induction.

4.3 Philosophical Interlude: Justifications for Path Induction

To begin, I shall briefly explain here why the question of justifying path induction might be appropriate for philosophy of mathematical practice. Recall the view of Gray and Ferreirós (2006), that philosophy of mathematical practice should concern itself with philosophical questions which are 'perceived as relevant and interesting' for practising mathematicians (2006, p. 10). The practising mathematicians, in this case, are homotopy type theorists, or the supporters of the UFP.

For homotopy type theorists, path induction is described as the definition of identity in HoTT (Univalent Foundations Program, 2013). To be more precise, it is a principle that shows how we can eliminate identity types. So, is justifying path induction something that is relevant and interesting for the homotopy type theorists? And what does it mean to *justify* a definition in mathematics? And what *kind* of definition is path induction? I shall answer these questions in reverse order. So, I first answer the question of what kind of definition path induction is, then consider what kind of justification it needs, and then finally, I show that such a justification would be of interest to the homotopy type theorists.

Definitions in mathematics or logic can be (but are not limited to being) of one of the following kinds: *extrinsic* definitions; *intrinsic* definitions. An *extrinsic* definition introduces a new *definiendum* in a formal or mathematical language. Usually, the definition is expressed in the following form

$$\forall x \; [P(x) \leftrightarrow \varphi(x)]$$

where P is a new predicate for the *definiendum* and φ is an open formula in a language L that does not contain the predicate P. For example, the definition of a group is an extrinsic definition since it introduces the *definiendum* 'group' in terms of the corresponding group axioms. An *intrinsic* definition, on the other hand, renders an informal notion into a formal language. Examples include second-order axioms of Peano arithmetic or Frege's definition of numbers.

Path induction, however, is neither an explicit nor an implicit definition. As a definition, it does not introduce a new predicate or symbol, hence it is not an explicit definition. Nor is it an implicit definition since it does not appeal to some informal notion of identity that is being formally rendered. Instead, path induction renders a general notion of identity into a very particular mathematical setting, i.e. into HoTT. I shall call definitions this kind 'contextualised', since they render a general notion into a particular mathematical context.[15]

Given that path induction is a contextualised definition, what kind of justification would it require? Let us briefly consider the kinds of justifications required for extrinsic and intrinsic definitions before turning to consider contextualised definitions. Broadly, a justification can be understood as an argument that supports our commitment. In epistemology, a justification is usually applied to beliefs, but in the context of mathematical definition, it is not clear whether we *believe* definitions. So I say that we have certain commitments to definitions, whether it be a belief or not. A justification for an extrinsic definition ought to show that the definition has certain characteristics. For example, the

[15] This is similar to the notion of *recasting* found in Panza and Sereni, 2016. Roughly, one can *recast* parts of mathematics in a particular theory if those parts are restated in that theory. Along this line, contextualised definitions are those which are recast in a particular mathematical theory.

definiendum P ought to be consistent with respect to the relevant L-theory, so that extending L with P does not result in an inconsistent theory. There may be other relevant characteristics, such as *fruitfulness* in producing results: i.e. the *definiendum* is apt to bring about new results in the $L \cup \{P\}$-theory. An intrinsic definition ought to also be consistent with the existing theories, but a justification also ought to show that the definition accurately characterises the informal notion. If a formal rendering does not accurately capture the informal notion in question, then it cannot be considered an acceptable definition.

For a contextualised definition, a different kind of justification is required. In particular, the justification ought to show what mathematical contexts are required for the rendering of the general notion. For example, take the notion of a continuous map. In the context of topology, a real continuous map $f : \mathbb{R} \to \mathbb{R}$ is defined in terms of open sets in $\mathbb{R}$ with an appropriate topology. But in the context of real analysis, a continuous map is defined by the ϵ-δ definition. These definitions are justified appropriately, since the mathematical contexts in which the definitions are given are clear.

When we treat path induction as a contextualised definition, a justification for it ought to clarify exactly what mathematical contexts are required, not only to state the definition, but also to understand it. While the homotopy type theorists state that homotopy theory is involved in understanding the definition, the type-theoretic expression of path induction does not make this obvious, and they do not explain (when giving the definition) exactly what homotopy theoretic notions are being involved in understanding path induction. So I believe that path induction is in need of a justification, because the mathematical context of the definition is not clear. Thus, I aim to clarify what path induction is by focusing on its mathematical context.

The need for clarification of path induction can also be found in the informal discussions among the mathematicians on GitHub (2013). GitHub is an online depository used by computer scientists and mathematicians. One of the leading mathematicians of HoTT, Vladmir Voevodsky, created a GitHub depository where mathematicians can develop HoTT collaboratively online (see Voevodskey, 2014). Given the relevance of GitHub in the development of HoTT, engaging in the informal discussions on GitHub would be 'engaging with the complexities of real mathematics' (Gray and Ferreirós, 2006, p. 6). Thus, I refer to one of the discussion found there to explain why justifying path induction is relevant and interesting for homotopy type theorists. Although the evidence I provide here comes from an informal online discussion between Mike Shulman and Robert Harper (two of the authors of the HoTT book), if philosophy of mathematical practice is to take mathematical *practice* seriously, then it should consider practice that goes beyond university (or other formal) settings. This should include consideration of discussions at conferences, online, and informal discussions among mathematicians. In some ways, online discussions are the modern variations of the letters exchanged between mathematicians of the past. The difference is that the historical letters were addressed to only a small number of scholars, but these online discussions are publicly available for everyone.

On the forum discussion, we find Mike Shulman and Robert Harper discussing the nature of path induction. When a GitHub user raises a question about path induction, Robert Harper writes:

> your concerns about the nature of path induction are understandable, and are related to the absence of a computational interpretation of the theory in the [Homotopy Type Theory book 2013]. (2013:User name: RobertHarper (Robert Harper))

Harper's comment suggests that path induction can be understood because of the computational interpretation of types. However, Shulman replies to Harper's comment as follows:

> As you can see, there are differences of opinion even among the authors of the book. Bob's [Robert Harper's] is one way of looking at it. I maintain that one can be completely satisfied with path induction as resulting from an *inductive definition of identity types*, together with HITs [Higher Inductive Types] and univalence, whether or not there is a computational interpretation of HoTT [...] (2013:User name: mikeshulman (Mike Shulman), emphasis added)

What is clear from Shulman's comment is that path induction should be understood as being based upon the 'inductive definition of identity types'. That is, it is not necessary to provide a computational interpretation of path induction, as Harper claims. Although it appears that both Harper and Shulman agree *that* path induction is legitimate for practice, the difference in their opinions implies that they do not agree on how they understand the definition. This discussion suggests that some philosophical clarification is needed for path induction.[16]

Some might argue that this is a *sociological* reason, rather than a *philosophical* one. However, if we consider philosophy of mathematical practice seriously, and if this informal online discussion between Harper and Shulman shows that there is a disagreement among the authors of the Univalent Foundations Program (2013), then this concern should be taken seriously from the perspective of *philosophy* of mathematical practice. And hence, some philosophical clarification for path induction is relevant for the homotopy type theorists.

Now, let us turn to the existing philosophical justifications for path induction, in particular, those endorsed by Ladyman and Presnell (2015) and P. Walsh (2017). Before discussing their views in more detail I would like to emphasise that I am not here going to endorse either of their views. In fact, I offer a criticism of Ladyman and Presnell's motivation. Ladyman and Presnell (2015) offer a justification (in their own sense) for path induction by claiming that HoTT must be an 'autonomous foundation' of mathematics. According to Ladyman and Presnell, an *autonomous* foundation is a mathematical theory that can be understood without appealing to any other mathematical theories. I will suggest that this goes against the motivation behind HoTT, and thus does not seem appropriate for path induction. P. Walsh (2017) offers a different view which is based on a philosophical position called 'inferentialism'. By supposing that a definition of a logical connective can be given by its inference rules and further supposing that the rules are 'harmonious', Walsh argues that path induction follows from harmonious inference rules of identity. I will suggest shortly that Walsh's justification is more appropriate for path induction than Ladyman and Presnell's, but I will also offer an alternative justification. Here, I shall explain and discuss both arguments briefly.

[16]While Harper's and Shulman's views seem to differ, I interpret this to show that there are different sub-communities of empathy among the homotopy type theorists. The community of empathy I highlight in this chapter is a particular one that follows Voevodsky's motivation.

4.3.1 Ladyman and Presnell's account of HoTT as an Autonomous Foundation

Ladyman and Presnell (2015) argue that a foundational theory must be autonomous in the sense of both conceptual autonomy and justificatory autonomy as defined by Linnebo and Pettigrew (2011, p. 241):

- A theory T_1 has *conceptual autonomy* with respect to T_2 if it is possible to understand T_1 without first understanding T_2, and
- T_1 has *justificatory autonomy* with respect to T_2 if it is possible to motivate and justify the claims of T_1 without appealing to T_2, or to justifications that belong to T_2.

Ladyman and Presnell suggest that HoTT ought to be autonomous both conceptually and justificatorily with respect to all of mathematics. Thus one should understand HoTT without understanding any other mathematical theory and one should motivate and justify the claims in HoTT (e.g. definitions in HoTT) without appealing to another mathematical theory:

> An autonomous foundation must therefore use only concepts that can be pre-mathematically understood, and rules that can be pre-mathematically motivated. (Ladyman and Presnell, 2015, p. 388)

Hence, (they argue that) path induction, as a definition in HoTT, can be understood and justified with the 'basic principles' of the 'pre-mathematical' concept of identity (2015, p. 388; see also Mayberry, 1994).[17]

One problem I find with Ladyman and Presnell's position is that their claim of autonomy of HoTT goes against the way path induction is understood in HoTT in terms of paths and spaces. They claim that the way path induction is justified in HoTT, as it stands, is not sufficient to show that HoTT is an autonomus foundation in their sense:

[17] P. Walsh (2017) argues that the basic principles of identity are not so 'pre-mathematical', because of the difficulty in understanding them without any mathematical background. Since this issue has already been touched on by Walsh, I shall focus on a different issue here.

> In the presentation in the HoTT Book identifications such as $a = b$ are thought of as paths in (homotopy) spaces, and the clearest and most accessible explanation and justification of path induction depends upon intuitions arising from homotopy theory[. . . .] If this were the only way of justifying path induction then that would of course undermine any claims for HoTT as an autonomous foundation for mathematics. (2015, p. 389)

While Ladyman and Presnell are committed to their notion of autonomy, I believe HoTT can be autonomous in its own sense, although I do maintain that, if HoTT is to be recognised as an autonomous foundation, we must clarify the sense in which HoTT is an autonomous foundation. This will involve fully understanding what identity is, as it is intended by the homotopy type theorists in terms of 'paths in (homotopy spaces)' (2015, p. 389). This is precisely what *Besinnung* aims to do. By *Besinnung*, we first clarify the meaning of identity in HoTT, and then we can further clarify HoTT's motivational goal, which includes that HoTT has a homotopical autonomy.

For instance, recall that the UFP considers HoTT as a theory that involves

> [. . .] a new conception of foundations of mathematics, with intrinsic homotopical content, [. . .] and convenient machine implementations, which can serve as a practical aid to the working mathematician. (UFP 2013, p. 3)

By 'homotopical content', they mean that

> Homotopy type theory (HoTT) interprets type theory from a homotopical perspective. (UFP, 2013, p. 3)

If HoTT is intended to be understood from the homotopical perspective, and that is included in its sense of foundation, then this is contrary to Ladyman and Presnell's claim that HoTT ought to be autonomous. Both homotopical and pre-theoretic ways of understanding path induction might be available, but HoTT is not an autonomous foundation in Ladyman and Presnell's sense. A better justification for path induction should actually show how this *homotopical perspective* explains path induction, and re-establish in what sense it still offers to be an autonomous foundation (if it does).

The interpretation of types as homotopical spaces is a *feature* of HoTT, rather than a *flaw*. To reject this is to refuse to empathise with the homotopy type theorists. Path induction is a definition *in* HoTT, and HoTT *necessarily* has homotopical content. At best, their justification does not actually concern path induction in HoTT. Rather, they are engaging with the existing discussions about justifying identity elimination rule in Martin-Löf type theory (see, e.g., Klev, 2019b).

Having challenged Ladyman and Presnell's motivation, I turn to Walsh's justification for path induction.

4.3.2 Walsh's Inferentialist Justification

Walsh's argument for path induction is motivated by the inferentialist view in philosophy of logic: 'the meaning of the logical concepts [...] is determined by the rules governing them' (P. Walsh, 2017, p. 1). Given this, inferentialists ask whether an inference rule is at least *meaning-bearing* (P. Walsh, (2017, p. 2). Walsh's goal then is to show that path induction is a meaning-bearing inference rule for identity. In particular, Walsh writes that his justification for path induction is 'established through a notion of *harmony* that ensures the rules are balanced and conservative with respect to provability' (2017, p. 4).

The need for harmony between the introduction rule and the corresponding elimination rule can be clarified by looking at Prior's example of '*tonk*' (Prior, 1960).

$$\frac{A}{A \textit{ tonk } B}\, \textit{tonk}\text{-I} \qquad\qquad \frac{A \textit{ tonk } B}{B}\, \textit{tonk}\text{-E}$$

When considered individually, the introduction and the elimination rules appear unproblematic. The introduction rule behaves similarly to the disjunction introduction rule, while the elimination rule behaves like the conjunction elimination rule. However, when we apply the introduction rule followed by the elimination rule, we can derive B from A, which might as well be independent statements. In this sense, the introduction and elimination rules of *tonk* are not balanced, as the elimination rule does not reverse the introduction rule, and they are not conservative with respect to provability, as we are able to prove new theorems such as B with the inference rules for *tonk*.

Walsh's argument is, then, that path induction can be justified because it is the elimination rule of identity types, and it is *categorically harmonious* to the corresponding introduction rule. Going into the details of categorical harmony is beyond the scope of this dissertation but, if Walsh's notion of categorical harmony is appropriate for an inferentialist notion of harmony (see, e.g., Read, 2016 or Klev, 2019a), then Walsh's argument does indeed show that path induction is harmonious to its corresponding introduction rule.

Walsh's argument is an interesting one and might even be appropriate from the perspective of homotopy type theorists. For instance, the UFP says

> one [virtue] of type theory is its computable character. In addition to being a foundation for mathematics, type theory is a formal theory of computation, and can be treated as a powerful programming language. From this perspective, the rules of the system cannot be chosen arbitrarily the way set-theoretic axioms can: there must be a *harmony* between them which allows all proofs to be 'executed' as programs. (UFP, 2013, p. 10; emphasis added)

What this passage suggests is that there is a community of homotopy type theorists, who would focus on this 'computable character' of HoTT. This is perhaps what Robert Harper was referring to earlier with 'the computational interpretation of the theory' (GitHub 2013).

Despite the possibility that Walsh's inferentialist argument might give an appropriate justification in relation to the HoTT community, I shall propose a different justification. My focus is to look at the mathematical context (of homotopy theory and topology) that is required to understand path induction, rather than looking at path induction as an identity elimination rule for the type-theoretic syntax. In this sense, my justification is a novel one that highlights the mathematical notions that are needed to understand path induction, and hence to understand identity in HoTT.

In the next section, I present the details of the mathematics required for the topological and homotopical interpretations of HoTT. Then we can fully understand the mathematical contexts involved in understanding path induction.

4.4 Topological and Homotopical Interpretations of HoTT

In this section, I introduce the mathematical definitions required to engage with the topological and homotopical interpretations of HoTT, with a particular focus on path induction. I shall also define what a homotopy equivalence is, and thus how identity is homotopically interpreted as a homotopy equivalence (Definition 4.8). With $p : x = y$ interpreted as a homotopy equivalence p from space x to space y, we can also interpret the identity type $x = y$ as a *homotopy type* (Definition 4.9). After providing the mathematical definitions, in Section 4.4.2, I briefly explain how we can use these definitions for homotopically interpreting types in HoTT. This interpretation helps us to understand what identity is in HoTT.

Recall that we are given a type family in path induction. The topological/homotopical interpretations tell us that a type family must be interpreted as a fibration (UFP, 2013, §3.3). In order to fully understand what that means, some mathematical premliminaries are necessary. I shall begin by defining paths and homotopies, and building from these notions to define what homotopy equivalences are (Section 4.4.1) and to show how a fibration can be defined (Section 4.4.3). Throughout the rest of the chapter, I shall use bold-font to highlight the mathematical term being defined.

Definition 4.3 (Path in topology)**.** *Let X be a topological space, and x and y be points in X. $p : [0, 1] \to X$ is a **path** from the starting point x to the end point y just in case p is a continuous map where $p(0) = x$ and $p(1) = y$.*

Simply put, a path is a continuous map from the interval $[0, 1]$ to a topological space, and the starting and end points of a path are points in the topological space. We can define certain properties using this notion of path. I shall define two of these properties here: the path lifting property and homotopies between paths. Eventually, we shall define what the *homotopy lifting property* is. But, in order to do so, we need to understand what homotopies are and what the lifting property is about.

Definition 4.4 (Path lifting)**.** *Let X be a topological space with a point x. Let $p : [0, 1] \to X$ be a path such that $p(0) = x$. Given a map $\pi : C \to X$ such that there is a point $\tilde{x} \in C$, where $\pi(\tilde{x}) = x$, we say that π has the **path lifting property with respect to** p just in case there is a path $\tilde{p} : [0, 1] \to C$ such that $\pi \circ \tilde{p} = p$.*

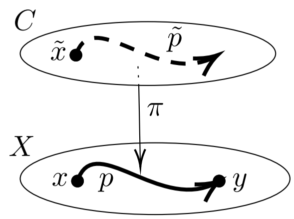


Figure 4.5: Path lifting

The definition tells us that a path p in the given space X can be lifted onto the space C via a projection map $\pi : C \to X$. Intuitively, as the project map takes points from C to X, the path in X is being lifted along this projection into the space C. This projection map π preserves the starting point of paths, so if $\tilde{p}(0) = \tilde{x}$ is the starting point of the lifted path, $\pi(\tilde{p}(0)) = p(0)$. See Figure 4.5. Since the whole path p is lifted into the space C, there must be an end point of the path $\tilde{p}$. Later, I shall show that this end point of the lifted path defines path induction in HoTT. In general, π has the **path lifting property** just in case it has the path lifting property with respect to *any* path p.

Another property that concerns paths in topology is homotopy. Understanding what a homotopy is is essential for understanding the homotopical interpretations in HoTT. Generally, a homotopy can be defined as a continuous map between continuous maps, but, for now, I define it only as a relation between paths. In the definition, we find that a path p_i is indexed with an $i \in [0, 1]$. This refers to continuum many paths, each indexed by i.

Definition 4.5 (Homotopy between paths)**.** *Let X be a topological space, and let each $p_i : [0, 1] \to X$ be a path such that $p_i(0) = x$ and $p_i(1) = y$ for all $i \in [0, 1]$. A* ***homotopy between paths*** *p_0 and p_1 is a continuous function $H : [0, 1] \times [0, 1] \to X$ such that*

$H(0, t) = p_0(t);$

$H(1, t) = p_1(t);$

$H(s, 0) = x,$ *and* $H(s, 1) = y.$

We say that p_0 and p_1 are ***homotopic*** *whenever there is a homotopy between them.*

The diagram (Figure 4.6) shows two paths $p_0, p_1 : [0,1] \to X$ that share the starting and end points. And $H : [0,1] \times [0,1] \to X$ is the homotopy (denoted as $\Rightarrow$) between the two paths p_0 and p_1. Intuitively, a homotopy indicates that there are many continuous paths between p_0 and p_1 such that p_0 can be morphed into p_1. Thus, a homotopy H can be visualised as if it were *filling the space* between the paths. So the purple area within the space X is filled by the homotopy H.

Homotopies between paths also define a relation $\simeq$ such that $p_0 \simeq p_1$ just in case there is a homotopy between p_0 and p_1. This relation is an equivalence relation (see Hatcher 2002:Proposition 1.2), so we can define an equivalence class of paths $[p_0]_{\simeq}$. Later we shall define homotopies between *maps* and a similar equivalence relation will be defined. This is necessary for defining *homotopy types*, and thus for the homotopical interpretation of types in HoTT. I use the double arrow $\Rightarrow$ notation for homotopies from now on.

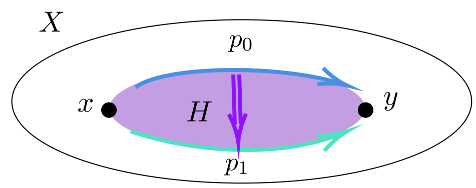


Figure 4.6: Homotopy between paths

Thus far, we have defined two properties concerning paths: path lifting and homotopies between paths. A map π satisfies the path lifting property just in case π lifts the path p to $\tilde{p}$. A homotopy 'fills the space' between two paths p_0 and p_1. Since a homotopy defines an equivalence relation, for any two paths that are homotopic, they are considered the 'same' (as in 'homos' in Greek).

We can also extend these properties to continuous maps. And, furthermore, we can define a homotopy lifting property, just as we had defined a lifting property for paths. For now, let us define a homotopy between maps.

Definition 4.6 (Homotopy between maps)**.** *Let X and Y be topological spaces. Let $f, g : X \to Y$ be continuous maps. A* ***homotopy between maps*** *f and g is a continuous map $H : X \times [0,1] \to Y$ such that, for all $x \in X$,*

$$H(x,0) = f(x) \qquad \textit{and} \qquad H(x,1) = g(x).$$

If there is a homotopy between f and g, we say that they are ***homotopic*** *to each other.*

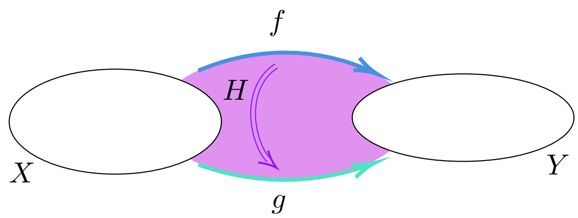


Figure 4.7: Homotopies between maps

Just as we described a homotopy between paths to be 'filling the space' between the paths, we can visualise the homotopy between maps in the same way. However, instead of filling the given *space*, the homotopy is filling something outside the spaces X and Y – in some ways, it creates a separate 'space' of its own that would take X and Y as points in that new space (see Figure 4.7). With the notion of homotopy between maps defined, the following defines the homotopy lifting property.

Definition 4.7 (Homotopy lifting property)**.** *Let $H : X \times [0,1] \to Y$ be a homotopy and $H_0 : X \to Y$ be a continuous map such that $H(x,0) = H_0(x)$. Let $\pi : C \to Y$ be a map and $\tilde{H}_0 : X \to C$ be a map such that $H_0 = \pi \circ \tilde{H}_0$. We say that (X, π) has the* ***homotopy lifting property*** *just in case there is a homotopy $\tilde{H} : X \times [0,1] \to C$, such that $\pi \circ \tilde{H} = H$.*

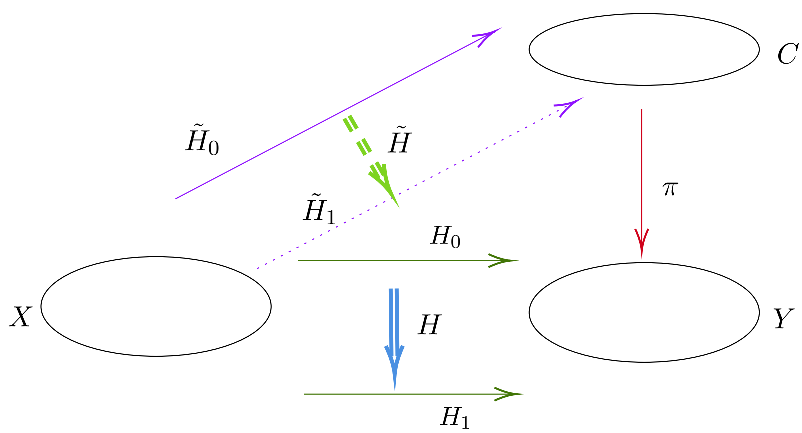


Figure 4.8: Homotopy lifting property

The diagram (Figure 4.8) shows the definition of the homotopy lifting property on (X, π). Given a homotopy H between H_0 and H_1, which are both maps from space X to Y, and also given a map

$\pi : C \to Y$, we can *lift* the homotopy H. We thereby obtain a new homotopy $\tilde{H}$, which is between $\tilde{H}_0$ and $\tilde{H}_1$, which are defined from H_0 and H_1.

In fact, we can restrict homotopy lifting property to path lifting property by taking the domain of $H_0 : X \to Y$ to be $[0,1]$. Then H_0 is a path in space Y. So $H : [0,1] \times [0,1] \to Y$ is a homotopy between paths in Y such that $H(y,0) = H_0(y)$. If $([0,1], \pi)$ satisfies the definition of homotopy lifting property, then the homotopy H gets lifted to a homotopy $\tilde{H} : [0,1] \times [0,1] \to C$ between paths in C. Hence we have paths $\tilde{H}_i : [0,1] \to C$ for $0 \le i \le 1$, which are lifted from H_i, as the homotopy H is lifted to $\tilde{H}$.

4.4.1 Homotopy Equivalences and Homotopy Types

Earlier we discussed how a homotopy between paths defines an equivalence relation. Similarly, a homotopy between maps defines an equivalence relation. For the following definition, let id_X denote an identity map from space X to itself.

Definition 4.8 (Homotopy equivalence)**.** *Let X and Y be topological spaces. We say that X and Y are* ***homotopy equivalent*** *just in case there are maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is homotopic to id_Y and $g \circ f$ is homotopic to id_X. We call f and g* ***homotopy equivalences.***

The homotopy equivalence relation is between spaces, rather than maps, and we denote that X and Y are homotopy equivalent by $X \simeq Y$. This relation is an equivalence relation (i.e. reflexive, symmetric, and transitive). Hence we can define an equivalence class $[X \simeq Y]$ on the maps from X to Y. Note that $f : X \to Y$ and $g : Y \to X$ do not have the same domain and codomain, but if there is an $f \in [X \simeq Y]$, then there is a $g \in [Y \simeq X]$. When the spaces X and Y are homotopy equivalent, we say that they have the *same homotopy type*.

Definition 4.9 (Homotopy type)**.** *Let X and Y be topological spaces. X and Y have the same* ***homotopy type*** *if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_Y. That is, X and Y have the same homotopy type just in case $X \simeq Y$.*

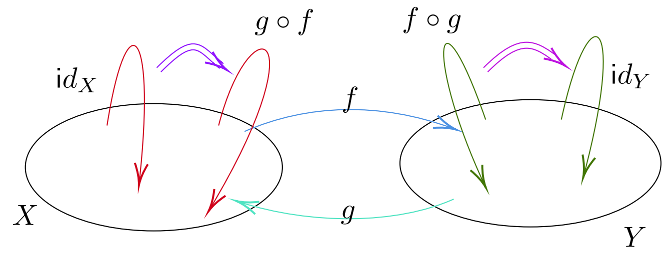


Figure 4.9: Homotopy type

The diagram (Figure 4.9) depicts the spaces X and Y as ovals, and the homotopy equivalences f and g as arrows between them. The other arrows id_X and $g \circ f$ are maps from X to itself, and they are depicted in the same colour because they are homotopic to each other. The homotopy between them is depicted by the purple double arrow $\Rightarrow$. Similarly, id_Y and $f \circ g$ are maps from Y to itself, and the homotopy between them is depicted by the pink double arrow.

The definition tells us that if X and Y have the same homotopy type, then there are equivalence classes $[X \simeq Y]$ and $[Y \simeq X]$. Further more, we have the homotopy equivalences $f \in [X \simeq Y]$ and $g \in [Y \simeq X]$. This is the key to the homotopical interpretation of types in HoTT: the judgement $p : x = y$ is interpreted as a homotopy equivalence p from space x to space y (UFP 2013, p. 5), i.e. $p \in [x \simeq y]$, where x and y are homotopy equivalent.

I shall expand on this to further clarify how types are understood in HoTT, before applying it to explicating path induction.

4.4.2 Homotopical Interpretations: Types as Homotopy Types and Identity as Homotopy Equivalence

Earlier, I mentioned that a type A in HoTT is interpreted as a homotopy type (See Section 4.2.2). This is confusing for (at least) two reasons. First, a homotopy type is a property between spaces: we defined what it is for X and Y to be of the *same* homotopy type, rather than saying what a homotopy type *is*. Second, a type A was interpreted as a space in the pictorial interpretation, rather than as a homotopy type $[X \simeq Y]$. In fact, I mentioned above that the identity type is interpreted as a homotopy type. In order to understand this, let me clarify a few things, namely that a type A is interpreted as a homotopy type means that there is an identity type $(x =_A y)$ on A. Since all types are interpreted as homotopy types, this generates a nested structure by identity types on every type, similar to an ∞-groupoid structure (see section 4.2.2).

Recall that a homotopy equivalence $f : X \to Y$ is a map such that there is a map $g : Y \to X$, and $f \circ g$ and $g \circ f$ are homotopic to id_Y and id_X, respectively. Another way to represent this is by using equivalence classes, so $f \in [X \simeq Y]$ and $g \in [Y \simeq X]$. Then, identity, $=$, in HoTT is homotopically interpreted as the relation $\simeq$, and a term $p : x = y$ is homotopically interpreted as a homotopy equivalence $f : X \to Y$ with x and y interpreted as spaces X and Y, respectively. Hence, a type A is topologically interpreted as a space consisting of points, e.g. x, y, such that these points x and y are homotopically interpreted as *spaces* with the same homotopy type. Thus, interpreting a type A as a homotopy type means treating A as having terms x and y, which have the same homotopy type. This means that, for any type A, the identity type $(x =_A y)$ can be introduced on any arbitrary A, then the identity type $(x = y)$ can be interpreted as the homotopy type $[X \simeq Y]$.

Another challenge is that the notations of homotopy theory and topology are very different from the type theoretic syntax. Here, I shall briefly explain how to interpret the type theoretic syntax, before fully adopting type theoretic syntax with topological/homotopical interpretations. To minimise confusion between the different interpretations, let us employ calligraphic and gothic fonts, respectively, when topological and homotopical interpretations are used. As we become more comfortable with the topological and homotopical interpretations of types, this will perhaps not be needed.

A judgement $a : A$ is topologically/homotopically interpreted as a space $\mathfrak{a}$, or a point $\mathfrak{a}$ in the space $\mathcal{A}$. After this clarification, I will not distinguish between topological and homotopical interpretations. Even though the topological interpretation looks simpler, one still needs to take the homotopical properties involved. In fact, the univalence axiom allows us to treat all of these interpretations as equal without mathematically privileging one over another.

Generally, for any types $A, B : \mathcal{U}$ and a term $p : A =_{\mathcal{U}} B$, p can be interpreted as a homotopy equivalence $\mathfrak{p} : \mathcal{A} \to \mathcal{B}$. At the same time, we can further interpret the terms a and a' of type A, as points $\mathfrak{a}$ and $\mathfrak{a}'$ in the space $\mathcal{A}$ and interpret the term $q : a = a'$ as a topological path $\mathfrak{q} : [0, 1] \to \mathcal{A}$ such that $\mathfrak{q}(0) = \mathfrak{a}$ and $\mathfrak{q}(1) = \mathfrak{a}'$. Note that in the type theoretic notation, the type $a = a$ indicates the starting point a and the end point a of the path q.

When we apply the topological interpretation with the type universe $\mathcal{U}$ as a space and types as points, it is important not to forget the homotopical features. Here, I shall briefly describe how the properties of homotopy equivalences are preserved when we treat them as paths. This shows that we can freely interchange between the homotopical and topological interpretations in HoTT. While, in our reasoning, we can think of $\mathfrak{q}$ as a topological path between points $\mathfrak{a}$ and $\mathfrak{a}'$, if we treat $\mathfrak{a}$ and $\mathfrak{a}'$ as spaces, $\mathfrak{q}$ is interpreted as homotopy equivalence. This detail is similar to the comparison between homotopies: a homotopy between paths is *filling the space* between paths, but a homotopy between maps is *filling the 'space'* between maps. For $\mathfrak{q}$ as a homotopy equivalence, there must be some $\mathfrak{q}' : \mathfrak{a}' \to \mathfrak{a}$ such that $\mathfrak{q} \circ \mathfrak{q}'$ and $\mathfrak{q}' \circ \mathfrak{q}$ are homotopic to $id_{\mathfrak{a}'}$ and $id_{\mathfrak{a}}$, respectively. So we can (topologically) think of $\mathfrak{q}'$ as a path that reverses the path $\mathfrak{q}$. So $\mathfrak{q}' : [0, 1] \to \mathcal{A}$ is such that $\mathfrak{q}'(0) = \mathfrak{a}'$ and $\mathfrak{q}'(1) = \mathfrak{a}$, where instead of composing, we simply follow one path after the other. Also, if we follow the path $\mathfrak{q}$ and then follow the path $\mathfrak{q}'$, the starting point and the end point of the concatenation of the paths are $\mathfrak{q}(0)$; and if we follow $\mathfrak{q}'$ first, then follow $\mathfrak{q}$, the starting point and end point are $\mathfrak{q}'(0)$. These give us new paths from $\mathfrak{q}(0)$ to itself and $\mathfrak{q}(1)$ to itself, i.e. the reflexive path $\mathsf{refl}_{(\cdot)}$. So the properties of homotopy equivalences are preserved when they are considered as paths between two points.

Having explained how a type in HoTT is interpreted as a homotopy type, let us turn to the homotopical interpretation of type families.

4.4.3 Families of Types as Fibrations

In the given conditions of path induction, we have an arbitrary family of types $C : \prod_{x,y:A}(x =_A y) \to \mathcal{U}$. A family of type in HoTT is homotopically interpreted as a *fibration* (UFP 2013, §2.3). The following defines a fibration in homotopy theory.

Definition 4.10 (Fibration). *$\pi : C \to Y$ is a **fibration** just in case for all X, (X, π) has the homotopy lifting property.*

Recall the definition of the homotopy lifting property, which said any map $H_i : X \to Y$ could be lifted by a map $\pi : C \to Y$ to a map $\tilde{H}_i : X \to C$ (just as the homotopy H between H_0 and H_1 is lifted to a homotopy $\tilde{H}$ between $\tilde{H}_0$ and $\tilde{H}_1$). A fibration generalises this property of π to any space X!

If we topologically interpret H_i as a path from point X to point Y in another space, e.g., U, then the homotopy lifting property of a fibration would also lift paths. We let $X = [0, 1]$, then the fibration $\pi : C \to Y$ lifts all topological paths in Y.

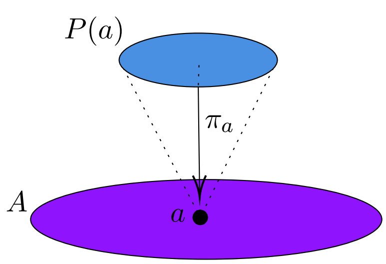


Figure 4.10: Type family as a fibration

Earlier, when we discussed path induction pictorially, we depicted a family of types $P : A \to \mathcal{U}$ as in Figure 4.4. Now, treating type family as a fibration, the pictorial interpretation must show the fibration map, as in Figure 4.10, rather than simply thinking of the spaces.

The claim that a family of types $P : A \to \mathcal{U}$ is said to be 'homotopically interpreted as a fibration' (UFP, 2013:§2.3) is actually misleading, since it is not the map P that is interpreted as a fibration, but rather it informs us the *existence* of a projection map, which is a fibration, $\mathsf{pr_1} : \sum_{x:A} P(x) \to A$ such that $\mathsf{pr_1}(x, y) = x$ (UFP, 2013:§2.3), or with the topological/homotopical notations, we have a fibration $\pi : \sum_{x \in \mathcal{A}} \mathcal{P}_x \to \mathcal{A}$, where the $\sum$ means the disjoint union. For any point $\mathfrak{d}$ in the homotopical interpretation $\mathcal{P}_\mathfrak{a}$ of $P(a)$, we have $\pi(\mathfrak{d}) = \mathfrak{a}$. This is because all elements of $\mathcal{P}_\mathfrak{a}$ would be a pair $(\mathfrak{a}, z)$ and the projection will output the first entry in the pair. So the fibration π goes from a disjoint union of spaces $\mathcal{P}_x$ (the interpretations of $P(x)$), to the interpretation $\mathcal{A}$ of A.

Hence the homotopical interpretation of a type family $P : A \to \mathcal{U}$ is the fibration $\pi : \sum_{x \in \mathcal{A}} \mathcal{P}_x \to \mathcal{A}$. Furthermore, we call the space $\mathcal{P}_x$, a *fibre* for each x.

Definition 4.11 (Fibre). *Given a fibration $\pi : \mathcal{P} \to Y$*[18] *and a point $y \in Y$, we call the subset $\pi^{-1}(y) \subseteq \mathcal{P}$ a **fibre**.*

This means a type family $P : A \to \mathcal{U}$ suggests that there is a fibration $\pi : \sum_{x \in \mathcal{A}} \mathcal{P}_x \to \mathcal{A}$, and the type $P(a)$ for each $a : A$ is interpreted as a fibre $\mathcal{P}_{\mathfrak{a}}$ for each $\mathfrak{a} \in \mathcal{A}$. Understanding type families as fibrations, $\sum_{x \in \mathcal{A}} P_x$ is called the **total space**, which is the disjoint union of spaces (i.e. fibres) P_x with the index set $\mathcal{A}$, and $\mathcal{A}$ is called the **base space**.

With the mathematical definitions clarified, we can now return to path induction. The rough idea behind my argument is as follows. From the topological/homotopical perspectives, path induction can be understood to follow from the path lifting property. In particular, $f(x, y, p) : C(x, y, p)$ in path induction is given by the end of point of the lifted path $\tilde{p}$. In the next section, I shall work through this idea step by step.

4.5 Path Induction from the Perspective of the Homotopy Type Theorists

In this section, we shall make free use of the type theoretic language and look for aid as regards the pictorial interpretation that carries the homotopical and topological interpretations of types. I shall explain how path induction mathematically follows from the path lifting property, so that, from the perspective of HoTT-ists, it ought to be understood via the path lifting property.[19]

[18] Henceforth, I will use $\mathcal{P} := \sum_{x:\mathcal{A}} \mathcal{P}_x$ to simplify notation. Regardless of the index set $\mathcal{A}$, I will use this notation for the disjoint union, unless it might be ambiguous.

[19] This argument was presented at the annual HoTT/UF Workshop in Vienna, Austria, in April 2023. I thank the audience for their questions and generous feedback.

The general idea is that, once we interpret type families as fibrations, any path in the base space can be lifted into the total space. The function $f(x, y, p)$, which generalises the function $c(x)$, can be defined as the end point of the lifted path $\tilde{p}$.

To be more precise, we begin by interpreting type family $C : \prod_{x,y:A}(x = y) \to \mathcal{U}$ as a fibration. That means that we have a fibration $\mathsf{pr}_1 : \prod_{x,y:A} \sum_{p:x=y} C(x, y, p) \to \prod_{x,y:A}(x = y)$, which satisfies the path lifting property. So, if we fix a point $a : A$, then the given conditions of path induction tell us that we have a refl_a such that $c(a) : C(a, a, \mathsf{refl}_a)$. So, the fibration pr_1 lifts the point $\mathsf{refl}_a : \prod_{x,y:A} x = y$ to the point $c(a)$ in the fibre $C(a, a, \mathsf{refl}_a)$. In the total space of the fibration, this is expressed as $(\mathsf{refl}_a, c(a))$.[20]

If, instead of the fibration pr_1 with the base space $\prod_{x,y:A} x = y$, we have another fibration

$$\pi : \prod_{x,y:A} \prod_{p:x=y} C(x, y, p) \to A,$$

we can prove path induction. This fibration π would lift the paths from the type (or space) A and lift them to a path in $C(x, y, p)$. For example, the path refl_a in A would be lifted as the path $\widetilde{\mathsf{refl}}_a$, and this path can be defined as $\mathsf{refl}_{c(a)}$, as in Figure 4.11.

Now, if π is indeed a fibration, it can lift an arbitrary path $q : a = b$ to a path $\tilde{q}$ in space $\prod_{x,y:A} \prod_{p:x=y} C(x, y, p)$. We can see this in Figure 4.12. Once we have all these conditions met, the consequent of path induction gives us a function

$$f : \prod_{x,y:A} \prod_{p:x=y} C(x, y, p),$$

such that $f(a, a, \mathsf{refl}_a) \equiv c(a)$. Here, $f(a, a, \mathsf{refl}_a)$ is precisely the end point $c(a)$ of the lifted path $\widetilde{\mathsf{refl}}_a$. Similarly, $f(a, b, q)$ can be defined as the end point of the lifted path $\tilde{q}$ in the space $\prod_{x,y:A} \prod_{p:x=y} C(x, y, p)$.

[20] It is my understanding that the map c can actually be understood as a section to the fibration π. A *section* is defined as follows.

Definition 4.12 (Section). *Given a map $\pi : C \to X$, a continuous map $\sigma : X \to C$ is a* **section** *of π just in case $\pi(\sigma(x)) = x$ for all $x \in X$.*

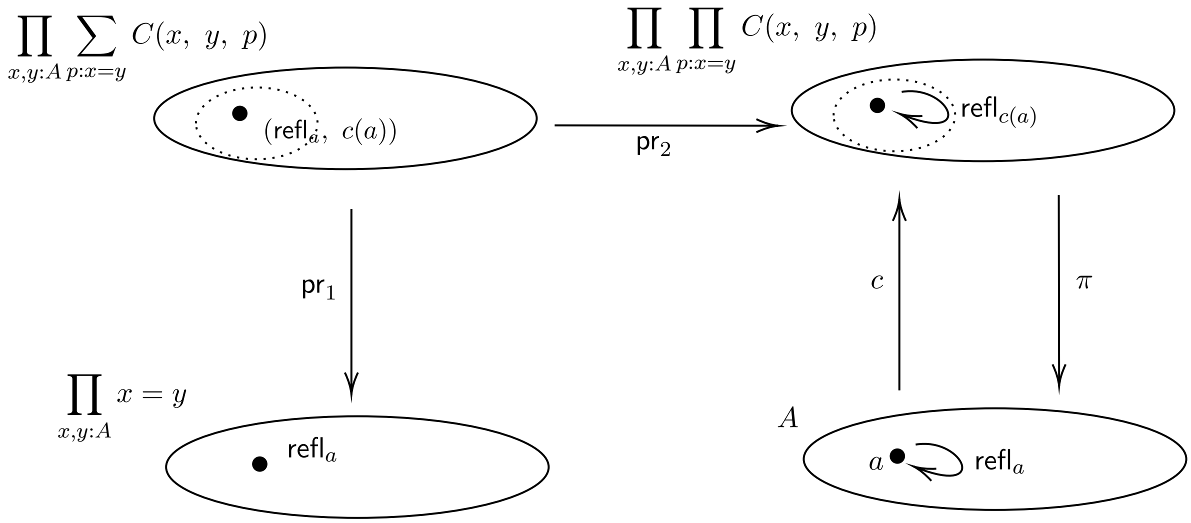


Figure 4.11: Fibration in Path Induction

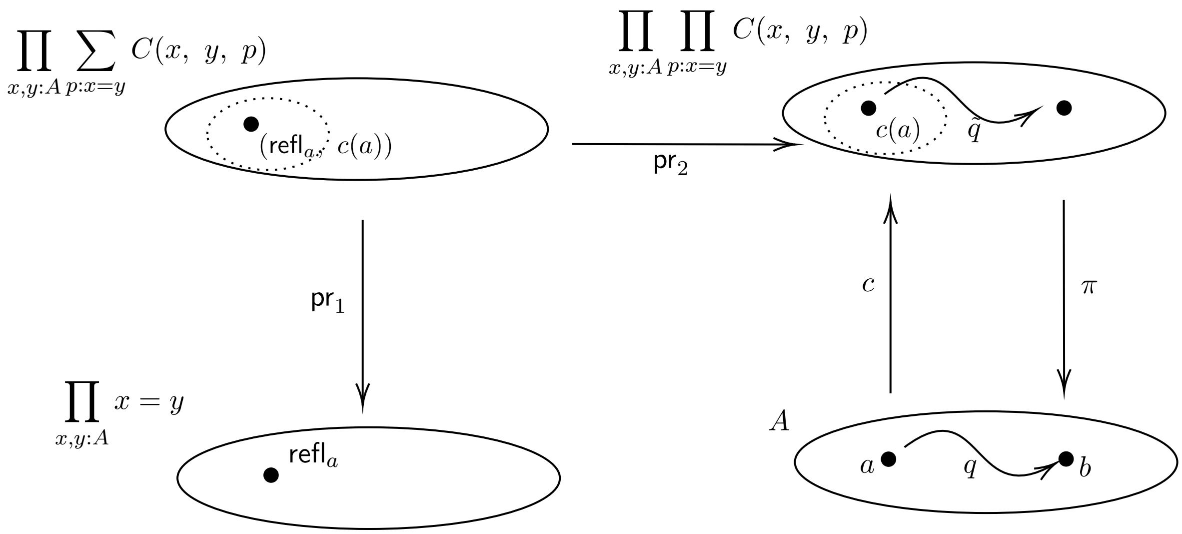


Figure 4.12: Fibration in Path Induction with Arbitrary Path

Interestingly, the converse of this result is given as a theorem in HoTT (see Lemma 2.1.2 in UFP, 2013). Roughly, the lemma states that, given any type family $P : A \to \mathcal{U}$ and an arbitrary $u : P(x)$, for any $p : x = y$, we have a *lifting map* $\mathsf{lift}(u, p)$ that lifts the path p with the starting point u in $P(x)$. So with the two results together, path induction can be seen as equivalent to the path lifting property.

With path induction clarified via the path lifting property from the perspective of homotopy type theorists, we can see that it is *internally* justified. We now have a better understanding of path induction and the mathematical contexts that are needed to understand it. This argument is *internal*, since we are simply clarifying the mathematical context that is already available *internally* in HoTT, without appealing to anything that goes beyond HoTT.

Having clarified path induction, now we can turn to what motivational goals can be found in HoTT, thereby completing *Besinnung*.

4.6 *Besinnung* and the *Zwecksinne* of HoTT

What can we say about the motivational goals of HoTT given this clarified understanding of path induction and the practice of HoTT? I suggest that there are two motivational goals that could be found within the community of empathy with the HoTT-ists: homotopical autonomy and rigour.

4.6.1 HoTT as a Homotopical Autonomous Foundation

Although I have argued against Ladyman and Presnell's account of autonomy for HoTT, I maintain that there is indeed a *sense* of autonomy, which is an aim of HoTT. My opposition to Ladyman and Presnell's view is not that it is incorrect to think that HoTT is an autonomous foundation. Ladyman and Presnell began with a *particular sense* of autonomy and insisted that HoTT be understood independently of other mathematical theories, e.g. homotopy theory or topology. But this, I hold, goes against one of the motivations behind HoTT, namely that it has an intrinsic homotopical content.

To answer whether HoTT is intended to be an autonomous foundation, we first ought to empathise with the homotopy type theorists, and then reflect on whether their practice aims at an autonomous foundation. And only if it does must we further clarify in what sense HoTT is an autonomous foundation. In fact, the three minimal requirements identified earlier suggest that HoTT *is* intended to be an autonomous foundation combining the logical and the practice-oriented aspects of mathematics (the former through the syntax of MLTT, and the latter through homotopy theory). By an '*autonomous foundation*', I mean a mathematical theory that can be understood as a *stand-alone* theory, in terms of which other mathematical theories can be interpreted – but I hold that this does not mean that it must be understood independently of other mathematical theories.

An example of such theory is ZFC-set theory, which has clearly defined syntax and axioms such that any given mathematical concept or definition could be interpreted in its language. Similarly, HoTT has a clearly defined syntax (MLTT) and axiom (Univalent Axiom). And various chapters in the HoTT Book (UFP 2013) attempt to interpret other mathematical concepts and definitions in the language of HoTT.

However, unlike set theory, HoTT is not a 'pre-mathematical theory' as required by Ladyman and Presnell. Instead, HoTT relies on the background *homotopical content* in order for the mathematician to interpret its syntax. Set theory, on the other hand, does not require a background in other areas of mathematics in order to understand its syntax. This difference is not intended to be a criticism against one theory over the other. While the two theories are both 'autonomous', they are 'autonomous' in different ways. Set theory is or can be understood as autonomous in the sense suggested by Ladyman and Presnell. It is possible to understand set theory, without a prior mathematical understanding of sets, via the axioms defining the intuitive understanding of sets required to practice set theory. HoTT is not such a theory. HoTT is a mathematical theory that arose out of existing contemporary mathematics, and shows the possibility of a homotopically understood foundation. Hence, we can characterise *homotopical autonomy* as a motivation goal, a *Zwecksinn*, of HoTT.

4.6.2 HoTT as a Rigorous Foundation

Another motivational goal of HoTT is that it be a rigorous mathematical theory. Traditional accounts of rigour claim that it can be obtained from the possibility of formalising the theorems into some formal language (see the survey of different accounts of rigour in mathematics in Burgess and Toffoli, 2022). More recently, alternative informal accounts of rigour have been proposed by Marfori (2010) and Tatton-Brown (2021). So, what is the account of rigour that can be read from the practice of HoTT?

To uncover a motivational goal (*Zwecksinn*) it is important to examine its historical development and consider its current practice. Recall that Voevodsky's motivation for using a formal language for his mathematics was to eliminate the sociological aspect from proof verification. HoTT is a product of this motivation. HoTT is a theory that can be written entirely in the syntax of MLTT, and this means that various definitions (including path induction) are given in terms of inference rules in MLTT, rather than in terms of axioms. This is not a common feature of a contemporary mathematical theory outside mathematical logic, particularly not in proof theory, where definitions are often given in terms of axioms.

HoTT being a theory written in MLTT also means that the sentences (or judgements) have to be written in the logical syntax of MLTT, and its proofs are given in accordance with the inference rules of MLTT. In mathematical logic generally, it is important to define the signature of a theory with appropriate axioms for the non-logical symbols, but, outside mathematical logic, it is not necessary to express every theorem in a formal syntax (e.g. first-order logic). This is also the case for algebraic topology, and in homotopy theory in particular. The attempt to write the homotopical content in the syntax of MLTT guarantees the kind of rigour that is independent of sociological influences. While the mathematicians can work with the homotopical content, when it comes to proving theorems, a rigorous account must be given that can be written in MLTT: the proof is given in accordance with the inference rules of MLTT, and can be verified by a computer program, independently of a mathematician's influence and reputation.

This does not mean that this is a universal account of rigour in all of mathematics, but *one* account of rigour in mathematics that can be found in HoTT. Importantly, this account of rigour does not sacrifice the other values or motivational goals (*Zwecksinne*) of HoTT. Whatever other motivational goals (*Zwecksinne*) there might be, HoTT preserves its rigour because it can be written in the syntax of MLTT.

It is worth reminding the readers of what the motivational goals (*Zwecksinne*) are in Husserlian terms. Husserlian '*Sinn*' refers to the meaning found in experience, or in practice. *Zwecksinn*, interpreted as 'final sense' in the case of subjective experience, means the clearest meaning of an intended object X, via which we can grasp X in an intuitive (i.e. immediate) act. In the context of *Besinnung*, the '*Zwecksinn*' refers to the motivational goal, i.e. the abstracted goal based on the historical motivation and the current practice of the community, via which we can understand different parts of the community's practice. The motivational goals we find within the community are then the *Sinn*, which guides the practice, and certain methods or concepts, etc., of the practice can be understood clearly as particular instantiations of the motivational goals. Thus, rigour and homotopical autonomy are the motivational goals of HoTT: the use of logical syntax (as demonstrated in the expression of path induction) is an example of how rigour is achieved; and the use of homotopical interpretation of the logical syntax along with the practice of interpreting other mathematical concepts in homotopy type theoretic terms is how homotopical autonomy is instantiated.

4.6.3 Radical *Besinnung* and HoTT

If we are to carry out radical *Besinnung*, some further investigation will need to be pursued. For instance, we would need to ask whether the current methods used in HoTT are actually appropriate to obtain its motivational goals (*Zwecksinne*), e.g. rigour. In particular, HoTT-ists have adopted machine implementation as a way to establish rigour in verifying proofs. Verifications of proofs are important to ensure that a proof is good (in some appropriate sense of goodness). Carrying out radical *Besinnung* would mean questioning whether this is an appropriate way to obtain rigour in mathematical proofs, and, if not, how the mathematicians ought consider a possible revision of the practice. For instance, in type theoretic syntax, definitions are given as inference rules. This contrasts with the more typical axiomatic practice of mathematics. So, the question is: Can we obtain rigour in proofs when the definitions are given as inference rules? I do not offer an answer here but, if I were to carry out radical *Besinnung*, this is one question that might be addressed.

Similarly, we might ask whether homotopical autonomy is something that is achievable in the current state of practice. While we have observed that HoTT-ists aim to interpret other mathematical theories (e.g. set theory) in HoTT, the question remains whether this is a viable pursuit for HoTT. My naïve perspective is that this is viable only with some restrictions. HoTT, as a theory, is a product of mathematical results showing similarities in MLTT-types, homotopy types, and ∞-groupoid structures. Given this, HoTT could offer an autonomous foundation for other mathematical concepts or theories that have similar structures, but not for *all* of mathematics. Mathematics is a huge discipline with many different applications, and I would think a more modest aim ought to be established for the HoTT-ists, and this is where another Husserlian method of zig-zagging between the motivational goal and the practice would be relevant.

By zig-zagging, one can look at the mathematics from the perspective of the practitioners, but also then turn to the clarified motivational goals (*Zwecksinne*) and critically evaluate them. Moving back and forth between the practice and the clarified sense is what Husserl calls *zigzagging*. This critical evaluation is not necessarily what is currently part of the practice, but rather something that *radical Besinnung* adds to the practice.

Although I shall not pursue radical *Besinnung* in much detail here, I believe it to be an important philosophical part of a Husserlian investigation. Questioning and analysing whether the clarified motivational goals are appropriate, and providing some guidance to mathematical practice could really make the collaboration between philosophers and mathematicians stronger. Philosophers can clarify mathematical goals that are to some extent vague into clarified motivational goals, and offer suggestions based on reflection on the practice and its goals, while the mathematicians can revise their practice so the methods used can be more appropriate for achieving their own goals. Husserlian *Besinnung* and, in particular, radical *Besinnung*, then offer a philosophical criticism and evaluation, based on the perspective of the practitioners. Thus, it can really be considered a method for the philosophy of mathematical practice.

4.7 Conclusion

In this chapter, I have demonstrated how *Besinnung* can be pursued in relation to HoTT. To stand in the community of empathy with the homotopy type theorists, I highlighted (a) the Univalent Axiom, (b) Logical Syntax, and (c) Intrinsic Homotopical Content as the minimal requirements of HoTT. These requirements are the unfulfilled senses *vaguely floating* in the practice of HoTT, and empathising with homotopy type theorists involves committing to these requirements. Only then can one carry out *Besinnung* by clarifying them – i.e. by fully engaging with what they mean in the practice – then one can explicate the motivational goals aimed at by the homotopy type theorists.

While empathising with the homotopy type theorists, I have also argued that, from the perspective of the homotopy type theorists, path induction can be understood in terms of path lifting property. This offers an alternative justification for path induction, challenging the views of Ladyman and Presnell (2015) and P. Walsh (2017), who argue, respectively, that path induction needs to be philosophically justified by pre-mathematical concepts or an inferentialist notion of harmony. From the homotopy type theorists' perspective, path induction is simply understood from the path lifting property, thus path induction can be internally justified.

I further clarified *homotopical autonomy* and *rigour* as motivational goals (*Zwecksinne*) of HoTT. Importantly, these values are identified and clarified through *Besinnung* rather than defined prior to empathising with homotopy type theorists. This contrasts with the account of autonomy given by Ladyman and Presnell (2015), who attempts to characterise what kind of *autonomy* is required for HoTT independently of a process of empathising with practitioners. We also saw how an appropriate understanding of rigour for HoTT could be found in consideration of the history and practice of HoTT. Unlike many mathematical theories outside mathematical logic, HoTT is written in a formal syntax, MLTT. This guarantees that every theorem of HoTT can be formalised and thus guarantees the rigour required for verifying the theorems.

Although rigour and homotopical autonomy could be the motivational goals of HoTT, we can further clarify the more general motivational goals for *mathematics* by the method of *Besinnung*. This means looking not only at the practice of HoTT, but also at other theories of mathematics and their practices, to establish their motivational goals and, in the light of identification of these, to consider whether further shared goals can be found in mathematical practice. This would be a continuation of the method of *Besinnung*, as we aim to clarify the motivational goals of *mathematics*, rather than a particular theory of mathematics. *Besinnung* must be carried out in different mathematical communities. Ultimately these communities could provide us with more general motivational goals for mathematical practice. In pursuing this broad project, we can carry on the *empathy-first* approach to philosophy of mathematical practice.

Chapter 5

Future Directions and Conclusion

In this concluding chapter, I outline three projects extending the work of my dissertation. The first two projects are applications of the Husserlian methods developed in the dissertation to the philosophy of mathematics. The third project offers an interpretation of HoTT as a geometrical foundation for mathematics, as opposed to an arithmetical foundation. (This suggestion will be clarified shortly.)

The first project (see Section 5.1) addresses a contemporary foundational debate in mathematics concerning set theory, category theory, and type theory (including HoTT and Univalent Foundations). By looking at each of the respective theories as understood by the foundational positions within the perspective of the community, the project aims to settle some ongoing debates. The second project (see Section 5.2), which is to be pursued during my post-doctoral position at the Czech Academy of Sciences, compares the intended informal semantics of HoTT and of Martin-Löf Type theory. The third project (see Section 5.3) offers new insight into how HoTT can be considered a geometrical foundation for mathematics, compared with set theory, which might be considered an arithmetical foundation. Then, in Section 5.4, I shall summarise the dissertation.

5.1 Project 1: Empathy and Contemporary Foundational Debate in Mathematics

Contemporary foundational debate in the philosophy of mathematics concerns in what sense different mathematical theories can serve as a foundation for mathematics. Many of the contributions, however, are offered by the advocates of particular theories (e.g. see Altenkirch, 2023), who are not fully engaged in the philosophical literature of the other theories. A general problem, then, is that the arguments offered by an advocate do not address the advocates of other theories. For productive philosophical discussion, or for there to be a *debate* concerning the foundation of mathematics, there ought to be a meaningful way in which different advocates can engage with each other. In this future project, I offer mathematical empathy as a method of engaging with advocates of different theories, by looking at their preferred theory from their perspective. This involves understanding the kinds of concepts that are taken to be primitive by them, rather than offering a logical translation between the syntax of the theories.

At the risk of stating the obvious, for type theorists, types and functions are considered to be primitive, and thus the rest of mathematics ought to be understood in terms of types and functions. For set theorists, sets are taken to be primitive. The empathy-first approach would explicate the conceptual differences in these approaches and in their motivation from the perspective of the advocates, and explain why certain questions are more concerning to them.

This will encourage genuine and open discussion between advocates of different theories and further the discussion on the foundations of mathematics generally.

5.2 Project 2: Intended Models of Homotopy Type Theory and Martin-Löf Type Theory

There has been some discussion on the status of path induction in HoTT and the identity elimination rule in Martin-Löf Type Theory. As discussed in Chapter 4, Ladyman and Presnell (2015) and

Walsh (2017) argue that path induction ought to be justified. I have suggested in the same chapter that this is not the case.

Ansten Klev (2019b), however, has argued that Ladyman and Presnell's justification is appropriate for justifying the identity elimination rule in MLTT, given Martin-Löf's *meaning explanation*: meaning explanation is offered as an intended interpretation of MLTT. This raises the question of *why* this justification is an appropriate one for identity elimination in MLTT, but not for path induction in HoTT. While Martin-Löf's meaning explanation offers an intended interpretation for MLTT, this is not the intended interpretation for HoTT. Thus, the two theories have different interpretations of identity.

This might seem surprising, because HoTT is an extension of MLTT: shouldn't the notion of identity in MLTT be preserved in HoTT? To answer this question, one must consider the philosophical motivations behind the two theories. My current answer is that, given Voevodsky's motivation for the development of HoTT, the two theories and their intended interpretations are independent of one another. For Voevodsky, MLTT was adopted as the syntax for HoTT because of its convenience for machine implementation (see UFP, 2013, p. 3). Hence MLTT's intended interpretation via meaning explanation is not considered for HoTT.

5.3 Project 3: Homotopy Type Theory as a Geometrical Foundation

As shown in Chapter 4, in empathising with the HoTT-ists, one must look at HoTT with the homotopical interpretation. This interpretation assumes that identity between terms can be understood as a continuous path from one term to the other. This underlying notion of continuity offers a possibility of HoTT as a geometric foundation for mathematics, as opposed to an arithmetical foundation.

Arithmetic and geometry are two fundamental branches of mathematics, which were historically taken to be the foundations of mathematics. On the one hand, arithmetic concerns discrete magnitudes, while geometry concerns continuous magnitudes. From a contemporary perspective, one might argue that set theory (sometimes called transfinite number theory) is an arithmetical foundation because it concerns finite and transfinite ordinals, which are *discrete* magnitudes. Along this line, HoTT can be considered a geometrical foundation concerning *continuous* magnitudes.

In order to pursue this project, one must clarify in what sense the two theories concern magnitudes. While the practice of ordinal arithmetic in set theory makes it obvious for its case, it is not immediately clear what this would look like in the case of HoTT. This project aims to investigate whether there is a notion of continuous magnitude in HoTT.[1]

5.4 Summary and Conclusion

In my dissertation, I have offered Husserlian methods for the contemporary philosophy of mathematics and mathematical practice. My general claim was that empathy, or understanding mathematics from the perspective of mathematicians via their historical motivation, is an important part of any Husserlian method.

In Chapter 1, I argued that Husserlian phenomenology can be considered as a methodological programme for philosophy of mathematical practice based on two general features. The first feature is that Husserlian phenomenology assumes the principle of pre-suppositionlessness. This means making no metaphysical presuppositions when it comes to considering mathematics, although the investigation may clarify what metaphysical presuppositions are being made. This corresponds to (what Stewart Shapiro describes as) the *mathematics-first* approach in philosophy of mathematics. In this approach, we are to look at mathematics and address philosophical (e.g. metaphysical or epistemological) questions arising from within mathematics. The second feature is the Husserlian method of *Besinnung*, which aims to clarify the goals of mathematicians. This method was compared with Maddy's method of Second Philosophy applied to mathematics.

The second chapter took a short, but natural, detour before explicating the method of *Besinnung*. The general aim was to show that a phenomenological method of first-person investigation can be used in interdisciplinary research concerning numerical cognition. It involves forming an empathic

[1]I thank John Mumma, both for raising this issue during my presentation at the UC Riverside Workshop on Goals and Values in Mathematics and Logic, and for subsequent follow-up discussions.

understanding of the sequence of numbers from the perspective of an ordinary natural language user. As I highlighted, at least three forms of non-arithmetical act are involved in how ordinary people (ordinary natural language users) understand the sequence of numbers. These acts were numeralising, predicting, and naming, all of which involve certain linguistic features of the numerals or simply the use of natural language. These acts, I argued, provide us with two distinct kinds of numbers in the ordinary understanding: arithmetical and non-arithmetical numbers.

The arithmetical numbers are those that, *in principle*, can be obtained only by counting. These numbers coincide with the natural numbers as understood by mathematicians. However, the non-arithmetical numbers are not considered as the natural numbers by mathematicians, yet they are present in our ordinary understanding of numbers.

The analysed concept of numbers (i.e. not the natural numbers) was then compared with the empirical evidence offered by Relaford-Doyle and Núñez. The comparison showed that undergraduate students who have not studied mathematical proofs convey a similar account of numbers, i.e. a sequence of numbers with arithmetical and non-arithmetical numbers.

I suggested that distinct concepts of numbers arise from different communal practices and that this could be clarified in the Husserlian notion of *community of empathy*, which is one of the first steps of carrying out the method of *Besinnung*.

The method of *Besinnung*, as analysed in Chapter 3, consists of (1) entering the community of empathy with the mathematicians, (2) identifying the vague concepts, goals, methods and/or questions in the practice, and (3) clarifying the vagueness in order to clarify the motivational goals of the disciplines. This method can be extended to the method of radical *Besinnung*, which adopts a critical perspective on the clarified motivational goals. This means evaluating whether the motivational goals are achievable given the mathematicians' current practice and then revising the practice accordingly.

In Chapter 4, I applied the method of *Besinnung* to the contemporary theory of Homotopy Type Theory (HoTT). By entering the community of empathy, I recognised that the notion of identity in HoTT was in need of further clarification. Importantly, empathising with the mathematicians, or looking at mathematics from their perspective and understanding their motivation, involved clarifying the use of univalence axioms, the logical syntax of MLTT and HoTT's intrinsic homotopical content. Clarifying these meant understanding the syntax of MLTT as topologically and homotopically interpreted. Through this interpreted picture, I argued that path induction is the end point of the lifted path based on the path lifting property in topology.

Having clarified what path induction is, I could further clarify the motivational goals of HoTT-ists, and thus I offered rigour and homotopical autonomy as the motivational goals. Rigour in HoTT is established by ensuring that the syntax of HoTT can be implemented by proof verification programs. Homotopical autonomy suggests that any mathematical concept could be understood in HoTT. Logically speaking, there will be a translation such that any given mathematical notion can be translated into the syntax of HoTT. Conceptually speaking, any mathematical concept can be understood with the appropriate homotopical/topological interpretations. This is distinct from offering a pre-mathematical autonomy, as suggested by Ladyman and Presnell (2015).

The four chapters together offer guidance on applying Husserlian methods to mathematics and mathematical practice. In contrast with the recent tendency of philosophers to prefer to adopt the methods of empirical and social sciences, I offer *philosophical* methods for studying mathematics and mathematical practice. With this dissertation, I hope that I have sufficiently demonstrated how fruitful such methods can be in philosophical investigations concerning mathematics.

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