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AN ALGEBRAIC SATO-TATE GROUP AND SATO-TATE CONJECTURE

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ABSTRACT. We make explicit a construction of Serre giving a definition of an algebraic Sato-Tate group associated to an abelian variety over a number field, which is conjecturally linked to the distribution of normalized L-factors as in the usual Sato-Tate conjecture for elliptic curves. The connected part of the algebraic Sato-Tate group is closely related to the Mumford-Tate group, but the group of components carries additional arithmetic information. We then check that in many cases where the Mumford-Tate group is completely determined by the endomorphisms of the abelian variety, the algebraic Sato-Tate group can also be described explicitly in terms of endomorphisms. In particular, we cover all abelian varieties (not necessarily absolutely simple) of dimension at most 3; this result figures prominently in the analysis of Sato-Tate groups for abelian surfaces given recently by Fité, Kedlaya, Rotger, and Sutherland.

1. INTRODUCTION

Let F be a number field with absolute Galois group G_F. Let A be an abelian variety of dimension $g \geq 1$ over F. Pick a prime number l and consider the action of G_F on the l-adic Tate module of A. A theorem of Weil implies that for each prime ideal $\mathfrak{p}$ of F at which A has good reduction, the characteristic polynomial $L(A/F, T)$ of any geometric Frobenius element of G_F corresponding to $\mathfrak{p}$ is a monic polynomial with integer coefficients, whose roots in the complex numbers all have absolute value $q^{1/2}$ for q the absolute norm of $\mathfrak{p}$.

One can ask how the renormalized characteristic polynomials $\overline{L}(A/F, T) = q^{-g}L(A/F, q^{1/2}T)$ are distributed in the limit as $q \to \infty$. For A an elliptic curve without complex multiplication, the *Sato-Tate conjecture* predicts equidistribution with respect to the image of the Haar measure on SU(2); this result is now known when F is a totally real field [BLGG], building upon recent advances in the modularity of Galois representations led by R. Taylor. For A an elliptic curve with complex multiplication, the situation is easier to analyze: for all F, one can prove equidistribution for the image of the Haar measure on a certain subgroup of SU(2). This group may be taken to be SO(2) when F contains the field of complex multiplication; otherwise, one must instead take the normalizer of SO(2) in SU(2)

With this behavior in mind, it is reasonable to formulate an analogous *Sato-Tate conjecture* for general A, in which the equidistribution is with respect to the image

of the Haar measure on a suitable compact Lie group. Such a conjecture seems to have been first formulated by Serre [Se4] in the language of motives. More recently, Serre has given an alternate formulation in terms of l-adic Galois representations [Se5, §8].

The purpose of this paper is to make explicit one aspect of Serre's l-adic construction of the putative *Sato-Tate group*, namely the role of an algebraic group closely related to (but distinct from) the classical *Mumford-Tate group* associated to A. This *algebraic Sato-Tate group* has the property that its connected part is conjecturally determined by the Mumford-Tate group; however, whereas the Mumford-Tate group is by construction connected, the algebraic Sato-Tate group has a component group with Galois-theoretic meaning. For example, in those cases where the Mumford-Tate group is determined entirely by the endomorphisms of A (e.g., the cases studied by the first author jointly with W. Gajda and P. Krasoń in [BGK1, BGK2]), we can determine the algebraic Sato-Tate group and interpret its component group as the Galois group of the minimal extension of F over which all geometric endomorphisms are defined.

One key motivation for this work is to establish the properties of the algebraic Sato-Tate group for all abelian varieties of dimension at most 3. (Note that includes only cases where the Mumford-Tate group is determined by endomorphisms, as the first counterexamples are Mumford's famous examples in dimension 4.) This result is applied by the second author in [**?**] in order to classify the possible Sato-Tate groups arising in dimension 2, and makes the corresponding classification in dimension 3 feasible as well.

In cases where the Mumford-Tate group is not determined by endomorphisms, one can still exhibit a natural candidate for the algebraic Sato-Tate group by following Serre's motivic construction [Se4]. Based on computations in §7 and §8, using the Tannakian formalism, we define in §9 the algebraic Sato-Tate group $AST_K(A)$ assuming that the Mumford-Tate group is the connected component of the motivic Mumford-Tate group (a hypothesis put forward by Serre in [Se4, sec. 3.4]). Then in §9 we prove (see Theorem 9.8) that the group $AST_K(A)$ satisfies the conditions (2.4)–(2.6) of Remark (2.5) which one naturally expects to hold for the algebraic Sato-Tate group. If a suitable motivic version of the Mumford-Tate conjecture holds, then we prove that the algebraic Sato-Tate conjecture also holds. We also show that the $AST_K(A)$ group defined in §9 agrees with the algebraic Sato-Tate group considered in sections §4 - §6, in particular in all cases where the Mumford-Tate group is determined by endomorphisms.

2. The algebraic Sato-Tate group and conjecture

We begin with the definition of the algebraic Sato-Tate group and the formulation of an associated conjecture refining the standard Mumford-Tate conjecture.

Definition 2.1. Let A/F be an abelian variety over a number field F. Choose an embedding $F \subset \mathbb{C}$, then put $V := V(A) := H_1(A(\mathbb{C}), \mathbb{Q})$ and $V^* := \mathrm{Hom}_{\mathbb{Q}}(V, \mathbb{Q})$. Put $D := D(A) := \mathrm{End}(A)^0 := \mathrm{End}_{\overline{F}}(A) \otimes \mathbb{Q}$.

We obtain a rational Hodge structure of weight 1 on V^* from the Hodge decomposition

$$V_{\mathbb{C}}^* := V^* \otimes_{\mathbb{Q}} \mathbb{C} = H^{1,0} \oplus H^{0,1},$$

where $H^{p,q} = H^p(A; \Omega^q_{A/\mathbb{C}})$ and $\overline{H^{p,q}} = H^{q,p}$. Observe that $H^{p,q}$ are invariant subspaces with respect to the action of D on $V^* \otimes_{\mathbb{Q}} \mathbb{C}$, so the $H^{p,q}$ are right D-modules. Put $H^{-p,-q} := \mathrm{Hom}_{\mathbb{C}}(H^{p,q}, \mathbb{C})$. The left action of D on $H^{-p,-q}$ is defined via the right action of D on $H^{p,q}$. Let

$$\psi = \psi_V : V \times V \to \mathbb{Q}$$

be the $\mathbb{Q}$-bilinear, nondegenerate, alternating form coming from the Riemann form of A.

Definition 2.2. Put $T_l := T_l(A)$ and $V_l := V_l(A) := T_l(A) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. It is well known [Md, §24, p. 237] (cf. [Mi, §16, p. 133]) that $V_l \cong V \otimes_{\mathbb{Q}} \mathbb{Q}_l$. The Galois group G_F acts naturally on V_l and because of the Weil pairing $\psi_l : V_l \times V_l \to \mathbb{Q}_l$ we have the l-adic representation

$$(2.1) \qquad \rho_l : G_F \to GSp(V_l)$$

induced by the natural representation $\rho_l : G_F \to GSp(T_l)$.

Let K/F be a finite extension. Let $G^{alg}_{l,K} \subseteq GSp_{(V_l,\psi_l)}$ be the Zariski closure of $\rho_l(G_K)$ in $GSp_{(V_l,\psi_l)}$. Define

$$\rho_l(G_K)_1 := \rho_l(G_K) \cap Sp_{(V_l,\psi_l)}.$$

Let $G^{alg}_{l,K,1} \subseteq Sp_{(V_l,\psi_l)}$ be the Zariski closure of $\rho_l(G_K)_1$ in $Sp_{(V_l,\psi_l)}$. The algebraic group $G^{alg}_{l,K}$ is reductive by [F, Theorem 3]. If $K = F$, we will put $G^{alg}_l := G^{alg}_{l,F}$ and $G^{alg}_{l,1} := G^{alg}_{l,F,1}$.

On the basis of the work of Mumford, Serre, and Tate, it has been conjectured for about 50 years (see Conjecture 4.7) that there is a connected reductive group scheme $G \subseteq GSp_{(V,\psi)}$ such that for all $l \gg 0$,

$$(2.2) \qquad G_{\mathbb{Q}_l} = (G^{alg}_{l,K})^{\circ},$$

where $(G^{alg}_{l,K})^{\circ}$ denotes the connected component of the identity. This implies that G does not depend on K (see §3). We propose the following refinement of this conjecture which does detect the field K.

Conjecture 2.3. *(Algebraic Sato-Tate conjecture) There is an algebraic group $AST_K(A) \subseteq Sp_{(V,\psi)}$ over $\mathbb{Q}$ such that $AST_K(A)^{\circ}$ is reductive and for each l,*

$$(2.3) \qquad AST_K(A)_{\mathbb{Q}_l} = G^{alg}_{l,K,1}.$$

Definition 2.4. The algebraic Sato-Tate group plays a key role in Serre's definition of the putative Sato-Tate group $ST_K(A)$. If Conjecture 2.3 holds, then base extension of $AST_K(A)_{\mathbb{Q}_l}$ along an embedding $\mathbb{Q}_l \subset \mathbb{C}$ gives a group $AST_K(A)_{\mathbb{C}}$ that does not depend on l or the embedding. Taking a maximal compact subgroup of $AST_K(A)(\mathbb{C})$, we obtain the *Sato-Tate group* $ST_K(A) \subseteq USp(V_{\mathbb{C}})$.

Remark 2.5. Besides Conjecture 2.3 and the usual Sato-Tate conjecture, we expect $AST_K(A)$ to have the following properties:

$$(2.4) \qquad AST_K(A) \subseteq DL_K(A)$$

$$(2.5) \qquad AST_K(A)^{\circ} = H(A)$$

$$(2.6) \qquad \pi_0(AST_K(A)) = \pi_0(ST_K(A)).$$

Here $H(A)$ is the Hodge group and $DL_K(A)$ is the twisted decomposable Lefschetz group, to be defined in §5. In §5, we exhibit a wide class of abelian varieties such that Conjecture 2.3 and conditions (2.4)–(2.6) hold (see Theorem 6.1).

In §9, we define a candidate group $AST_K(A)$ based on computations in §7 and §8 using the Tannakian formalism. Assuming that the Mumford-Tate group is the connected component of the motivic Mumford-Tate group (a hypothesis put forward by Serre in [Se4, sec. 3.4]), we prove (see Theorem 9.8) that the group $AST_K(A)$ satisfies conditions (2.4)–(2.6).

3. Some remarks on Zariski closure. A theorem of Serre.

Before proceeding further, we collect some observations about the groups we have just defined.

Remark 3.1. We have the following commutative diagram with exact columns:

$$
\begin{array}{ccccc}
\rho_l(G_K)_1 & \longrightarrow & G_{l,K,1}^{alg} & \longrightarrow & Sp_{(V_l,\psi_l)} \\
\downarrow & & \downarrow & & \downarrow \\
\rho_l(G_K) & \longrightarrow & G_{l,K}^{alg} & \longrightarrow & GSp_{(V_l,\psi_l)} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Q}_l^\times & \longrightarrow & \mathbb{G}_m & \overset{=}{\longrightarrow} & \mathbb{G}_m
\end{array}
$$

By the theorem of Bogomolov [Bo] on homotheties, $\rho_l(G_K)$ is an open subgroup of $G_{l,K}^{alg}$. Since $Sp_{(V_l,\psi_l)}$ is closed in $GSp_{(V_l,\psi_l)}$, $\rho_l(G_K)_1 = \rho_l(G_K) \cap Sp_{(V_l,\psi_l)}$ is an open subgroup of $G_{l,K}^{alg} \cap Sp_{(V_l,\psi_l)}$. It follows that

$$(3.1) \qquad G_{l,K,1}^{alg} = G_{l,K}^{alg} \cap Sp_{(V_l,\psi_l)}$$

is closed in $G_{l,K}^{alg}$, and that there is an exact sequence

$$(3.2) \qquad 1 \to G_{l,K,1}^{alg} \to G_{l,K}^{alg} \to \mathbb{G}_m \to 1.$$

Since $G_{l,K,1}^{alg}$ is the kernel of a homomorphism from a reductive group to a torus, it is also reductive.

Remark 3.2. Let $F \subseteq K \subseteq L \subset \overline{F}$ be a chain of extensions such that K/F and L/K are finite. Let $L' \subset \overline{F}$ be the Galois closure of L over K. It is clear that $G_{l,L'}^{alg} \subseteq G_{l,L}^{alg} \subseteq G_{l,K}^{alg}$ and $G_{l,L',1}^{alg} \subseteq G_{l,L,1}^{alg} \subseteq G_{l,K,1}^{alg}$. On the other hand, there is a surjective homomorphism $\rho_l(G_K)/\rho_l(G_{L'}) \to G_{l,K}^{alg}/G_{l,L'}^{alg}$, so $G_{l,L'}^{alg}$ is a subgroup of $G_{l,K}^{alg}$ of finite index, as then is $G_{l,L}^{alg}$. In particular, $(G_{l,L}^{alg})^\circ = (G_{l,K}^{alg})^\circ$. By a similar argument, $(G_{l,L,1}^{alg})^\circ = (G_{l,K,1}^{alg})^\circ$.

Proposition 3.3. *There is a finite Galois extension L_0/K such that $G_{l,L_0}^{alg} = (G_{l,K}^{alg})^\circ$ and $G_{l,L_0,1}^{alg} = (G_{l,K,1}^{alg})^\circ$.*

Proof. Let Z be the set of $\mathbb{Q}_l$-points of the closed subscheme $(G_{l,K}^{alg} - (G_{l,K}^{alg})^\circ) \cup (G_{l,K,1}^{alg} - (G_{l,K,1}^{alg})^\circ)$ of $GSp_{(V_l,\psi_l)}$. Since ρ_l is continuous, we can find a finite Galois

extension L_0/K such that $\rho_l(G_{L_0}) \cap Z = \emptyset$. Such an extension satisfies $G_{l,L_0}^{alg} \subseteq (G_{l,K}^{alg})^\circ$ and $G_{l,L_0,1}^{alg} \subseteq (G_{l,K,1}^{alg})^\circ$. Since we already have the reverse inclusions, we obtain the desired equalities. $\qquad\square$

The following theorem is a special case of a result of Serre [Se5, §8.3.4].

Theorem 3.4. *The following map is an isomorphism:*

$$i_{CC} \ : \ G_{l,K,1}^{alg}/(G_{l,K,1}^{alg})^\circ \xrightarrow{\cong} G_{l,K}^{alg}/(G_{l,K}^{alg})^\circ.$$

Proof. Choose L_0 as in Proposition 3.3. Then the following commutative diagram has exact rows:

The first two columns are also exact by (3.2), so a diagram chase (as in the snake lemma) shows that the third column is also exact. $\qquad\square$

Remark 3.5. Observe that the natural action by left translation

$$(3.3) \qquad\qquad G_K \times G_{l,K}^{alg}/(G_{l,K}^{alg})^\circ \to G_{l,K}^{alg}/(G_{l,K}^{alg})^\circ$$

is continuous. By Theorem 3.4, we also have a continuous action

$$(3.4) \qquad\qquad G_K \times G_{l,K,1}^{alg}/(G_{l,K,1}^{alg})^\circ \to G_{l,K,1}^{alg}/(G_{l,K,1}^{alg})^\circ.$$

Choose a suitable field embedding $\mathbb{Q}_l \to \mathbb{C}$ and put $G_{l,K,1_\mathbb{C}}^{alg} := G_{l,K,1}^{alg} \otimes_{\mathbb{Q}_l} \mathbb{C}$. By Theorem 3.4 and [**?**, Lemma 2.8], we have the following.

Proposition 3.6. *Let K/F be any finite extension such that $F \subseteq K \subset \overline{F}$. Then there are natural isomorphisms*

$$(3.5) \qquad G_{l,K,1}^{alg}/(G_{l,K,1}^{alg})^\circ \quad\cong\quad G_{l,K,1_\mathbb{C}}^{alg}/(G_{l,K,1_\mathbb{C}}^{alg})^\circ \quad\cong\quad ST_A/ST_A^\circ.$$

If the algebraic Sato-Tate conjecture (Conjecture 2.3) holds, then all of the groups in (3.5) are isomorphic to $AST_K(A)/AST_K(A)^\circ$.

Remark 3.7. Consider the continuous homomorphism

$$\epsilon_{l,K} \ : \ G_K \to G_{l,K}^{alg}(\mathbb{Q}_l).$$

Serre [Se3], [Se2] proved that $\epsilon_{l,K}^{-1}((G_{l,K}^{alg})^\circ(\mathbb{Q}_l))$ is independent of l. Since $(G_{l,K}^{alg})^\circ$ is open in $G_{l,K}^{alg}$, this preimage equals $G_{L_0} = G(\overline{K}/L_0)$ for some finite Galois extension

L_0/K. This theorem of Serre can also be seen from Propositions 3.3 and 3.6 and Theorem 3.4. In fact, L_0/K is the minimal extension such that G_{l,L_0}^{alg} and $G_{l,L_0,1}^{alg}$ are connected for all l. Let $H_{l,K,1} := \rho_l^{-1}(\rho_l(G_K)_1)$. Put $K_1 := \overline{K}^{H_{l,K,1}}$. Observe that

$$\epsilon_{l,K}^{-1}((G_{l,K,1}^{alg})^{\circ}(\mathbb{Q}_l)) = \epsilon_{l,K}^{-1}(G_{l,L_0,1}^{alg}(\mathbb{Q}_l)) = \epsilon_{l,K}^{-1}((G_{l,L_0}^{alg} \cap Sp_{(V_l,\psi_l)})(\mathbb{Q}_l)) =$$
$$= \epsilon_{l,K}^{-1}(G_{l,L_0}^{alg}(\mathbb{Q}_l)) \cap \epsilon_{l,K}^{-1}(Sp_{(V_l,\psi_l)}(\mathbb{Q}_l)) = G_{L_0} \cap G_{K_1} = G_{LK_1}.$$

4. Mumford-Tate group and Mumford-Tate conjecture

We next recall some standard definitions and results concerning the Mumford-Tate group. We continue to assume that A is an abelian variety over a number field F equipped with a fixed embedding into $\mathbb{C}$.

Definition 4.1. For $V = H_1(A, \mathbb{Q})$, define the cocharacter

$$\mu_{\infty,A} : \mathbb{G}_m(\mathbb{C}) \to GL(V_{\mathbb{C}})$$

such that for any $z \in \mathbb{C}^{\times}$, the automorphism $\mu_{\infty,A}(z)$ acts as multiplication by z on $H^{-1,0}$ and as the identity on $H^{0,-1}$. Observe that by definition we have $\mu_{\infty,A}(\mathbb{C}^{\times}) \subset GSp_{(V,\psi)}(\mathbb{C})$.

Definition 4.2. (Mumford-Tate and Hodge groups)

(1) The *Mumford-Tate group* of A/F is the smallest algebraic subgroup $MT(A) \subseteq GL_V$ over $\mathbb{Q}$ such that $MT(A)(\mathbb{C})$ contains $\mu_{\infty}(\mathbb{C})$.
(2) The *decomposable Hodge group* is $DH(A) := MT(A) \cap SL_V$.
(3) The *Hodge group* $H(A) := DH(A)^{\circ}$ is the connected component of the identity in $DH(A)$.

By definition, if K/F is a field extension such that $K \subset \overline{F}$, then $MT(A)$ and $H(A)$ for A/F and for A/K are the same. That is, we do not need to include K in the notation.

Remark 4.3. Note that $MT(A)$ is a reductive subgroup of $GSp_{(V,\psi)}$ [D2, Prop. 3.6], $H(A) \subseteq Sp_{(V,\psi)}$, and $MT(A) \subseteq C_D(GSp_{(V,\psi)})$. Hence

$$(4.1) \qquad\qquad H(A) \subseteq C_D(Sp_{(V,\psi)}).$$

Definition 4.4. The algebraic group $L(A) = C_D^{\circ}(Sp_{(V,\psi)})$ is called the *Lefschetz group*[1] of A. By (4.1) and the connectedness of $H(A)$, we have $H(A) \subseteq L(A)$.

Definition 4.5. For a field extension $L/\mathbb{Q}$ put

$$MT(A)_L := MT(A) \otimes_{\mathbb{Q}} L, \quad DH(A)_L := DH(A) \otimes_{\mathbb{Q}} L,$$
$$H(A)_L := H(A) \otimes_{\mathbb{Q}} L, \quad L(A)_L := L(A) \otimes_{\mathbb{Q}} L.$$

Theorem 4.6. *(Deligne* [D2, I, Prop. 6.2]*) For any prime number l,*

$$(4.2) \qquad\qquad (G_l^{alg})^{\circ} \subseteq MT(A)_{\mathbb{Q}_l}.$$

Conjecture 4.7. *(Mumford-Tate) For any prime number l,*

$$(4.3) \qquad\qquad (G_l^{alg})^{\circ} = MT(A)_{\mathbb{Q}_l}.$$

[1]This terminology is due to Murty [My].

Remark 4.8. Observe that (4.2) is equivalent to the inclusion

$$(4.4) \qquad (G_{l,K,1}^{alg})^{\circ} \subseteq H(A)_{\mathbb{Q}_l},$$

while the Mumford-Tate conjecture is equivalent to the equality

$$(4.5) \qquad (G_{l,K,1}^{alg})^{\circ} = H(A)_{\mathbb{Q}_l}.$$

This follows immediately from the following commutative diagram in which every column is exact and every horizontal arrow is a containment of corresponding group schemes.

$$
\begin{array}{ccccc}
1 & & 1 & & 1 \\
\downarrow & & \downarrow & & \downarrow \\
G_{l,K,1}^{alg} & \longrightarrow & DH(A)_{\mathbb{Q}_l} & \longrightarrow & Sp_{V_l,\psi_l} \\
\downarrow & & \downarrow & & \downarrow \\
G_{l,K}^{alg} & \longrightarrow & MT(A)_{\mathbb{Q}_l} & \longrightarrow & GSp_{V_l,\psi_l} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{G}_m & \overset{=}{\longrightarrow} & \mathbb{G}_m & \overset{=}{\longrightarrow} & \mathbb{G}_m \\
\downarrow & & \downarrow & & \downarrow \\
1 & & 1 & & 1
\end{array}
$$

It is known that the Mumford-Tate and Hodge groups do not behave well in general with respect to products of abelian varieties [G, p. 316]. However one can prove the following theorem.

Theorem 4.9. *The Mumford-Tate group of abelian varieties have the following properties.*

1. *An isogeny $\phi : A_1 \to A_2$ of abelian varieties induces isomorphisms $MT(A_1) \cong MT(A_2)$ and $H(A_1) \cong H(A_2)$.*
2. *If A is an abelian variety and $A^s := \prod_{i=1}^{s} A$, then $MT(A^s) \cong MT(A)$ and $H(A^s) \cong H(A)$.*

Proof. 1. This follows because an isogeny induces an isomorphism of associated polarised rational Hodge structures.

2. Observe that there is a natural isomorphism $H^1(A^s(\mathbb{C}), \mathbb{Q}) \cong \oplus_{i=1}^{s} H^1(A(\mathbb{C}), \mathbb{Q})$ or simply $V(A^s) \cong V(A)^s$. So in this case, the Hodge structure on $H^1(A^s(\mathbb{C}), \mathbb{Q})$ is isomorphic to the direct sum of the Hodge structures on $H^1(A(\mathbb{C}), \mathbb{Q})$. Via this identification, we have $\mu_{\infty,A^s} = \Delta(\mu_{\infty,A})$ where $\Delta : GSp(V_{\mathbb{C}}) \to GSp(V_{\mathbb{C}})^s \subseteq GL(V_{\mathbb{C}}^s)$ is the diagonal homomorphism. So we can consider $MT(A^s)$ as a subgroup of $MT(A)^s := \prod_{i=1}^{s} MT(A)$. Consider the diagonal homomorphism $\Delta : MT(A) \to MT(A)^s$. Since $\mu_{\infty,A^s}(\mathbb{G}_m(\mathbb{C})) = \Delta(\mu_{\infty,A})(\mathbb{G}_m(\mathbb{C})) \subseteq \Delta(MT(A))(\mathbb{C})$, we have $MT(A^s) \subseteq \Delta(MT(A)) \cong MT(A)$. Since $MT(A^s)(\mathbb{C})$ contains $\Delta(\mu_{\infty,A}(\mathbb{G}_m(\mathbb{C})))$ then the image of $MT(A^s)$ via projection $\pi : MT(A)^s \to MT(A)$ on every factor contains the image of $\mu_{\infty,A}(\mathbb{G}_m(\mathbb{C}))$. By the definition of $MT(A)$, this means that the projection π must surject onto $MT(A)$. On the other hand, since $MT(A^s) \subseteq \Delta(MT(A))$, it is clear that for π to be onto, we must have $MT(A^s) =$

$\Delta(MT(A))$. To get the isomorphism of Hodge groups, we note that the isomorphism $V(A^s) \cong V(A)^s$ gives an isomorphism $SL_{V(A^s)} \cong SL_{V(A)^s}$. We thus have the following isomorphisms of algebraic groups over $\mathbb{Q}$:

$$MT(A^s) \cap SL_{V(A^s)} \cong \Delta(MT(A)) \cap SL_{V(A)^s} \cong$$
$$\cong \Delta(MT(A)) \cap (SL_{V(A)})^s \cong MT(A) \cap SL_{V(A)}.$$

Taking connected components, we get the desired isomorphism of Hodge groups.
□

One can make a corresponding calculation also on the Galois side.

Theorem 4.10. *We have the following results.*

1. *An isogeny $\phi : A_1 \to A_2$ of abelian varieties over a number field K induces isomorphisms $G_{l,K}^{alg}(A_1) \cong G_{l,K}^{alg}(A_2)$ and $G_{l,K,1}^{alg}(A_1) \cong G_{l,K,1}^{alg}(A_2)$.*
2. *If A is an abelian variety over a number field K, then for any positive integer s, $G_{l,K}^{alg}(A^s) \cong G_{l,K}^{alg}(A)$ and $G_{l,K,1}^{alg}(A^s) = G_{l,K,1}^{alg}(A)$.*

Proof. 1. Observe that an isogeny ϕ induces an isomorphism of Galois modules $V_l(A_1) \cong V_l(A_2)$.

2. The natural isomorphism $V_l(A^s) \cong V_l(A)^s$ of G_K-modules shows that $\rho_{l,A^s} \cong \Delta\rho_{l,A}$, where $\Delta\rho_{l,A} : G_K \to GSp(V_l(A))^s$ is the natural diagonal representation $\Delta\rho_{l,A} = \mathrm{diag}(\rho_{l,A}, \ldots, \rho_{l,A})$.   □

Corollary 4.11. *Let A/K be an abelian variety such that the Mumford-Tate conjecture holds for A. Then the Mumford-Tate conjecture holds for A^s for any positive integer s.*

Proof. It follows from Theorems 4.9 and 4.10.   □

Remark 4.12. Observe that if the Mumford-Tate conjecture holds for A and K is a finite extension of F for which $G_{l,K}^{alg}$ is connected, then

$$(4.6) \qquad\qquad G_{l,K,1}^{alg}(A^s) = H(A^s)_{\mathbb{Q}_l}.$$

for any $s \geq 1$. Hence the algebraic Sato-Tate conjecture holds for A^s for any $s \geq 1$ with

$$(4.7) \qquad\qquad AST_K(A^s) = H(A^s).$$

5. Twisted Lefschetz groups and the algebraic Sato-Tate conjecture

We next use the Lefschetz group to define an upper bound on the algebraic Sato-Tate group. In those cases where the Mumford-Tate group is determined entirely by endomorphisms, this allows us to deduce the algebraic Sato-Tate conjecture from the Mumford-Tate conjecture. (An analogous construction for modular forms is the *twisting field* appearing in the work of Momose [Mo] and Ribet [R].) We continue to take A to be an abelian variety over a number field F embedded in $\mathbb{C}$ and K to be a finite extension of F.

Definition 5.1. Note that we have a continuous representation

$$(5.1) \qquad\qquad \rho_e : G_K \to Aut_{\mathbb{Q}}(D).$$

Put $G_{L_e} :=$ Ker ρ_e, so that L_e/K is a finite Galois extension. For $\tau \in G(L_e/K)$, define

$$(5.2) \qquad DL_K^\tau(A) := \{g \in Sp_V : g\beta g^{-1} = \rho_e(\tau)(\beta) \; \forall \beta \in D\}.$$

It is enough to impose the condition for β running over a $\mathbb{Q}$-basis of D; therefore, $DL_K^\tau(A)$ is a closed subscheme of Sp_V for each τ.

Definition 5.2. Define the *twisted decomposable algebraic Lefschetz group* of A over K to be the closed algebraic subgroup of $Sp_{V,\psi}$ given by

$$(5.3) \qquad DL_K(A) = \bigsqcup_{\tau \in G(L_e/K)} DL_K^\tau(A).$$

For any subextension L/K of $\overline{F}/K$ we have $DL_L(A) \subseteq DL_K(A)$ and $DL_L^{id}(A) = DL_K^{id}(A)$. Hence the classical Lefschetz group of A can be written as:

$$(5.4) \qquad L(A) = DL_K^{id}(A)^\circ = DL_K(A)^\circ.$$

and $L(A) = DL_L^{id}(A)^\circ = DL_L(A)^\circ$, for any subextension L/K of $\overline{F}/K$. In particular $DL_{L_e}^{id}(A) = DL_{L_e}(A) = DL_{\overline{F}}(A) = DL_{\overline{F}}^{id}(A)$ and $L(A) = DL_{L_e}(A)^\circ = DL_{\overline{F}}(A)^\circ$.

Theorem 5.3. *The twisted decomposable Lefschetz groups of abelian varieties have the following properties.*

1. *An isogeny $\phi : A_1 \to A_2$ of abelian varieties over K induces an isomorphism $DL_K^\tau(A_1) \cong DL_K^\tau(A_2)$ for every $\tau \in G(L_e/K)$, and consequently an isomorphism $DL_K(A_1) \cong DL_K(A_2)$.*

2. *If A/K is an abelian variety, then $DL_K^\tau(A^s) \cong DL_K^\tau(A)$ for every $\tau \in G(L_e/K)$.*

3. *If $A = \prod_{i=1}^t A_i$ with A_i/K simple for all i and $\mathrm{Hom}_{\overline{F}}(A_i, A_j) = 0$ for all $i \neq j$, then $DL_K^\tau(A) \cong \prod_{i=1}^t DL_K^\tau(A_i)$ for every $\tau \in G(L_e/K)$.*

4. *If $A = \prod_{i=1}^t A_i^{s_i}$ with A_i/K for all i and $\mathrm{Hom}_{\overline{F}}(A_i, A_j) = 0$ for all $i \neq j$, then $DL_K^\tau(A) \cong \prod_{i=1}^t DL_K^\tau(A_i)$ for every $\tau \in G(L_e/K)$.*

5. *If $A = \prod_{i=1}^t A_i^{s_i}$ with all A_i/K simple, pairwise nonisogenous then $DL_K^\tau(A) \cong \prod_{i=1}^t DL_K^\tau(A_i)$ for every $\tau \in G(L_e/K)$.*

Proof. 1. An isogeny induces the isomorphism $(V(A_1), \psi_{V(A_1)}) \cong (V(A_2), \psi_{V(A_2)})$ of $\mathbb{Q}$-bilinear spaces. It also induces an isomorphism $D(A_1) = \mathrm{End}_{\overline{K}}(A_1) \otimes_{\mathbb{Z}} \mathbb{Q} \cong D(A_2) = \mathrm{End}_{\overline{K}}(A_2) \otimes \mathbb{Q}$ of $\mathbb{Q}[G_K]$-modules, and so the first claim follows.
2. There is a natural isomorphism:

$$(V(A^s), \psi_{V(A^s)}) \cong (V(A), \psi_{V(A)})^s := \bigoplus_{i=1}^s (V(A), \psi_{V(A)})$$

and an isomorphism $D(A^s) \cong M_s(D(A))$ of $\mathbb{Q}$-algebras. Let Δ be the homomorphism that maps $Sp_{(V(A), \psi_{V(A)})}$ onto $\mathrm{diag}(Sp_{(V(A), \psi_{V(A)})}, \ldots, Sp_{(V(A), \psi_{V(A)})}) \subseteq Sp_{(\oplus_{i=1}^s (V(A), \psi_{V(A)}))}$. Since $\mathbb{Q} \subseteq D(A)$, we have $M_s(Q) \subseteq M_s(D(A))$; from the definition of the twisted decomposable Lefschetz group, we get $DL_K^\tau(A^s) \cong \Delta(DL_K^\tau(A)) \cong DL_K^\tau(A)$.
3. The proof is similar to the proof of 2. under the observation that $D(A) \cong \prod_{i=1}^t D(A_i)$ and $(V(A), \psi_{V(A)}) \cong \bigoplus_{i=1}^t (V(A_i), \psi_{V(A_i)})$.
4. This follows immediately from 2. and 3.
5. Immediate corollary from 4. $\qquad\qquad\square$

Remark 5.4. Theorem 5.3 remains true if we replace $DL_K^\tau(B)$ with $DL_K^\tau(B)^\circ$ for all abelian varieties B that appear in the theorem. Since $L(B) = DL_K^{id}(B)^\circ$, then the Lefschetz group satisfies properties 1-5 of Theorem 5.3. The property 5. of this theorem in the case of the Lefschetz group was obtained previously by K. Murty [My, Lemma 2.1], cf. [G, Lemma B.70].

Remark 5.5. Observe that we have

$$DL_K(A) := \{g \in Sp_{(V,\psi)} : \exists_{\tau \in G_K} \forall_{\beta \in D} \quad g\beta g^{-1} = \rho_e(\tau)(\beta)\}$$

Changing quantifiers we get another group scheme

(5.5) $$\widetilde{DL_K}(A) := \{g \in Sp_{(V,\psi)} : \forall_{\beta \in D} \exists_{\tau \in G_K} \quad g\beta g^{-1} = \rho_e(\tau)(\beta)\}$$

Observe that $DL_K(A) \subseteq \widetilde{DL_K}(A)$.

We now use the twisted Lefschetz group to obtain an upper bound on the algebraic Sato-Tate group.

Remark 5.6. Observe that (4.1) implies that

(5.6) $$DH(A) \subseteq DL_K^{id}(A) \subseteq DL_K(A).$$

On the other hand, for any $P \in A(\overline{F})$, any $\beta \in End_{\overline{F}}(A)$, and any $\sigma \in G_K$, we have $\sigma\beta\sigma^{-1}(P) = \sigma(\beta)(P)$. Hence $\rho_l(G_K)_1 \subseteq DL_K(A)(\mathbb{Q}_l)$. In particular,

(5.7) $$G_{l,K,1}^{alg} \subseteq DL_K(A)_{\mathbb{Q}_l}.$$

We will be particularly interested in cases where $G_{l,K,1}^{alg} = DL_K(A)_{\mathbb{Q}_l}$. We next check that this condition does not depend on the field K.

Remark 5.7. Let $\tilde\tau \in G_K$ be a lift of $\tau \in G(L_e/K)$. The coset $\tilde\tau G_{L_e}$ does not depend on the lift. The Zariski closure of $\rho_l(\tilde\tau G_{L_e}) = \rho_l(\tilde\tau)\rho_l(G_{L_e})$ in Sp_{V_l} is

(5.8) $$\rho_l(\tilde\tau) G_{l,L_e,1}^{alg} \subseteq DL_K^\tau(A)_{\mathbb{Q}_l}.$$

We observe that

(5.9) $$G_{l,K,1}^{alg} = \bigsqcup_{\tau \in G(L_e/K)} \rho_l(\tilde\tau) G_{l,L_e,1}^{alg}.$$

Since $DL_K^{id}(A) = DL_{L_e}(A) = DL_{\overline{F}}(A)$, we get

(5.10) $$G_{l,L_e,1}^{alg} \subseteq DL_{L_e}(A)_{\mathbb{Q}_l}.$$

Hence the following two equalities are equivalent:

(5.11) $$G_{l,L_e,1}^{alg} = DL_{L_e}(A)_{\mathbb{Q}_l},$$

(5.12) $$G_{l,K,1}^{alg} = DL_K(A)_{\mathbb{Q}_l}.$$

6. Some cases of the algebraic Sato-Tate conjecture

In general, the containment $H(A) \subseteq L(A)$ can be strict, which makes the Mumford-Tate conjecture a somewhat subtle problem; for instance, Mumford exhibited examples of simple abelian fourfolds for which $H(A) \neq L(A)$. Fortunately, when A has a large endomorphism algebra as compared to its dimension, such pathologies do not occur: one has $H(A) = L(A)$ and one can often show that the Mumford-Tate conjecture holds. When this happens, we will say that the Mumford-Tate conjecture for A is *explained by endomorphisms*.

The following theorem asserts that in cases where the Mumford-Tate conjecture is explained by endomorphisms *and* the twisted decomposable Lefschetz group over $\overline{F}$ is connected, the algebraic Sato-Tate conjecture is in a sense also explained by endomorphisms.

Theorem 6.1. *Assume that the following conditions hold.*

1. *We have $H(A) = L(A)$.*
2. *We have $(G_{l,K}^{alg})^\circ = MT(A)_{\mathbb{Q}_l}$.*
3. *The group $DL_{\overline{F}}(A)$ is connected.*

Then (5.12) holds for every l. Consequently, the algebraic Sato-Tate conjecture (Conjecture 2.3) holds for A/K with

$$(6.1) \qquad AST_K(A) = DL_K(A).$$

Proof. It is enough to prove (5.11). By our assumptions and Remark 4.8 we get $(G_{l,L_e,1}^{alg})^\circ = H(A)_{\mathbb{Q}_l} = L(A)_{\mathbb{Q}_l} = DL_{L_e}(A)_{\mathbb{Q}_l}$. Hence $DL_{L_e}(A)_{\mathbb{Q}_l}$ is also connected for every l and by (5.10) we obtain $(G_{l,L_e,1}^{alg})^\circ = G_{l,L_e,1}^{alg}$ for every l.   □

Remark 6.2. Under assumptions of Theorem 6.1 the results of Theorems 4.9, 4.10 and 5.3 show that the algebraic Sato-Tate conjecture holds for A^s for all $s \geq 1$ with $AST_K(A^s) = DL_K(A^s) \cong DL_K(A) = AST_K(A)$.

Conversely, if the algebraic Sato-Tate conjecture for A is explained by endomorphisms, so is the Mumford-Tate conjecture.

Theorem 6.3. *Assume that (5.12) and (6.1) hold for every l (so in particular, the algebraic Sato-Tate conjecture holds). We then have the following.*

1. *We have $H(A) = L(A)$.*
2. *We have $(G_{l,K}^{alg})^\circ = MT(A)_{\mathbb{Q}_l}$.*

Proof. By Theorem 4.6 and Remark 4.8 (see (4.4)), we have

$$(6.2) \qquad (G_{l,K,1}^{alg})^\circ \subseteq H(A)_{\mathbb{Q}_l} \subseteq L(A)_{\mathbb{Q}_l} = DL_K(A)_{\mathbb{Q}_l}^\circ.$$

By (6.1) we get

$$(6.3) \qquad (G_{l,K,1}^{alg})^\circ = H(A)_{\mathbb{Q}_l} = L(A)_{\mathbb{Q}_l} = DL_K(A)_{\mathbb{Q}_l}^\circ.$$

Hence by Remark 4.8, we obtain $(G_{l,K}^{alg})^\circ = MT(A)_{\mathbb{Q}_l}$. Moreover, since $H(A)$ is closed in $L(A)$, (6.3) gives

$$(6.4) \qquad H(A) = L(A)$$

as desired.   □

Remark 6.4. Recall that $L(A) = DL_K(A)^\circ \triangleleft DL_K^{id}(A) \triangleleft DL_K(A)$. Consider the following epimorphism of groups:

$$(6.5) \qquad DL_K(A)/L(A) \to DL_K(A)/DL_K^{id}(A) \cong G(L_e/K).$$

If A satisfies the assumptions of Theorem 6.1, then the epimorphism (6.5) is an isomorphism. This gives an identification $G(L_e/K) \cong AST_K(A)/AST_K(A)^\circ$.

We now assemble some cases where the Mumford-Tate and algebraic Sato-Tate conjecture are explained by endomorphisms. We start with some cases with A simple.

Definition 6.5. Put $g := \dim(A)$. We say A is of *CM type* if D contains a commutative $\mathbb{Q}$-subalgebra of dimension $2g$. If A is simple, then $DL_{\overline{F}}(A)$ is a torus of dimension at most g; equality holds if $g \leq 3$ [R, Example 3.7]. In this case, $L(A) = DL_{\overline{F}}(A)$.

Theorem 6.6. *Let A/F be an abelian variety of CM type. Then A satisfies the conditions of Theorem 6.1, so the algebraic Sato-Tate conjecture holds with $AST_K(A) = DL_K(A)$.*

Proof. This was shown by Serre using results of Pohlman; see [Se1].   □

Lemma 6.7. *Let A/F be an absolutely simple abelian variety for which the endomorphism algebra D is of type I, II, or III in the Albert classification. Then $C_D Sp_{(V,\psi)}$ is connected.*

Proof. This follows from [BGK1, §7] in the type I and II cases and [BGK2, §5] in the type III case.   □

Theorem 6.8. *Let A/F be an absolutely simple abelian variety of dimension g for which the endomorphism algebra D is of type I, II, or III in the Albert classification. Let E be the center of D and put $e := [E : \mathbb{Q}]$, $d^2 = [D : E]$. Assume that $\frac{g}{de}$ is odd. Then A satisfies the conditions of Theorem 6.1, so the algebraic Sato-Tate conjecture holds with $AST_K(A) = DL_K(A)$.*

Proof. Recall that $MT(A)$, $(G_{l,K,}^{alg})^\circ$, $(G_{l,K,1}^{alg})^\circ$ are independent of the field extension K/F. We may thus take K large enough that $G_{l,K}^{alg}$ is connected. In this case, conditions 1 and 2 of Theorem 6.1 are established in [BGK1] (in the type I and II case) and [BGK2] (in the type III case). Condition 3 holds by Lemma 6.7.   □

One can also show that the Mumford-Tate conjecture and the algebraic Sate-Tate conjecture are explained by endomorphisms for all abelian varieties of dimension at most 3. This result is the starting point of the analysis of the Sato-Tate conjecture for abelian surfaces given in [?]; the corresponding analysis for threefolds has not yet been carried out.

Lemma 6.9. *For $i = 1, 2$, let A_i/F be an abelian variety satisfying the Mumford-Tate conjecture. Suppose that over $\overline{F}$, A_1 has no factors of type IV while A_2 either is of CM type or has no factors of type IV. Suppose in addition over $\overline{F}$, there is no nontrivial homomorphism from A_1 to A_2. Then $A = A_1 \times A_2$ also satisfies the Mumford-Tate conjecture and $H(A) = H(A_1) \times H(A_2)$.*

Proof. Let G, G_1, G_2 be the groups $(G_{l,F,1}^{alg})^\circ$ associated to A, A_1, A_2. By hypothesis, we have $G_1 = H(A_1)_{\mathbb{Q}_l}$ and $G_2 = H(A_2)_{\mathbb{Q}_l}$. Since A_1 has no factors of type IV, G_1 is semisimple [T, Lemma 1.4]. Similarly, if A_2 has no factors of type IV, then G_2 is semisimple; if instead A_2 is of CM type, then G_2 is a torus.

There is a natural inclusion $G \subseteq G_1 \times G_2$ with the property that the induced projection maps $\pi_1 : G \to G_1$ and $\pi_2 : G \to G_2$ are surjective. Since G is reductive, we can write G as an almost direct product $G' \cdot T$ with G' semisimple and T a torus.

In case A_2 is of CM type, we argue following Imai [I]. In this case, $p(G') = p([G, G]) = [p(G), p(G)] = G_1$ because G_1 is semisimple, while $q(T) = q(G) = G_2$ because G' has no nontrivial character. Hence $\dim(G) \geq \dim(G_1 \times G_2)$, forcing $G = G_1 \times G_2$. By the same reasoning, $H(A_1) = H(A_1) \times H(A_2)$.

In case A_2 has no factors of type IV, then an argument of Hazama [Ha, Proposition 1.8] shows that $H(A) = H(A_1) \times H(A_2)$. The same argument implies that if $G \neq G_1 \times G_2$, there is a nonzero G-equivariant homomorphism $V_l(A_1) \to V_l(A_2)$. However, the latter would imply the existence of a nonzero homomorphism $A_1 \to A_2$ of abelian varieties over $\overline{F}$, by virtue of Faltings's proof of the Tate conjecture for abelian varieties over number fields [F].

In both cases, we conclude $G = H(A)_{\mathbb{Q}_l} = H(A_1)_{\mathbb{Q}_l} \times H(A_2)_{\mathbb{Q}_l}$, as desired.   □

Theorem 6.10. *Let A/F be an abelian variety of dimension at most 3. Then A satisfies the conditions of Theorem 6.1, so the Mumford-Tate conjecture holds with $H(A) = L(A)$, and the algebraic Sato-Tate conjecture holds with $AST_K(A) = DL_K(A)$.*

Proof. Since we are free to enlarge F, we may assume that all simple factors of A are absolutely simple. We may invoke [MZ, Theorem 0.1] to establish condition 1 of Theorem 6.1 in all cases we are considering, so we focus on conditions 2 and 3.

We start with the case where A is simple. If A is a CM elliptic curve, it is straightforward to check condition 2 (or one can apply Theorem 6.6) and condition 3 holds because $DL_{\overline{F}}(A)$ is a torus. If A is a non-CM elliptic curve, condition 2 follows from Serre's open image theorem or from Theorem 6.8, and condition 3 holds because $DL_{\overline{F}}(A) = SL_2$. In the other cases, the dimension is prime, so we may deduce condition 2 from [Ch, Concluding Remark] and condition 3 from a theorem of Tankeev [MZ, Theorem 2.7].

We next consider the case where A is not simple. By Theorem 5.3, condition 3 for A reduces to the corresponding condition for each of the simple factors of A. We thus need only check condition 2. In case A is isogenous to a product of elliptic curves, this is a result of Imai [I]. The only other possibility is that A is isogenous to the product of an elliptic curve E with a simple abelian surface B; this case may be treated using Lemma 6.9.   □

Remark 6.11. As noted earlier, Theorem 6.10 cannot be extended to dimension 4, due in part to Mumford's examples of absolutely simple abelian fourfolds for which the Hodge conjecture is not explained by endomorphisms. However, there are also some nonsimple examples with the same property. For a complete classification of Hodge groups for abelian varieties of dimension at most 5, see [MZ].

7. Category of motives for absolute Hodge cycles

Even for abelian varieties for which the Mumford-Tate conjecture is not explained by endomorphisms, one can infer from Serre [Se4] a proposed construction of the general algebraic Sato-Tate group in terms of motivic Galois groups and absolute[2] Hodge cycles [D1, D2, DM]. In the remainder of the paper, we make the ℓ-adic interpretation of this construction explicit, and show that if a suitably motivic form of the Mumford-Tate conjecture holds for a particular abelian variety, then the algebraic Sato-Tate conjecture holds as well.

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[2]Since we consider only the motives associated to abelian varieties, the adjective *absolute* is rendered unnecessary by Deligne's theorem equating Hodge cycles with absolute Hodge cycles [D1]. However, we prefer to speak of *absolute Hodge cycles* in order to emphasize the action of Galois on them, which is crucial for our construction of the disconnected part of the algebraic Sato-Tate group.

To begin with, we recall some terminology and set some notation concerning Tannakian categories. Let $\mathcal{C}$ be a unital, $\mathbb{Q}$-linear Tannakian category equipped (as in [DM, Definition 2.19]) with a $\mathbb{Q}$-linear, faithful tensor functor $\omega : \mathcal{C} \to \text{Vec}_{\mathbb{Q}}$, called the *fiber functor* of $\mathcal{C}$.

Definition 7.1. For $X, Y \in \text{obj}(\mathcal{C})$, let $\underline{\text{Hom}}_{\mathcal{C}}(X, Y) \in \text{obj}(\mathcal{C})$ be the object which represents the functor

$$\mathcal{C}^{\circ} \to \text{Set}, \qquad T \mapsto \text{Hom}_{\mathcal{C}}(T \otimes X, Y).$$

That is, for every $T \in \text{obj}(\mathcal{C})$ there is a natural bijection

$$\text{Hom}_{\mathcal{C}}(X \otimes T, Y) \cong \text{Hom}_{\mathcal{C}}(T, \underline{\text{Hom}}_{\mathcal{C}}(X, Y)).$$

One puts $X^{\vee} := \underline{\text{Hom}}_{\mathcal{C}}(X, \underline{1})$ and one gets $\underline{\text{Hom}}_{\mathcal{C}}(X, Y) \cong X^{\vee} \otimes Y$.

Proposition 7.2. ([DM, Proposition 1.9]) *For a functor $F : \mathcal{C} \to \mathcal{C}'$ of unital Tannakian categories and $X, Y \in \text{obj}(\mathcal{C})$, there is a natural isomorphism*

$$(7.1) \qquad F(\underline{\text{Hom}}_{\mathcal{C}}(X, Y)) \cong \underline{\text{Hom}}_{\mathcal{C}'}(F(X), F(Y)).$$

Let us consider now the case of motives.

Definition 7.3. Let K be a number field and let $\mathcal{M}_K$ be the motivic category for absolute Hodge cycles: $\text{Hom}_{\mathcal{M}_K}(M, N) := \text{Mor}^0_{AH}(X, Y)$ (see [DM]). Define $\mathcal{M}_{\overline{K}}$ similarly. Let H_B be the fiber functor given by Betti realization:

$$(7.2) \qquad H_B : \mathcal{M}_K \to \text{Vec}_{\mathbb{Q}}.$$

The functor H_B factors through the functor

$$(7.3) \qquad \mathcal{M}_K \to \mathcal{M}_{\overline{K}}, \qquad M \mapsto \overline{M} := M \otimes \overline{K}.$$

Definition 7.4. Let $\{X_i\}$ be a family of varieties over K and let $\mathcal{C} \subset \mathcal{M}_K$ be the full Tannakian subcategory generated by $\{h(X_i)\}$. Let $\overline{\mathcal{C}}$ be the smallest full Tannakian subcategory of $\mathcal{M}_{\overline{K}}$ containing the image of $\mathcal{C}$ via the functor (7.3).

The category $\overline{\mathcal{C}}$ is the full Tannakian subcategory of $\mathcal{M}_{\overline{K}}$ generated by $\{h(\overline{X_i})\}$, c.f. [DM, pp. 215-216]. For $M, N \in \text{obj}(\mathcal{C})$ the natural map

$$(7.4) \qquad \text{Hom}_{\overline{\mathcal{C}}}(\overline{M}, \overline{N}) \times \overline{M} \to \overline{N}$$

gives the natural map

$$(7.5) \qquad \text{Hom}_{\overline{\mathcal{C}}}(\overline{M}, \overline{N}) \to \underline{\text{Hom}}_{\overline{\mathcal{C}}}(\overline{M}, \overline{N})$$

where $\text{Hom}_{\overline{\mathcal{C}}}(\overline{M}, \overline{N}) = \text{Hom}_{\mathcal{M}_{\overline{K}}}(\overline{M}, \overline{N})$ is a finite dimensional vector space which is also a discrete G_K-module, so it can be considered as an Artin motive in $\mathcal{M}_K$. Moreover by (7.1) we get $\underline{\text{Hom}}_{\overline{\mathcal{C}}}(\overline{M}, \overline{N}) \cong \underline{\text{Hom}}_{\mathcal{M}_{\overline{K}}}(\overline{M}, \overline{N})$. Hence the map (7.5) is naturally isomorphic to the map

$$(7.6) \qquad \text{Hom}_{\mathcal{M}_{\overline{K}}}(\overline{M}, \overline{N}) \to \underline{\text{Hom}}_{\mathcal{M}_{\overline{K}}}(\overline{M}, \overline{N}).$$

Let L/K be a field extension with $L \subset \overline{K}$. For any variety X/L of pure dimension m, one sets $h(X)^{\vee} := h(X)(m)$, which extends to the contravariant functor

$$\mathcal{M}_L \to \mathcal{M}_L, \qquad M \mapsto M^{\vee}$$

and one sets (see [DM, p. 205])

$$(7.7) \qquad \underline{\text{Hom}}_{\mathcal{M}_L}(M, N) := M^{\vee} \otimes N.$$

Since the map (7.4) commutes with the G_K-action, using (7.7) the maps (7.5) and (7.6) also commute with the G_K-action.

Proposition 7.5. *There is a natural morphism of motives in $\mathcal{M}_K$:*

$$\text{(7.8)} \qquad \text{Hom}_{\mathcal{M}_{\overline{K}}}(\overline{M}, \overline{N}) \to \underline{\text{Hom}}_{\mathcal{C}}(M, N),$$

which is an embedding of motives.

Proof. The existance of the morphism (7.8) follows from the definition of morphisms for the category of motives for absolute Hodge cycles. Let $\underline{1}$ be the unit of $\mathcal{M}_{\overline{K}}$. Applying $\text{Hom}_{\mathcal{M}_{\overline{K}}}(\underline{1}, \,.\,)$ to the map (7.6) one immediately gets the identity map $\text{Hom}_{\mathcal{M}_{\overline{K}}}(\overline{M}, \overline{N}) \to \text{Hom}_{\mathcal{M}_{\overline{K}}}(\overline{M}, \overline{N})$. This shows that the map (7.8) is an embedding of motives. $\qquad\square$

8. Motivic Mumford-Tate and Sato-Tate groups

Using the Tannakian formalism, we now define a motivic Sato-Tate group by reprising the setup from §3.

Definition 8.1. Let $G_{\mathcal{M}_K} := \text{Aut}^{\otimes}(H_B)$ and let $\mathcal{M}_K(A)$ be the smallest Tannakian subcategory of $\mathcal{M}_K$ containing A. Let $H_B|\mathcal{M}_K(A)$ be the restriction of H_B to the subcategory $\mathcal{M}_K(A)$. Put $G_{\mathcal{M}_K(A)} := \text{Aut}^{\otimes}(H_B|\mathcal{M}_K(A))$.

The algebraic groups $G_{\mathcal{M}_K(A)}$ are reductive but not necessarily connected (see [Se4, p. 379]). Observe that the ring of endomorphisms $D := \text{End}_{\overline{K}}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a discrete G_K-module which is also a finite dimensional $\mathbb{Q}$-vector space. Moreover (see [DM, p. 213]),

$$\text{(8.1)} \qquad \text{End}_{\overline{K}}(A) \otimes_{\mathbb{Z}} \mathbb{Q} = \text{End}_{\overline{K}}(\overline{A}) \otimes_{\mathbb{Z}} \mathbb{Q} = \text{End}_{\mathcal{M}_{\overline{K}}}(h^1(\overline{A})).$$

We may thus consider D as an Artin motive, as follows. Recall that $\mathcal{M}_K^0$ is equivalent to $\text{Rep}_{\mathbb{Q}}(G_K)$, the category of finite dimensional $\mathbb{Q}$-vector spaces with continuous actions of G_K.

Definition 8.2. Let $h^0(D)$ denote the Artin motive corresponding to D. Let $\mathcal{M}_K^0(D)$ be the smallest Tannakian subcategory of $\mathcal{M}_K^0$ containing $h^0(D)$ and put $G_{\mathcal{M}_K^0(D)} := \text{Aut}^{\otimes}(H_B^0|\mathcal{M}_K^0(D))$.

By (7.1), Proposition 7.5, and (8.1), we have a natural embedding of motives

$$\text{(8.2)} \qquad h^0(D) \subset \underline{\text{End}}_{\mathcal{M}_K(A)}(h^1(A)) = \underline{\text{End}}_{\mathcal{M}_K}(h^1(A)).$$

Note that $\underline{\text{End}}_{\mathcal{M}_K(A)}(h^1(A)) = h^1(A) \otimes h^1(A)^{\vee} \in \mathcal{M}_K(A)$. Recall that $G_{\mathcal{M}_K^0} \cong G_K$ so we observe that

$$G_{\mathcal{M}_K^0(D)} \cong G(L_e/K).$$

Since $\mathcal{M}_K$ is semisimple [DM, Prop. 6.5], the motive $h^0(D)$ splits off of $\underline{\text{End}}_{\mathcal{M}_K(A)}(h^1(A))$ in $\mathcal{M}_K$. Moreover the semisimplicity of $\mathcal{M}_K$, together with the observation that $\mathcal{M}_K^0$ and $\mathcal{M}_K(A)$ are full subcategories of $\mathcal{M}_K$, shows that the top horizontal and left vertical maps in the following diagram are faithfully flat (see [DM, (2.29)]):

$$\text{(8.3)} \qquad \begin{array}{ccc} G_{\mathcal{M}_K} & \longrightarrow & G_K \\ \downarrow & & \downarrow \\ G_{\mathcal{M}_K(A)} & \longrightarrow & G(L_e/K) \end{array}$$

In particular all homomorphisms in (8.3) are surjective.

Since $V := H_1(A(\mathbb{C}), \mathbb{Q})$ admits a $\mathbb{Q}$-rational polarized Hodge structure of weight 1, we have

$$(8.4) \qquad G_{\mathcal{M}_K(A)} \subset GSp_{(V,\psi)}.$$

Definition 8.3. Define the following algebraic groups:

$$G_{\mathcal{M}_K(A),1} := G_{\mathcal{M}_K(A)} \cap Sp_V$$
$$G^\circ_{\mathcal{M}_K(A),1} := (G_{\mathcal{M}_K(A)})^\circ \cap Sp_V.$$

Remark 8.4. In [Se4, p. 396] the group $G_{\mathcal{M}_K(A),1}$ is denoted $G^1_{\mathcal{M}_K(A)}$.

Definition 8.5. For any $\tau \in G(L_e/K)$, put

$$(8.5) \qquad GSp^\tau_{(V,\psi)} := \{g \in GSp_{(V,\psi)} : \ g\beta g^{-1} = \rho_e(\tau)(\beta) \ \forall \beta \in D\}.$$

We have the following equality:

$$(8.6) \qquad GSp_{(V,\psi)} = \bigsqcup_{\tau \in G(L_e/K)} GSp^\tau_{(V,\psi)}.$$

Observe that

$$(8.7) \qquad GSp^{Id}_{(V,\psi)} = C_D(GSp_{(V,\psi)}).$$

Remark 8.6. The bottom horizontal arrow in the diagram (8.3) is

$$(8.8) \qquad G_{\mathcal{M}_K(A)} \to G_{\mathcal{M}_K(D)} \cong G(L_e/K).$$

Let $g \in G_{\mathcal{M}_K(A)}$ and let $\tau := \tau(g)$ be the image of g via the map (8.8). Hence for any element $\beta \in D$ considered as an endomorphism of V we have:

$$(8.9) \qquad g\beta g^{-1} = \rho_e(\tau)(\beta).$$

Definition 8.7. For any $\tau \in G(L_e/K)$, put

$$(8.10) \qquad G^\tau_{\mathcal{M}_K(A)} := \{g \in G_{\mathcal{M}_K(A)} : \ g\beta g^{-1} = \rho_e(\tau)(\beta), \ \forall \beta \in D\}.$$

It follows from (8.9), (8.10), and the surjectivity of (8.8) that

$$(8.11) \qquad G_{\mathcal{M}_K(A)} = \bigsqcup_{\tau \in G(L_e/K)} G^\tau_{\mathcal{M}_K(A)}$$

It is clear from (8.4) and (8.5) that

$$(8.12) \qquad G^\tau_{\mathcal{M}_K(A)} \subset GSp^\tau_{(V,\psi)}.$$

Hence (8.9) and (8.11) give

$$(8.13) \qquad (G_{\mathcal{M}(A)})^\circ \lhd G^{Id}_{\mathcal{M}_K(A)} \lhd G_{\mathcal{M}_K(A)}.$$

The map (8.8) gives the following natural map:

$$(8.14) \qquad G_{\mathcal{M}_K(A),1} \to G(L_e/K).$$

Definition 8.8. For any $\tau \in G(L_e/K)$ put

$$(8.15) \qquad G^\tau_{\mathcal{M}_K(A),1} := \{g \in G_{\mathcal{M}_K(A),1} : \ g\beta g^{-1} = \rho_e(\tau)(\beta), \ \forall \beta \in D\}.$$

It follows that there is the following equality

$$(8.16) \qquad G^{\tau}_{\mathcal{M}_K(A),1} = G_{\mathcal{M}_K(A),1} \cap G^{\tau}_{\mathcal{M}_K(A)}.$$

Hence for $\tau = \tau(g)$, we have

$$(8.17) \qquad G^{\tau}_{\mathcal{M}_K(A),1} \subset DL^{\tau}_K(A)$$
$$(8.18) \qquad G_{\mathcal{M}_K(A),1} \subset DL_K(A).$$

Naturally we also have

$$(8.19) \qquad (G_{\mathcal{M}(A),1})^{\circ} \lhd G^{Id}_{\mathcal{M}_K(A),1} . \lhd G_{\mathcal{M}_K(A),1}$$

Hence by (8.16) we get

$$(8.20) \qquad G_{\mathcal{M}(A),1}/G^{Id}_{\mathcal{M}_K(A),1} \subset G_{\mathcal{M}(A)}/G^{Id}_{\mathcal{M}_K(A)}.$$

We have the following analogue of Theorem 3.4.

Theorem 8.9. *Assume that $G^{\circ}_{\mathcal{M}_K(A),1}$ is connected. Then the following map is an isomorphism:*

$$i_M \; : \; G_{\mathcal{M}_K(A),1}/(G_{\mathcal{M}_K(A),1})^{\circ} \xrightarrow{\cong} G_{\mathcal{M}_K(A)}/(G_{\mathcal{M}_K(A)})^{\circ}.$$

Proof. Consider the following diagram, in which we write $\mathcal{M}(A)$ for $\mathcal{M}_K(A)$ to make the notation simpler.

By definition the rows are exact and the middle column is exact. Hence the map i_M is surjective. Since $G^{\circ}_{\mathcal{M}_K(A),1}$ has the same dimension as $G_{\mathcal{M}_K(A),1}$ and by assumption $G^{\circ}_{\mathcal{M}_K(A),1}$ is connected, we then have $G^{\circ}_{\mathcal{M}_K(A),1} = (G_{\mathcal{M}_K(A),1})^{\circ}$. Hence the left column is also exact. This shows that i_M is an isomorphism. $\qquad \square$

Remark 8.10. Since $G_{\mathcal{M}(A)}$ is reductive, the middle vertical column of the diagram of the proof of Theorem 8.9) shows that $G_{\mathcal{M}(A),1}$ is also reductive.

Corollary 8.11. *Assume that $G^\circ_{\mathcal{M}_K(A),1}$ is connected. Then there are natural isomorphisms*

$$(8.21) \qquad G_{\mathcal{M}_K(A),1}/G^{Id}_{\mathcal{M}_K(A),1} \xrightarrow{\cong} G_{\mathcal{M}_K(A)}/G^{Id}_{\mathcal{M}_K(A)},$$

$$(8.22) \qquad G^{Id}_{\mathcal{M}_K(A),1}/(G_{\mathcal{M}_K(A),1})^\circ \xrightarrow{\cong} G^{Id}_{\mathcal{M}_K(A)}/(G_{\mathcal{M}_K(A)})^\circ,$$

$$(8.23) \qquad G_{\mathcal{M}_K(A),1}/G^{Id}_{\mathcal{M}_K(A),1} \xrightarrow{\cong} DL_K(A)/DL_K(A)^{Id} \xrightarrow{\cong} G(L_e/K).$$

In particular the natural map (8.14) is surjective.

Proof. This follows from (8.13), (8.19), (8.20), the surjectivity of (8.8), and Theorem 8.9. $\qquad\square$

Remark 8.12. The l-adic representation $\rho_l : G_K \to G_{\mathcal{M}_K(A)}(\mathbb{Q}_l)$ associated with A factors through $G_{\mathcal{M}_K(A)}(\mathbb{Q}_l)$ (see [Se4, p. 386]). Hence

$$(8.24) \qquad\qquad\qquad G^{alg}_{l,K} \subset G_{\mathcal{M}_K(A)_{\mathbb{Q}_l}}$$

where $G_{\mathcal{M}_K(A)_{\mathbb{Q}_l}} := G_{\mathcal{M}_K(A)} \otimes_{\mathbb{Q}} \mathbb{Q}_l$. By the commutativity of the diagram

$$
\begin{array}{ccc}
G^{alg}_{l,K,1} & \longrightarrow & G_{\mathcal{M}_K(A),1_{\mathbb{Q}_l}} \\
\downarrow & & \downarrow \\
G^{alg}_{l,K} & \longrightarrow & G_{\mathcal{M}_K(A)_{\mathbb{Q}_l}} \\
\downarrow & & \downarrow \\
\mathbb{G}_m & \xrightarrow{\;=\;} & \mathbb{G}_m
\end{array}
$$

it follows that

$$(8.25) \qquad\qquad\qquad G^{alg}_{l,K,1} \subset G_{\mathcal{M}_K(A),1_{\mathbb{Q}_l}}.$$

Definition 8.13. The algebraic groups

$$MMT_K(A) := G_{\mathcal{M}_K(A)}$$
$$MST_K(A) := G_{\mathcal{M}_K(A),1}$$

will be called the *motivic Mumford-Tate group* and *motivic Sato-Tate group* for A.

Conjecture 8.14. *(Motivic Mumford-Tate) For any prime number l,*

$$(8.26) \qquad\qquad\qquad G^{alg}_{l,K} = MMT_K(A)_{\mathbb{Q}_l}.$$

By the diagram above, Conjecture 8.14 is equivalent to the following.

Conjecture 8.15. *(Motivic Sato-Tate) For any prime number l,*

$$(8.27) \qquad\qquad\qquad G^{alg}_{l,K,1} = MST_K(A)_{\mathbb{Q}_l}.$$

Remark 8.16. Conjecture 8.14 is equivalent to the conjunction of the following equalities:

$$(8.28) \qquad\qquad\qquad (G^{alg}_{l,K})^\circ = (MMT_K(A)_{\mathbb{Q}_l})^\circ$$

$$(8.29) \qquad\qquad\qquad \pi_0(G^{alg}_{l,K}) = \pi_0(MMT_K(A)_{\mathbb{Q}_l}).$$

Similarly, Conjecture 8.15 is equivalent to the conjunction of the following equalities:

$$(8.30) \qquad (G_{l,K,1}^{alg})^\circ = (MST_K(A)_{\mathbb{Q}_l})^\circ$$

$$(8.31) \qquad \pi_0(G_{l,K,1}^{alg}) = \pi_0(MST_K(A)_{\mathbb{Q}_l}).$$

Theorem 8.17. *We have the following results.*

1. *An isogeny $\phi : A_1 \to A_2$ of abelian varieties over a number field K induces isomorphisms $G_{\mathcal{M}_K(A_1)} \cong G_{\mathcal{M}_K(A_2)}$ and $G_{\mathcal{M}_K(A_1),1} \cong G_{\mathcal{M}_K(A_2),1}$.*
2. *If A is an abelian variety over a number field K, then for every positive integer s, $\quad G_{\mathcal{M}_K(A^s)} \cong G_{\mathcal{M}_K(A)}$ and $G_{\mathcal{M}_K(A^s),1} \cong G_{\mathcal{M}_K(A),1}$.*

Proof. 1. Observe that an isogeny ϕ induces a natural isomorphism $V(A_1) \cong V(A_2)$ and a natural equivalence of categories $\mathcal{M}_K(A_1) \cong \mathcal{M}_K(A_2)$. This leads to a natural equivalence of fiber functors $H_B \,|\, \mathcal{M}_K(A_1) \cong H_B \,|\, \mathcal{M}_K(A_2)$.

2. By the Künneth formula, there is an isomorphism of motives $h^1(A^s) \cong h^1(A)^s$. In addition for any abelian variety B over K we have $h(B) = \wedge h^1(B)$. This shows that the diagonal map $\Delta : A \to A^s$ induces a natural equivalence of categories $\mathcal{M}_K(A) \cong \mathcal{M}_K(A^s)$. Moreover there is a natural isomorphism $V(A^s) \cong V(A)^s$. Hence there is a natural equivalence of fiber functors $H_B \,|\, \mathcal{M}_K(A^s) \cong H_B \,|\, \mathcal{M}_K(A)$. $\qquad \square$

Corollary 8.18. *Let A/K be an abelian variety such that the motivic Mumford-Tate and motivic Sato-Tate conjectures hold for A. Then the motivic Mumford-Tate and motivic Sato-Tate conjectures hold for A^s for every positive integer s.*

Proof. This follows from Theorems 4.10 and 8.17. $\qquad \square$

9. The algebraic Sato-Tate group

To conclude, we introduce a candidate for the algebraic Sato-Tate group for an arbitrary abelian variety.

Remark 9.1. One observes (cf. [Se4, p. 379]) that

$$(9.1) \qquad MT(A) \subset (G_{\mathcal{M}_K(A)})^\circ$$

and it was proven by Deligne that

$$(9.2) \qquad (G_{l,K}^{alg})^\circ \subset MT(A)_{\mathbb{Q}_l}.$$

It follows by (8.17), (8.19), and (9.1) that for every abelian variety A/K there is a sequence of natural containments

$$(9.3) \qquad H(A) \subset (G_{\mathcal{M}_K(A),1})^\circ \subset G_{\mathcal{M}_K(A),1}^{Id} \subset C_D(Sp_{(V,\psi)}).$$

In the same way, it follows by (8.7), (8.12), (8.13), and (9.1) that for every abelian variety A/K there is a sequence of natural containments

$$(9.4) \qquad MT(A) \subset (G_{\mathcal{M}_K(A)})^\circ \subset G_{\mathcal{M}_K(A)}^{Id} \subset C_D(GSp_{(V,\psi)}).$$

We observe that the equality

$$(9.5) \qquad H(A) = C_D(Sp_{(V,\psi)})$$

is equivalent to the following equality:

$$(9.6) \qquad MT(A) = C_D(GSp_{(V,\psi)}).$$

Remark 9.2. It is conjectured [Se4, sec. 3.4] that $MT(A) = MMT_K(A)^\circ = (G_{\mathcal{M}_K(A)})^\circ$, and in [Se4, p. 380] J.-P. Serre gave examples where this conjecture holds. Theorems 9.3 and 9.4 below gives more evidence for this conjecture.

Theorem 9.3. *Let A/F be an absolutely simple abelian variety of dimension g for which the endomorphism algebra D is of type I, II, or III in the Albert classification. Let E be the center of D and put $e := [E : \mathbb{Q}]$, $d^2 = [D : E]$. Assume that $\frac{g}{de}$ is odd. Then*

$$(9.7) \qquad H(A) = MST_K(A)^\circ = (G_{\mathcal{M}_K(A),1})^\circ = C_D(Sp_{(V,\psi)}),$$

$$(9.8) \qquad MT(A) = MMT_K(A)^\circ = (G_{\mathcal{M}_K(A)})^\circ = C_D(GSp_{(V,\psi)}).$$

Proof. By [BGK1, Cor. 7.19] and [BGK2, Cor. 5.19], for abelian varieties satisfying the assumptions of the theorem, the equality (9.5) follows. $\square$

Theorem 9.4. *Let A/F be an abelian variety of dimension at most 3. Then the equalities (9.7) and (9.8) hold also for A.*

Proof. For abelian varieties of dimension at most 3 the equality (9.5) holds by Theorem 6.10. $\square$

Remark 9.5. To obtain $G_{l,K}^{alg}$ as an extension of scalars to $\mathbb{Q}_l$ of an expected algebraic Sato-Tate group defined over $\mathbb{Q}$, the assumption in the following definition is natural in view of (9.1), (9.2), and Theorem 8.9.

Definition 9.6. Assume that $MT(A) = MMT_K(A)^\circ$. Then the *algebraic Sato-Tate group* $AST_K(A)$ is defined as follows:

$$(9.9) \qquad AST_K(A) := MST_K(A) = G_{\mathcal{M}_K(A),1}.$$

Every maximal compact subgroup of $AST_K(A)(\mathbb{C})$ will be called a *Sato-Tate group* associated with A/K and denoted $ST_K(A)$.

Corollary 9.7. *Let A/K be an abelian variety such that $MT(A) = MMT_K(A)^\circ$ and the algebraic Sato-Tate conjecture holds for A. Then the algebraic Sato-Tate conjecture holds for A^s for every positive integer s.*

Proof. It follows immediately from Corollary 8.18. $\square$

We now prove that the group $AST_K(A)$ satisfies the expected properties stated in Remark 2.5.

Theorem 9.8. *Assume that $MT(A) = MMT_K(A)^\circ$. Then the group $AST_K(A)$ is reductive and:*

$$(9.10) \qquad AST_K(A) \subset DL_K(A),$$

$$(9.11) \qquad AST_K(A)^0 = H(A),$$

$$(9.12) \qquad \pi_0(AST_K(A)) = \pi_0(MMT_K(A)),$$

$$(9.13) \qquad \pi_0(AST_K(A)) = \pi_0(ST_K(A)).$$

In addition, there are the following exact sequences:

$$(9.14) \qquad 0 \to H(A) \to AST_K(A) \to \pi_0(AST_K(A)) \to 0$$

$$(9.15) \qquad 0 \to \pi_0(G_{\mathcal{M}_K(A),1}^{Id}) \to \pi_0(AST_K(A)) \to G(L_e/K) \to 0.$$

Proof. The group $AST_K(A)$ is reductive by Remark 8.10, and (9.10) follows by (8.18). Equality (9.13) follows since $AST_K(A)^\circ(\mathbb{C})$ is a connected complex Lie group and any maximal compact subgroup of a connected complex Lie group is a connected real Lie group. Assuming that $MT(A) = (G_{\mathcal{M}_K(A)})^\circ$, we have

$$(9.16) \qquad G^\circ_{\mathcal{M}_K(A),1} = (G_{\mathcal{M}_K(A)})^\circ \cap Sp_{(V,\psi)} = MT(A) \cap Sp_{(V,\psi)} = H(A),$$

so $G^\circ_{\mathcal{M}_K(A),1}$ is connected. Hence

$$(9.17) \qquad AST_K(A)^0 = (G_{\mathcal{M}_K(A),1})^\circ = G^\circ_{\mathcal{M}_K(A),1},$$

so (9.11) follows. Hence the equality (9.12) and the exact sequence (9.14) may be extracted from the top row of the diagram from the proof of Theorem 8.9. The exactness of the sequence (9.15) follows immediately from Corollary 8.11. $\square$

Remark 9.9. The containments (9.3) give some approximation for $\pi_0(G^{Id}_{\mathcal{M}_K(A),1})$. Moreover, under the assumption $MT(A) = (G_{\mathcal{M}_K(A)})^\circ$ of Definition 9.6, we have the following commutative diagrams:

$$(9.18) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & H(A) & \longrightarrow & AST_K(A) & \longrightarrow & \pi_0\, AST_K(A) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L(A) & \longrightarrow & DL_K(A) & \longrightarrow & \pi_0\, DL_K(A) & \longrightarrow & 0 \end{array}$$

$$(9.19) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \pi_0\, G^{Id}_{\mathcal{M}_K(A),1} & \longrightarrow & \pi_0\, AST_K(A) & \longrightarrow & G(L_e/K) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \pi_0 DL^{Id}_K(A) & \longrightarrow & \pi_0\, DL_K(A) & \longrightarrow & G(L_e/K) & \longrightarrow & 0 \end{array}$$

Corollary 9.10. *Asume that* $H(A) = C_D(Sp_{(V,\psi)})$. *Then*

$$(9.20) \qquad AST_K(A) = DL_K(A).$$

Proof. It follows by the assumption and (9.3) that $\pi_0(G^{Id}_{\mathcal{M}_K(A),1}) = \pi_0(DL^{Id}_K(A)) = 1$. Hence the middle vertical arrow in the diagram (9.19), which is the right vertical arrow in the diagram (9.18), is an isomorphism. Since $L(A) := (C_D Sp_{(V,\psi)})^\circ$, by our assumption we have $H(A) = L(A)$. Hence the left vertical arrow in the diagram (9.18) is an isomorphism, so the middle vertical arrow in the diagram (9.18) is an isomorphism. $\square$

Corollary 9.11. *If* $H(A) = C_D(Sp_{(V,\psi)})$ *and the Mumford-Tate conjecture holds for* A *then the algebraic Sato-Tate conjecture holds for* $AST_K(A) := G_{\mathcal{M}_K(A),1}$, *namely*

$$G^{alg}_{l,K,1} = AST_K(A)_{\mathbb{Q}_l}.$$

Proof. By (8.25) and Corollary 9.10, we get

$$G^{alg}_{l,K,1} \subset AST_K(A)_{\mathbb{Q}_l} = DL_K(A)_{\mathbb{Q}_l}.$$

By the assumption $H(A) = DL_{L_e}(A)$, the equivalence of (5.11) and (5.12) show that we need to prove that $(G^{alg}_{l,K,1})^\circ = H(A)_{\mathbb{Q}_l}$. However, this is equivalent to the Mumford-Tate conjecture by Remark 4.8. $\square$

Remark 9.12. The results of this and the previous chapter, in particular Theorem 9.8 and Corollaries 9.10 and 9.11, show that the group $AST_K(A)$ from Definition 9.6 is a natural candidate for the algebraic Sato-Tate group for the abelian variety A.

Remark 9.13. For absolutely simple A/F for which the endomorphism algebra D is of type I, II, or III in the Albert classification with $\frac{g}{de}$ odd and for A/F of dimension at most 3 we explicitly determined the motivic Mumford-Tate and motivic Sato-Tate groups (see (9.12) of Theorem 9.8, Corollary 9.10, Theorems 9.3 and 9.4).

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