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RISKY BUSINESS: THE ALLOCATION OF CAPITAL

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Key words: risk, expected return

Abstract

This paper examines the effect of risk on the allocation of capital in a general equilibrium model. In models of the firm a mean-preserving spread in the price of output increases the expected value of profits and frequently the firm's demand for capital. These results seem to contradict the conventional wisdom from financial asset pricing models where an increase in an asset's risk decreases the demand for that asset. This paper presents a resolution to the apparent contradiction. In general equilibrium an increase in exogenous risk (a mean-preserving spread in the state of nature) usually increases expected output in that technology (the result from the theory of the firm) but it may also increase the risk of that technology (the results from finance). An increase in exogenous risk reallocates capital toward less risky businesses.

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Introduction:

This paper examines the effect of risk on the allocation of capital in a simple general equilibrium model.

Hartman (1972,1976), Pindyck (1982), and Abel (1983,1984,1985) examined the effect of output price uncertainty on the firm's demand for capital. They showed that a mean-preserving spread in the distribution of the price of output (which makes the future profit stream more variable) increases the firm's demand for capital. And Oi (1961) showed that a mean-preserving spread in commodity and/or factor prices increases the firm's expected profits. These results seem to contradict the conventional wisdom from financial asset pricing models where an increase in an asset's risk decreases the demand for that asset.

This paper presents a resolution to the apparent contradiction. Assets are a claim on a stream of payoffs. The current value of an asset is the expected value of the discounted payoff stream. The covariance of the asset's return with the discount factor measures the asset's risk. In equilibrium a riskier asset requires a higher expected return. The theory of the firm and the theory of finance are partial equilibrium analyses that make complementary assumptions about the relationship between an asset's payoffs and the discount factor.

Financial asset pricing models assume an exogenous distribution

of asset payoffs. Household behavior affects the endogenous discount factor. A mean-preserving spread in the distribution of an asset's payoff stream cannot (by definition) increase the expected value of the payoff stream, but it can change the asset's risk.

Traditional specifications of the firm assume that the firm's profits are distributed independently of the exogenous discount factor. The firm is a risk free asset. Firm behavior affects the endogenous stream of asset payoffs, ie, profits. A change in the distribution of firm profits cannot (by assumption) increase the risk of the firm, but it can increase the expected value of the payoff stream. Oi, Abel, Hartman and Pindyck showed that the firm's optimal response to a mean-preserving spread in the distribution of the price of output increases the expected value of the payoff stream and the value of the asset.

In a general equilibrium, the discount factor and the payoff to assets are interdependent endogenous variables. The random states of nature which affect firms' technologies and/or household preferences are exogenous. A mean-preserving spread in the distribution for the exogenous states of nature makes the economy riskier. An increase in exogenous risk usually increases expected payoffs in a technology (the Abel, Hartman, Pindyck and Oi result), but it can also increase the risk of that technology. An increase in exogenous risk reallocates resources toward less

risky businesses.

The paper is organized as follows: Section 1 shows the asset equilibrium conditions in the consumption-capital asset pricing model and a model of the firm. It shows that the definition of risk in the consumption-capital asset pricing model applies to any (physical or financial) asset, and gives a general condition for the equilibrium comparative static response to a change in risk. Section 2 shows the asset equilibrium conditions in a simple general equilibrium model and illustrates the results with an example based on the closed-form general equilibrium solutions in Brock (1982) and Long & Plosser (1983).

Section 1 Partial Equilibrium Models

1.1 The Consumption-Capital Asset Pricing Model

In the consumption-capital asset pricing model¹ asset prices satisfy a Euler equation from the household's maximization problem,

$$1.1.1 \quad U_{ct}V(i)_t = \beta E_t[U_{ct+1}\{V(i)_{t+1} + d(i)_{t+1}\}].$$

Here U_c denotes the marginal utility of consumption, $V(i)$ the price of the i^{th} security, $d(i)$ the dividend on the security, β the agent's time discount factor. The expectation is conditional on information at t . The Euler equation states that the agent purchases the security until the decrease in current utility from purchasing an additional share equals the increase in expected utility from owning an additional share of the security.

Rearranging 1.1.1 and solving the difference equation gives,

$$1.1.2 \quad V(i)_t = E_t\left[\sum_{\tau=1}^{\infty} D_{t+\tau}d(i)_{t+\tau}\right], \quad i=1, \dots, N$$

an expression that equates the current asset price to the expected value of the discounted payoff stream. The distribution of the payoff stream distinguishes one asset from another. Partial equilibrium financial asset pricing models assume an exogenous distribution of asset payoffs, $d(i)$. Rewriting 1.1.1 in rates of return gives,

¹ See the outstanding seminal papers by Lucas (1978), and Breeden (1979), or Sargent (1987), Chapter 3, for an excellent survey that includes recent work.

$$1.1.3 \quad 1 = E_t[D_{t+1}R(i)_{t+1}] = E_t\left[\sum_{\tau=1}^{\infty} D_{t+\tau}d(i)_{t+\tau}\right]/V(i)_t$$

$$\text{where } R(i)_{t+1} \equiv (V(i)_{t+1} + d(i)_{t+1})/V(i)_t,$$

the expectation of the discounted return factor equals one at a maximum.

The discount factor,

$$1.1.4 \quad D_{t+\tau} \equiv \beta^\tau U_{ct+\tau}/U_{ct},$$

is the marginal intertemporal rate of substitution for consumption discounted for time preference. The discount factor is random since the agent's consumption depends on the random asset payoffs. The expected value of the discount factor is,

$$1.1.5 \quad E_t D_{t+\tau} \equiv \beta^\tau E_t U_{ct+\tau}/U_{ct} \equiv \text{RRF}_{t+\tau}^{-1},$$

the reciprocal of the risk free return factor (RRF). The reciprocal of the risk free return factor is the current price of a pure discount bond (if one exists) that promises one unit of consumption with certainty in $t+\tau$.

At an interior maximum the asset prices equal the expected value of their discounted payoff stream, or the expected discounted return factor equals one. So equation 1.1.2, or 1.1.3, is an equilibrium condition.

Expected Return and Risk

Using the fact that the expectation of a product of random variables equals the product of the expectations plus their

covariance one can rewrite the expectation of the discounted return as,

$$\begin{aligned}
 1.1.6 \quad 1 &= E_t D_{t+1} E_t R(i)_{t+1} + \text{cov}_t(D_{t+1}, R(i)_{t+1}), \\
 &= \sum_{\tau=1}^{\infty} \{E_t D_{t+\tau} E_t d(i)_{t+\tau}\} + \text{cov}_t(D_{t+\tau}, d(i)_{t+\tau}) / V(i)_t,
 \end{aligned}$$

the expected return discounted by the risk free return factor plus a risk adjustment. The covariance between the discount factor and the asset's return factor measures the asset's risk.

Define the asset's "beta" as,

$$\begin{aligned}
 1.1.7 \quad B(i)_{t+1} &\equiv \text{cov}_t(D_{t+1}, R(i)_{t+1}) \\
 &= \sum_{\tau=1}^{\infty} \text{cov}_t[(D_{t+\tau}, d(i)_{t+\tau})] / V(i)_t
 \end{aligned}$$

In equilibrium,

$$1.1.8 \quad E_t R(i)_{t+1} = RRF_{t+1} (1 - B(i)_{t+1}),$$

the expected return on the asset equals the risk free rate adjusted by the asset's beta. If the asset has a negative beta (a high return in states of nature where consumption is high so the marginal utility of consumption is low) then the asset is relatively more risky and its expected return contains a positive risk premium. If the asset's return is uncorrelated with the marginal utility of consumption, then the asset is a risk free asset. In equilibrium the expected return on a risk free asset equals the risk free return. Positive beta assets have a negative risk premium.

Response to a Change in Risk

Define an increase in risk as a mean-preserving spread in the distribution of the exogenous random asset payoffs, $d(i)$. The comparative static results are that an increase in exogenous risk increases (decreases) the demand for an asset if the product of the discount factor with the asset return factor is a convex (concave) function of the random variable.² The response to an increase in risk follows directly from applying Jensen's inequality to the equilibrium condition 1.1.2. If the function is convex in $d(i)$, then a mean-preserving spread in the probability distribution of dividends increases the expected value of the discounted payoff stream. Equilibrium requires a higher asset price.

1.2 A Model of the Firm

The firm produces a stream of output which yields a stream of (random) profits for the owners. The owners instruct the firm manager to maximize the expected value of the discounted stream of profits. The traditional specification assumes a deterministic discount factor, D , or a discount factor that is

² If the discount factor, equation 1.1.5, is a nonlinear function of the asset's payoffs, then a change in exogenous risk changes the risk free rate in equation 1.1.6 so the asset's beta is not a sufficient statistic to describe the comparative static response to a change in risk.

distributed independently of the firm's profits.³ Let,

$$1.2.1 \quad W(i)_t = \max \sum_{\tau=0}^{\infty} D_{t+\tau} E_t P(i)_{t+\tau},$$

define the firm's objective function, where,

$$P(i)_t = f(k(i)_t, z(i)_t) s(i)_t - g(I(i)_t) - w_t z(i)_t$$

$$\text{and,} \quad I(i) = k(i)_{t+1} - (1-\delta)k(i)_t; \quad 0 \leq \delta \leq 1,$$

define the firm's profit, $P(i)$, and investment, $I(i)$. The production function, f , is homogeneous of degree one in the factor inputs, and g is a convex function that includes a cost to adjusting capital. Abel, Hartman, and Pindyck essentially use the specification of the firm in equation 1.2.1.

The firm chooses the current labor input, $z(i)_t$, and next period's capital, $k(i)_{t+1}$, to maximize the objective function. The wage, w , and the commodity price, $s(i)$, are strictly positive random variables which are exogenous to the firm.

The firm's objective function is an asset evaluation equation. Let the firm pay out all profits in dividends, $P(i) = d(i)$, then,

$$1.2.2 \quad W(i)_t = \max \sum_{\tau=0}^{\infty} D_{t+\tau} E_t P(i)_{t+\tau} = V(i)_t + d(i)_t,$$

the equity value of the firm plus the current dividend equals the

³ For example Chirinko (1987) uses a deterministic model and Shapiro (1986) assumes a random discount factor that is distributed independently of the firm's profits. Abel and Blanchard (1986) do not assume independence between the discount factor and the firm's payoffs. They are an exception to the traditional specification.

maximum of the discounted expected profit stream.⁴

Expected Return and Risk

Assuming that the firm's profits are distributed independently of the discount factor seems like a natural assumption for a firm in competitive equilibrium. But it implies that the firm is a risk free asset for its owners, ie, only idiosyncratic shocks hit the firm's profits, so a well-diversified portfolio eliminates the risk,

$$1.2.3 \quad W(i)_t = \max_{\tau=0}^{\infty} \sum E_t [D_{t+\tau} P(i)_{t+\tau}] = \max_{\tau=0}^{\infty} \sum RRF_{t+\tau}^{-1} E_t P(i)_{t+\tau}.$$

The firm is a zero beta asset.

Response to a Change in Risk

If profits are distributed independently of the discount factor, then a mean-preserving spread in wages or the commodity price only affects the value of the firm if it changes the expected value of the firm's profits. Oi showed the firm's indirect profit function is convex in commodity and factor prices, so an increase in exogenous risk increases expected profits. Abel, Hartman, and Pindyck showed the marginal revenue product of

⁴ The owners of the firm receive the current dividend. The notation for stocks, V , follows the convention that stocks are valued ex dividend. If the Modigliani-Miller theorem holds then V equals the market value of the firm.

capital⁵ is convex in the price of output, so a mean-preserving spread in the price of output increases the expected payoff to investment and the firm's demand for capital.

Convexity follows from the fact that the firm varies its labor input, $z(s,w)$, optimally after observing the realization of the random variables. Consider a one-period problem and let $P(z(s,w))$ denote the maximum of the indirect profit function. Now maximum expected profits must be greater than or equal to maximum profits valued at the expected value of the random variables, $E[P(z(s,w))] \geq P(z(Es, Ew))$, since the firm could always choose the constant labor input $z(Es, Ew)$ for all realizations of the random variables, see Varian p46. The argument generalizes to the dynamic maximization problem in equation 1.2.1. As a result, a mean-preserving spread in factor prices or the commodity price increases the expected value of the firm when the discount factor is independent of profits. If the expected return to owning the firm exceeds the risk free return,

$$1.2.4 \quad E_t W(i)_{t+1} / \{W(i)_t - P(i)_t\} = E_t \{V(i)_{t+1} + d(i)_{t+1}\} / V(i)_t \\ = E_t R(i)_{t+1} > RRF_{t+1},$$

then equilibrium requires devoting more resources to the firm to drive the expected return to the risk free return.

⁵ Hartman (1976) shows the marginal revenue product of capital is convex in the price of output for all linearly homogeneous production functions. Abel and Pindyck use linearly homogeneous production functions.

Capital Allocation

Abel, Hartman, and Pindyck examine the effect of an increase in output price uncertainty on the firm's demand for capital. The firm's Euler equation for capital accumulation is,

$$\begin{aligned}
 1.2.4 \quad g_{It} &= D_{t+1} E_t [f_{kt+1} s(i)_{t+1} + (1-\delta) g_{It+1}], \\
 &= \sum_{\tau=0}^{\infty} D_{t+\tau} (1-\delta)^{\tau} E_t [f_{kt+\tau} s(i)_{t+\tau}].
 \end{aligned}$$

At a maximum the marginal cost of an additional unit of capital equals the value of the discounted expected payoff stream to the last unit of capital. If the marginal revenue product of capital is convex in a random variable, then a mean-preserving spread in that random variable increases the expected value of the payoff to investment and the firm's demand for capital.

Section 2 A Simple General Equilibrium Model

The theory of the competitive firm treats prices as exogenous and assumes that payoffs to the firm are independent of the discount factor. Financial asset pricing models treat asset payoffs as exogenous. In a general equilibrium, output, the decomposition of output into consumption and investment, relative prices, and the payoffs to assets are endogenous. Random shocks--the so-called states of nature--are exogenous to the economy. A mean-preserving spread in the distribution of a state of nature increases risk.

This Section examines the effect of risk on the general equilibrium allocation of capital in a simple model and illustrates the allocation with an example.

The Model

The representative household wants to maximize expected (infinite) lifetime utility,

$$2.1.1 \quad \sum_{j=0}^{\infty} \beta^j E_t U(C_{t+j}, 1-Z_{t+j}, s_{t+j}); \quad 0 < \beta, Z < 1,$$

where β is the household discount factor, C is consumption, $1-Z$ is leisure, and Z labor, and s is an independently and identically distributed strictly positive vector of random shocks. The vector s is the "state of nature". The household's instantaneous utility function is concave in consumption and leisure.

Society's resource constraint limits current consumption and capital accumulation, I , to current production, Y ,

$$2.1.2 \quad Y_t \equiv C_t + I_t.$$

N technologies (firms or industries) exist that produce a single commodity that can be consumed or added to productive capital,

$$2.1.3 \quad Y_t \equiv \sum_{i=1}^N f(i, k(i)_t, z(i)_t, s(i)_t),$$

$$I_t \equiv K_{t+1} - (1-\delta)K_t$$

$$K_{t+1} \equiv \sum k(i)_{t+1}$$

$$Z_t \equiv \sum z(i)_t.$$

The production processes are homogeneous of degree one in the factor inputs, labor and capital, k . Labor is a variable input; capital is predetermined¹. The shocks $s(i)$ are industry-specific shocks, but they may be correlated with shocks in other industries.

The Central Planning Solution

The central planner chooses a set of contingent plans for labor, $z(i)_{t+j}$, and capital, $k(i)_{t+1+j}$, to maximize the utility function 2.1.1 subject to the resource constraint 2.1.2. If a solution exists the allocation is Pareto optimal. And since the constraint set is convex, a competitive equilibrium supports the Pareto

¹ This specification omits the cost to adjusting capital. Adding a cost to adjusting capital will not change the qualitative results, but it drives a wedge between the returns to the marginal and average unit of capital.

optimal allocation. I assume an interior solution exists.²

Labor

The central planner chooses the labor input after observing the realizations of the shocks, s . Labor is allocated so that the marginal product of labor in each industry equals the shadow wage,

$$2.1.4 \quad f(i)_{zt} = U_{1-zt}/U_{ct}; \quad i=1, \dots, N.$$

Capital

Capital is an asset the planner uses to transfer consumption between periods and to diversify risk across technologies. Current capital accumulation adds to next period's production, so the planner must choose capital before observing the realization of the states of nature. Capital is a risky asset. At a maximum the planner allocates capital to a technology until,

$$2.1.5 \quad U_{ct} = \beta E_t[U_{ct+1}\{f(i)_{kt+1} + (1-\delta)\}], \text{ or}$$

$$2.1.5' \quad 1 = E_t[D_{t+1}\{f(i)_{kt+1} + (1-\delta)\}], \quad i = 1, \dots, N,$$

where, $D_{t+1} \equiv \beta U_{ct+1}/U_{ct}$,

the expected discounted return to capital equals one.

Expected Return and Risk

The condition for a maximum, equation 2.1.5', can be rewritten as,

² The existence of a solution requires some additional technical conditions, see Brock or Prescott and Merha (1980).

$$2.1.6 \quad E_t[f(i)_{kt+1} + (1-\delta)] = RRF_{t+1}\{1 - \text{cov}_t(D_{t+1}, f(i)_{kt+1})\}$$

the expected return to capital in technology i equals the risk free rate adjusted for capital's risk.³ Riskier technologies require a higher expected return.

Comparison with the Partial Equilibrium Models

The Euler equation in the general equilibrium model highlights the *ceteris paribus* conditions in the partial equilibrium formulas in Section 1. A mean-preserving spread in the distribution of a state of nature increases risk in the economy. The increase in exogenous risk implies a different allocation of resources and distribution of profits.

Partial equilibrium models of finance emphasize the relationship between an asset's expected return and risk, but they omit the effect of variable inputs on the asset's expected payoffs. Define the payoffs to equities as the firm's profits,

$$2.1.7 \quad d(i)_t = f(i, k(i)_t, z(i)_t) s(i)_t - I(i)_t - \{U_{1-zt}/U_{ct}\} z(i)_t = P(i)_t.$$

Oi showed that the firm's indirect profit function evaluated at the optimal labor input is convex in the price of output, or, in this case the multiplicative productivity shock, $s(i)$. A mean-preserving spread in the distribution of a state of nature, $s(i)$,

³ If there is a cost to adjusting capital substitute the marginal return to capital, $RM(i)_{t+1} \equiv \{f(i)_{kt+1} + (1-\delta)g_{It+1}\}/g_{It}$, in equation 2.1.6 giving, $E_t RM(i)_{t+1} = RRF_{t+1}\{1 - \text{cov}_t(D_{t+1}, RM(i)_{t+1})\}$.

increases the asset's expected payoff stream and affects its risk. Ferson and Merrick (1987) found that conditioning the joint distribution of consumption changes and asset returns on business cycle variables improved the fit in a consumption-capital asset pricing equation.

Traditional models of the firm focus on resource reallocation in response to a change in risk, but they assume the firm is a risk free asset. The model in this Section (where there is no cost to adjusting capital) illustrates the fragility of the assumption that the firm's profits are independent of the discount factor. Suppose profits, and therefore the marginal products of capital, are distributed independently of the discount factor. Now since the production function is homogeneous of degree one, the marginal product of capital evaluated at the optimal labor input is also independent of capital. So the Euler equation, 2.1.6, does not depend on capital, or give an equilibrium condition for capital allocated to the i^{th} technology.⁴ If the expected return to capital in technology i exceeds the risk free return,

$$E_t\{f(i)_{kt+1} + (1-\delta)\} = E_t R(i)_{t+1} > RRF_{t+1},$$

then the planner, or private agents, should put all their assets in the i^{th} technology. But if society devotes all its resources to a single technology, then the payoff is highly correlated with aggregate consumption (and the discount factor) since,

⁴ This is simply the traditional model of the firm with a linearly homogeneous production function and no cost to adjusting capital; the firm size is indeterminate.

$$d_t + (U_{1-z_t}/U_{c_t})Z_t = f(i)_t - I_t = C_t,$$

profits plus the wage bill equal consumption, contradicting the supposition that the discount factor is independent of the firm's profits.

In the general equilibrium model, risk limits resources devoted to any technology in the same way that risk limits the share of a portfolio devoted to a single asset in financial theory.

2.2 An Example with a Closed-Form Solution

This example illustrates the allocation of capital in a general equilibrium model with a closed-form solution. The solution is based on the examples in Brock, and Long and Plosser. I changed the model in Long and Plosser by making labor a variable input. In my example, and in theirs, a mean-preserving spread in a state of nature has no effect on aggregate investment, but it alters the allocation of capital and labor among technologies. The share of capital devoted to less risky technologies increases.

Assume the household's instantaneous utility is,

$$2.2.1 \quad U(C, 1-Z, s) = \ln C + u(1-Z),$$

the logarithm of consumption plus a concave function of leisure. Society's resource constraint,

$$2.2.2 \quad Y_t = C_t + K_{t+1},$$

limits consumption plus capital accumulation to current output. The resource constraint follows Long and Plosser's specification

that capital has a one-period life, ie, $\delta = 1$. Current aggregate output is,

$$2.2.3 \quad Y_t = \Sigma y(i)_t,$$

$$\text{where, } y(i)_t = k(i)_t^a z(i)_t^{1-a} s(i)_t$$

the sum of output in each technology. The production function in each technology is Cobb-Douglas with a multiplicative productivity shock. I assume the shocks are independently distributed. The distribution of the shocks distinguishes one technology from another.

Labor is a variable input selected after observing the realization of the vector of shocks, s . At a maximum the marginal product of labor in each technology,

$$2.2.4 \quad f(i)_{zt} = (1-a)y(i)_t/z(i)_t = U_{1-z}/U_{ct} = U_{1-z}C_t,$$

equals the shadow wage.

Capital is a predetermined input. Society selects the capital allocation before observing the realizations of the states of nature. At a maximum society invests until,

$$2.2.5 \quad U_{ct} = \beta E_t[U_{ct+1}f(i)_{kt+1}]$$

$$1/C_t = \beta E_t[ay(i)_{t+1}/k(i)_{t+1}C_{t+1}], \quad i = 1, \dots, N,$$

the decrease in current utility from the marginal unit of investment equals the increase in expected utility from having an additional unit of capital in the i^{th} technology.

Equilibrium

Conjecture a solution of the form in Brock, or Long and Plosser,

$$2.2.6(a) \quad C_t = \{1 - a\beta\}Y_t$$

$$(b) \quad K_{t+1} = a\beta Y_t.$$

The conjectured solution is that each period the household consumes a constant fraction of income, $1-a\beta$, and invests the remaining fraction, $a\beta$. Notice that if the conjecture is correct, then a mean-preserving spread in the distribution of a state of nature makes the time path for consumption and aggregate investment more variable, but it does not affect the aggregate investment decision.

Verification

To verify the solution, substitute the consumption conjecture 2.2.6a into the equilibrium condition for capital accumulation, equation 2.2.5, giving,

$$2.2.7 \quad 1/(1-a\beta)Y_t = \beta E_t[ay(i)_{t+1}/k(i)_{t+1}\{1-a\beta\}Y_{t+1}], \text{ or,}$$

$$k(i)_{t+1} = E_t[h(i)_{t+1}]a\beta Y_t,$$

where, $h(i)_{t+1} \equiv y(i)_{t+1}/Y_{t+1}.$

The weight, $h(i)_{t+1}$, is the share of next period's output generated by the i^{th} technology. Since the expectation of the sum equals the sum of the expectations,

$$2.2.8 \quad \sum_i E_t h(i)_{t+1} = E_t[\sum_i y(i)_{t+1}/Y_{t+1}] = 1,$$

the conjecture is verified for aggregate capital, $K_{t+1} = a\beta Y_t$. The share of capital in any technology equals the expectation of the weight, $k(i)/K = E_t h(i)$.

Now, substituting the consumption conjecture into the labor equilibrium condition, equation 2.2.4, gives,

$$2.2.9 \quad (1-a)y(i)_t/z(i)_t = u_z(1-Z_t)(1-a\beta)Y_t, \text{ or}$$

$$2.2.9' \quad (1-a)\{1-a\beta\}^{-1}h(i)_t = u_z(1-Z_t)z(i)_t.$$

Summing over technologies gives a constant aggregate labor input,

$$2.2.10 \quad (1-a)\{1-a\beta\}^{-1} = u_z(1-Z)Z,$$

completing the description of the aggregate allocation.⁵

Dividing equation 2.2.9' by the aggregate labor equation 2.2.10 gives the share of labor in technology i ,

$$2.2.11 \quad z(i)_t/Z = h(i)_t,$$

which completes the description of the allocations to each technology.

The Resource Shares: $h(i)$

In this example the weights are analogous to the fractions of wealth invested in particular assets in a traditional portfolio problem in finance. Resources, capital and labor, are society's wealth. In each period aggregate wealth is fixed; aggregate capital is predetermined and aggregate labor is constant. The weights give the resource shares allocated to each technology. The share of capital devoted to the i^{th} technology equals the

⁵ If the utility of leisure is state-dependent, then the aggregate labor input varies with the realization of the states of nature. Equation 2.2.10 gives the aggregate labor input as a function of the states of nature. The closed-form solution still holds since Y can be written as a function of predetermined capital and the states of nature.

expected share of output from that technology, $E_t[Y(i)_{t+1}/Y_{t+1}]$, since capital is allocated before the realization of the shocks. And, since society chooses the labor shares after observing the realizations of the states of nature, the share of labor in the i^{th} technology equals the realized share of output from the i^{th} technology, $y(i)_t/Y_t$.

Response to a Change in Risk

The response to a change in risk depends on the convexity of the product of the discount factor with the return factor in a random variable. In this example, the condition collapses to the convexity of the weights in a technology shock since,

$$\begin{aligned} 2.2.12 \quad E_t[D_{t+1}R(i)_{t+1}] &= \beta a Y_t / k(i)_{t+1} E_t[Y(i)_{t+1}/Y_{t+1}] \\ &= K_{t+1} / k(i)_{t+1} E_t[Y(i)_{t+1}/Y_{t+1}] = 1, \end{aligned}$$

where,

$$D_{t+1} = \beta C_t / C_{t+1} = \beta Y_t / Y_{t+1}$$

$$R(i)_{t+1} = f(i)_{kt+1} = a y(i)_{t+1} / k(i)_{t+1}.$$

Using the first-order condition 2.2.4 to eliminate labor gives the set of nonlinear simultaneous equations,

$$2.2.13 \quad h(i) = k(i) s(i)^{1/a} / \left\{ \sum_j k(j) s(j)^{1/a} \right\}, \quad i = 1, \dots, N$$

$$k(i)/K = E[h(i)], \quad i = 1, \dots, N.$$

Even though the model is fairly simple, the weights are neither uniformly convex nor concave in a technology shock. Output in each technology is a convex function of the productivity shock to that sector, $y(i) = k(i) s(i)^{1/a}$. So expected output is an increasing function of the exogenous risk, as Oi, Abel, Hartman

and Pindyck recognized. But aggregate output, and the discount factor, cannot be distributed independently of shocks to technology i . The equilibrium allocation depends on risk and expected return as financial asset pricing models emphasize.

The second derivative of the weight, $h(i)$, with respect to the $s(i)$, equals,

$$2.2.14 \quad \frac{d^2 h(i)}{ds(i)^2} = \frac{[(1-a)Y-y(i)]\{Y-y(i)\}}{Y^3} \frac{dy(i)}{as(i) ds(i)}.$$

The sign of the derivative depends on a parameter, the elasticity of output with respect to labor, $1-a$, and a random variable, $(1-a)Y-y(i)$, which is essentially technology i 's importance in aggregate output. In general the sign is indeterminate.

When the elasticity of output with respect to labor is zero, ($a=1$), the weight is strictly concave in $s(i)$. Setting $a=1$ is equivalent to the assumption made in partial equilibrium financial asset pricing models. The payoff per unit of capital in technology i , $R(i) = y(i)/k(i) = s(i)$ is exogenous. A mean-preserving spread in $s(i)$ increases the asset's risk, but it does not increase the expected return. Technology i is riskier and receives a small share of aggregate resources. This is the standard result in financial asset pricing models with independently distributed asset payoffs.

When the elasticity of output with respect to labor is greater than zero, ($0 < a < 1$), and $y(i)$ is sufficiently small the

weight is strictly convex in $s(i)$. Making $y(i)$ sufficiently small is equivalent to assuming the payoffs are (almost) independent of the discount factor, ie, the assumption in partial equilibrium models of the firm. A mean-preserving spread in $s(i)$ increases expected output in technology i , but it does not increase the risk (very much).

The appendix presents the results from computer simulations for this example which give some indication of the general trade-offs. With eleven or more technologies I found some convex regions.

Section 3 Summary

This paper examines the effect of risk on the allocation of capital in a simple general equilibrium model. It presents a resolution of the apparently contradictory results between financial asset pricing models and the theory of the firm. Models of the firm assume the firm is a risk free asset. A mean-preserving spread in an exogenous variable can increase the expected payoffs to the firm, but not the risk. Financial asset pricing models assume the payoffs to the asset are exogenous. A mean-preserving spread in the exogenous payoffs can increase the asset's risk, but not the expected payoff.

In a general equilibrium the asset payoffs and the discount factor are endogenous. A mean-preserving spread in an exogenous state of nature usually increases the expected payoffs to that technology, but it can also increase the risk of that technology. An increase in exogenous risk reallocates resources toward less risky businesses.

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Appendix

This appendix presents numerical solutions to the nonlinear equations,

$$h(i) = k(i)s(i)^{1/a} / \left\{ \sum_j k(j)s(j)^{1/a} \right\}, \quad i = 1, \dots, N$$

$$k(i)/K = E[h(i)], \quad i = 1, \dots, N,$$

for the share of capital in technology one, $k(1)/K = E[h(1)]$.

The technology shocks, $s(j)$, $j \neq 1$, were drawn from independent log-normal distributions with a mean and variance of one. The objective of the simulations is to map out the response to a change in risk. I varied three parameters:

- 1 The variance of the shock to technology one, σ^2 .
- 2 The elasticity of output with respect to labor, $1-a$.
- 3 The number of technologies, N .

For each set of parameter values I searched for a solution to the nonlinear equations. I iterated until the distance between the initial "guess", $k(i)_0$, and the computed average was less than one one-thousandth, $|k(i)_0/K - E[h(i)]| < .001$. I used 5000 draws at each iteration.

The Figure plots the share of capital in technology one as a function of the variance of the technology shock when $1-a = 1/2$. The top line shows the share of capital devoted to technology 1 when there are only two technologies. The share is a strictly decreasing function of the variance to shocks to technology one. Each lower line represents an increase in the total number of

technologies by five. The plot shows that an increase in exogenous risk only increases the share of capital in that technology when the technology is "small". In these simulations small meant less than 10% which is certainly not tiny.

The Table gives more detailed results. A simple reference point is where $\sigma^2=1$ so the errors are independently and identically distributed and capital's share equals one over the number of technologies, $1/N$. The rows show the response to an increase in exogenous risk holding the other parameters constant. The columns show the response to an increase in the elasticity of output with respect to labor.

Table
Capital Shares

N = 2				
$1-a \backslash \sigma^2$	0.5	1.0	1.5	2.0
1.0	0.76	0.50	0.35	0.26
0.75	0.73	0.50	0.37	0.28
0.50	0.67	0.49	0.40	0.32
0.25	0.65	0.51	0.42	0.36

N = 6				
$1-a \backslash \sigma^2$	0.5	1.0	1.5	2.0
1.0	0.21	0.17	0.11	0.07
0.75	0.20	0.17	0.13	0.10
0.50	0.20	0.17	0.14	0.12
0.25	0.16	0.17	0.16	0.14

N = 11

$1-a\sigma^2$	0.5	1.0	1.5	2.0
1.0	0.23	0.09	0.07	0.04
0.75	0.10	0.09	0.08	0.07
0.50	0.06	0.09	0.09	0.09
0.25	0.04	0.09	0.10	0.10

N = 16

$1-a\sigma^2$	0.5	1.0	1.5	2.0
1.0	0.07	0.06	0.05	0.04
0.75	0.06	0.06	0.06	0.05
0.50	0.02	0.06	0.07	0.07
0.25	0.01	0.06	0.08	0.08

RESOURCE SHARES

