Title
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Journal
Economic Theory, 58(3)

ISSN
0938-2259

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Publication Date
2015-04-01

DOI
10.1007/s00199-015-0865-z

Peer reviewed

Econ Theory (2015) 58:413–440
DOI 10.1007/s00199-015-0865-z

RESEARCH ARTICLE

Nash bargaining for log-convex problems

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Received: 20 February 2014 / Accepted: 3 February 2015 / Published online: 25 February 2015

Abstract We introduce log-convexity for bargaining problems. With the requirement of some basic regularity conditions, log-convexity is shown to be necessary and sufficient for Nash's axioms to determine a unique single-valued bargaining solution up to choices of bargaining powers. Specifically, we show that the single-valued (asymmetric) Nash solution is the unique solution under Nash's axioms without that of symmetry on the class of regular and log-convex bargaining problems, but this is not true on any larger class. We apply our results to bargaining problems arising from duopoly and the theory of the firm. These problems turn out to be log-convex but not convex under familiar conditions. We compare the Nash solution for log-convex bargaining problems with some of its extensions in the literature.

Professor Shuzhong Shi passed away before the completion of this paper. He was a pleasure and an inspiration to work with and will be deeply missed.

We gratefully acknowledge helpful comments from Rabah Amir, Kim Border, Juan Carrillo, Bo Chen, Matthew Jackson, Ehud Kalai, Joel Sobel, Walter Trockel, Simon Wilkie, Adam Wong, Lin Zhou, and seminar and conference participants at Beijing University, Ohio State University, Southwest Economic Theory Conference, the Third Congress of the Game Theory Society, Shanghai University of Finance and Economics, University of Arizona, University of Bielefeld, University of California at Irvine, Riverside, and San Diego, University of Southern California, and Zhejiang University. We also thank an anonymous referee for comments that helped to improve the paper.

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Keywords Bargaining problem · Non-convexity · Log-convexity · Nash solution · Nash product

JEL Classification C78 · D21 · D43

1 Introduction

The bargaining theory introduced in the seminal papers of Nash (1950, 1953) postulates that a group of players chooses a payoff allocation from a set of feasible payoff allocations. The implementation of a payoff allocation requires unanimous agreement among the players. In case of disagreement, the players end up getting some prespecified payoff allocation known as the status quo or the *threat point*. A bargaining problem (henceforth, a problem in short) in the sense of Nash is thus represented by a pair consisting of a choice set and a threat point in the payoff space. A solution on a class of problems is a rule that assigns a feasible payoff allocation to each problem in the class.

Nash adopted an axiomatic approach to resolve the bargaining problem:

> One states as axioms several properties that it would seem natural for the solution to have, and then one discovers that the axioms actually determine the solution uniquely (Nash 1953, p. 129).

He postulated a set of axioms that are deemed to be natural for a solution to satisfy. These axioms are Pareto optimality (PO), symmetry (SYM), invariance to equivalent utility representations (INV), and independence of irrelevant alternatives (IIA). It is remarkable that these axioms determine a unique solution, to be called the Nash solution, on the class of all *compact* and *convex* problems. Furthermore, the Nash solution assigns to each problem the unique maximizer of the (symmetric) *Nash product* over the choice set. Nash bargaining model has become one of the most fruitful paradigms in game theory.[1]

When SYM is removed, Kalai (1977) showed that an asymmetric Nash solution *unique up to* (henceforth unique) specifications of payoff allocations for a "normalized problem" is characterized by the remaining axioms on the class of compact convex problems. A payoff allocation for the normalized problem has been customarily interpreted as representing players' bargaining powers.[2] Furthermore, given players' bargaining powers, the asymmetric Nash solution assigns to each problem the unique maximizer of the Nash product weighted by the bargaining powers. The Nash product weighted by players' bargaining powers is also known as the *generalized Nash*

[1] Thomson (2009) provides an excellent survey of the literature on the axiomatic approach to bargaining as pioneered by Nash. An alternative approach is to model bargaining explicitly as a non-cooperative game. Nash (1953, p. 129) argued that the two approaches "are complementary; each helps to justify and clarify the other." See Binmore et al. (1986) for a formal establishment of the relationship between the Nash axiomatic and the sequential non-cooperative approaches to bargaining.

[2] This interpretation can be traced to Shubik (1959, p. 50).

product.[3] These properties consist of the standard axiomatic characterization of the asymmetric Nash solution on the class of compact convex problems.

For applications, the uniqueness of the solution up to specifications of players' bargaining powers is useful, since there is no a priori reason why players' bargaining powers are not case-specific. In addition, the choice set is not directly given, but often derived from more primitive information. As a result, the choice set needs not to be convex. Indeed, the non-convexity of the choice set of feasible profit shares arising from duopolies with asymmetric constant marginal costs and concave monopoly profit functions has been well recognized in the industrial organization literature. See, for example, Bishop (1960, p. 948), Schmalensee (1987, pp. 354–356), and Tirole (1988, p. 242, 271). As discovered in Aoki (1980), McDonald and Solow (1981), Miyazaki (1984), among other ones, non-convexity also often arises in employer–employee bargaining problems.[4]

The non-convexity is usually removed by allowing for randomized payoff allocations and by assuming that players are expected utility maximizers. Randomized allocations can cause conflicts of interest at the post-realization stage and may not be realistic in some applications. Randomization was also considered as impractical for employer–employee bargaining (see e.g., Miyazaki 1984, p. 484 for discussions).[5] It is therefore desirable to analyze the extent to which the Nash bargaining theory can be extended to allow for non-convexity.

Our purposes in this paper are threefold. First, we characterize the complete class of regular problems allowing for non-convexity, on which the axioms of IIA, INV, and SIR uniquely determine the asymmetric Nash solution as in the case with convex and compact problems. Second, we apply our results to analyze non-convex problems arising from duopoly and employer–employee bargaining. Third, we compare our approach with two extensions of Nash bargaining theory in the literature.

A notion of log-convexity plays a key role in characterizing problems in our class. With arbitrary bargaining powers, our first main result shows that the maximizer of the generalized Nash product over the choice set of a regular problem is unique *if and only if* the problem is log-convex (Theorem 1). Regularity of a problem in this paper means that the threat point is strictly dominated and the choice set is closed, comprehensive relative to the threat point, and bounded above. Log-convexity, on the other hand, requires that the strict Pareto frontier of the set of the natural log-transformed allocations of positive payoff gains be *strictly* concave. The regularity requirement is mild while the log-convexity of a problem is strictly *weaker* than the convexity of the choice set. As illustrations of log-convexity but not convexity

[3] By the Lemma in Roth (1977, p. 65), a bargaining solution is Pareto optimal whenever the solution satisfies IIA, INV, and strict individual rationality (SIR). Thus, IIA, INV, and PO can be replaced by IIA, INV, and SIR. Since bargaining problems in our class satisfy all the assumptions in Roth (1977) except for the convexity, we will consider IIA, INV, and SIR as the Nash's axioms for the asymmetric Nash bargaining solution.

[4] Bargaining problems arising from the principal–agent problems need not be convex unless randomized contracts are allowed (see Ross 1973; Hougaard and Tvede 2003).

[5] Similarly, Hougaard and Tvede (2003, p. 82) argue that it is not reasonable to allow for randomized contracts in the context of a principal–agent problem.

that arises in application, we show that duopoly and employer–employee bargaining problems are log-convex but not convex under familiar conditions.

Given a specific bargaining problem, one might simply equate the Nash solution for the problem with the unique maximizer (if exists) of the Nash product. We wish to remark that the Nash axioms, which determine the Nash solution, are defined on an exogenously specified class of bargaining problems. It follows that one cannot simply treat the unique maximizer of the Nash product as the Nash solution for a given problem, unless the problem belongs to the class of problems on which the Nash axioms are defined. The Nash product maximization provides a method to compute, but not a way to define the Nash solution.

With convexity, the establishment of the axiomatic characterization of the asymmetric Nash solution relies on the following separation property: The unique maximizer of the generalized Nash product over a convex choice set can be separated from any other feasible point by the tangent line to the indifference curve of the generalized Nash product at the maximizer. With log-convexity, however, this separation is not always valid. Nevertheless, our second main result shows that each non-maximizer is separable from the unique maximizer by the tangent line to the indifference curve of the generalized Nash product at some feasible point, which may depend on but differ from the non-maximizer (Lemma 1). As shown by our third main result (Theorem 2), this separation, interesting in its own right, turns out to be strong enough for IIA, INV, and SIR to uniquely determine the asymmetric Nash solution on our class of all regular and log-convex bargaining problems. Moreover, our class is complete in the sense that the three axioms do not uniquely determine single-valued bargaining solutions on any class containing ours as a proper subclass (Theorem 3). We formally establish these results for two-person problems. However, it follows from the bilateral stability of the Nash solution that parallel results hold for the n-person case. We provide a brief discussion along this line at the end of Sect. 3.2.

Several extensions of Nash bargaining theory allowing for non-convexity have appeared in recent literature. Zhou (1997) showed that on the class of all regular problems any single-valued solution satisfying IIA, INV, and SIR must be a selection of the maximizers of the generalized Nash product. However, such single-valued solution on this broad class of problems is not unique.[6] Denicolò and Mariotti (2000) associated a social welfare ordering with each single-valued solution on a class of problems, which includes those with the comprehensive hulls of two-point sets as choice sets. Their work introduces techniques from social choice theory useful for extending Nash bargaining theory. Peters and Vermeulen (2012) established new techniques to characterize all multi-valued bargaining solutions satisfying Nash's three axioms on a class of problems, which includes problems with choice sets consisting of finitely many payoff allocations. They showed that a multi-valued solution satisfying Nash's axioms can be found by an iterated maximization of generalized Nash products.[7]

[6] Zhou's characterization of single-valued solutions does not specify how to select among multiple Nash product maximizers. In a recent paper, Qin et al. (2014) provide a complete characterization of exact single-valued solutions satisfying Nash axioms under more general conditions on bargaining problems.

[7] They replaced PO with weak Pareto optimality (WPO). See Peters and Vermeulen (2012, pp. 26–27) for discussions about their refinements of multi-valued extensions in Kaneko (1980), Mariotti (1998a) and

Results in the preceding papers are based on large classes of problems that are not all log-convex. For example, the comprehensive hull of two payoff allocations is not log-convex. Intuitively, solutions are subject to stronger requirements on a larger class of problems than on a smaller one. Thus, although a solution on a larger class always induces a solution on a smaller class satisfying the same axioms, such induced solutions do not necessarily exhaust all possible solutions on the smaller class. Consequently, the existing results do not imply the unique axiomatic characterization of the asymmetric Nash solution by IIA, INV, and SIR on our class of all regular and log-convex problems, nor do they imply that our class is the complete one on which the three axioms characterize a unique single-valued solution.

Conley and Wilkie (1996) established a single-valued extension of the Nash solution by extending Nash's axioms. Their approach involves a two-step procedure for finding the solution. First, convexify the choice set via randomization and consider the Nash solution for the convexified problem. Second, use the intersection point of the original Pareto frontier with the segment between the threat point and the Nash solution for the convexified problem as the solution.[8] The uniqueness and single-valuedness of their extension enable an interesting comparison with the asymmetric Nash solution for log-convex problems. Herrero (1989) introduced an equal distance property to characterize her multi-valued extension with equal bargaining powers. When adjusted by players' bargaining powers, this property turns out to be also closely related to the Nash product maximization for log-convex problems.

Boche and Schubert (2011) also applied the notion of log-convexity to bargaining problems and extended the symmetric Nash solution to log-convex problems. Their approach is to consider the counterparts of the Nash axioms for the log-transformation of bargaining problems in their class, and then show that these counterparts jointly determined a unique solution, provided that the log-transformed problems are strictly convex and regular. That is, their results imply that the log-convexity together with their regularity requirement is *sufficient* for the standard axiomatic characterization of the symmetric Nash solution to hold. A significant disadvantage with their approach is the requirement that the threat point must be normalized to the origin; otherwise, the translation form of INV for their class of log-transformed problems is invalid (see Boche and Schubert 2011, equation (5), p. 3392). As such, one cannot apply their extension to problems with nonzero threat point including bargaining problems with endogenous threats and non-convexity. In comparison, we show that log-convexity is *both* sufficient and necessary for the standard axiomatic characterization of the asymmetric Nash bargaining solution to hold.[9] Furthermore, we directly work with Nash axioms instead of their counterparts on log-transformed problems. The proof of necessity of log-convexity is much involved. Nonetheless, the necessity is a useful technical result for it makes the class of bargaining problems complete, in the sense

Footnote 7 continued
Xu and Yoshihara (2006). For further references and discussions on non-convex bargaining problems, see the recent survey by Thomson (2009).

[8] See Mariotti (1998b) for an alternative characterization of Conley and Wilkie's (1996) extension.

[9] We have recently become aware of the Boche and Schubert (2011) paper. Our analysis is carried out independently of their work.

that there will be multiple solutions satisfying Nash axioms when bargaining powers are not a priori definite and bargaining problems are not all log-convex. Finally, we apply our results to duopoly and employer–employee bargaining problems which are shown to be log-convex but not convex.

The rest of the paper is organized as follows. The next section briefly reviews Nash axioms and Nash solution with convexity. Section 3 introduces log-convexity and presents our main results. Section 4 applies our results to non-convex bargaining problems arising from duopoly and the theory of the firm. Section 5 compares our approach with some other extensions. Section 6 concludes.

2 Nash bargaining with convexity

A two-person problem is composed of a choice set $S \subset \Re^2$ of feasible payoff allocations the players can jointly achieve with agreement, and a threat point $d \in S$ the players end up getting in case of disagreement. Let (S°, d°) be the normalized problem where

$$S^\circ = \left\{u \in \Re^2 | u_1 + u_2 \leq 1\right\} \text{ and } d^\circ = (0, 0). \tag{1}$$

A solution on a class $\mathcal{B}$ of bargaining problems is a rule f assigning a feasible allocation $f(S, d) = (f_1(S, d),\ f_2(S, d)) \in S$ to each problem $(S, d) \in \mathcal{B}$.[10] A positive affine transformation for player i's payoff is a mapping $\tau_i : \Re \longrightarrow \Re$ such that for some two real numbers $a_i > 0$ and b_i, $\tau_i(u_i) = a_i u_i + b_i$ for all $u_i \in \Re$. Given τ_1 and τ_2, $\tau(u) = (\tau_1(u_1), \tau_2(u_2))$ for all $u \in \Re^2$.

Nash's axioms can now be specified as follows (see footnote 3 for references on the replacement of PO with SIR).

SIR: For any $(S, d) \in \mathcal{B}$, $f_i(S, d) > d_i$, $i = 1, 2$.
SYM: For any $(S, d) \in \mathcal{B}$ with $d_1 = d_2$ and $(u_2, u_1) \in S$ whenever $(u_1, u_2) \in S$, $f_1(S, d) = f_2(S, d)$.
INV: For any $(S, d) \in \mathcal{B}$ and for any positive affine transformation $\tau : \Re^2 \longrightarrow \Re^2$, $f(\tau(S), \tau(d)) = \tau(f(S, d))$.
IIA: For any $(S, d), (S', d) \in \mathcal{B}$ with $S \subseteq S'$, $f(S', d) \in S$ implies $f(S, d) = f(S', d)$.

When $\mathcal{B}$ is composed of compact convex problems with strictly Pareto dominated threat points, there is a unique solution $N : \mathcal{B} \longrightarrow \Re^2$ satisfying the above four axioms, known as the symmetric Nash solution, which assigns payoff allocation

$$N(S, d) = \arg \max_{u \in S, u \geq d} (u_1 - d_1)(u_2 - d_2) \tag{2}$$

to $(S, d) \in \mathcal{B}$. See Nash (1953) and Roth (1977) for details.

When SYM is removed, Kalai (1977) showed that the solution on the class of compact convex bargaining problems $\mathcal{B}$ satisfying IIA, INV, SIR is unique up to the choices of players' bargaining powers [i.e., up to the specifications of solutions for

[10] Unless noticed otherwise, bargaining solutions are single-valued throughout the rest of the paper.

(S°, d°)]. More precisely, Kalai showed that if a solution f satisfies IIA, INV, and SIR, it must be given by

$$f(S, d) = \arg \max_{u \in S, u \geq d} (u_1 - d_1)^\alpha (u_2 - d_2)^{1-\alpha} \tag{3}$$

for all $(S, d) \in \mathcal{B}$, where $(\alpha, 1-\alpha) = f(S^\circ, d^\circ)$. Notice that $0 < f_1(S^\circ, d^\circ), f_2(S^\circ, d^\circ) < 1$ because of SIR. Thus, the solution on the class $\mathcal{B}$ of compact convex bargaining problems is unique whenever $f(S^\circ, d^\circ)$ is specified.

For later references in this paper, we say that the asymmetric Nash solution on a class of bargaining problems is unique up to choices of the players' bargaining powers whenever IIA, INV, SIR together with given solution for (S°, d°) uniquely determine the solution on the class. When it is unique, we use N^α to denote the asymmetric Nash solution that assigns payoff allocation $(\alpha, 1-\alpha)$ to (S°, d°).

3 Nash bargaining with log-convexity

The convexity of bargaining problems plays a crucial role in the establishment of the uniqueness and representation of the solution under Nash axioms. When the convexity is dropped, the solutions under the axioms of IIA, INV, and SIR can still be represented by the Nash product maximization, but may fail to be unique. In this section, we characterize the complete class of bargaining problems allowing for non-convexity, on which the uniqueness as well as the representation of the asymmetric solution under the three Nash axioms holds.

3.1 Regularity and log-convexity

We first introduce log-convexity of a bargaining problem. Fix a problem (S, d) and take the natural logarithmic transformation of the positive payoff gains to get

$$V(S, d) = \left\{ v \in \Re^2 \middle| \exists u \in S: \begin{array}{l} u_1 > d_1, u_2 > d_2, \\ v_1 \leq \ln(u_1 - d_1), \\ v_2 \leq \ln(u_2 - d_2). \end{array} \right\}.$$

Then, the maximization (3) is equivalent to

$$\max_{v \in V(S,d)} \alpha v_1 + (1-\alpha) v_2. \tag{4}$$

Since $\ln(u_i - d_i)$ is increasing and concave over $u_i \in (d_i, \infty)$ for $i = 1, 2$, the convexity of S implies the convexity of $V(S, d)$. For the rest of the paper we consider (S, d) that satisfies the following regularity condition:

> Regular Problems ($\mathcal{R}$): *A problem (S, d) is regular if S is closed, bounded above, d-comprehensive (i.e., $u \in S$ whenever $d \leq u \leq u'$ for some $u' \in S$), and d is strictly Pareto dominated in S.*

The following log-convexity of (S, d) is critical for the uniqueness of the asymmetric Nash solution:[11]

> Log-Convex Problems ($\mathcal{L}$): *For any two different points v' and v'' on the strict Pareto frontier of $V(S, d)$, $\lambda v' + (1-\lambda)v''$ is in the interior of $V(S, d)$ for all* $0 < \lambda < 1$.

Before presenting our main results, we make the following observations about log-convexity that will be useful later. First, since $V(S, d)$ is comprehensive, log-convexity implies that $V(S, d)$ is convex. Second, log-convexity of $V(S, d)$ is equivalent to the conditions that $V(S, d)$ is convex and it does not contain segments with normal vectors $(\theta, 1-\theta)$ for all $\theta \in (0, 1)$. Third, let (S, d) and (S', d) be two regular problems with the same threat point d. Then, $V(S \cap S', d) = V(S, d) \cap V(S', d)$. This implies that $(S \cap S', d)$ is regular and log-convex whenever both (S, d) and (S', d) are.[12]

Theorem 1 *For $(S, d) \in \mathcal{R}$, (4) has a unique solution for all $\alpha \in (0, 1)$ if and only if $(S, d) \in \mathcal{L}$.*

For $\alpha \in (0, 1)$, the log-convexity of (S, d) implies that solution for (4), if exists, must be unique. Next, the existence of a solution for (4) is guaranteed by the regularity condition of (S, d). This establishes the sufficiency. We prove the necessity in the "Appendix". The essential part of the proof is the convexity of $V(S, d)$. Our idea is to show that the closure of the convex hull $\text{co}V(S, d)$ of $V(S, d)$ equals $V(S, d)$.

When $V(S, d) = C - \Re^2_+$ for some compact set $C \subseteq \Re^2$ (i.e., when $V(S, d)$ is compactly generated), we can apply Caratheodory theorem to first show that the closure of $\text{co}V(S, d) = \text{co}C - \Re^2_+$. Next, by applying separating hyperplane theorem to the boundary points of the closure of $\text{co}C - \Re^2_+$, the condition that problem (4) has a unique solution implies that they must be contained in C.[13] Thus, $\text{co}V(S, d) = \text{co}C - \Re^2_+ = C - \Re^2_+ = V(S, d)$, implying that $V(S, d)$ is convex. However, $V(S, d)$ is not necessarily compactly generated, although $V(S, d)$ is closed, comprehensive, and bounded from above. For example,

$$V = \left\{ v \in \Re^2 | v_1 < 0,\ v_2 \le 1/v_1 \right\}$$

is closed, comprehensive, and bounded above, but $V \neq C - \Re^2_+$ for any compact set $C \subset \Re^2$.

The fact that problem (4) has a unique solution for all $\alpha \in (0, 1)$ implies that $V(S, d)$ is contained in the intersection of the lower closed half-spaces determined by the tangent lines of $V(S, d)$ with normal vectors $(\alpha, 1-\alpha)$, $\alpha \in (0, 1)$. Since the intersection of these half-spaces is convex, the convexity of $V(S, d)$ would be

[11] See Boche and Schubert (2011) for an application of log-convexity to extend the symmetric Nash solution with zero threat point.

[12] Indeed, it is straightforward to verify that $(S \cap S', d)$ is regular and $V(S \cap S', d) \subseteq V(S, d) \cap V(S', d)$. On the other hand, take any $v \in V(S, d) \cap V(S', d)$, $u \in S$, and $u' \in S'$ such that $u_i > d_i$, $u'_i > d_i$, $v_i \le \ln(u_i - d_i)$, and $v_i \le \ln(u'_i - d_i)$ for $i = 1, 2$. Set $\underline{u}_i = \min\{u_i, u'_i\}$. Then, $v_i \le \ln(\underline{u}_i - d_i)$ and $d << \underline{u} \le u, u'$. By the regularity, $\underline{u} \in S \cap S'$. This shows $v \in V(S \cap S', d)$. Hence, $V(S \cap S', d) \supseteq V(S, d) \cap V(S', d)$.

[13] Since $V(S, d)$ is comprehensive, it suffices to consider boundary points that are Pareto optimal.

established if it can be shown that $V(S, d)$ coincides with the intersection. However, to show this coincidence using the fact that (4) has a unique solution for all $\alpha \in (0, 1)$, we need to show that there exists a tangent line at each point on the Pareto frontier of $V(S, d)$, which is not automatic unless $V(S, d)$ is convex.

Our proof proceeds as follows. First, we apply the separating hyperplane theorem to each boundary point of the convex hull of $V(S, d)$. Next, we approximate the boundary point by a sequence of points in the convex hull and apply Caratheodory theorem to represent these points as convex combinations of points in $V(S, d)$. Finally, by considering the limits of the representing points, the uniqueness of the solution for (4) enables us to show that the boundary point is either on the boundary of $V(S, d)$ or is Pareto dominated in $V(S, d)$. This together with the comprehensiveness of $V(S, d)$ implies that the closure of the convex hull of $V(S, d)$ equals $V(S, d)$; hence, $V(S, d)$ is convex.

3.2 Uniqueness

We now show that IIA, INV, and SIR uniquely determine the asymmetric Nash solution on the class of all regular and log-convex bargaining problems.

Theorem 2 *A bargaining solution f on $\mathcal{R} \cap \mathcal{L}$ satisfies IIA, INV, and SIR if and only if $f = N^{\alpha}$ with $(\alpha, 1-\alpha) = f(S^{\circ}, d^{\circ})$.*

To prove this theorem, we need the following separation result.

Lemma 1 *Let $(T, d^{\circ}) \in \mathcal{R} \cap \mathcal{L}$ and let $x = \arg\max_{u \in T : u \geq d^{\circ}} u_1^{\alpha} u_2^{1-\alpha}$ with $\alpha \in (0, 1)$. Then, for any $y \in T$ with $y \gg d^{\circ}$ and $y \neq x$, there exists $z \in T$ such that $z \neq y$ and*

$$\alpha z_2 x_1 + (1-\alpha) z_1 x_2 \geq z_1 z_2 \tag{5}$$

and

$$\alpha z_2 y_1 + (1-\alpha) z_1 y_2 < z_1 z_2. \tag{6}$$

Proof Since $(T, d^{\circ}) \in \mathcal{R} \cap \mathcal{L}$, it follows from Theorem 1 that $x = \arg\max_{u \in T : u \geq d^{\circ}} u_1^{\alpha} u_2^{1-\alpha}$ is uniquely determined. Let $y \in T$ be such that $y \gg d^{\circ}$ and $y \neq x$. Then, from the uniqueness of the maximizer,

$$x_1^{\alpha} x_2^{1-\alpha} > y_1^{\alpha} y_2^{1-\alpha},$$

or equivalently,

$$\alpha(\ln y_1 - \ln x_1) < (1-\alpha)(\ln x_2 - \ln y_2).$$

Assume without loss of generality $x_1 < y_1$ (the other case can be analogously proved). The locations of x and y are illustrated in Fig. 1a. Notice with $x_1 < y_1$, the previous inequality implies

$$\frac{\alpha}{1-\alpha} < \frac{\ln x_2 - \ln y_2}{\ln y_1 - \ln x_1}. \tag{7}$$

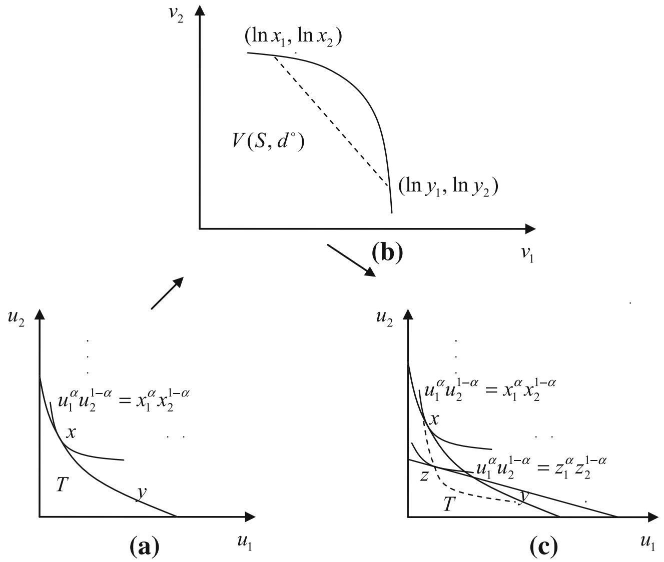

Fig. 1 Separation between the maximizer x of Nash product $u_1^\alpha u_2^{1-\alpha}$ over choice set T and point $y \in T$ by the tangent line of Nash product at point $z = z(t) \in T$ with $z(t) \neq y$

Since $x,\ y \in T$ and $d^\circ = (0, 0)$, it follows from the log-convexity

$$t(\ln x_1, \ln x_2) + (1-t)(\ln y_1, \ln y_2) = \left(\ln x_1^t y_1^{1-t}, \ln x_2^t y_2^{1-t}\right) \in V\left(T, d^\circ\right)$$

for $t \in [0, 1]$. This is the segment between $(\ln x_1, \ln x_2)$ and $(\ln y_1, \ln y_2)$ in Fig. 1b. Thus, by the definition of $V(T, d^\circ)$, there exists $u(t) \in T$ such that $(\ln x_1^t y_1^{1-t}, \ln x_2^t y_2^{1-t}) \leq (\ln u_1(t), \ln u_2(t))$, or equivalently, $(x_1^t y_1^{1-t}, x_2^t y_2^{1-t}) \leq (u_1(t), u_2(t))$. Since $x, y \gg d^\circ$ implies $(x_1^t y_1^{1-t}, x_2^t y_2^{1-t}) \gg d^\circ$, it follows from the regularity that

$$z(t) \equiv (x_1^t y_1^{1-t}, x_2^t y_2^{1-t}) \in T$$

for all $t \in [0, 1]$ as illustrated in Fig. 1c.

For $t \in (0, 1)$, the absolute value of the slope of the tangent line of $u_1^\alpha u_2^{1-\alpha}$ at $z(t)$ is

$$\frac{\alpha x_2^t y_2^{1-t}}{(1-\alpha) x_1^t y_1^{1-t}}$$

and that of the segment between $z(t)$ and y is

$$\frac{x_2^t y_2^{1-t} - y_2}{y_1 - x_1^t y_1^{1-t}}.$$

Thus, the tangent line of $u_1^{\alpha}u_2^{1-\alpha}$ at $z(t)$ is flatter than the segment between $z(t)$ and y if and only if[14]

$$\frac{\alpha x_2^t y_2^{1-t}}{(1-\alpha)x_1^t y_1^{1-t}} < \frac{x_2^t y_2^{1-t} - y_2}{y_1 - x_1^t y_1^{1-t}} \Leftrightarrow \frac{\alpha x_2^t}{(1-\alpha)x_1^t} < \frac{x_2^t - y_2^t}{y_1^t - x_1^t}.$$

By L'Hôpital's rule and (7), the above inequalities hold as $t \longrightarrow 0$. Now, choose $t \in (0, 1)$ such that the tangent line of $u_1^{\alpha}u_2^{1-\alpha}$ at point $z(t)$,

$$\alpha z_2(t)u_1 + (1-\alpha)z_1(t)u_2 = z_1(t)z_2(t), \tag{8}$$

is flatter than the segment between $z(t)$ and y. Since x is Pareto optimal, $x_1 < y_1$ necessarily implies $x_2 > y_2$. Thus, $z_1(t) < y_1$ and $z_2(t) > y_2$. Consequently, y is below the tangent line in (8). On the other hand, by definition of x, x cannot lie below the indifference curve of $u_1^{\alpha}u_2^{1-\alpha}$ at $z(t)$ (see Fig. 1c for an illustration). Thus, (5) and (6) are established by setting $z = z(t)$. □

Lemma 1 extends the separation between the maximizer of the generalized Nash product over a convex choice set and the rest of the choice set. Specifically, with convexity, the maximizer is separable from any other point of the choice set by the tangent line of the indifference curve of the generalized Nash product through the maximizer. In the case with a regular and log-convex choice set, the separation in Lemma 1 is weaker in the sense that it may be point-dependent (i.e., z may change as point y changes in the choice set), but it is nonetheless strong enough for IIA, INV, and SIR together to determine a unique solution.

Proof of Theorem 2 Let f be a solution on $\mathcal{R} \cap \mathcal{L}$ satisfying IIA, INV, SIR and let $f(S^{\circ}, d^{\circ}) = (\alpha, 1-\alpha)$. Suppose $f \neq N^{\alpha}$. Then, there exists $(T, d^{\circ}) \in \mathcal{R} \cap \mathcal{L}$ such that

$$f(T, d^{\circ}) \neq \arg \max_{u \in T : u \geq d^{\circ}} u_1^{\alpha}u_2^{1-\alpha}. \tag{9}$$

Set $y = f(T, d^{\circ})$. By SIR, $y >> d^{\circ}$. By Lemma 1 and (9), there exists $z \in T$ such that $z \neq y$ and y is below the tangent line of $u_1^{\alpha}u_2^{1-\alpha}$ at z:

$$\alpha z_2 u_1 + (1-\alpha)z_1 u_2 = z_1 z_2, \quad u \in \Re^2. \tag{10}$$

Set

$$S = \left\{ u \in \Re_+^2 | \alpha z_2 u_1 + (1-\alpha)z_1 u_2 \leq z_1 z_2 \right\}.$$

[14] Since $x_1 < y_1$, the condition is equivalent to y lying below the tangent line.

Clearly, $(S, d^\circ) \in \mathcal{R} \cap \mathcal{L}$ and $y \in S$ (see Fig. 1c). By INV and the assumption $f(S^\circ, d^\circ) = (\alpha, 1-\alpha)$, we have[15]

$$f(S, d^\circ) = z. \tag{11}$$

Since $(S \cap T, d^\circ) \in \mathcal{R} \cap \mathcal{L}$ and since $z \in S \cap T$, it follows from IIA and (11) that $f(S \cap T, d^\circ) = z$. On the other hand, since $y \in S \cap T$ and $y = f(T, d^\circ)$, it follows from IIA that $f(S \cap T, d^\circ) = y$. This shows $y = z$, which is a contradiction. □

Our next result shows that the class of all regular and log-convex problems is the complete class for IIA, INV, and SIR to uniquely determine the asymmetric Nash solution.

Theorem 3 *The axioms of IIA, INV, and SIR do not uniquely characterize solutions up to choices of bargaining powers on any class $\mathcal{B}$ such that $\mathcal{R} \cap \mathcal{L} \subsetneq \mathcal{B} \subseteq \mathcal{R}$.*

Proof Let $(S', d') \in \mathcal{B}$ be such that $(S', d') \notin \mathcal{R} \cap \mathcal{L}$. Since $\mathcal{B} \subseteq \mathcal{R}$, it must be $(S', d') \in \mathcal{R} \setminus \mathcal{L}$ (set theoretic minus). By Theorem 1, there exists $\alpha' \in (0, 1)$ such that (S', d') has multiple maximizers of the generalized Nash product with bargaining power α' for player 1. Now, consider two solutions f^1 and f^2 on $\mathcal{B}$ given by $f^1(S^\circ, d^\circ) = f^2(S^\circ, d^\circ) = (\alpha', 1-\alpha')$,

$$f^1(S, d) = \arg \max_{u \in \mathcal{N}^{\alpha'}(S,d)} u_1,$$

and

$$f^2(S, d) = \arg \max_{u \in \mathcal{N}^{\alpha'}(S,d)} u_2$$

where $\mathcal{N}^{\alpha'}(S, d) = \arg\max_{u \in S: u \geq d} u_1^{\alpha'} u_2^{1-\alpha'}$, for $(S, d) \in \mathcal{B}$. By the multiplicity of $\mathcal{N}^{\alpha'}(S', d')$, $f^1(S', d') \neq f^2(S', d')$. Moreover, by construction, the usual proof implies that f^1 and f^2 are solutions on $\mathcal{B}$ satisfying IIA, INV, and SIR. This together with $f^1(S^\circ, d^\circ) = f^2(S^\circ, d^\circ) = (\alpha', 1-\alpha')$ shows that solutions on $\mathcal{B}$ satisfying IIA, INV, and SIR are not unique up to choices of bargaining powers. □

Since the class of regular and log-convex problems is a proper subset of the regular problems, Theorem 3 implies that solutions under IIA, INV, and SIR are not unique on the broad class of bargaining problems in Zhou (1997).

[15] Consider affine transformation $\tau : \Re^2 \longrightarrow \Re^2$:

$$\tau(u) = \left(\frac{\alpha}{z_1} u_1, \frac{1-\alpha}{z_2} u_2\right), \ u \in \Re^2.$$

Then, by (10), $\tau(S) = S^\circ$. Since $f(S^\circ, d^\circ) = (\alpha, 1-\alpha)$, $f(\tau(S), \tau(d^\circ)) = (\alpha, 1-\alpha)$. By INV, $\tau(f(S, d^\circ)) = f(\tau(S), \tau(d^\circ))$. Hence, $f(S, d^\circ) = \tau^{-1}(\alpha, 1-\alpha) = z$.

3.3 Generalization to n-person case

The notions of regularity and log-convexity and Theorem 1 can all be naturally generalized to the n-person case. The asymmetric Nash solution for the n-person case satisfies bilateral stability (consistency), which states that the solution for a n-person problem agrees with the solution for each two-person subproblem obtained by taking the other players' utility levels in the original solution as given.[16] The two-person subproblems of a n-person regular and log-convex problem are again regular and log-convex. Thus, by applying Lemma 1 to two-person subproblems, it can be shown that Theorem 2 can be generalized to n-person regular and log-convex problems. Finally, the method of proof of Theorem 3 is directly applicable to n-person regular and log-convex problems. It follows that Theorem 3 can also be generalized to the n-person case.

3.4 A subclass of log-convex problems

By the IR frontier of a bargaining problem (S, d) we mean the portion of the frontier of S that is individually rational (i.e., satisfies $u_i \geq d_i$ for $i = 1, 2$). For Nash product maximization with generic bargaining powers to have a unique solution, log-convexity quantifies how non-convex the individually rational (IR) portion of a bargaining problem can be allowed. When the frontier of the choice set is smooth, log-convexity is implied by a simple condition as shown in the following proposition.

Proposition 1 *Let $(S, d^\circ) \in \mathcal{R}$ be such that the IR frontier of S is given by a C^2 function $u_j = \psi(u_i)$, $u_i \in [0, \bar{u}_i]$ for some number $\bar{u}_i < \infty$ with $\psi(\bar{u}_i) = 0$ and for $i \neq j$. If*

$$(C) \quad \psi'(u_i) < 0 \text{ and } e(u_i) \equiv \frac{u_i \psi'(u_i)}{\psi(u_i)} \text{ is strictly decreasing over } (0, \bar{u}_i),$$

then (S, d°) is log-convex.

Proof Notice first that for any $v = (v_i, v_j)$ in the frontier of $V(S, d^\circ)$, there exists $u_i \in (0, \bar{u}_i)$ such that $v_i = \ln(u_i)$ and $v_j = \ln[\psi(u_i)]$. That is, the frontier of $V(S, d^\circ)$ is parameterized by $u_i \in (0, \bar{u}_i)$. Thus, the slope of the frontier at v is given by

$$e(u_i) = \frac{u_i \psi'(u_i)}{\psi(u_i)}.$$

Hence, the frontier of $V(S, d^\circ)$ is downward sloping due to $\psi'(u_i) < 0$ and is strictly concave if $e(u_i)$ is strictly decreasing. □

16 Bilateral stability of the n-person symmetric Nash solution was introduced in Harsanyi (1959). Bilateral stability also holds for the asymmetric Nash solution. See Lensberg (1988) for general results on multilateral as well as bilateral stabilities of the symmetric Nash solution.

Note that, $e(u_i)$ in Proposition 1 is the elasticity of the IR frontier, $u_j = \psi(u_i)$, of a bargaining problem with respect to u_i. Thus, in applications, it is relatively easier to check (C) than checking the log-convexity.

A stronger sufficient condition is either ψ is log-concave, or $u_i\psi'(u_i)$ is decreasing, in addition to ψ being decreasing. With a generic threat point d, (C) takes the following form,

$$\psi'(u_i) < 0 \text{ and } e(u_i) \equiv \frac{(u_i - d_i)\psi'(u_i)}{\psi(u_i) - d_j} \text{ is strictly decreasing over } (d_i, \bar{u}_i).$$

Moreover, as d varies in S, the log-convexity of (S, d) puts further restrictions on S. A natural question is whether S is necessarily convex when (S, d) is log-convex for all strictly Pareto dominated threat points $d \in S$. Proposition 2 in the next section establishes a class of problems that is not convex but log-convex for all threat points that are strictly Pareto dominated.

4 Applications

Nash bargaining theory has been widely applied to employer–employee bargaining and bargaining in duopoly, among many other areas. We refer the reader to Bishop (1960), Aoki (1980), McDonald and Solow (1981), Miyazaki (1984), Schmalensee (1987) and Tirole (1988) for applications involving non-convexity in previous literature. In this section, we demonstrate that bargaining problems arising in these two areas are log-convex but not convex under mild familiar conditions. By our results, Nash bargaining theory can be applied to these problems, even though it was regarded as nonapplicable due to the non-convexity in the preceding papers.

4.1 Bargaining in duopoly

We now apply our results to duopoly bargaining (or collusion) problems.[17] Suppose, there are two firms in an industry supplying a homogeneous product with a market demand function $D(p)$. Each firm i has zero fixed cost and constant marginal cost of production given by c_i, where $c_i \geq 0$ for $i = 1, 2$.

Let $p^\circ = \inf\{p : D(p) = 0\}$. We make the following assumptions on the demand function and marginal costs: (i) $0 \leq c_1 < c_2 < p^\circ$; (ii) $D(p)$ is twice continuously differentiable and $D'(p) < 0$ on $(0, p^\circ)$; and (iii) firm 1's *monopoly profit* function $\bar{\pi}_1(p) \equiv (p - c_1)D(p)$ is strictly concave on $(0, p^\circ)$ and $\lim_{p \to p^\circ} \bar{\pi}_1(p) = 0$. Assumption (iii) implies that there exists a unique monopoly price $\bar{p}_1$ that maximizes $\bar{\pi}_1(p)$ and that $c_1 < \bar{p}_1 < p^\circ$. To make bargaining nontrivial, we also assume (iv) $c_2 < \bar{p}_1$.

[17] Binmore (2007, pp. 480–481) illustrated the use of Nash bargaining solution as a tool for studying collusive behavior in a Cournot duopoly with asymmetric constant marginal costs. However, the profit choice set without side payments is drawn as if it is convex (Figure 16.13(a)), which is not with respect to the assumed demand and cost functions.

Suppose, the two firms bargain over their production outputs (or equivalently price and market shares) without side payments.[18] We first describe the feasible set of profit allocations for the firms, which is denoted as Π. Assuming free disposability, we only need to characterize the Pareto frontier of Π, which we denote by PF. Let profit allocation $\pi = (\pi_1, \pi_2) \in$ PF be resulted from an output agreement (q_1, q_2). Since $\pi_i = (p - c_i)q_i$ with for $i = 1, 2$ with p being the price such that $q_1 + q_2 = D(p)$, $\pi \in$ PF satisfies

$$\frac{\pi_1}{p - c_1} + \frac{\pi_2}{p - c_1} = D(p). \tag{12}$$

That is, feasible profit allocations on PF can be parameterized by a range of prices instead of output allocations. We shall narrow down the price range that completely characterizes PF.

Notice that, profit allocations associated with prices $p \le c_2$ or $p \ge p^\circ$ are Pareto dominated by those with prices in (c_2, p°). Given $p \in (c_2, p^\circ)$ and π_2, it follows from (12) that

$$h(p, \pi_2) \equiv \bar{\pi}_1(p) - \frac{\pi_2(p - c_1)}{p - c_2} \tag{13}$$

is the profit for firm 1 should they agree to let firm 2 receive profit π_2. Thus, $\pi \in$ PF if and only if

$$\pi_1 = \max_{c_2 < p < p^\circ} h(p, \pi_2).$$

The supporting price, $p(\pi_2)$, is determined by the first-order condition

$$\bar{\pi}_1'(p) + \frac{(c_2 - c_1)\pi_2}{(p - c_2)^2} = 0$$

and hence, the Pareto frontier is represented by $\pi_1 = h(p(\pi_2), \pi_2)$ for $\pi_2 \in [0, \bar{\pi}_2]$, where $\bar{\pi}_2$ is the maximum monopoly profit for firm 2. Note that, the second-order condition is satisfied due to Assumption (iii).

To illustrate non-convexity of Π, consider a simple example in which $D(p) = 20 - p$, $c_1 = 0$, and $c_2 = 10$. In this case, the Pareto frontier can be easily computed and is given by $\pi_1 = 100 - \pi_2 - 3(5\pi_2)^{\frac{2}{3}}$ for $0 \le \pi_2 \le 25$. Simple calculation shows that the PF is a strictly convex curve. Consequently, the feasible choice set of the resulting bargaining problem is not convex. Intuitively, without side payments, production arrangements are necessarily inefficient in order for firm 2, the inefficient one, to produce positive quantities and receive positive profits, so that one unit of profit gain by firm 2 results in more than one unit of profit loss for firm 1.

We next show that under our assumptions, the PF curve $\pi_1 = h(p(\pi_2), \pi_2)$ is strictly decreasing and convex in $\pi_2 \in [0, \bar{\pi}_2]$.[19] Nevertheless, we show that for any $d \ge 0$ that is strictly Pareto dominated in Π, (Π, d) satisfies Condition (C) for log-convexity with generic threat point mentioned in Sect. 3.4 (pp. 15–16).

[18] Explicit transfers or briberies between firms may be too risky due to antitrust scrutiny.

[19] The non-convexity of Π was shown in Tirole (1988, p. 242, 271) under one extra assumption that the monopoly profit function for firm 2 is concave. We will show the non-convexity without such an assumption.

Proposition 2 *Suppose* (i)–(iv) *hold. Then,* Π *is non-convex, but* $(\Pi, d) \in \mathcal{R} \cap \mathcal{L}$ *for all* $d \geq 0$ *that are strictly Pareto dominated in* Π.

Proposition 2 and Theorem 2 together imply that the duopoly problem (Π, d) has a unique Nash bargaining solution under the assumed conditions. The uniqueness of the solution makes it easier for one to perform comparative static analysis of the Nash solution with respect to bargaining powers, marginal costs, threat points, and demand parameters.

4.2 A bargaining model of the firm

Aoki (1980) analyzes a bargaining model of the firm as an organization of stockholders and employees. In his model, the employees, who are assumed to be homogenous, work with assets supplied by the stockholders to generate economic gains that are not possible through casual combination of marketed factors of production. As a result, by acquiring firm-specific skills and knowledge, the employees may exert bargaining power over the disposition of the economic gains. In what follows, we first consider single-product monopoly profit maximization parameterized by per product unit wage, and we show that the bargaining problem with Pareto frontier determined by such profit maximization is log-convex but not convex, under familiar conditions on demand and cost functions. We then show that the bargaining problem associated with the Aoki's model can be reduced to such a bargaining problem.

To that end, let $D(p)$ be the market demand for a monopoly firm's product and consider the following profit maximization problem

$$\pi(w) = \max_p (p - c - w) D(p), \tag{14}$$

where c is the component of the firm's marginal cost on production factors other than labor and w is the other component. We take both c and w to be constant. This could be due to the firm's technology being of constant returns to scale and perfect competitiveness of the factor markets. Alternatively, it could also be resulted from the fact that the firm is in short run with labor as the only variable input (in which case $c = 0$) and w as the outcome of wage bargaining.

Assume (i) $D(c) > 0$ and $D(p)$ is twice continuously differentiable with $D'(p) < 0$ whenever $D(p) > 0$. Set $\mu(p) = -D(p)/D'(p)$. This term has useful applications to the analysis of monopoly pricing decisions among many others (see for example, Weyl and Fabinger 2013 for details). Assume further (ii) $\mu'(p) < 1$ whenever $D(p) > 0$; and (iii) there exists a price $0 < \bar{p} < \infty$ such that $\mu(\bar{p}) - \bar{p} + c < 0$. Assumptions (i) and (iii) are self-explanatory. Assumption (ii) is equivalent to the monopoly firm's marginal revenue being downward sloping. Hence, the objective function in (14) is single-peaked in price p. It follows that maximization problem (14) has a unique solution that is determined by the first-order condition

$$D(p) + (p - c - w) D'(p) = 0 \Leftrightarrow \mu(p) - p + c + w = 0. \tag{15}$$

Denote by $p(w)$ the solution for (15). Define $\bar{w}$ by $[p(\bar{w}) - c - \bar{w}] D(p(\bar{w})) = 0$.

Lemma 2 *Assume* (i)–(iii). *Then, the bargaining problem with Pareto frontier determined by* (14) *and threat point* d° *is non-convex; but it is log-convex if in addition,* $0 < \bar{w} < \infty$ *and the absolute value of price elasticity of* $D(p)$ *is monotonically increasing in price.*

Proof By (15),

$$p'(w) = \frac{1}{1 - \mu'(p(w))} \text{ and } \pi'(w) = -D(p(w)). \tag{16}$$

Consequently, $\pi''(w) = -D'(p(w))p'(w)$. Assumption (ii) together with (16) then implies $p'(w) > 0$ and hence, $\pi''(w) > 0$. This shows that the PO frontier of the choice set is strictly convex. As a result, the bargaining problem is non-convex.

Observe next that from (14) and (16) it follows

$$e(w) = -\frac{w}{p(w) - c - w},$$

where $e(w) = w\pi'(w)/\pi(w)$. Consequently,

$$\begin{aligned} e'(w) &= w\frac{-(-c-w) + w(p'-1)}{(-c-w)^2} \\ &= \frac{w}{(p-c-w)^2}\Big[wp' - (p-c)\Big]. \end{aligned}$$

By (15) and (16),

$$\begin{aligned} e'(w) < 0 &\Leftrightarrow wp' - (p-c) < 0 \\ &\Leftrightarrow w < (p-c)(1 - \mu'(w)) \\ &\Leftrightarrow (p-c) - \mu < (p-c)(1-\mu') \\ &\Leftrightarrow \mu - (p-c)\mu' > 0. \end{aligned}$$

If $\mu' \leq 0$, then $\mu - (p-c)\mu' > 0$ because $\mu > 0$ and $p \geq c$. If $\mu' > 0$, however, $\mu - (p-c)\mu' \geq \mu - p\mu'$. This together with the preceding analysis implies

$$e'(w) < 0 \quad \text{whenever } \frac{p\mu'(p)}{\mu(p)} < 1. \tag{17}$$

Now, let $\epsilon(p) = -pD'(p)/D(p)$. Then,

$$\mu(p)\epsilon(p) \equiv p \Rightarrow \frac{p\mu'(p)}{\mu(p)} + \frac{p\epsilon'(p)}{\epsilon(p)} = 1.$$

Applying the assumption on $\epsilon(p)$ to the preceding equation and then combining with (17) would imply

$$e'(w) < 0, \quad w \in [0, \bar{w}].$$

With $\psi(\cdot)$ replaced by $\pi(\cdot)$ and $\bar{u}_i$ by $\bar{w}$, the assumptions in Lemma 2 together with the preceding analysis shows that all the conditions in Proposition 1 are satisfied. It follows that the bargaining problem with disagreement point d° is log-convex. □

Assumptions (i)–(iii) are satisfied by many familiar demand functions. Examples include $D(p) = a - bp$ and $D(p) = e^{-\eta p}$, where $a, b, \eta > 0$ are constant.

Remark 1 It is worth pointing out that the non-convexity of the choice set can arise under much weaker conditions. To see this, notice that as the maximum value function, $\pi(w)$ is convex. As such, the bargaining problem is non-convex if $\pi(w)$ is nonlinear, which holds under more general conditions on the demand function than what are required in the lemma.

In Aoki's (1980) model, the firm incurs an average growth expenditure $T = \phi(g)$ per unit of current sales in order to realize planned growth rate g of sales from current period to the next one. Payment to an employee consists of a fixed wage ($\bar{w}$) and differential earning (ω). While the fixed wage is determined by competitive external labor market, the differential earning is settled through bargaining.

Following Aoki (1980), assume ϕ is increasing and convex. Let p and $x(p)$ denote the price and sale of the firm's product in the current period. Furthermore, let l denote the number of employees needed for producing one unit of the product, which is assumed to be constant in Aoki (1980). The stockholders and employees jointly share the firm's organizational rent R, where

$$R = [p - l\bar{w} - \phi(g)]\,x(p). \tag{18}$$

Let θ be the stockholders' share of the rent. Then,

$$\omega = \frac{(1-\theta)R}{lx(p)}. \tag{19}$$

Aoki assumed that investors follow a simple Keynesian "convention," in the sense that they believe on average that the existing state and firm's policies will continue indefinitely. With this in place, the rate of stock value appreciation equals g and the dividends at the end of the period equals θR. It follows that competitive valuation π of the firm's stock satisfies

$$g\pi + \theta R = \rho\pi,$$

where ρ is the one-period interest rate. The preceding equation implies

$$\pi = \frac{\theta R}{\rho - g}. \tag{20}$$

The stockholders and employees jointly bargain over the shares of the firm's organizational rent R with their objectives given in (19) and (20). As such, the set of feasible choices for the parties is the set of (ω, π) reached by feasibly varying (g, p, θ). We refer the reader to Aoki (1980) for more details of the model.

In the rest of this subsection, we first show that the bargaining problem for Aoki's model can be transformed into one with Pareto frontier determined by (14). Correspondingly, it follows from Lemma 2 that the problem is log-convex but not convex under familiar conditions. To this end, observe first that by (18), (19), and (20),

$$\pi = \frac{R - l\omega x(p)}{\rho - g} = [p - l\bar{w} - l\omega - \phi(g)]\,\frac{x(p)}{\rho - g}.$$

Thus, the Pareto frontier of the choice set is determined by the following maximization problem:

$$\pi(\omega) = \max_{p,g}\,[p - l\bar{w} - l\omega - \phi(g)]\,\frac{x(p)}{\rho - g}. \tag{21}$$

Next, denote by $y = p - \phi(g)$ and $D(y) = x(y + \phi(g(y)))/(\rho - g(y))$, where $g(y)$ solves

$$\max_{g}\,\frac{x(y + \phi(g))}{\rho - g}. \tag{22}$$

By (22), it can be shown that given differential earning ω, the maximum value in (21) satisfies

$$\pi(\omega) = \max_{y}(y - l\bar{w} - l\omega)D(y).$$

Letting $w = l\omega$ and $c = l\bar{w}$, we have from the preceding equation,

$$\pi(w) = \max_{y}(y - c - w)D(y)$$

which is identical to maximization problem (14).

Finally, Aoki assumes that there exists a critical value $\tilde{g} < \rho$ such that it can never be profitable for the firm to plan a growth rate larger than $\tilde{g}$. We maintain this assumption below. Set $\mu_x(p) = -x(p)/x'(p)$.

Proposition 3 *Assume $x(p)$ is twice differentiable with $x'(p) < 0$. Assume further $\mu_x'(p) \le 0$ and $(\rho - g)\phi''(g) \ge 2\phi'(g)$ for $g < \tilde{g}$. Then, the bargaining problem with d° is log-convex but not convex.*

Proposition 3 implies that the choice set is log-convex but not convex under the assumed conditions and the threat point normalized at d°. The log-convexity for cases with general threat points can be similarly established. Notice also that the condition $\mu_x'(p) \le 0$ is satisfied by the demand functions such as $x(p) = a - bp,\ a - bp^\eta$ with $0 < \eta < 1$, and $a - be^p$ in the relevant price ranges.

5 Comparison

We now compare our extension of Nash bargaining with the two other extensions by Conley and Wilkie (1996) and Herrero (1989), respectively. To simplify our comparison, we focus on log-convex problems (S, d°) satisfying condition (*C*) in Sect. 3.4, which include the duopoly bargaining problem in Sect. 4.

Under Conley and Wilkie's (1996) extension, the solution is determined by the intersection of the Pareto frontier of the choice set with the segment connecting the threat point and the Nash solution for the convexified problem. Although they only consider extensions of the symmetric Nash solution, their method can be also applied to extend the asymmetric Nash solution. Given (S, d) and $\alpha \in (0, 1)$, we denote by $N^{\alpha}(S, d)$ the Nash solution with bargaining powers α for player 1 and $1-\alpha$ for player 2 as before and by $CW^{\alpha}(S, d)$ the Conley–Wilkie's extension.

Proposition 4 *Let $(S, d^{\circ}) \in \mathcal{R}$ be such that the IR frontier of S is given by a C^2 function $u_2 = \psi(u_1)$, $u_1 \in [0, \bar{u}_1]$ for some number $\bar{u}_1 < \infty$ with $\psi(\bar{u}_1) = 0$. Suppose ψ satisfies (C) for $i = 1$ and $\psi'' > 0$.*[20] *Then, there exists $\hat{\alpha} \in (0, 1)$ such that (a) $N^{\hat{\alpha}}(S, d^{\circ}) = CW^{\hat{\alpha}}(S, d^{\circ})$, (b) $N_1^{\alpha}(S, d^{\circ}) < CW_2^{\alpha}(S, d^{\circ})$ and $N_2^{\alpha}(S, d^{\circ}) > CW_2^{\alpha}(S, d^{\circ})$ for $\alpha < \hat{\alpha}$, and (c) $N_1^{\alpha}(S, d^{\circ}) > CW_2^{\alpha}(S, d^{\circ})$ and $N_2^{\alpha}(S, d^{\circ}) < CW_2^{\alpha}(S, d^{\circ})$ for $\alpha > \hat{\alpha}$.*

Proposition 4 implies that the Nash solution without convexification is more responsive to the relative bargaining powers than the Conley–Wilkie solution. The intuition appears to be that the Nash solution without convexification utilizes more local curvature of the Pareto frontier of the choice set than the Conley–Wilkie solution does. For the class of duopoly problems with $c_1 < c_2$ as discussed in Sect. 4, we can further show that $\hat{\alpha} > 1/2$, implying that in the presence of asymmetric marginal costs and equal bargaining powers, the Nash solution favors the less efficient firm as compared to the Conley–Wilkie solution.

Condition (C) together with the regularity condition also guarantees a geometric property of the asymmetric Nash solution given in Proposition 4 below. Given $\alpha \in (0, 1)$, let $I_1(\alpha)$ and $I_2(\alpha)$ denote the horizontal and vertical intercepts of the tangent line to the Pareto frontier of S at $N^{\alpha}(S, d^{\circ})$, respectively. We use $|| \cdot ||$ to denote the Euclidean norm.

Proposition 5 *Let $(S, d^{\circ}) \in \mathcal{R}$ be such that the IR frontier of S is given by a C^1 function $u_2 = \psi(u_1)$, $u_1 \in [0, \bar{u}_1]$ for some number $\bar{u}_1 < \infty$ with $\psi(\bar{u}_1) = 0$. Suppose ψ satisfies* (C) *for $i = 1$. Then, for any $\alpha \in (0, 1)$,*

$$\alpha||N^{\alpha}(S, d^{\circ}) - I_2(\alpha)|| = (1-\alpha)||N^{\alpha}(S, d^{\circ}) - I_1(\alpha)||.$$

Proposition 5 states that the distances between the asymmetric Nash solution and the two intercepts are proportional to the associated bargaining powers. In the case of equal bargaining powers, this geometric (equal distance) property is the main feature in Herrero's (1989) characterization of the multi-valued solution under a different set of axioms and has also been noted in Mas-Colell et al. (1995, Example 22.E.3, p. 842) for compact convex problems. The *proportional distance property* of the asymmetric Nash solution in Proposition 5 represents an extension of the equal distance property to asymmetric cases and to a large class of regular problems satisfying (C).

[20] If ψ is concave, then S is convex in which case both extended Nash solutions reduce to the standard Nash solution.

6 Conclusion

It is well-known that on the class of compact convex bargaining problems, Nash axioms without that of symmetry uniquely determine the asymmetric Nash solution, which is parameterized by players' bargaining powers. The parametrization by players' bargaining powers is useful in applications, for it offers flexibility in fitting data by choosing suitable bargaining powers. In this paper, we have extended the standard axiomatic characterization of the unique asymmetric Nash solution to the class of all regular and log-convex problems. We have shown that this is the complete class for such an extension to hold.

As well recognized in the IO literature, bargaining between duopolists with asymmetric constant marginal costs and concave monopoly profit function for the efficient firm results in non-convex problems. Nevertheless, they are shown to be regular and log-convex in this paper. Similarly, it is also shown in this paper that the bargaining problem for Aoki's model of the firm is regular and log-convex but not convex. These results illustrate that more applications of log-convex bargaining problems may be possible. Further explorations will be conducted in a different project.

The uniqueness and single-valuedness of the asymmetric Nash solution make it possible to conduct comparative static analysis with respect to bargaining powers, threat points, and determinants of the choice sets such as the marginal costs for duopoly problems. They also enable interesting comparisons with other single-valued solutions appeared in the literature.

Appendix: Proofs

Before proceeding with the proof of Theorem 1, we make the following observations. First, the logarithmic transformation $(v_1, v_2) = (\ln(u_1 - d_1), \ln(u_2 - d_2))$ is a homeomorphism from $S \cap \{(u_1, u_2) \mid u_1 > d_1, u_2 > d_2\}$ onto $V(S, d)$. It follows that the boundary $\partial V(S, d)$ of $V(S, d)$ must be the homeomorphic image of that of $S \cap \{(u_1, u_2) \mid u_1 > d_1, u_2 > d_2\}$; that is,

$$\partial V(S, d) = \left\{ \left(\ln(u_1 - d_1), \ln(u_2 - d_2) \right) \middle| u \in \partial S : \begin{array}{l} u_1 > d_1, \\ u_2 > d_2. \end{array} \right\}.$$

Thus, $V(S, d)$ is closed. Second, because S is bounded above, $V(S, d)$ is bounded above and $\partial V(S, d) \neq \emptyset$. Third, for any $(u_1, u_2) \in S \cap \{(u_1, u_2) \mid u_1 > d_1, u_2 > d_2\}$, the intervals $((d_1, u_2), (u_1, u_2)]$ and $((u_1, d_2), (u_1, u_2)]$ are transformed into $((-\infty, \ln(u_2 - d_2)), (\ln(u_1 - d_1), \ln(u_2 - d_2))]$ and $((\ln(u_1 - d_1), -\infty), (\ln(u_1 - d_1), \ln(u_2 - d_2))]$ in $V(S, d)$, respectively. As a result, $V(S, d) = V(S, d) - \Re^2_+$. Thus, by moving the origin properly, we can assume, without loss of generality, that for some constant $h > 0$,

$$V(S, d) \subset -(h, h) - \Re^2_+, \tag{23}$$

so that the closed cone generated by $V(S, d)$ and any line passing through the origin with a normal vector in $\Re^2_+$ has no point in common other than the origin.

Lemma 3 *Let $U \subseteq \Re^2$ be closed, convex, and comprehensive (i.e., $U = U - \Re^2_+$). If the boundary ∂U of U is nonempty, then $U = \partial U - \Re^2_+$.*

Proof Notice $\partial U - \Re^2_+ \subseteq U - \Re^2_+ = U$ is automatic. Conversely, let u be any interior point of U. Then, there exists a number $\epsilon > 0$ such that $(u_i + \delta, u_j) \in U$ for $i \neq j$ and for all $\delta \in [0, \epsilon)$. Set $\delta_i = \sup\{\delta | (u_i + \delta, u_j) \in U\}$. Since U is convex and comprehensive, $\delta_i < \infty$ for at least one i, for otherwise $U = \Re^2$ which contradicts $\partial U \neq \emptyset$. Assume without loss of generality $\delta_1 < \infty$. Then, $u^1 = (u_1 + \delta_1, u_2) \in \partial U$ because U is closed. It follows that $u = u^1 - (\delta_1, 0) \in \partial U - \Re^2_+$. This shows $U \subseteq \partial U - \Re^2_+$.

Proof of Theorem 1 As observed earlier, log-convexity is equivalent to conditions that $V(S, d)$ is convex and it does not contain segments with normal vectors $(\alpha, 1 - \alpha)$ for all $\alpha \in (0, 1)$. To prove the necessity, we only need to show that $V = V(S, d)$ is convex because the necessity of the rest is obvious.

Let $\bar{V}$ denote the closure of the convex hull of V. Notice $\bar{V}$ also satisfies $\bar{V} = \bar{V} - \Re^2_+$; hence, from Lemma 3, $\bar{V} = \partial\bar{V} - \Re^2_+$. For any boundary point $\bar{v} \in \partial\bar{V}$, it follows from the separation theorem and the comprehensiveness of $\bar{V}$ that there exists $a \in \Re^2_+$ such that

$$a \cdot \bar{v} = \max_{v \in \bar{V}} a \cdot v = \max_{v \in V} a \cdot v. \tag{24}$$

Without loss of generality, we may take $a = (\theta, 1 - \theta)$ for some $\theta \in [0, 1]$.

There exist sequences $\{v^k(n)\}_n$ in V for $k = 1, 2, 3$, $\{\lambda(n)\}_n$ in $\Re^3_+$, and $\{\epsilon(n)\}_n$ in $\Re$ such that[21]

$$\bar{v} = \lambda_1(n)v^1(n) + \lambda_2(n)v^2(n) + \lambda_3(n)v^3(n) + \varepsilon(n), \tag{25}$$

$$\lambda_1(n) + \lambda_2(n) + \lambda_3(n) = 1, \tag{26}$$

$$\varepsilon(n) \to 0. \tag{27}$$

By the nonnegativity of $\lambda(n)$ and (26), we may assume

$$\lambda(n) \to \bar{\lambda} \in \Re^3_+ \text{ with } \bar{\lambda}_1 + \bar{\lambda}_2 + \bar{\lambda}_3 = 1. \tag{28}$$

By (23), the sequences $\{\lambda_k(n)v^k(n)\}_n$, $k = 1, 2, 3$, are bounded above and by (25), (27), and (28), they are also bounded below. Hence, we may assume

$$\lambda_k(n)v^k(n) \to \bar{w}^k \in \Re^2, \quad k = 1, 2, 3 \Rightarrow \bar{v} = \bar{w}^1 + \bar{w}^2 + \bar{w}^3. \tag{29}$$

Suppose first $\bar{\lambda}_k > 0$ for $k = 1, 2, 3$. Then, since V is closed, (25) and (27)–(29) imply

$$v^k(n) \to \bar{v}^k \in V, \quad k = 1, 2, 3, \quad \bar{v} = \bar{\lambda}_1\bar{v}^1 + \bar{\lambda}_2\bar{v}^2 + \bar{\lambda}_3\bar{v}^3. \tag{30}$$

[21] The reason is as follows. Since $\bar{v}$ is in $\bar{V}$, there exists a sequence $\{\bar{v}(n)\}_n$ in the convex hull of V such that $\bar{v}(n) \to \bar{v}$. By Carathéodory Theorem, for each n, there exist $v^1(n), v^2(n), v^3(n)$ in V and $\lambda(n) \in \Re^3_+$ satisfying (26) such that $\bar{v}(n) = \lambda_1(n)v^1(n) + \lambda_2(n)v^2(n) + \lambda_3(n)v^3(n)$. Finally, (25) and (27) are established by setting $\epsilon(n) = \bar{v} - \bar{v}(n)$ for all n.

Suppose now $\bar{\lambda}_k = 0$ for some k but $\bar{\lambda}_{k'} > 0$ for $k' \neq k$. Without loss of generality, assume $\bar{\lambda}_1 = 0$, $\lambda_2 > 0$, and $\lambda_3 > 0$. In this case,

$$\bar{v} = \bar{w}^1 + \bar{\lambda}_2 \bar{v}^2 + \bar{\lambda}_3 \bar{v}^3, \quad \bar{\lambda}_2, \bar{\lambda}_3 > 0, \quad \bar{\lambda}_2 + \bar{\lambda}_3 = 1. \tag{31}$$

By (24) and (29),

$$a \cdot \bar{w}^1 = \lim_{n\to\infty} a \cdot \lambda_1(n) v^1(n) = \lim_{n\to\infty} \lambda_1(n) a \cdot v^1(n) \leq \lim_{n\to\infty} \bar{\lambda}_1 a \cdot \bar{v} = 0.$$

On the other hand, by (24) and (31),

$$a \cdot \bar{w}^1 = a \cdot \bar{v} - \left[\bar{\lambda}^2 (a \cdot \bar{v}^2) + \bar{\lambda}^3 (a \cdot \bar{v}^3)\right] \geq 0.$$

It follows that $a \cdot \bar{w}^1 = 0$. This shows that $\bar{w}^1$ lies on the line having normal vector $a = (\theta, 1-\theta) \in \Re^2_+$ and passing though the origin. Furthermore, as the limit of $\{\lambda_1(n) v^1(n)\}$, $\bar{w}^1$ is in the closed cone generated by V. Thus, by (23), we must have $\bar{w}^1 = 0$. Suppose finally $\bar{\lambda}_k = 1$ for some k. Assume without loss of generality $\bar{\lambda}_3 \neq 0$. In this case, a similar proof as before shows $\bar{w}^1 = \bar{w}^2 = 0$.

In summary, by letting $\bar{v}^k$ be arbitrary element in V when $\bar{\lambda}_k = 0$, the preceding analysis establishes

$$\bar{v} = \bar{\lambda}_1 \bar{v}^1 + \bar{\lambda}_2 \bar{v}^2 + \bar{\lambda}_3 \bar{v}^3. \tag{32}$$

When $\bar{\lambda}_k > 0$, (24) and (32) together imply

$$a \cdot \bar{v}^k = \max_{v \in V} a \cdot v,$$

which implies $\bar{v}^k \in \partial V$. If $a \in \Re^2_{++}$, then the above equality and the assumption that (4) has a unique solution imply $\bar{v}^k = \bar{v}$ whenever $\bar{\lambda}_k > 0$. Hence, $\bar{v} \in \partial V \subset V$. If $a = (1, 0)$, then

$$\bar{v}_1 = a \cdot \bar{v} = a \cdot \bar{v}^k = \bar{v}^k_1.$$

Assume without loss of generality $\bar{\lambda}_1 > 0$ and $\bar{v}^1_2 = \max\{\bar{v}^k_2 \mid \bar{\lambda}_k > 0\}$. Then, by (32),

$$\bar{v}_2 = \bar{\lambda}_1 \bar{v}^1_2 + \bar{\lambda}_2 \bar{v}^2_2 + \bar{\lambda}_3 \bar{v}^3_2 \leq \bar{v}^1_2.$$

Thus, $\bar{v} \leq \bar{v}^1$. This shows $\bar{v} \in V$. If $a = (0, 1)$, then a similar proof establishes $\bar{v} \in V$.

We have shown that $\bar{v} \in V$ for any $\bar{v} \in \partial \bar{V}$. Thus, by Lemma 3,

$$V \subseteq \bar{V} = \partial \bar{V} - \Re^2_+ \subseteq V - \Re^2_+ = V.$$

This concludes $V = \bar{V}$ which implies that V is convex. □

Proof of Proposition 2 To show the non-convexity of Π, notice that the PF of Π is given by the function

$$\pi_1 = \psi(\pi_2) = h(p(\pi_2), \pi_2)), \quad \pi_2 \in [0, \bar{\pi}_2].$$

By (13),

$$h_2(p,\pi_2) = -\frac{p-c_1}{p-c_2},\ h_{22}(p,\pi_2) = 0, \quad h_{12}(p,\pi_2) = \frac{c_2-c_1}{(p-c_2)^2}, \tag{33}$$

and

$$h_{11}(p,\pi_2) = \bar{\pi}_1''(p) - \frac{2(c_2-c_1)\pi_2}{(p-c_2)^3},\ \ p'(\pi_2) = -\frac{h_{12}(p(\pi_2),\pi_2)}{h_{11}(p(\pi_2),\pi_2)}. \tag{34}$$

It follows from the envelope theorem, (33), and (34) that

$$\psi'(\pi_2) = h_2(p(\pi_2),\pi_2), \tag{35}$$

and

$$\psi''(\pi_2) = h_{12}(p(\pi_2),\pi_2)p'(\pi_2) = -\frac{[h_{12}(p(\pi_2),\pi_2)]^2}{h_{11}(p(\pi_2),\pi_2)} > 0, \tag{36}$$

where the strict inequality holds since $h_{11}(p,\pi_2) < 0$ due to the strict concavity of $\bar{\pi}_1(p)$.

Next, we show that $(\Pi, d) \in \mathcal{L}$ for all $d \geq 0$ that are strictly Pareto dominated in Π. By the discussions following Proposition 1, it suffices to show that $\psi'(\pi_2) < 0$ and $e(\pi_2) \equiv \frac{(\pi_2-d_2)\psi'(\pi_2)}{\psi(\pi_2)-d_1}$ is strictly decreasing in π_2 over $(d_2, \bar{\pi}_2)$.

By (33) and (35), $\psi'(\pi_2) < 0$ if $p(\pi_2) > c_2$. Suppose $p(\pi_2) < \bar{p}_1$. The strict concavity of $\bar{\pi}_1(p)$ implies that $\bar{\pi}_1'(p) > 0$ for $p \in [p(\pi_2), \bar{p}_1]$, which in turn implies $h_1(p,\pi_2) > 0$ for $p \in [p(\pi_2), \bar{p}_1]$. Consequently, $p(\pi_2)$ cannot be Pareto optimal. Thus, $p(\pi_2) \geq \bar{p}_1 > c_2$. By the strict concavity of $\bar{\pi}_1(p)$ and (34),

$$-h_{11}(p,\pi_2) > \frac{2(c_2-c_1)\pi_2}{(p-c_2)^3} > 0 \tag{37}$$

for any $\pi_2 \in (0, \bar{\pi}_2)$. It then follows from (33)-(37) that

$$\begin{aligned}\psi'(\pi_2) + \pi_2\psi''(\pi_2) &= h_2(p(\pi_2),\pi_2) + \pi_2\frac{[h_{12}(p(\pi_2),\pi_2)]^2}{-h_{11}(p(\pi_2),\pi_2)} \\ &< -\frac{p(\pi_2)-c_1}{p(\pi_2)-c_2} + \frac{c_2-c_1}{2[p(\pi_2)-c_2]} \\ &= -\frac{2p(\pi_2)-c_1-c_2}{2[p(\pi_2)-c_2)]} \\ &< 0\end{aligned}$$

for any $\pi_2 \in (0, \bar{\pi}_2)$, where the last inequality holds since $p(\pi_2) \geq \bar{p}_1 > c_2 > c_1$. Since the derivative of $e(\pi_2)$ has the same sign as

$$[\psi(\pi_2) - d_1]\left[\psi'(\pi_2) + \pi_2\psi''(\pi_2) - d_2\psi''(\pi_2)\right] - (\pi_2 - d_2)[\psi'(\pi_2)]^2 < 0,$$

where the inequality follows from $\pi_2 \geq d_2 \geq 0$, $\psi(\pi_2) \geq d_1$, $\psi''(\pi_2) > 0$, and $\psi'(\pi_2) + \pi_2\psi''(\pi_2) < 0$ for any $\pi_2 \in (d_2, \bar{\pi}_2)$. □

Proof of Proposition 3 The first-order condition for maximization problem (22) is

$$(\rho - g)x'\phi' + x = 0 \Leftrightarrow \mu_x(y + \phi(g)) - (\rho - g)\phi'(g) = 0. \tag{38}$$

Recall that $D(y) = x(y + \phi(g(y)))/(\rho - g(y))$. Thus, simple calculation shows that (38) implies

$$D'(y) = \frac{x'(y + \phi(g(y)))}{\rho - g(y)} \text{ and } \frac{D(y)}{D'(y)} = \frac{x(y + \phi(g(y)))}{x''(y + \phi(g(y)))}.$$

Consequently, assumptions (i) and (iii) in Lemma 2 are automatically satisfied due to the assumptions in Proposition 3. Next, since $g(y) < \rho$ and $\mu_x(y + \phi(g(y))) = (\rho - g(y))\phi'(g(y))$, there exists $\bar{y}$ such that $\mu_x(\bar{y} + \phi(g(\bar{y}))) - \bar{y} < 0$. This together with $\mu(y) = \mu_x(y+\phi(g(y)))$ as shown above implies that assumption (ii) in Lemma 2 is also automatically satisfied.

Let $\epsilon(y) = -yD'(y)/D(y)$. Then, from the derivation of $D(y)$ it follows that

$$\epsilon(y) = -\frac{yx'(y)}{x(y)} = \frac{y}{\mu_x(y + \phi(g(y)))}.$$

Hence,

$$\epsilon'(y) > 0 \Leftrightarrow \mu_x > y\frac{d\mu_x}{dy}. \tag{39}$$

By (38),

$$\frac{d\mu_x}{dy} = \left[(\rho - g)\phi'' - \phi'\right]g'$$

and

$$g' = \frac{\mu'_x}{(\rho - g)\phi'' - \phi' - \mu'_x\phi'}.$$

Notice that the dominator on the right-hand side of the last equation is positive under the assumptions. Combining these last two equations with (39) would then imply

$$\epsilon'(y) > 0 \Leftrightarrow (\mu_x - y\mu'_x)[(\rho - g)\phi'' - \phi'] - \mu_x\mu'_x\phi' > 0$$

which holds under the assumed conditions. This completes the proof that all the conditions in Lemma 2 are satisfied. □

Proof of Proposition 4 Notice that $(S, d^{\circ}) \in \mathcal{R}$ and by Proposition 1, $(S, d) \in \mathcal{L}$. Thus, by Theorem 2, there is a unique Nash solution given by (2), where the first-order condition is given by

$$-\frac{u_1 \psi'(u_1)}{\psi(u_1)} = \frac{\alpha}{1-\alpha}. \tag{40}$$

The Nash solution $(N_1^{\alpha}(S, d^{\circ}), N_2^{\alpha}(S, d^{\circ}))$ is determined by (40) and $u_2 = \psi(u_1)$. It follows that the ratio of the two payoff gains is

$$R^N(\alpha) \equiv \frac{N_2^{\alpha}(S, d^{\circ})}{N_1^{\alpha}(S, d^{\circ})} = -\frac{1-\alpha}{\alpha}\psi'(N_1^{\alpha}(S, d^{\circ})).$$

Moreover, the left-hand side of (40) is strictly increasing in u_1 by (C) and the right-hand side is strictly increasing in α. It follows that $N_1^{\alpha}(S, d^{\circ})$ is strictly increasing in α for any interior solution.

The frontier of the convex hull of S is a straight line given by

$$u_2 = \frac{\bar{u}_2}{\bar{u}_1}(\bar{u}_1 - u_1).$$

Applying Nash solution on the convex hull yields a solution $u = (\alpha\bar{u}_1, (1-\alpha)\bar{u}_2)$. This solution and the Conley–Wilkie's extended Nash solution have the same ratio of the payoff gains for the two players, which is given by

$$R^{CW}(\alpha) \equiv \frac{CW^{\alpha}(S, d^{\circ})}{CW_1^{\alpha}(S, d^{\circ})} = \frac{(1-\alpha)\bar{u}_2}{\alpha\bar{u}_1}.$$

Comparing the two solutions yields

$$\frac{R^N(\alpha)}{R^{CW}(\alpha)} = \frac{-\psi'\left(N_1^{\alpha}(S, d^{\circ})\right)}{\bar{u}_2/\bar{u}_1}.$$

The strict convexity of ψ and the monotonicity of $N_1^{\alpha}(S, d^{\circ})$ at the interior solution imply that $-\psi'(N_1^{\alpha}(S, d^{\circ}))$ is strictly decreasing in α. Thus, there exists $\hat{\alpha} \in (0, 1)$ such that (a) $R^N(\hat{\alpha}) = R^{CW}(\hat{\alpha})$, (b) $R^N(\alpha) > R^{CW}(\alpha)$ if $\alpha < \hat{\alpha}$, and (c) $R^N(\alpha) < R^{CW}(\alpha)$ if $\alpha > \hat{\alpha}$. The claims then follow since both solutions are on the frontier. □

Proof of Proposition 5 Notice first that $(S, d^{\circ}) \in \mathcal{R} \cap \mathcal{L}$. Set $z = N^{\alpha}(S, d^{\circ})$. Then, by Theorem 1, z is the only tangent point between the indifference curve of Nash product $u_1^{\alpha}u_2^{1-\alpha}$ and the Pareto frontier of S. The corresponding tangent line is represented by

$$u_2 = \psi(z_1) + \psi'(z_1)(u_1 - z_1).$$

It intersects the horizontal axis at $I_1 = (z_1 + \psi(z_1)/\psi'(z_1), 0)$ and it intersects the vertical axis at $I_2 = (0, \psi(z_1) - z_1\psi'(z_1))$. The distances $\|z - I_1\|$ from z to I_1 and $\|z - I_2\|$ from z and I_2 are given by

$$\|z - I_1\|^2 = (\psi(z_1)/\psi'(z_1))^2 + (\psi(z_1))^2 = \frac{(\psi(z_1))^2[1 + (\psi'(z_1))^2]}{(\psi'(z_1))^2}$$

and

$$\|z - I_2\|^2 = (z_1)^2 + (z_1\psi'(z_1))^2 = (z_1)^2 \left[1 + (\psi'(z_1))^2\right].$$

It follows that

$$\frac{\|z - I_2\|^2}{\|z - I_1\|^2} = \left(\frac{z_1\psi'(z_1)}{\psi(z_1)}\right)^2. \tag{41}$$

Recall that the proportional distance property states that

$$\frac{\|z - I_2\|}{\|z - I_1\|} = \frac{1-\alpha}{\alpha}.$$

By (41), the preceding equality is equivalent to

$$\left(\frac{1-\alpha}{\alpha} - \frac{z_1\psi'(z_1)}{\psi(z_1)}\right)\left(\frac{1-\alpha}{\alpha} + \frac{z_1\psi'(z_1)}{\psi(z_1)}\right) = 0.$$

Since $\psi'(z_1) < 0$, the above equality in turn is equivalent to (40), which is automatic because it is the first-order condition for Nash bargaining solution $z = N^\alpha(S, d^\circ)$. □

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