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Abstract

This paper briefly explains the Gale-Shapley algorithm for assigning divorce-proof marriages. It is demonstrated that in general, if an assignment mechanism leads to divorce-proof marriages, some participants can gain by misrepresenting their preferences.

David Cale and Lloyd Shap~ey (1962)[1] devised a courtship algorithm for arranging a "divorce-proof" assignment of marriages. Their idea is this. Suppose that there is a population of n boys and n girls who are to be paired off. Each of the boys has a preference ranking over the set of girls and each of the girls has a preference ranking of the boys as potential mates. Different people may have different rankings of the opposite sex. An assignment of partners is "divorce-proof" if no two people of opposite sex would prefer to have each other rather than their assigned partners. Stated another way, an assignment is divorce-proof if when a man covets his neighbor's wife, she does not covet him in return.[2]

The Gale-Shapley algorithm works like this. Each boy writes a note addressed it to his favorite girl and signs it. The notes are delivered to the addressees. If each girl receives a note, the process is finished and the girls are paired with their respective suitors. Usually some girls won't get notes and some girls will get more than one note. If a girl receives more than one note, she examines her list of suitors. To the one she likes best on the list she says "maybe". To the others, she says "no". Rejected suitors are informed of their misfortune. Each rejected suitor then addresses a message to his second-choice girl. If some girls have still not received proposals, then there will be other girls with more than one suitor. A girl who has more than one offer which she has not as yet declined compares these pending offers and decides which boy she likes best of the lot. To him she says "maybe". The others are told "no". Rejected suitors then propose to their first choice among the girls who have not as yet rejected them. The process continues until every girl has received a proposal. Each girl is then assigned to the one boy who has proposed to her and whom she hasn't rejected. It is not hard to show that the process must terminate after a finite number of steps and that the outcome is "divorce-proof".

Of course, a dual process could be employed, where the girls propose to the boys. In general the equilibrium assignment when the girls propose is different from that when the boys propose. To see this, consider the following example. There are two boys and two girls. Their preferences are listed in Table 1.

Table 1

Person	Favorite Partner	Second Favorite
Alan	Alice	Betsy
Bill	Betsy	Alice
Alice	Bill	Alan
Betsy	Alan	Bill

If the boys propose, Alan proposes to Alice and Bill proposes to Betsy. Since both girls have proposals, the process ends. Alan and Alice are paired as are Bill and Betsy. If the girls propose, Alice proposes to Bill and Betsy proposes to Alan. Since both boys have proposals, the process is finished. Alice marries Bill and Betsy marries Alan. Either outcome is divorce-proof. When the boys propose, both boys get their first choices. When the girls propose, both girls get their first choices. In this example, girls-propose is better for both girls and boys-propose is better for both boys. There are configurations of preferences such that boys-propose is better for some girls than girls-propose. However, there is a sense in which the proposing sex is, in general, favored. Gale and Shapley define a divorce-proof assignment to be "boy-optimal" if there is no other divorce-proof assignment that is better for some boys and worse for no boys. "Girl-optimal" is defined dually. They then prove the following theorem. The boys-propose assignment is always boy-optimal and the girls-propose assignment is always girl-optimal.

Any reader of Econometrica, (Gibbard (1973)), Satterthwaite (1975)), or follower of the soap operas will notice that so far we have neglected the rich possibilities for duplicity in courtship. A natural question, in the language of Gibbard, is whether courtship a la Gale and Shapley is cheatproof. The answer is "no" as we illustrate with the following simple example. There are 3 boys and 3 girls. Their preferences are described in Table 2.

Table 2

Person	Favorite Partner	Second Favorite	Third Favorite
Alan	Alice	Betsy	Clara
Bill	Betsy	Alice	Clara
Charlie	Alice	Clara	Betsy
Alice	Bill	Alan	Charlie
Betsy	Alan	Bill	Charlie
Clara	Alan	Charlie	Bill

Suppose that the boys propose and the girls answer truthfully. On the first round, Alice will receive proposals from Alan and Charlie and Betsy will receive a proposal from Bill. Alice will reject Charlie who then proposes to Clara. At this stage all girls have proposals so that Alice marries Alan, Betsy marries Bill and Clara marries Charlie.

Suppose that before the proposals were sent, the girls learned about each other's preferences and about the boys' preferences.[3] Could it be that Alice, who winds up with her second choice if she plays truthfully, can usefully employ a bit of feminine guile. Suppose that Alice thinks that everyone else will play truthfully. If when Alan and Charlie propose to her, she pretends to like Charlie better than Alan, then the rejected Alan will propose to Betsy on the second round. Betsy will then turn down Bill whom Alice really wants. On the next round, Bill, who is now unattached, will choose Alice, who is his second choice. Alice then accepts Bill and rejects Charlie, to whom she had previously said "maybe". Charlie proposes to his second choice, Clara. At this point all girls have proposals and the assignments are Alice marries Bill, Betsy marries Alan, and Clara marries Charlie. For her duplicity, Alice has been rewarded with Bill, her first choice, instead of her second choice.

Both Alice and Betsy are gainers because Alice lied and Clara is unaffected. Alan and Bill are made worse off. In fact, the outcome with the boys proposing and Alice lying is the same as the truthful girls-propose outcome. Of course, in order to be able to safely accomplish this maneuver Alice had to know that someone, in this case, Betsy, would give up Bill if Alan became available and she would also have to know that Bill preferred her to Clara. Otherwise she would be stuck with Charlie. Betsy, like Alice is a net gainer from the transaction so Betsy would have willingly revealed her preferences to Alice. On the other hand, Bill is the worse off for Alice's ploy and Alice needs to know that Bill likes her better than Clara in order to be sure that she won't be left holding Charlie.[4]

This example shows that the Gale-Shapley courtship algorithm, although divorce-proof is not cheatproof. Might there be some other assignment mechanism that is both divorce-proof and cheatproof? It turns out that there is not. To establish the sense in which this claim is true, we need some definitions. The structure used here is essentially the same as that of Gibbard (1973). More formal definitions are found in the appendix of this paper.

An assignment mechanism asks each of n boys and n girls to express a preference ranking over members of the opposite sex. Since true preferences are regarded as privately held information, the mechanism must operate with expressed preferences which need not be the same as true preferences. For any specification of the expressed preferences of each person, an assignment mechanism must determine a marriage partner of the opposite sex for everyone. An assignment mechanism is cheatproof if, no matter what the preferences of individuals are, no one can get a better partner from the mechanism by lying about his preferences than by telling the truth. Our result is the following.

Theorem - If the number of persons of each sex is at least 3, then there are no cheatproof assignment mechanisms that guarantee divorce-proof assignments.

The proof of this theorem takes the form of a counterexample based on the preference profiles expressed in Table 2.[5] We proceed by a series of steps.

Step 1: There are only two divorce-proof assignments for the preference profile in Table 2. These are:

(1) Alan-Alice, Bill-Betsy, Charlie-Clara
(2) Alan-Betsy, Bill-Alice, Charlie-Clara.

Proof: Any divorce-proof mechanism must assign Charlie to Clara. If Alan were assigned to Clara, then Alan and Betsy would prefer each other to their assigned partners, since Alan is Betsy's first choice and Clara is Alan's last choice. If Bill were assigned to Clara, then Bill and Alice would prefer each other to their assigned partners, by similar reasoning.

Step 2: If the preference profile is as in Table 3, then the only divorce-proof assignment is Alan-Betsy, Alice-Bill, Charlie-Clara.

Table 3

Person	Favorite Partner	Second Favorite	Third Favorite
Alan	Alice	Betsy	Clara
Bill	Betsy	Alice	Clara
Charlie	Alice	Clara	Betsy
Alice	Bill	Charlie	Alan
Betsy	Alan	Bill	Charlie
Clara	Alan	Charlie	Bill

Proof: The same argument used in Step 1 will imply that a divorce-proof outcome must pair Charlie and Clara. Suppose Alice is paired with Alan. Then Alice prefers Charlie to her spouse and Charlie prefers Alice to his spouse. Therefore Alice must be paired with Bill and Betsy and Alan are left for each other. This assignment is divorce-proof since Alice and Betsy have their first choices and if Clara is to be made better off, she must have Alan who ranks her last.

Step 3: A cheatproof and divorce-proof assignment mechanism could not assign Alan to Alice when preferences are as in Table 2.

Proof: Alice could improve on this assignment by pretending to have the ordering-Bill, Charlie, Alan. If she did then according to Step 2, any cheat-proof, divorce proof mechanism would give her Bill whom she prefers to Alan.

Step 4: A cheatproof and divorce-proof mechanism could not assign Bill to Alice when preferences are in Table 2.

Proof: The proof of Step 4 is exactly parallel to the proof of Step 3. Just as any cheatproof divorce-proof assignment mechanism would reward Alice with Bill if she pretended to like Charlie better than Allen, so will any such mechanism reward Bill with Betsy if he pretends to like Clara better than Alice. Since Bill prefers Betsy to Alice, no cheatproof, divorce-proof assignment mechanism can assign Bill to Alice.

Step 5: From Step 1 we see that any cheatproof, divorceproof assignment mechanism must assign Alice to Alan or Bill. From Steps 3 and 4 we see that such a mechanism can not assign Alice to either Alan or Bill. Therefore there can be no such mechanism.

It is evidently too much to ask that an assignment mechanism be both cheatproof and divorce-proof. This suggests that there is abundant room for further exploration by dramatists and choice theorists of the complexities involved in even the simplest entanglements of affections.

Appendix

Let there be n males and n females. Members of each sex are indexed from 1 to n. Let be the set of all permutations of the integers 1 to n. Let $P_i \in \theta$ and $P_i^* \in \theta$ represent respectively boy i's preferences and girl i's preferences over members of the opposite sex. Thus, for example, if j appears before k in the permutation P_i, then boy i likes girl j better than he likes girl k. If this is the case we will write $j\ P_i k$. Similarly we will write jP_i^*k if girl i likes boy j better than boy k. Let $S_i \in \theta$ and $S_i^* \in \theta$ denote respectively the preferences signalled by boy i and girl i. The set of all possible marriages can be identified with the set θ by a simple convention. Let $M \in \theta$ denote the assignment of marriages in which if girl k appears in the jth place of M, then girl k marries boy j. Since individuals are assumed to be concerned only about who their own partners will be, the preference relations P_i and P_i^* induce preferences on marriages assignments in an obvious way. Let M(i) be the girl that assignment M assigns to boy i, and M*(i) the boy that assignment M assigns to girl i. Where M and $\hat{M}$ are two different assignments define $MP_i\hat{M}$ to mean $M(i)P_i\hat{M}(i)$ and define $MP_i^*\hat{M}$ to mean $M^*(i)P_i\hat{M}^*(i)$.

An assignment mechanism is a function $A(\cdot)$ from θ^{2M} to θ which is interpreted as follows:

$$A(S_1,\ldots,S_n,S_1^*,\ldots,S_n^*) = M$$

means that if the signals sent are $S_1,\ldots,S_n,S_n^*,\ldots,S_n^*$, then the marriage assignment will be M. An assignment mechanism, A, is <u>cheatproof</u> if for all i and for all $(P_1,\ldots,P_n,P_1^*,\ldots,P_n^*) \in \theta^{2n}$, there is no $S_i \in \theta$ such that $A(P_1,\ldots,S_i,\ldots,P_n,P_1^*,\ldots,P_n^*)P_i\ \ A(P_1,\ldots,P_i,\ldots,P_n,P_1^*,\ldots,P_n^*)$ and if there is no $S_i^* \in \theta$ such that $A(P_1,\ldots,P_n,P_1^*,\ldots,S_i^*,\ldots,P_n^*)P_i\ \ A(P_1,\ldots,P_n,P_1^*,\ldots,P_i^*,\ldots,P_n^*)$. A marriage assignment M is <u>divorce-proof</u> if there is no boy i and girl j such that $jP_iM(i)$ and $iP_j^*M^*(j)$.

Footnotes

[1] This is an article of great elegance and charm. We strongly recommend it, even if finding it means a trip to the mathematics library.

[2] Game theorists will recognize a divorce-proof assignment as being the core of the game without transferable utility in which the only coalitions that can accomplish anything interesting are twosomes of opposite sexes.

[3] Perhaps this is what teen-agers discuss on the telephone.

[4] More generally if a girl gains by lying, it must be that the boy she winds up with is worse off than he would be if everyone told the truth. This follows from the fact that the Gale-Shapley assignment is divorce-proof when people play truthfully. But in order to be sure that she can get this boy by lying, she typically must be sure that she isn't his last choice. If he understands her game, it is in his interest to pretend he doesn't like her at all.

[5] This result can not be obtained as a direct corollary of Gibbard's theorem because Gibbard requires that the domain of preferences include all possible orderings of the alternatives. In our example, preferences on marriage assignment patterns are restricted by the requirement that individuals are concerned only about whom they, themselves, marry.