Title
The Horizontal Electric Dipole in a Conducting Half-Space

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Publication Date
1953-09-01

University of California

Marine Physical Laboratory of the

Scripps Institution of Oceanography

THE HORIZONTAL ELECTRIC DIPOLE IN A CONDUCTING HALF-SPACE

by

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Sponsored by

Bureau of Ships

Contract NObsr - 43356

NE - 120221-5

SIO Reference 53-33

September, 1953

Approved for distribution:

Roger Revelle

ACKNOWLEDGEMENT

Grateful appreciation is here expressed to Professor Carl Eckart, Director of the Marine Physical Laboratory of the Scripps Institution of Oceanography, who initiated this research project with us, for his constant encouragement and for the frequent illuminating discussions held in the course of these studies. In addition, the authors wish to make patent their debt of gratitude to Dr. Horace G. Ferris of Scripps for bringing to their attention several important references and for many discussions and criticisms of our own work which proved most useful.

ABSTRACT

This report gives a thorough and complete account of the mathematical problems involved in the determination of the electromagnetic field components generated by a horizontal electric dipole embedded in a conducting half-space whose plane boundary is also horizontal. The problem is formulated by introducing the Hertzian vectors or polarization potentials and employing the technique of triple Fourier transforms in Cartesian coordinates, in configuration space as well as in transform space. Suitable integral representations are obtained for the components of the Hertzian vectors.

It is shown that this formulation is fundamental in the sense that it contains 'per se' all other known formulations of the problem. Thus, by suitable transformations of the variable or variables of integration one readily obtains the formulations of Sommerfeld (1909), Weyl (1919), Ott (1942), etc. Further, by correctly specifying the original path of integration in Sommerfeld's formulation of the problem and by carefully analyzing the class of permissible deformations of the original path, the whole moot question of poles and residues is clarified in a straightforward manner.

The report also presents the complete independent solution of the static problem and it is shown that all solutions for the alternating case converge uniformly to the static solutions as the frequency is made to vanish. Further, the static solution is applied to an extended source pointing out the way for a similar extension of the alternating dipolar solution.

The Cartesian components of the Hertzian vectors and the cylindrical components of the field vectors (E and H) are given, for both media, in terms of four fundamental integrals, which are expanded in asymptotic series by saddle point methods, two of these integrals belonging to the conducting medium and the other two to the free space above. It is shown, in the treatment of each of the four integrals mentioned, that there are two distinct asymptotic contributions arising from two saddle points and the notable feature of the results is that one of the saddle points yields a solution which is not exponentially attenuated in the horizontal direction in accordance with known experimental results. Thus, the possibility of large ranges of the field in the horizontal direction at depths which are not too great is clearly established.

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NOTATION

A_{2m}	Coefficient in the power series expansion of $\Phi(x)$, Eq. (5.11).
$A^{2m}_{2(n-m)}$	Coefficient in the double power series expansion of $\Phi(x,y)$, Eq. (5.53).
a_{2n}	Coefficient in the power series expansion of $w(x)$, Eq. (5.22).
$B^{2m}_{2(n-m)}$	Coefficient in the double power series expansion of $g(x,y)$, Eq. (5.72).
c_{2n}	Coefficient in the power series expansion of $x^2/2$ qua function of w, Eq. (5.21).
h	Depth of dipole source and height of dipole image (Fig. 1).
$K(n,\theta_2)$	A function introduced for convenience in notation, Eq. (6.67).
k_o	Value of the transform variable λ associated with the zeros of the Sommerfeld denominator, $N(\lambda)$, Eq. (2.96).
k_1	Propagation constant for plane homogeneous waves in medium (1); $k_1^2 = \omega^2\mu_o\epsilon_1 + i\omega\mu_o\sigma_1 \approx i\omega\mu_o\sigma$.
k_2	Propagation constant for plane homogeneous waves in medium (2); $k_2^2 = \omega^2\mu_o\epsilon_o + i\omega\mu_o\sigma_2 \approx \omega^2\mu_o\epsilon_o$.
M_1	Fundamental integral for medium (1), Eq. (3.18).
$M_1^{(1)}$	Fundamental integral M_1 evaluated over path C_1 (Fig. 4).
$M_1^{(2)}$	Fundamental integral M_1 evaluated over path C_2 (Fig. 4).
$N(\lambda)$	The Sommerfeld denominator, Eq. (2.94).
n	Complex index of refraction; $n = k_2/k_1$.
$Q(n)$	A function introduced for convenience in notation, Eqs. (6.133), (6.134).
R_1	Distance from dipole source to point of observation (Fig. 1), Eq. (2.1).
R_2	Distance from dipole image to point of observation (Fig. 1), Eq. (2.1).
U	Fundamental integral for $(h-z) = 0$, Eq. (2.72).
U_1	Fundamental integral for medium (1), Eq. (2.67).

U_2	Fundamental integral for medium (2), Eq. (2.69).
$U_1^{(1)}$	Fundamental integral U_1 evaluated over path C_1 (Fig. 4).
$U_1^{(2)}$	Fundamental integral U_1 evaluated over path C_2 (Fig. 4).
V	Fundamental integral for $(h-z) = 0$, Eq. (2.73).
V_1	Fundamental integral for medium (1), Eq. (2.68).
V_2	Fundamental integral for medium (2), Eq. (2.70).
$V_1^{(1)}$	Fundamental integral V_1 evaluated over path C_1 (Fig. 4).
$V_1^{(2)}$	Fundamental integral V_1 evaluated over path C_2 (Fig. 4).
$W^{(p)}$	Component of $V_1^{(2)}$ representing the contribution from the pole, Eq. (6.125).
$W^{(s)}$	Component of $V_1^{(2)}$ representing the passage through the saddle point, Eq. (6.125).
α	Transform variable, variable of integration, parameter.
α_1	Variable of integration introduced in the conformal transformation $\lambda = k_1 \sin \alpha_1$ (Section 6.1a).
α_2	Variable of integration introduced in the conformal transformation $\lambda = k_2 \sin \alpha_2$ (Section 6.2a).
β	Transform variable, variable of integration, parameter.
γ_1	Attenuation factor associated with k_1, Eqs. (2.32), (2.58).
γ_2	Attenuation factor associated with k_2, Eqs. (2.35), (2.58).
ζ	Transform variable, variable of integration.
ζ_1	Intrinsic impedance for medium (1).
ζ_2	Intrinsic impedance for medium (2).
η	Transform variable, variable of integration.
η_1	Intrinsic admittance for medium (1).
η_2	Intrinsic admittance for medium (2).

θ_1 Angle between positive z direction and R_1 (Fig. 1), Eq. (2.1).

θ_2 Angle between negative z direction and R_2 (Fig. 1), Eq. (2.1).

λ Transform variable, variable of integration.

ξ Transform variable, variable of integration.

Ψ_1 Source function, Eq. (2.65).

Ψ_2 Image function, Eq. (2.66).

I. INTRODUCTION

The problem originally proposed to us has to do with the complete determination of the electromagnetic field generated by a horizontal antenna embedded in a conducting half-space, the antenna consisting of an insulated wire terminated by bare electrodes (Fig. 2). We were asked to determine the near field, the far field, the current distribution along the antenna wire and the input impedance of such a device submerged in a conducting half-space and located close to the horizontal boundary. It soon became clear to us that the problem of determining the input impedance and the current distribution along the wire was in essence tied up with the solution of an extremely difficult antenna problem and we abandoned all efforts to answer these two questions.

There remained for us the alternative of considering the current distribution along the wire as prescribed; and, granting that the Green's function for an elementary, horizontal electric dipole, embedded in the conducting half-space, had been obtained, the problem of the extended source with prescribed current distribution could then be solved by integrating over the source using Green's theorem. Thus, it looked to us that a necessary step towards the complete solution of this complicated antenna problem was the determination of the electromagnetic field of a horizontal electric dipole in a conducting half-space (Fig. 1), which is precisely the problem that we have undertaken in a lengthy investigation covering nearly two years and culminating in the present report which covers the most essential details of our calculations.

As is well known, the problem in question was first discussed in a brilliant memoir by Arnold Sommerfeld* in 1909 and since then a considerable number of papers have appeared on various aspects of the problem as studied by several authors. In attacking this problem we have been led to examine some of the pertinent references and, thus, this report contains also a comprehensive review of such papers. In this Introduction we wish to stress those results of ours which are new or which go beyond the work of all our predecessors.

In Chapter II we undertake the complete formulation of the two-medium problem for a dipolar source by employing the technique of Fourier integral representations using Cartesian coordinates in both transform and configuration spaces; and, by introducing suitable transformations of the variables of integration we obtain the known formulations of Sommerfeld (1909), Weyl (1919), Ott (1942), and others. We examine in particular the Cartesian components of the Hertzian vectors or polarization potentials and we exhibit them in terms of four fundamental integrals, two of which correspond to points of observation in air and the other two to points of observation in the conducting medium. Because of the magnitude of the present project this report is concerned mainly with the evaluation of the integrals for points of observation in the conducting medium, and we reserve the evaluation of the integrals for points of observation in air to a future publication. None of the results presented in Chapter II are essentially new, except our complete treatment

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* See Bibliography at the end of this report.

of the triple Fourier integral representations and our discussion of the regions of analiticity for the transform variables which has an important bearing on the whole question of poles and residues. In addition, we give a detailed description of the Riemann surface of four sheets in the λ-plane (Sommerfeld's plane of integration) showing how to draw the branch cuts and indicating clearly how to determine the nature of the poles of the integrand, whether real or virtual, on the various sheets of the Riemann surface.

In Chapter III we deduce the electric and magnetic field components in cylindrical coordinates, expressing our results in terms of the Cartesian components of the Hertzian vectors which in turn are given in terms of the fundamental integrals and their derivatives as mentioned in the preceding paragraph. Making use of certain differential equations which connect the various fundamental integrals among themselves, we are able to exhibit the cylindrical components of the field vectors in various forms more suitable for computational purposes. For example, the Cartesian components of the Hertzian vector and the cylindrical components of the field vectors for points of observation in the conducting medium can all be expressed in terms of a single fundamental integral and its derivatives, and similarly for points of observation in air.

Chapter IV contains the solution of the two-medium dipolar problem in the static limit ($\omega \to 0$). We find that all of our integral representations converge uniformly to the static solution (as obtained independently by elementary methods) for $\omega \to 0$, which affords an important partial check on our formulation of the problem. Thus, we present the independent solution for the electric field based on the method of images which in turn allows the determination of the current distribution everywhere. From a knowledge of this current distribution we then determine the complete magnetic field and we discover that the major contribution (to the magnetic field) comes from the surface layer discontinuity in $\nabla \times J$ which exists at the interface separating the two media. We believe that the independent solution for the magnetic field is being presented here for the first time.

In Chapter V we undertake a general discussion of the saddle point method of integration which we apply in this report to the asymptotic evaluation of the fundamental integrals and their derivatives. First, we consider the saddle point method for a single integral and we discuss the necessary and sufficient conditions for the application of Watson's lemma. It is clearly pointed out that the "asymptotic convergence" of the resulting series is governed by the radius of convergence of the power series expansion of the integrand about the origin in the complex plane of integration, which is the distance from the origin to the nearest singularity. And we discovered that, when the nearest singularity is a simple pole (or a pole of any order), the "asymptotic convergence" of the series could be greatly enhanced by the removal of the pole from the integrand, a process which was discovered independently by van der Waerden, but which we feel we have developed in the simplest possible fashion. We wish to call attention to this achievement, for we feel that it constitutes one of our major original contributions.

In addition, we have developed the saddle point method for a double integral which arose when we replaced the Hankel function appearing in the integrand by a suitable integral representation (thus leading to a double integral), in turn to be treated by the saddle point method of integration. We believe that it was this extension of the method to a double integral that allowed us to determine the asymptotic expansions of the fundamental integrals and their derivatives to three terms, which had never been attained by any of the previous authors and which proved absolutely necessary in order to clearly delimit the range of applicability of various approximations undertaken later.

In Chapter VI we present the evaluation of the fundamental integrals U_1 and V_1 and their derivatives for points of observation in the conducting medium. As shown in Chapter II, each typical integral can be resolved into the sum of two integrals, $I = I_1 + I_2$, by a suitable deformation of the original path of integration. Integrals of the type I_1 are evaluated asymptotically by the saddle point method for single integration and, because they are shown later to be of negligible magnitude in comparison with the contributions of the integrals of type I_2, we present only the leading terms of the asymptotic expansions for the fundamental integrals. On the other hand, integrals of the type I_2 and their derivatives are evaluated by the saddle point method for a double integral employing the technique of the removal of the pole from the integrand whenever necessary. The reason for the independent evaluation of the higher order derivatives is clear: asymptotic series can not in general be differentiated term by term to yield the asymptotic series of the derivative as we confirmed by actual comparison. Thus, we felt all along that it was not sufficient to undertake the asymptotic evaluation of the Cartesian components of the Hertzian vectors, from which the field components can be obtained by applying differential operators, but that to obtain accurate results it was necessary to examine the asymptotic expansion of each derivative. We feel that in this respect we have again gone beyond all of our predecessors, for in all the papers that we have studied the authors content themselves with the asymptotic evaluation of the fundamental integrals, which they then proceed to differentiate to obtain, sometimes in error, the electric and magnetic field components.

Chapter VII contains the results for the conducting medium and represents the culmination of the present research project. First of all, we undertake in this Chapter to give a clear-cut and unambiguous definition of the various ranges in which it is possible to obtain much simpler formulas than the ones presented in Chapter VI. We consider the asymptotic range $\rho \rightarrow \infty$ and we present the Cartesian components of the Hertzian vector and the cylindrical components of the electric and magnetic field vectors in this limit. We recognize that this range is of no practical value at low frequencies, but the results given, which are new, are used here to describe completely the nature of the electromagnetic field as $\rho \rightarrow \infty$. Next, we take up the range of parameters for which the horizontal range is large in comparison with a wavelength in air but for which, at the same time, we have the condition that Sommerfeld's numerical distance is very small in comparison with unity. This range is of interest because it applies to the well-known Sommerfeld - van der Pol "attenuation formulas" with which we have compared our asymptotic results with complete agreement. And, finally, we consider the range or parameters which is of practical value in the low frequency case; namely, when the horizontal range is small in comparison with one wavelength in air but large in comparison with one wavelength in the conducting medium. For this important range of parameters we present again the simpler forms assumed by the components of the Hertzian vector and by the electric and magnetic field components.

Next, we take up the study of the limiting forms of our results when the source dipole and the point of observation both lie on the surface separating the two media. In this manner we are able to compare directly our results with those of Sommerfeld and van der Pol. Furthermore, we undertake a thorough review of the various papers published on Sommerfeld's electromagnetic surface wave and by a critical analysis of the errors committed by several authors we are able, we trust, to reinstate the work of Sommerfeld to the esteem and respect which it deserves. We point out that the Zenneck type surface wave first encountered by Sommerfeld in his 1909 solution of the problem is a legitimate part of the solution in the range of parameters for which it is valid, but that the contribution of this surface wave is of negligible magnitude in all cases of practical interest, e.g., the low frequency case.

Next, we discuss the limiting form of our results when we assume that the wavelength in air is infinite. This case was treated by Lien and we examine in detail Lien's approximation with the conclusion that it constitutes an excellent approximation in the low frequency case. In fact, we are able to justify Lien's approximation, which he failed to do, and in so doing we are able to show the exact nature of the approximation and the magnitude of the errors incurred.

Finally, we take up a numerical example to illustrate the application of our formulas in the low frequency case. Considering realistic data we obtain approximate expressions for the electric and magnetic field components which are valid, at a frequency of 900 c.p.s., for horizontal ranges between 50 and 5000 meters. It is shown that the field vectors vary as the inverse cube of the horizontal range and are exponentially attenuated with the aggregate depth of source and point of observation.

II. FORMULATION OF THE TWO-MEDIUM PROBLEM FOR A DIPOLAR SOURCE

In this Chapter we are concerned with the problem of setting up suitable integral representations for the Cartesian components of the Hertzian vectors which will be employed in Chapter III in the computation of the electric and magnetic field components. It will be assumed that the sole source of the electromagnetic field is an oscillating dipole embedded in a conducting half-space and oriented parallel to its plane boundary. It is shown that the most convenient formulation of the problem is based on the application of triple Fourier transforms employing Cartesian coordinates in both configuration and transform spaces.

2.1 STATEMENT OF THE PROBLEM

As shown in Fig. 1, we shall assume that the horizontal plane $z = 0$ coincides with the interface between two homogeneous and isotropic media of different conductivities. In the present instance, medium (1) is a conducting half-space with conductivity $\sigma_1 = \sigma$, and medium (2) is the air above with conductivity $\sigma_2 = 0$. However, for the sake of symmetry which facilitates the discussion of the static limit (Chapter IV), and to secure the convergence of the integrals for points of observation in air (cf. post, Section 2.3), it will be assumed that both conductivities are finite with σ_2 as small as desired. It will be further assumed that both media have the magnetic inductive capacity of free space, $\mu_1 = \mu_2 = \mu_0$; and that, as regards electric inductive capacity, we can put $\epsilon_1 = \kappa\epsilon_0$ and $\epsilon_2 = \epsilon_0$, where κ is the dielectric constant of the conducting medium and ϵ_0 is the electric inductive capacity of free space.

The source, consisting of a horizontal electric dipole, is located at the point (0,0,-h) in medium (1) and is oriented parallel to the x axis. It will be convenient to introduce here, as shown in Fig. 1, the image point located symmetrically in medium (2) at (0,0,h). As shown below, the whole problem can be formulated in terms of functions which exhibit axial symmetry about the z axis passing through the source point and its image. Therefore, it will be convenient to identify points of observation by their cylindrical coordinates (ρ, ϕ, z). Figure 1 displays a point of observation in medium (1), which lies on the plane of the paper (x = 0), illustrating the distances R_1 and R_2, from the point of observation to the source point and its image respectively, and the angles θ_1 and θ_2 which these distances make with the z axis. As here defined,

$$R_1 = \left[\rho^2 + (h + z)^2\right]^{\frac{1}{2}}, \quad \tan\theta_1 = \rho/(h + z) ;$$
$$R_2 = \left[\rho^2 + (h - z)^2\right]^{\frac{1}{2}}, \quad \tan\theta_2 = \rho/(h - z) , \tag{2.1}$$

from which it is clear that, with finite depth of source $(h > 0)$, we have $0 \leq \theta_1 < \pi/2$ for points of observation in medium (2) and $0 \leq \theta_2 < \pi/2$ for points of observation in medium (1).

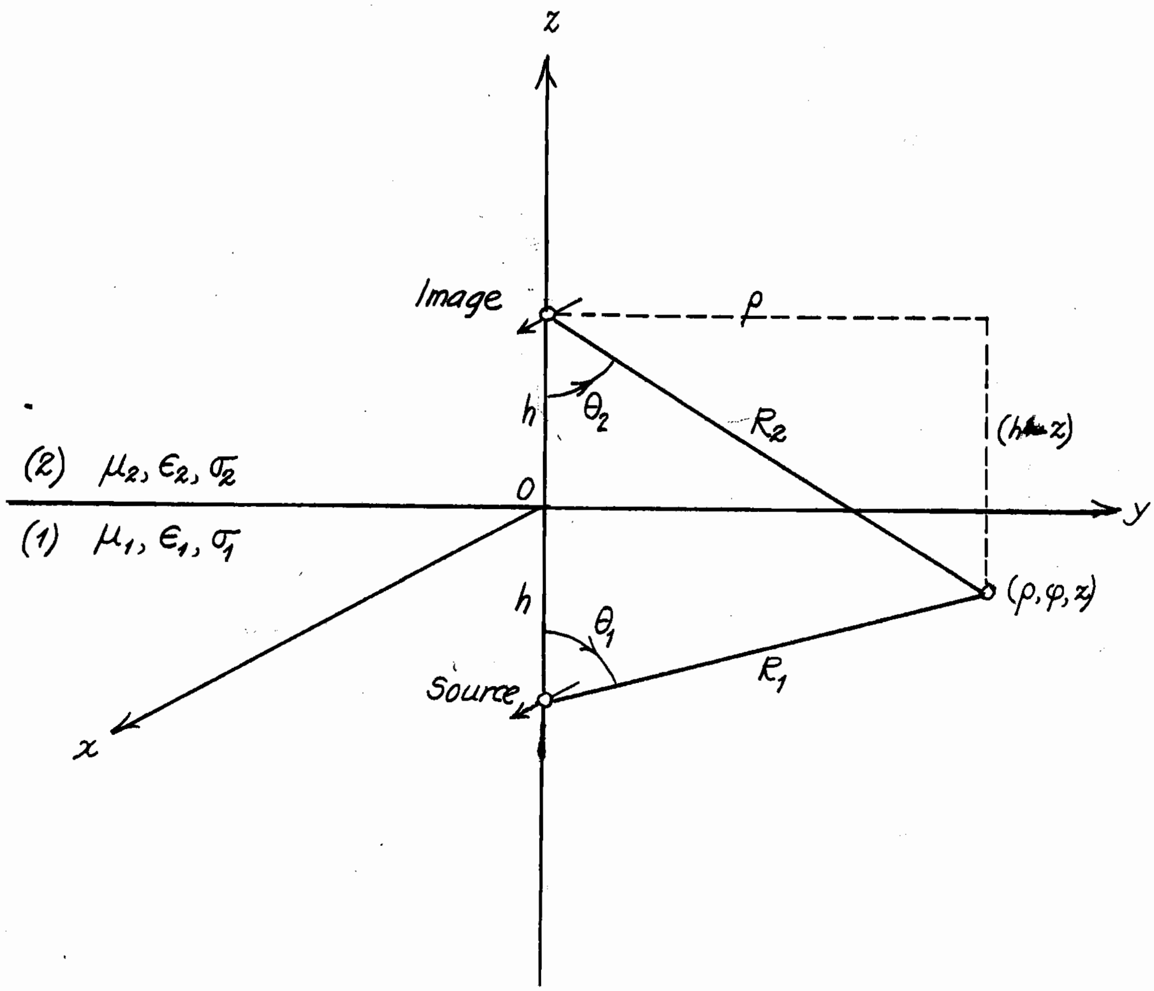

Fig. 1.- Coordinate system showing the position and orientation of the source and its image and a point of observation in medium (1), chosen for convenience on the **yz**-plane.

2.2 FUNDAMENTAL EQUATIONS

The formulation of the present two-medium problem implies the solution of Maxwell's equations for each medium subject to the classical boundary conditions at the interface. The solution of this problem is facilitated by the introduction of the Hertz vectors or polarization potentials from which the field components are readily computed. The imposition of the boundary conditions at the interface then furnishes the necessary relations which render the solution determinate and unique.

2.2a. The field equations.- Consider first a homogeneous and isotropic conducting medium of infinite extent which is fully characterized by the macroscopic parameters μ, ϵ, and σ. Assume next that the source function and all the field variables exhibit the common time dependence $e^{-i\omega t}$, where ω is the fixed angular frequency of the driving source. In terms of the propagation constant k for plane homogeneous waves, which is defined by

$$k^2 = \omega^2 \mu \epsilon + i\omega\mu\sigma, \tag{2.2}$$

and in terms of the intrinsic impedance ζ and intrinsic admittance η of the medium, as given by the relations

$$k\zeta = \omega\mu; \qquad k\eta = \omega\epsilon + i\sigma, \tag{2.3}$$

the set of Maxwellian equations, when expressed in rationalized m.k.s. units, assume the form

$$\begin{aligned} &\text{I.}\ \nabla \times E = ik\zeta H && \text{III.}\ \nabla \cdot H = 0 \\ &\text{II.}\ \nabla \times H = -ik\eta E + J^0 && \text{IV.}\ \nabla \cdot E = \frac{\nabla \cdot J^0}{ik\eta} \end{aligned} \tag{2.4}$$

in which J^o denotes the vector of prescribed current density distribution which characterizes the source function.

As is well known, the set I - IV given by Eqs. (2.4) admits a solution in terms of the Hertzian vector or electric polarization potential Π as follows:

$$E = \nabla \nabla \cdot \Pi + k^2 \Pi; \qquad H = -ik\eta \nabla \times \Pi, \tag{2.5}$$

where the Hertzian vector Π is a particular integral of the inhomogeneous, vector Helmholtz equation

$$(\nabla^2 + k^2)\Pi = -iJ^o/k\eta. \tag{2.6}$$

2.2b. The nature of the source.- In general, the prescribed current density distribution J^o appearing in the inhomogeneous term of Eq. (2.6) is assumed to be confined within a finite region at a finite distance from the origin. In this case, the so-called radiation condition demands that the solution of Eq. (2.6) consist of outgoing waves on the surface of the sphere at infinity. In the present instance, the source consists essentially of an insulated horizontal wire of length 2ℓ (Fig. 2), terminated by suitable bare electrodes at the extremities of the wire, and located in the conducting medium at a depth h below the horizontal interface. The generator leads are assumed to be inserted at some suitable point along the wire, e.g., between one electrode and the immediate extremity of the wire, thus driving a current $Ie^{-i\omega t}$ along the length of the wire.

It is recognized at the outset that the amplitude factor I is an unknown function of position along the length of the wire which depends on the nature of the insulation and on the complicated propagation characteristics along the wire as affected by its finite length and its proximity to the boundary surface separating the two media. As a first approximation we have assumed, as indicated in Fig. 2, that the amplitude current I is a constant, independent of position along the wire. Thus, the prescribed current density vector J^o, corresponding to this idealized source, may be conveniently written as

$$J^o = e_x I \left\{u(x+\ell) - u(x-\ell)\right\} \delta(y)\, \delta(z+h)\,, \tag{2.7}$$

where e_x is the unit vector in the direction of the x axis and where $u(x)$ denotes the unit step function, which is defined as identically zero for negative argument and unity for positive argument, while $\delta(x)$ is Dirac's δ-function which we here regard as the derivative of the unit step function.

In this report we are mainly concerned with points of observation at distances which are large compared with the length of the extended source. In this case we may regard the source as a dipole which is generated from the extended source by letting $\ell \to 0$. Thus, defining the electric moment* of the dipole as

$$p = \mathrm{Lim}(2\ell I) \quad \text{as} \quad \ell \to 0 \quad \text{and} \quad I \to \infty\,, \tag{2.8}$$

we readily obtain for the prescribed source function the compact expression

$$J^o = e_x p\, \delta(x)\, \delta(y)\, \delta(z+h)\,, \tag{2.9}$$

which corresponds to a Hertzian dipole located on the z axis, at a depth h, and oriented in the direction of the positive x axis (Fig. 1).

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* This definition differs from the conventional definition of the electric dipole moment of a Hertzian dipole by a constant factor; in fact, $p = -i\omega p^{(1)}$ where $p^{(1)}$ is the conventional electric dipole moment.

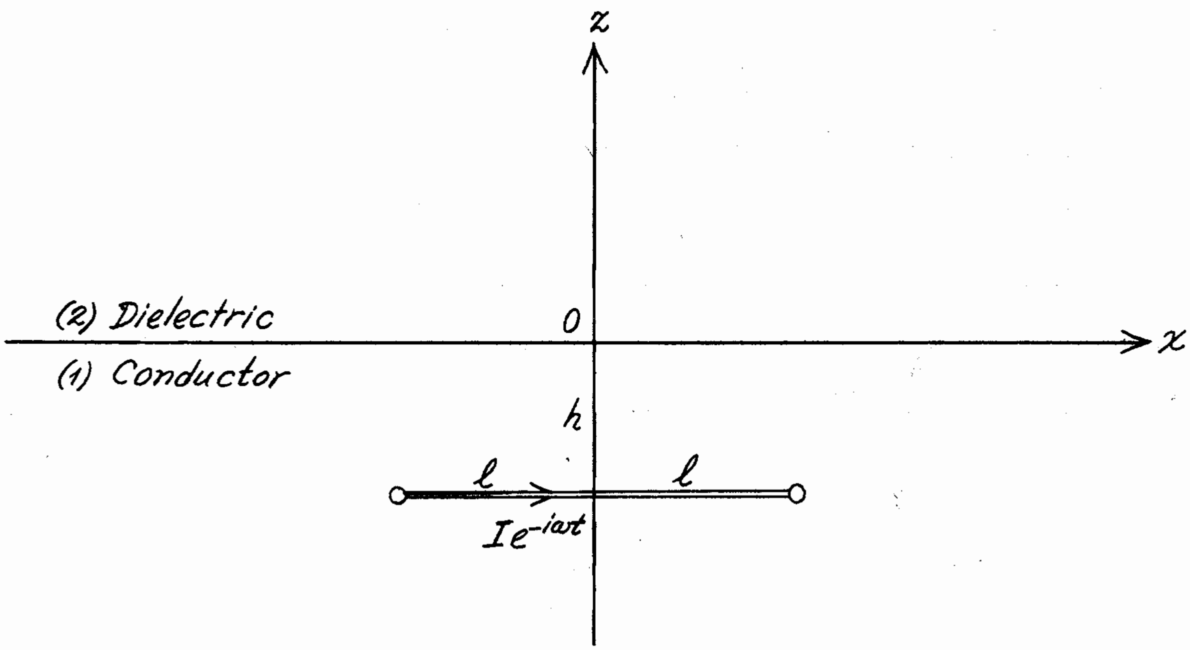

Fig. 2.- Diagram of an extended source.

Besides the assumption of a dipole source being an accurate approximation to the problem at hand, it is a logical first step in obtaining a completely rigorous solution to the problem of an extended source; since, as is well known, the rigorous solution of the problem of a prescribed extended source may be given as the superposition of properly weighted solutions of the dipole problem by the application of Green's theorem.

2.2c. Formulation of solution in terms of Π-vectors.- We have shown that the source vector J^o may be regarded as a dipole singularity in the x direction; therefore, in accordance with Eq. (2.6), the Π-vectors for both media must have at least an x component. It will be proved later (Section 4.5) that the interface $z = 0$ separating the two media is in fact a surface singularity for $\nabla \times J$ (where $J = \sigma E$ denotes the conduction current) which acts as a secondary source; therefore, we must also have the z components of the Π-vectors. The boundary conditions and the symmetry of the problem clearly point out that no y components are needed. In consequence, we exhibit at once the Π-vectors for both media as

$$\begin{aligned} \Pi^{(1)} &= e_x \Pi_{x1} + e_z \Pi_{z1} , \qquad z \leqslant 0 , \\ \Pi^{(2)} &= e_x \Pi_{x2} + e_z \Pi_{z2} , \qquad z \geqslant 0 ; \end{aligned} \tag{2.10}$$

furthermore, because the source singularity is embedded in medium (1), it will be convenient to exhibit Π_{x1} as the sum of two components,

$$\Pi_{x1} = \Pi^o_{x1} + \Pi^1_{x1} , \qquad z \leq 0 , \tag{2.11}$$

where Π^o_{x1}, in accordance with Eq. (2.6), is a particular integral of the inhomogeneous, scalar Helmholtz equation

$$(\nabla^2 + k_1^2)\Pi_{x1}^{o} = -(ip/k_1\eta_1)\,\delta(x)\,\delta(y)\,\delta(z+h), \tag{2.12}$$

and Π_{x1}^{1} is a solution of the corresponding homogeneous equation, namely

$$(\nabla^2 + k_1^2)\Pi_{x1}^{1} = 0, \qquad z \leqslant 0. \tag{2.13}$$

The remaining components, Π_{z1}, Π_{x2} and Π_{z2}, likewise satisfy identical homogeneous, scalar Helmholtz equations in their respective media; that is,

$$\begin{aligned} (\nabla^2 + k_1^2)\Pi_{z1} &= 0, \qquad z \leqslant 0, \\ (\nabla^2 + k_2^2)\Pi_{x2} &= 0, \qquad z \geqslant 0, \\ (\nabla^2 + k_2^2)\Pi_{z2} &= 0, \qquad z \geqslant 0. \end{aligned} \tag{2.14}$$

The boundary conditions imposed on the Cartesian components of the Π-vectors, as defined by Eqs. (2.10), are readily deduced from the continuity of the tangential components of E and H, at the interface $z = 0$, which are computed for each medium in accordance with Eqs. (2.5). Thus, omitting details, we obtain at $z = 0$ the boundary conditions

$$k_1^2\Pi_{x1} = k_2^2\Pi_{x2}, \qquad k_1^2\frac{\partial\Pi_{x1}}{\partial z} = k_2^2\frac{\partial\Pi_{x2}}{\partial z}; \tag{2.15}$$

$$k_1^2\Pi_{z1} = k_2^2\Pi_{z2}, \qquad \frac{\partial\Pi_{x1}}{\partial x} + \frac{\partial\Pi_{z1}}{\partial z} = \frac{\partial\Pi_{x2}}{\partial x} + \frac{\partial\Pi_{z2}}{\partial z}. \tag{2.16}$$

In consequence, we are to choose solutions of the homogeneous Eqs. (2.13) and (2.14) which, when combined through Eqs. (2.10) and (2.11) with a particular integral of Eq. (2.12), satisfy simultaneously the four boundary conditions given above.

2.3 FOURIER INTEGRAL REPRESENTATIONS IN CARTESIAN COORDINATES

The formulation of the present boundary value problem and the satisfaction of the boundary conditions are more readily effected by expressing the Cartesian components of the Π-vectors in terms of their triple Fourier integral representations displayed in Cartesian coordinates in transform space as well as in configuration space. According to Fourier integral theory such a representation necessarily yields a complete and unique solution; therefore, as shown in Section 2.4, all proper formulations of the problem may be obtained by merely making suitable coordinate transformations in one or both configuration and transform spaces. Furthermore, the triple Fourier integral representation in Cartesian coordinates, after performing two integrations, allows the unequivocal determination of the path of integration in the complex plane of the third transform variable which must lie within a certain strip of analiticity.* Thus, the choice of the path of integration in the third transform variable is quite independent of the boundary conditions of the problem and, once chosen, dictates the proper interpretation to be ascribed to partial results deduced from certain permissible deformations of the original path. This matter is intimately connected with the correct interpretation of Sommerfeld's electromagnetic surface wave to which we return in Section 7.3.

<u>2.3a. The particular integral corresponding to the source.</u>— The component Π^{o}_{x1}, which is a particular integral of Eq. (2.12), can be conveniently chosen as the Green's function for a horizontal dipole embedded in the unbounded conducting medium; that is,

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* E. C. Titchmarsh, "Introduction to the Theory of Fourier Integrals," (Oxford Press, London, 1948), 2nd ed., p. 44, Sec. 1.27.

$$\Pi^{o}_{x1} = \frac{ip}{4\pi k_1 \eta_1} \frac{e^{ik_1R_1}}{R_1} , \tag{2.17}$$

where R_1 is the distance from the point of observation to the source dipole as defined by Eq. (2.1) and where, in accordance with Eqs. (2.2) and (2.3), we have for the conducting medium

$$k_1^2 = \omega^2\mu_1\epsilon_1 + i\omega\mu_1\sigma_1 = \omega^2\mu_o\kappa\epsilon_o + i\omega\mu_o\sigma \tag{2.18}$$

and

$$k_1\eta_1 = \omega\epsilon_1 + i\sigma_1 = \omega\kappa\epsilon_o + i\sigma . \tag{2.19}$$

For the purpose at hand, we now seek the triple Fourier integral représentation of the particular integral (2.17), which is most readily obtained by going back to the original differential equation (2.12). First, we define the triple Fourier transform

$$F(\xi,\eta,\zeta) = \iiint_{-\infty}^{\infty} \Pi^{o}_{x1}(x,y,z)e^{-i(\xi x+\eta y+\zeta z)}dxdydz , \tag{2.20}$$

which represents an analytic function of the three transform variables (ξ, η, ζ) for limited domains of their respective complex planes as pointed out below. To compute $F(\xi,\eta,\zeta)$, one multiplies both sides of Eq. (2.12) by the exponential factor $e^{-i(\xi x+\eta y+\zeta z)}$ and integrates with respect to the real variables x, y and z between $-\infty$ and $+\infty$. The right hand side of Eq. (2.12) yields at once

$$\iiint_{-\infty}^{\infty} \delta(x)\,\delta(y)\,\delta(z + h)e^{-i(\xi x+\eta y+\zeta z)}dxdydz = e^{i\zeta h} , \tag{2.21}$$

while the term $k_1^2 \Pi_{x1}^o$ on the left hand side yields simply $k_1^2 F(\xi,\eta,\zeta)$ in accordance with Eq. (2.20).

The remaining terms on the left of Eq. (2.12) involve the three second order partial derivatives of the Laplacian operator in Cartesian coordinates. Each of the three terms must be treated separately. Thus, for example, consider the first integral,

$$\iiint_{-\infty}^{\infty} \frac{\partial^2 \Pi_{x1}^o}{\partial x^2} e^{-i(\xi x+\eta y+\zeta z)} dxdydz = \iint_{-\infty}^{\infty} \left\{ e^{-i\xi x} \left[\frac{\partial \Pi_{x1}^o}{\partial x} + i\xi \Pi_{x1}^o \right] \right\} \Bigg|_{x=-\infty}^{x=+\infty} e^{-i(\eta y+\zeta z)} dydz$$

$$-\xi^2 \iiint_{-\infty}^{\infty} \Pi_{x1}^o \, e^{-i(\xi x+\eta y+\zeta z)} dxdydz \,, \qquad (2.22)$$

where we have twice carried out an integration by parts in the x variable. It is clear that the vanishing of the <u>integrated part</u> at the upper and lower limits in the double integral on the right of Eq. (2.22) is guaranteed for all <u>real</u> values of the transform variable ξ by the asymptotic behavior of the function Π_{x1}^o itself; for we know from Eq. (2.17) that, with y and z fixed,

$$\Pi_{x1}^o \longrightarrow \frac{1}{|x|} e^{ik_1|x|} \quad \text{as} \quad |x| \longrightarrow \infty \,; \qquad (2.23)$$

and, thus, with $k_1 = \beta_1 + i\alpha_1$ $(\alpha_1 > 0)$, Π_{x1}^o vanishes exponentially at the upper and lower limits. In consequence, the right hand side of Eq. (2.22) becomes merely $-\xi^2 F(\xi,\eta,\zeta)$ under the sufficient condition that ξ be real.

Identical considerations applied to the two remaining terms of the Laplacian operator yield similar results, whence we obtain from the original Helmholtz equation the analytic expression

$$F(\xi,\eta,\zeta) = \frac{ip}{k_1\eta_1}\frac{e^{i\zeta h}}{\xi^2 + \eta^2 + \zeta^2 - k_1^2} \tag{2.24}$$

for the triple Fourier transform defined by Eq. (2.20). Hence, we have established that a sufficient condition for the existence of the transform $F(\xi,\eta,\zeta)$, defined by the triple integral (2.20) as an analytic function of the three complex variables ξ, η and ζ, is that these variables be rigorously real.

We now propose to show that this condition is too stringent. To this end, we transform to spherical coordinates in both transform and configuration spaces. Writing, with reference to Fig. 1,

$$\begin{aligned} x &= R_1 \sin\theta_1 \cos\varphi & \xi &= K \sin\alpha \cos\beta \\ y &= R_1 \sin\theta_1 \sin\varphi & \eta &= K \sin\alpha \sin\beta \\ z + h &= R_1 \cos\theta_1 & \zeta &= K \cos\alpha \end{aligned}$$

we obtain, instead of (2.24),

$$F(K,\alpha,\beta) = \frac{ip}{k_1\eta_1}\frac{e^{iKh\cos\alpha}}{K^2 - k_1^2} . \tag{2.25}$$

Applying the inversion theorem to the transform (2.25) we have

$$\Pi^o_{x1}(R_1,\theta_1,\varphi) = \frac{ip}{8\pi^3 k_1\eta_1}\int_0^\infty\int_0^\pi\int_0^{2\pi}\frac{e^{iKR_1\cos\alpha}}{K^2 - k_1^2}K^2 dK \sin\alpha\, d\alpha\, d\beta , \tag{2.26}$$

which can be written in this form by merely rotating the polar axis in transform space until it coincides with the direction of R_1.

Carrying out the integrations with respect to the angular variables α and β and extending to negative values the range of integration in the K variable, we deduce

$$\Pi^{o}_{x1} = \frac{p}{4\pi^2 k_1 \eta_1} \frac{1}{R_1} \int_{-\infty}^{\infty} \frac{e^{iKR_1}}{K^2 - k_1^2} K\, dK = \frac{ip}{4\pi k_1 \eta_1} \frac{e^{ik_1 R_1}}{R_1}, \qquad (2.27)$$

in which the last result, obtained by the method of residues, is in full accord with Eq. (2.17). The essential point to observe is that the path of integration in (2.27) need not coincide with the real axis in the K-plane, but that it is sufficient for the path to lie entirely within the strip

$$-\alpha_1 < \operatorname{Im}\{K\} < \alpha_1 \qquad (2.28)$$

where $\alpha_1 = \operatorname{Im}\{k_1\} > 0$. We have then shown that the triple integral (2.20), regarded as an analytic function of the complex variable K and the rigorously real variables α and β, as expressed by Eq. (2.25), is so defined only so long as K remains within the so-called "strip of analiticity" defined by Eq. (2.28). This means, returning to our original transform variables ξ, η and ζ, that the triple integral (2.20) defines an analytic function of the three complex variables ξ, η and ζ, as given by Eq. (2.24), only so long as

$$K = (\xi^2 + \eta^2 + \zeta^2)^{\frac{1}{2}} \qquad (2.29)$$

remains within the strip defined by Eq. (2.28). Thus, for example, if we choose to keep the variables ξ and η rigorously real, then the third variable ζ may be allowed to wander off its real axis but just so long as

$$-\alpha_1 < \mathrm{Im}\{\zeta\} < \alpha_1 \tag{2.30}$$

which now defines the strip of analiticity in the ζ variable when ξ and η and both real.

Applying next the inversion theorem to the transform (2.24) we obtain for the particular integral π^o_{x1} the representation

$$\pi^o_{x1} = \frac{ip}{8\pi^3 k_1 \eta_1} \iiint_{-\infty}^{\infty} \frac{e^{i[\xi x + \eta y + \zeta(z+h)]}}{\xi^2 + \eta^2 + \zeta^2 - k_1^2}\, d\xi d\eta d\zeta \,, \tag{2.31}$$

in which the path of integration for each of the three complex variables ξ, η and ζ is the corresponding real axis. Introducing the attenuation factor

$$\gamma_1 = (\xi^2 + \eta^2 - k_1^2)^{\frac{1}{2}} \xrightarrow[\xi,\eta \to 0]{} -ik_1 \,, \tag{2.32}$$

where the sign of the radical is so chosen that $\mathrm{Re}\{\gamma_1\} > 0$ for all real values of ξ and η, and carrying out the integration with respect to ζ in (2.31) by the method of residues, we obtain the desired representation in the form of a double integral,

$$\pi^o_{x1} = \frac{ip}{8\pi^2 k_1 \eta_1} \iint_{-\infty}^{\infty} \frac{1}{\gamma_1} e^{-\gamma_1|z+h| + i(\xi x + \eta y)}\, d\xi d\eta \,, \tag{2.33}$$

where, in accordance with the final conclusions of the last paragraph, the paths of integration in the ξ and η variables are taken along their respective real axes.

Comparison of Eqs. (2.17) and (2.33) shows that we have established the following integral representation for the source function,

$$\frac{e^{ik_1R_1}}{R_1} = \frac{1}{2\pi}\iint_{-\infty}^{\infty} \frac{1}{\gamma_1} e^{-\gamma_1|z+h|+i(\xi x+\eta y)}\, d\xi d\eta, \tag{2.34}$$

where $\gamma_1(\xi,\eta;k_1)$ has the unique definition given by Eq. (2.32). The form of the integral representation (2.34) is clear: it represents the superposition of elementary harmonic waves in the x and y directions, exponentially attenuated in the z direction away from the source and so combined through the "amplitude" factor $1/\gamma_1$ that the double integral over all real values of ξ and η yields the elementary spherical wave on the left of Eq. (2.34). That we have indeed a superposition of elementary solutions of the homogeneous Helmholtz equation is seen at once by noting that $\gamma_1 - \xi^2 - \eta^2 = -k_1^2$, which states that the integral representation (2.34) is a solution of the homogeneous Helmholtz equation in the unbounded conducting medium except at the point occupied by the source, $R_1 = 0$ or $(0,0,-h)$, at which point the integral diverges as $1/R_1$.

2.3b. Representation of the x components of the Π-vectors.— The double Fourier integral representation (2.34) for the source function suggests at once the form of similar representations for the remaining x components, Π^1_{x1} and Π_{x2}, which satisfy the homogeneous Helmholtz equation in their respective media in accordance with Eqs. (2.13) and the second of (2.14). First, we introduce, as in Eq. (2.32), the second attenuation factor

$$\gamma_2 = (\xi^2 + \eta^2 - k_2^2)^{\frac{1}{2}} \xrightarrow[\xi,\eta \to 0]{} - ik_2 , \tag{2.35}$$

where again the sign of the radical is chosen in such a way that $\mathrm{Re}\left\{\gamma_2\right\} > 0$ for all values of ξ and η on their paths of integration and where k_2, the propagation constant for medium (2), is deduced in accordance with Eq. (2.2) from

$$k_2^2 = \omega^2 \mu_2 \epsilon_2 + i\omega\mu_2\sigma_2 = \omega^2 \mu_0 \epsilon_0 + i\omega\mu_0\sigma_2 , \tag{2.36}$$

with σ_2 as small as desired. Similarly, making use of (2.3), we put

$$k_2\eta_2 = \omega\epsilon_2 + i\sigma_2 = \omega\epsilon_0 + i\sigma_2 . \tag{2.37}$$

Then, following the pattern set by Eq. (2.33), the desired integral representations may be written down at once,

$$\Pi^1_{x1} = \frac{ip}{4\pi^2 k_1\eta_1} \iint\limits_{-\infty}^{\infty} f_1(\xi,\eta) e^{-\gamma_1(z-h)+i(\xi x+\eta y)}\, d\xi d\eta, \quad z \leqslant 0 , \tag{2.38}$$

and

$$\Pi_{x2} = \frac{ip}{4\pi^2 k_2\eta_2} \iint\limits_{-\infty}^{\infty} f_2(\xi,\eta) e^{-\gamma_2 z-\gamma_1 h+i(\xi x+\eta y)}\, d\xi d\eta, \quad z \geqslant 0 , \tag{2.39}$$

where in each case the paths of integration for the ξ and η variables are taken along their respective real axes. The amplitude functions appearing in the proposed integral representations contain the common convergence factor

$e^{-\gamma_1 h}$, introduced here for convenience, and the unknown functions f_1 and f_2 which are to be determined from the boundary conditions. It is seen by inspection that the proposed expansions for Π^1_{x1} and Π_{x2} satisfy the homogeneous Helmholtz equation in their respective media.

2.3c. Imposition of the boundary conditions satisfied by the x components.-- The x components of the Π-vectors, $\Pi_{x1} = \Pi^o_{x1} + \Pi^1_{x1}$ and Π_{x2}, as given by the integral representations (2.33), (2.38) and (2.39), must comply at $z = 0$ with the boundary conditions given by Eqs. (2.15). Differentiating with respect to z under the signs of integration, as called for by the second of Eqs. (2.15), putting $z = 0$ and recalling that both media are assumed to have the same magnetic inductive capacity, which means that we are allowed to write, by virtue of Eq. (2.3),

$$k_1\eta_1 = k_1^2/\omega\mu_o \quad \text{and} \quad k_2\eta_2 = k_2^2/\omega\mu_o , \tag{2.40}$$

we obtain the equality of two pairs of double Fourier integral representations. Making use of the uniqueness property of such expansions, we are allowed to equate the corresponding double Fourier transforms, obtaining, after dropping common factors,

$$\begin{aligned} 1 + \gamma_1 f_1 &= \gamma_1 f_2 ; \\ -1 + \gamma_1 f_1 &= -\gamma_2 f_2 , \end{aligned} \tag{2.41}$$

as a pair of simultaneous equations in the unknown functions f_1 and f_2. The solution of Eqs. (2.41) yields readily

$$f_1 = \frac{\gamma_1 - \gamma_2}{\gamma_1(\gamma_1 + \gamma_2)} = -\frac{1}{\gamma_1} + \frac{2}{\gamma_1 + \gamma_2} ; \qquad f_2 = \frac{2}{\gamma_1 + \gamma_2} , \tag{2.42}$$

in which γ_1 and γ_2 have the definitions given by Eqs. (2.32) and (2.35) respectively.

Substituting the above results into the corresponding integral representations, we obtain for the x components of the Π-vectors the expressions which we set out to establish,

$$\Pi_{x1} = \frac{ip}{8\pi^2 k_1 \eta_1} \iint_{-\infty}^{\infty} \left\{ \frac{e^{-\gamma_1 |z+h|}}{\gamma_1} - \frac{e^{\gamma_1 (z-h)}}{\gamma_1} + \frac{2e^{\gamma_1 (z-h)}}{\gamma_1 + \gamma_2} \right\} e^{i(\xi x + \eta y)} d\xi d\eta, \quad z \leq 0 \qquad (2.43)$$

$$\Pi_{x2} = \frac{ip}{8\pi^2 k_2 \eta_2} \iint_{-\infty}^{\infty} \frac{2}{\gamma_1 + \gamma_2} e^{-\gamma_2 z - \gamma_1 h + i(\xi x + \eta y)} d\xi d\eta, \quad z \geq 0, \qquad (2.44)$$

in which an essential point to remember is that the paths of integration in the ξ and η variables are their respective real axes.

<u>2.3d. Representation of the z components of the Π-vectors.</u>– The z components of the Π-vectors, Π_{z1} and Π_{z2}, satisfy in accordance with the first and last of Eqs. (2.14) the homogeneous Helmholtz equation in their respective media and have Fourier integral representations of a form analogous to Eqs. (2.38) and (2.39). Thus, we propose at once the representations

$$\Pi_{z1} = \frac{ip}{4\pi^2 k_1 \eta_1} \iint_{-\infty}^{\infty} i\xi g_1(\xi,\eta) e^{\gamma_1 (z-h) + i(\xi x + \eta y)} d\xi d\eta, \quad z \leq 0 ; \qquad (2.45)$$

$$\Pi_{z2} = \frac{ip}{4\pi^2 k_2 \eta_2} \iint_{-\infty}^{\infty} i\xi g_2(\xi,\eta) e^{-\gamma_2 z - \gamma_1 h + i(\xi x + \eta y)} \, d\xi d\eta, \quad z \geq 0 \tag{2.46}$$

in which the only new feature is the explicit factor $i\xi$ in both integrands, introduced here for convenience, and in which g_1 and g_2 denote the unknown amplitude functions to be determined from the boundary conditions. It is seen by inspection that the proposed expansions for Π_{z1} and Π_{z2} satisfy the homogeneous Helmholtz equation in their respective media.

2.3e. Imposition of the boundary conditions satisfied by the z components.- Proceeding as before, we now substitute the integral representations (2.45) and (2.46) into the boundary conditions given by Eqs. (2.16). Carrying out under the signs of integration the differentiations with respect to x and z, as called for by the second of Eqs. (2.16), and evaluating at $z = 0$, we obtain the pair of simultaneous equations

$$g_1 = g_2 ; \qquad \frac{1}{k_1^2}\left\{\frac{2}{\gamma_1 + \gamma_2} + \gamma_1 g_1\right\} = \frac{1}{k_2^2}\left\{\frac{2}{\gamma_1 + \gamma_2} - \gamma_2 g_2\right\} , \tag{2.47}$$

from which, noting from Eqs. (2.32) and (2.35) that $\gamma_1^2 - \gamma_2^2 = k_2^2 - k_1^2$, we readily obtain

$$g_1 = g_2 = -2(\gamma_1 - \gamma_2)/(k_2^2\gamma_1 + k_1^2\gamma_2) , \tag{2.48}$$

which exhibits the famous Sommerfeld denominator, $N = k_2^2\gamma_1 + k_1^2\gamma_2$, about which we have a great deal more to say in this report.

Substituting the above results into the proposed integral representations, Eqs. (2.45) and (2.46), and noting that the explicit factor $i\xi$ in both integrands can be made implicit through the partial differential operator $\partial/\partial x$ outside the integral signs, we obtain finally

$$\Pi_{z1} = - \frac{ip}{8\pi^2 k_1 \eta_1} \frac{\partial}{\partial x} \iint_{-\infty}^{\infty} \frac{2(\gamma_1 - \gamma_2)}{k_2^2 \gamma_1 + k_1^2 \gamma_2} e^{\gamma_1(z-h)+i(\xi x+\eta y)} d\xi d\eta, \quad z \leq 0; \qquad (2.49)$$

$$\Pi_{z2} = - \frac{ip}{8\pi^2 k_2 \eta_2} \frac{\partial}{\partial x} \iint_{-\infty}^{\infty} \frac{2(\gamma_1 - \gamma_2)}{k_2^2 \gamma_1 + k_1^2 \gamma_2} e^{-\gamma_2 z-\gamma_1 h+i(\xi x+\eta y)} d\xi d\eta, \quad z \geq 0, \qquad (2.50)$$

in which once again an essential point to remember is the fact that the paths of integration in the ξ and η variables are their respective real axes.

<u>2.3f. Transformation to cylindrical coordinates in configuration and transform spaces: Sommerfeld's formulation.</u>– All four integral representations derived above are surface integrals over the entire ξ-η plane of a form which lends itself readily to a transformation to cylindrical coordinates in configuration space as well as in transform space. Thus, putting

$$x = \rho \cos \varphi, \quad y = \rho \sin \varphi, \quad \rho = (x^2 + y^2)^{\frac{1}{2}}, \qquad (2.51)$$

for the space coordinates and

$$\xi = \lambda \cos \beta, \quad \eta = \lambda \sin \beta, \quad \lambda = (\xi^2 + \eta^2)^{\frac{1}{2}}, \qquad (2.52)$$

for the (real) transform variables, ξ and η, and replacing the element of area $d\xi d\eta$ by $\lambda d\lambda d\beta$, we note that all the double integrals appearing in Eqs. (2.43), (2.44), (2.49) and (2.50) are of the general form

$$\frac{1}{2\pi}\iint_{-\infty}^{\infty} f(\xi^2+\eta^2;z)e^{i(\xi x+\eta y)}d\xi d\eta = \frac{1}{2\pi}\int_0^{\infty} f(\lambda^2;z)\lambda d\lambda \int_{-\pi}^{\pi} e^{i\lambda\rho\cos(\beta-\varphi)}d\beta$$

$$= \int_0^{\infty} f(\lambda^2;z)J_o(\lambda\rho)\lambda d\lambda \,, \qquad (2.53)$$

in which we have made use of a well-known integral representation for the Bessel function of order zero. It may be noted that this transformation proves advantageous because the function $f(\xi^2+\eta^2;z)$ contains the variables ξ and η in the form $\lambda = (\xi^2 + \eta^2)^{\frac{1}{2}}$ in all the instances encountered.

Applying the result embodied in Eq. (2.53) to our four integral representations and noting that in transforming to cylindrical coordinates, the operator $\partial/\partial x$ is to be replaced by $\cos\varphi\,(\partial/\partial\rho)$, we obtain finally the integral representations

$$\Pi_{x1} = \frac{ip}{4\pi k_1\eta_1}\int_0^{\infty}\left\{\frac{1}{\gamma_1}e^{-\gamma_1|z+h|} - \frac{1}{\gamma_1}e^{\gamma_1(z-h)} + \frac{2}{\gamma_1+\gamma_2}e^{\gamma_1(z-h)}\right\}J_o(\lambda\rho)\lambda d\lambda, \quad z \leqslant 0, \quad (2.54)$$

$$\Pi_{x2} = \frac{ip}{4\pi k_2\eta_2}\int_0^{\infty}\frac{2}{\gamma_1+\gamma_2}e^{-\gamma_2 z-\gamma_1 h}J_o(\lambda\rho)\lambda d\lambda\,, \quad z \geqslant 0\,; \qquad (2.55)$$

$$\Pi_{z1} = -\frac{ip\cos\varphi}{4\pi k_1\eta_1}\frac{\partial}{\partial\rho}\int_0^{\infty}\frac{2(\gamma_1-\gamma_2)}{k_2^2\gamma_1+k_1^2\gamma_2}e^{\gamma_1(z-h)}J_o(\lambda\rho)\lambda d\lambda\,, \quad z \leqslant 0\,, \qquad (2.56)$$

$$\Pi_{z2} = -\frac{ip\cos\varphi}{4\pi k_2\eta_2}\frac{\partial}{\partial\rho}\int_0^\infty \frac{2(\gamma_1-\gamma_2)}{k_2^2\gamma_1 + k_1\gamma_2^2} e^{-\gamma_2 z-\gamma_1 h} J_o(\lambda\rho)\lambda d\lambda, \quad z \geqslant 0, \qquad (2.57)$$

where an essential point to remember is that the variable of integration $\lambda = (\xi^2 + \eta^2)^{\frac{1}{2}}$ is a positive definite quantity and, thus, the path of integration is rigorously along the positive half of the real axis in the complex λ-plane. The parameters γ_1 and γ_2, originally introduced by Eqs. (2.32) and (2.35) respectively, are now defined as

$$\gamma_1 = (\lambda^2 - k_1^2)^{\frac{1}{2}} \xrightarrow[\lambda\to 0]{} -ik_1 ; \qquad \gamma_2 = (\lambda^2 - k_2^2)^{\frac{1}{2}} \xrightarrow[\lambda\to 0]{} -ik_2 , \quad (2.58)$$

in which the sign of the square roots is to be chosen in such a way that $\mathrm{Re}\left\{\gamma_1\right\} > 0$ and $\mathrm{Re}\left\{\gamma_2\right\} > 0$ along the entire path of integration, $0 \leq \lambda < \infty$.

The above integral representations, Eqs. (2.54) to (2.58), embody the results contained in Sommerfeld's formulation of the problem. Originally, Sommerfeld considered only the problem of the vertical electric dipole located at the boundary surface separating the two media (h = 0) in a classical memoir[1] which even today remains as the fundamental basis of all subsequent investigations. The horizontal electric dipole was first considered by Hörschelmann[2] in his doctoral dissertation dealing with the theoretical investigation of the directional properties of the original Marconi antenna. The horizontal electric dipole located at the interface between the two media

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[1] A. Sommerfeld, Ann. Physik 28, 665-737 (1909).

[2] H. von Hörschelmann, Jahrb. draht. Telegr. u. Teleph. 5, 14-34, 188-211 (1911).

is discussed by Sommerfeld in his excellent summary of the whole problem of propagation of radio waves over the surface of the earth in Riemann-Weber[3] and, finally, the case of the horizontal dipole located at a height h above the surface of the conducting medium is again treated by Sommerfeld in a more recent book,[4] where he gives substantially the same integral representations that we derived above. We feel that our approach, starting with Cartesian coordinates in both configuration and transform spaces, is fundamentally simpler, for it embodies all known formulations of the problem which can now be derived by merely imposing suitable coordinate transformations. In the present instance, we attain Sommerfeld's form of the integrals in the λ-plane by merely going over to cylindrical coordinates.

2.4 VARIOUS FORMS OF THE FUNDAMENTAL INTEGRALS

So far we have established, in Eqs. (2.54) to (2.58), the proposed integral representations for the Cartesian components of the π-vectors for the two media, which satisfy the requisite boundary conditions at $z = 0$ and which represent solutions of their respective Helmholtz equations. Then, making use of Eqs. (2.5), we deduce in Chapter III the electric and magnetic field components which involve the application of various differential operators under the sign of integration. To facilitate the operations indicated and to give a systematic presentation of the results, it is desirable at this time to give a tabulation and discussion of the minimum number of fundamental

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3 Philipp Frank and Richard von Mises, "Die Differential- und Integralgleichungen der Mechanik und Physik," (Friedr. Vieweg, Braunschweig, 1935), 2nd edition, Vol. II, p. 943.

4 A. Sommerfeld, "Partial Differential Equations in Physics," (Academic Press, Inc., New York, 1949), p. 257.

integrals in terms of which the Cartesian components of the Π-vectors and the cylindrical components of the field vectors can be expressed.

2.4a. Tabulation of essential integrals.— It is readily seen from Eqs. (2.54) to (2.58) that our integral representations involve under the sign of integration two distinct factors,

$$f = \frac{2}{\gamma_1 + \gamma_2} \qquad \text{and} \qquad g = -\frac{2(\gamma_1 - \gamma_2)}{k_2^2\gamma_1 + k_1^2\gamma_2}, \tag{2.59}$$

in terms of which we now define a number of functions which prove useful later,

$$v_1 = f + \gamma_1 g = \frac{2k_1^2}{k_2^2\gamma_1 + k_1^2\gamma_2}, \tag{2.60}$$

$$v_2 = f - \gamma_2 g = \frac{2k_2^2}{k_2^2\gamma_1 + k_1^2\gamma_2}, \tag{2.61}$$

$$k_1^2 g/\gamma_1 = -v_1(1 - \gamma_2\gamma_1^{-1}) = -(1 + n^2)v_1 + 2\gamma_1^{-1}, \tag{2.62}$$

$$k_2^2 g/\gamma_2 = v_2(1 - \gamma_1\gamma_2^{-1}) = (1 + n^{-2})v_2 - 2\gamma_2^{-1}, \tag{2.63}$$

in which

$$n = k_2/k_1 \tag{2.64}$$

is the so-called complex index of refraction.

The integral representation (2.54) for Π_{x1} is seen to consist of three distinct integrals, the first two of which are readily identified with the source and its image. With R_1 and R_2 as defined by Eqs. (2.1), we denote the source and image integrals respectively by

$$\Psi_1 = \frac{e^{ik_1R_1}}{R_1} = \int_0^\infty \frac{1}{\gamma_1} e^{-\gamma_1|z+h|} J_o(\lambda\rho)\lambda d\lambda , \quad -\infty < z < \infty , \tag{2.65}$$

and

$$\Psi_2 = \frac{e^{ik_1R_2}}{R_2} = \int_0^\infty \frac{1}{\gamma_1} e^{\gamma_1(z-h)} J_o(\lambda\rho)\lambda d\lambda , \quad z \leq 0 . \tag{2.66}$$

The last integral in Eq. (2.54) and the remaining integral representations, (2.55), (2.56) and (2.57), can all be expressed, as shown in Section 3.1, in terms of the following four fundamental integrals:

$$U_1 = \int_0^\infty f(\lambda)e^{\gamma_1(z-h)} J_o(\lambda\rho)\lambda d\lambda , \quad z \leq 0 , \tag{2.67}$$

$$V_1 = \int_0^\infty v_1(\lambda)e^{\gamma_1(z-h)} J_o(\lambda\rho)\lambda d\lambda , \quad z \leq 0 , \tag{2.68}$$

$$U_2 = \int_0^\infty f(\lambda)e^{-\gamma_2 z-\gamma_1 h} J_o(\lambda\rho)\lambda d\lambda , \quad z \geq 0 , \tag{2.69}$$

$$V_2 = \int_0^\infty v_2(\lambda)e^{-\gamma_2 z-\gamma_1 h} J_o(\lambda\rho)\lambda d\lambda , \quad z \geq 0 , \tag{2.70}$$

in which the functions f, v_1 and v_2 have the definitions given in the preceding paragraph. As indicated, the integrals U_1 and V_1 are defined only for $z \leq 0$ and, therefore, correspond to points of observation in the conducting medium; whereas the integrals U_2 and V_2 are defined only for $z \geq 0$ and, therefore, pertain to points of observation in the air above, medium (2).

The fundamental character of the integrals listed above is more forcefully brought to attention by noting that, as first shown by Sommerfeld,[5] the integrals V_1 and V_2 are the only integrals appearing in the solution for a vertical electric dipole; while the integrals U_1 and U_2, as first shown by Elias[6] and later by Sommerfeld[7] correspond in the same manner to the vertical magnetic dipole or frame antenna. Thus, the solution of the problem for the horizontal electric dipole combines, in effect, the solutions for the vertical electric dipole and the vertical magnetic dipole.

The major portion of this investigation is devoted to the evaluation of the fundamental integrals U_1 and V_1 for the conducting medium, since we are primarily interested in the electric and magnetic field components as observed in the conducting medium. As it happens in all but the simplest diffraction problems, the real difficulties arise when an effort is made to reduce the formal solution to a form suitable for numerical computations. The evaluation of the integrals U_1 and V_1 has proved a major undertaking, as evinced by the fact that more than a dozen investigators have published upward of thirty papers over a period of forty years, mainly on the mathematical aspects of the reduction of the integrals by approximate methods.* We feel that the asymptotic expansions presented here go beyond the work of all our predecessors

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5 Loc. cit., reference 3, p. 925.

6 G. J. Elias, Physica 2, 207-217 and 361-375 (1922).

7 A. Sommerfeld, Ann. Physik 81, 1135-1153 (1926).

* See Bibliography at the end of this Report.

in that we have considered second and third order terms that are essential for the correct estimate of the errors involved in our asymptotic solutions.

As a further point of interest, it should be remarked that the integrals U_1 and V_1, Eqs. (2.67) and (2.68), are not really independent of each other. In fact, it can be shown that U_1 can be expressed in terms of V_1, thus

$$V_1 - U_1 = \frac{2}{k_1^2}\frac{\partial^2 \Psi_2}{\partial z^2} - \frac{1+n^2}{k_1^2}\frac{\partial^2 V_1}{\partial z^2}, \quad z \leq 0, \tag{2.71}$$

which is readily deduced by making use of Eqs. (2.59), (2.60) and (2.62). This would appear to indicate that all our results for the conducting medium could be expressed in terms of V_1 and its derivatives, which is true but unfortunately not useful for practical computational purposes, for it is still easier to deal with the integral U_1 on its own merits. A similar relationship between U_2 and V_2 may be deduced from Eqs. (2.69), (2.70), (2.59) and (2.61) if one introduces derivatives with respect to h, namely

$$U_2 = V_2 + \frac{1}{k_2^2}\frac{\partial}{\partial z}\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial h}\right)V_2, \quad z \geq 0. \tag{2.71a}$$

Finally, it should be noted that the distinction between the integrals belonging respectively to either medium disappears in the important practical case when the source is placed at the interface between the two media, $h = 0$, and the points of observation are confined to the surface, $z = 0$. In this case we have merely, instead of U_1 and U_2,

$$U(\rho,0) = \int_0^\infty \frac{2}{\gamma_1 + \gamma_2} J_0(\lambda\rho)\lambda d\lambda, \tag{2.72}$$

and, instead of V_1 and V_2,

$$V(\rho,0) = \int_0^\infty \frac{2k_1^2}{k_2^2\gamma_1 + k_1^2\gamma_2} J_o(\lambda\rho)\lambda d\lambda . \qquad (2.73)$$

These integrals have received the attention of many investigators among which B. van der Pol[8] deserves special mention for his ingenious method of attack, which consists of replacing the amplitude functions in (2.72) and (2.73) by suitable elementary definite integrals and, then, eliminating the Bessel function $J_o(\lambda\rho)$ by inverting the order of integrations. This procedure yields for $U(\rho,0)$ an exact expression which we rededuce in Section 7.2a as a check on our asymptotic treatment of the more general integral $U_1(\rho,z)$; and, for $V(\rho,0)$, van der Pol obtains an approximate (asymptotic) result which is identical to Sommerfeld's original formula[9] obtained by a considerably more elaborate method. Similarly, as a check on our asymptotic treatment of the more general integral $V_1(\rho,z)$, we rededuce in Section 7.2b the approximate Sommerfeld - van der Pol result.[10] Finally, Wise[11] and Rice[12] treated the integral (2.73) by expanding the integrand into two distinct power series

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[8] B. van der Pol, Z. Hochfrequenz-Tech. 37, 152-157 (1931).

[9] A. Sommerfeld, Ann. Physik 28, 665-737 (1909). Eq. (47), p. 711, of this classical paper, gives essentially the formula deduced by van der Pol except for the famous error in sign that has been so often referred to in the literature and which was corrected by Sommerfeld in Ann. Physik 81, 1135-1153 (1926).

[10] This result was also obtained, using substantially the method of van der Pol, by L. H. Thomas, Proc. Cambridge Phil. Soc. 26, 123-126 (1929); F. H. Murray, Proc. Cambridge Phil. Soc. 28, 433-422 (1932), and K. F. Niessen, Ann. Physik 16, 810-820 (1933).

[11] W. H. Wise, Bell System Tech. J. 16, 35-44 (1937).

[12] S. O. Rice, Bell System Tech. J. 16, 101-109 (1937).

and integrating term by term to obtain asymptotic expansions.

2.4b. Weyl's formulation.- It was originally recognized by Sommerfeld[13] that the source function, Eq. (2.34), can also be interpreted as a bundle of plane waves whose wave normals are characterized by complex direction cosines. This point of view was taken up by Weyl[14] who based his whole formulation on a fundamental integral which, for our source function, becomes

$$\Psi_1 = \frac{e^{ik_1R_1}}{R_1} = \frac{ik_1}{2\pi}\int e^{ik_1(\ell x+my+n|z+h|)}d\Omega, \tag{2.74}$$

where ℓ,m,n denote the (complex) direction cosines of the elementary plane wave normals and where $d\Omega = \sin\alpha\, d\alpha\, d\beta$ is the element of solid angle in which α represents the (complex) colatitude and β the (real) longitude, the direction of the polar axis being wholly arbitrary.

To establish the integral representation (2.74) and to define more precisely the ranges of integration in the angular variables α and β, we note first of all that our integral representation (2.34) can be written in the form

$$\Psi_1 = \frac{e^{ik_1R_1}}{R_1} = \frac{1}{2\pi}\int_0^{\infty}\frac{1}{\gamma_1}e^{-\gamma_1|z+h|}\lambda d\lambda\int_{-\pi}^{\pi}e^{i\lambda(x\cos\beta+y\sin\beta)}d\beta, \tag{2.75}$$

by changing to cylindrical coordinates in transform space in accordance with

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[13] Loc. cit., reference 1, Section 11.

[14] H. Weyl, Ann. Physik 60, 481-500 (1919).

Eqs. (2.52). Next we introduce the conformal transformation

$$\lambda = k_1 \sin\alpha, \qquad d\lambda = k_1 \cos\alpha \, d\alpha;$$
$$\gamma_1 = (\lambda^2 - k_1^2)^{\frac{1}{2}} = -ik_1 \cos\alpha, \qquad (2.76)$$

from which we deduce that the path of integration in the λ-plane, along the positive half of the real axis $(0 \leq \lambda < \infty)$, transforms into the curve shown in the α-plane in Fig. 3, from $\alpha = 0$ to $\alpha = (\pi/2) - \kappa_1 - i\infty$, where $\kappa_1 = \arg\{k_1\}$, $0 \leq \kappa_1 < \pi/4$. Thus, making the indicated change of the variable of integration in (2.75), we obtain for the function $h_o^{(1)}(k_1R_1)$ the representation[15]

$$h_o^{(1)}(k_1R_1) = \frac{e^{ik_1R_1}}{ik_1R_1} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{0}^{\pi/2-\kappa_1-i\infty} e^{ik_1(\ell x+my+n|z+h|)}\sin\alpha \, d\alpha \, d\beta, \qquad (2.77)$$

which is identical to Eq. (2.74) and where the complex direction cosines ℓ, m and n are given by

$$\ell = \sin\alpha\cos\beta, \qquad m = \sin\alpha\sin\beta, \qquad n = \cos\alpha. \qquad (2.78)$$

The particular advantage of the integral representation (2.77), as pointed out by Weyl, is that the direction of the polar axis in transform space is wholly arbitrary; and, thus, one has the freedom to rotate the transform coordinate axes into an orientation that may facilitate the evaluation of the integrals. For example, if the polar axis in transform space is

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[15] See, for example, J. A. Stratton, "Electromagnetic Theory," (McGraw-Hill Book Co., New York, 1941), p. 410, Eq. (66).

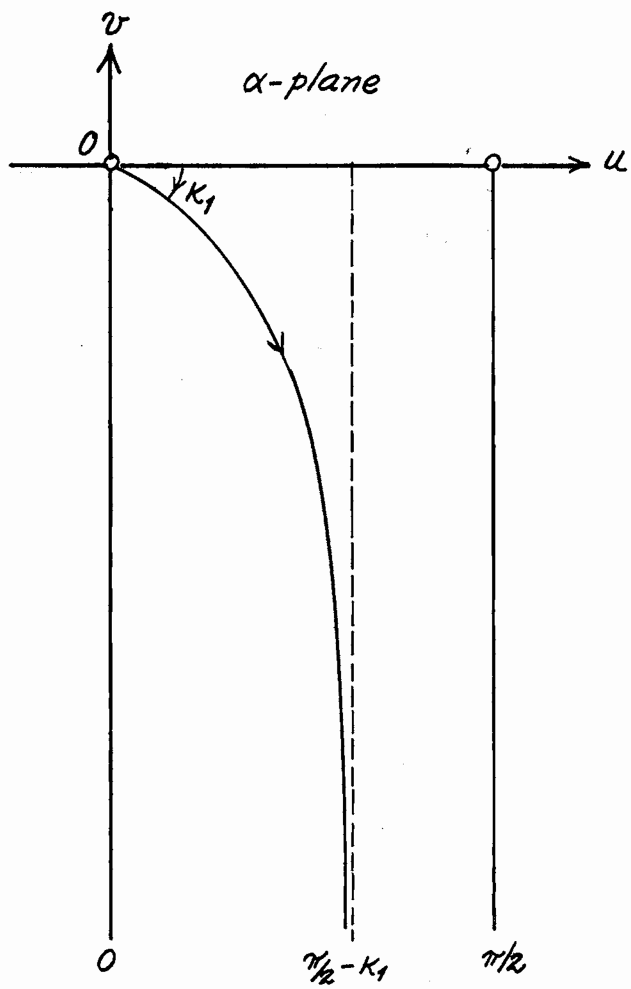

Fig. 3.- Path of integration in the α-plane corresponding to the positive half of the real axis in the λ-plane, $0 \leq \lambda < \infty$.

rotated until it coincides with the vector $\vec{R_1}$ in configuration space, we have that the exponent in Eq. (2.77) can be written as

$$i\,\vec{k_1}\cdot\vec{R_1} = ik_1R_1\cos\vartheta = ik_1(\ell x + my + n\,|z+h|)\,, \qquad (2.79)$$

where ϑ is the (complex) colatitude of the elementary wave normal referred to the direction of the position vector of the point of observation. Thus, if ψ denotes the longitude angle associated with the colatitude ϑ, we have for $h_o^{(1)}(k_1R_1)$ the simpler form

$$h_o^{(1)}(k_1R_1) = \frac{e^{ik_1R_1}}{ik_1R_1} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{0}^{\pi/2-\kappa_1-i\infty} e^{ik_1R_1\cos\vartheta}\sin\vartheta\,d\vartheta\,d\psi, \qquad (2.80)$$

in which, one notes, the limits of integration are still the same as in (2.77), though the variables of integration are not.

Finally, to illustrate the form that our fundamental integrals assume in Weyl's formulation, consider, for example, the integral V_1 as given by Eq. (2.68). Restoring the integral representation of the Bessel function and changing the variable of integration from λ into α, in accordance with the conformal transformation (2.76), the integral in question assumes the form, with $v_1(\lambda)$ as in Eq. (2.60),

$$V_1 = \frac{ik_1}{\pi}\int_{-\pi}^{\pi}\int_{0}^{\pi/2-\kappa_1-i\infty}\frac{\cos\alpha}{n^2\cos\alpha + (n^2-\sin^2\alpha)^{\frac{1}{2}}}\,e^{ik_1R_1\cos\vartheta}\sin\alpha\,d\alpha\,d\beta\,,\ z \lessgtr 0, \qquad (2.81)$$

in which $n = k_2/k_1$ and, as in Eq. (2.79),

$$\cos\vartheta = \sin\theta_2 \sin\alpha \cos(\beta - \varphi) + \cos\theta_2 \cos\alpha, \tag{2.82}$$

which is readily deduced by reference to Fig. 1 making use of Eqs. (2.1); that is, ϑ denotes here the (complex) angle between the elementary wave normal and the direction of R_2 from the image to the point of observation in medium (1). According to Weyl's method, one would rotate the polar axis in transform space until it coincides with the direction of R_2, using ϑ and ψ as new variables of integration, replacing the element of solid angle $\sin\alpha\, d\alpha\, d\beta$ by the new element $\sin\vartheta\, d\vartheta\, d\psi$ and noting that in the remainder of the integrand one must express α in terms of ϑ and ψ by means of the relation

$$\cos\alpha = \cos\theta_2 \cos\vartheta - \sin\theta_2 \sin\vartheta \cos\psi, \tag{2.83}$$

which results from the rotation of axes.

This procedure was carried out successfully by Weyl in his treatment of Sommerfeld's classical problem of the vertical electric dipole located at the interface between the two media. Other investigators, notably Strutt[16] and Krueger,[17] have endeavored to extend Weyl's method to more general situations and have obtained alternative asymptotic expansions for the fundamental integrals. In the opinion of the present writers, however, the method of Weyl does not lend itself readily to further extension, mainly because there are still two integrations to be performed as against only one integration in Sommerfeld's formulation. We believe that this difficulty completely offsets the advantage accruing from the possibility of rotating the polar axis at will in transform space.

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[16] M. J. O. Strutt, Ann. Physik 1, 721-772 (1929); Ann. Physik 4, 1-16 (1930), and Ann. Physik 9, 67-91 (1931).

[17] M. Krueger, Z. Physik 121, 377-438 (1943).

2.4c. Ott's formulation.— To overcome the above difficulty, H. Ott[18] undertook a formulation of the problem which combines the methods of Weyl and Sommerfeld. In effect, Ott adopted the conformal transformation (2.76) and applied it to Sommerfeld's original form of the integrals. Thus, to illustrate Ott's method of attack consider again the source function (2.65) which we rewrite as follows:

$$\Psi_1 = \frac{e^{ik_1R_1}}{R_1} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{\gamma_1} e^{-\gamma_1|z+h|} H_o^1(\lambda\rho)\lambda d\lambda , \tag{2.84}$$

where the path of integration has been extended to the negative real axis in the λ-plane by making use of the formula[19]

$$2J_o(z) = H_o^1(z) - H_o^1(-z) \tag{2.85}$$

and changing the variable of integration from λ into $-\lambda$ in the second integral. Applying Weyl's conformal transformation (2.76), one obtains the following integral representation for the function $h_o^{(1)}(k_1R_1)$,

$$h_o^{(1)}(k_1R_1) = \frac{e^{ik_1R_1}}{ik_1R_1} = \frac{1}{2}\int_{-\pi/2+\kappa_1+i\infty}^{\pi/2-\kappa_1-i\infty} H_o^1(k_1\rho\sin\alpha)e^{ik_1|z+h|\cos\alpha}\sin\alpha\, d\alpha, \tag{2.86}$$

in which the path of integration corresponding to the positive λ-axis is

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[18] H. Ott, Ann. Physik 41, 443-467 (1942), Ann. Physik 43, 393-404 (1943).

[19] G. N. Watson, "A Treatise on the Theory of Bessel Functions," (The MacMillan Company, New York, 1944), 2nd edition, p. 75, Eq. (5).

again as in Fig. 3, while for negative λ it is merely the curve symmetric about the origin in the α-plane. Here $\kappa_1 = \arg\{k_1\}$, with $0 \leq \kappa_1 < \pi/4$ in general.

To illustrate the form assumed by our fundamental integrals in Ott's method, consider again, as an example, the integral V_1 given by Eq. (2.68). Extending the path of integration to negative values of λ, as above, and changing the variable of integration from λ into α, in accordance with (2.76), one obtains the integral

$$V_1 = ik_1 \int_{-\pi/2+\kappa_1+i\infty}^{\pi/2-\kappa_1-i\infty} \frac{\cos\alpha \sin\alpha \, d\alpha}{n^2 \cos\alpha + (n^2 - \sin^2\alpha)^{\frac{1}{2}}} H_0^1(k_1 \rho \sin\alpha) e^{-ik_1(z-h)\cos\alpha}, \quad z \leq 0 \qquad (2.87)$$

which may be compared with Weyl's form, Eq. (2.81). Ott next applies the saddle point method of integration to permissible deformations of the path in Eq. (2.87) and thereby obtains the leading terms for the asymptotic expressions of the integrals. We have adopted Ott's formulation and have extended his methods in this investigation. We have also computed second and third order terms which are necessary for the correct interpretation of the asymptotic results obtained.

2.4d. Other forms of the integrals.- In the preceding sections we have shown that our formulation of the problem, employing Cartesian coordinates in configuration and transform spaces, contains per se all other known formulations of the problem. Thus, by suitable transformations of the variables of integration we have easily derived the formulations of Sommerfeld (1909), Weyl (1919) and Ott (1942). Another formulation of the general problem which deserves special mention is due to B. van der Pol,[20] who starts from the integral representations of Sommerfeld and Weyl and, by making use of rather ingenious though intricate transformations of the variables of integrations, is able to express the fundamental integrals U_1 and V_1, Eqs. (2.67) and (2.68), in the form of volume integrals extended over a certain domain in real space which are susceptible of heuristic interpretation. Further, by making permissible approximations in the case of high conductivity for medium (1), van der Pol obtains in a physical way the purely mathematical approximation obtained by Sommerfeld and Weyl. The reader is referred to the original paper by van der Pol for details of these interesting transformations.

Finally, the form of the fundamental integrals which we have adopted in the present investigation is obtained from Eqs. (2.67) through (2.70) by merely extending the path of integration to negative values of λ, making use of Eq. (2.85) as in Ott's formulation. Thus, we have

$$U_1 = \frac{1}{2}\int_{-\infty}^{\infty} f(\lambda)e^{\gamma_1(z-h)}H_0^1(\lambda\rho)\lambda d\lambda\,, \quad z \leq 0\,, \tag{2.88}$$

$$V_1 = \frac{1}{2}\int_{-\infty}^{\infty} v_1(\lambda)e^{\gamma_1(z-h)}H_0^1(\lambda\rho)\lambda d\lambda\,, \quad z \leq 0\,, \tag{2.89}$$

$$U_2 = \frac{1}{2}\int_{-\infty}^{\infty} f(\lambda)e^{-\gamma_2 z-\gamma_1 h}H_0^1(\lambda\rho)\lambda d\lambda, \quad z \geq 0\,, \tag{2.90}$$

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[20] B. van der Pol, Physica 2, 843-854 (1935).

$$V_2 = \frac{1}{2}\int_{-\infty}^{\infty} v_2(\lambda)e^{-\gamma_2 z-\gamma_1 h}H_0^1(\lambda\rho)\lambda d\lambda, \quad z \geqslant 0, \tag{2.91}$$

where the functions f, v_1 and v_2 are defined by Eqs. (2.59), (2.60) and (2.61) respectively and where the path of integration is along the real axis in the λ-plane.

2.5 THE RIEMANN SURFACE OF FOUR SHEETS IN THE λ-PLANE

As stated in the Introduction, we confine our attention in this Report to the evaluation of the integrals U_1 and V_1 which correspond to points of observation in the conducting medium; and we obtain in Chapter VI, using the saddle point method of integration, asymptotic expansions in a form suitable for numerical computations. The method of attack, as already indicated, is an extension of Ott's formulation involving the approximate (asymptotic) evaluation of the contour integrals around the branch cuts in the upper half of the λ-plane which are deduced from a study of the permissible deformations of the original path of integration. Thus, the integrals in question, (2.88) and (2.89), are both of the form

$$I = \frac{1}{2}\int_{-\infty}^{\infty} v(\lambda)e^{\gamma_1(z-h)}H_0^1(\lambda\rho)\lambda d\lambda, \quad z \leq 0, \tag{2.92}$$

where $v(\lambda)$ stands for $f(\lambda)$ in the case of U_1 and for $v_1(\lambda)$ in the case of V_1. In either case, $v(\lambda)$ is a function of $\gamma_1 = (\lambda^2 - k_1^2)^{\frac{1}{2}}$ and $\gamma_2 = (\lambda^2 - k_1^2)^{\frac{1}{2}}$ as defined in Eqs. (2.58), whereas the exponential function contains only γ_1, but not γ_2. Hence, the integrand in (2.92) exhibits the following singularities: (1) a pair of branch points at $\lambda = \pm k_1$ arising from γ_1; (2) a pair of branch points at $\lambda = \pm k_2$ arising from γ_2; (3) branch points at $\lambda = 0$ and $\lambda = \infty$ arising from $H_0^1(\lambda\rho)$; and (4) a possible pair of poles in the case of the function $v_1(\lambda)$ which we discuss below.

The original path of integration in (2.92) is taken along the real axis in the λ-plane from $-\infty$ to $+\infty$. Before discussing permissible deformations of the original path, it will be necessary to choose a cut λ-plane in which the integrand of (2.92) is defined as a single-valued, regular, analytic function of the variable of integration which vanishes exponentially as $z \rightarrow \infty$. Furthermore, because of the presence of the Hankel function, $H_0^1(\lambda\rho)$, in the integrand of (2.92), it is seen that all permissible deformations of the original path are then confined to the upper half plane in order to guarantee the convergence of the integrals as $\rho \rightarrow \infty$.

<u>2.5a. Range of parameters in the low frequency case</u>.- With the view in mind of possible applications of our results to low frequencies, it is pertinent to introduce at this point some numerical values for the essential parameters of the problem, k_1, k_2 and the ratio $n = k_2/k_1$. We take for the conductivity of medium (1) the representative value $\sigma_1 = \sigma = 4$ mhos/meter and for medium (2) we take $\sigma_2 = 0$, except when mathematical expediency demands that we ascribe to medium (2) a finite though arbitrarily small conductivity. If we confine our attention to frequencies under 10^5 c.p.s., then, in consequence of the high conductivity assumed for medium (1), we have from Eq. (2.18) that $k_1^2 \approx i\omega\mu_0\sigma$ to an extremely high degree of approximation. Thus, we put

$$k_1 = (i\omega\mu_0\sigma)^{\frac{1}{2}} = |k_1|e^{i\pi/4}, \quad |k_1| = (\omega\mu_0\sigma)^{\frac{1}{2}};$$

$$k_2 = \omega(\mu_0\epsilon_0)^{\frac{1}{2}} = \omega/c \quad \text{(real)}; \tag{2.93}$$

$$n = k_2/k_1 = |n|e^{-i\pi/4}, \quad |n| = (\omega\epsilon_0/\sigma)^{\frac{1}{2}}.$$

In Table I we give, with the aid of (2.93), the values of the wavelengths λ_1 and λ_2 in the two media, respectively, and of the parameters $|n|$ and $|n|^2$ for a selected set of frequency values within the specified range. It is shown in this study that our asymptotic formulas are valid at distances from the source which exceed a few wavelengths in the conducting medium; hence, the importance of the second column in Table I is apparent. Furthermore, the parameter $|n|^2$, given in the last column, plays a prominent part in all of our results; and the fact that in the frequency range of interest $|n|^2 < 10^{-6}$ implies considerable simplification in numerical results at low frequencies.

Table I.- Range of Essential Parameters at Low Frequencies

ν (sec^{-1})	λ_1 (meter)	λ_2 (km)	$\|n\|$	$\|n\|^2$
10	500	30,000	1.18×10^{-5}	1.39×10^{-10}
10^2	158	3,000	3.73×10^{-5}	1.39×10^{-9}
10^3	50	300	1.18×10^{-4}	1.39×10^{-8}
10^4	15.8	30	3.73×10^{-4}	1.39×10^{-7}
10^5	5.0	3	1.18×10^{-3}	1.39×10^{-6}

2.5b. Choice of cuts for γ_1 and γ_2.— As deduced from the exponential factor in the integrand of (2.92), in which $(z-h) < 0$, we must choose the branch cut for γ_1 in the λ-plane in such a way that $\text{Re}\{\gamma_1\} > 0$ for all values of λ on the original path of integration, $-\infty < \lambda < \infty$, and on the corresponding sheet of the Riemann surface. This is achieved, as shown in Fig. 4, by drawing the cuts for $\gamma_1 = (\lambda^2 - k_1^2)^{\frac{1}{2}}$ from $\lambda = k_1$ to $\lambda \to i\infty$ along the half-branch of an equilateral hyperbola and from $\lambda = -k_1$ to $\lambda \to -i\infty$ along the symmetric half-branch. This procedure guarantees that $\text{Re}\{\gamma_1\} > 0$ everywhere on the cut λ-plane and $\text{Re}\{\gamma_1\} = 0$ along the branch cuts, with $\arg\{\gamma_1\} = -\pi/2$ on the side of the branch cuts facing the axis of imaginaries and $\arg\{\gamma_1\} = \pi/2$ on the opposite side. Furthermore, it is clear that the locus, $\arg\{\gamma_1\} = 0$, is given by the dotted half-branches of the equilateral hyperbola in Fig. 4. Finally, the straight line through the origin passing through the points $\pm k_1$ has the property that, for points lying on this line between $+k_1$ and $-k_1$, $\arg\{\gamma_1\} = -\pi/4$, while for points lying outside of this segment, $\arg\{\gamma_1\} = +\pi/4$. Thus it is seen that, as λ varies from 0 to ∞ along the positive real axis, the phase of γ_1 varies between $-\pi/4$ and 0, with a completely symmetric behavior for negative values of λ.

Since $\gamma_2 = (\lambda^2 - k_2^2)^{\frac{1}{2}}$ does not enter into the exponential factor in the integrand of (2.92), but only in the amplitude function $v(\lambda)$, we are not required to draw the cuts for γ_2 as in the case for γ_1, though we must still adhere to the agreement, called for by Eq. (2.58), that $\text{Re}\{\gamma_2\} > 0$ over the original path of integration along the real axis in the λ-plane. In this case it proves convenient to choose the cuts in such a way that $\text{Im}\{\gamma_2\} < 0$ everywhere on the cut λ-plane with $\text{Re}\{\gamma_2\} > 0$ along the real axis. This is achieved, as depicted in Fig. 4, by drawing

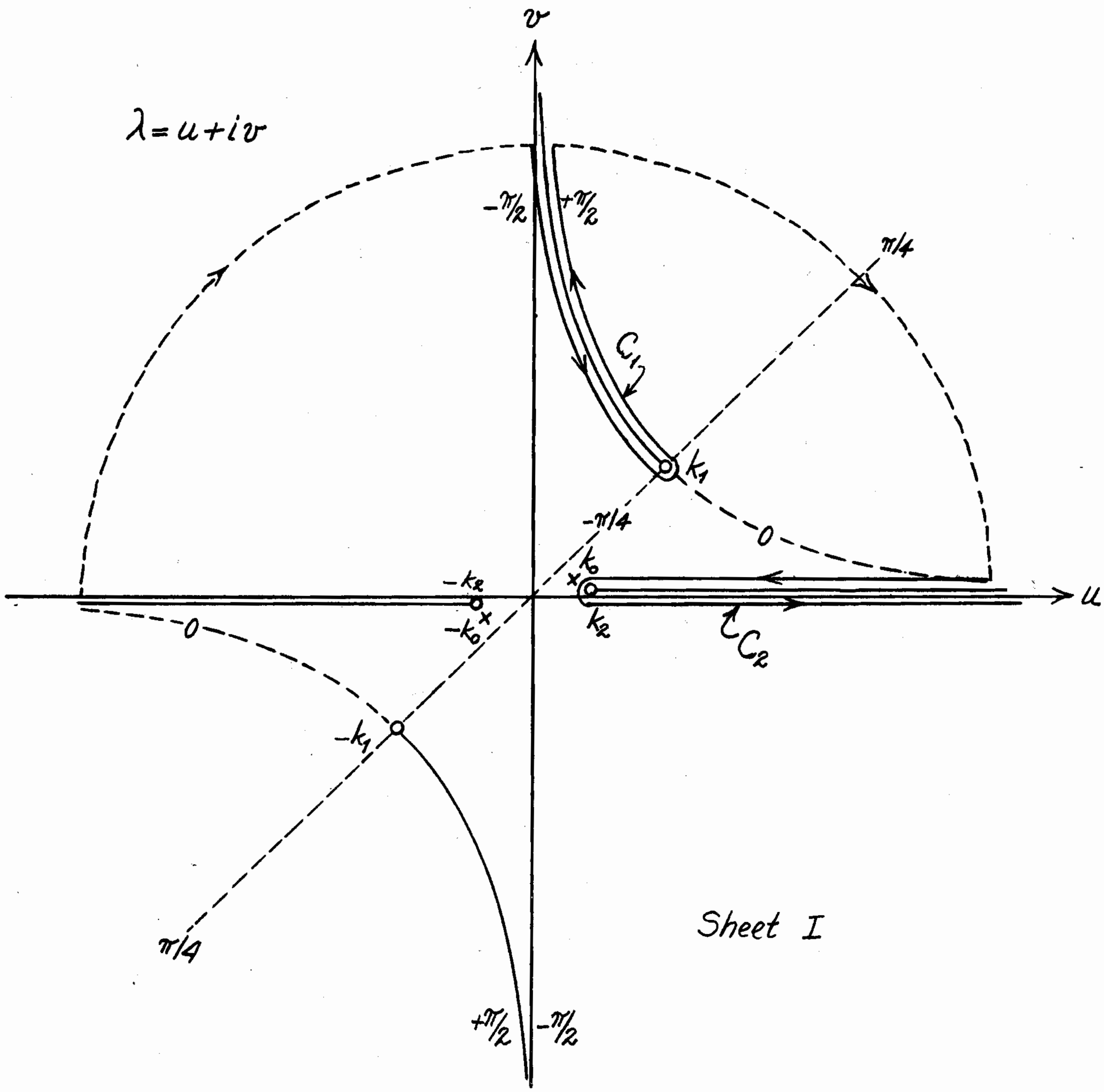

Fig. 4.-- The λ-plane showing the cuts for γ_1 and γ_2 and the deformed path of integration for the conducting medium.

the cuts for γ_2 in precisely the converse situation as regards γ_1; that is, from $\lambda = k_2$ (where k_2 is assumed to have an arbitrarily small positive imaginary part) to $\lambda \to \infty$ along the half-branch of an equilateral hyperbola in the upper half-plane and from $\lambda = -k_2$ to $\lambda \to -\infty$ along the symmetric half-branch in the lower half-plane. Because $\mathrm{Im}\{k_2\}$ is arbitrarily small, the branch cuts for γ_2 in Fig. 4 are undistinguishable from the corresponding segments of the real axis: $-\infty < \lambda \leq -\mathrm{Re}\{k_2\}$ for the left hand cut and $\mathrm{Re}\{k_2\} \leq \lambda < \infty$ for the right hand cut. This procedure guarantees that $\mathrm{Im}\{\gamma_2\} < 0$ everywhere on the cut λ-plane and $\mathrm{Im}\{\gamma_2\} = 0$ along the branch cuts, with $\arg\{\gamma_2\} = 0$ on the side of the branch cuts facing the real axis and $\arg\{\gamma_2\} = -\pi$ on the opposite side. Furthermore, as in the case of γ_1, the straight line through the origin passing through the points $\pm k_2$ has the property that, for points lying on the segment between $-k_2$ and $+k_2$, $\arg\{\gamma_2\} = -\pi/2 + \kappa_2$, where $\kappa_2 = \arg\{k_2\}$, while for points lying outside of this segment, $\arg\{\gamma_2\} = -\pi + \kappa_2$. Thus it is seen that, as λ varies from 0 to ∞ along the positive real axis, the phase of γ_2 varies between $(-\pi/2 + \kappa_2)$ and 0, with a completely symmetric behavior for negative values of λ.

Summarizing, we have chosen in Fig. 4 a sheet of the Riemann surface, which henceforth will be referred to as Sheet I, on which we have $-\pi/2 < \arg\{\gamma_1\} < \pi/2$, as regards γ_1, and $-\pi < \arg\{\gamma_2\} < 0$ or, better, $-\pi/2 < \arg\{i\gamma_2\} < \pi/2$, as regards γ_2. Furthermore, we have guaranteed by the present choice of cuts that $\mathrm{Re}\{\gamma_1\} > 0$ and $\mathrm{Re}\{\gamma_2\} > 0$ over the original path of integration along the real axis in the cut λ-plane, in accordance with the demands imposed by the present boundary value problem which requires that all our Fourier-Bessel transforms have a common region of analiticity, namely, the strip $|\mathrm{Im}\{\lambda\}| < \mathrm{Im}\{k_2\}$.

It is clear that the presence of two pairs of branch points in the integrand of (2.92) implies that the Riemann surface in the λ-plane consists of four sheets as indicated schematically in Fig. 5. Sheet I corresponds to the sheet described above and depicted in Fig. 4. Access from one sheet to another can be effected by crossing the branch cuts as indicated in Fig. 5, with the consequent changes in the signs of the real parts of γ_1 and $i\gamma_2$ as listed in Table II.

Table II.- Arguments of γ_1 and $i\gamma_2$ and nature of poles on the various sheets of the Riemann surface

Sheet	$\mathrm{Re}\{\gamma_1\}$	$\mathrm{Re}\{i\gamma_2\}$	Poles of $v_1(\lambda)$
I	+	+	virtual
II	-	+	real
III	+	-	real
IV	-	-	virtual

It is seen from Fig. 5 and Table II that to guarantee the convergence of the integrals (2.88) and (2.89) as $z \to -\infty$ all permissible deformations of the original path of integration must remain on Sheets I and III, thus allowing the crossing of the cuts for γ_2, but not the cuts for γ_1. Furthermore, to guarantee the convergence of the integrals as $\rho \to \infty$, the permissible deformations must be confined to the upper half-planes on Sheets I and III. Because the original path of integration coincides with the real axis on Sheet I, any proposed deformation of the path must start and end on Sheet I at $\lambda \to \pm\infty$.

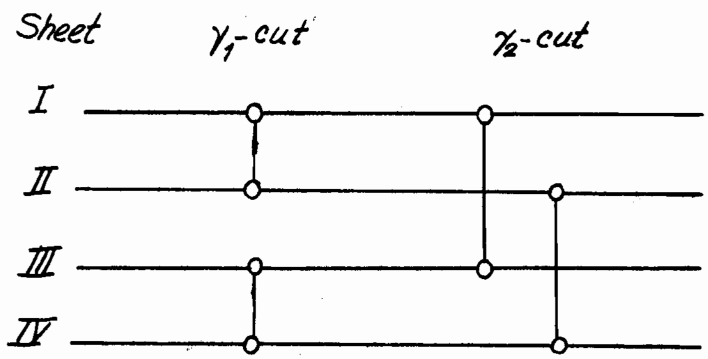

Fig. 5.- The Riemann surface of four sheets in the λ-plane.

<u>2.5c. Discussion of the poles of $v_1(\lambda)$.</u>— In addition to the branch points discussed above, the integrand of V_1, Eq. (2.89), may exhibit a pair of first order poles depending on the choice of cuts and on the particular sheet of the Riemann surface under consideration. This is due to the fact that the amplitude function $v_1(\lambda)$, as defined by Eq. (2.60), contains the famous Sommerfeld denominator

$$N(\lambda) = k_2^2\gamma_1 + k_1^2\gamma_2 , \tag{2.94}$$

which may vanish, depending on the arguments of γ_1 and γ_2, for $\lambda = \pm k_o$ where

$$k_o^2 = \frac{k_1^2 k_2^2}{k_1^2 + k_2^2} = \frac{k_2^2}{1 + n^2} \tag{2.95}$$

is given as a symmetric function of k_1 and k_2.

It is important, first of all, to locate on the various sheets of the Riemann surface the correct positions of the poles, real or virtual, as the case might be. To this end we define from Eq. (2.95)

$$k_o = k_2/(1 + n^2)^{\frac{1}{2}} , \tag{2.96}$$

as a complex number in the first quadrant with k_2 essentially real and $n = k_2/k_1$ as given by the last equation in (2.93). It is clear from (2.96) that we have for the phase of k_o,

$$\kappa_2 < \arg\{k_o\} < \pi/4 ; \tag{2.97}$$

and because $|n|^2 << 1$, it turns out that $\arg\{k_o\}$ is extremely small with $|k_o| < |k_2|$. Thus, the roots of (2.94) are located, as shown in

Fig. 4, just above and to the left of $+k_2$ and just below and to the right of $-k_2$.

To determine whether $N(\pm k_o)$ vanishes or not on the various sheets of the Riemann surface, it is necessary to examine carefully the arguments of γ_1 and γ_2 at $\lambda = \pm k_o$. To this end, let us first compute $\gamma_1(k_o) = (k_o^2 - k_1^2)^{\frac{1}{2}}$. Making use of Eq. (2.96) we have

$$\gamma_1(k_o) = \left(\frac{k_2^2}{1+n^2} - k_1^2\right)^{\frac{1}{2}} = k_1\left(\frac{n^2}{1+n^2} - 1\right)^{\frac{1}{2}} = \frac{\pm ik_1}{(1+n^2)^{\frac{1}{2}}} = \pm\frac{ik_o}{n}, \tag{2.98}$$

where the choice of sign depends on the particular sheet of the Riemann surface in which we wish to operate. Thus, from the discussion of the equiphase curves for $\gamma_1(\lambda)$, as given in Section 2.5b, we deduce that the phase of $\gamma_1(k_o)$, by virtue of (2.97), is subject to the inequalities

$$\begin{aligned} -\pi/4 < \arg\left\{\gamma_1(k_o)\right\} < 0 \qquad &\text{on Sheets I and III;} \\ 3\pi/4 < \arg\left\{\gamma_1(k_o)\right\} < \pi \qquad &\text{on Sheets II and IV,} \end{aligned} \tag{2.99}$$

from which we conclude that, in Eq. (2.98), the choice of sign yields

$$\gamma_1(k_o) = \begin{cases} -ik_o/n\,, & \text{Sheets I and III} \\ +ik_o/n\,, & \text{Sheets II and IV} \end{cases} \tag{2.100}$$

where k_o has been defined as a complex number in the first quadrant. In fact, if we write $k_o = |k_o|\, e^{i\delta}$ where $\delta \approx |n|^2/2$ when $\kappa_2 \approx 0$, then

$$\arg\left\{\gamma_1(k_o)\right\} = \begin{cases} -\pi/4 + \delta\,, & \text{Sheets I and III} \\ 3\pi/4 + \delta\,, & \text{Sheets II and IV} \end{cases} \tag{2.101}$$

as called for by the inequalities (2.99).

Proceeding similarly, we now compute $\gamma_2(k_o) = (k_o^2 - k_2^2)^{\frac{1}{2}}$ and, again making use of Eq. (2.96), we have

$$\gamma_2(k_o) = \left(\frac{k_2^2}{1+n^2} - k_2^2\right)^{\frac{1}{2}} = k_2\left(\frac{1}{1+n^2} - 1\right)^{\frac{1}{2}} = \frac{\pm\, ink_2}{(1+n^2)^{\frac{1}{2}}} = \pm\, ink_o, \quad (2.102)$$

where once more the choice of sign depends on the sheet of the Riemann surface. From an analysis of the equiphase curves for $\gamma_2(\lambda)$, as given above, we deduce that the phase of $\gamma_2(k_o)$ is subject to the inequalities

$$-\pi + K_2 < \arg\left\{\gamma_2(k_o)\right\} < -\pi/2 \quad \text{on Sheets I and II ;}$$
$$(2.103)$$
$$K_2 < \arg\left\{\gamma_2(k_o)\right\} < \pi/2 \quad \text{on Sheets III and IV ,}$$

where $K_2 = \arg\left\{k_2\right\}$ is arbitrarily small, but finite. Thus we conclude that, in Eq. (2.102), the choice of sign yields

$$\gamma_2(k_o) = \begin{cases} -\, ink_o\,, & \text{Sheets I and II} \\ +\, ink_o\,, & \text{Sheets III and IV ;} \end{cases} \quad (2.104)$$

and, hence, with $k_o = |k_o|\, e^{i\delta}$ as above, we have

$$\arg\left\{\gamma_2(k_o)\right\} = \begin{cases} -3\pi/4 + \delta\,, & \text{Sheets I and II} \\ \pi/4 + \delta\,, & \text{Sheets III and IV} \end{cases} \quad (2.105)$$

as called for by the inequalities (2.103).

With the values of $\gamma_1(k_o)$ and $\gamma_2(k_o)$, as given in Eqs. (2.100) and (2.104) respectively, we now quickly ascertain that the Sommerfeld denominator $N(\lambda) = k_2^2\gamma_1 + k_1^2\gamma_2$, when evaluated at $\lambda = \pm k_o$, assumes the values

$$N(\pm k_o) = \begin{cases} -2ik_ok_1k_2 \;, & \text{Sheet I} \\ 0 \;, & \text{Sheets II and III} \\ +2ik_ok_1k_2 \;, & \text{Sheet IV .} \end{cases} \qquad (2.106)$$

Hence, we have established that $v_1(k_o)$ remains finite on Sheets I and IV and, therefore, the points $\lambda = \pm k_o$ are not singularities of the integrand of V_1, Eq. (2.89). On the other hand, on Sheets II and III the integrand of V_1 exhibits a pair of first order poles at $\lambda = \pm k_o$. As indicated in the last column of Table II, we denominate these points, $\lambda = \pm k_o$, "virtual" poles on Sheets I and IV to distinguish them from the "real" or actual poles that do occur on Sheets II and III.

2.5d. Deformation of the original path of integration.- It has been shown that on Sheet I the only singularities of the integrand in (2.92), apart from the branch points which the Hankel function $H_o^1(\lambda\rho)$ exhibits at the origin and at infinity, are the two pairs of branch points associated with the functions γ_1 and γ_2 and that, even in the special case of the function $v_1(\lambda)$ in the integrand of (2.89), we have no further singularities on the chosen sheet. Thus, we may now proceed to discuss permissible deformations of the original path of integration.

In an effort to evaluate (2.92) by contour integration, we first deform the original path as indicated in Fig. 4. Starting on the real axis at $\lambda \rightarrow -\infty$, just above the left branch cut for γ_2, the proposed path follows, first, the semi-circle at infinity in the second quadrant, then the contour C_1 completely around the upper branch cut for γ_1, thence along the semi-circle at infinity in the first quadrant and, finally, the contour C_2 completely around the right hand branch cut for γ_2, terminating on the real axis at $\lambda \to +\infty$, just below the branch cut. By Cauchy's theorem, the proposed path of integration is completely equivalent to the original path along the real axis, for there are no singularities of the integrand between the two paths. Furthermore, it can readily be shown that the contribution over the semi-circle at infinity in the upper half-plane vanishes, with the result that we can express our original integral (2.92) as the sum of two integrals,

$$I = I_1 + I_2 , \tag{2.107}$$

where I_1 is the integral along the contour C_1 around the upper branch cut for γ_1 and I_2 denotes the integral along the contour C_2 around the right hand branch cut for γ_2.

This resolution into two contour integrals differs from Sommerfeld's original resolution in that he writes[21]

$$I = I_1 + I_2' + P , \tag{2.108}$$

where P stands for the contribution from the <u>real</u> pole at $\lambda = + k_o$ which his integrand exhibits by virtue of a different choice of cuts; thus,

$$I_2 = I_2' + P . \tag{2.109}$$

The value of P is readily computed as the residue of the integrand at $\lambda = + k_o$ and is seen to exhibit the characteristics of the Zenneck[22] surface

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[21] A. Sommerfeld, Ann. Physik 28, (1909), p. 649, Eq. (23).

[22] J. Zenneck, Ann. Physik 23, 846-866, (1907).

wave. Much has been written in the last twenty years on the existence or non-existence of these so-called surface waves, and it appeared to us that this unfortunate confusion should be cleared once and for all. This we have achieved in Section 7.3 by showing that our resolution (2.107) is completely equivalent to Sommerfeld's and that there is a contribution from the pole quite irrespective of the choice of cuts.

III. FIELD COMPONENTS IN CYLINDRICAL COORDINATES

In the preceding Chapter we have given the complete formulation of the two-medium boundary value problem for a horizontal electric dipole embedded in the conducting medium. In Section 2.4a we indicated that the Cartesian components of the π-vectors as well as the cylindrical components of the field vectors can be expressed in terms of a minimum number of fundamental integrals, which we now propose to establish in this Chapter.

3.1 CARTESIAN COMPONENTS OF THE HERTZIAN VECTORS

The Cartesian components of the Hertzian vectors, as first introduced by Eqs. (2.10), are given by the integral representations (2.54) through (2.57) which were obtained by a transformation into cylindrical coordinates in configuration and transform spaces. We now wish to exhibit these representations in terms of the fundamental integrals tabulated in Section 2.4a.

3.1a. The components π_{x1} and π_{z1} and their derivatives.-- The component π_{x1} for the conducting medium has the integral representation

(2.54) which can be compactly written as

$$\Pi_{x1} = \frac{ip}{4\pi k_1 \eta_1} \left\{ \Psi_1 - \Psi_2 + U_1 \right\} , \quad z \leq 0 , \tag{3.1}$$

by making use of Eqs. (2.65), (2.66) and (2.67). Similarly, the component Π_{z1}, which has the integral representation (2.56), can be rewritten as

$$\Pi_{z1} = \frac{ip \cos\phi}{4\pi k_1 \eta_1} \frac{\partial}{\partial \rho} \int_0^\infty g e^{\gamma_1 (z-h)} J_0(\lambda\rho) \lambda d\lambda = \frac{ip \cos\phi}{4\pi k_1 \eta_1} \frac{\partial}{\partial \rho} \int_h^\infty \left\{ V_1 - U_1 \right\} dh, \quad z \leq 0 \tag{3.2}$$

in which we have made use of Eqs. (2.59), (2.60), (2.67) and (2.68). Eq. (3.1) exhibits Π_{x1} in terms of the source function Ψ_1, the image function Ψ_2 and the fundamental integral U_1; while Eq. (3.2) gives Π_{z1} in terms of the two fundamental integrals U_1 and V_1 integrated with respect to h from the position of the image at $z = h$ to infinity. Thus, this latter integral may be interpreted as giving, for points of observation in the conducting medium, the contribution of a continuous distribution of images of strength $p dh$ located in medium (2) along the z-axis from $z = h$ to $z \to \infty$.

Other important representations are similarly obtained. Thus, from (3.2) we have at once for the derivative

$$\frac{\partial \Pi_{z1}}{\partial z} = \frac{ip \cos\phi}{4\pi k_1 \eta_1} \frac{\partial}{\partial \rho} \left\{ V_1 - U_1 \right\} , \quad z \leq 0 . \tag{3.3}$$

By making use of the important connection between the fundamental integrals U_1 and V_1, given in Eq. (2.71), we obtain from (3.2) the alternative form

$$k_1^2 \pi_{z1} = \frac{ip \cos\phi}{4\pi k_1 \eta_1} \frac{\partial^2}{\partial\rho\, \partial z} \left\{ 2\, \Psi_2 - (1 + n^2) V_1 \right\}, \quad z \leq 0; \tag{3.4}$$

wherein we note that in (3.3) and (3.4) we have made use of the fact that $\partial/\partial h = - \partial/\partial z$ when operating upon the factor $\exp\left\{\gamma_1(z - h)\right\}$ which is common to all the integrals for the conducting medium. Finally, with the aid of (3.1) and (3.3) we construct the divergence of the π-vector in medium (1),

$$\nabla \cdot \pi^{(1)} = \frac{\partial \pi_{x1}}{\partial x} + \frac{\partial \pi_{z1}}{\partial z} = \frac{ip \cos\phi}{4\pi k_1 \eta_1} \frac{\partial}{\partial \rho} \left\{ \Psi_1 - \Psi_2 + V_1 \right\}, \quad z \leq 0, \tag{3.5}$$

which completes our presentation as regards medium (1).

3.1b. The components π_{x2} and π_{z2} and their derivatives.-- Proceeding similarly for the non-conducting medium, we have at once from Eqs. (2.55), (2.59) and (2.69)

$$\pi_{x2} = \frac{ip}{4\pi k_2 \eta_2} U_2, \quad z \geq 0, \tag{3.6}$$

and from Eq. (2.57), making use of (2.61), (2.69) and (2.70), we obtain

$$\pi_{z2} = \frac{ip \cos\phi}{4\pi k_2 \eta_2} \frac{\partial}{\partial \rho} \int_0^\infty g e^{-\gamma_2 z - \gamma_1 h} J_0(\lambda\rho) \lambda d\lambda$$

$$= - \frac{ip \cos\phi}{4\pi k_2 \eta_2} \frac{\partial}{\partial \rho} \int_z^\infty \left\{ V_2 - U_2 \right\} dz, \quad z \geq 0, \tag{3.7}$$

in complete analogy with Eq. (3.2). From (3.7) we have for the derivative

$$\frac{\partial \Pi_{z2}}{\partial z} = \frac{ip\cos\phi}{4\pi k_2 \eta_2} \frac{\partial}{\partial \rho} \left\{ V_2 - U_2 \right\}, \quad z \geqslant 0 . \tag{3.8}$$

Proceeding as in the case of Eq. (3.4), we have the alternative form for the z component,

$$k_2^2 \Pi_{z2} = -\frac{ip\cos\phi}{4\pi k_2 \eta_2} \frac{\partial}{\partial \rho} \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial h} \right) V_2, \quad z \geqslant 0, \tag{3.9}$$

which is deduced from (3.7) by writing

$$k_2^2 \Pi_{z2} = -\frac{ip\cos\phi}{4\pi k_2 \eta_2} \frac{\partial^2}{\partial \rho\, \partial z} \int_0^\infty (k_2^2 g \gamma_2^{-1}) e^{-\gamma_2 z - \gamma_1 h} J_0(\lambda\rho) \lambda d\lambda, \quad z \geqslant 0, \tag{3.10}$$

and then making use of Eq. (2.63). Finally, resorting to Eqs. (3.6) and (3.8) we construct the divergence of the Π-vector in medium (2), obtaining

$$\nabla \cdot \Pi^{(2)} = \frac{\partial \Pi_{x2}}{\partial x} + \frac{\partial \Pi_{z2}}{\partial z} = \frac{ip\cos\phi_0}{4\pi k_2 \eta_2} \frac{\partial V_2}{\partial \rho}, \quad z \geqslant 0, \tag{3.11}$$

thus completing our presentation for medium (2).

3.2 VARIOUS FORMS OF THE ELECTRIC FIELD COMPONENTS

The electric field components are readily derived from Eqs. (2.5) with the aid of the results presented in the last Section. Thus, exhibiting the Π-vector in terms of its components we have, in general

$$\vec{\Pi} = e_x \Pi_x + e_z \Pi_z = e_r \Pi_x \cos\phi - e_\phi \Pi_x \sin\phi + e_z \Pi_z , \tag{3.12}$$

the last form of which expresses the Π-vector in terms of its cylindrical components. Accordingly, from the first of Eqs. (2.5), we have for the cylindrical components of the electric field intensity the general expressions

$$E_\rho = \frac{\partial}{\partial \rho}(\nabla \cdot \Pi) + k^2 \Pi_\rho = \frac{\partial}{\partial \rho}(\nabla \cdot \Pi) + k^2 \Pi_x \cos\phi \; ;$$

$$E_\phi = \frac{1}{\rho}\frac{\partial}{\partial \phi}(\nabla \cdot \Pi) + k^2 \Pi_\phi = \frac{1}{\rho}\frac{\partial}{\partial \phi}(\nabla \cdot \Pi) - k^2 \Pi_x \sin\phi \; ; \qquad (3.13)$$

$$E_z = \frac{\partial}{\partial z}(\nabla \cdot \Pi) + k^2 \Pi_z \, .$$

<u>3.2a. Electric field components for medium (1).</u>-- The electric field components for the conducting medium, $z \leq 0$, are now written down from (3.13) by making use of Eqs. (3.1), (3.4) and (3.5),

$$E_{\rho 1} = \frac{ip\cos\phi}{4\pi k_1 \eta_1}\left\{\frac{\partial^2}{\partial \rho^2}\left[\Psi_1 - \Psi_2 + V_1\right] + k_1^2\left[\Psi_1 - \Psi_2 + U_1\right]\right\} \; ; \quad (3.14)$$

$$E_{\phi 1} = -\frac{ip\sin\phi}{4\pi k_1 \eta_1}\left\{\frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\Psi_1 - \Psi_2 + V_1\right] + k_1^2\left[\Psi_1 - \Psi_2 + U_1\right]\right\} \; ; \quad (3.15)$$

$$E_{z1} = \frac{ip\cos\phi}{4\pi k_1 \eta_1}\left\{\frac{\partial^2}{\partial z \partial \rho}\left[\Psi_1 + \Psi_2 - n^2 V_1\right]\right\} , \qquad (3.16)$$

which exhibit the electric field components in medium (1) in terms of the source function Ψ_1, the image function Ψ_2 and our fundamental integrals U_1 and V_1.

Other forms of the electric field components can be readily derived from the above equations by making various transformations. Thus, for example, a useful transformation consists of eliminating entirely the integral U_1 by making use of the important connection given in Eq. (2.71) between our fundamental integrals U_1 and V_1. In this way we obtain for the transverse components of electric intensity,

$$E_{\rho 1} = -\frac{ip\cos\phi}{4\pi k_1 \eta_1}\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\Psi_1 - \Psi_2 + V_1\right] + \frac{\partial^2}{\partial z^2}\left[\Psi_1 + \Psi_2 - n^2 V_1\right]\right\}; \qquad (3.14a)$$

$$E_{\phi 1} = -\frac{ip\sin\phi}{4\pi k_1 \eta_1}\left\{\left(\frac{1}{\rho}\frac{\partial}{\partial\rho} + k_1^2\right)\left[\Psi_1 - \Psi_2 + V_1\right] + \frac{\partial^2}{\partial z^2}\left[(1+n^2)V_1 - 2\Psi_2\right]\right\}, \qquad (3.15a)$$

in which use has been made of the fact that V_1, Ψ_1 and Ψ_2 are axially symmetric solutions of the scalar Helmholtz equation, while recognizing that Ψ_1 is otherwise singular at the source.

Another useful transformation is obtained by noting that our fundamental integral U_1, Eq. (2.67), can be resolved as follows:

$$k_1^2 U_1 = k_1^2 M_1 - \frac{2}{1-n^2}\frac{\partial^2 \Psi_2}{\partial z^2}, \quad z \leq 0, \qquad (3.17)$$

in terms of a new integral, M_1, which is defined as

$$M_1 = \frac{2}{k_1^2 - k_2^2}\int_0^\infty \gamma_2 e^{\gamma_1(z-h)} J_o(\lambda\rho)\lambda d\lambda, \quad z \leq 0. \qquad (3.18)$$

The resolution (3.17) is quickly obtained by noting that the amplitude function $f(\lambda)$ in Eq. (2.67) can be written as

$$f(\lambda) = \frac{2}{\gamma_1 + \gamma_2} = \frac{2(\gamma_2 - \gamma_1)}{\gamma_2^2 - \gamma_1^2} = \frac{2(\gamma_2 - \gamma_1)}{k_1^2 - k_2^2}, \tag{3.19}$$

where use has been made of Eqs. (2.58). Replacing $k_1^2 U_1$ by (3.17), we now obtain for the transverse components of electric intensity for $z \leq 0$,

$$E_{\rho 1} = - \frac{ip \cos\phi}{4\pi k_1 \eta_1} \left\{ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left[\Psi_1 - \Psi_2 + V_1 \right] + \frac{n^2 k_1^2}{1 + n^2} \left[V_1 - M_1 \right] + \frac{\partial^2}{\partial z^2} \left[\Psi_1 + \frac{1 + n^4}{1 - n^4} \Psi_2 \right] \right\}; \tag{3.14b}$$

$$E_{\phi 1} = - \frac{ip \sin\phi}{4\pi k_1 \eta_1} \left\{ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left[\Psi_1 - \Psi_2 + V_1 \right] + k_1^2 \left[\Psi_1 - \Psi_2 + M_1 \right] - \frac{2}{1 - n^2} \frac{\partial^2 \Psi_2}{\partial z^2} \right\}, \tag{3.15b}$$

in which, as before, use has been made of the fact that Ψ_2 is an axially symmetric solution of the scalar Helmholtz equation.

<u>3.2b. Electric field components for medium (2)</u>.- The electric field components for the non-conducting medium, $z \geq 0$, are likewise written down from (3.13). Making use of Eqs. (3.6), (3.9) and (3.11) we have

$$E_{\rho 2} = \frac{ip \cos\phi}{4\pi k_2 \eta_2} \left\{ \frac{\partial^2 V_2}{\partial \rho^2} + k_2^2 U_2 \right\}; \tag{3.20}$$

$$E_{\phi 2} = - \frac{ip \sin\phi}{4\pi k_2 \eta_2} \left\{ \frac{1}{\rho} \frac{\partial V_2}{\partial \rho} + k_2^2 U_2 \right\}; \tag{3.21}$$

$$E_{z2} = \frac{ip \cos\phi}{4\pi k_2 \eta_2} \frac{\partial^2 V_2}{\partial h \, \partial \rho}, \tag{3.22}$$

which express the electric field components for medium (2) in terms of our fundamental integrals U_2 and V_2.

Making use of Eq. (2.71a) and noting that V_2 is an axially symmetric solution to the scalar Helmholtz equation, we obtain for the transverse components of the electric intensity, similar to Eqs. (3.14a) and (3.15a),

$$E_{\rho 2} = - \frac{ip \cos\phi}{4\pi k_2 \eta_2} \left\{ \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z \partial h} \right\} V_2 \; ; \qquad (3.20a)$$

$$E_{\phi 2} = - \frac{ip \sin\phi}{4\pi k_2 \eta_2} \left\{ \frac{1}{\rho} \frac{\partial}{\partial \rho} + k_2^2 + \frac{\partial}{\partial z} \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial h} \right) \right\} V_2 \; , \qquad (3.21a)$$

thus indicating that besides the source and image functions it is possible in theory to express the electric field in both medium (1) and medium (2) in terms of just two functions, V_1 and V_2. In the next Section it will be shown that these same two functions together with the source and image functions are also sufficient to determine the magnetic field everywhere; thus, the entire field is determined by V_1, V_2, Ψ_1, and Ψ_2.

3.3 VARIOUS FORMS OF THE MAGNETIC FIELD COMPONENTS

The magnetic field components are likewise computed from Eqs. (2.5) with the aid of the results presented in Section 3.1. Accordingly, we have in general from Eq. (3.12),

$$H_\rho = -ik\eta\left\{\sin\phi\,\frac{\partial\Pi_z}{\partial z} + \frac{1}{\rho}\frac{\partial\Pi_z}{\partial\phi}\right\};$$

$$H_\phi = -ik\eta\left\{\cos\phi\,\frac{\partial\Pi_x}{\partial z} - \frac{\partial\Pi_z}{\partial\rho}\right\}; \qquad (3.23)$$

$$H_z = ik\eta\sin\phi\,\frac{\partial\Pi_x}{\partial\rho},$$

which exhibit the cylindrical components of magnetic field intensity.

3.3a. Magnetic field components for medium (1).-- The magnetic field components for the conducting medium are now written down from Eqs. (3.23) by making use of Eqs. (3.1) and (3.4). We thus have for $z \leq 0$,

$$H_{\rho 1} = \frac{p\sin\phi}{4\pi}\frac{\partial}{\partial z}\left\{\Psi_1 - \Psi_2 + U_1 + \frac{1}{k_1^2\rho}\frac{\partial}{\partial\rho}\left[(1+n^2)V_1 - 2\Psi_2\right]\right\}; \quad (3.24)$$

$$H_{\phi 1} = \frac{p\cos\phi}{4\pi}\frac{\partial}{\partial z}\left\{\Psi_1 - \Psi_2 + U_1 + \frac{1}{k_1^2}\frac{\partial^2}{\partial\rho^2}\left[(1+n^2)V_1 - 2\Psi_2\right]\right\}; \quad (3.25)$$

$$H_{z1} = -\frac{p\sin\phi}{4\pi}\frac{\partial}{\partial\rho}\left\{\Psi_1 - \Psi_2 + U_1\right\}, \qquad (3.26)$$

which exhibit the magnetic field components in medium (1) in terms of the source function Ψ_1, the image function Ψ_2 and our fundamental integrals U_1 and V_1.

As in the case of the electric field components for the conduction medium, it is possible to express the above results exclusively in terms of Ψ_1, Ψ_2 and V_1, by eliminating U_1 with the aid of Eq. (2.71), thus obtaining for $z \leq 0$,

$$H_{\rho 1} = \frac{p \sin\phi}{4\pi k_1^2} \frac{\partial}{\partial z}\left\{\left(\frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2}\right)\left[(1 + n^2)V_1 - 2\Psi_2\right] + k_1^2\left[\Psi_1 - \Psi_2 + V_1\right]\right\}; \quad (3.24a)$$

$$H_{\phi 1} = \frac{p \cos\phi}{4\pi k_1^2} \frac{\partial}{\partial z}\left\{\frac{1}{\rho}\frac{\partial}{\partial \rho}\left[2\Psi_2 - (1 + n^2)V_1\right] + k_1^2\left[\Psi_1 + \Psi_2 - n^2 V_1\right]\right\}; \quad (3.25a)$$

$$H_{z1} = -\frac{p \sin\phi}{4\pi k_1^2} \frac{\partial}{\partial \rho}\left\{\frac{\partial^2}{\partial z^2}\left[(1 + n^2)V_1 - 2\Psi_2\right] + k_1^2\left[\Psi_1 - \Psi_2 + V_1\right]\right\}, \quad (3.26a)$$

in the derivation of which we have made use of the fact that the functions Ψ_2 and V_1 satisfy the homogeneous Helmholtz equation for $z \leq 0$.

Finally, making use of Eq. (3.17), which expresses the fundamental integral in terms of the new integral M_1, Eq. (3.18), we obtain still another useful form for the magnetic field components for $z \leq 0$, namely,

$$H_{\rho 1} = \frac{p \sin\phi}{4\pi k_1^2} \frac{\partial}{\partial z}\left\{\frac{1}{\rho}\frac{\partial}{\partial \rho}\left[(1 + n^2)V_1 - 2\Psi_2\right] + k_1^2\left[\Psi_1 - \Psi_2 + M_1\right] - \frac{2}{1 - n^2}\frac{\partial^2 \Psi_2}{\partial z^2}\right\}; \quad (3.24b)$$

$$H_{\phi 1} = \frac{p \cos\phi}{4\pi k_1^2} \frac{\partial}{\partial z}\left\{\frac{1}{\rho}\frac{\partial}{\partial \rho}\left[2\Psi_2 - (1 + n^2)V_1\right] + k_1^2\left[\Psi_1 + \Psi_2 - n^2 V_1\right]\right\}; \quad (3.25b)$$

$$H_{z1} = -\frac{p \sin\phi}{4\pi k_1^2} \frac{\partial}{\partial \rho}\left\{k_1^2\left[\Psi_1 - \Psi_2 + M_1\right] - \frac{2}{1 - n^2}\frac{\partial^2 \Psi_2}{\partial z^2}\right\}, \quad (3.26b)$$

in which again use has been made of the fact that the function Ψ_2 is an axially symmetric solution of the scalar Helmholtz equation.

3.3b. Magnetic field components for medium (2).- The magnetic field components for the non-conducting medium are similarly written down from (3.23). Making use of Eqs. (3.6) and (3.9) we have for $z \geq 0$,

$$H_{\rho 2} = \frac{p \sin\phi}{4\pi} \left\{ \frac{\partial U_2}{\partial z} + \frac{1}{k_2^2 \rho} \frac{\partial}{\partial \rho} \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial h} \right) V_2 \right\} ; \tag{3.27}$$

$$H_{\phi 2} = \frac{p \cos\phi}{4\pi} \left\{ \frac{\partial U_2}{\partial z} + \frac{1}{k_2^2} \frac{\partial^2}{\partial \rho^2} \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial h} \right) V_2 \right\} ; \tag{3.28}$$

$$H_{z2} = - \frac{p \sin\phi}{4\pi} \frac{\partial U_2}{\partial \rho} , \tag{3.29}$$

which exhibit the magnetic field components for medium (2) in terms of our fundamental integrals U_2 and V_2.

As in the case of the electric field components for the non-conducting medium, it is possible to obtain the magnetic field components for medium (2) in terms of the single fundamental integral V_2 by using Eq. (2.71a) and the fact that V_2 is an axially symmetric solution of the scalar Helmholtz equation; thus,

$$H_{\rho 2} = \frac{p \sin\phi}{4\pi k_2^2} \left\{ k_2^2 \frac{\partial}{\partial z} + \left(\frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} \right) \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial h} \right) \right\} V_2 ; \tag{3.27a}$$

$$H_{\phi 2} = \frac{p \cos\phi}{4\pi k_2^2} \left\{ k_2^2 \frac{\partial}{\partial h} - \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial h} \right) \right\} V_2 ; \tag{3.28a}$$

$$H_{z2} = -\frac{p \sin\phi}{4\pi k_2^2} \frac{\partial}{\partial \rho} \left\{ k_2^2 + \frac{\partial}{\partial z} \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial h} \right) \right\} V_2 , \qquad (3.29a)$$

which completes our demonstration that the entire solution may be given in terms of Ψ_1, Ψ_2, V_1 and V_2.

IV. SOLUTION OF DIPOLAR PROBLEM IN THE STATIC LIMIT

An important special case of the present two-medium problem arises when we consider the static limit $(\omega \rightarrow 0)$ of the preceding results. It is shown that our integral representations converge uniformly to the static solution as $\omega \rightarrow 0$, thus affording a partial check on the whole formulation since the static problem can be solved independently by elementary methods. Furthermore, the study of the static case has clarified the rôle played by the interface separating the two media which, as shown below, constitutes a source of secondary waves. In this Chapter we deduce the limiting form of our fundamental integrals by letting $\omega \rightarrow 0$, process which is simplified by assuming that σ_2 is finite. Finally, we consider the independent solution of the static problem which is elementary as far as the electric field is concerned, but which is considerably more involved in the case of the magnetic field.

4.1 STATIC LIMIT OF THE FUNDAMENTAL INTEGRALS

Referring to Section 2.4a, we establish readily by letting $\omega \to 0$, which implies $k_1\eta_1 \to i\sigma_1$ and $k_2\eta_2 \to i\sigma_2$ with $k_1 = k_2 = 0$, the following limiting forms:

$$\Psi_1 = U_2 = \frac{1}{R_1}; \tag{4.1}$$

$$\Psi_2 = U_1 = \frac{1}{R_2}; \tag{4.2}$$

$$V_1 = \frac{2\sigma_1}{\sigma_1 + \sigma_2}\frac{1}{R_2}; \tag{4.3}$$

$$V_2 = \frac{2\sigma_2}{\sigma_1 + \sigma_2}\frac{1}{R_1}, \tag{4.4}$$

all of which follow immediately from the corresponding integral representations.

4.2 CARTESIAN COMPONENTS OF THE HERTZIAN VECTORS

We now proceed to establish the limiting forms in the static case for the Cartesian components of the Π-vectors. Considering first the conducting medium, we obtain immediately from Eq. (3.1)

$$\Pi_{x1} = \frac{p}{4\pi\sigma_1}\frac{1}{R_1}, \quad z \leq 0, \tag{4.5}$$

where we have made use of Eqs. (4.1) and (4.2). Similarly, making use of (4.2) and (4.3), we have at once from the second integral in (3.2)

$$\Pi_{z1} = \frac{p\cos\phi}{4\pi\sigma_1}\,\frac{\sigma_1-\sigma_2}{\sigma_1+\sigma_2}\,\frac{\partial}{\partial\rho}\int_h^\infty \frac{dh'}{\left[\rho^2+(z-h')^2\right]^{\frac{1}{2}}}, \quad z \leq 0, \tag{4.6}$$

which, after carrying out the integration, may be written in the convenient form

$$\Pi_{z1} = -\frac{p\cos\phi}{4\pi\sigma_1}\,\frac{\sigma_1-\sigma_2}{\sigma_1+\sigma_2}\,\frac{\partial}{\partial z}\left\{\frac{R_2+(z-h)}{\rho}\right\}, \quad z \leq 0. \tag{4.7}$$

Then, either from (4.7) or else taking the limiting form of (3.3), we have

$$\frac{\partial\Pi_{z1}}{\partial z} = \frac{p\cos\phi}{4\pi\sigma_1}\,\frac{\sigma_1-\sigma_2}{\sigma_1+\sigma_2}\,\frac{\partial}{\partial\rho}\,\frac{1}{R_2}, \quad z \leq 0, \tag{4.8}$$

with the aid of which, together with (4.5), we have for the divergence

$$\nabla\cdot\Pi^{(1)} = \frac{\partial\Pi_{x1}}{\partial x} + \frac{\partial\Pi_{z1}}{\partial z} = \frac{p\cos\phi}{4\pi\sigma_1}\,\frac{\partial}{\partial\rho}\left\{\frac{1}{R_1} + \frac{\sigma_1-\sigma_2}{\sigma_1+\sigma_2}\,\frac{1}{R_2}\right\}, \quad z \leq 0, \tag{4.9}$$

which completes our presentation of the static limit for the components of the Π-vector and their derivatives in the conducting medium.

Proceeding similarly for the non-conducting medium, we have at once from Eqs. (3.6) and (4.1)

$$\Pi_{x2} = \frac{p}{4\pi\sigma_2}\,\frac{1}{R_1}, \quad z \geq 0; \tag{4.10}$$

and, in complete analogy with (4.7), making use of Eqs. (3.7), (4.1) and (4.4),

$$\Pi_{z2} = \frac{p\cos\phi}{4\pi\sigma_2}\frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2}\frac{\partial}{\partial z}\left\{\frac{R_1 - (z + h)}{\rho}\right\}, \quad z \geqslant 0, \tag{4.11}$$

from which we compute the derivative

$$\frac{\partial \Pi_{z2}}{\partial z} = -\frac{p\cos\phi}{4\pi\sigma_2}\frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2}\frac{\partial}{\partial\rho}\frac{1}{R_1}, \quad z \geqslant 0. \tag{4.12}$$

Combining the preceding results we have, finally, for the divergence

$$\nabla \cdot \Pi^{(2)} = \frac{\partial \Pi_{x2}}{\partial x} + \frac{\partial \Pi_{z2}}{\partial z} = \frac{p\cos\phi}{4\pi\sigma_2}\frac{2\sigma_2}{\sigma_1 + \sigma_2}\frac{\partial}{\partial\rho}\frac{1}{R_1}, \quad z \geqslant 0, \tag{4.13}$$

with which we complete our presentation of the static limit for the components of the Π-vector and their derivatives in the non-conducting medium.

4.3 FIELD COMPONENTS IN CYLINDRICAL COORDINATES

We have already obtained in Section 3.2 various forms of the electric field components in cylindrical coordinates in terms of our fundamental integrals whose limiting forms in the static case are listed in Section 4.1. Inserting these limiting forms in lieu of the fundamental integrals in the corresponding equations, we have for $z \leqslant 0$

$$E_{\rho 1} = \frac{p\cos\phi}{4\pi\sigma_1}\frac{\partial^2}{\partial\rho^2}\left\{\frac{1}{R_1} + \frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2}\frac{1}{R_2}\right\}; \tag{4.14}$$

$$E_{\phi 1} = -\frac{p \sin\phi}{4\pi\sigma_1} \frac{1}{\rho} \frac{\partial}{\partial\rho} \left\{ \frac{1}{R_1} + \frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2} \frac{1}{R_2} \right\} ; \tag{4.15}$$

$$E_{z1} = \frac{p \cos\phi}{4\pi\sigma_1} \frac{\partial^2}{\partial z \partial\rho} \left\{ \frac{1}{R_1} + \frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2} \frac{1}{R_2} \right\} , \tag{4.16}$$

which exhibit the static limit of the cylindrical components of the electric field intensity in the conducting medium. It may be seen from the first of Eqs. (2.5) that in the static limit

$$E = \nabla \nabla \cdot \Pi = -\nabla \psi ; \qquad \psi = -\nabla \cdot \Pi , \tag{4.17}$$

where ψ is the scalar potential. Clearly, then, we see from the above equations that for $z \leq 0$ the electrostatic potential becomes

$$\psi_1 = -\nabla \cdot \Pi^{(1)} = -\frac{p \cos\phi}{4\pi\sigma_1} \frac{\partial}{\partial\rho} \left\{ \frac{1}{R_1} + \frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2} \frac{1}{R_2} \right\} , \tag{4.18}$$

which yields directly, using (4.17), the cylindrical components listed above.

Proceeding likewise for $z \geq 0$, we have for the cylindrical components of the electric field intensity in the non-conducting medium the limiting expressions

$$E_{\rho 2} = \frac{2p \cos\phi}{4\pi(\sigma_1 + \sigma_2)} \frac{\partial^2}{\partial\rho^2} \frac{1}{R_1} ; \tag{4.19}$$

$$E_{\phi 2} = -\frac{2p \sin\phi}{4\pi(\sigma_1 + \sigma_2)} \frac{1}{\rho} \frac{\partial}{\partial\rho} \frac{1}{R_1} ; \tag{4.20}$$

$$E_{z2} = \frac{2p\cos\phi}{4\pi(\sigma_1 + \sigma_2)} \frac{\partial^2}{\partial z \partial \rho} \frac{1}{R_1}, \qquad (4\cdot21)$$

which are seen to remain finite as $\sigma_2 \to 0$, the actual case for the non-conducting medium. Finally, as in the preceding paragraph, we deduce from the above equations or directly from (4·13) that the electrostatic potential in medium (2) is given by

$$\psi_2 = -\nabla\cdot\Pi^{(2)} = -\frac{2p\cos\phi}{4\pi(\sigma_1 + \sigma_2)} \frac{\partial}{\partial \rho} \frac{1}{R_1}. \qquad (4\cdot22)$$

The computation of the static limit for the electric field components given in the preceding paragraphs can be carried out either by substituting the limiting forms of the fundamental integrals in the expressions of the electric field components deduced in the general alternating case or else by computing the components anew from Eqs. (3·13) and making use of the limiting forms for the Cartesian components of the Π-vectors displayed in Section 4·2. The same two possibilities do not occur, however, in the case of the transverse magnetic field components; for, as seen by Eqs. (3·24), (3·25), (3·27) and (3·28), there are terms containing k_1^{-2} or k_2^{-2} which makes it necessary to examine the static limit more carefully.

In consequence, as far as the static limit of the magnetic field components is concerned, it is more expedient to make use of Eqs. (3·23) to compute the magnetic field components directly in terms of the static limiting forms for the Cartesian components of the Π-vectors as exhibited in Section 4·2. Thus we have for the conducting medium, $z \leq 0$, the limiting forms

$$H_{\rho 1} = \frac{p \sin\phi}{4\pi} \frac{\partial}{\partial z}\left\{\frac{1}{R_1} + \frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2} \frac{R_2 + (z - h)}{\rho^2}\right\} ; \qquad (4.23)$$

$$H_{\phi 1} = \frac{p \cos\phi}{4\pi} \frac{\partial}{\partial z}\left\{\frac{1}{R_1} + \frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2}\left[\frac{1}{R_2} - \frac{R_2 + (z - h)}{\rho^2}\right]\right\} ; \qquad (4.24)$$

$$H_{z1} = -\frac{p \sin\phi}{4\pi} \frac{\partial}{\partial \rho} \frac{1}{R_1} ; \qquad (4.25)$$

and for the non-conducting medium, $z \geqslant 0$, the corresponding limiting forms are

$$H_{\rho 2} = \frac{p \sin\phi}{4\pi} \frac{\partial}{\partial z}\left\{\frac{1}{R_1} - \frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2} \frac{R_1 - (z + h)}{\rho^2}\right\} ; \qquad (4.26)$$

$$H_{\phi 2} = \frac{p \cos\phi}{4\pi} \frac{\partial}{\partial z}\left\{\frac{1}{R_1} - \frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2}\left[\frac{1}{R_1} - \frac{R_1 - (z + h)}{\rho^2}\right]\right\} ; \qquad (4.27)$$

$$H_{z2} = -\frac{p \sin\phi}{4\pi} \frac{\partial}{\partial \rho} \frac{1}{R_1} . \qquad (4.28)$$

Putting $\sigma_2 = 0$ in the above equations leads to considerable simplification in our results. In this important practical case, we have for the electric field components in the conducting medium for $z \leqslant 0$

$$E_{\rho 1} = \frac{p \cos\phi}{4\pi\sigma} \frac{\partial^2}{\partial \rho^2}\left\{\frac{1}{R_1} + \frac{1}{R_2}\right\} ; \qquad (4.14a)$$

$$E_{\phi 1} = -\frac{p \sin\phi}{4\pi\sigma} \frac{1}{\rho} \frac{\partial}{\partial \rho} \left\{ \frac{1}{R_1} + \frac{1}{R_2} \right\} ; \tag{4.15a}$$

$$E_{z1} = \frac{p \cos\phi}{4\pi\sigma} \frac{\partial^2}{\partial z \partial \rho} \left\{ \frac{1}{R_1} + \frac{1}{R_2} \right\} , \tag{4.16a}$$

where we have written $\sigma_1 = \sigma$ for the conductivity of medium (1). Similarly we have for the electric field components in the non-conducting medium for $z \geq 0$

$$E_{\rho 2} = \frac{2p \cos\phi}{4\pi\sigma} \frac{\partial^2}{\partial \rho^2} \frac{1}{R_1} ; \tag{4.19a}$$

$$E_{\phi 2} = -\frac{2p \sin\phi}{4\pi\sigma} \frac{1}{\rho} \frac{\partial}{\partial \rho} \frac{1}{R_1} ; \tag{4.20a}$$

$$E_{z2} = \frac{2p \cos\phi}{4\pi\sigma} \frac{\partial^2}{\partial z \partial \rho} \frac{1}{R_1} . \tag{4.21a}$$

The magnetic field components become for the conducting medium, $z \leq 0$, with $\sigma_1 = \sigma$ and $\sigma_2 = 0$,

$$H_{\rho 1} = \frac{p \sin\phi}{4\pi} \frac{\partial}{\partial z} \left\{ \frac{1}{R_1} + \frac{R_2 + (z - h)}{\rho^2} \right\} ; \tag{4.23a}$$

$$H_{\phi 1} = \frac{p \cos\phi}{4\pi} \frac{\partial}{\partial z} \left\{ \frac{1}{R_1} + \frac{1}{R_2} - \frac{R_2 + (z - h)}{\rho^2} \right\} ; \tag{4.24a}$$

$$H_{z1} = -\frac{p \sin\phi}{4\pi} \frac{\partial}{\partial \rho} \frac{1}{R_1} ; \qquad (4.25a)$$

and for the non-conducting medium, $z \geq 0$, the corresponding magnetic field components are

$$H_{\rho 2} = \frac{p \sin\phi}{4\pi} \frac{\partial}{\partial z} \left\{ \frac{1}{R_1} - \frac{R_1 - (z + h)}{\rho^2} \right\} ; \qquad (4.26a)$$

$$H_{\phi 2} = \frac{p \cos\phi}{4\pi} \frac{\partial}{\partial z} \left\{ \frac{R_1 - (z + h)}{\rho^2} \right\} ; \qquad (4.27a)$$

$$H_{z2} = -\frac{p \sin\phi}{4\pi} \frac{\partial}{\partial \rho} \frac{1}{R_1} . \qquad (4.28a)$$

It is important to note that the magnetic field components, as given above, are independent of the conductivity σ, which of course is not true of the electric field. This indicates that the body currents that flow throughout the conducting medium do not contribute to the observed magnetic field in the present case of $\sigma_1 = \sigma$ and $\sigma_2 = 0$. The proof of this statement is given in Section 4.5.

4.4 INDEPENDENT SOLUTION FOR THE ELECTRIC FIELD

In the preceding sections we have examined the static limit of the results for the dipolar problem in the general alternating case as $\omega \to 0$. As a check on the previous work and because it proves illuminating, we now consider the independent solution of the static problem. We observe, first, that in the static case the electric and magnetic fields are uncoupled; that is, we can determine the electric field everywhere independently of any knowledge of the magnetic field. Thus, consider our isotropic and homogeneous conductor of infinite extent for which the Maxwellian equations (2.4) become in the static limit $(k = 0, \; k\eta = i\sigma)$

$$\begin{aligned} &\text{I.} \quad \nabla \times E = 0 && \text{III.} \quad \nabla \cdot H = 0 \\ &\text{II.} \quad \nabla \times H - \sigma E = J^{o} && \text{IV.} \quad \nabla \cdot E = -(\nabla \cdot J^{o})/\sigma \end{aligned} \qquad (4.29)$$

where J^{o} denotes the prescribed current density distribution characterizing the source. In the present instance, the static dipolar source is still expressed as in Eq. (2.9); that is,

$$J^{o} = e_x p \, \delta(x) \, \delta(y) \, \delta(z + h) \, , \qquad (4.30)$$

which describes an elementary current element in the x direction located at $(0,0,-h)$ and where p, the dipole moment of the current source, is still defined by Eq. (2.8).

We now introduce the scalar potential ψ and the Hertzian vector Π, related to each other by the so-called Lorentz condition,

$$\psi = -\nabla \cdot \Pi \, , \qquad (4.31)$$

which we already referred to in Eq. (4.17). In terms of (4.31), we have from I and III of (4.29)

$$E = -\nabla \psi = \nabla \nabla \cdot \Pi \quad \text{and} \quad H = \sigma \nabla \times \Pi \, . \qquad (4.32)$$

Making use of the expressions for the electric and magnetic field vectors in II and IV of Eqs. (4.29), we deduce readily that the Hertzian vector Π and the scalar potential ψ satisfy respectively the following vector and scalar Poisson's equations:

$$\nabla^2 \Pi = -J^o/\sigma \quad \text{and} \quad \nabla^2 \psi = (\nabla \cdot J^o)/\sigma . \tag{4.33}$$

In the present instance J^o is given by (4.30), from which $\nabla \cdot J^o$ becomes simply

$$\nabla \cdot J^o = p\ \delta'(x)\ \delta(y)\ \delta(z+h) , \tag{4.34}$$

where $\delta'(x)$ denotes the derivative of Dirac's delta function. Thus, in this case the scalar potential satisfies the Poisson's equation

$$\nabla^2 \psi = (p/\sigma)\ \delta'(x)\ \delta(y)\ \delta(z+h) , \tag{4.35}$$

which admits the particular integral

$$\psi = -\frac{p}{4\pi\sigma} \frac{\partial}{\partial x} \frac{1}{R_1} , \tag{4.36}$$

where R_1 denotes the distance from the elementary current dipole at (0,0,-h) to the point of observation (Fig. 1). It is seen that Eq. (4.36) has precisely the structure of the electrostatic potential of a dipole oriented in the x direction.

Restricting our attention to the electric field, we now have for the two-medium problem the following fundamental equations:

$$J_1 = \sigma_1 E_1 , \quad E_1 = -\nabla \psi_1 , \quad \nabla^2 \psi_1 = (p/\sigma)\ \delta'(x)\ \delta(y)\ \delta(z+h), \tag{4.37}$$

which apply in medium (1) for $z \leq 0$, and

$$J_2 = \sigma_2 E_2 , \quad E_2 = -\nabla \psi_2 , \quad \nabla^2 \psi_2 = 0 , \tag{4.38}$$

which apply in medium (2), $z \geq 0$, with finite conductivity σ_2. We note that, according to (4.36), the electrostatic potential ψ_1 can be written for $z \leq 0$ as

$$\psi_1 = \psi_1^o + \psi_1^1 , \tag{4.39}$$

where ψ_1^o is a particular integral of Poisson's equation (4.37), for which we take the solution given in (4.36), and ψ_1^1 is a solution of Laplace's equation. For $z \geq 0$ the electrostatic potential ψ_2 is merely a solution of Laplace's equation.

The boundary conditions in the present instance require the continuity of the tangential components of E and of the normal components of J upon crossing the interface at $z = 0$ separating the two media. In terms of the potentials, the boundary conditions read

$$\psi_1 = \psi_2 \quad \text{and} \quad \sigma_1 \frac{\partial \psi_1}{\partial z} = \sigma_2 \frac{\partial \psi_2}{\partial z} \quad \text{at} \quad z = 0 . \tag{4.40}$$

Next, resorting to the method of images, we postulate for the potentials ψ_1 and ψ_2, according to Eq. (4.39), the expressions

$$\psi_1 = - \frac{1}{4\pi\sigma_1} \frac{\partial}{\partial x} \left\{ \frac{p}{R_1} + \frac{p'}{R_2} \right\} , \quad z \leq 0 ; \tag{4.41}$$

$$\psi_2 = - \frac{1}{4\pi\sigma_2} \frac{\partial}{\partial x} \frac{p''}{R_1} , \quad z \geq 0 , \tag{4.42}$$

where p' and p'' are the dipole moments of image current sources located at (0,0,h) and (0,0,-h) respectively (see Fig. 1) and whose values are to be determined from the boundary conditions at $z = 0$. Thus, in this way we obtain

$$p' = \frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2} p \quad \text{and} \quad p'' = \frac{2\sigma_2}{\sigma_1 + \sigma_2} p , \tag{4.43}$$

with which the solution of the electrostatic problem yields for the potentials

$$\psi_1 = - \frac{p}{4\pi\sigma_1} \frac{\partial}{\partial x} \left\{ \frac{1}{R_1} + \frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2} \frac{1}{R_2} \right\} , \quad z \leq 0 , \tag{4.44}$$

and

$$\psi_2 = - \frac{2p}{4\pi(\sigma_1 + \sigma_2)} \frac{\partial}{\partial x} \frac{1}{R_1} , \quad z \geq 0 , \tag{4.45}$$

which are seen to be identical to Eqs. (4.18) and (4.22), respectively, after putting $\partial/\partial x = \cos\phi(\partial/\partial\rho)$. Thus, we have shown that the static solution for the electric field obtained independently by the method of images coincides exactly, as it should, with the limiting forms of the alternating solution obtained by letting $\omega \rightarrow 0$.

The special case $\sigma_2 = 0$ offers no difficulty, for it is readily appreciated that in Eq. (4.42) the ratio p''/σ_2 is independent of the conductivity of medium (2). In this important case, $\sigma_2 = 0$, the electric field may be deduced from the potentials

$$\psi_1 = - \frac{p}{4\pi\sigma} \frac{\partial}{\partial x} \left\{ \frac{1}{R_1} + \frac{1}{R_2} \right\} , \quad z \leq 0 , \tag{4.44a}$$

and

$$\psi_2 = - \frac{p}{4\pi o} \frac{\partial}{\partial x} \frac{2}{R_1} , \quad z \geq 0 , \tag{4.45a}$$

where we have written $\sigma_1 = \sigma$. These results admit the following interpretation: the potential in the conducting medium, $z \leq 0$, is that due to the source current dipole at $(0,0,-h)$ and an equal current dipole located at the image point $(0,0,h)$ in medium (2), whereas the potential in the non-conducting medium is that of a current source of twice the dipole moment of the source located at $(0,0,-h)$ in medium (1). The electric fields deduced from Eqs. (4.44a) and (4.45a) have the cylindrical components listed in Eqs. (4.14a), (4.19a) *et seq*. The structure of the electric field in a vertical plane containing the source dipole is readily deduced from the above considerations. It is seen that, in accordance with (4.44a) and the boundary conditions, the lines of electric intensity in medium (1) and, hence, the lines of current density are purely tangential at the interface $z = 0$; while the lines of electric intensity in medium (2) have a normal as well as a tangential component at $z = 0$, the normal component terminating on the surface charge density appearing at the interface between the two media.

4.5 INDEPENDENT SOLUTION FOR THE MAGNETIC FIELD

The solution for the electric field in the static limit has furnished us the complete current distribution set up throughout both media by the elementary dipole current embedded in medium (1); and, in principle, we should be able to compute the magnetic field directly from the knowledge of the current distribution. To this end we note that it is more convenient to deal with the vector potential A which we define from (4.31) and (4.32) in terms of our old Hertzian vector as follows:

$$A = \sigma \Pi, \qquad \nabla \cdot A = -\sigma \psi, \qquad H = \nabla \times A. \tag{4.46}$$

In consequence of this definition, we note from Eqs. (4.5) and (4.10) that the x components of the vector potential in each medium become merely

$$A_{x1} = \sigma_1 \pi_{x1} = p/4\pi R_1 , \quad z \leq 0 , \tag{4.47}$$

and

$$A_{x2} = \sigma_2 \pi_{x2} = p/4\pi R_1 , \quad z \geq 0 , \tag{4.48}$$

which are one and the same expression for both media. Therefore, we conclude that, irrespective of the presence of the boundary separating the two media of different conductivities, we can write the x component of the vector potential <u>everywhere</u> as

$$A_x = p/4\pi R_1 , \tag{4.49}$$

where R_1 is the distance from the point of observation to the source dipole in medium (1). Noting that (4.49) is the particular integral of the Poisson's equation

$$\nabla^2 A_x = - p\, \delta(x)\, \delta(y)\, \delta(z + h) \tag{4.50}$$

for the <u>unbounded</u> medium, we conclude from (4.49) and (4.50) that the elementary current dipole gives rise, all by itself, to the magnetic field derived from A_x.

It follows, therefore, that the remainder of the magnetic field which we associate with the z components of the π-vectors and, hence, with A_z must be due to the current distribution set up throughout the two conducting media. In fact, we have from Eqs. (4.7) and (4.11) that the z components of the vector potential in each medium can be written as

$$A_{z1} = \sigma_1 \pi_{z1} = -\frac{p\cos\phi}{4\pi}\frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2}\frac{1}{\rho}\left\{1 - \frac{h - z}{R_2}\right\}, \quad z \leq 0, \tag{4.51}$$

and

$$A_{z2} = \sigma_2 \pi_{z2} = -\frac{p\cos\phi}{4\pi}\frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2}\frac{1}{\rho}\left\{1 - \frac{h + z}{R_1}\right\}, \quad z \geq 0, \tag{4.52}$$

where R_1 and R_2 are defined by Eqs. (2.1). It is noted that the two preceding results can be combined into a single expression for A_z, valid <u>everywhere</u>, by merely writing

$$A_z = -\frac{p\cos\phi}{4\pi}\frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2}\frac{1}{\rho}\left\{1 - \frac{h + |z|}{\left[\rho^2 + (h + |z|)^2\right]^{\frac{1}{2}}}\right\}, \tag{4.53}$$

which says that A_z is an even function of z completely symmetric about the plane $z = 0$ separating the two media.

We now propose to derive A_z, as given by Eq. (4.53), directly from the current distribution. To this end we observe first that the two media may be conveniently regarded as a <u>single</u> medium of discontinuous conductivity $\sigma(z)$ which is expressed by

$$\sigma(z) = \sigma_1 - (\sigma_1 - \sigma_2)u(z) = \begin{cases} \sigma_1, & z \leq 0 \\ \sigma_2, & z \geq 0 \end{cases} \tag{4.54}$$

where $u(z)$ denotes the unit step function defined as zero for negative argument and unity for positive argument. Thus, we have

$$\nabla \sigma = -(\sigma_1 - \sigma_2)\,\delta(z)\,e_z \tag{4.55}$$

as a surface vector singularity occurring at the interface $z = 0$ as a result of the sudden discontinuity in the conductivity of the medium. We observe, next, that the magnetostatic field everywhere, in accordance with II and III of Eqs. (4.29), is governed by the equations

$$\nabla \times H = J + J^o, \quad \nabla \cdot H = 0, \quad J = \sigma E, \tag{4.56}$$

in which J^o is the source current dipole in medium (1), Eq. (4.30), and $J = \sigma E$ is the known current distribution set up throughout the infinite medium of discontinuous conductivity; that is,

$$\text{for} \quad z \leq 0, \quad J = \sigma_1 E_1 = -\sigma_1 \nabla \psi_1,$$

and (4.57)

$$\text{for} \quad z \geq 0, \quad J = \sigma_2 E_2 = -\sigma_2 \nabla \psi_2,$$

where ψ_1 and ψ_2 are the known electrostatic potentials already given in Eqs. (4.44) and (4.45).

To obtain a solution of the first of Eqs. (4.56), we note that the total magnetic field may be resolved into the sum of two components, $H = H^o + H^*$,

$$\nabla \times H^o = J^o \quad \text{and} \quad \nabla \times H^* = J. \tag{4.58}$$

The first of Eqs. (4.58) has already been solved for J^o as given by Eq. (4.30); whence, placing $H^o = \nabla \times A^o$, $\nabla \cdot A^o = 0$, we note at once that $A^o \equiv e_x A_x$ where A_x is given by (4.49); that is, $H^o = \nabla \times (e_x A_x)$ $= -e_x \times \nabla A_x$, and the magnetic field due to the source dipole is everywhere perpendicular to the x axis. The second of Eqs. (4.58) may now be solved by putting, from (4.46),

$$H' = \nabla \times A', \quad \nabla \cdot A' = -\sigma\psi = \begin{cases} -\sigma_1 \psi_1, & z \leq 0 \\ -\sigma_2 \psi_2, & z \geq 0 \end{cases} \tag{4.59}$$

from which we deduce that the vector potential A' must satisfy the vector Poisson's equation

$$\nabla \times \nabla \times A' = J = -\sigma \nabla \psi ; \tag{4.60}$$

or, making use of the vector identity $\nabla \times \nabla \times A' = \nabla \nabla \cdot A' - \nabla^2 A'$ and of the divergence condition in (4.59), we see that the vector potential A' becomes $A' = e_z A_z$ where A_z is a solution of the scalar Poisson's equation

$$\nabla^2 A_z = (\sigma_1 - \sigma_2)\, \delta(z)\, \psi , \tag{4.61}$$

which was derived by noting from the divergence condition that

$$\nabla \nabla \cdot A' = \nabla(-\sigma\psi) = -\sigma \nabla \psi + \psi \nabla \sigma , \tag{4.62}$$

where $\nabla \sigma$ is given by Eq. (4.55). We may note that the above procedure is permissible because ψ is a continuous function of position defined by Eqs. (4.44) and (4.45).

Thus we have, from (4.61), that $A_z(\rho, \phi, z)$ satisfies the equation

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial A_z}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 A_z}{\partial \phi^2} + \frac{\partial^2 A_z}{\partial z^2} = \frac{2p \cos\phi}{4\pi}\, \frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2}\, \frac{\rho\, \delta(z)}{(\rho^2 + h)^{3/2}} , \tag{4.63}$$

where we have made use of either (4.44) or (4.45) to evaluate $\psi(\rho, \phi, \theta)$.
To solve Eq. (4.63) we first write

$$A_z(\rho, \phi, z) = \frac{p \cos\phi}{4\pi} \frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2} F(\rho, z) , \tag{4.64}$$

in which the function $F(\rho,z)$ is seen, from Eq. (4.63), to be a solution of

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial F}{\partial\rho}\right) - \frac{F}{\rho^2} + \frac{\partial^2 F}{\partial z^2} = \frac{2\rho\,\delta(z)}{(\rho^2 + h)^{3/2}} . \tag{4.65}$$

In order to obtain a solution of Eq. (4.65), we first eliminate the z variable by introducing the Fourier transform

$$G(\rho,\zeta) = \int_{-\infty}^{\infty} F(\rho,z)e^{-i\zeta z}dz, \qquad F(\rho,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\rho,\zeta)e^{i\zeta z}d\zeta , \tag{4.66}$$

with the aid of which, using standard procedures which will not be detailed here, one deduces that the transform $G(\rho,\zeta)$ must satisfy the inhomogeneous Bessel's equation of order one,

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G}{\partial\rho}\right) - \left(\zeta^2 + \frac{1}{\rho^2}\right)G = \frac{2\rho}{(\rho^2 + h^2)^{3/2}} . \tag{4.67}$$

A particular integral of Eq. (4.67) is most readily deduced with the aid of the Fourier-Bessel transform

$$H(\lambda,\zeta) = \int_0^{\infty} G(\rho,\zeta)J_1(\lambda\rho)\rho\,d\rho, \qquad G(\rho,\zeta) = \int_0^{\infty} H(\lambda,\zeta)J_1(\lambda\rho)\lambda d\lambda , \tag{4.68}$$

according to which we obtain for the right hand term of (4.67)

$$\int_0^\infty J_1(\lambda\rho)\,\frac{\rho^2 d\rho}{(\rho^2+h^2)^{3/2}} = e^{-\lambda h}$$

by using a formula in Watson,[23] and for the left hand terms of Eq. (4.67), using again standard procedures, we obtain merely $-(\lambda^2+\zeta^2)H(\lambda,\zeta)$, whence there results the double transform

$$H(\lambda,\zeta) = -\frac{2e^{-\lambda h}}{\lambda^2+\zeta^2}\,. \tag{4.69}$$

Inverting this expression for $H(\lambda,\xi)$ with the aid of the corresponding inversion formulas in Eqs. (4.66) and (4.68), we obtain for the function $F(\rho,z)$ the double integral

$$F(\rho,z) = -\frac{1}{\pi}\int_{-\infty}^{\infty} e^{i\zeta z}d\zeta \int_0^\infty J_1(\lambda\rho)\,\frac{\lambda d\lambda}{\lambda^2+\zeta^2}\,, \tag{4.70}$$

from which, by inverting the order of integrations and carrying out the integration with respect to ζ by the method of residues, we have finally

$$F(\rho,z) = -\int_0^\infty e^{-\lambda(h+|z|)}J_1(\lambda\rho)d\lambda = -\frac{1}{\rho}\left\{1-\frac{h+|z|}{\left[\rho^2+(h+|z|)^2\right]^{\frac{1}{2}}}\right\}, \tag{4.71}$$

where again we have evaluated the last integral by making use of a formula in Watson.[24]

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[23] G. N. Watson, "A Treatise on the Theory of Bessel Functions," (The Macmillan Company, New York, 1944), second edition, p. 434, Eq. (2), with $\nu = 1$ and $\mu = 1/2$, and then p. 80, Eq. (13).

[24] Loc. cit., p. 386, Eq. (8) and further algebraic reductions.

Substituting the above result into (4.64), we have at last for the component A_z the expression

$$A_z = -\frac{p\cos\phi}{4\pi}\frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2}\frac{1}{\rho}\left\{1 - \frac{h + |z|}{\left[\rho^2 + (h + |z|)^2\right]^{\frac{1}{2}}}\right\}, \qquad (4.72)$$

which, of course, is identical to Eq. (4.53), thus proving our contention that the magnetic field derived from z components of the Π-vectors is indeed due to the current distribution set up throughout both media and can, in fact, be computed directly from the current distribution. The above computation is illuminating on many counts, for it shows that the magnetic field due to the current distribution is a "surface" phenomenon, having its source at the interface between the two media, $z = 0$, where $\nabla \times J$ is discontinuous.

To prove that the body of the current distribution itself gives rise to no magnetic field, we first note that our vector potential **A** can be expressed in general, in terms of the current distribution, by means of the Poisson's integral

$$\mathbf{A}(r) = \frac{1}{4\pi}\int_{V'}\frac{J(r')}{R}\,dv', \quad R = |r - r'|, \qquad (4.73)$$

where r denotes the position vector of the point of observation and r' denotes the position vector of the variable point of integration. Computing the magnetic field from (4.73), we have

$$H = \nabla \times \mathbf{A} = -\frac{1}{4\pi}\int_{V'} J(r') \times \nabla\frac{1}{R}\,dv' = \frac{1}{4\pi}\int_{V'} J(r') \times \nabla'\frac{1}{R}\,dv', \qquad (4.74)$$

in which ∇' denotes differentiation with respect to the primed variables. Making use of the vector identity

$$\nabla' \times \frac{J}{R} = \frac{\nabla' \times J}{R} - J \times \nabla' \frac{1}{R}$$

and assuming that the volume of integration V' is for the moment restricted to the finite region enclosed by a bounding surface S', we have from (4.74)

$$H = \frac{1}{4\pi} \int_{V'} \frac{\nabla' \times J}{R} dv' - \frac{1}{4\pi} \int_{S'} \frac{n' \times J}{R} da' , \tag{4.75}$$

where n' denotes the outward normal to the bounding surface S'. Eq. (4.75) shows that if the current distribution is such that $\nabla' \times J = 0$, which is the situation in our present problem except at $z = 0$ where $\nabla' \times J$ is a surface singularity, then the only contribution to the magnetic field comes from the surface integral of the tangential component of current on the bounding surface S'.

It is clear, then, from Eq. (4.75) that to compute the magnetic field due to the current distribution in medium (1) we need only apply the above result to a volume consisting of a hemisphere with center at the origin of coordinates and with its equatorial plane coinciding with the interface at $z = 0$, the hemisphere being drawn into medium (1). Allowing the radius of the hemisphere to become infinite, we have, because $\nabla' \times J = 0$ everywhere except at $z = 0$, that

$$H_1 = - \frac{1}{4\pi} \int_{S'} \frac{e_z \times J_1}{R} da' , \tag{4.76}$$

where the surface of integration is the plane $z = 0$ and J_1 denotes the current density in medium (1) evaluated at the interface. A similar computation for the current distribution in medium (2) gives rise to an identical result,

$$H_2 = + \frac{1}{4\pi} \int_{S'} \frac{e_z \times J_2}{R} \, da' , \qquad (4.78)$$

where the change in sign arises from the change in direction of the outward normal. Combining the above results, we have finally for the total magnetic field <u>anywhere</u> due to the totality of the current distribution throughout both media the surface integral

$$H = - \frac{1}{4\pi} \int_{S'} \frac{e_z \times (J_1 - J_2)}{R} \, da' , \qquad (4.79)$$

which proves that this is a surface phenomenon, the magnetic field being due to the discontinuity in the tangential component of J which occurs at the interface $z = 0$ and which results in a surface layer singularity for $\nabla \times J$, the sole source of the magnetic field component presently discussed. It is clear that the magnetic field given by Eq. (4.79) must be identical to the magnetic field component $\mathbf{H}'$ defined earlier in (4.59) and which is given by $\mathbf{H}' = \nabla \times e_z \mathbf{A}_z = - e_z \times \nabla \mathbf{A}_z$ where $\mathbf{A}_z$ is given by Eq. (4.64).

The above analysis has given a physical explanation for the necessity of postulating the z components of the Π-vectors in the static as well as in the general alternating case. It is true that they were needed to satisfy the boundary conditions, but their real physical significance was only made clear after we understood that the fields derived from the z components of the Π-vectors have as a source the surface singularity of $\nabla \times J$ which occurs at the interface separating the two media.

V. ON THE SADDLE POINT METHOD OF INTEGRATION

The deformation of the original path of integration proposed in Section 2.5d has reduced the problem to the evaluation of the typical integrals I_1 and I_2 defined by Eq. (2.107) along the contours C_1 and C_2 illustrated in Fig. 4. As stated in the Introduction, we have succeeded in evaluating these integrals asymptotically by the saddle point method of integration; and, because it was found necessary to introduce several new elements into the theory, we have collected in this Chapter a number of theorems and formulas which facilitate the systematic evaluation of the integrals discussed in Chapter VI. In addition, it is pertinent to present at this juncture some concepts which help to clarify the following three points that are frequently neglected or improperly treated in the literature:

1. The asymptotic evaluation of an integral by means of the saddle point method of integration is independent of the choice of path through the saddle point, the only restriction being that the path be so chosen as to guarantee the convergence of the original integral for all allowable values of the parameters. The so-called path of steepest descents is not

at all necessary,[25] though it offers some distinct theoretical advantages.

2. The order of magnitude of the remainder after a fixed number of terms in the asymptotic expansion is either omitted or else erroneously estimated.

3. The first few terms of the asymptotic expansion may not be dropping off for a given range of parameters and should then be grouped together for computational purposes.

5.1 SADDLE POINT METHOD FOR SINGLE INTEGRATION

For points of observation in the conducting medium both integrals I_1 and I_2, as defined by Eq. (2.107), are of the general form

$$I = \frac{1}{2}\int_C v(\lambda)e^{\gamma_1(z-h)}H_0^1(\lambda\rho)\lambda d\lambda , \quad z \leqslant 0 , \tag{5.1}$$

where the path of integration in the λ-plane becomes either C_1 or C_2 as illustrated in Fig. 4. In both cases, however, it is possible to introduce a transformation of the variable of integration which reduces the typical integral (5.1) to the form

$$I = \int_C F(w)e^{\phi(w)}dw , \tag{5.2}$$

where C denotes a symmetric path in the w-plane which originates at infinity, passes through the saddle point $w = 0$, defined by

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[25] See, for example, B. L. van der Waerden, Applied Sci. Res. B2, 33-46 (1951).

$$\phi'(o) = 0 , \tag{5.3}$$

and terminates at infinity, altogether within the strip

$$\left|\arg\left\{\phi(o) - \phi(w)\right\}\right| < \pi/2 , \tag{5.4}$$

which guarantees the convergence of the integral for all allowable values of the parameters. In particular, the path C becomes the path of steepest descents when the condition

$$\arg\left\{\phi(o) - \phi(w)\right\} = 0 \tag{5.5}$$

is satisfied on C.

In the present instance it turns out that $\phi(w)$ is an even function of w,

$$\phi(-w) = \phi(w) , \tag{5.6}$$

with the result that, introducing the new variable of integration defined by

$$x^2/2 = \phi(o) - \phi(w) , \tag{5.7}$$

and considering only paths of integration C which are symmetric about the origin in the w-plane, we obtain for the integral (5.2) the expression

$$e^{-\phi(o)} I = \int_0^{\infty \exp(i\beta)} \Phi(x) e^{-x^2/2} dx , \tag{5.8}$$

where

$$\Phi(x) = \left\{F(w) + F(-w)\right\} \frac{dw}{dx} , \quad w = w(x) , \tag{5.9}$$

and in which $|\beta| < \pi/4$ and $|\arg\{x\}| < \pi/4$ on the entire path of integration in the x-plane. In particular, for the path of steepest descents we have from Eq. (5.5) that

$$e^{-\phi(0)}I = \int_0^\infty \Phi(x)e^{-x^2/2}dx \,, \tag{5.10}$$

in which the path of integration is rigorously along the positive half of the real axis in the x-plane. It is clear that the integrals (5.8) and (5.10) lead to precisely the same asymptotic expansion; for, by hypothesis, there are no singularities of the integrand between the path of steepest descents and any other permissible path within the circle of convergence of the power series expansion for $\Phi(x)$ about the saddle point $x = 0$.

5.1a. Watson's lemma as applied to the present problem.– To evaluate the integral (5.10) asymptotically, we note first that, by virtue of (5.6) and (5.7), the function $\Phi(x)$ defined by Eq. (5.9) is an even function of x and, therefore, admits the power series expansion

$$\Phi(x) = \sum_{m=0}^{\infty} A_{2m}x^{2m} \,, \tag{5.11}$$

which is valid for $|x| < \lambda^{\frac{1}{2}}$ where $\lambda^{\frac{1}{2}}$ denotes the radius of convergence; that is, the function $\Phi(x)$ exhibits its nearest singularity to the origin at $x = x_0$ where $|x_0| = \lambda^{\frac{1}{2}}$. Further, to bring the integral (5.10) within the domain of Watson's lemma,[26] it is necessary to make a change of scale in the variable of integration by putting

[26] G. N. Watson, "A Treatise on the Theory of Bessel Functions," (The Macmillan Company, New York, 1944), 2nd ed., p. 236.

$$x = \lambda^{\frac{1}{2}}u \quad \text{and} \quad \Phi(x) = \Psi(u). \tag{5.12}$$

Our typical integral (5.10) then becomes

$$e^{-\phi(o)}I = \lambda^{\frac{1}{2}} \int_o^\infty \Psi(u) e^{-\lambda u^2/2} du , \tag{5.13}$$

in which, by virtue of (5.11) and (5.12), $\Psi(u)$ has the power series expansion

$$\Psi(u) = \sum_{m=o}^{\infty} A_{2m}\lambda^m u^{2m} , \tag{5.14}$$

which is valid for $|u| < 1$. In terms of this new variable of integration Watson's lemma as applied to the problem at hand may be phrased as follows:

<u>Lemma</u>:- Let $\Psi(u)$ be analytic within the unit circle $|u| < 1$, i.e., let $\Psi(u)$ have the power series expansion (5.14); further, assume that

$$|\Psi(u)| < Au^{2p} \tag{5.15}$$

where A is a positive number independent of u and p is a positive integer or zero, when u is real and $u \geq 1$. Then the asymptotic expansion

$$e^{-\phi(o)}I = \lambda^{\frac{1}{2}} \int_o^\infty \Psi(u) e^{-\lambda u^2/2} du \sim \sum_{m=o}^{\infty} 2^{m-\frac{1}{2}} \Gamma(m + \tfrac{1}{2}) A_{2m} \tag{5.16}$$

is valid in the sense of Poincaré when λ is sufficiently large, i.e., $\lambda > 1$.

Note that the asymptotic expansion (5.16) may also be conveniently written as

$$e^{-\phi(o)}I \sim \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \sum_{m=o}^{\infty} \frac{(2m)!}{2^m\, m!} A_{2m} = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} (A_o + A_2 + 3A_4 + 15A_6 + \cdots) \tag{5.16a}$$

in terms of the expansion coefficients A_{2m} which, of course, are themselves functions of λ. In fact, as shown in the sequel, it turns out that in the present study we need only consider two cases characterized by the fact that the functions

$$\text{(a)}\quad \lambda^m A_{2m} \qquad \text{and} \qquad \text{(b)}\quad \lambda^{m+1} A_{2m} \tag{5.17}$$

remain bounded as $\lambda \to \infty$.

To establish the above expansion we note from Eqs. (5.14) and (5.15) that, if $M \geq p$ is a <u>fixed</u> integer, a constant B can be found such that

$$\left| \Psi(u) - \sum_{m=o}^{M-1} A_{2m} \lambda^m u^{2m} \right| \leq B\, u^{2M} \tag{5.18}$$

whenever $u \geq 0$, whether $u \leq 1$ or $u \geq 1$; and therefore

$$\lambda^{\frac{1}{2}} \int_o^{\infty} \Psi(u) e^{-\lambda u^2/2} du = \sum_{m=o}^{M-1} A_{2m} \lambda^{m+\frac{1}{2}} \int_o^{\infty} u^{2m} e^{-\lambda u^2/2} du + R_M$$

where R_M, the remainder <u>after</u> M terms, is bounded as follows:

$$|R_M| \leq \lambda^{\frac{1}{2}} B \int_o^{\infty} u^{2M} e^{-\lambda u^2/2} du = B \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{(2M)!}{2^M\, M!} \lambda^{-M} = O(\lambda^{-M}). \tag{5.19}$$

The analysis remains valid even when the path of integration in the x-plane does not coincide with the path of steepest descents and so we have proved that, with $|\beta| < \pi/4$ and $|\arg\{x\}| < \pi/2$ on the path of integration,

$$\int_0^{\infty \exp(i\beta)} \Phi(x) e^{-x^2/2} dx = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left\{ \sum_{m=0}^{M-1} \frac{(2m)!}{2^m\, m!} A_{2m} + O(\lambda^{-M}) \right\} \qquad (5.20a)$$

whenever the expansion coefficients satisfy the condition (5.17a), or else, that

$$\int_0^{\infty \exp(i\beta)} \Phi(x) e^{-x^2/2} dx = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left\{ \sum_{m=0}^{M-1} \frac{(2m)!}{2^m\, m!} A_{2m} + O(\lambda^{-M-1}) \right\} \qquad (5.20b)$$

whenever the expansion coefficients satisfy the condition (5.17b). This last result is readily established from the foregoing analysis by merely carrying the summation in Eq. (5.18) to one more term and then grouping the last term with the remainder, since both are now of the same order of magnitude.

It is of interest to remark that the form of the remainder as given by Eq. (5.19) has been established for $M \geq p$. This is a sufficient condition only, for it can be readily seen that Eq. (5.19) is valid for $M \geq 1$ regardless of the value of p. However, it turns out that, for moderately small values of $\lambda > 1$, it is sometimes necessary to group the first p terms in order to guarantee that the remainder is sufficiently small and, in fact, the first $(p + 1)$ terms must be computed before there can be any assurance that the remainder has the proper order of magnitude.

5.1b. Inversion of the power series expansion for $x^2/2$.- The practical application of the results embodied in Eqs. (5.20) require the evaluation of the expansion coefficients A_{2m} in Eq. (5.11). This in turn necessitates the inversion of Eq. (5.7) to express w qua function of x, and in many cases this inversion and the evaluation of the coefficients A_{2m} constitute the most laborious part of the computation. To facilitate the inversion of Eq. (5.7), it is noted that, by virtue of (5.3) and (5.6), the function $x^2/2$ admits the power series expansion

$$x^2/2 = w^2(c_o + c_2 w^2 + c_4 w^4 + \dots) \qquad (5.21)$$

which is certainly valid for sufficiently small values of w. Inverting the power series expansion (5.21), we obtain

$$w = a_o x + \frac{1}{3} a_2 x^3 + \frac{1}{5} a_4 x^5 + \dots \qquad (5.22)$$

and

$$\frac{dw}{dx} = a_o + a_2 x^2 + a_4 x^4 + \dots \qquad (5.23)$$

where the coefficients a_{2n} are given, according to Watson,[27] by the expressions

$$a_o = (2c_o)^{-\frac{1}{2}} ,$$

$$a_2 = (2c_o)^{-3/2} \left\{ -\frac{3}{2} \frac{c_2}{c_o} \right\}, \qquad (5.24)$$

$$a_4 = (2c_o)^{-5/2} \left\{ -\frac{5}{2} \frac{c_4}{c_o} + \frac{35}{8} \left(\frac{c_2}{c_o} \right)^2 \right\} .$$

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[27] Loc. cit., p. 242, Eqs. (1).

Thus, in practice, one first determines the coefficients c_{2n} in Eq. (5.21) by expanding the function on the right of Eq. (5.7) into a power series in w, which of course is valid only within its own circle of convergence as determined by the nearest singularity of $\phi(w)$. Once in possession of the c's, the computation of the a's in the inverted power series (5.22) is effected by applying the formulas given in Eqs. (5.24). In the present instance we found it unnecessary to go beyond the coefficient a_4.

<u>5.1c. Calculation of the coefficients A_{2m}</u>.— According to Eq. (5.9) the function $\Phi(x)$ may be regarded as the product of two functions, $\{F(w) + F(-w)\}$ and dw/dx, which are themselves even functions of x. Thus, expanding each factor into a power series in x^2 and multiplying out the two series, we obtain, in accordance with (5.11),

$$\Phi(x) = \{F(w) + F(-w)\} \frac{dw}{dx} = A_0 + A_2 x^2 + A_4 x^4 + A_6 x^6 + \dots , \qquad (5.25)$$

where, making use of (5.23), the coefficients of the power series expansion about the saddle point become

$$\begin{aligned}
A_0 &= 2a_0 F(0) \\
A_2 &= \frac{2a_0^3}{2!}\left\{F^{II}(0) + 2F(0)\frac{a_2}{a_0^3}\right\} \\
A_4 &= \frac{2a_0^5}{4!}\left\{F^{IV}(0) + 20F^{II}(0)\frac{a_2}{a_0^3} + 24F(0)\frac{a_4}{a_0^5}\right\} \\
A_6 &= \frac{2a_0^7}{6!}\left\{F^{VI}(0) + 70F^{IV}(0)\frac{a_2}{a_0^3} + 504F^{II}(0)\frac{a_4}{a_0^5} + 280F^{II}(0)\frac{a_2^2}{a_0^6} + 720F(0)\frac{a_6}{a_0^7}\right\}
\end{aligned} \qquad (5.26)$$

where the primes over the F's denote differentiation with respect to w and subsequent evaluation at $w = 0$.

Thus, by virtue of Eqs. (5.20), the asymptotic evaluation of our primitive integral (5.2) has been reduced to the evaluation of the expansion coefficients A_{2m}, given by Eqs. (5.26), which in turn are expressed in terms of the coefficients a_{2n} in the power series expansion (5.22) and in terms of the function $F(w)$ and its higher derivatives of even order, evaluated at the saddle point $w = 0$.

5.1d. Subtraction of a first order pole in the neighborhood of the saddle point.- It is clear from the foregoing analysis and in particular from Eqs. (5.20) that the magnitude of the remainder, as given by Eq. (5.19), limits the applicability of our asymptotic expansions to fairly large values of λ, where $\lambda^{\frac{1}{2}}$ represents the radius of convergence of the power series expansion of the function $\Phi(x)$, defined by Eq. (5.9), about the origin in the x-plane. Expressed otherwise, it is the singularity occurring nearest to the origin in the x-plane which governs the nature of the asymptotic expansion. If it happens, however, that the nearest singularity of $\Phi(x)$ is a pole of the first order (or of higher order), then it turns out that the radius of convergence can be enlarged to the next nearest singularity by the subtraction of the pole, thus improving the range of applicability of the asymptotic expansion which now exhibits a remainder R_M with a lower upper bound. This method was employed by van der Waerden[28] in a paper which was not available to us at the time that we developed independently the method of subtraction of a first order pole presented here and which, we believe, has the merit of greater simplicity.

Thus, we assume that the function $\Phi(x) = \{F(w) + F(-w)\} dw/dx$ has a pair of simple poles at $x = \pm x_0$, where $|x_0| = \lambda^{\frac{1}{2}}$ denotes the radius of

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[28] Loc. cit. reference (25). See also H. Ott, Ann. Physik 43, 393-404 (1943).

convergence of the power series expansion (5.11) and that the next nearest singularity of $\Phi(x)$, presumably an algebraic singularity and not a pole, occurs at $x = x_1$ with $|x_1| = \mu^{\frac{1}{2}}$ and $\mu > \lambda$ by hypothesis. Then, the function

$$\Psi(x) = \Phi(x) - \frac{2x_o C}{x^2 - x_o^2} , \qquad (5.27)$$

where C is defined as

$$C = \lim_{x \to x_o} \left\{ (x - x_o) F(w) \frac{dw}{dx} \right\} , \qquad (5.28)$$

is analytic for $|x| < \mu^{\frac{1}{2}}$. Hence, in accordance with (5.11), $\Psi(x)$ admits the power series expansion

$$\Psi(x) = \sum_{m=o}^{\infty} \left\{ A_{2m} + \frac{2C}{x_o^{2m+1}} \right\} x^{2m} \qquad (5.29)$$

which converges for $|x| < \mu^{\frac{1}{2}}$.

Solving for $\Phi(x)$ in Eq. (5.27) and substituting into Eq. (5.10), we obtain

$$e^{-\phi(o)} I = \int_o^{\infty} \left\{ \Psi(x) + \frac{2x_o C}{x^2 - x_o^2} \right\} e^{-x^2/2} dx = W_s + W_p , \qquad (5.30)$$

where W_p, the contribution arising from the pole, is given by

$$W_p = 2x_o C \int_o^{\infty} \frac{e^{-x^2/2}}{x^2 - x_o^2} dx , \qquad (5.31)$$

while W_s, the contribution over the path through the saddle point, now becomes, with $|\beta| < \pi/4$ and $|\arg\{x\}| < \pi/4$ along the entire path of integration,

$$W_s = \int_0^{\infty \exp(i\beta)} \Psi(x) e^{-x^2/2} dx = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left\{ \sum_{m=0}^{M-1} \frac{(2m)!}{2^m\, m!} \left[A_{2m} + \frac{2C}{x_0^{2m+1}} \right] + R_M \right\}, \qquad (5.32)$$

where the order of magnitude of the remainder <u>after</u> M terms, R_M, is given by

$$\text{(a)} \quad |R_M| \doteq O(\mu^{-M}) \qquad \text{or} \qquad \text{(b)} \quad |R_M| \doteq O(\mu^{-M-1}) \qquad (5.33)$$

in accordance with Eqs. (5.20).

<u>5.1e. Evaluation of the contribution from the pole</u>.- It is to be noted that the path of integration in the integral (5.31) for W_p, the contribution from the pole, must be specified without ambiguity in relation to the pole occurring at $x = x_0$ and that it must not go through the pole in order to avoid the necessity of computing principal value integrals. In the present case we have chosen the path of integration along the positive real axis in the x-plane; and, as it turns out, the pole at $x = x_0$ occurs in the first quadrant with $0 < \arg\{x_0\} << \pi/4$. This precaution concerning the choice of path is necessary because, as shown below, the value of the integral <u>itself</u> depends on whether the pole is above or below the path of integration.

The integral (5.31) may be expressed in closed form in terms of the error function defined by

$$\mathrm{erf}(z) = \frac{2}{(\pi)^{\frac{1}{2}}} \int_0^z e^{-t^2} dt \,. \tag{5.33a}$$

To this end, consider the associated integral

$$W(\lambda) = 2x_0 C \int_0^\infty \frac{e^{-\lambda x^2/2}}{x^2 - x_0^2} dx \,, \tag{5.34}$$

in which λ is a parameter that we will eventually set equal to unity. It is readily shown that $W(\lambda)$ satisfies the following first order, inhomogeneous differential equation:

$$\frac{dW}{d\lambda} + \frac{x_0^2}{2} W = - x_0 C(\pi/2\lambda)^{\frac{1}{2}} \,, \tag{5.35}$$

which admits a particular integral of the form

$$W(\lambda) = f(\lambda) e^{-\lambda x_0^2/2} \,. \tag{5.36}$$

Substituting (5.36) into (5.35), we find for $f'(\lambda)$ the expression

$$f'(\lambda) = - x_0 C(\pi/2\lambda)^{\frac{1}{2}} \, e^{\lambda x_0^2/2} \,, \tag{5.37}$$

from which we deduce by integration that

$$\begin{aligned} f(1) &= f(0) - x_0 C(\pi/2)^{\frac{1}{2}} \int_0^1 \lambda^{-\frac{1}{2}} e^{\lambda x_0^2/2} d\lambda \\ &= f(0) - i\pi C \; \mathrm{erf}\left[-ix_0/(2)^{\frac{1}{2}}\right]; \end{aligned} \tag{5.38}$$

whence, from Eqs. (5.31) and (5.36),

$$W_p = W(1) = f(1)e^{-x_o^2/2} = e^{-x_o^2/2}\left\{f(0) - i\pi C \operatorname{erf}\left[-ix_o/(2)^{\frac{1}{2}}\right]\right\} , \qquad (5.39)$$

there remaining for us only the evaluation of the constant of integration $f(0)$.

To compute $f(0)$ we note from (5.34) and (5.36) that

$$f(0) = W(0) = 2x_oC\int_0^\infty \frac{dx}{x^2 - x_o^2} = C \ln \frac{x - x_o}{x + x_o}\Bigg|_0^\infty , \qquad (5.40)$$

in which the evaluation at the lower limit $x = 0$ must be performed with due regard to the phase of the logarithmic argument, $(x - x_o)/(x + x_o)$, as $x \rightarrow 0$. In this way we find

$$f(0) = \begin{cases} i\pi C, & \text{when } \pi/4 > \arg\{x_o\} > 0 \\ -i\pi C, & \text{when } -\pi/4 < \text{are}\{x_o\} < 0 . \end{cases} \qquad (5.40a)$$

In the present instance, as already stated, the path of integration in (5.34) is along the positive real axis in the x-plane and we have assumed that the pole at $x = x_o$ occurs in the first quadrant with $0 < \arg\{x_o\} \ll \pi/4$. In consequence, the final evaluation of W_p, making use of (5.39) and (5.40a), yields

$$W_p = 2x_oC\int_0^\infty \frac{e^{-x^2/2}}{x^2 - x_o^2}\,dx = i\pi Ce^{-x_o^2/2}\left\{1 - \operatorname{erf}\left[-ix_o/(2)^{\frac{1}{2}}\right]\right\}$$

$$= \tfrac{1}{2}\pi\left\{1 - \operatorname{erf}\left[-ix_o/(2)^{\frac{1}{2}}\right]\right\} , \qquad (5.41)$$

where

$$\Pi = 2x_0 C \int^{(x_0+)} \frac{e^{-x^2/2}}{x^2 - x_0^2}\,dx = 2\pi i C e^{-x_0^2/2} \tag{5.42}$$

represents simply the contribution from the residue of the first order pole at $x = x_0$.

It now becomes of interest to investigate the effect of deforming the path of integration in (5.31) from the positive real axis in the x-plane to, say, a straight line in the first quadrant starting at the origin and inclined at an angle β with respect to the real axis and such that

$$0 < \arg\{x_0\} < \beta < \pi/4 . \tag{5.43}$$

That is, we would like to evaluate directly the new integral defined by

$$W_p' = 2x_0 C \int_0^{\infty \exp(i\beta)} \frac{e^{-x^2/2}}{x^2 - x_0^2}\,dx \tag{5.44}$$

in which the path of integration, in accordance with (5.43), now lies <u>above</u> the pole at $x = x_0$. To evaluate W_p', we first rotate the real axis until it coincides with the new path of integration by introducing the new variable of integration

$$\xi = xe^{-i\beta} , \quad \xi_0 = x_0 e^{-i\beta} = |x_0| e^{-i(\beta - \delta)} , \tag{5.45}$$

where $\delta = \arg\{x_0\}$, and obtain

$$W_p^{\prime} = 2\xi_o C \int_o^{\infty} \frac{e^{-\exp(2i\beta)\xi^2/2}}{\xi^2 - \xi_o^2} d\xi \, , \tag{5.46}$$

where now the path of integration is the positive real axis in the ξ-plane. Proceeding as before, we readily conclude from Eqs. (5.34), (5.36) and (5.46) that

$$W_p^{\prime} = W(e^{2i\beta}) = f(e^{2i\beta})e^{-x_o^2/2} = e^{-x_o^2/2}\left\{f(0) - i\pi C \,\text{erf}\left[-ix_o/(2)^{\frac{1}{2}}\right]\right\} \, , \tag{5.47}$$

where, in accordance with (5.40a), noting that $\arg\{\xi_o\}$ now is negative, $f(0)$ becomes $-i\pi C$ and we obtain finally

$$W_p^{\prime} = -\tfrac{1}{2}\Pi\left\{1 + \text{erf}\left[-ix_o/(2)^{\frac{1}{2}}\right]\right\} \, , \tag{5.48}$$

where Π denotes the contribution from the residue of the pole at $\xi = \xi_o$ as given by (5.42).

This last expression for $W_p^{\prime}$ differs, of course, from the original expression for W_p. To reconcile the above results one merely observes that, according to Cauchy's theorem, we ought to have

$$W_p - W_p^{\prime} = \Pi \qquad \text{or} \qquad W_p = \Pi + W_p^{1} \, , \tag{5.49}$$

which is certainly verified by Eqs. (5.41) and (5.48). The essential point to observe is that, if we have once computed the contribution due to the pole by evaluating the integral (5.31) over the positive real axis in the x-plane, and then we undertake a deformation of the path of integration which "sweeps" past the pole picking up its residue, we still come out with the same result, namely $\Pi + W_p^{+} = W_p$, showing that the complete contribution from the pole integral is certainly independent of the choice of path starting from the origin, so long as $|\beta| < \pi/4$ and $|\arg\{x\}| < \pi/4$ along the path; that is, so long as due care is taken to include the contribution from the residue of the pole when such accrues.

Since W_s, as given by Eq. (5.32), is likewise independent of the choice of path in the x-plane, subject only to the restrictions already annotated, we conclude, as already stated, that the original integral (5.1) in the w-plane may be taken over by any (symmetric) path passing through the saddle point at $w = 0$ and extending to infinity in both directions away from the origin, the only restriction being that the path be permissible in the sense that it guarantees the convergence of the original integral for all allowable values of the parameters. This important point has a definite bearing on the whole question of the existence or non-existence of the so-called Zenneck surface waves, which we take up again in Section 7.3 where we settle the whole debate, we trust, in a definitive manner.

5.2 SADDLE POINT METHOD FOR DOUBLE INTEGRATION

The introduction of the Hankel function in our typical integral (5.1) has made it possible to resort to contour integration, but it has also introduced the branch point of the Hankel function at $\lambda = 0$. The application of the saddle point method of integration discussed in the preceding section is therefore limited by the presence of this additional singularity at $\lambda = 0$, particularly in the case of the integral I_2 which is to be evaluated over the contour C_2 (Fig. 4). In this case the singularity nearest to the origin in the x-plane, save for the possible pair of first order poles at $x = \pm x_o$ which we know how to subtract (Section 5.1d), is precisely the branch point of the Hankel function occurring at $\lambda = 0$ in the λ-plane and it is the presence of this singularity that fixes the radius of convergence of the expansion (5.11) and consequently dictates the character of the ensuing asymptotic expansions (5.20). Notwithstanding this limitation, we expanded asymptotically the Hankel function and its derivatives, appearing in the expansion coefficients (5.26), and discovered that the resulting asymptotic series seemed to correspond to a function $\Phi(x)$ with a power series expansion having a larger radius of convergence as dictated by the next nearest singularity. In other words, the process of expanding the Hankel function and its derivatives asymptotically is tantamount to the removal of the branch point at $\lambda = 0$.

To justify the above interpretation, to obtain the correct estimate for the remainder of the asymptotic series and to further facilitate the asymptotic evaluation of our integrals, not in terms of Hankel functions and their derivatives, but in terms of their corresponding asymptotic expansions, we replaced the Hankel functions appearing in (5.1) and similar integrals by the integral representation[29]

$$H^1_\nu(z) = \frac{4e^{i(z-\nu\pi)}}{(\pi)^{\frac{1}{2}}2^{3\nu}\Gamma(\nu+\frac{1}{2})z^\nu}\int_0^{\infty\exp(i\alpha)} y^{2\nu}(4iz - y^2)^{\nu-\frac{1}{2}}e^{-y^2/2}dy \,, \qquad (5.50)$$

which is valid when $|\alpha| < \pi/4$ and $-\frac{\pi}{4} + \alpha < \frac{1}{2}\arg\{z\} < \frac{3\pi}{4} + \alpha$,

[29] G. N. Watson, loc. cit., paragraph 7.2, p. 196, after making the substitution $u = y^2/2$ for the variable of integration.

provided that Re $\{\nu + \frac{1}{2}\} > 0$. Thus, replacing the Hankel function which is implicit in the integrand of (5.8) by its corresponding integral representation, there results the double integral

$$e^{-\phi(o)}I = \int_0^{\infty \exp(i\beta)} \int_0^{\infty \exp(i\alpha)} \Phi(x,y) e^{-(x^2+y^2)/2} dxdy \, , \qquad (5.51)$$

which we now propose to evaluate asymptotically by the double saddle point method of integration. In particular, if the paths of integration in the x and y planes are made to coincide with their respective positive real axes, then we obtain

$$e^{-\phi(o)}I = \int_0^{\infty} \int_0^{\infty} \Phi(x,y) e^{-(x^2+y^2)/2} dxdy \qquad (5.52)$$

wherein $0 \leq x < \infty$ and $0 \leq y < \infty$, which means that in each plane we are using the path of steepest descents. It is clear that both integrals (5.51) and (5.52) lead to the same asymptotic expansion for, by hypothesis, one goes from the paths of steepest descents in (5.52) to the permissible paths in (5.51) through a continuous set of allowed deformations without encountering additional singularities within the domain of allowable complex values of x and y for which the function $\Phi(x,y)$ is analytic and, therefore, has a valid double power series expansion.

5.2a. Extension of Watson's lemma to a double integral.- To evaluate the double integral (5.52) asymptotically, we note first that, by virtue of (5.9) and (5.50), the function $\Phi(x,y)$ appearing in the integrand of (5.52) is an even function of both x and y and, therefore, admits the double power series expansion

$$\Phi(x,y) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} A_{2(n-m)}^{2m} \, x^{2(n-m)} y^{2m} \, , \tag{5.53}$$

which we shall assume is valid for $|x| < \lambda^{\frac{1}{2}}$ and $|y| < \nu^{\frac{1}{2}}$ in the sense explained below. For example, the trivial case in which the extension of Watson's lemma to a double integral is immediate is obtained when $\Phi(x,y)$ is of the form

$$\Phi(x,y) = f(x)\, g(y) \tag{5.54}$$

where

$$f(x) = \sum_{s=0}^{\infty} A_{2s} x^{2s}, \quad |x| < \lambda^{\frac{1}{2}} \, , \tag{5.55}$$

and

$$g(y) = \sum_{r=0}^{\infty} A_{2r} y^{2r} \, , \quad |y| < \nu^{\frac{1}{2}} \, ; \tag{5.56}$$

so that multiplying out the two series (5.55) and (5.56) and collecting terms of the same power, we obtain a series of the form (5.53) with coefficients $A_{2(n-m)}^{2m} = A_{2(n-m)} B_{2m}$. To bring the integral (5.52) within the domain of Watson's lemma (extended to a double integral), it is necessary to change the variables of integration as follows:

$$x = \lambda^{\frac{1}{2}} u \, , \quad y = \nu^{\frac{1}{2}} v \, , \quad \text{whence} \quad \Phi(x,y) = \Psi(u,v) \, , \tag{5.57}$$

with which the double integral (5.52) becomes

$$e^{-\phi(o)}I = (\lambda\nu)^{\frac{1}{2}} \int_{o}^{\infty} \int_{o}^{\infty} \Psi(u,v)e^{-(\lambda u^2+\nu v^2)/2}dudv \ , \qquad (5.58)$$

where, by virtue of Eqs. (5.53) and (5.57), $\Psi(u,v)$ has the double power series expansion

$$\Psi(u,v) = \sum_{n=o}^{\infty} \sum_{m=o}^{n} A^{2m}_{2(n-m)}(\lambda u^2)^{n-m}(\nu v^2)^m \ , \qquad (5.59)$$

which is valid for $|u| < 1$ and $|v| < 1$. In terms of these new variables of integration, u and v, Watson's lemma as extended to a double integral now reads:

Lemma:- Let $\Psi(u,v)$ be analytic for both u and v when $|u| < 1$ and $|v| < 1$; i.e., let $\Psi(u,v)$ have the power series expansion (5.59). Assume further that

$$|\Psi(u,v)| < A\ u^{2p}v^{2q} \qquad (5.60)$$

where A is a positive number independent of u and v and p and q are positive integers or zero, when both u and v are real and such that $u \geq 1$ and $v \geq 1$. Then, the asymptotic expansion

$$e^{-\phi(o)}I = (\lambda\nu)^{\frac{1}{2}} \int_{o}^{\infty} \int_{o}^{\infty} \Psi(u,v)e^{-(\lambda u^2+\nu v^2)/2}dudv$$

$$\sim \frac{\pi}{2} \sum_{n=o}^{\infty} \sum_{m=o}^{n} \frac{(2m)!\ [2(n-m)]!}{2^n m!\ (n-m)!} A^{2m}_{2(n-m)} \qquad (5.61)$$

is valid in the sense of Poincaré provided λ and ν are sufficiently large, i.e., provided $\lambda > 1$ and $\nu > 1$.

To establish the above expansion we note that, if N is a <u>fixed</u> integer such that $N \geq (p+q)$, a constant B can be found for which

$$\left| \Psi(u,v) - \sum_{n=o}^{N-1} \sum_{m=o}^{n} A_{2(n-m)}^{2m} (\lambda u^2)^{n-m} (\nu v^2)^m \right| \leqslant B\,(u^2+v^2)^N, \qquad (5.62)$$

whenever u and v are real and positive, whether less or greater than unity; and, therefore, reverting to the original variables of integration in (5.61), we have by limiting the summation

$$e^{-\phi(o)} I = \sum_{n=o}^{N-1} \sum_{m=o}^{n} A_{2(n-m)}^{2m} \int_o^\infty x^{2(n-m)} e^{-x^2/2} dx \int_o^\infty y^{2m} e^{-y^2/2} dy + R_N, \qquad (5.63)$$

where R_N, the remainder after N (grouped) terms, is bounded as follows:

$$|R_N| \leqslant (\lambda\nu)^{\frac{1}{2}} B \int_o^\infty \int_o^\infty (u^2+v^2)^N e^{-(\lambda u^2 + \nu v^2)/2} du dv$$

$$= B\,\frac{\pi}{2} \sum_{s=o}^{N} \frac{N!}{(N-s)!\,s!} \left\{ \frac{(2s)!\,[2(N-s)]!}{2^N\,s!\,(N-s)!} \right\} \lambda^{-N+s} \nu^{-s}$$

from which, replacing the bracket by its largest value attained when $s=0$ or $s=N$, we obtain

$$|R_N| < B\,\frac{\pi}{2}\,\frac{(2N)!}{2^N\,N!} \left(\frac{1}{\lambda} + \frac{1}{\nu}\right)^N = O\left(\frac{1}{\lambda} + \frac{1}{\nu}\right)^N. \qquad (5.64)$$

Thus, we have established the asymptotic expansion

$$\int_{0}^{\infty}\int_{0}^{\infty} \Phi(x,y)e^{-(x^2+y^2)/2}dxdy = \frac{\pi}{2}\left\{\sum_{n=0}^{N-1}\sum_{m=0}^{n} \frac{(2m)!\,[2(n-m)]!}{2^n\, m!\,(n-m)!} A_{2(n-m)}^{2m} + R_N\right\} \tag{5.65}$$

where, in accordance with the discussion leading to Eqs. (5.20), we are to write

$$\text{(a)}\quad |R_N| = 0\left(\frac{1}{\lambda}+\frac{1}{\nu}\right)^{N} \qquad \text{or} \qquad \text{(b)}\quad |R_N| = 0\left(\frac{1}{\lambda}+\frac{1}{\nu}\right)^{N+1} \tag{5.66}$$

depending upon which of the conditions (5.17) is satisfied by the grouped expansion terms.

For convenience in tabulating our results we write the asymptotic series (5.61) in the form

$$e^{-\phi(o)}I = \int_{0}^{\infty}\int_{0}^{\infty} \Phi(x,y)e^{-(x^2+y^2)/2}dxdy \sim \frac{\pi}{2}\sum_{n=0}^{\infty}\Phi^{(n)} \quad , \tag{5.67}$$

by introducing the expansion terms

$$\Phi^{(n)} = \sum_{m=0}^{n} \frac{(2m)!\,[2(n-m)]!}{2^n\, m!\,(n-m)!} A_{2(n-m)}^{2m} \,, \tag{5.68}$$

from which the first five terms become

$$\Phi^{(0)} = A_0^0 ,$$
$$\Phi^{(1)} = A_2^0 + A_0^2 ,$$
$$\Phi^{(2)} = 3A_4^0 + A_2^2 + 3A_0^4 , \qquad (5.69)$$
$$\Phi^{(3)} = 15A_6^0 + 3A_4^2 + 3A_2^4 + 15A_0^6 ,$$
$$\Phi^{(4)} = 105A_8^0 + 15A_6^2 + 9A_4^4 + 15A_2^6 + 105A_0^8 .$$

It should be remarked that the above treatment leading to the asymptotic expansion (5.67) was based on rather stringent conditions. We assumed that the paths of integration were along the positive real axes in both the x and y planes and we assumed that the function $\Phi(x,y)$ was analytic in the region defined by $|x| < \lambda^{\frac{1}{2}}$ and $|y| < \nu^{\frac{1}{2}}$. It will be shown in Section 6.3, when we come to the actual application of the present theory, that these conditions, although sufficient, are not necessary.

5.2b. Calculation of the terms $\Phi^{(n)}$ in the asymptotic expansion of a double integral.-- The asymptotic series (5.67), expressed in terms of the expansion coefficients $A_{2(n-m)}^{2m}$ as listed in Eqs. (5.69), constitutes, so far, a purely formal solution. In the practical applications of the theory discussed in Section 6.3 we encounter two types of functions $\Phi(x,y)$. In the first type we have

$$\Phi(x,y) = f(x)\, g(x,y) , \qquad (5.70)$$

where $f(x)$ is an even function of x which happens to vanish at the origin and, therefore, has an expansion of the form (5.25) with $A_0 = 0$, that is,

$$f(x) = A_2x^2 + A_4x^4 + A_6x^6 + \dots , \tag{5.71}$$

and where $g(x,y)$, as deduced from (5.50), is likewise an even function of both x and y and, therefore, may be expanded in the vicinity of $x = 0$ and $y = 0$ in a double power series of the form

$$g(x,y) = B_o^o + (B_2^o x^2 + B_o^2 y^2) + (B_4^o x^4 + B_2^2 x^2 y^2 + B_o^4 y^4) + \dots . \tag{5.72}$$

Thus, multiplying out the series (5.71) and (5.72) and collecting terms of the same power in the manner of (5.53), we obtain

$$\begin{aligned} \Phi(x,y) = A_2B_o^o x^2 &+ \left[(A_2B_2^o + A_4B_o^o)x^4 + A_2B_o^2 x^2y^2\right] \\ &+ \left[(A_2B_4^o + A_4B_2^o + A_6B_o^o)x^6 + (A_2B_2^2 + A_4B_o^2)x^4y^2 + A_2B_o^4 x^2y^4\right] + \dots , \end{aligned} \tag{5.73}$$

which upon comparison with (5.53) allows the immediate identification of the expansion coefficients $A_{2(n-m)}^{2m}$. Hence, finally, noting that all coefficients with a subscript zero are missing, we obtain directly from (5.69) for the terms $\Phi^{(n)}$ of the asymptotic expansion corresponding to the function $\Phi(x,y)$ defined by Eq. (5.70) the expressions

$$\begin{aligned} \Phi^{(o)} &= 0 , \\ \Phi^{(1)} &= A_2B_o^o , \\ \Phi^{(2)} &= 3(A_2B_2^o + A_4B_o^o) + A_2B_o^2 , \\ \Phi^{(3)} &= 15(A_2B_4^o + A_4B_2^o + A_6B_o^o) + 3(A_2B_2^2 + A_4B_o^2) + 3A_2B_o^4 . \end{aligned} \tag{5.74}$$

The second type of function $\Phi(x,y)$ in which we are interested is of the form

$$\Phi(x,y) = y^2 f(x)\, g(x,y) \,, \tag{5.75}$$

where again $f(x)$ and $g(x,y)$ admit, respectively, power series expansions of the form (5.71) and (5.72). In consequence, we can write at once for $\Phi(x,y)$ in this case the double power series expansion

$$\begin{aligned}\Phi(x,y) = {} & A_2B_o^ox^2y^2 + \left[(A_2B_2^o + A_4B_o^o)x^4y^2 + A_2B_o^2x^2y^4\right] \\ & + \left[(A_2B_4^o + A_4B_2^o + A_6B_o^o)x^6y^2 + (A_2B_2^2 + A_4B_o^2)x^4y^4 + A_2B_o^4x^2y^6\right] + \dots \,,\end{aligned} \tag{5.76}$$

which upon comparison with (5.53) allows the immediate identification of the coefficients $A^{2m}_{2(n-m)}$ that abide in this case. Thus, finally, noting that all coefficients with subscript or superscript zero are now missing, we obtain directly from (5.69) for the expansion terms $\Phi^{(n)}$ corresponding to the function $\Phi(x,y)$ defined by Eq. (5.75) the expressions

$$\begin{aligned}\Phi^{(o)} &= 0 \,, \\ \Phi^{(1)} &= 0 \,, \\ \Phi^{(2)} &= A_2B_o^o \,, \\ \Phi^{(3)} &= 3(A_2B_2^o + A_4B_o^o) + 3A_2B_o^2 \,, \\ \Phi^{(4)} &= 15(A_2B_4^o + A_4B_2^o + A_6B_o^o) + 9(A_2B_2^2 + A_4B_o^2) + 15A_2B_o^4 \,.\end{aligned} \tag{5.77}$$

<u>5.2c. Subtraction of a first order pole in the neighborhood of the saddle point for a double integral.</u>— As indicated in Eqs. (5.66), the order of magnitude of the remainder R_N depends on the parameters λ and ν. When it happens that $\Phi(x,y)$ has a pair of first order poles at $x = \pm x_o$ where

$|x_o| = \lambda^{\frac{1}{2}}$, independent of y, and this pair of poles is very close to the origin in the x-plane, then the resulting asymptotic expansion (5.65) is worthless for all practical purposes. It happens, however, that we can remove these singularities by extending the methods of Section 5.1d to the present double integral. Thus, if the next nearest singularity in the x-plane, independent of y, occurs at $x = x_1$ where $|x_1| = \mu^{\frac{1}{2}}$ and (by hypothesis) $\mu > \lambda$, then the removal of the first order poles improves the behavior of the resulting asymptotic series by replacing λ with μ in the form of the remainder (5.66). To this end we note that the function

$$\Psi(x,y) = \Phi(x,y) - \frac{2x_o\, C(y)}{x^2 - x_o^2}, \tag{5.78}$$

where $C(y)$ is defined by

$$C(y) = \frac{1}{2x_o} \lim_{x \to x_o} \left\{ (x^2 - x_o^2)\, \Phi(x,y) \right\}, \tag{5.79}$$

no longer has first order poles at $x = \pm x_o$. Hence, in accordance with (5.53) and noting that $C(y)$ is an even function of y,

$$C(y) = \sum_{m=o}^{\infty} B_{2m}\, y^{2m}, \quad |y| < \nu^{\frac{1}{2}}, \tag{5.80}$$

we see that $\Psi(x,y)$ admits the double power series expansion

$$\Psi(x,y) = \sum_{n=o}^{\infty} \sum_{m=o}^{n} \left\{ A^{2m}_{2(n-m)} + \frac{2B_{2m}}{x_o^{2(n-m)+1}} \right\} x^{2(n-m)} y^{2m}, \tag{5.81}$$

which we may assume valid for $|x| < \mu^{\frac{1}{2}}$ and $|y| < \nu^{\frac{1}{2}}$.

In consequence, solving for $\Phi(x,y)$ in Eq. (5.78) and substituting into Eq. (5.52) we obtain as before

$$e^{-\phi(o)}I = \int_o^\infty \int_o^\infty \left\{ \Psi(x,y) + \frac{2x_o C(y)}{x^2 - x_o^2} \right\} e^{-(x^2+y^2)/2} dxdy = W_s + W_p, \qquad (5.82)$$

where W_p, the contribution arising from the pole, is now given by

$$W_p = 2x_o \int_o^\infty \frac{e^{-x^2/2}}{x^2 - x_o^2} dx \int_o^\infty C(y) e^{-y^2/2} dy , \qquad (5.83)$$

and where W_s, the contribution over the path through the saddle point in the x-plane becomes, according to (5.65),

$$W_s = \int_o^\infty \int_o^\infty \Psi(x,y) e^{-(x^2+y^2)/2} dxdy$$

$$= \frac{\pi}{2} \left\{ \sum_{n=o}^{N-1} \sum_{m=o}^{n} \frac{(2m)!\,[2(n-m)]!}{2^n\, m!\,(n-m)!} \left[A_{2(n-m)}^{2m} + \frac{2B_{2m}}{x_o^{2(n-m)+1}} \right] + R_N \right\} \qquad (5.84)$$

where R_N, the remainder <u>after</u> N grouped terms, is bounded as indicated by Eqs. (5.66). That is, briefly,

$$W_s \sim \frac{\pi}{2} \sum_{n=o}^{\infty} \Psi^{(n)} , \qquad (5.85)$$

where the terms of the asymptotic expansion, in accordance with (5.66) and (5.84), are given by

$$\Psi^{(n)} = \sum_{m=o}^{n} \frac{(2m)!\,\left[2(n-m)\right]!}{2^n\, m!\,(n-m)!} \left\{ A^{2m}_{2(n-m)} + \frac{2B_{2m}}{x_o^{2(n-m)+1}} \right\}, \tag{5.86}$$

from which, with the aid of Eqs. (5.69), one readily computes the first few terms.

The evaluation of the integral (5.83), at least insofar as the x variable is concerned, has already been carried out in Section 5.le. Thus, assuming as before that $\arg\left\{x_o\right\} > 0$ so that the path of integration in the x-plane runs below the pole at $x = x_o$, we have immediately from Eq. (5.41) that

$$W_p = i\pi e^{-x_o^2/2}\left\{1 - \mathrm{erf}\left[-ix_o/(2)^{\frac{1}{2}}\right]\right\} \int_o^{\infty} C(y)e^{-y^2/2}dy , \tag{5.87}$$

or simply

$$W_p = i\pi C e^{-x_o^2/2}\left\{1 - \mathrm{erf}\left[-\,ix_o/(2)^{\frac{1}{2}}\right]\right\} , \tag{5.88}$$

exactly as in (5.41) and where the constant C is given by

$$C = \int_o^{\infty} C(y)e^{-y^2/2}dy . \tag{5.89}$$

The actual evaluation of the constant C is deferred until Section 6.3e where we discuss the specific instance of the subtraction of a first order pole.

VI. EVALUATION OF THE INTEGRALS PERTAINING TO THE CONDUCTING MEDIUM

The fundamental integrals corresponding to points of observation in the conducting medium, $z \leq 0$, have been defined as U_1 and V_1 by Eqs. (2.88) and (2.89) respectively, and are here rewritten for convenience as

$$U_1(\rho,z) = \int_{-\infty}^{\infty} \frac{1}{\gamma_1 + \gamma_2} e^{\gamma_1(z-h)} H_0^1(\lambda\rho)\lambda d\lambda \tag{6.1}$$

and

$$V_1(\rho,z) = \int_{-\infty}^{\infty} \frac{k_1^2}{k_2^2\gamma_1 + k_1^2\gamma_2} e^{\gamma_1(z-h)} H_0^1(\lambda\rho)\lambda d\lambda \, . \tag{6.2}$$

According to Eq. (3.17), however, the fundamental integral U_1 can be resolved into two terms, as follows:

$$k_1^2 U_1 = k_1^2 M_1 - \frac{2}{1-n^2} \frac{\partial^2 \Psi_2}{\partial z^2} , \quad z \leq 0 , \tag{6.3}$$

where $\Psi_2(\rho,z)$ is given by Eq. (2.66) and where M_1, according to Eq. (3.18), is defined as a new fundamental integral from

$$(k_1^2 - k_2^2)M_1(\rho,z) = \int_{-\infty}^{\infty} \gamma_2 e^{\gamma_1(z-h)} H_0^1(\lambda\rho)\lambda d\lambda , \quad z \leq 0 , \tag{6.4}$$

which is actually more convenient to deal with than U_1.

Both integrals of interest, M_1 and V_1, as already indicated in Eq. (2.92) are of the general form

$$I(\rho,z) = \frac{1}{2}\int_{-\infty}^{\infty} v(\lambda) e^{\gamma_1(z-h)} H_0^1(\lambda\rho)\lambda d\lambda , \quad z \leq 0 , \tag{6.5}$$

where, in this case

$$v(\lambda) = \begin{cases} \dfrac{2\gamma_2}{k_1^2 - k_2^2} & \text{for} \quad M_1(\rho,z) , \\ \\ \dfrac{2k_1^2}{k_2^2\gamma_1 + k_1^2\gamma_2} & \text{for} \quad V_1(\rho,z) . \end{cases} \tag{6.6}$$

Furthermore, it was shown in Section 2.5d and illustrated in Fig. 4 that the typical integral (6.5) can be resolved by a suitable deformation of the original path of integration into the sum of two integrals,

$$I = I_1 + I_2 , \tag{6.7}$$

where I_1 is the integral along the contour C_1 around the upper branch cut for γ_1 and I_2 denotes the integral along the contour C_2 around the right hand branch cut for γ_2. In this Chapter we apply the methods developed in the preceding one to the evaluation of the integrals I_1 and I_2, corresponding to M_1 and V_1, by the saddle point method of integration. It will be shown that the contribution from I_1 is generally negligible as compared with I_2 due to the fact that I_1, at low frequencies, is of the order of magnitude of the uncertainty in the asymptotic evaluation of I_2. Notwithstanding, we present in the next section the asymptotic evaluation of I_1 for completeness sake and because the results may prove of interest to other workers.

6.1 EVALUATION OF I_1 BY THE SADDLE POINT METHOD

According to Section 5.1, to evaluate I_1 asymptotically we must transform the variable of integration in such a way that the integral assumes the form given in Eq. (5.2), that is,

$$I_1 = \int_{C_1^{\prime}} F(w)e^{\phi(w)}dw \,, \tag{6.8}$$

where $C_1^{\prime}$ is the path passing through the saddle point $w = 0$ and is obtained by a permissible deformation from the transformed path corresponding to C_1 in the λ-plane. This transformation to the w-plane is obtained by considering first the conformal transformation

$$\lambda = k_1 \sin \alpha_1, \tag{6.9}$$

already introduced in Section 2.4b, Eqs. (2.76), which we discuss at greater length in the next section.

6.1a. Transformation to the α_1-plane.- The conformal transformation (6.9) takes us from the λ-plane depicted in Fig. 4 to the α_1-plane shown in Fig. 6. It is seen that the entire Sheet I of the λ-plane maps into the half-period strip of width π in the α_1-plane with curvilinear boundaries passing through the points $\alpha_1 = \pm\,\pi/2$ corresponding to the branch points at $\lambda = \pm\, k_1$, $k_1 = |k_1| e^{i\pi/4}$, in the λ-plane. To determine the equations of the boundaries of the half-period strip we note from (6.9), making use of the first of Eqs. (2.58), that

$$\gamma_1 = (\lambda^2 - k_1^2)^{\frac{1}{2}} = -\, ik_1 \cos\alpha_1 , \qquad (6.10)$$

whence, recalling that γ_1 is pure imaginary along the corresponding branch cuts, we see that the equation of the boundaries is given by demanding that

$$\text{Re}\left\{-ik_1 \cos\alpha_1\right\} = 0 . \qquad (6.11)$$

Writing $\alpha_1 = u + iv$ and recalling that $\arg\left\{k_1\right\} = \pi/4$, the above condition results in the equation

$$\cos u \cosh v - \sin u \sinh v = 0 , \qquad (6.12)$$

from which, solving for v, one readily deduces

$$v = -\tfrac{1}{2} \log \tan\left[u + (4m - 1)\pi/4\right] , \quad m = 0, \pm 1, \pm 2, \ldots , \qquad (6.13)$$

as the equations for the denumerable infinite set of periodic boundaries which divide the α_1-plane into alternate strips corresponding to Sheets I and III. In particular, putting $m = 0$ in (6.13) yields

$$v = -\tfrac{1}{2} \log \tan\left(u - \frac{\pi}{4}\right) \qquad (6.14)$$

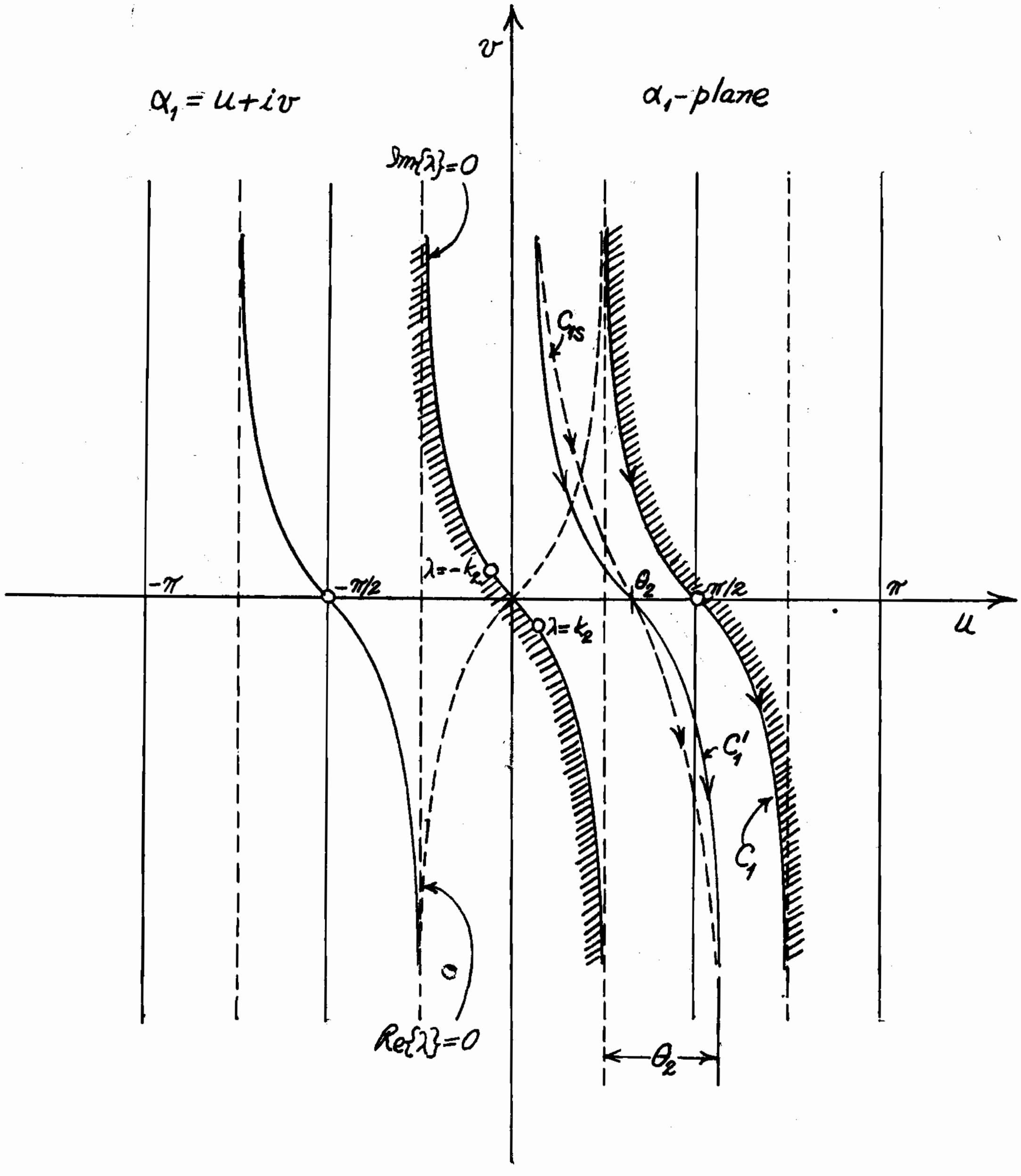

Fig. 6.- The α_1-plane illustrating the half-period strip corresponding to Sheet I in the λ-plane and the paths of integration C_1, C_1' and C_{1s}.

as the equation for the boundary C_1 corresponding in the λ-plane to the upper branch cut for γ_1. It is readily seen from (6.14) that, as $v \to \pm\infty$, $u \to \pi/4, 3\pi/4$, respectively, and that the slope of the curve, as it crosses the real axis at $\alpha_1 = \pi/2$, is -1 as indicated by the symmetric curve sketched in Fig. 6. Likewise, putting $m = 1$ in (6.13) yields

$$v = -\frac{1}{2}\log\tan\left(u + \frac{3\pi}{4}\right) \tag{6.15}$$

as the equation for the left boundary of the (principal) half-period strip corresponding to the contour around the lower branch cut for γ_1. It is seen that the curve given by (6.15) is identical to the contour C_1 except that it crosses the real axis at $\alpha_1 = -\pi/2$.

To examine further the mapping of Sheet I of the λ-plane unto the principal half-period strip in the α_1-plane, it is of interest to study how the real and imaginary axes in the λ-plane map into the α_1-plane. The equation of the real axis, $\operatorname{Im}\left\{k_1 \sin\alpha_1\right\} = 0$, deduced by the identical methods indicated above becomes

$$v = -\frac{1}{2}\log\tan\left(u + \frac{\pi}{4}\right), \tag{6.16}$$

which corresponds to a symmetric curve passing through the origin, of exactly the same shape as the boundaries as indicated in Fig. 6; that is, we have from (6.16) that, as $v \to \pm\infty$, $u \to \mp\pi/4$ and the curve crosses the origin $\alpha_1 = 0$ with a slope of -1. Similarly, the equation of the axis of imaginaries, $\operatorname{Re}\left\{k_1 \sin\alpha_1\right\} = 0$, becomes

$$v = -\frac{1}{2}\log\tan\left(\frac{\pi}{4} - u\right), \tag{6.17}$$

which is obtained from (6.16) by merely changing u into $-u$. Thus, the mapping into the α_1-plane of the real and imaginary axes in the λ-plane results in two curves which are symmetric about the $u = 0$ axis as indicated in Fig. 6.

It is clear from the discussion of Section 2.5 that crossing the boundaries passing through $\alpha_1 = \pm \pi/2$ in Fig. 6 is equivalent to crossing the branch cuts for γ_1 onto Sheet II which according to Section 2.5d is not accessible to us. Furthermore, the presence of the Hankel function in the integrands under discussion forbids any excursion of the path of integration into the lower half of the λ-plane, which means that in the α_1-plane the permissible deformations of the original path of integration C_1 must be confined to the quarter period strip lying between the mapping of the real axis, $\text{Im}\left\{k_1 \sin\alpha_1\right\} = 0$, and the mapping of the contour C_1 itself; that is, the region between the shaded boundaries in Fig. 6. It is recalled that these limitations are imposed by the requirement that our integrals vanish in the limits $\rho \to \infty$ and $(z - h) \to -\infty$.

Finally, making the substitution (6.9) into the integrand of (6.5) and writing for convenience

$$v(k_1 \sin\alpha_1) = \frac{2i}{k_1} G(\alpha_1), \tag{6.18}$$

we obtain for the integral I_1 along the contour C_1 the expression

$$I_1 = \tfrac{1}{2} ik_1 \int_{C_1} G(\alpha_1) \sin 2\alpha_1 H_0^1(k_1 \rho \sin\alpha_1) e^{-ik_1(z-h)\cos\alpha_1} d\alpha_1, \tag{6.19}$$

in which, making use of Eqs. (6.6) and (6.18), and noting that

$$\gamma_2 = (\lambda^2 - k_2^2)^{\frac{1}{2}} = - ik_1(n^2 - \sin^2\alpha_1)^{\frac{1}{2}} , \tag{6.20}$$

we obtain for $G(\alpha_1)$ the two expressions

$$G(\alpha_1) = \begin{cases} -\dfrac{(n^2 - \sin^2\alpha_1)^{\frac{1}{2}}}{1 - n^2} & \text{for } M_1^{(1)} \\ \\ \left[n^2\cos\alpha_1 + (n^2 - \sin^2\alpha_1)^{\frac{1}{2}}\right]^{-1} & \text{for } V_1^{(1)} \end{cases} \tag{6.21}$$

where the superscripts (1) on M_1 and V_1 denote the evaluation of the corresponding integrals along the contour C_1.

<u>6.1b. Path of integration through the saddle point.</u>- To bring the integral (6.19) into the required form (6.8) we first rewrite it in the form

$$I_1 = \tfrac{1}{2}\, ik_1 \int_{C_1} G(\alpha_1)\sin 2\alpha_1 \left\{H_0^1(k_1\rho\sin\alpha_1)e^{-ik_1\rho\sin\alpha_1}\right\} e^{ik_1R_2\cos(\alpha_1-\theta_2)}d\,\alpha_1, \tag{6.22}$$

where R_2 and θ_2 are defined by Eqs. (2.1). This form, which is obtained from (6.19) by merely extracting from the Hankel function $H_0^1(k_1\rho\sin\alpha_1)$ its asymptotic exponential behavior, now has an exponential factor which exhibits a saddle point at $\alpha_1 = \theta_2$, for which the derivative vanishes. Thus, from the discussion leading to Section 5.1a and from our knowledge of the permissible deformations of the original path of integration C_1, discussed in the preceding Section, we may take for the path of integration C_1' through the saddle point $\alpha_1 = \theta_2$ any curve lying within the permissible quarter-period strip defined above. In particular, we choose C_1' as the path of integration, given by $\text{Im}\left\{k_1 \sin(\alpha_1 - \theta_2)\right\} = 0$, as shown in Fig. 6, which is obtained by translating the path C_1 parallel to the axis of reals in the α_1-plane until it crosses the real axis at the saddle point $\alpha_1 = \theta_2$. Thus, either from (6.14) or from (6.16), one readily deduces that the equation of the chosen path of integration through the saddle point, C_1', becomes

$$v = -\tfrac{1}{2} \log \tan(u - \theta_2 + \tfrac{\pi}{4}) , \qquad (6.23)$$

which is seen to cross the saddle point at $\alpha_1 = \theta_2$ with a slope of -1 and which has asymptotes $u \rightarrow \theta_2 \mp \pi/4$ as $v \rightarrow \pm \infty$, respectively. This path of integration has the virtue that, for $0 \leq \theta_2 \leq \pi/2$, the entire path lies within the admissible quarter-period strip.

The path of steepest descent, however, which was employed by Ott[30] in similar calculations is obtained from (6.22) by demanding that

$$\text{Im}\left\{ik_1R_2 \cos(\alpha_1 - \theta_2)\right\} = \text{Im}\left\{ik_1R_2\right\} , \qquad (6.24)$$

which leads to the path labelled C_{1s} in Fig. 6 and which exhibits the same vertical asymptotes as C_1' in Eq. (6.23), the only difference being that the slope at the saddle point now corresponds to an angle of $-3\pi/8$ instead of $-\pi/4$. It is seen that, as a consequence of this steeper slope, the entire length of the path C_{1s} does not lie within the permissible region for all values of θ_2.

To complete the picture in the α_1-plane we must indicate the branch points $\lambda = \pm k_2$ and the corresponding branch cuts for γ_2. Putting $\lambda = \pm k_2$ in (6.9), we obtain for the branch points

$$\alpha_1 = \pm \sin^{-1} n , \qquad (6.25)$$

[30] H. Ott, Ann. Physik 41, 443-467 (1942).

where $n = k_2/k_1$ is, in general, very small. As indicated in Fig. 6, the branch points occur on the curve $\mathrm{Im}\left\{k_1 \sin\alpha_1\right\} = 0$ which corresponds to the real axis in the λ-plane (k_2, real); and, because $|n| << 1$, they appear very close to the origin. The corresponding branch cuts for γ_2 coincide with the above curve except for the portion between the branch points. It is clear from Fig. 6 that crossing this set of cuts takes us onto Sheet III which in this problem is still accessible to us. More explicitly, the α_1-plane is a Riemann surface of two sheets, the top sheet depicted in Fig. 6 corresponding by alternate half-period strips to Sheets I and II of the λ-plane, while the bottom sheet similarly corresponds to Sheets III and IV (see Fig. 5).

Finally, to exhibit our integral in the required form (6.8), we transform the origin to the saddle point by the substitution

$$w = \alpha_1 - \theta_2, \quad \alpha_1 = \theta_2 + w, \tag{6.26}$$

whence (6.22) becomes

$$I = \tfrac{1}{2}\, ik_1 \int_{C_1'} F(w)e^{\phi(w)}dw, \tag{6.27}$$

where C_1' denotes the <u>symmetric</u> path of integration through the saddle point $w = 0$, as chosen above, and where

$$\phi(w) = ik_1R_2 \cos w \tag{6.28}$$

and

$$F(w) = G(\theta_2 + w)\sin 2(\theta_2 + w)\left\{H_o^1\left[k_1\rho \sin(\theta_2 + w)\right] e^{-ik_1\rho\sin(\theta_2+w)}\right\}, \tag{6.29}$$

with $G(\theta_2 + w)$ as prescribed by Eqs. (6.21).

6.1c. Transformation to the x-plane.— In accordance with the prescriptions of Section 5.1, as embodied in Eq. (5.7), we introduce the new variable of integration x defined, in the present instance, by

$$\frac{1}{2} x^2 = \phi(o) - \phi(w) = ik_1R_2(1 - \cos w) = 2ik_1R_2\sin^2\frac{w}{2}, \qquad (6.30)$$

which, in this case, can be inverted at once without resorting to the procedure outlined in Section 5.1b. Thus, we have from (6.30)

$$w = 2 \sin^{-1}\left[x/(4ik_1R_2)^{\frac{1}{2}}\right], \qquad (6.31)$$

which, for $|x| < 2(|ik_1R_2|)^{\frac{1}{2}}$ can be expanded into a power series as in Eq. (5.22),

$$w = a_o x + \frac{1}{3} a_2 x^3 + \frac{1}{5} a_4 x^5 + \cdots, \qquad (6.32)$$

where

$$a_o = (ik_1R_2)^{-\frac{1}{2}}, \quad a_2 = a_o^3/8, \quad a_4 = 3a_o^5/128, \quad \cdots. \qquad (6.33)$$

In consequence of the transformation (6.30) and making use of Eqs. (5.10) and (6.27), the integral I_1 may now be expressed as

$$I_1 = \frac{1}{2} ik_1 e^{ik_1R_2} \int_o^\infty \Phi(x) e^{-x^2/2} dx, \qquad (6.34)$$

where $\Phi(x)$ is defined by (5.9) in terms of the function $F(w)$, with $w = w(x)$, as given in the present case by Eqs. (6.29) and (6.30). The path of integration in (6.34), as already pointed out (Section 5.1), coincides with the positive half of the real axis in the x-plane for the path of steepest descents C_{1s}; however, our chosen path of integration C_1' which has the same termini as the path of steepest descents (Fig. 6) no longer coincides with the real axis except at $x = 0$ and $x \rightarrow \infty$. In either case, according to Eq. (5.20b), the integral I_1 admits the asymptotic expansion

$$I_1 = \tfrac{1}{2}\, ik_1 e^{ik_1 R_2} \cdot \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left\{ \sum_{m=0}^{M-1} \frac{(2m)!}{2^m\, m!}\, A_{2m} + O(\lambda^{-M-1}) \right\} \tag{6.35}$$

in terms of the expansion coefficients A_{2m} and in which $\lambda^{\frac{1}{2}} = |x_1|$, where x_1 corresponds to the singularity nearest to the origin in the power series expansion (5.11).

In the present instance, except for values of θ_2 near zero, when the singularity of the Hankel function must be considered, the nearest singularity of the integrand in (6.34) arises from the function $G(\alpha_1)$, Eq. (6.21), and occurs at the branch point corresponding to $\lambda = k_2$ or $\alpha_1 = \sin^{-1} n$. Since we are here concerned with the order of magnitude of the remainder in (6.35) and because $|n| \ll 1$, we incur little error by assuming that the nearest singularity occurs at $\alpha_1 = 0$. Thus from (6.30), putting $w = -\theta_2$, we have approximately

$$\lambda \approx 4|k_1|R_2 \sin^2(\theta_2/2) \xrightarrow[\theta_2 \to \pi/2]{} 2|k_1|R_2\,, \tag{6.36}$$

which is valid provided only that $\theta_2 \gg |\sin^{-1} n|$, as is certainly true in practice.

6.1d. Evaluation of the expansion coefficients.- Limiting the asymptotic expansion (6.35) to two terms $(M = 2)$, we have

$$I_1 = \tfrac{1}{2}\, ik_1 e^{ik_1R_2} \cdot \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left\{ A_o + A_2 + O(\lambda^{-3}) \right\} , \tag{6.37}$$

where λ is given by (6.36). Thus, the problem has been reduced to the evaluation of the expansion coefficients A_o and A_2 as taken from Eqs. (5.26) in which the coefficients a_o and a_2 are given by (6.33) in the present case; and we still need to evaluate $F(0)$ and $F''(0)$ where $F(w)$ is here given by Eq. (6.29). To this end we note that $F(w)$ can be written as

$$F(w) = G(\theta_2 + w)\, \sin 2(\theta_2 + w)\, g(w) , \tag{6.38}$$

where

$$g(w) = H_o^1\left[k_1\rho \sin(\theta_2 + w)\right] e^{-ik_1\rho\sin(\theta_2+w)} , \tag{6.39}$$

from which one obtains readily

$$F(o) = G(\theta_2)\, \sin 2\theta_2\, g(o) ,$$

$$F''(o) = G(\theta_2)\, \sin 2\theta_2 \left\{ \left[\frac{G''(\theta_2)}{G(\theta_2)} + 4\,\frac{G'(\theta_2)}{G(\theta_2)}\cot 2\theta_2 - 4\right] g(o) + \left[2\,\frac{G'(\theta_2)}{G(\theta_2)} + 4\cot 2\theta_2\right] g'(o) + g''(o) \right\} , \tag{6.40}$$

where $g(o)$, $g'(o)$ and $g''(o)$ are to be computed from Eq. (6.39). Carrying out the differentiations in (6.39) and replacing the Hankel functions and their derivatives by their asymptotic expansions which are valid for $\left|k_1\rho\,\sin\theta_2\right| \gg 1$, we obtain

$$g(o) = (2/\pi i k_1 \rho \sin\theta_2)^{\frac{1}{2}} \left\{ 1 + \frac{1}{8ik_1\rho\sin\theta_2} + O(k_1\rho\sin\theta_2)^{-2} \right\} ,$$

$$g'(o) = -\tfrac{1}{2}\cot\theta_2 (2/\pi i k_1 \rho \sin\theta_2)^{\frac{1}{2}} \left\{ 1 + O(k_1\rho\sin\theta_2)^{-1} \right\} , \tag{6.41}$$

$$g''(o) = (2/\pi i k_1 \rho \sin\theta_2)^{\frac{1}{2}} \left\{ \tfrac{1}{2} + \tfrac{3}{4}\cot^2\theta_2 + O(k_1\rho\sin\theta_2)^{-1} \right\} .$$

Substituting the results (6.41) into (6.40) and collecting terms we have,

$$F(o) = G(\theta_2)\, \sin 2\theta_2 (2/\pi i k_1 \rho \sin\theta_2)^{\frac{1}{2}} \left\{ 1 + \frac{1}{8ik_1\rho\sin\theta_2} + O(k_1\rho\sin\theta_2)^{-2} \right\} ,$$

$$F''(o) = G(\theta_2)\, \sin 2\theta\, (2/\pi i k_1 \rho \sin\theta_2)^{\frac{1}{2}} \left\{ \frac{G''(\theta_2)}{G(\theta_2)} + \frac{G'(\theta_2)}{G(\theta_2)} (\cot\theta_2 - 2\tan\theta_2) - \frac{1}{4}\cot\theta_2 - \frac{5}{2} + O(k_1\rho\sin\theta_2)^{-1} \right\} . \tag{6.42}$$

Finally, making use of the first two equations in (5.26), with the coefficients a_o and a_2 as given in (6.33) and substituting the above results, we have for the expansion terms A_o and A_2 the following asymptotic expressions:

$$\left(\frac{\pi}{2}\right)^{\frac{1}{2}} A_o = 4G(\theta_2)\cos\theta_2 \left\{ \frac{1}{ik_1R_2} + \frac{\csc^2\theta_2}{8(ik_1R_2)^2} + O\left(\frac{\csc^4\theta_2}{(k_1R_2)^3}\right) \right\} ,$$

$$\left(\frac{\pi}{2}\right)^{\frac{1}{2}} A_2 = 2G(\theta_2)\cos\theta_2 \left\{ \frac{1}{(ik_1R_2)^2} \left[\frac{G''(\theta_2)}{G(\theta_2)} + \frac{G'(\theta_2)}{G(\theta_2)} (\cot\theta_2 - 2\tan\theta_2) - \frac{1}{4}(9 + \cot^2\theta_2) \right] + O\left(\frac{\csc^2\theta_2}{(k_1R_2)^3}\right) \right\} . \tag{6.43}$$

Substituting these results into the two-term asymptotic expansion (6.37) and noting that for $\theta_2 \gg |\sin^{-1} n|$ we have essentially $\csc\theta_2 = O(1)$, we obtain*

$$I_1 = 2ik_1 G(\theta_2)\cos\theta_2 e^{ik_1R_2}\left\{\frac{1}{ik_1R_2} + \frac{1}{2(ik_1R_2)^2}\left[\frac{G''(\theta_2)}{G(\theta_2)} + \frac{G'(\theta_2)}{G(\theta_2)}(\cot\theta_2 - 2\tan\theta_2) - 2\right] + O(\lambda^{-3})\right\} \qquad (6.44)$$

where λ is given in the present instance by (6.36) and where we have lumped together terms in like powers of $(ik_1R_2)^{-1}$.

An interesting check of the above formula is obtained when applied to the integral, [Eq. (2.66)],

$$\Psi_2 = \frac{e^{ik_1R_2}}{R_2} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{\gamma_1}e^{\gamma_1(z-h)}H_0^1(\lambda\rho)\lambda d\lambda, \quad z \leq 0, \qquad (6.45)$$

which represents the elementary spherical wave emanating from the image source. In this case, in accordance with (6.18), $G(\alpha_1) = \frac{1}{2}\sec\alpha_1$ and the integrand of (6.45) when expressed in the form (6.22) exhibits no singularity in the finite plane. Therefore, the first term of the expansion (6.35) must represent the function ψ_2. Substituting $G(\alpha_1) = \frac{1}{2}\sec\alpha_1$ into (6.44) is seen to yield the required function (6.45) with zero for the second order term; and the computation of higher order terms would, of course, also yield zero.

6.1e. Asymptotic expansions for the integrals $M_1^{(1)}$ and $V_1^{(1)}$.- Applying the above results to the integrals of interest, M_1 and V_1, as

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* This result should be compared with Ott's own computation of the second order term, loc. cit., paragraph 4, p. 450. Ott did not obtain the correct second order term and, therefore, arrived at erroneous conclusions.

evaluated asymptotically over the path C_1' through the saddle point in the α_1-plane, we obtain the following results:

For $V_1^{(1)}$ we substitute into (6.44) the expression for $G(\theta_2)$ which is deduced from the second of Eqs. (6.21) and putting $M = 1$ we obtain at once the leading term of the expansion

$$V_1^{(1)} \sim \frac{2\cos\theta_2}{n^2\cos\theta_2 + (n^2 - \sin^2\theta_2)^{\frac{1}{2}}} \frac{e^{ik_1R_2}}{R_2} \tag{6.46}$$

Similarly, for $M_1^{(1)}$ we substitute into (6.44) the expression for $G(\theta_2)$ given by the first of Eqs. (6.21) and, putting $M = 2$, we obtain the two-term expansion

$$M_1^{(1)} \sim -\frac{2\cos\theta_2(n^2 - \sin^2\theta_2)^{\frac{1}{2}}}{(1 - n^2)} \frac{e^{ik_1R_2}}{R_2}\left\{1 - \frac{i(1 - 6\sin^2\theta_2) + O(n^2)}{2\,k_1R_2\sin^2\theta_2}\right\} \tag{6.47}$$

in which the first term is exact and the second term is given only to terms of $O(n^2)$, $|n| \ll 1$, under the assumption that $\theta_2 \gg |\sin^{-1}n|$. In both, Eqs. (6.46) and (6.47), we have omitted writing the remainder which appears in Eq. (6.44) and is $O(\lambda^{-M-1})$ in accordance with Eq. (5.20a).

6.2 EVALUATION OF I_2 BY THE SADDLE POINT METHOD FOR SINGLE INTEGRATION

The evaluation of I_2 by the saddle point method for a single integration now requires, in accordance with the method developed in the preceding Section, that we transform the integral over the contour C_2 in the λ-plane (Fig. 4) into an integral of the form (5.2) with a suitable path of integration passing through the saddle point $w = 0$ in the w-plane. To this end, we discovered that the conformal transformation

$$\lambda = k_2 \sin\alpha_2 , \tag{6.48}$$

which apparently had been overlooked by earlier authors, proved most convenient in the present problem.

6.2a. Transformation to the α_2-plane.- The conformal transformation (6.48) takes us from the λ-plane depicted in Fig. 4 to the α_2-plane shown in Fig. 7. It is seen that the entire Sheet I of the λ-plane now maps into the vertical half-period strip of width π in the α_2-plane which, for k_2 real, has straight line boundaries parallel to the axis of imaginaries, passing through the points $\alpha_2 = \pm \pi/2$ which correspond to the branch points at $\lambda = \pm k_2$ in the λ-plane. To determine these boundaries we note from (6.48), making use of the second of Eqs. (2.58), that

$$\gamma_2 = (\lambda^2 - k_2^2)^{\frac{1}{2}} = - ik_2 \cos\alpha_2 , \tag{6.49}$$

from which, recalling that γ_2 is real along the chosen branch cuts, we see that the equation of the boundaries is obtained by demanding that

$$\mathrm{Im} \left\{ - ik_2 \cos\alpha_2 \right\} = 0 , \tag{6.50}$$

or, with k_2 real, $\mathrm{Re}\left\{\cos\alpha_2\right\} = 0$. Writing $\alpha_2 = u + iv$, this condition results in

$$\cos u = 0 \quad \text{or} \quad u = \frac{\pi}{2} + m\pi, \quad m = 0, \pm 1, \ldots , \tag{6.51}$$

which are the equations for the denumerable infinite set of periodic boundaries which divide the α_2-plane into alternate vertical strips of

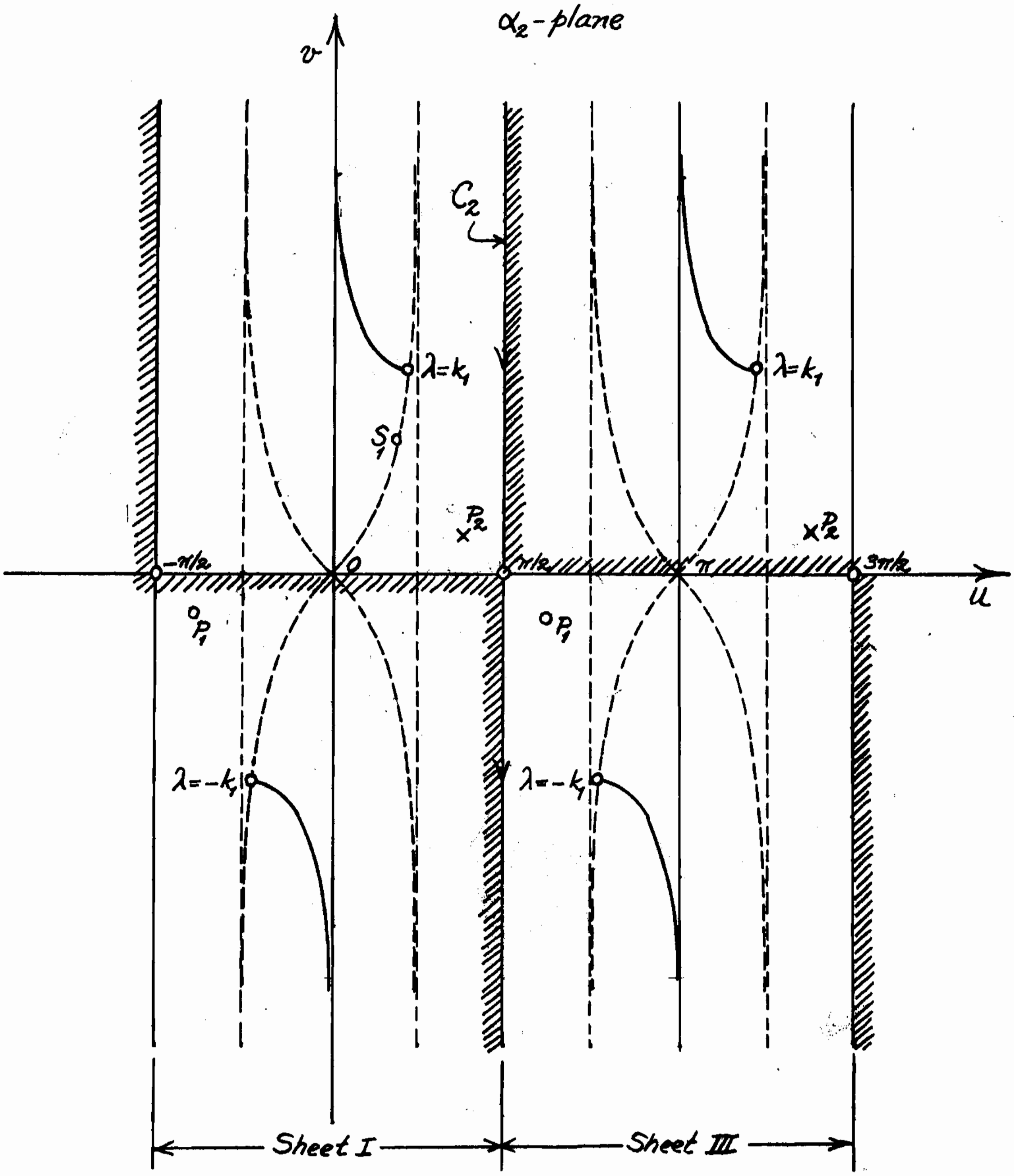

Fig. 7.— The α_2-plane illustrating the half-period strips corresponding to Sheets I and III in the λ-plane and the chosen path of integration C_2.

width π corresponding to Sheets I and III. In particular, putting $m = 0$ in (6.51) yields $u = \pi/2$ which is the equation for the boundary C_2 corresponding in the λ-plane to the contour around the right hand branch cut for γ_2. Similarly, putting $m = -1$ in (6.51) yields $u = -\pi/2$, which is the equation for the left boundary corresponding to the contour around the left hand branch cut for γ_2.

Confining our attention to the principal half-period strip $-\pi/2 \leq u \leq \pi/2$, corresponding to Sheet I, it is seen that the axis of imaginaries in the λ-plane maps simply into the axis of imaginaries in the α_2-plane; while the segment of the real axis in the λ-plane between the branch points at $\lambda = \pm k_2$ maps into the segment of the real axis in the α_2-plane between the corresponding points at $\alpha_2 = \pm \pi/2$. Furthermore, it is clear from the discussion of Section 2.5 that crossing the boundaries $u = \pm \pi/2$ in Fig. 7 away from the principal half-period strip is equivalent to crossing the branch cuts for γ_2 onto Sheet III which, according to Section 2.5d, is still accessible to us. Since we must guarantee the convergence of the integral I_2 as $\rho \rightarrow \infty$, which in the λ-plane means that the permissible deformations of the path of integration must be confined to the upper half-plane, we see that in the α_2-plane the permissible deformations of the original path of integration C_2 must be confined to the upper half of the (principal) half-period strip for Sheet I and to the lower half of the half-period strip, say $\pi/2 \leq u \leq 3\pi/2$, for Sheet III. This region for permissible deformations is bordered by shaded boundaries in Fig. 7. Finally, it must be noted that to insure the convergence of our integral I_2 as $(z - h) \rightarrow -\infty$ we cannot cross the branch cuts for γ_1, shown in Fig. 7, onto the forbidden Sheets II and IV.

Substituting (6.48) into the integrand of (6.5), writing for convenience

$$v(k_2 \sin \alpha_2) = \frac{2i}{k_2} G_2(\alpha_2), \tag{6.52}$$

and noting that

$$\gamma_1 = (\lambda^2 - k_1^2)^{\frac{1}{2}} = -ik_1(1 - n^2 \sin^2 \alpha_2)^{\frac{1}{2}}, \tag{6.53}$$

we obtain for the integral I_2 along the contour C_2 the expression

$$I_2 = \tfrac{1}{2} ik_2 \int_{C_2} G_2(\alpha_2) \sin 2\alpha_2 \, H_0^1(k_2 \rho \sin \alpha_2) e^{-ik_1(z-h)(1-n^2\sin^2\alpha_2)^{\frac{1}{2}}} \, d\alpha_2, \tag{6.54}$$

in which the path of integration in the α_2-plane is along the right hand boundary of the principal half-period strip, $u = \pi/2$, from $\pi/2 + i\infty$ to $\pi/2 - i\infty$ and in which, making use of (6.6) and (6.52), we obtain for $G_2(\alpha_2)$ the two expressions

$$G_2(\alpha_2) = \begin{cases} -n^2(1 - n^2)^{-1} \cos \alpha_2 & \text{for } M_1^{(2)} \\[2ex] \left(n(1 - n^2 \sin^2\alpha_2)^{\frac{1}{2}} + \cos \alpha_2\right)^{-1} & \text{for } V_1^{(2)} \end{cases} \tag{6.55}$$

where the superscripts (2) on M_1 and V_1 denote the evaluation of the corresponding integrals along the contour C_2.

To complete the picture in the α_2-plane we must indicate the branch points $\lambda = \pm k_1$ and the corresponding cuts for γ_1. Putting $\lambda = \pm k_1$ into

(6.48), we obtain for the branch points

$$\alpha_2 = \pm \sin^{-1}(1/n) , \qquad (6.56)$$

where $n = |n|e^{-i\pi/4}$. The positions of the periodic set of branch points (6.56) and the corresponding branch cuts for γ_1 are shown in Fig. 7. As indicated, the branch points occur in the α_2-plane on the curves representing the mapping of the straight line through the points $\lambda = \pm k_1$ in the λ-plane. It is clear from Fig. 7 that crossing the branch cut for γ_1 takes us from Sheet I onto Sheet II or from Sheet III onto Sheet IV (Fig. 5) depending on the cut crossed. This is due to the fact that the α_2-plane is in reality a Riemann surface of two sheets, the top sheet depicted in Fig. 7 corresponding by alternate half-period strips to Sheets I and III of the λ-plane, while the bottom sheet similarly corresponds to Sheets II and IV.

6.2b. Path of integration through the saddle point.- To bring the integral (6.54) into the required form (5.2) we first extract the asymptotic exponential behavior of the Hankel function, as was done in Eq. (6.22), and next we introduce a second transformation of the variable of integration,

$$\alpha_2 = w + \pi/2 \quad \text{or} \quad w = \alpha_2 - \pi/2 . \qquad (6.57)$$

The integral I_2 is finally written in the form

$$I_2 = -\tfrac{1}{2} ik_2 \int_{C_2} F(w)e^{\phi(w)}\, dw , \qquad (6.58)$$

where C_2 now becomes the axis of imaginaries in the w-plane traversed in the negative direction, that is, from $i\infty$ to $-i\infty$, and where

$$F(w) = G(w)\ \sin 2w\ H_0^1(k_2\rho \cos w)e^{-ik_2\rho\cos w}, \tag{6.59}$$

and

$$\phi(w) = ik_1\rho\left[n\cos w + \cot\theta_2(1 - n^2\cos^2 w)^{\frac{1}{2}}\right], \tag{6.60}$$

with $G(w)$, as deduced from (6.55) and (6.57), given by the two distinct expressions

$$G(w) = \begin{cases} n^2(1-n^2)^{-1}\sin w & \text{for } M_1^{(2)}, \\ \left[n(1-n^2\cos^2 w)^{\frac{1}{2}} - \sin w\right]^{-1} & \text{for } V_1^{(2)}. \end{cases} \tag{6.61}$$

This form of the integral I_2, Eq. (6.58), exhibits an exponent $\phi(w)$, as defined by (6.60), which has two stationary points within the principal half-period strip; that is, $\phi'(w) = 0$ for $w = 0$ and $w = \cos^{-1}(n^{-1}\sin\theta_2)$. The latter saddle point is seen to correspond to the saddle point S_1 at $\alpha_1 = \theta_2$ in the α_1-plane (Section 6.1b) and, hence, has already been treated in the evaluation of I_1. Therefore, we choose $w = 0$ as the saddle point S_2 in the w-plane through which must pass the path of integration C_2. As already indicated, we have chosen for the path of integration C_2 the axis of imaginaries in the w-plane or the vertical line $u = \pi/2$ in the α_2-plane (Fig. 7) traversed in the negative direction.

The path of steepest descents C_{2s} is obtained from (6.60) by demanding that

$$\operatorname{Im}\left\{\phi(w)\right\} = \operatorname{Im}\left\{\phi(o)\right\}, \tag{6.62}$$

which leads to a somewhat complicated path that offers no particular advantage in the present study. The only important point to observe is that, for all values of θ_2, the path of steepest descents C_{2s} crosses the saddle point $w = 0$ at an angle of inclination with respect to the negative axis of imaginaries which never exceeds $\pi/4$ and that the termini of the path tend asymptotically to $\mathrm{Re}\left\{w\right\} = \pm\,\theta_2$.

6.2c. Transformation to the x-plane.-- In accordance with Eq. (5.7), we now introduce the new variable of integration x defined from (6.60) as

$$\tfrac{1}{2}\,x^2 = \phi(o) - \phi(w) = ik_1\rho\left\{n(1 - \cos w) + \cot\theta_2\left[(1 - n^2)^{\frac{1}{2}} - (1 - n^2\cos^2 w)^{\frac{1}{2}}\right]\right\}, \quad (6.63)$$

which we now proceed to invert, following the prescriptions of Section 5.1b. Thus, we observe that $x^2/2$ admits, at least for sufficiently small values of $|x|$, the power series expansion

$$x^2/2 = w^2(c_o + c_2w^2 + c_4w^4 + \dots), \quad (6.64)$$

where

$$c_o = ik_2\rho \cdot \frac{1}{2!}\left\{1 - \frac{n}{(1 - n^2)^{\frac{1}{2}}}\cot\theta_2\right\},$$

$$c_2 = -\,ik_2\rho \cdot \frac{1}{4!}\left\{1 - \frac{n}{(1 - n^2)^{\frac{1}{2}}}\,\frac{4 - n^2}{1 - n^2}\cot\theta_2\right\}, \quad (6.65)$$

$$c_4 = ik_2\rho \cdot \frac{1}{6!}\left\{1 - \frac{n}{(1 - n^2)^{\frac{1}{2}}}\,\frac{16 + 28n^2 + n^4}{(1 - n^2)^2}\cot\theta_2\right\}.$$

According to Section 5.1b, the power series (6.64) can be inverted in the form

$$w = a_o x + \frac{1}{3} a_2 x^3 + \frac{1}{5} a_4 x^5 + \dots \tag{6.66}$$

in which the coefficients a_{2n} $(n = 0,1,2)$ are computed by means of Eqs. (5.24) in terms of the c's defined by (6.65). To facilitate the computations we define the function

$$K = K(n, \theta_2) = \frac{n \cot\theta_2}{(1 - n^2)^{\frac{1}{2}} - n \cot\theta_2} = \frac{n(h - z)}{\rho(1 - n^2)^{\frac{1}{2}} - n(h - z)} , \tag{6.67}$$

in terms of which the coefficients of the expansion (6.66) assume the simpler forms:

$$\begin{aligned} a_o &= \left(\frac{1 + K}{ik_2\rho}\right)^{\frac{1}{2}} , \\ a_2 &= \frac{1}{8} a_o^3 \left(1 - \frac{3K}{1 - n^2}\right) , \\ a_4 &= \frac{1}{128} a_o^5 \left(3 - \frac{50K}{1 - n^2} + \frac{5K(8 + 7K)}{(1 - n^2)^2}\right) . \end{aligned} \tag{6.68}$$

It is noted from (6.67) that, when $(h - z) = 0$ or $\theta_2 = \pi/2$, we have $K = 0$ and the above coefficients assume the same form as the corresponding coefficients in Eqs. (6.33).

To complete the transformation to the x-plane we note from (6.60) that

$$\phi(o) = ik_2\rho - ik_1(1 - n^2)^{\frac{1}{2}}(z - h) ; \tag{6.69}$$

and introducing this expression into Eq. (5.10), making use of (6.58), the integral I_2 now becomes

$$I_2 = -\tfrac{1}{2}\, ik_2 e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)} \int_0^{\infty \exp(-i\theta_2/2)} \Phi(x) e^{-x^2/2}\, dx, \qquad (6.70)$$

where $\Phi(x)$ is defined by Eq. (5.9) in terms of $F(w)$ and dw/dx, with $w = w(x)$, as given in the present instance by Eqs. (6.59) and (6.66). In (6.70) the path of integration does not coincide with the positive real axis, nor is it even along a straight line, for it corresponds to the path C_2 which in the w-plane becomes the axis of imaginaries traversed in the negative direction. The phase of the upper limit in (6.70) is readily determined from (6.63) by letting $w \rightarrow -i\infty$. Thus, since the conditions of Watson's lemma are satisfied, we have from Eq. (5.20a) the following asymptotic expansion:

$$I_2 = -\tfrac{1}{2}\, ik_2 e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)} \cdot \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left\{ \sum_{m=0}^{M-1} \frac{(2m)!}{2^m\, m!} A_{2m} + O(\lambda^{-M-1}) \right\}, \qquad (6.71)$$

in which $\lambda^{\frac{1}{2}} = |x_0|$ where x_0 corresponds to the singularity nearest to the origin of the function $\Phi(x)$; that is, $\lambda^{\frac{1}{2}}$ denotes the radius of convergence of the power series expansion (5.11) about the origin in the x-plane.

6.2d. Evaluation of the expansion coefficients.- Limiting the asymptotic expansion (6.71) to three terms ($M = 3$), we have

$$I_2 = -\tfrac{1}{2}\, ik_2 e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)} \cdot \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left\{ A_0 + A_2 + 3A_4 + O(\lambda^{-4}) \right\}, \qquad (6.72)$$

where λ will be determined when we consider a specific example. Thus, the problem has been reduced to the evaluation of the expansion coefficients A_o, A_2 and A_4 as computed from Eqs. (5.26) in which we still need to evaluate $F(o)$, $F^{II}(o)$ and $F^{IV}(o)$ where $F(w)$ is here given by Eq. (6.59). Thus, we have

$$F(o) = 0, \qquad F^{II}(o) = 4G'(o)e^{-ik_2\rho} H_o^1(k_2\rho) ,$$

$$F^{IV}(o) = 8G'(o)e^{-ik_2\rho}\left\{ H_o^1(k_2\rho)\frac{G'''(o)}{G'(o)} + 3ik_2\rho\left[H_o^1(k_2\rho) - iH_1^1(k_2\rho)\right] - 4H_o^1(k_2\rho)\right\} . \tag{6.73}$$

Assuming that $k_2\rho >> 1$, we expand the Hankel functions appearing in (6.73) asymptotically, obtaining

$$F(o) = 0, \quad F^{II}(o) = 4G'(o)(2/\pi\ ik_2\rho)^{\frac{1}{2}}\left\{1 + \frac{1}{8ik_2\rho} + O(k_2\rho)^{-2}\right\} ,$$

$$F^{IV}(o) = 8G'(o)(2/\pi\ ik_2\rho)^{\frac{1}{2}}\left\{\frac{G'''(o)}{G'(o)} - \frac{5}{2} + O(k_2\rho)^{-1}\right\} . \tag{6.74}$$

Finally, making use of the first three equations in (5.26), with the coefficients a_o and a_2 as given in (6.68), we obtain for the expansion coefficients the expressions

$$A_o = 0, \quad \left(\frac{\pi}{2}\right)^{\frac{1}{2}} A_2 = 4G'(o)\,\frac{(1 + K)^{3/2}}{(ik_2\rho)^2}\left\{1 + \frac{1}{8ik_2\rho} + O(k_2\rho)^{-2}\right\} ,$$

$$\left(\frac{\pi}{2}\right)^{\frac{1}{2}} A_4 = \frac{1}{6}\,G'(o)\,\frac{(1 + K)^{5/2}}{(ik_2\rho)^3}\left\{4\,\frac{G'''(o)}{G'(o)} - 5 - \frac{15K}{1 - n^2} + O(k_2\rho)^{-1}\right\} , \tag{6.75}$$

where K has been defined by Eq. (6.67). Substituting these expressions into

(6.72) and collecting terms of the same order, we obtain the asymptotic expansion

$$I_2 = -2ik_2G'(o)(1+K)^{3/2}e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)}\left\{\frac{1}{(ik_2\rho)^2} + \frac{1}{8(ik_2\rho)^3}\left[4(1+K)\frac{G'''(o)}{G'(o)} - \frac{15K(1+K)}{1-n^2} - 5K - 4\right] + O(k_2\rho)^{-4} + O(\lambda^{-4})\right\}, \tag{6.76}$$

which is valid so long as $k_2\rho > 1$ and $\lambda > 1$, and in which we have lumped together terms in $(ik_2\rho)^{-1}$. The form of the remainder in (6.76) deserves special attention. It is seen to consist of two terms: the first term, $O(k_2\rho)^{-4}$, arose from the asymptotic expansions of the Hankel functions and implies $k_2\rho > 1$; the second term containing the factor $O(\lambda^{-4})$ arose from the original asymptotic expansion as given in Eq. (6.72) and is seen to contain explicitly the radius of convergence $\lambda^{\frac{1}{2}}$ of the power series expansion (5.11). Thus it may happen that one or the other of the terms in the remainder predominates depending on whether $\lambda \ll k_2\rho$ or vice versa.

6.2e. Asymptotic expansion for $V_1^{(2)}$.— To illustrate the foregoing developments, we will now apply the formula (6.76) to deduce the asymptotic expansion $V_1^{(2)}$ for the integral $V_1(\rho,z)$, Eq. (6.2), as evaluated over the path C_2 in the α_2-plane. To this end, we note from the second of Eqs. (6.61) that the derivatives of $G(w)$, evaluated at $w = 0$, become

$$G'(o) = \frac{1}{n^2(1-n^2)} \quad \text{and} \quad G'''(o) = \frac{6-n^2-5n^4}{n^4(1-n^2)^2}. \tag{6.77}$$

Furthermore, it is recalled from the discussion of Section 2.5c, that the function $G(w)$ exhibits first order poles on Sheets II and III of the λ-plane. Thus, in Fig. 7 which displays in the α_2-plane alternate half-period strips corresponding to Sheets I and III of the λ-plane we find from Eqs. (2.96) and (6.61) that the periodic set of real poles (labelled P_1 in Fig. 7) occur in the α_2-plane at $\alpha_2 = \pi/2 + w_o$ where

$$w_o = \tan^{-1} n + m\pi \quad (m = 0, \pm 1, \pm 2, \ldots) , \tag{6.78}$$

corresponding to Sheet III; whereas the periodic set (labelled P_2 in Fig. 7) given by $\alpha_2 = \pi/2 - w_o$ and occurring in Sheet I corresponds to the so-called virtual poles for which $G(w)$ actually remains finite.

Since $|n| << 1$, we see from (6.78) that the pole occurring at $w = w_o$ in the w-plane is the singularity nearest to the origin and, hence, also in the x-plane. Thus, we have that the radius of convergence of the expansion (5.11) is given by $\lambda^{\frac{1}{2}} = |x_o|$ where x_o is determined from Eq. (6.63) by putting $w = w_o$, yielding

$$\lambda = \left| 2ik_1 \rho \left\{ n \left[1 - \frac{1}{(1 + n^2)^{\frac{1}{2}}} \right] + \left[(1 - n^2)^{\frac{1}{2}} - \frac{1}{(1 + n^2)^{\frac{1}{2}}} \right] \cot\theta_2 \right\} \right| , \tag{6.79}$$

which, neglecting higher powers of n, can be written approximately as

$$\lambda \approx \left| n^2 k_2 \left[\rho + n(z - h) \right] \right| \tag{6.80}$$

or, for all practical purposes, simply as

$$\lambda \approx | n^2 k_2 \rho | .$$

Thus, substituting the derivatives (6.77) into the formula (6.76), we obtain finally for the integral $V_1^{(2)}$ the expression

$$V_1^{(2)} = -2ik_2 n^2(1-n^2)(1+K)^{3/2} e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)} \left\{ \frac{1}{\left[n^2(1-n^2)ik_2\rho\right]^2} \right. \tag{6.82}$$

$$\left. + \frac{24(1+K) - n^2(8 + 24K + 15K^2) - n^4(16 + 15K)}{8\left[n^2(1-n^2)ik_2\rho\right]^3} + O(n^2k_2\rho)^{-4} \right\},$$

where we have retained only the largest term of the remainder. The leading term in (6.82) is seen to agree exactly with Ott's so-called "Flankenwelle"[31] except for differences in notation, and thus we have extended Ott's results to the next higher order term.

The above result, Eq. (6.82), is essentially useless in the present instance, because $|n| \ll 1$ and Watson's lemma requires that $\lambda > 1$. The difficulty arises from the fact that the singularity nearest to the saddle point $w = 0$ is the first order pole at $w = w_o = \tan^{-1} n$; and, thus, the radius of convergence for the power series (5.11) turns out to be $\lambda^{\frac{1}{2}} = |x_o|$ which is extremely small. The difficulty, however, can be resolved by extracting this singularity from the integrand as already explained in Section 5.1d, but which we undertake by the saddle point method for double integration in Section 6.3e. Likewise, the evaluation of the integral $M_1^{(2)}$ is handled more conveniently by the saddle point method for double integration which was explained in Section 5.2 and which we now proceed to apply in the remainder of this Chapter.

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[31] Loc. cit., Section 4, Eq. (24), p. 455.

6.3 EVALUATION OF I_2 BY THE SADDLE POINT METHOD FOR DOUBLE INTEGRATION

In accordance with the prescriptions of Section 5.2, we now proceed to evaluate I_2 asymptotically by applying the double saddle point method of integration. Thus, the first step consists of replacing the function $H_0^1(k_2\rho\cos w)e^{-ik_2\rho\cos w}$ which is implicit in the integrand of (6.58) by its integral representation

$$H_0^1(k_2\rho\cos w)e^{-ik_2\rho\cos w} = \frac{4}{\pi}\int_0^{\infty\exp(i\alpha)} (4ik_2\rho\cos w - y^2)^{-\frac{1}{2}} e^{-y^2/2}\, dy; \qquad (6.83)$$

where, in accordance with Eq. (5.50) after putting $\nu = 0$ and $z = k_2\rho\cos w$, we must have $|\alpha| < \pi/4$ and $-\pi/4 + \alpha < \frac{1}{2}\arg\left\{k_2\rho\cos w\right\} < (3\pi/4) + \alpha$ and where we take as the path of integration in the y-plane the straight line from the origin to $\infty e^{i\alpha}$. With this substitution in (6.58), we obtain from (6.70) the integral I_2 in the form of a double integral

$$I_2 = -(2ik_2/\pi)e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)} \int_0^{\infty\exp(-i\theta_2/2)} \int_0^{\infty\exp(i\alpha)} \Phi(x,y)e^{-(x^2+y^2)/2} dx dy, \qquad (6.84)$$

with

$$\Phi(x,y) = f(x)\, g(x,y), \qquad (6.85)$$

where we have from Eqs. (5.9) and (6.59), with $w = w(x)$,

$$f(x) = \left[G(w) - G(-w)\right] \sin 2w \frac{dw}{dx}, \qquad (6.86)$$

$G(w)$ being given by Eqs. (6.61), and

$$g(x,y) = (4ik_2\rho\cos w - y^2)^{-\frac{1}{2}} . \qquad (6.87)$$

Thus, we have exhibited our integral I_2 in the form required by Eq. (5.51) and there remains for us to show that the function $\Phi(x,y)$, as defined by (6.85), leads to results which fall within the jurisdiction of Watson's lemma, Section 5.2a, although the function $g(x,y)$, Eq. (6.87), is obviously not uniformly analytic in both x and y in an arbitrary neighborhood of the origin.

6.3a. Extension of Watson's lemma to include the case at hand.– Since $g(x,y)$ is not analytic in both x and y in an arbitrary neighborhood of the origin, Watson's lemma, as stated in Section 5.2a, cannot be applied directly. However, as is often the case, the integration of a series which is apparently not properly convergent will yield the correct result, which we intend to show is the present situation. The singularities of $g(x,y)$ which make the Taylor series expansion for $g(x,y)$, Eq. (5.72), and therefore the expansion of $\Phi(x,y)$, Eq. (5.53), invalid in any fixed region are given by the vanishing of the radical in Eq. (6.87); that is,

$$y = \left(4ik_2\rho\cos w(x)\right)^{\frac{1}{2}} \qquad (6.89)$$

or

$$x = H(y) = (2ik_2\rho)^{\frac{1}{2}}\left\{1 - \frac{y^2}{4ik_2\rho} + \frac{\cot\theta_2}{n}\left[(1-n^2)^{\frac{1}{2}} - \left(1 - \frac{n^2y^4}{(4ik_2\rho)^2}\right)^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}}, \qquad (6.90)$$

where the latter expression is obtained by substitution the value of $\cos w$

as obtained from Eq. (6.89) into Eq. (6.63). By examining Eq. (6.87) it is seen that the Taylor series expansion (5.72) will be valid in the following variable regions and in the following sense: If x is fixed, $x = \xi$, then the expansion will be valid for $|y| < |4ik_2\rho \cos w(\xi)|^{\frac{1}{2}}$; and if y is fixed, $y = \eta$, then the expansion will be valid for $|x| < |H(\eta)|$ where $H(y)$ is defined by Eq. (6.90). This immediately suggests that it may be possible to break up the integral in such a way that it will always be taken over regions in which the expansion (5.72) is valid. This is achieved by showing that it is possible to find a function $y = s(x)$ such that the double integral in Eq. (6.89) may be written as

$$\int_0^{\infty\exp(-i\theta_2/2)} \int_0^{\infty\exp(i\alpha)} \Phi(x,y)e^{-(x^2+y^2)/2}dxdy = \int_0^{\infty\exp(-i\theta_2/2)} dx \int_0^{s(x)} dy\, \Phi(x,y)e^{-(x^2+y^2)/2}$$

$$+ \int_0^{\infty\exp(i\alpha)} dy \int_0^{s^{-1}(y)} dx\, \Phi(x,y)e^{-(x^2+y^2)/2} \tag{6.91}$$

where for the first integral on the right $|y| < |4ik_2\rho \cos w(x)|^{\frac{1}{2}}$ and for the second integral on the right $|x| < |H(y)|$. The power series expansion for $\Phi(x,y)$ when substituted into the first integral on the right of Eq. (6.91) will then be valid for all values of y which are present and will be limited to values of x given by $|x| < \lambda^{\frac{1}{2}}$ where λ is determined by the singularity nearest to the origin of the function $f(x)$ given by (6.86). This singularity, save for a possible pair of first order poles which we know how to subtract, is seen to be the branch point which occurs in the λ-plane at $\lambda = k_1$ or in the w-plane at $\cos w = 1/n$. From this we deduce that the parameter λ is given as $\lambda^{\frac{1}{2}} = |x_1|$ where x_1 is the value of x corresponding to the above branch point; that is, from (6.63),

$$\lambda^{\frac{1}{2}} = |x_1| = \left|2ik_1\rho(1 - n - (1 - n^2)^{\frac{1}{2}}\cot\theta_2)\right|^{\frac{1}{2}} . \qquad (6.92)$$

The power series expansion for $\Phi(x,y)$ when substituted into the second integral on the right of Eq. (6.91) will then be valid for values of x given by $|x| < \lambda^{\frac{1}{2}}$ and for values of y such that $|y| < \nu^{\frac{1}{2}}$ where $\nu^{\frac{1}{2}}$ is given by

$$\nu^{\frac{1}{2}} = \left|4ik_2\rho\cos w(x_1)\right|^{\frac{1}{2}} = \left|4k_1\rho\right|^{\frac{1}{2}} , \qquad (6.93)$$

as obtained from (6.89). Thus, if we can demonstrate that it is possible to find a function $y = s(x)$ such that the resolution into two integrals, Eq. (6.91), is fulfilled, then the power series expansion for $\Phi(x,y)$, Eq. (5.53), is valid insofar as the integration is concerned and we may apply the results of Watson's lemma to the present case.

In particular, we need only show that it is possible to choose a region of integration, Region (I), for the first integral on the right of Eq. (6.91) which overlaps a region of integration, Region (II), for the second integral on the right of Eq. (6.91); any function $y = s(x)$ contained in the region of overlap will then be sufficient to satisfy all requirements. Consider a fixed value of x, say $x = \xi$, where ξ lies on C_2 and $0 \leq |\xi| < \lambda^{\frac{1}{2}}$; then the function $\Phi(\xi,y)$ is an analytic function of y for all positive real values of y within Region (I) shown in Fig. 8; that is, as deduced from Eq. (6.89), for all values of y which lie <u>below</u> the boundary given by

$$y = h(x) = (4k_2\rho\cos w)^{\frac{1}{2}} , \qquad (6.94)$$

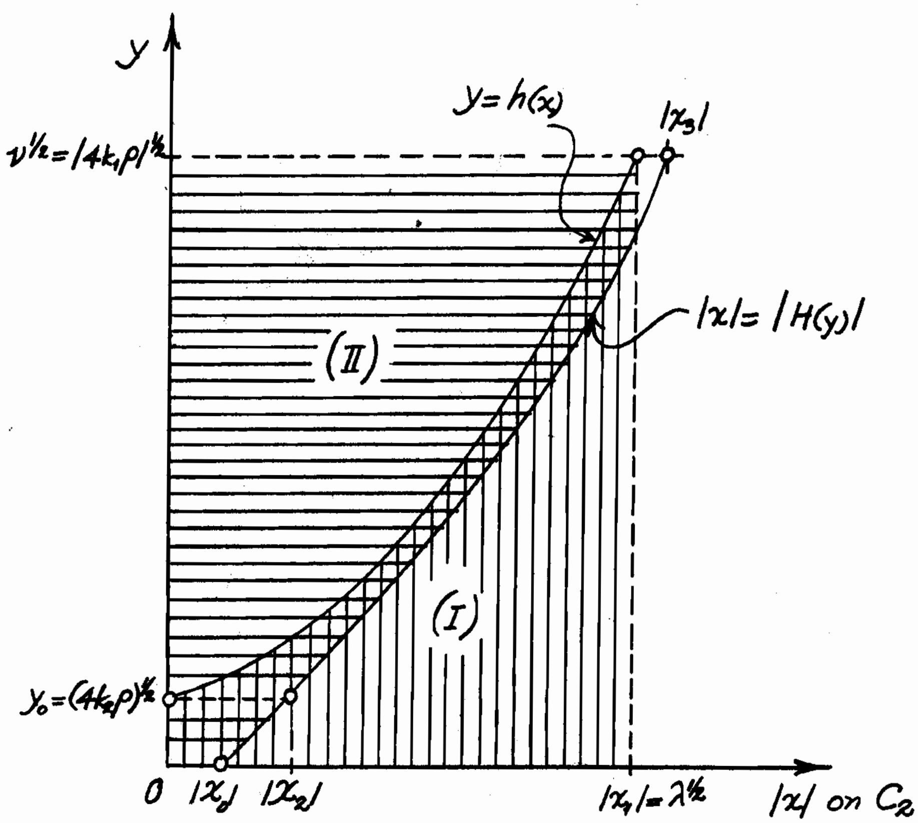

Fig. 8.- Regions of analiticity of the function $\Phi(x,y)$.

where k_2 is real and in which it is understood that $w = w(x)$, $|x| < \lambda^{\frac{1}{2}}$, lies on C_2 and therefore $\cos w$ is also real. The upper limit, $y = y_1$, for this region of analyticity of $\Phi(\xi,y)$ is obtained by putting $\cos w = 1/|n|$ into Eq. (6.94) which yields $y_1 = \nu^{\frac{1}{2}} = |4k_1\rho|^{\frac{1}{2}}$, as given by Eq. (6.93). Putting $x = 0$ or $\cos w = 1$ into Eq. (6.94), we obtain the intercept $y_o = (4k_2\rho)^{\frac{1}{2}}$. Thus, for intermediate values of $|x|$ on C_2, $0 \leq |x| < \lambda^{\frac{1}{2}}$, the upper boundary for Region (I), as given by Eq. (6.94), is a monotonically increasing curve between the points $(0, y_o)$ and $(\lambda^{\frac{1}{2}}, \nu^{\frac{1}{2}})$.

On the other hand, for fixed y, say $y = \eta$ and $0 \leq \eta < \nu^{\frac{1}{2}}$ on the chosen path of integration in the y-plane, the function $\Phi(x,\eta)$ is an analytic function of x for all values of x on C_2 which lie within Region (II) in Fig. 8; that is, as deduced from Eq. (6.90), for all values of $|x| < \lambda^{\frac{1}{2}}$ which lie to the <u>left</u> of the boundary given by

$$|x| = |H(y)| \tag{6.95}$$

where $H(y)$ is defined by Eq. (6.90). Putting $y = 0$ into (6.95), we determine the intercept of the curve on the $|x|$-axis given approximately by $|x_o| \approx |2k_2\rho(1 - \frac{1}{2}n\cot\theta_2)|^{\frac{1}{2}}$ where we have neglected n^2 and higher powers. Next, putting $y = y_o = (4k_2\rho)^{\frac{1}{2}}$ into Eq. (6.95), we obtain $|x_2| \approx |2k_2\rho(1 + i - n\cot\theta_2)|^{\frac{1}{2}}$ where again we have neglected higher powers of n. And finally putting $y = \nu^{\frac{1}{2}} = |4k_1\rho|^{\frac{1}{2}}$ into Eq. (6.95), which corresponds to the upper limit of y, we obtain

$$|x_3| = \left|2k_1\rho\left\{e^{i\pi/4} + n + \cot\theta_2\left[(1 - n^2)^{\frac{1}{2}} - (1 - i)^{\frac{1}{2}}\right]\right\}\right|^{\frac{1}{2}}.$$

Assuming temporarily that θ_2 is limited to the range $\pi/4 < \theta_2 \leq \pi/2$, so that $\cot\theta_2 < 1$, we see from Eq. (6.63) and from the equations of the preceding paragraph that we have $0 < |x_o| < |x_2| < \lambda^{\frac{1}{2}} < |x_3|$

which states, as shown in Fig. 8, that the right boundary of Region (II), given by the monotonically increasing curve $|x| = |H(y)|$, lies entirely to the right of the upper boundary for Region (I) as given by the curve $y = h(x)$. Since these two regions, therefore, have a common strip, we have demonstrated that a monotonic curve lying in this strip may be used for $y = s(x)$ as called for by Eq. (6.91). Thus we may use the results of Watson's lemma to obtain the asymptotic evaluation of the integral I_2 which, according to Eqs. (5.65), (5.66a) and (6.84), becomes

$$I_2 = -ik_2 e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)} \left\{ \sum_{n=0}^{N-1} \Phi^{(n)} + O\left(\frac{1}{\lambda} + \frac{1}{\nu}\right)^{N+1} \right\} \tag{6.96}$$

where the expansion terms $\Phi^{(n)}$ are defined by Eq. (5.68) and λ is given by Eq. (6.92) and ν by Eq. (6.93) under the assumption that the poles, if present, have been removed.

<u>6.3b. Evaluation of the expansion terms</u>.- The problem has now been reduced to the computation of the expansion terms $\Phi^{(n)}$ as listed in Eqs. (5.74) in terms of the A's and B's which correspond, respectively, to the power series expansions for the factors $f(x)$ and $g(x,y)$ in accordance with Eqs. (5.71) and (5.72). The A's are determined from Eqs. (5.26) in terms of the coefficients a_{2n}, listed by Eqs. (6.68), and in terms of the derivatives of the function

$$F(w) = G(w) \sin 2w \tag{6.97}$$

which appears in Eq. (6.86), evaluated at $w = 0$. In this way we obtain

$$A_2 = 4\left(\frac{1+K}{ik_2\rho}\right)^{3/2} G'(o) ,$$

$$A_4 = \frac{1}{3!}\left(\frac{1+K}{ik_2\rho}\right)^{5/2} G'(o)\left\{4\,\frac{G'''(o)}{G'(o)} - \left(11 + \frac{15K}{1-n^2}\right)\right\} ,$$

$$A_6 = \frac{1}{5!}\left(\frac{1+K}{ik_2\rho}\right)^{7/2} G'(o)\left\{4\,\frac{G^V(o)}{G'(o)} - 10\left(3 + \frac{7K}{1-n^2}\right)\frac{G'''(o)}{G'(o)} - \frac{1}{4}\left(31 + \frac{70K}{1-n^2} - \frac{105K(8+9K)}{(1-n^2)^2}\right)\right\} , \tag{6.98}$$

in which K has been defined by Eq. (6.67). Similarly, the B's are found by expanding the function $g(x,y)$, given by Eq. (6.87), into a double power series of the form (5.72), yielding

$$B_o^o = \tfrac{1}{2}\,(ik_2\rho)^{-\frac{1}{2}} ,$$

$$B_2^o = \frac{1}{8}\,(1+K)(ik_2\rho)^{-3/2} , \quad B_o^2 = \frac{1}{16}\,(ik_2\rho)^{-3/2} ,$$

$$B_4^o = \frac{1}{64}\,(1+K)^2\left(3 - \frac{2K}{1-n^2}\right)(ik_2\rho)^{-5/2}, \quad B_2^2 = \frac{3}{64}\,(1+K)(ik_2\rho)^{-5/2} , \tag{6.99}$$

$$B_o^4 = \frac{3}{256}\,(ik_2\rho)^{-5/2} .$$

Substituting the A's and B's listed in (6.98) and (6.99) into Eqs. (5.74), we obtain the expansion terms

$$\Phi^{(0)} = 0,$$

$$\Phi^{(1)} = 2\,\frac{(1+K)^{3/2}}{(ik_2\rho)^2}\,G'(o),$$

$$\Phi^{(2)} = \frac{1}{4}\,\frac{(1+K)^{3/2}}{(ik_2\rho)^3}\,G'(o)\left\{4(1+K)\,\frac{G'''(o)}{G'(o)} - (4+5K) - \frac{15K(1+K)}{1-n^2}\right\}, \qquad (6.100)$$

$$\Phi^{(3)} = \frac{1}{64}\,\frac{(1+K)^{3/2}}{(ik_2\rho)^4}\,G'(o)\left\{16(1+K)^2\,\frac{G^V(o)}{G'(o)} - 8(1+K)\left[4+5K+\frac{35K(1+K)}{1-n^2}\right]\frac{G'''(o)}{G'(o)}\right.$$

$$\left. - (48+128K+71K^2) - 10\,\frac{K(1+K)(52+49K)}{1-n^2} + 105\,\frac{K(1+K)^2(8+9K)}{(1-n^2)^2}\right\}.$$

6.3c. Asymptotic expansion for the integrals $M_1^{(2)}$ and $\partial M_1^{(2)}/\partial z$.– We now apply the results of the preceding Section to the evaluation of $M_1^{(2)}$ and its derivative $\partial M_1^{(2)}/\partial z$, where M_1 is defined by (6.4) and where the superscript (2) denotes evaluation of the integral along the contour C_2. Thus, for $M_1^{(2)}$ we have from (6.61) that $G(w) = n^2(1-n^2)^{-1}\sin w$; and, hence, the derivatives of $G(w)$, evaluated at $w = 0$, which must be inserted in Eqs. (6.100) are simply

$$G'(o) = \frac{n^2}{1-n^2}, \qquad G'''(o) = -\frac{n^2}{1-n^2}, \qquad G^V(o) = \frac{n^2}{1-n^2}. \qquad (6.101)$$

Since we wish to examine the higher order terms in the asymptotic expansion of $M_1^{(2)}$ we take $N = 4$ and, thus, substituting (6.101) into (6.100) and introducing the results in (6.96), we obtain the three-term asymptotic expansion

$$M_1^{(2)} = -2ik_1(1-n^2)(1+K)^{3/2} e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)} \left\{ \phi^{(1)} + \phi^{(2)} + \phi^{(3)} + 0\left(\frac{1}{\lambda} + \frac{1}{\nu}\right)^5 \right\} \tag{6.102}$$

where λ and ν are given by Eqs. (6.92) and (6.93) respectively, and where

$$\phi^{(0)} = 0 ,$$

$$\phi^{(1)} = n \left[(1 - n^2)ik_1\rho\right]^{-2} , \tag{6.103}$$

$$\phi^{(2)} = -\frac{1}{8}\left[(8 + 24K + 15K^2) - n^2(8 + 9K)\right] \left[(1 - n^2)ik_1\rho\right]^{-3} ,$$

$$\phi^{(3)} = \frac{3K}{128n}\left[3(64 + 240K + 280K^2 + 105K^3) + 2n^2(48 + 80K + 35K^2) - n^4(8 + 5K)\right] \left[(1 - n^2)ik_1\rho\right]^{-4} .$$

Similarly, the integral $\partial M_1^{(2)}/\partial z$ may be obtained from (6.96) by taking a new definition for $G(w)$. Thus we have in the original λ-plane, from Eq. (6.5),

$$\frac{\partial I_2}{\partial z} = \frac{1}{2}\int_{C_2} \gamma_1 v(\lambda) e^{\gamma_1(z-h)} H_0^1(\lambda\rho)\lambda d\lambda, \quad z \leq 0 ; \tag{6.104}$$

whence, we deduce at once from (6.53) and (6.61) that for $\partial M_1^{(2)}/\partial z$ we have, instead of $G(w)$, the new function

$$G_z(w) = -ik_1n^2(1 - n^2)^{-1} \sin w(1 - n^2 \cos^2 w)^{\frac{1}{2}} . \tag{6.105}$$

Computing the odd derivatives of $G_z(w)$ and evaluating at $w = 0$, we obtain

$$G_z'(o) = -\frac{ik_1 n^2}{(1-n^2)^{\frac{1}{2}}} ,$$

$$G_z'''(o) = \frac{ik_1 n^2}{(1-n^2)^{\frac{1}{2}}} \frac{1-4n^2}{1-n^2} , \tag{6.106}$$

$$G_z^V(o) = -\frac{ik_1 n^2}{(1-n^2)^{\frac{1}{2}}} \frac{1-32n^2+16n^4}{(1-n^2)^2} .$$

Proceeding as before, we now substitute the derivatives (6.106) into the expansion terms (6.100) and introducing the results into (6.96) with $N = 4$, we obtain the three-term asymptotic expansion

$$\frac{\partial M_1^{(2)}}{\partial z} = -2k_1^2(1-n^2)^{3/2}(1+K)^{3/2} e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)} \left\{ \phi_z^{(1)} + \phi_z^{(2)} + \phi_z^{(3)} + 0\left(\frac{1}{\lambda}+\frac{1}{\nu}\right)^5 \right\} \tag{6.107}$$

where λ and ν are again given by Eqs. (6.92) and (6.93) respectively and where

$$\phi_z^{(o)} = 0 ,$$

$$\phi_z^{(1)} = n\left[(1-n^2)ik_1\rho\right]^{-2} ,$$

$$\phi_z^{(2)} = -\frac{1}{8}\left[(8+24K+15K^2) - n^2(20+21K)\right]\left[(1-n^2)ik_1\rho\right]^{-3} , \tag{6.108}$$

$$\phi_z^{(3)} = \frac{3}{128}\Big[(3K/n)(64+240K+280K^2+105K^3) - 6n(32+96K+100K^2+35K^3) + n^3(112+224K+115K^4)\Big]\left[(1-n^2)ik_1\rho\right]^{-4} .$$

It is of interest to compare the asymptotic expansions (6.102) and (6.107) which correspond to the integrals $M_1^{(2)}$ and $\partial M_1^{(2)}/\partial z$ respectively.

It is seen that the <u>first</u> term in the expansion of the derivative $\partial M_1^{(2)}/\partial z$ is obtained by merely taking the derivative with respect to z of the exponential factor contained in the <u>first</u> term of the expansion for $M_1^{(2)}$, without regard to the fact that the parameter z also appears in the term K as shown in Eq. (6.67). It is noted that differentiating the factor $(1 + K)^{3/2}$ would result in a term of higher order; and, hence, it is correct to state that, to the same order, the derivative of the first term in the expansion for $M_1^{(2)}$ yields precisely the first term in the expansion of the derivative $\partial M_1^{(2)}/\partial z$. It is evident, however, by inspection of the remaining terms in (6.103) and (6.108) that the same statement does not hold true for higher order terms. Therefore, it is concluded that, in the present instance, differentiation of an asymptotic series term by term is not permitted <u>except</u> for the first term.

As a further point of interest, it is seen by comparing the expansion terms (6.103) and (6.108) that in both cases successive terms appear to be multiplied by the factor $(ik_1\rho)^{-1}$ and thus we have obtained asymptotic series in inverse powers of $ik_1\rho$ where k_1 is the propagation constant for the conducting medium. This is in accord with the form of the remainder which is $O\left(\frac{1}{\lambda} + \frac{1}{\nu}\right)^{N+1}$, wherein $\nu = |4k_1\rho|$ and λ, as given by (6.92), can be written $\lambda \approx |2k_1\rho|$ provided $|1 - \cot\theta_2| > 0$. That is, from (6.92), if $|n| << 1$ we can write

$$\lambda \approx \left|2k_1\rho\left\{(1 - \cot\theta_2) - n(1 - \tfrac{1}{2}\cdot n\cot\theta_2)\right\}\right| , \qquad (6.109)$$

which confirms the above statement. However, when $\cot\theta_2 = 1$, we see that $\lambda \approx |2k_2\rho|$ and the magnitude of the remainder in (6.102) and (6.107) becomes so large as to render useless the corresponding asymptotic series. In practice the range most commonly employed corresponds to $\cot\theta_2 < 1$; and, hence, this limitation imposed already in the analysis at the end of Section 6.3a proves to be not a serious one.

6.3d. Asymptotic expansion for the integral $\partial M_1^{(2)}/\partial\rho$.-- In the original λ-plane the derivative of I_2 with respect to ρ becomes, from (6.5),

$$\frac{\partial I_2}{\partial\rho} = -\frac{1}{2}\int_{C_2} v(\lambda)e^{\gamma_1(z-h)}H_1^1(\lambda\rho)\lambda^2 d\lambda, \quad z \leq 0, \tag{6.110}$$

which in accordance with (6.58) and (6.70) becomes in the x-plane

$$\frac{\partial I_2}{\partial\rho} = \frac{1}{2}\, ik_2^2 e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)}\int_0^{\infty\exp(-i\theta_2/2)} \Phi(x)e^{-x^2/2}dx, \tag{6.111}$$

where $\Phi(x) = \left\{F(w) + F(-w)\right\}(dw/dx)$ with

$$F(w) = G(w)\sin 2w\cos w\left\{H_1^1(k_2\rho\cos w)e^{-ik_2\rho\cos w}\right\}, \tag{6.112}$$

in which $G(w)$ is given by the first of Eqs. (6.61).

Next, replacing the Hankel function and exponential factor by the integral representation deduced from Eq. (5.50),

$$H_1^1(k_2\rho\cos w)e^{-ik_2\rho\cos w} = -(\pi k_2\rho\cos w)^{-1}\int_0^{\infty\exp(i\alpha)} y^2(4ik_2\rho\cos w - y)^{\frac{1}{2}}e^{-y^2/2}dy, \tag{6.113}$$

where $|\alpha| < \pi/4$ and $-\frac{\pi}{4} + \alpha < \frac{1}{2} \arg\{k_2 \rho \cos w\} < \frac{3\pi}{4} + \alpha$, we transform (6.111) into the double integral

$$\frac{\partial I_2}{\partial \rho} = (k_2^2/2\pi i k_2 \rho) e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)} \int_0^{\infty \exp(-i\theta_2/2)} \int_0^{\infty \exp(i\alpha)} \Phi(x,y) e^{-(x^2+y^2)/2} dx dy, \tag{6.114}$$

where we now have

$$\Phi(x,y) = y^2 f(x)\, g(x,y), \tag{6.115}$$

with $f(x)$ precisely as before, Eq. (6.86), and with

$$g(x,y) = (4ik_2 \rho \cos w - y^2)^{\frac{1}{2}}, \tag{6.116}$$

which now differs from (6.87).

The conditions imposed by Watson's lemma are still the same as in Section 6.3a; and, thus, we can proceed immediately to the computation of the expansion terms $\Phi^{(n)}$ which this time, because of the factor y^2 in (6.115), are given by Eqs. (5.77) in terms of A's corresponding to the expansion of the same $f(x)$, already listed in Eqs. (6.98), and in terms of new B's which correspond to the double power series expansion for the function $g(x,y)$ given in (6.116), namely

$$B_0^0 = 2(ik_2\rho)^{\frac{1}{2}}, \quad B_2^0 = -\tfrac{1}{2}(1+K)(ik_2\rho)^{-\frac{1}{2}},$$

$$B_0^2 = -\tfrac{1}{4}(ik_2\rho)^{-\frac{1}{2}}, \quad B_4^0 = \tfrac{1}{16}(1+K)^2\left[\frac{2K}{1-n^2} - 1\right](ik_2\rho)^{-3/2}, \tag{6.117}$$

$$B_2^2 = -\tfrac{1}{16}(1+K)(ik_2\rho)^{-3/2}, \quad B_0^4 = -\tfrac{1}{64}(ik_2\rho)^{-3/2}.$$

Substituting the A's and B's listed in (6.98) and (6.117) into Eqs. (5.77), we obtain the expansion terms

$$\Phi^{(o)} = 0, \qquad \Phi^{(1)} = 0,$$

$$\Phi^{(2)} = 8\,\frac{(1+K)^{3/2}}{ik_2\rho}\,G'(o),$$

$$\Phi^{(3)} = \frac{(1+K)^{3/2}}{(ik_2\rho)^2}\,G'(o)\left\{4(1+K)\,\frac{G'''(o)}{G'(o)} - 20 - 17K - \frac{15K(1+K)}{1-n^2}\right\}, \qquad (6.118)$$

$$\Phi^{(4)} = \frac{1}{16}\,\frac{(1+K)^{3/2}}{(ik_2\rho)^3}\,G'(o)\left\{16(1+K)^2\,\frac{G^{V}(o)}{G'(o)} - 8(1+K)\left[28 + 25K + \frac{35K(1+K)}{1-n^2}\right]\frac{G'''(o)}{G'(o)}\right.$$

$$\left. + 3(48 + 96K + 43K^2) + \frac{10K(1+K)(44+35K)}{1-n^2} + \frac{105K(1+K)^2(8+9K)}{(1-n^2)^2}\right\}.$$

To apply the above results to the integral $\partial M_1^{(2)}/\partial\rho$ we need only substitute into (6.118) the derivatives of $G(w)$, evaluated at $w = 0$, which are listed in (6.101). Next, putting $N = 5$, and substituting the results from (6.118) into (6.114) we obtain the three-term asymptotic expansion for $\partial M_1^{(2)}/\partial\rho$ which we write in the form

$$\frac{\partial M_1^{(2)}}{\partial\rho} = 2k_1^2(1-n^2)(1+K)^{3/2}e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)}\left\{\phi_\rho^{(2)} + \phi_\rho^{(3)} + \phi_\rho^{(4)} + O\left(\frac{1}{\lambda} + \frac{1}{\nu}\right)^5\right\}, \qquad (6.119)$$

where λ and ν are given respectively by Eqs. (6.92) and (6.93) and where

$$\phi_\rho^{(o)} = 0, \qquad \phi_\rho^{(1)} = 0, \qquad \phi_\rho^{(2)} = n^2\left[(1-n^2)ik_1\rho\right]^{-2},$$

$$\phi_\rho^{(3)} = -\frac{3n}{8}\left[8 + 12K + 5K^2 - n^2(8+7K)\right]\left[(1-n^2)ik_1\rho\right]^{-3}, \qquad (6.120)$$

$$\phi_\rho^{(4)} = \frac{3}{128}\left[128 + 768K + 1440K^2 + 1120K^3 + 315K^4 - 2n^2(128 + 368K + 348K^2 + 105K^3) + n^4(128 + 248K + 115K^2)\right]\left[(1-n^2)ik_1\rho\right]^{-4}.$$

Once more, comparing the asymptotic series for $M_1^{(2)}$ and $\partial M_1^{(2)}/\partial\rho$, it is readily ascertained that the first term of the derivative is equal, to the same order, to the derivative of the first term of the function itself which is obtained by differentiating the exponential factor alone, but that the same statement does not hold true for higher order terms. Further, we note again that successive terms of (6.120) are multiplied by the factor $(ik_1\rho)^{-1}$; and, thus, we have again an asymptotic series in reciprocal powers of $k_1\rho$. As already explained in Section 5.2, the use of the double saddle point method of integration which employs the Hankel function in its integral representation is equivalent to the removal of the branch point for zero argument of the Hankel function; and, therefore, the series obtained are expressed in reciprocal powers of $k_1\rho$ rather than $k_2\rho$. It is, of course, this feature that makes the present computations of practical value, for the series obtained are valid at distances from the source dipole which measure a few wavelengths in the conducting medium. Thus, for example, referring to Table I, at a frequency of only 1000 sec^{-1}, a wavelength in the conducting medium is 50 meters; and, hence, at this frequency, measurements starting at 500 meters or more ought to agree very well with our computed results.

6.3e. Subtraction of the pole from the integrand of $V_1^{(2)}$.— As already explained in Section 6.2e, the poor asymptotic behavior of the expansion (6.82) for $V_1^{(2)}$ which is valid for $|n^2k_2\rho| > 1$ stems from the presence of a first order pole in the immediate vicinity of the saddle point at $w = 0$ in the w-plane. This difficulty can be resolved, as shown in Section 5.2c, by extracting the pole from the integrand of the double integral for $V_1^{(2)}$ which, according to Eqs. (5.82) and (6.84), we now write in terms of the new integrals $W^{(s)}$ and $W^{(p)}$ as

$$V_1^{(2)} = -(2ik_2/\pi)e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)}\left\{W_s + W_p\right\} = W^{(s)} + W^{(p)}, \qquad (6.121)$$

where W_s is evaluated from the asymptotic expansion (5.84) and W_p is given by (5.87).

The integral W_p is given by Eq. (5.88) in terms of the parameter x_o and of the constant C, defined by Eq. (5.89), which is still to be evaluated. The point $x = x_o$ in the x-plane corresponds to the position of the first order pole of $G(w)$, as given by the second of Eqs. (6.61), which occurs at $w = w_o$ as defined by (6.78). Inserting this value into (6.63), we obtain for x_o the expression

$$\tfrac{1}{2}x_o^2 = ik_1\rho\left[n\left(1 - \frac{1}{(1+n^2)^{\frac{1}{2}}}\right) + \left((1-n^2)^{\frac{1}{2}} - \frac{1}{(1+n^2)^{\frac{1}{2}}}\right)\cot\theta_2\right] \qquad (6.122)$$

which we have already encountered in (6.79). Next, to compute $C(y)$, as defined by (5.79), we note that with $\Phi(x,y) = f(x)g(x,y)$, given by Eqs. (6.85), (6.86) and (6.87), $C(y)$ becomes

$$C(y) = \frac{g(x_o,y)}{2x_o}\lim_{x\to x_o}\left\{(x^2 - x_o^2)f(x)\right\}$$

$$= \frac{2n}{1+n^2}\left(\frac{4ik_2\rho}{(1+n^2)^{\frac{1}{2}}} - y^2\right)^{-\frac{1}{2}}\cdot\frac{1}{2x_o}\lim_{w\to w_o}\left\{(x^2 - x_o^2)\left[G(w) - G(-w)\right]\frac{dw}{dx}\right\},$$

from which, substituting for $G(w)$ the expression for $V_1^{(2)}$ in (6.61) and applying L'Hospital's rule, we obtain

$$C(y) = -\frac{2n}{(1+n^2)^{\frac{1}{2}}(1-n^4)}\left(\frac{4ik_2\rho}{(1+n^2)^{\frac{1}{2}}} - y^2\right)^{-\frac{1}{2}}. \qquad (6.123)$$

Inserting this expression into (5.89) and making use of (6.83), we have

$$C = -\frac{2n}{(1+n^2)^{\frac{1}{2}}(1-n^4)} \int_0^\infty \left(\frac{4ik_2\rho}{(1+n^2)^{\frac{1}{2}}} - y^2 \right)^{-\frac{1}{2}} e^{-y^2/2}\, dy$$

$$= -\frac{2n}{(1+n^2)^{\frac{1}{2}}(1-n^4)} \cdot \frac{\pi}{4} H_o^1\left(\frac{k_2}{(1+n^2)^{\frac{1}{2}}} \right) e^{ik_2\rho/(1+n^2)^{\frac{1}{2}}} \qquad (6.124)$$

Thus, finally, substituting (6.124) into (5.87) and making use of (6.121), we obtain, after some reductions,

$$W^{(p)} = -(2ik_2/\pi)e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)}\, W_p$$

$$= -\frac{\pi k_1 n^2}{(1+n^2)^{\frac{1}{2}}(1-n^4)} H_o^1\left(\frac{k_2\rho}{(1+n^2)^{\frac{1}{2}}} \right) e^{-ik_1(z+h)/(1+n^2)^{\frac{1}{2}}} \left\{ 1 - \mathrm{erf}(-ix_o/2^{\frac{1}{2}}) \right\} . \qquad (6.125)$$

To check this last result, it may be readily verified that the factor in front of the bracket is precisely one-half of the residue of the pole as computed in either the λ or w planes, in complete accord with the demands of Eq. (5.41) and subsequent discussion.

Next, we proceed to the evaluation of W_s by the double saddle point method of integration. First, we consider the function

$$\Psi(x,y) = \Phi(x,y) + \chi(x,y), \qquad (6.126)$$

which now replaces Eq. (5.78). Here $\Phi(x,y)$ is given by Eqs. (6.85), (6.86) and (6.87) wherein $G(w)$ corresponding to $V_1^{(2)}$ is given by the second of Eqs. (6.61) and $\chi(x,y)$ becomes, from (5.78) and (6.123),

$$\chi(x,y) = -\frac{2x_oC(y)}{x^2 - x_o^2} = \frac{2nx_o(x^2 - x_o^2)^{-1}}{(1+n^2)^{\frac{1}{2}}(1-n^4)} \left(\frac{4ik_2\rho}{(1+n^2)^{\frac{1}{2}}} - y^2 \right)^{-\frac{1}{2}}, \qquad (6.127)$$

which is of the form (5.54) and, therefore, has a readily computed double power series expansion. To facilitate the algebra, we rewrite the expression (6.122) in terms of our parameter K, Eq. (6.67), in the form

$$x_o^2 = \frac{ik_1 n^3 \rho \; Q}{(1+n^2)^{\frac{1}{2}}(1+K)}, \quad Q = \frac{2}{n^2}\left[(1+n^2)^{\frac{1}{2}} - 1\right] + \frac{2K}{n^4}\left[(1+n^2)^{\frac{1}{2}} - (1-n^2)^{\frac{1}{2}} - n^2\right], \quad (6.128)$$

in which the factor Q can be conveniently expanded as

$$Q = 1 - \frac{n^2}{4}(1-K) + \frac{n^4}{8} - \frac{n^6}{64}(5-7K) + \frac{7n^8}{128} + O(n^{10}). \quad (6.129)$$

Thus, to evaluate W_s in accordance with Eq. (5.84), we choose $N = 3$ and proceed to the calculation of the expansion terms

$$\Psi^{(n)} = \Phi^{(n)} + \chi^{(n)}, \quad n = 0,1,2, \quad (6.130)$$

by applying Eq. (5.86). In this manner we obtain for the expansion coefficients $\Phi^{(n)}$ the expressions

$$\Phi^{(o)} = 0,$$

$$\Phi^{(1)} = \frac{2}{n^4(1-n^2)}\,\frac{(1+K)^{3/2}}{(ik_1\rho)^2}, \quad (6.131)$$

$$\Phi^{(2)} = \frac{1}{4}\,\frac{1}{n^7(1-n^2)^2}\,\frac{(1+K)^{3/2}}{(ik_1\rho)^3}\left\{24(1+K) - n^2(8+24K+15K^2) - n^4(16+15K)\right\},$$

which were obtained from (6.100) after inserting for $G'(o)$ and $G'''(o)$ the expressions given in (6.77). Proceeding similarly, we obtain for the corresponding expansion coefficients $\chi^{(n)}$ the expressions

$$\chi^{(o)} = -\frac{2(1+K)^{\frac{1}{2}}\,Q^{-\frac{1}{2}}}{n(1-n^4)(ik_1\rho)},$$

$$\chi^{(1)} = -\frac{(1+n^2)^{\frac{1}{2}}(1+K)^{\frac{1}{2}}Q^{-3/2}}{4n^4(1-n^4)}\left[8(1+K)+n^2Q\right](ik_1\rho)^{-2}, \qquad (6.132)$$

$$\chi^{(2)} = -\frac{(1+n^2)(1+K)^{\frac{1}{2}}Q^{-5/2}}{64n^7(1-n^4)}\left[384(1+K)^2+16n^n(1+K)Q+9n^4Q^2\right](ik_1\rho)^{-3},$$

which were obtained by first expanding $\chi(x,y)$ into a double power series and then inserting the coefficients into the first three of Eqs. (5.69).

Adding corresponding terms of (6.131) and (6.132) to obtain the expansion terms $\Psi^{(n)}$ for W_s, as given by (6.130), and inserting the corresponding asymptotic expansion (5.84) into (6.121), we obtain, after considerable agebraic manipulation, the three-term expansion for $W^{(s)}$, namely

$$\begin{aligned} W^{(s)} &= -(2ik_2/\pi)e^{ik_2\rho-ik_1(1-n^2)^{\frac{1}{2}}(z-h)}W_s \\ &= 2ik_1(1+K)^{\frac{1}{2}}e^{ik_2\rho-ik_1(1-n^2)^{\frac{1}{2}}(z-h)}\left\{w^{(o)}+w^{(1)}+w^{(2)}+O\left(\frac{1}{\lambda}+\frac{1}{\nu}\right)^4\right\} \end{aligned} \qquad (6.133)$$

where λ and ν are given respectively by Eqs. (6.92) and (6.93) and where

$$w^{(o)} = \left[1+\frac{n^2}{8}(1-K)+O(n^4)\right](ik_1\rho)^{-1},$$

$$w^{(1)} = -\frac{1}{8}\left[(K/n)(4+3K)-\frac{3n}{16}(3-19K-13K^2+5K^3)+O(n^3)\right](ik_1\rho)^{-2}, \qquad (6.134)$$

$$\begin{aligned} w^{(2)} = \frac{3}{128}\Big[&(K/n)^2(48+80K+35K^2)+\frac{5}{8}(5+72K+274K^2+299K^3+77K^4-21K^5) \\ &-\frac{1}{8}K(1+K)+O(n^2)\Big](ik_1\rho)^{-3}, \end{aligned}$$

which were obtained to $O(n^2)$ by expanding the resulting terms in powers of n and neglecting higher order terms.

6.3f. Asymptotic evaluation of the derivatives of $V_1^{(2)}$.– Applying the methods developed in the preceding sections, we now proceed to the asymptotic evaluation of the derivatives $\partial V_1^{(2)}/\partial z$, $\partial V_1^{(2)}/\partial \rho$ and $\partial^2 V_1^{(2)}/\partial\rho\,\partial z$ which we need in Chapter VII in the computation of the electric and magnetic field components.

The derivative $\partial V_1^{(2)}/\partial z$, according to Eq. (6.104), can be obtained by the method of the preceding section if we merely replace $G(w)$ by $G_z(w)$ defined as

$$G_z(w) = \frac{-ik_1(1-n^2\cos^2 w)^{\frac{1}{2}}}{n(1-n^2\cos^2 w)^{\frac{1}{2}} - \sin w} \tag{6.135}$$

whose odd derivatives, evaluated at $w = 0$, become

$$G_z'(o) = \frac{-ik_1}{n^2(1-n^2)^{\frac{1}{2}}}, \qquad G_z'''(o) = -ik_1\frac{6 - n^2 - 2n^4}{n^4(1-n^2)^{3/2}}, \tag{6.136}$$

and replace $C(y)$ in (6.123) by $C_z(y) = \left(-ik_1/(1+n^2)^{\frac{1}{2}}\right)C(y)$, i.e.,

$$C_z(y) = \frac{2ink_1}{(1+n^2)(1-n^4)}\left(\frac{4ik_2\rho}{(1+n^2)^{\frac{1}{2}}} - y^2\right)^{-1/2}. \tag{6.137}$$

As before, we write our integral $\partial V_1^{(2)}/\partial z$ as the sum of two terms, as in Eq. (6.121), thus

$$\frac{\partial V_1^{(2)}}{\partial z} = -(2ik_2/\pi)e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)}\left\{W_{zs} + W_{zp}\right\} = W_z^{(s)} + W_z^{(p)}, \tag{6.138}$$

where the subscripts z on the W's do not mean differentiation with respect to z, since W_{zs} and W_{zp} are not the derivatives of W_s and W_p. From Eq. (5.87) we see at once that $W^{(p)}$ in Eq. (6.125) can be converted into $W_z^{(p)}$ by merely multiplying by the factor $\left[-ik_1/(1+n^2)^{\frac{1}{2}}\right]$ which is the factor that converts C(y) into $C_z(y)$. Thus,

$$W_z^{(p)} = \frac{im^2k_1^2}{(1+n^2)(1-n^4)} H_o^1\left(\frac{k_2\rho}{(1+n^2)^{\frac{1}{2}}}\right) e^{-ik_1(z-h)/(1+n^2)^{\frac{1}{2}}}\left\{1 - \mathrm{erf}(-ix_o/2^{\frac{1}{2}})\right\}, \quad (6.139)$$

which can also be obtained from $W^{(p)}$, Eq. (6.125), by merely taking the derivative with respect to z of the exponential factor. Similarly, the expansion terms $\chi^{(n)}$ listed in (6.132) are readily converted into the terms $\chi_z^{(n)}$ through multiplication by the factor $\left[-ik_1/(1+n^2)^{\frac{1}{2}}\right]$, yielding

$$\chi_z^{(o)} = \frac{2ik_1(1+K)^{\frac{1}{2}}\, Q^{-\frac{1}{2}}}{n(1-n^4)(1+n^2)^{\frac{1}{2}}}(ik_1\rho)^{-1},$$

$$\chi_z^{(1)} = \frac{ik_1(1+K)^{\frac{1}{2}}\, Q^{-3/2}}{4n^4(1-n^4)}\left[8(1+K) + n^2Q\right](ik_1\rho)^{-2}, \quad (6.140)$$

$$\chi_z^{(2)} = \frac{ik_1(1+n^2)^{\frac{1}{2}}(1+K)^{\frac{1}{2}}\, Q^{-5/2}}{64\, n^7\, (1-n^4)}\left[384(1+K)^2+16n^2(1+K)Q+9n^4Q^2\right](ik_1\rho)^{-3},$$

while the new expansion terms $\Phi_z^{(n)}$ are again determined from (6.100) by replacing G(w) with $G_z(w)$ and using the evaluated derivatives given in (6.136). In this manner, we obtain

$$\Phi_z^{(o)} = 0, \qquad \Phi_z^{(1)} = -\frac{2ik_1(1+K)^{3/2}}{n^4(1-n^2)^{\frac{1}{2}}}(ik_1\rho)^{-2},$$

$$\Phi_z^{(2)} = -\frac{ik_1(1+K)^{3/2}}{4\, n^7(1-n^2)^{3/2}}\left[24(1+K) - n^2(8+24K+15K^2) - n^4(4+3K)\right](ik_1\rho)^{-3}. \quad (6.141)$$

Combining the above expansion terms in accordance with (6.130) and substituting the resulting asymptotic expansion (5.84) into (6.138), we obtain, after considerable simplification,

$$W_z^{(s)} = \frac{2k_1^2(1+K)^{\frac{1}{2}}}{(1+n^2)^{\frac{1}{2}}} e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)} \left\{ w_z^{(o)} + w_z^{(1)} + w_z^{(2)} + O\left(\frac{1}{\lambda} + \frac{1}{\nu}\right)^4 \right\} \quad (6.142)$$

where again λ and ν are given by Eqs. (6.92) and (6.93), respectively, and where

$$w_z^{(o)} = \left[1 + \frac{n^2}{8}(1-K) + O(n^4)\right](ik_1\rho)^{-1},$$

$$w_z^{(1)} = -\frac{1}{8}\left[\frac{K}{n}(4+3k) - \frac{n}{16}(73+7K-39K^2+15K^3) + O(n^3)\right](ik_1\rho)^{-2}, \quad (6.143)$$

$$w_z^{(2)} = \frac{1}{128}\left[3\left(\frac{K}{n}\right)^2(48+80K+35K^2) - \frac{1}{8}(437+971K-1611K^2-3525K^3-1155K^4+315K^5) + O(n^2)\right](ik_1\rho)^{-3}.$$

Comparing the asymptotic expansion (6.133) for $W^{(s)}$ with (6.143) for $W_z^{(s)}$, we see that the latter is not the term by term derivative of the former and that not even the leading term of $W_z^{(s)}$ is the derivative of the leading term of the primitive function except to within terms of $O(n^4)$.

The asymptotic expansion for the derivative $\partial V_1^{(2)}/\partial\rho$ can again be written, from Eq. (6.114), as the sum of two terms

$$\frac{\partial V_1^{(2)}}{\partial\rho} = (k_2^2/2\pi i k_2\rho)e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)}\left\{W_{\rho s} + W_{\rho p}\right\} = W_\rho^{(s)} + W_\rho^{(p)}, \quad (6.144)$$

where the subscripts ρ do not mean differentiation with respect to ρ.

Thus, from Eq. (5.78) as applied to $\Phi(x,y)$ in Eq. (6.115), we have

$$C_\rho(y) = -\frac{2ny^2}{(1+n^2)^{\frac{1}{2}}(1-n^4)}\left(\frac{4ik_2\rho}{(1+n^2)^{\frac{1}{2}}} - y^2\right)^{\frac{1}{2}} ; \tag{6.145}$$

and from Eqs. (5.86) and (5.89), making use of (6.144), we then obtain

$$W_\rho^{(p)} = \frac{m^3k_1^2}{(1+n^2)(1-n^4)} H_1^1\left(\frac{k_2\rho}{(1+n^2)^{\frac{1}{2}}}\right) e^{-ik_1(z-h)/(1+n^2)^{\frac{1}{2}}}\left\{1 - \mathrm{erf}(-ix_o/2^{\frac{1}{2}})\right\} , \tag{6.146}$$

which can be deduced from $W^{(p)}$, Eq. (6.125), by merely taking the derivative with respect to ρ of the Hankel function.

Proceeding as before, the expansion terms $\chi_\rho^{(n)}$ are deduced by first expanding $\chi_\rho(x,y)$, obtained from (6.127) after replacing $C(y)$ by $C_\rho(y)$, Eq. (6.145), into a double power series and then making use of Eqs. (5.69). In this way we obtain

$$\chi_\rho^{(o)} = 0 , \qquad \chi_\rho^{(1)} = -\frac{8(1+K)^{\frac{1}{2}}}{(1+n^2)^{\frac{1}{2}}(1-n^4)} Q^{-\frac{1}{2}} ,$$

$$\chi_\rho^{(2)} = -\frac{(1+K)^{\frac{1}{2}} Q^{-3/2}}{n^3(1-n^4)}\left[8(1+K) - 3n^2Q\right](ik_1\rho)^{-1} , \tag{6.147}$$

$$\chi_\rho^{(3)} = -\frac{3}{16}\frac{(1+n^2)^{\frac{1}{2}}(1+K)^{\frac{1}{2}} Q^{-5/2}}{n^6(1-n^4)}\left[128(1+K)^2 - 16n^2(1+K)Q - 5n^4Q^2\right](ik_1\rho)^{-2} ;$$

and the expansion terms $\Phi_\rho^{(n)}$ are next obtained from (6.118) with $G'(o)$ and $G'''(o)$ as taken from (6.77), yielding

$$\Phi_\rho^{(o)} = 0, \qquad \Phi_\rho^{(1)} = 0, \qquad \Phi_\rho^{(2)} = \frac{8(1+K)^{3/2}}{n^3(1-n^2)}(ik_1\rho)^{-1},$$

$$\Phi_\rho^{(3)} = \frac{3(1+K)^{3/2}}{n^6(1-n^2)^2}\left[8(1+K) - n^2(8+12K+5K^2) - n^4K\right](ik_1\rho)^{-2}. \qquad (6.148)$$

Combining the above expansion terms in accordance with (6.130) and inserting the resulting asymptotic expansion (5.84) into Eq. (6.144), we obtain, after some simplification,

$$W_\rho^{(s)} = -\frac{2k_1^2(1+K)^{\frac{1}{2}}}{(1+n^2)^{\frac{1}{2}}} e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)}\left\{w_\rho^{(o)} + w_\rho^{(1)} + w_\rho^{(2)} + w_\rho^{(3)} + 0\left(\frac{1}{\lambda}+\frac{1}{\nu}\right)^4\right\}, \qquad (6.149)$$

where λ and ν are again given by Eqs. (6.92) and (6.93), respectively, and where

$$w_\rho^{(o)} = 0, \qquad w_\rho^{(1)} = n\left[1 + \frac{n^2}{8}(1-K) + 0(n^4)\right](ik_1\rho)^{-1},$$

$$w_\rho^{(2)} = -\frac{1}{8}\left[8 + 8K + 3K^2 + \frac{n^2}{16}(79 + 97K + 39K^2 - 15K^3) + 0(n^4)\right](ik_1\rho)^{-2}, \qquad (6.150)$$

$$w_\rho^{(3)} = \frac{3}{128}\left[\frac{K}{n}(64 + 144K + 120K^2 + 35K^3) - \frac{n}{8}(35 - 1059K - 2489K^2 - 2035K^3 - 385K^4 + 105K^5) + 0(n^3)\right](ik_1\rho)^{-3}.$$

The asymptotic expansion for the mixed second derivative $\partial^2 V_1^{(2)}/\partial z\partial\rho$ can also be written, as in Eq. (6.144), as the sum of two terms

$$\frac{\partial v_1^{(2)}}{\partial\rho\,\partial z} = (k_2^2/2\pi i k_2\rho)e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)}\left\{W_{\rho zs} + W_{\rho zp}\right\} = W_{\rho z}^{(s)} + W_{\rho z}^{(p)}, \qquad (6.151)$$

where the subscripts ρ and z do not mean differentiation. From Eq. (5.78) as applied to $\Phi(x,y)$ in Eq. (6.115) where $G(w)$ is replaced by $G_z(w)$, Eq. (6.135), we have

$$C_{\rho z}(z) = \frac{2ik_1ny^2}{(1+n^2)(1-n^4)}\left(\frac{4ik_2\rho}{(1+n^2)^{\frac{1}{2}}} - y^2\right)^{\frac{1}{2}}; \qquad (6.152)$$

and, thus, comparing (6.152) with (6.145), $W_{\rho z}^{(p)}$ may be obtained from $W_{\rho}^{(p)}$ by multiplying by the factor $\left[-ik_1/(1+n^2)^{\frac{1}{2}}\right]$, yielding

$$W_{\rho z}^{(p)} = -\frac{\pi i k_1^3 n^3}{(1+n^2)^{3/2}(1-n^4)} H_1^1\left(\frac{k_2\rho}{(1+n^2)^{\frac{1}{2}}}\right)e^{-ik_1(z-h)/(1+n^2)^{\frac{1}{2}}}\left\{1 - \mathrm{erf}(-ix_o/2^{\frac{1}{2}})\right\}. \quad (6.153)$$

Proceeding as before, the expansion terms $\chi_{\rho}^{(n)}$ are obtained from (6.147) by simply multiplying by the same factor, $\left[-ik_1/(1+n^2)^{\frac{1}{2}}\right]$; thus

$$\chi_{\rho z}^{(o)} = 0, \qquad \chi_{\rho z}^{(1)} = \frac{8ik_1(1+K)^{\frac{1}{2}}}{(1+n^2)^{\frac{1}{2}}(1-n^4)}Q^{-\frac{1}{2}},$$

$$\chi_{\rho z}^{(2)} = \frac{ik_1(1+K)^{\frac{1}{2}}Q^{-3/2}}{n^3(1+n^2)^{\frac{1}{2}}(1-n^4)}\left[8(1+K) - 3n^2Q\right](ik_1\rho)^{-1}, \qquad (6.154)$$

$$\chi_{\rho z}^{(3)} = \frac{3}{16}\frac{ik_1(1+K)^{\frac{1}{2}}Q^{-5/2}}{n^6(1-n^4)}\left[128(1+K)^2 - 16n^2(1+K)Q - 5n^4Q^2\right](ik_1\rho)^{-2};$$

and the expansion terms $\Phi_{\rho z}^{(n)}$ are next obtained from (6.118) with the odd derivatives of $G(w)$ evaluated at $w = 0$ given for $G_z(w)$ in Eqs. (6.136), yielding

$$\Phi^{(o)}_{\rho z} = 0, \qquad \Phi^{(1)}_{\rho z} = 0, \qquad \Phi^{(2)}_{\rho z} = -\frac{8ik_1(1+K)^{3/2}}{n^3(1-n^2)^{\frac{1}{2}}}(ik_1\rho)^{-1},$$

$$\Phi^{(3)}_{\rho z} = -\frac{3ik_1(1+K)^{3/2}}{n^6(1-n^2)^{3/2}}\left[8(1+K) - n^2(8 + 12K + 5K^2) + n^4(4 + 3K)\right](ik_1\rho)^{-2}. \tag{6.155}$$

Combining the above expansion terms in accordance with (6.130) and inserting the resulting asymptotic expansion (5.84) into Eq. (6.151), we obtain, after some simplification,

$$W^{(s)}_{\rho z} = \frac{2ik_1^3 n(1+K)^{\frac{1}{2}}}{(1+n^2)(1-n^4)} e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)}\left\{w^{(o)}_{\rho z} + w^{(1)}_{\rho z} + w^{(2)}_{\rho z} + w^{(3)}_{\rho z} + O\left(\frac{1}{\lambda} + \frac{1}{\nu}\right)^4\right\} \tag{6.156}$$

where λ and ν are given by Eqs. (6.92) and (6.93), respectively, and where

$$w^{(o)}_{\rho z} = 0, \qquad w^{(1)}_{\rho z} = n\left[1 + \frac{n^2}{8}(1 - K) + O(n^4)\right](ik_1\rho)^{-1},$$

$$w^{(2)}_{\rho z} = -\frac{1}{8}\left[8 + 8K + 3K^2 + \frac{3}{16}n^2(5 + 11K + 13K^2 - 5K^3) + O(n^4)\right](ik_1\rho)^{-2}, \tag{6.157}$$

$$w^{(3)}_{\rho z} = \frac{3}{128}\left[\frac{K}{n}(64 + 144K + 120K^2 + 35K^3) - \frac{n}{8}(547 + 221K - 1501K^2 - 1715K^3 - 385K^4 + 105K^5) + O(n^3)\right](ik_1\rho)^{-3}.$$

VII. RESULTS FOR THE CONDUCTING MEDIUM

In the present Chapter we discuss the approximate expressions for the fundamental integrals, the components of the Hertzian vector, and the electric and magnetic field components pertaining to points of observation in the conducting medium. In particular, it is important to note that all of our results are necessarily restricted to $|n| < 1$, $|k_1 \rho| > 1$, and $|k_1 \rho (\cot\theta_2 - 1)| > 1$ when $k_2 \rho < 1$, where the first condition was imposed initially and the latter two arise from the form of the remainder in the asymptotic evaluation of the integrals. Rather than merely cataloguing the results which could be obtained directly from our expressions for the fundamental integrals and their derivatives, as given in the previous Chapter, we present also further approximations of our results appropriate to different ranges of the parameters and obtain simpler and more useful expressions. We consider the three principal ranges: $\rho \rightarrow \infty$, $|n^2 k_2 \rho| < 1 < k_2 \rho$, and $k_2 \rho < 1 < |k_1 \rho|$, the only range of practical interest for the present low frequency investigation being for $k_2 \rho < 1 < |k_1 \rho|$, particularly when $\theta_2 > 5^{\circ}$; other ranges are included in order to compare our results with the results obtained by other workers. In addition, we consider the two important limiting cases in which $(h-z) = 0$ and $n = 0$ that have been considered by other authors. In so doing, we have been led to examine anew the whole question of the existence or non-existence of the so-called Zenneck surface waves; and, as mentioned earlier, we trust that we have settled the question beyond further controversy.

7.1 APPROXIMATE RESULTS FOR VARIOUS RANGES OF THE PARAMETERS

In this Section we consider the form assumed by our results in various ranges of the parameters. First, we consider the effect of imposing the condition $|k_1 \rho| > 1$ required by our asymptotic series which means that the point of observation must be at least a few wavelengths in the conducting medium away from the source and we show that, under this condition, all exponentially attenuated terms can be neglected which results in considerable simplification. Next, we take up the analysis of our asymptotic results in the limit $\rho \to \infty$ and we show that the interface at $z = 0$ separating the two media acts as a source of secondary waves. In order to be able to compare our results with those of Sommerfeld we then consider the range $|n^2 k_2 \rho| < 1 < k_2 \rho$ which implies that the point of observation is at least several wavelengths in air away from the source while Sommerfeld's "numerical distance" (measured by $|n^2 k_2 \rho|$) is small; and, finally, we take up the range of parameters which is the only one applicable to the low frequency case; namely, $k_2 \rho < 1 < |k_1 \rho|$ and which implies that the distance of the point of observation from the source is at least a few wavelengths in the conducting medium but only a small fraction of a wavelength in air.

7.1a. Imposition of the condition $|k_1 \rho| > 1$.— The equations developed in Chapters II and III were obtained without any assumption as regards the order of magnitude of the parameters other than $|n| < 1$; however, some simplification can be achieved by now imposing the condition $|k_1 \rho| > 1$ which was found necessary in order to obtain the asymptotic series evaluation of the integrals. Thus, for $|k_1 \rho| > 1$ we may neglect terms which are exponentially attenuated which include Ψ_1 and Ψ_2, Eqs. (2.65) and (2.66), respectively, as well as all integrals of the type I_1 and their derivatives, all of which may be regarded as arising from contributions over the path C_1 around the cut for γ_1 in the λ-plane. Thus, we contend that all integrals of the type I_1 are of the order of magnitude of the error committed in the asymptotic evaluation of the integrals of the type I_2 which are computed over the path C_2 around the branch cut for γ_2 in the λ-plane (see Fig. 4). To see this, consider an integral of the form (5.10) in which $\Phi(x)$ possesses a branch point at $x = x_1$ and is analytic for $|x| < \lambda^{\frac{1}{2}}$ where $\lambda^{\frac{1}{2}} = |x_1|$ and let the branch cut extend from $x = x_1$ to infinity within the sector $|\arg\{x\}| < \pi/4$ as shown in Fig. 9. Then, according to Watson's lemma, Section 5.1a, the integral yields identical results for path (1) and for path (2) which differ from each other only in that they go to infinity on different sides of the chosen branch cut. The difference between the integral taken over path (1) and the integral taken over path (2) is clearly the contribution around the branch cut. Thus, this contribution itself must be of the order of the magnitude of the uncertainty in the asymptotic evaluation of the integral over either path (1) or path (2).

Applying the results of the previous paragraph, we obtain from Eqs. (6.3) and (6.7)

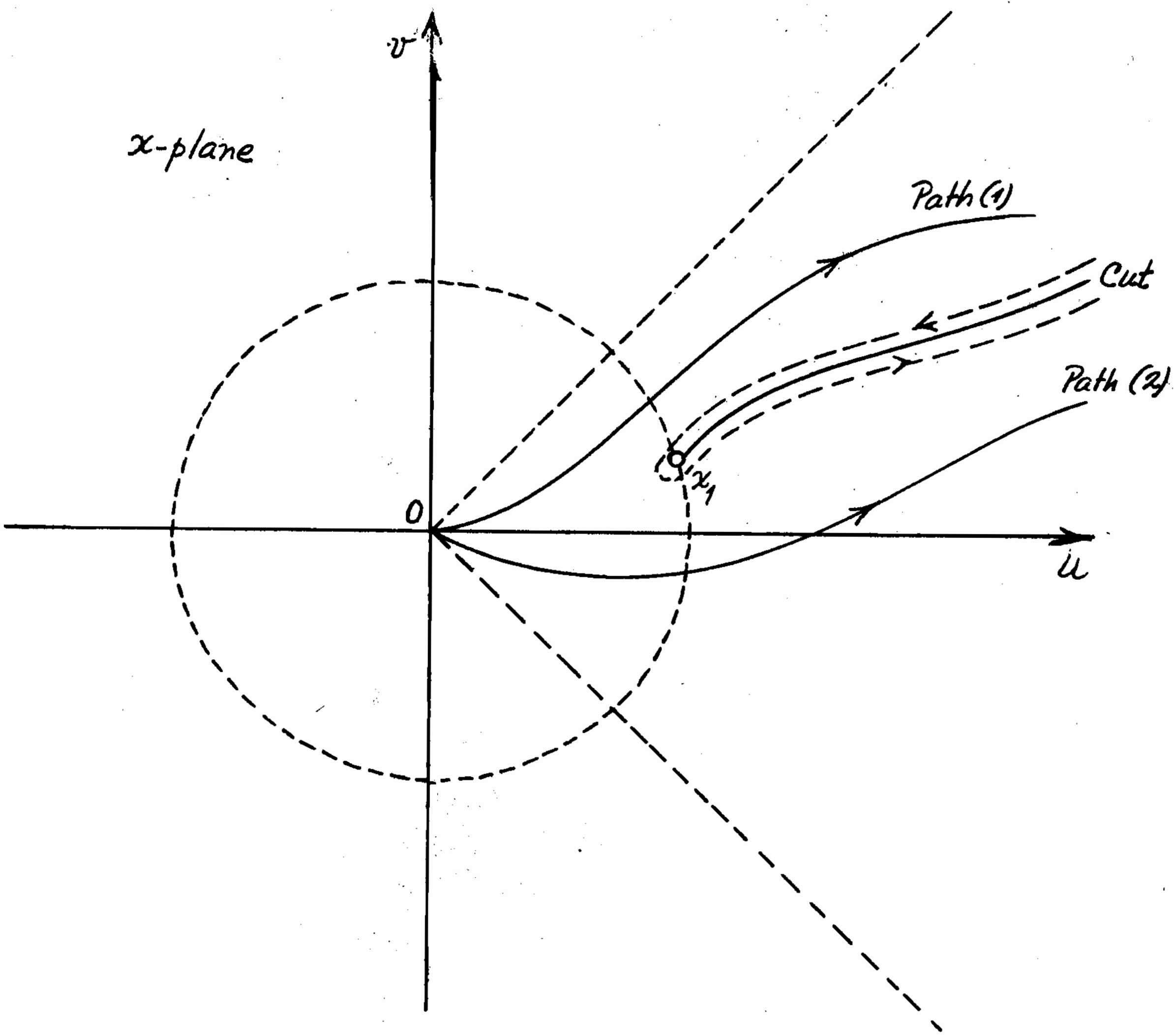

Fig. 9.- Diagram showing that the difference between the integral over path (1) and the integral over path (2) is the contribution around the branch cut.

$$U_1 \approx M_1^{(2)} ,$$
$$V_1 \approx V_1^{(2)} , \qquad (7.1)$$

with similar relations for the derivatives. Using (7.1), the Cartesian components of the Hertzian vector for the conducting medium given by Eqs. (3.1) and (3.4) become

$$\Pi_{x1} \approx \frac{ip}{4\pi k_1 \eta_1} M_1^{(2)} ,$$
$$\Pi_{z1} \approx - \frac{ip \cos\phi}{4\pi\, k_1 \eta_1} \frac{(1 + n^2)}{k_1^2} \frac{\partial^2 V_1^{(2)}}{\partial\rho\, \partial z} . \qquad (7.2)$$

Similarly, the electric field components from Eqs. (3.14a), (3.15a) and (3.16) become

$$E_{\rho 1} \approx - \frac{ip \cos\phi}{4\pi\, k_1 \eta_1} \left\{ \frac{1}{\rho} \frac{\partial V_1^{(2)}}{\partial\rho} + \frac{n^2 k_1^2}{1 + n^2} \left[V_1^{(2)} - M_1^{(2)} \right] \right\} ,$$
$$E_{\phi 1} \approx - \frac{ip \sin\phi}{4\pi\, k_1 \eta_1} \left\{ \frac{1}{\rho} \frac{\partial V_1^{(2)}}{\partial\rho} + k_1^2 M_1^{(2)} \right\} , \qquad (7.3)$$
$$E_{z1} \approx - \frac{ip \cos\phi}{4\pi\, k_1 \eta_1} \left\{ n^2 \frac{\partial^2 V_1^{(2)}}{\partial\rho\, \partial z} \right\} ;$$

and finally the magnetic field components from Eqs. (3.24b), (3.25b), and (3.26b) become

$$H_{\rho 1} \approx \frac{p \sin\phi}{4\pi k_1^2} \frac{\partial}{\partial z} \left\{ (1+n^2) \frac{1}{\rho} \frac{\partial V_1^{(2)}}{\partial \rho} + k_1 M_1^{(2)} \right\},$$

$$H_{\phi 1} \approx - \frac{p \cos\phi}{4\pi k_1^2} \frac{\partial}{\partial z} \left\{ (1+n^2) \frac{1}{\rho} \frac{\partial V_1^{(2)}}{\partial \rho} + n^2 k_1^2 V_1^{(2)} \right\}, \tag{7.4}$$

$$H_{z1} \approx - \frac{p \sin\phi}{4\pi} \left\{ \frac{\partial M_1^{(2)}}{\partial \rho} \right\}.$$

All of the integrals appearing in the above equations have been evaluated subject only to the conditions $|n| < 1$, $|k_1 \rho| > 1$, $|k_1 \rho (\cot\theta_2 - 1)| > 1$ when $k_2 \rho < 1$ and are presented in Chapter VI.

Before proceeding with the investigation of the results for the three principal ranges of parameters, $\rho \to \infty$, $|n^2 k_2 \rho| < 1 < k_2 \rho$ and $k_2 \rho < 1 < |k_1 \rho|$, it is important to note that our results are not discontinuous, even though the approximate expressions derived for one region of the parameters may not fit smoothly with the approximate expressions derived for another region. If one wishes to investigate the transition from one region to another, it is only necessary to return to the exact expressions of the asymptotic series and choose the appropriate approximation for this transition region to obtain the continuity which actually exists.

In order to illustrate the type of additional approximations which are appropriate for particular ranges of the parameters, we single out for special study the fundamental integral $V_1^{(2)}$. To facilitate later discussion we collect together in one set of formulas the most general expression that we have been able to obtain for $V_1^{(2)}$ which is valid for all ranges of the parameters that are considered; thus, from Eqs. (6.121), (6.125), (6.133), (6.131) and (6.137), using the definition (6.123) for Q, we may write

$$V_1^{(2)} = W^{(p)} + W^{(s)} \sim \frac{2(1+K)^{\frac{1}{2}}}{(1-n^4)\,Q^{1/2}} \frac{1}{\rho} e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)} \left\{ W_1^{(p)} + W_1^{(s)} \right\} \qquad (7.5)$$

where

$$W_1^{(p)} = -\left[\left(\frac{\pi i k_2}{2(1+n^2)^{\frac{1}{2}}}\right)^{\frac{1}{2}} e^{-ik_2\rho/(1+n^2)^{\frac{1}{2}}} H_0^1\left(\frac{k_2\rho}{(1+n^2)^{\frac{1}{2}}}\right)\right]\left[\pi^{\frac{1}{2}}(-ix_0/2^{\frac{1}{2}})e^{-x_0^2/2}\left(1-\mathrm{erf}(-ix_0/2^{\frac{1}{2}})\right)\right] \qquad (7.5a)$$

is an exact expression exhibiting the term as the product of two distinct factors, and where

$$W_1^{(s)} \sim 1 - \frac{(1+n^2)^{\frac{1}{2}}}{8n^3}\left[8(1+n^2)^{\frac{1}{2}}(1+K) - 8(1+K)Q^{-3/2} - n^2Q^{-\frac{1}{2}}\right] Q^{\frac{1}{2}}(ik_1\rho)^{-1}$$

$$+ \frac{3}{128}\left[\left(\frac{K}{n}\right)^2 (48 + 80K + 35K^2) + \frac{5}{8}(5 + 72K + 274K^2 + 299K^3 \qquad (7.5b)$$

$$+ 77K^4 - 21K^5) - \frac{1}{8}K(1+K) + O(n^2)\right](ik_1\rho)^{-2}$$

is a three-term asymptotic expansion. Here K is defined by Eq. (6.67) and x_0 by Eq. (6.122). A similar general expression for $M_1^{(2)}$ is given by Eq. (6.102) which, accordingly, is not reproduced here.

Despite the fact that most of our attention is given to $V_1^{(2)}$, it is extremely important to note that the components of the Hertzian vector and the field components, in general, are not obtainable by differentiating the asymptotic series expressions for $M_1^{(2)}$ and $V_1^{(2)}$, since an asymptotic series may not, in general, be differentiated term by term (only for the case $\rho \to \infty$ is it found to be a valid process). The components of the Hertzian vector and the field components must be obtained by applying the additional approximations which are appropriate for a particular range of the parameters to the asymptotic series evaluation of the derivatives as given in the previous Chapter.

7.1b. Asymptotic results for $\rho \to \infty$.— Although the investigation of this case has no direct practical application for the present low frequency case, it is of interest to further demonstrate that the interface between the conducting and non-conducting media acts as a distributed source of secondary waves, as was demonstrated for the static case, Section 4.5, and to illustrate the nature of the transition in the form of $V_1^{(2)}$ between the ranges of the parameters $\rho \to \infty$ or $|n^2 k_2 \rho| > 1$ and $|n^2 k_2 \rho| < 1 < k_2 \rho$ which is of particular interest when we wish to compare our results with those of Sommerfeld for $(h-z) = 0$.

The integral $V_1^{(2)}$ for the case $\rho \to \infty$ or in particular for $|n^2 k_2 \rho| > 1$ may be obtained directly from the evaluation of $V_1^{(2)}$ without first removing the pole; thus, from the first term of Eq. (6.82), we obtain

$$V_1^{(2)} \sim \frac{2i(1+K)^{3/2}}{n^2 k_2 (1-n^2)} \frac{1}{\rho^2} e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)} \tag{7.6}$$

where K is given by Eq. (6.67). This result may also be obtained from the evaluation of $V_1^{(2)}$ by first removing the pole, as given by Eq. (7.5), by first expanding asymptotically both the Hankel function and the function $\left\{1 - \mathrm{erf}(-ix_o/2^{\frac{1}{2}})\right\}$ arising from $W^{(p)}$, process which is justified since $k_2\rho > |x_o^2| > 1$ where $|x_o^2| \approx |n^2 k_2 \rho|$ from (6.122). To two terms,[32] we

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[32] E. T. Whittaker and G. N. Watson, "A Course of Modern Analysis," (The Macmillan Co., New York, 1948), Am. Ed., pp. 340-343, Section 16.3.

have for the second factor in (7.5a)

$$\pi^{\frac{1}{2}}(-ix_o/2^{\frac{1}{2}})e^{(-ix_o/2^{\frac{1}{2}})^2}\left\{1 - \mathrm{erf}(-ix_o/2^{\frac{1}{2}})\right\} = 1 - \tfrac{1}{2}(-ix_o/2^{\frac{1}{2}})^{-2} + \cdots \qquad (7.7)$$

Similarly, expanding the Hankel function in Eq. (7.5a) asymptotically we obtain to two terms for the contribution from the pole

$$W^{(p)} \sim -\frac{2(1+K)^{\frac{1}{2}}}{(1-n^4)Q^{\frac{1}{2}}}\,\frac{1}{\rho}\,e^{ik_2\rho-ik_1(1-n^2)^{\frac{1}{2}}(z-h)}\left\{1 + \frac{(1+n^2)^{\frac{1}{2}}}{8n^2}\left[8(1+K)Q^{-1} + n^2\right]\frac{1}{ik_2\rho}\right\} \qquad (7.8)$$

where use has been made of Eq. (6.128) to express x_o in terms of K and Q. Combining (7.8) with $W^{(s)}$, given by Eq. (7.5) and (7.5b), and neglecting the third term of the series for $W^{(s)}$, we again obtain the result (7.6) which was to be proved.

The derivatives of $V_1^{(2)}$ in the limit $\rho \to \infty$ may be obtained by differentiating (7.6), noting from Eq. (6.67) that

$$\frac{\partial K}{\partial \rho} = -\frac{K(1+K)}{\rho} \quad \text{and} \quad \frac{\partial K}{\partial z} = \frac{n(1+K)^2}{(1-n^2)^{\frac{1}{2}}\rho}\,, \qquad (7.9)$$

and retaining only the leading term, process which may be verified by either the saddle point method before the removal of the pole or after the removal, as demonstrated above.

The integral $M_1^{(2)}$ for the case $\rho \to \infty$ is given directly by the first term of Eq. (6.102), thus

$$M_1^{(2)} \sim \frac{2in(1+K)^{3/2}}{k_1(1-n^2)}\,\frac{1}{\rho^2}\,e^{ik_2\rho-ik_1(1-n^2)^{\frac{1}{2}}(z-h)}\,. \qquad (7.10)$$

The derivatives of $M_1^{(2)}$ in the limit $\rho \to \infty$ may be obtained by differentiating

(7.10) and retaining the leading term, as may be verified by examining the leading terms of Eqs. (6.107) and (6.119).

The components of the Hertzian vector as given by Eqs. (7.2) become in the limit $\rho \to \infty$, making use of Eqs. (7.6) and (7.10),

$$\Pi_{x1} \sim -\frac{p(1+K)^{3/2}}{2\pi k_1^2 \eta_1} \frac{n}{1-n^2} \frac{1}{\rho^2} e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)} ,$$

$$\Pi_{z1} \sim \frac{p(1+K)^{3/2}}{2\pi k_1^2 \eta_1} \frac{(1+n^2)\cos\phi}{n^2(1-n^2)^{\frac{1}{2}}} \frac{1}{\rho^2} e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)} . \tag{7.11}$$

Apart from the $\cos\phi$ factor, it is apparent that we have $|\Pi_{x1}/\Pi_{z1}| = O(n^3)$, which indicates that in the low frequency case the field, as $\rho \to \infty$, may be described primarily in terms of the z component of the Hertzian vector, which in the static limit was associated with a secondary source distributed over the boundary between the conducting and non-conducting media.

Next, we compute the field components and further demonstrate that they are essentially produced by a secondary source distributed over the surface $z = 0$ by showing that the power flow, as represented by Poynting's vector, is essentially a vector in the negative z direction and therefore not a vector along the line from the original source. Thus, by inserting Eqs. (7.6) and (7.10) into Eqs. (7.3), we obtain for the electric field components as $\rho \to \infty$

$$E_{\rho 1} \sim \frac{p\cos\phi\,(1+K)^{3/2}}{2\pi\eta_1} \frac{1}{n} \frac{1}{\rho^2} e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)} ,$$

$$E_{\phi 1} \sim \frac{p\sin\phi\,(1+K)^{3/2}}{2\pi\eta_1} \frac{n}{1-n^2} \frac{1}{\rho^2} e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)} , \tag{7.12}$$

$$E_{z1} \sim \frac{p\cos\phi\,(1+K)^{3/2}}{2\pi\eta_1} \frac{1}{(1-n^2)^{\frac{1}{2}}} \frac{1}{\rho^2} e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)} .$$

These same results to within $O(n^2)$ may be obtained from Eqs. (3.13) by inserting Eqs. (7.11), thus

$$E_{\rho 1} \approx \frac{\partial^2 \Pi_{z1}}{\partial \rho \, \partial z} , \qquad E_{\phi 1} \approx - k_1^2 \sin\phi \, \Pi_{x1} ,$$

$$E_{z1} \approx - \frac{\partial^2 \Pi_{z1}}{\partial \rho^2} . \tag{7.12a}$$

It is seen that apart from the factors $\sin\phi$ and $\cos\phi$, the electric field components stand in the relative order of magnitude $|E_{\rho 1}/E_{z1}/E_{\phi 1}| = O(1/n/n^2)$.

Similarly, by inserting Eqs. (7.6) and (7.10) into Eqs. (7.4) we obtain for the magnetic field components as $\rho \to \infty$

$$H_{\rho 1} \sim \frac{p \sin\phi \, (1+K)^{3/2}}{2\pi \, (1-n^2)^{\frac{1}{2}}} \, n \, \frac{1}{\rho^2} \, e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)} ,$$

$$H_{\phi 1} \sim - \frac{p \cos\phi \, (1+K)^{3/2}}{2\pi \, (1-n^2)^{\frac{1}{2}}} \, \frac{1}{n} \, \frac{1}{\rho^2} \, e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)} , \tag{7.13}$$

$$H_{z1} \sim \frac{p \sin\phi \, (1+K)^{3/2}}{2\pi \, (1-n^2)^{\frac{1}{2}}} \, \frac{n^2}{(1-n^2)^{\frac{1}{2}}} \, \frac{1}{\rho^2} \, e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)} ,$$

which can also be obtained to within $O(n^2)$ from Eqs. (3.23) by inserting Eqs. (7.11), thus

$$H_{\rho 1} \approx - ik_1\eta_1 \sin\phi \, \frac{\partial \Pi_{x1}}{\partial z} , \qquad H_{\phi 1} \approx ik_1\eta_1 \frac{\partial \Pi_{z1}}{\partial \rho} ,$$

$$H_{z1} \approx ik_1\eta_1 \sin\phi \, \frac{\partial \Pi_{x1}}{\partial \rho} ; \tag{7.13a}$$

and, therefore, we obtain the relative order of magnitudes $|H_{\phi 1}/H_{\rho 1}/H_{z1}| = O(1/n^2/n^3)$, apart from the factors $\sin\phi$ and $\cos\phi$.

Next, we examine the components of the complex Poynting's vector, $S_1 = \frac{1}{2}(E_1 \times H_1^*)$, and readily determine from the relative orders of magnitude of the field components given in the previous paragraph that $|S_{z1}/S_{\rho 1}/S_{\phi 1}| = O(1/n/n^3)$. Neglecting the ϕ component as being negligible the remaining components can be expressed to an accuracy of $O(n^4)$ simply as

$$S_{z1} \approx \tfrac{1}{2}E_{\rho 1}H_{\phi 1}^* \quad \text{and} \quad S_{\rho 1} \approx -\tfrac{1}{2}E_{z1}H_{\phi 1}^* , \tag{7.14}$$

where it may be seen from Eqs. (7.12) and (7.13) that to an accuracy of $O(n^2)$ the Poynting's vector is associated with Π_{z1} only. To obtain the net power flow we consider the time average Poynting's vector[33] given by

$$\langle S_{z1} \rangle \approx \tfrac{1}{2}\mathrm{Re}\left\{E_{\rho 1}H_{\phi 1}^*\right\} , \qquad \langle S_{\rho 1} \rangle \approx \tfrac{1}{2}\mathrm{Re}\left\{-E_{z1}H_{\phi 1}^*\right\} . \tag{7.15}$$

Substituting the first and third of Eqs. (7.12) and the second of Eqs. (7.13) into Eqs. (7.15), neglecting n^2 as compared with unity, recalling that $1/\eta_1 = \zeta_1 = \omega\mu_0/k_1$, and putting $k_1 = (1+i)/\delta$ where $\delta = (2/\omega\mu_0\sigma)^{\frac{1}{2}}$ is the so-called "skin depth", we obtain

$$\begin{aligned} \langle S_{z1} \rangle &\approx -\frac{p^2\,|1+K|^3}{8\pi^2\,\sigma\delta|n|^2}\,\frac{\cos^2\phi}{\rho^4}\,e^{-2(h-z)/\delta} , \\ \langle S_{\rho 1} \rangle &\approx \mathrm{Re}\left\{\frac{i|n|}{2^{\frac{1}{2}}}\,\frac{p^2|1+K|^3}{8\pi^2\,\sigma\delta|n|^2}\,\frac{\cos^2\phi}{\rho^4}\,e^{-2(h-z)/\delta}\right\} = 0 , \end{aligned} \tag{7.16}$$

where K is given by Eq. (6.67). Since $\langle S_{\rho 1} \rangle$ was set equal to zero by

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[33] J. A. Stratton, "Electromagnetic Theory," (McGraw-Hill Book Co., New York, 1941), p. 137, Eq. (29).

neglecting n^2 as compared with unity, we conclude that to an accuracy of $O(n^3)$ the time average Poynting's vector for $\rho \to \infty$ is given exclusively by the z component.

Thus, in conclusion, since the time average Poynting's vector (7.16) (and therefore the net power flow) is essentially a vector in the negative z direction which arises from the z component of the Hertzian vector, Π_{z1}, the asymptotic field may be regarded as arising essentially from a secondary source distributed over the boundary between the conducting and non-conducting media, at least as far as the conducting medium is concerned. These conclusions are in complete accord with our findings for the static limit, Sections 4.4 and 4.5.

7.1c. Results for the range $|n^2 k_2 \rho| < 1 < k_2 \rho$.-- This range, which implies that the point of observation is several wavelengths in air away from the source, is again of no practical interest for the present low frequency case and is included here in order to be able to compare our results with those of Sommerfeld and others when we consider the limiting case $(h-z) = 0$. Thus we limit ourselves to the consideration of the fundamental integrals $V_1^{(2)}$ and $M_1^{(2)}$ and leave to those who are particularly interested the problem of computing the field components from Eqs. (7.3) and (7.4) by substituting the results of the previous Chapter and making the appropriate approximations for this range of the parameters.

By examining Eqs. (6.103) we see that in this range, $|k_1 \rho| > k_2 \rho > 1$, $M_1^{(2)}$ may be satisfactorily represented by the first term; and, therefore, Eq. (7.10) may be used to represent $M_1^{(2)}$.

The situation is different for $V_1^{(2)}$; we are no longer permitted to use Eq. (7.6), since it was derived under the assumption that $|n^2 k_2 \rho| > 1$.

Thus, we must consider the complete expression for $V_1^{(2)}$ as given by the set of Eqs. (7.5), (7.5a) and (7.5b) which were obtained by the removal of the pole from the integrand. In the present range, $k_2\rho > 1$, we distinguish two cases: (1) when $k_2\rho \gg 1$, which allows the asymptotic expansion of the Hankel function in the first factor of Eq. (7.5a); and (2) when $k_2\rho \sim 1$, which permits the power series expansion of the error function factor in Eq. (7.5a).

Thus in case (1), with $k_2\rho \gg 1$, we expand the first factor in Eq. (7.5a) asymptotically retaining only the leading term which is just equal to unity; and, to the same approximation, we retain only the leading term for $W_1^{(s)}$ in Eq. (7.5b) which is again unity. In this manner we obtain for $V_1^{(2)}$, from Eq. (7.5),

$$V_1^{(2)} \sim \frac{2(1+K)^{\frac{1}{2}}}{(1-n^4)Q^{\frac{1}{2}}} \frac{1}{\rho} e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)} \left\{1 - \pi^{\frac{1}{2}}(-ix_o/2^{\frac{1}{2}}) e^{-x_o^2/2}\left[1-\mathrm{erf}(-ix_o/2^{\frac{1}{2}})\right]\right\} \quad (7.17)$$

which is the desired approximate expression valid in the range $|n^2k_2\rho| < 1 \ll k_2\rho$. In obtaining the above result we retained only the leading terms of the asymptotic expansions for the Hankel function in $W_1^{(p)}$, Eq. (7.5a), and for the term $W_1^{(s)}$, Eq. (7.5b). Since the neglected terms will certainly become smaller as $\rho \to \infty$ we conclude that the above result must also be valid as $\rho \to \infty$ and, hence, that Eq. (7.17) is restricted only by the condition $k_2\rho \gg 1$. In fact, if we allow $\rho \to \infty$ which implies $x_o \to \infty$, we can expand the bracket in Eq. (7.17) asymptotically by making use of Eq. (7.7), obtaining zero to $O(1/\rho^2)$ in agreement with Eq. (7.6).

In case (2), with $k_2\rho$ close to unity, we retain the Hankel function in the first factor of $W_1^{(p)}$, Eq. (7.5a), and since $|n^2k_2\rho| \ll 1$ we expand the second factor into a power series in x_o. Thus, retaining only the first terms in $W_1^{(s)}$ and $W_1^{(p)}$ we obtain for $V_1^{(2)}$, from Eq. (7.5), the useful expression

$$V_1^{(2)} \sim \frac{2\,(1+K)^{\frac{1}{2}}}{(1-n^4)Q^{\frac{1}{2}}}\frac{1}{\rho}\,e^{ik_2\rho - ik_1(1-n^2)^{\frac{1}{2}}(z-h)}\left\{1\right.$$

$$\left. - \frac{\pi n k_2 \rho Q^{\frac{1}{2}}}{2(1+n^2)^{\frac{1}{2}}(1+K)^{\frac{1}{2}}}\,e^{-ik_2\rho}H_0^1\left(\frac{k_2\rho}{(1+n^2)^{\frac{1}{2}}}\right)\right\} \qquad (7.18)$$

which is valid for $|nk_2\rho| \ll 1 < k_2\rho$ when $k_2\rho$ is close to unity.

7.1d. Results for $k_2\rho < 1 < |k_1\rho|$ and $|k_1\rho\,(\cot\theta_2 - 1)| > 1$.— This range is the only one of practical interest in the present low frequency case and implies that the point of observation is away from the source at least several wavelengths in the conducting medium but only a fraction of an air wavelength. In addition to the restrictions $|k_1\rho| > 1$ and $|k_1\rho\,(\cot\theta_2 - 1)| > 1$ which are imposed by the form of the remainder in our asymptotic series we impose the additional condition $\theta_2 > 5^\circ$ in order to be able to neglect K^2 and higher powers, where K is defined by Eq. (6.67), and thus further simplify our results.

To examine the order of magnitude of the function $K(n,\theta_2)$, we obtain from Eq. (6.67)

$$|K| \approx |n|\cot\theta_2\left\{|n|^2\cot^2\theta_2 - 2^{\frac{1}{2}}|n|\cot\theta_2 + 1\right\}^{-\frac{1}{2}} \qquad (7.19)$$

where we have placed

$$n = |n|\,e^{-i\pi/4} \qquad (7.20$$

in accordance with previous assumptions and where we neglect n^2 in comparison with unity. Thus, we find approximately that $0 \leq |K| \leq 2^{\frac{1}{2}}$ where the maximum is attained when $|n|\cot\theta_2 = 2^{\frac{1}{2}}$ and where $|K| = 1$ for $\theta_2 = 0$. In addition, it may be seen from Table I and from the fact that $\cot 5^\circ \approx 11$ that $|K| < 2 \times 10^{-2}$ for $\theta_2 > 5^\circ$, whence we neglect K^2 and higher powers. Furthermore, from the definition of K, Eq. (6.67) we may expand K as follows:

$$K = \frac{n}{(1-n^2)^{\frac{1}{2}}} \cot\theta_2 \left(1 - \frac{n}{(1-n^2)^{\frac{1}{2}}} \cot\theta_2\right)^{-1} = \sum_{s=1}^{\infty} n^s \left(\frac{\cot\theta_2}{(1-n^2)^{\frac{1}{2}}}\right)^s \approx n \cot\theta_2 \qquad (7.21)$$

with an error of less than 4×10^{-4} for the largest value of n given in Table I and assuming as above $\theta_2 > 5^\circ$.

Since for the present range of parameters we are considering $k_2\rho < 1$ we will expand our functions, including the exponential factor $e^{ik_2\rho}$, to second powers of $k_2\rho$ neglecting third and higher powers.

Finally we are content to consider only the first power of n neglecting n^2 and higher powers where K must be regarded as O(n). Thus, three additional approximations beyond those used to obtain the evaluations of the integrals in Chapter VI are to be applied to the results of Chapter VI in order to obtain results appropriate to the present range of the parameters. If higher accuracy is required, additional terms in powers of K, n, $k_2\rho$, $1/k_1\rho$ may be obtained from the results of Chapter VI.

To obtain $M_1^{(2)}$ for the present range of parameters we must use the first two terms of Eq. (6.102); and carrying out the approximations indicated in the preceding paragraphs, we obtain the result

$$M_1^{(2)} \sim -\frac{2}{k_1^2}\left\{1 - \tfrac{1}{2}(ik_2\rho)^2 + \frac{3}{2}\, n(3 + 2ik_2\rho)\cot\theta_2\right\}\frac{1}{\rho^3}\, e^{-ik_1(z-h)} \qquad (7.22)$$

where the bracket may be usually chosen as unity for the interesting range of the parameters.

An expression for $V_1^{(2)}$, adequate for the present range of parameters, is given already by Eq. (7.18), which was derived from Eq. (7.5) by retaining only the leading term of the asymptotic expansion for $W_1^{(s)}$, Eq. (7.5b), and retaining only the first term in the power series expansion of the second factor in $W_1^{(p)}$, Eq. (7.5a). For $k_2\rho \ll 1$, we may replace the Hankel function by the leading term of its expansion about the origin to obtain

$$W^{(p)} \sim 2ik_2\, n \log(2/\gamma k_2\rho)e^{-ik_1(z-h)} \tag{7.23}$$

where $\gamma = 1.78107...$ and where n^2 has been neglected in comparison with unity. To the present approximation $W^{(s)}$ becomes

$$W^{(s)} \sim 2\left\{1 + ik_2\rho + \tfrac{1}{2}(ik_2\rho)^2 + \tfrac{1}{2}n \cot\theta_2\right\} e^{-ik_1(z-h)} . \tag{7.24}$$

Adding Eqs. (7.24) and (7.23), we obtain

$$V_1^{(2)} \sim 2\left\{1 + ik_2\rho + \tfrac{1}{2}(ik_2\rho)^2 + n\left[ik_2\rho \log(2/\gamma k_2\rho) + \tfrac{1}{2}\cot\theta_2\right]\right\} \frac{1}{\rho} e^{-ik_1(z-h)} \tag{7.25}$$

where again the bracket may be taken as unity for most practical purposes for the present range of parameters $k_2\rho < 1 < |k_1\rho|$.

The Cartesian components of the Hertzian vector in the present range of parameters and to the same approximation may be obtained from Eqs. (7.2) and the integrals evaluated in Chapter VI. Thus, in this manner we obtain

$$\Pi_{x1} \sim -\frac{p}{2\pi\sigma k_1^2}\left\{1 - \tfrac{1}{2}(ik_2\rho)^2 + \frac{3}{2}n(3 + 2ik_2\rho)\cot\theta_2\right\}\frac{1}{\rho^3} e^{-ik_1(z-h)}$$

$$\Pi_{z1} \sim -\frac{ip\cos\phi}{2\pi\sigma k_1}\left\{1 - \tfrac{1}{2}(ik_2\rho)^2 + n\left[ik_2\rho + \frac{3}{2}\cot\theta_2 + ik_2\rho\cot\theta_2\right]\right\}\frac{1}{\rho^2} e^{-ik_1(z-h)} \tag{7.26}$$

where we have put $k_1\eta_1 \approx i\sigma$ in accordance with the second of Eqs. (2.3). To derive Eq. (7.26) for Π_{x1} we merely inserted the result (7.22) into the first of Eqs. (7.3); and to derive Eq. (7.26) for Π_{z1} we made use of the expression for $\partial^2 V_1^{(2)}/\partial\rho\,\partial z$, Eq. (6.151), noting that

$$W_{\rho z}^{(p)} \sim -2k_2^2 \frac{1}{\rho} e^{-ik_1(z-h)} , \qquad (7.27)$$

and retained only the first two nonvanishing terms of the expression for $W_{\rho z}^{(s)}$, Eq. (6.156). For most practical purposes the brackets in both Eqs. (7.26) and (7.27) may be set equal to unity. And, finally, it is seen that for the present range of parameters Π_{z1} is again of greater importance than Π_{x1}; since, apart from the $\cos\phi$ factor, we have $|\Pi_{z1}/\Pi_{x1}| = 0|k_1\rho|$.

To obtain the electric and magnetic field components for the present range of parameters we proceed similarly; that is, we neglect n^2, K^2, $(k_2\rho)^3$ and $n(k_2\rho)^2$ in comparison with unity. Thus making use of the formulas of the preceding Chapter, without other approximations than the ones indicated here and taking due care to retain all terms of the same order, we obtain for the electric field components, from Eqs. (7.3), the leading terms

$$E_{\rho 1} \sim \frac{p\cos\phi}{2\pi\sigma}\left\{1 + \tfrac{1}{2}(ik_2\rho)^2 + \frac{n}{2}\left[2ik_2\rho + 3\cot\theta_2 + ik_2\rho\cot\theta_2\right]\right\}\frac{1}{\rho^3}\, e^{-ik_1(z-h)}$$

$$E_{\phi 1} \sim 2\,\frac{p\sin\phi}{2\pi\sigma}\left\{1 - \tfrac{1}{2}(ik_2\rho)^2 + \frac{n}{2}\left[ik_2\rho + 6\cot\theta_2 + 4ik_2\rho\cot\theta_2\right]\right\}\frac{1}{\rho^3}\, e^{-ik_1(z-h)} \qquad (7.28)$$

$$E_{z1} \sim -\frac{p\cos\phi}{2\pi\sigma}\, n(ik_2\rho)\left\{1 - \tfrac{1}{2}(ik_2\rho)^2 + \frac{n}{2}\left[2ik_2\rho + 3\cot\theta_2 + 2ik_2\rho\cot\theta_2\right]\right\}\frac{1}{\rho^3}\, e^{-ik_1(z-h)}$$

and for the magnetic field components we obtain, from Eqs. (7.4), the leading terms

$$H_{\rho 1} \sim 2\,\frac{ip\,\sin\phi}{2\pi k_1}\left\{1 - \tfrac{1}{2}(ik_2\rho)^2 + \frac{n}{2}\left[ik_2\rho + 6\cot\theta_2 + 4ik_2\rho\cot\theta_2\right]\right\}\frac{1}{\rho^3}\,e^{-ik_1(z-h)}$$

$$H_{\phi 1} \sim -\frac{ip\,\cos\phi}{2\pi k_1}\left\{1 + \tfrac{1}{2}(ik_2\rho)^2 + \frac{n}{2}\left[2ik_2\rho + 3\cot\theta_2 + ik_2\rho\cot\theta_2\right]\right\}\frac{1}{\rho^3}\,e^{-ik_1(z-h)} \qquad (7{\cdot}29)$$

$$H_{z1} \sim -\frac{ip\,\sin\phi}{2\pi k_1}\left(\frac{3}{ik_1\rho}\right)\left\{1 - \frac{1}{6}(ik_2\rho)^2 + \frac{3}{2}\,n\left[5 + 3ik_2\rho\right]\cot\theta_2\right\}\frac{1}{\rho^3}\,e^{-ik_1(z-h)}\ .$$

A further approximation of the above results may be obtained by neglecting $(k_2\,\rho)^2$ and n as compared with unity yielding for the components of the Hertzian vector from Eqs. (7·26),

$$\Pi_{x1} \sim -\frac{p}{2\pi\sigma\,k_1^2}\,\frac{1}{\rho^3}\,e^{-ik_1(z-h)}$$

$$\Pi_{z1} \sim -\frac{ip\,\cos\phi}{2\pi\sigma\,k_1}\,\frac{1}{\rho^2}\,e^{-ik_1(z-h)}\ . \qquad (7{\cdot}26a)$$

The same approximation yields for the electric field components from Eqs. (7·28)

$$E_{\rho 1} \sim \frac{p}{2\pi\sigma}\,\frac{\cos\phi}{\rho^3}\,e^{-ik_1(z-h)}$$

$$E_{\phi 1} \sim \frac{p}{2\pi\sigma}\,\frac{2\sin\phi}{\rho^3}\,e^{-ik_1(z-h)} \qquad (7{\cdot}28a)$$

$$E_{z1} \sim -\frac{p}{2\pi\sigma}\,\frac{\cos\phi}{\rho^2}\,ik_2 n\,e^{-ik_1(z-h)}\ ;$$

and the magnetic field components from Eqs. (7·29) become

$$H_{\rho 1} \sim \frac{ip}{2\pi k_1} \frac{2\sin\phi}{\rho^3} e^{-ik_1(z-h)}$$

$$H_{\phi 1} \sim - \frac{ip}{2\pi k_1} \frac{\cos\phi}{\rho^3} e^{-ik_1(z-h)} \qquad (7.29a)$$

$$H_{z1} \sim - \frac{ip}{2\pi k_1} \frac{\sin\phi}{\rho^3}\left(\frac{3}{ik_1\rho}\right) e^{-ik_1(z-h)} \quad .$$

The electric field components in the form given by Eq. (7.28a) except for an exponential factor $e^{ik_2\rho}$ were recently derived by Ferris[34] by an ingenious application of Green's theorem employing the known results of Wise and Rice for the fundamental integrals in the case $(h-z) = 0$, which he extended to arbitrary values of h and z by approximate methods; although he failed to point out the exact range of validity of his method. The above derivation of Eqs. (7.28a), which save for matters of notation and the exponential factor $e^{ik_2\rho}$ are equivalent to Ferris's Eqs. (62), establishes quite precisely the domain of applicability of his formulas.

Examining the order of magnitude of the electric field components as given by Eqs. (7.28) or Eqs. (7.28a), we see that E_{z1} is of $O(n^2)$ as compared with $E_{\rho 1}$ or $E_{\phi 1}$; and, since E_{z1} is less than the order of the error in the expressions for $E_{\rho 1}$ and $E_{\phi 1}$, it may be altogether neglected when considering the field as a whole. Similarly, examining the order of magnitude of the magnetic field components, we find that H_{z1} is negligible and of $O(1/k_1\rho)$ as compared with $H_{\rho 1}$ and $H_{\phi 1}$. The remaining field components which are all horizontal satisfy the vector equation

$$E_1 \times e_z = \zeta_1 H_1 \qquad (7.30)$$

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[34] Horace G. Ferris, Scripps Institution of Oceanography Report 53-14, (October, 1952), Eqs. (62).

where e_z is a unit vector directed in the positive z direction and $\zeta_1 = 1/\eta_1 \approx -ik_1/\sigma$, as defined by Eq. (2.3). Thus the entire field is essentially described by only two quantities, e.g., $H_{\rho 1}$ and $H_{\phi 1}$, the other two quantities being given by (7.30). In addition, Eq. (7.30) tells us that the field may be regarded essentially as a plane wave propagated[35] and attenuated in the negative z direction whose amplitude is a function of ρ. Thus for the present range of parameters the field in the conducting medium may be described essentially as arising from a secondary source distributed over the boundary between the conducting and non-conducting media, as was demonstrated for the static case and for $\rho \to \infty$. A complete analysis for all of the regions characterized by the condition $|k_1\rho| > 1$ shows that Eq. (7.30) may be used for the transverse components whenever n^2 is negligible in comparison with unity.

7.2 THE LIMITING CASE $(h-z) = 0$

When the dipole source is placed at the interface between the two media we have $h = 0$ in Fig. 1; when, furthermore, the point of observation is on the bounding surface, $z = 0$, we have the important practical case $(h-z) = 0$ that was originally discussed by Sommerfeld and subsequently by other investigators. Therefore, in the present Section we discuss the limiting forms of our general results in the case $(h-z) = 0$ which we find to be in complete agreement with the results of earlier writers, thus affording a necessary check on our asymptotic formulas.

7.2a. Evaluation of $U_1(\rho, 0)$.- As pointed out in Section 2.4a, the integral in question, $U_1(\rho, 0)$, becomes simply

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[35] J. A. Stratton, loc. cit., p. 272, Eq. (27).

$$U_1(\rho,0) = \int_0^\infty \frac{2}{\gamma_1 + \gamma_2} J_o(\lambda\rho)\,\lambda d\lambda, \tag{7.31}$$

which, it is recalled, was evaluated exactly by van der Pol,[36] who started from an elementary integral and by a very ingenious transformation obtained the result

$$U_1(\rho,0) = \frac{2}{k_1^2(1-n^2)}\,\frac{1}{\rho}\,\frac{\partial}{\partial\rho}\left(\left[\frac{1}{\rho}\right]e^{ik_2\rho} - \left[\frac{1}{\rho}\right]e^{ik_1\rho}\right) \tag{7.32}$$

which has been rewritten in our notation.

Considering our resolution of the integral U_1 into two terms, Eq. (6.3), we have, letting $(h-z) \rightarrow 0$,

$$U_1(\rho,0) = M_1(\rho,0) - \frac{2}{k_1^2(1-n^2)} \lim_{(h-z)\rightarrow 0}\left\{\frac{\partial^2 \Psi_2}{\partial z^2}\right\}; \tag{7.33}$$

rewriting $\partial^2\Psi_2/\partial z^2$ in terms of derivatives with respect to R_2, according to Eqs. (2.1) and (2.66) and proceeding to the limit, we obtain

$$\lim_{(h-z)\rightarrow 0}\left\{\frac{\partial^2\Psi_2}{\partial z^2}\right\} = \frac{1}{\rho}\,\frac{\partial}{\partial\rho}\left[\frac{1}{\rho}\right]e^{ik_1\rho}, \tag{7.34}$$

which yields precisely the second term of van der Pol's exact result (7.32). The limiting form of $M_1 = M_1^{(1)} + M_1^{(2)}$ is obtained by examining the limiting forms of the individual integrals $M_1^{(1)}$ and $M_1^{(2)}$. From (6.47) we have at once $M_1^{(1)}(\rho,0) = 0$, since the whole asymptotic series vanishes when $\theta_2 = \pi/2$ which corresponds to $(h-z) = 0$. From (6.102), putting $(h-z) = 0$, which according to Eq. (6.67) implies $K = 0$, we obtain

[36] B. van der Pol, Z. Hochfrequenz-Tech. 37, 152-157 (1931).

$$M_1^{(2)}(\rho,0) = \frac{2}{k_1^2(1-n^2)}\left\{\frac{ik_2}{\rho^2} - \frac{1}{\rho^3}\right\}e^{ik_2\rho} = \frac{2}{k_1^2(1-n^2)}\frac{1}{\rho}\frac{\partial}{\partial\rho}\frac{d}{\rho}e^{ik_2\rho}, \quad (7.35)$$

which is seen to be identical to the first term of the exact solution, Eq. (7.32). Thus, we have shown that our asymptotic results for the integral $U_1(\rho, h-z)$ reduce to van der Pol's exact formula, Eq. (7.32), when we put $(h-z) = 0$. The fact that the first two expansion terms, $\phi^{(1)}$ and $\phi^{(2)}$ in Eqs. (6.103), yield the exact solution for $M_1^{(2)}(\rho,0)$ while the third term, $\phi^{(3)}$, vanishes in this limit implies that all expansion terms $\phi^{(n)}$, $n \geq 3$, must vanish with K. The fact that our solution to three terms does reduce to the exact result when both source and point of observation are on the surface has led us to accept this as further evidence that our results must certainly be correct for other values of $(h-z) > 0$.

<u>7.2b. Evaluation of $V_1(\rho,0)$</u>.- To obtain the limiting form of the integral $V_1 = V_1^{(1)} + V_1^{(2)} = V_1^{(1)} + W^{(s)} + W^{(p)}$ as $(h-z) \rightarrow 0$ we need to examine the limiting forms of the individual terms. Thus, from (6.46) we have at once $V_1^{(1)}(\rho,0) = 0$, since the whole asymptotic expansion again vanishes when $\theta_2 = \pi/2$, which corresponds to $(h-z) = 0$. Next, from Eq. (6.125), putting $(z-h) = 0$ we have

$$W^{(p)}(\rho,0) = -\frac{\pi k_1 n^2}{(1+n^2)^{\frac{1}{2}}(1-n^4)} H_o^1\left(\frac{k_2\rho}{(1+n^2)^{\frac{1}{2}}}\right)\left\{1 - \text{erf}(-ix_o/2^{\frac{1}{2}})\right\}, \quad (7.36)$$

where x_o, originally defined by Eq. (6.122), now becomes with $\theta_2 = \pi/2$

$$x_o^2 = 2ik_2\rho\left\{1 - (1+n^2)^{-\frac{1}{2}}\right\} = in^2k_2\rho\left[1 - \frac{3}{4}n^2 + \ldots\right]. \quad (7.37)$$

Similarly, putting $(h-z) = 0$ and $K = 0$ in the expansion (6.133) for $W^{(s)}$ and the exact expressions for the first two terms as obtained directly from (6.131) and (6.132), we obtain in this limit

$$W^{(s)}(\rho,0) \sim \frac{2ik_1}{(1-n^4)} e^{ik_2\rho}\left\{\frac{Q^{-\frac{1}{2}}}{ik_1\rho} - \frac{(1+n^2)^{\frac{1}{2}}}{8n^3}\left[8(1+n^2)^{\frac{1}{2}} - 8Q^{-3/2} - n^2Q^{-\frac{1}{2}}\right]\frac{1}{(ik_1\rho)^2} + \frac{74}{1024}\frac{1}{(ik_1\rho)^3} + \cdots\right\}, \tag{7.38}$$

where, from Eq. (6.128),

$$Q = 2n^{-2}\left((1+n^2)^{\frac{1}{2}}-1\right) = 1 - n^2/4 + n^4/8 - \cdots \tag{7.39}$$

and where the third term in the bracket is given to $O(n^2)$. Thus, our complete result for $V_1(\rho,0)$, valid in the range $|k_1\rho| > 1$, is obtained by adding the individual terms

$$V_1(\rho,0) = W^{(p)}(\rho,0) + W^{(s)}(\rho,0) \tag{7.40}$$

as given by Eqs. (7.36) and (7.38).

<u>7.2c. Results for $(h-z) = 0$ as $\rho \to \infty$.</u>-- The results for $(h-z) = 0$ as $\rho \to \infty$ may be obtained directly from Section 7.1b by the substitutions $(h-z) = 0$ and $K = 0$. However, as indicated in Sections 7.2a and 7.2b the results take on an added element of accuracy as a result of the fact that the contributions over the path C_1 may be shown to be zero. Thus, from either (7.32) and (7.33) or from (7.10) and Section 7.1a we obtain

$$U_1(\rho,0) \approx M_1^{(2)}(\rho,0) \sim \frac{2\,in}{k_1(1-n^2)}\frac{1}{\rho^2} e^{ik_2\rho} \tag{7.41}$$

for $\rho \to \infty$. Similarly, from (7.40), (7.38) and (7.36) using the expansion (7.7), or by putting $(h-z) = 0$ directly into (7.6), we obtain

$$V_1(\rho,0) = V_1^{(2)}(\rho,0) \sim \frac{2i}{k_2 n^2(1-n^2)} \frac{1}{\rho^2} e^{ik_2\rho} \tag{7.42}$$

for $\rho \to \infty$. The components of the Hertzian vector and the field components may be obtained from Eqs. (7.11), (7.12) and (7.13) by letting $(h-z) = 0$.

<u>7.2d. Results for $(h-z) = 0$ when $|n^2 k_2 \rho| < 1 < k_2\rho$.</u>-- The results for $(h-z) = 0$ valid in the range $|n^2 k_2 \rho| < 1 < |k_2 \rho|$ may be obtained directly from Section 7.1c by substituting $(h-z) = 0$ or from Sections 7.2a and 7.2b for $U_1(\rho,0)$ and $V_1(\rho,0)$, respectively. Thus, for this range of parameters we use the exact expression for $U_1(\rho,0)$ as given by Eq. (7.32); and, for $V_1(\rho,0)$ we make use of Eq. (7.40), retaining the leading term of $W^{(s)}(\rho,0)$, Eq. (7.39), and replacing the Hankel function in $W^{(p)}(\rho,0)$, Eq. (7.36), by its leading term or, else, we merely set $(z-h) = 0$ in Eq. (7.17) to obtain

$$V_1(\rho,0) = V_1^{(2)}(\rho,0)$$

$$\sim \frac{2\left[\left(2/n^2\right)\left((1+n^2)^{\frac{1}{2}}-1\right)\right]^{-\frac{1}{2}}}{(1-n^4)} \frac{1}{\rho} e^{ik_2\rho}\left\{1 - \pi^{\frac{1}{2}}(-ix_o/2^{\frac{1}{2}})e^{-x_o^2/2}\left[1-\mathrm{erf}(-ix_o/2^{\frac{1}{2}})\right]\right\} \tag{7.43}$$

where x_o is given by Eq. (7.37).

To within terms of $O(n^2)$ for the constant in front it may be shown that Eq. (7.43) for $V_1(\rho,0)$ is identical to the result obtained by Sommerfeld.[37] To do this we introduce Sommerfeld's "numerical distance" ρ_s defined as

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[37] A. Sommerfeld, Ann. Physik <u>28</u>, 665-737 (1909), Eq. (47).

$$\rho_s = i(k_2 - k_o)\rho \tag{7.44}$$

where $k_o = k_2(1+n^2)^{-\frac{1}{2}}$ as given by Eq. (2.96); that is,

$$\rho_s = ik_2\rho\left\{1 - (1+n^2)^{-\frac{1}{2}}\right\} = x_o^2/2 \tag{7.45}$$

in which the last result is obtained by making use of Eq. (7.37). Thus, putting $x_o/2^{\frac{1}{2}} = (\rho_s)^{\frac{1}{2}}$, the error function in (7.43) becomes

$$\operatorname{erf}\left(-i(\rho_s)^{\frac{1}{2}}\right) = \frac{2}{\pi^{\frac{1}{2}}}\int_0^{-i(\rho_s)^{\frac{1}{2}}} e^{-t^2}dt = -\frac{2i}{\pi^{\frac{1}{2}}}\int_0^{(\rho_s)^{\frac{1}{2}}} e^{y^2}dy \tag{7.45a}$$

in which the last form is obtained by putting $t = -iy$ in the integral definition of the error function. Next, we see that the factor in front of Eq. (7.43) reduces to 2 to an accuracy of $O(n^2)$ where it is assumed that $|n|^2 \ll 1$.

Thus, finally, we obtain in this notation the celebrated Sommerfeld-van der Pol formula

$$V_1(\rho,0) \approx \frac{2}{\rho}e^{ik_2\rho}\left\{1 + i(\pi\rho_s)^{\frac{1}{2}}e^{-\rho_s} - 2(\rho_s)^{\frac{1}{2}}e^{-\rho_s}\int_0^{(\rho_s)^{\frac{1}{2}}} e^{y^2}\,dy\right\}, \tag{7.46}$$

which, it is important to recall, is valid for $k_2\rho \gg 1$ but $|n^2k_2\rho| \ll 1$. The first condition allowed us to expand the Hankel function asymptotically and the second condition implies, from (7.37) and (7.45), that the numerical distance ρ_s is very small, $|\rho_s| \ll 1$, which was explicitly imposed by Sommerfeld in his original expansion.[38] The same formula was obtained by

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[38] Loc. cit.

van der Pol,[39] employing his ingenious scheme of starting with an elementary integral. In contradistinction to his derivation of $U_1(\rho,0)$ mentioned in Section 2.4a which is exact, Eq. (7.35), the procedure leading to Eq. (7.46) in van der Pol's method implied two approximations, one of which is equivalent to our requirement $|n^2k_2\rho| << 1$. Other workers, notably Thomas,[40] Murray,[41] and Niessen[42] have re-derived the same formula by different treatments of the fundamental integral (2.73).

7.2e. Results for $(h-z) = 0$ when $k_2\rho < 1 < |k_1\rho|$.— The results for $(h-z) = 0$ in the range of parameters $k_2\rho < 1 < |k_1\rho|$ may be obtained from the corresponding equations for $(h-z) \geq 0$ given in Section 7.1d by substituting $(h-z) = 0$ which implies $K = 0$ from Eq. (6.67). The results for $M_1 = M_1^{(2)}$ and $V_1 = V_1^{(2)}$ may also be obtained from Sections 7.2a and 7.2b, respectively. The results of this Section also apply to the case $(h-z) > 0$ whenever K can be neglected as compared with unity; in order to convert the results of this Section to the case $(h-z) > 0$, we need only multiply by the factor $e^{-ik_1(z-h)}$. This fact may be utilized when we consider a numerical example in the last Section.

Accordingly, from Eq. (7.32) or from Eq. (7.22), putting $(h-z) = 0$, we obtain

$$U_1(\rho,0) \approx M_1^{(2)}(\rho,0) \sim -\frac{2}{k_1^2}\left\{1 - \tfrac{1}{2}(ik_2\rho)^2\right\}\frac{1}{\rho^3}\,; \qquad (7.47)$$

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[39] B. van der Pol, loc. cit.

[40] L. H. Thomas, Proc. Cambridge Phil. Soc. 26, 123-126 (1929).

[41] F. H. Murray, Proc. Cambridge Phil. Soc. 28, 433-442 (1932).

[42] K. F. Niessen, Ann. Physik 16, 810-820 (1933).

and similarly from Eqs. (7.40), (7.38) and (7.37) or Eq. (7.25), we find

$$V_1^{(2)} \sim \frac{2}{\rho}\left\{1 + ik_2\rho + \tfrac{1}{2}(ik_2\rho)^2 + n(ik_2\rho)\log 2/\gamma k_2\rho\right\}. \qquad (7.48)$$

The components of the Hertzian vector become from Eqs. (7.26)

$$\Pi_{x1} \sim -\frac{p}{2\pi\sigma k_1^2}\left\{1 - \tfrac{1}{2}(ik_2\rho)^2\right\}\frac{1}{\rho^3}$$

$$\Pi_{z1} \sim -\frac{ip\cos\phi}{2\pi\sigma k_1}\left\{1 - \tfrac{1}{2}(ik_2\rho)^2 + n(ik_2\rho)\right\}\frac{1}{\rho^2}. \qquad (7.50)$$

The electric field components, obtained from Eqs. (7.28) are as follows:

$$E_{\rho 1} \sim \frac{p}{2\pi\sigma}\frac{\cos\phi}{\rho^3}\left\{1 + \tfrac{1}{2}(ik_2\rho)^2 + n(ik_2\rho)\right\}$$

$$E_{\phi 1} \sim \frac{p}{2\pi\sigma}\frac{2\sin\phi}{\rho^3}\left\{1 - \tfrac{1}{2}(ik_2\rho)^2 + \tfrac{1}{2}n(ik_2\rho)\right\} \qquad (7.51)$$

$$E_{z1} \sim -\frac{p}{2\pi\sigma}\frac{\cos\phi}{\rho^3}n(ik_2\rho)\left\{1 - \tfrac{1}{2}(ik_2\rho)^2 + n(ik_2\rho)\right\}.$$

The transverse components of the magnetic field, $H_{\rho 1}$ and $H_{\phi 1}$, may be obtained from $E_{\rho 1}$ and $E_{\phi 1}$, as given by the first two equations in Eq. (7.51), by using Eq. (7.30). The z component of the magnetic field may be obtained from the last of Eqs. (7.29), thus

$$H_{z1} \sim -\frac{ip}{2\pi k_1}\frac{\sin\phi}{\rho^3}\left(\frac{3}{ik_1\rho}\right)\left\{1 - \frac{1}{6}(ik_2\rho)^2\right\}. \qquad (7.52)$$

7.3 SOMMERFELD'S ELECTROMAGNETIC SURFACE WAVE

Consider again the fundamental integral $V_1(\rho, h-z)$ as defined by Eq. (6.2) which, in accordance with our choice of cuts and consequent selection of Sheet I of the Riemann surface (Fig. 4), can be resolved into the sum of two integrals

$$V_1(\rho, h-z) = V_1^{(1)} + V_1^{(2)}, \quad z \leq 0, \tag{7.53}$$

around the respective branch cuts. As pointed out at the end of Section 2.5d, Sommerfeld chose his branch cuts for γ_2 in the same manner as our common choice of cuts for γ_1, with the result that in Sommerfeld's Sheet I of the Riemann surface the points $\lambda = \pm k_o$, where k_o is defined by Eq. (2.96), are real poles of the integrand of (6.2). Hence, according to Sommerfeld, we should write

$$V_1(\rho, h-z) = V_{1s}^{(1)} + V_{1s}^{(2)} + P, \quad z \leq 0, \tag{7.54}$$

where the subscript s denotes the evaluation of the respective contour integrals on Sommerfeld's Sheet I of the Riemann surface and where P denotes the contribution from the residue of the pole at $\lambda = + k_o$; that is, from (6.2)

$$P(\rho, h-z) = \int^{(k_o+)} \frac{k_1^2}{k_2^2\gamma_1 + k_1^2\gamma_2} e^{\gamma_1(z-h)} H_o^1(\lambda\rho)\, \lambda d\lambda, \quad z \leq 0, \tag{7.55}$$

which when properly evaluated with due regard to the phases of $\gamma_1(k_o)$ and $\gamma_2(k_o)$ in the new Sheet I of the Riemann surface yields

$$P(\rho, h-z) = -\frac{2\pi n^2 k_1}{(1+n^2)^{\frac{1}{2}}(1-n^4)} H_o^1\left(\frac{k_2\rho}{(1+n^2)^{\frac{1}{2}}}\right) e^{-ik_1(z-h)/(1+n^2)^{\frac{1}{2}}}, \quad z \leq 0 \tag{7.56}$$

and, as pointed out by Sommerfeld, it has all the characteristics of an electromagnetic surface wave.

We are not here concerned with details of the interpretation of the term P, but merely wish to point out the correctness of Sommerfeld's resolution (7.54) which agrees in every respect, as shown below, with our own resolution (7.53). Thus, we must prove that

$$V_1^{(1)} + V_1^{(2)} \equiv V_{1s}^{(1)} + V_{1s}^{(2)} + P ;$$

and, since it is clear that the integrals $V_1^{(1)}$ and $V_{1s}^{(1)}$, being taken around the identical cut for γ_1 and being independent of choice of cut for γ_2, must be identical to each other, we need only prove that

$$V_1^{(2)} \equiv V_{1s}^{(2)} + P .$$

But this we have already accomplished in Section 5.1d where we showed that the value of the integral

$$e^{-\phi(o)} I = W_s + W_p$$

is independent of the choice of path through the saddle point in the α_2-plane. Briefly, then, we have shown that

$$V_{1s}^{(2)} + P \equiv V_1^{(2)} = W^{(s)} + W^{(p)} , \qquad (7.58)$$

where $W^{(s)}$, the contribution of the integral over the path (any permissible path) through the saddle point is given asymptotically by (6.133) and $W^{(p)}$, the "contribution" from the pole integral taken over any permissible path through the saddle point, is given exactly by (6.125). The situation is most clearly seen for the case $|n^2 k_2 \rho| < 1 < k_2 \rho$, for whichever way V_1 is computed, either by Sommerfeld's method or our own, the complete and correct result for $(V_{1s}^{(2)} + P)$ or for $V_1^{(2)}$ contains the term $\frac{1}{2}P$ as shown by comparing Eqs. (7.56) and (6.125) and by noting the origin of the second term of Sommerfeld's formula for $(h-z) = 0$ as given by Eq. (7.46), the term identified by Sommerfeld as coming from $\frac{1}{2}P$. Similarly, agreement is obtained for $|n^2k_2\rho| > 1$ or for $\rho \to \infty$ where it is seen from either Section 7.1c or Section 7.2d that the term $\frac{1}{2}P$ disappears in this limit; that is, asymptotically, $W^{(s)}$ in (7.58) contains precisely the terms corresponding to the asymptotic expansion for $(-\frac{1}{2}P)$, which is evident from the very method that led to the subtraction of the pole in the first place. Finally, agreement between the Sommerfeld results and our own occurs for $k_2\rho < 1 < |k_1\rho|$ where the contribution $\frac{1}{2}P$ becomes negligible.

The situation can be summarized as follows:

1) Our resolution of the integral $V_1(\rho, h-z)$ into two terms, Eq. (7.53), is identical to Sommerfeld's resolution into three terms, Eq. (7.54).

2) For the range of parameters $k_2\rho < 1 < |k_1\rho|$ the solution by either method yields a so-called Zenneck-type surface wave, an explicit contribution from the residue of the pole, of value $\frac{1}{2}P$, which is negligible as compared with the remainder of the solution.

3) For $|n^2k_2\rho| < 1 < k_2\rho$ the solution by either method again yields the same Zenneck-type surface wave, associated with the term $\frac{1}{2}P$, which is retained and contributes to the result.

4) For $|n^2k_2\rho| > 1$ or $\rho \to \infty$ the solution by either method yields no Zenneck-type surface wave, the contribution $\frac{1}{2}P$ being cancelled asymptotically in this region by the remainder of the solution.

5) Although a Zenneck-type surface wave or contribution $\frac{1}{2}P$ does appear explicitly in the solution for $|n^2k_2\rho| < 1$ and although we have

been able to show that our entire solution is essentially a surface wave, the so-called Zenneck surface wave is never a major part of the solution.

The above summary of conclusions might be the proper ending for the present Section, except for the fact that in the past ten years a number of writers have pretended to disqualify the Sommerfeld-van der Pol results by claiming the non-existence of Sommerfeld's electromagnetic surface wave and it seems appropriate to indicate here wherein these writers have erred. Thus, for example, reading Stratton[43] one finds an excellent summary on the mathematical aspects of the effect of a plane earth on the propagation of radio waves with a fairly complete bibliography up to 1941 which, however, contains a summary on surface waves that is unfortunately not entirely accurate. Specifically, Stratton opens up the discussion of surface waves with the statement: "Doubt as to the validity of Sommerfeld's resolution was first raised by Weyl in his 1919 paper, etc." That the Sommerfeld and Weyl surface wave terms are not identical is merely a consequence of two widely different methods of attack leading to results which are valid for <u>different</u> ranges of parameters, Weyl's applying for $|n^2 k_2 \rho| > 1$ or $\rho \to \infty$ while Sommerfeld's apply for $|n^2 k_2 \rho| < 1 < k_2 \rho$. When the two complete solutions are examined in a common region of validity they are seen to agree with each other, since they are both correct.

Stratton goes on to say: "Burrows[44] has pointed out that numerically the transmission formulas based on Sommerfeld's results differ from those of Weyl by just the surface wave term P and has made careful measurements which

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[43] J. A. Stratton, "Electromagnetic Theory." (McGraw-Hill Book Co., Inc., New York, 1941); Chapter IX, Section 9.28 et seq.

[44] C. R. Burrows, Nature <u>138</u>, 284 (1936); Proc. I.R.E. <u>25</u>, 219-229 (1937).

support the results of Weyl. The discrepancy is large only when the displacement current in the ground is comparable to the conduction current."* This is a consequence of the fact that Sommerfeld's transmission formula, specifically Eq. (7.54), is strictly valid only for $|n|^2 << 1$ and $|n^2 k_2 \rho| << 1$ as specified by Sommerfeld himself; and, thus, Sommerfeld's formula is not valid in the range where the displacement current is comparable to the conduction current in the earth. In other words, Burrows had no right to apply Sommerfeld's formula outside its range of applicability. Further, Stratton states that "the results of Sommerfeld and the charts of Rolf[45] which are based upon them, can not be relied upon when the displacement current is appreciable, as is the case at ultrahigh frequencies," which is of course true, but not due to any error in Sommerfeld's work or to the non-existence of the surface wave term, but merely due to the limitations already annotated of Sommerfeld's formula. Finally, Stratton makes a statement that may easily be misinterpreted; namely, that the asymptotic expansions given by Wise[46] and Rice[47] show that the term P in Sommerfeld's solution, Eq. (7.54), is cancelled when all the terms of the series for $V_{1s}^{(1)}$ and $V_{1s}^{(2)}$ and P are taken into account. The statement is of course true asymptotically in accordance with our conclusion (4) above; but it does not mean, as one might erroneously read by implication, that the surface wave term P does not belong in Sommerfeld's formula in the first place.

The argument has not ended here. In 1947 Epstein[48] published a paper in

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* Italics are ours.

45 B. Rolf, Proc. I.R.E. 18, 391-402 (1930).

46 W. H. Wise, Bell System Tech. J. 16, 35-44 (1937).

47 S. O. Rice, Bell System Tech. J. 16, 101-109 (1937).

48 P. S. Epstein, Proc. Nat. Acad. Sci. 33, 195-199 (1947).

which he disqualified Sommerfeld's original formulation of the problem and proposed a new solution which was shortly afterwards shown to be in error by Bouwkamp.[49] Epstein alleged that the term P arising from the residue of the pole did not belong in the final answer at all because such a singularity was not contained, say, in the original Fourier integral representation for the source function, Eqs. (2.34) or (2.65). And he proposed a new solution by deforming the path of integration in the λ-plane from the positive real axis, $0 \leq \lambda < \infty$, to another path in the first quadrant having the same termini but curving upward beyond the point $\lambda = + k_0$ in Fig. 4 and, then, proceeded to satisfy the boundary conditions with integral representations over such a path. Naturally, his final result is the Sommerfeld answer minus the term P; but, as pointed out by Bouwkamp, he overlooked the fact that his proposed solution is singular for $\rho = 0$ everywhere on the z axis, which is, of course, incompatible with the physics of the situation. The error committed by Epstein is simple to point out. In accordance with our lengthy discussion of Section 2.3a, it is clear that all our integral representations in the λ-plane have a common region of analiticity, namely, the strip $|\mathrm{Im}\{\lambda\}| < \mathrm{Im}\{k_2\}$; and that, in applying boundary conditions, we must confine the path of integration in all of the integral representations, including the source function, to the prescribed strip of analiticity. Epstein's path of integration violates this requirement and therefore it is not surprising that it leads to erroneous results.

Lastly, Kahan and Eckart[50] in a series of papers which have been justly criticized by Bouwkamp[51] attempted to show that Sommerfeld's well-known

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[49] C. J. Bouwkamp, Math. Rev. 9, 126, 637 (1948).

[50] T. Kahan and G. Eckart, Comptes Rendus 226, 1513-1515 (1948); Comptes Rendus 227, 969-971 (1948); J. de phys. et rad. 10, 165-177 (1949); Phys. Rev. 76, 406-411 (1949).

[51] C. J. Bouwkamp, Phys. Rev. 80, 294 (L), (1950).

electromagnetic surface wave, P, does not exist in the radiation of a Hertzian dipole over a plane earth because it does not fulfill the so-called "radiation condition". Since the radiation condition applies only to the case $\rho \to \infty$ and Sommerfeld's complete solution contains no such P term for $\rho \to \infty$, their argument is irrelevant; for it matters not whether the term P alone satisfies or does not satisfy the radiation condition, the behavior of the complete solution being conclusive. Actually, their conclusion that P alone does not satisfy the radiation condition which is undoubtedly correct can only be interpreted as supporting Sommerfeld's results; although, according to Bouwkamp, the authors failed to prove their point.

Thus, finally, we trust that there will be no more papers attempting to disprove the existence of the term P in Sommerfeld's solution and that our summary of conclusions together with the foregoing discussion has sufficiently clarified the issue so that there will be no more controversy on a subject which being both physical and mathematical is capable of complete determination with no room for debate.

7.4 THE LIMITING CASE $k_2 = 0$

For small frequencies it is of interest to consider the approximation obtained by putting $k_2 = 0$; that is, by assuming that the wavelength in air is infinite. Two limiting processes are considered in this Section and compared with each other. The first method consists of letting $k_2 \to 0$ in the integrand of $V_1(\rho, h-z)$, Eqs. (2.60) and (2.68), and then integrating the resulting approximate integral, which is essentially the method employed by Lien as described in a posthumous paper.[52] The second method consists of

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[52] R. H. Lien, Jour. App. Physics 24, 1-5 (1953).

letting $k_2 \rightarrow 0$ in our general asymptotic results for $V_1 = V_1^{(1)} + V_1^{(2)}$ which were originally obtained with $k_2 > 0$. The two limiting processes do not yield the same result, for the integral $V_1(\rho, h-z; k_2)$ does not converge uniformly in the vicinity of $k_2 = 0$; that is, we have from (2.60) and (2.68),

$$\int_0^\infty \left\{ \lim_{k_2 \to 0} v_1(\lambda, k_2) \right\} e^{\gamma_1(z-h)} J_0(\lambda\rho)\lambda d\lambda \neq \lim_{k_2 \to 0} \int_0^\infty v_1(\lambda, k_2) e^{\gamma_1(z-h)} J_0(\lambda\rho)\lambda d\lambda \, . \tag{7.59}$$

However, it is shown that the two limits differ only beginning with the third term when expanded asymptotically. The limit which represents V_1 for $k_2 = 0$ correctly is the limit obtained by the second method where k_2 is set equal to zero <u>after</u> the asymptotic evaluation of the integral. The first method yields the result in a closed form, which in certain ranges of parameters and frequency proves to be a useful approximation.

<u>7.4a. First limiting process: Lien's approximation</u>.- Putting $k_2 = 0$ in $v_1(\lambda)$, Eq. (2.60), and inserting this result into (2.68) we obtain an integral which leads immediately to Lien's integral,

$$\Lambda_1(\rho, h-z) = -2 \frac{\partial}{\partial(h-z)} \int_{-ik_1}^{\infty} (\gamma_1^2 + k_1^2)^{-\frac{1}{2}} J_0\left((\gamma_1^2 + k_1^2)^{\frac{1}{2}}\rho\right) e^{-\gamma_1(h-z)} d\gamma_1 \, , \tag{7.60}$$

by changing the variable of integration from λ to $\gamma_1 = (\lambda^2 - k_1^2)^{\frac{1}{2}}$. In (7.60) the path of integration is the contour from $\gamma_1 = -ik_1$ to $\gamma_1 = \infty$ on the real axis, along the curve defined by $\mathrm{Im}\left\{\gamma_1^2 + k_1^2\right\} = 0$ which corresponds to the original path of integration $0 \leq \lambda < \infty$ in the λ-plane.

The integral (7.60) has received considerable attention in the past. It was apparently first considered by Foster[53] who gave the result without sufficient references. Lien refers to Foster's paper and adds that, by extending the Laplace transformation to the complex plane and making use of a formula in Magnus and Oberhettinger,[54] he obtains (in our notation)

$$\Lambda_1(\rho,h-z) = -i\pi \frac{\partial}{\partial (h-z)} \left\{ J_0\left(\tfrac{1}{2}k_1\left[R_2 - (h-z)\right]\right) H_0^1\left(\tfrac{1}{2}k_1\left[R_2 + (h-z)\right]\right) \right\}, \quad (7.61)$$

where $R_2 = \left[\rho^2 + (h-z)^2\right]^{\frac{1}{2}}$ as originally defined by Eq. (2.1). The formula given in Magnus and Oberhettinger is strictly valid only for a real variable of integration with a real lower limit and Lien does not explain how he justified the extension of the formula to the complex plane, nor have we been able to find a satisfactory justification. On the other hand, Wolf[55] comments that the result given by Foster and hence by Lien, Eq. (7.61), can be verified by observing that the right hand sides of (7.60) and (7.61) satisfy the scalar Helmholtz equation, $(\nabla^2 + k_1^2)\Lambda_1 = 0$, and that for $(h-z) = 0$, the result (7.61) is correct according to Watson (Op. cit.), Sec. 13.6, Eq. (3), page 435. Thus, we feel confident that the result given by Lien, Eq. (7.61), which is valid for $\rho \geq 0$ and $(h-z) \geq 0$, represents the correct evaluation of the integral (7.60) in closed form.

Differentiating (7.61) as indicated and replacing the Bessel function by the sum of two Hankel functions we obtain

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[53] R. M. Foster, Bell System Tech. J. 10, 408-419 (1931).

[54] W. Magnus and F. Oberhettinger, "Formulas and Theorems for the Special Functions of Mathematical Physics," (Chelsea Publishing Co., New York, 1949), p. 133.

[55] A. Wolf, Geophysics 11, 518-537 (1946).

$$\Lambda_1 = \Lambda_1^{(1)} + \Lambda_1^{(2)} , \tag{7.62}$$

where

$$\Lambda_1^{(1)} = \frac{i\pi k_1}{4R_2}\left\{x_2 H_0^1(\tfrac{1}{2}k_1x_1)\ H_1^1(\tfrac{1}{2}k_1x_2) - x_1 H_1^1(\tfrac{1}{2}k_1x_1)\ H_0^1(\tfrac{1}{2}k_1x_2)\right\} , \tag{7.63}$$

and

$$\Lambda_1^{(2)} = \frac{i\pi k_1}{4R_2}\left\{x_2 H_0^2(\tfrac{1}{2}k_1x_1)\ H_1^1(\tfrac{1}{2}k_1x_2) - x_1 H_1^2(\tfrac{1}{2}k_1x_1)\ H_0^1(\tfrac{1}{2}k_1x_2)\right\} , \tag{7.64}$$

in which

$$x_1 = R_2 - (h-z) \qquad \text{and} \qquad x_2 = R_2 + (h-z). \tag{7.65}$$

The functions $\Lambda_1^{(1)}$ and $\Lambda_1^{(2)}$ correspond respectively to our integrals $V_1^{(1)}$ and $V_1^{(2)}$ evaluated for $k_2 = 0$. In order to compare the results with the correct second limiting process, where it has been assumed that $|k_1\rho| > 1$, we expand asymptotically the Hankel functions appearing in (7.63) and (7.64) to obtain

$$\Lambda_1^{(1)} = -\frac{i(x_2-x_1)}{R_2(x_1x_2)^{\frac{1}{2}}}\, e^{ik_1(x_2+x_1)/2}\left\{1 + \frac{1}{8}\left[\frac{2(x_2+x_1)}{ik_1x_1x_2}\right] + \frac{9}{128}\left[\frac{2(x_2+x_1)}{ik_1x_1x_2}\right]^2 + 0\left|\frac{2(x_2+x_1)}{ik_1x_1x_2}\right|^3\right\} \tag{7.66}$$

and

$$\Lambda_1^{(2)} = \frac{x_1+x_2}{R_2(x_1x_2)^{\frac{1}{2}}}\, e^{ik_1(x_2-x_1)/2}\left\{1 + \frac{1}{8}\left[\frac{2(x_1-x_2)}{ik_1x_1x_2}\right] + \frac{9}{128}\left[\frac{2(x_1-x_2)}{ik_1x_1x_2}\right]^2 + 0\left|\frac{2(x_1-x_2)}{ik_1x_1x_2}\right|^3\right\} . \tag{7.67}$$

Substituting the values of x_1 and x_2 according to Eqs. (7.65), the above expressions become

$$\Lambda_1^{(1)} = -\frac{2i\cos\theta_2}{\rho} e^{ik_1R_2}\left\{1 + \frac{1}{8}\left[\frac{4\csc\theta_2}{ik_1\rho}\right] + \frac{9}{128}\left[\frac{4\csc\theta_2}{ik_1\rho}\right]^2 + 0\left|\frac{4\csc\theta_2}{ik_1\rho}\right|^3\right\} \quad (7.68)$$

and

$$\Lambda_1^{(2)} = \frac{2}{\rho} e^{ik_1(h-z)}\left\{1 + \frac{1}{8}\left[\frac{4\cot\theta_2}{-ik_1\rho}\right] + \frac{9}{128}\left[\frac{4\cot\theta_2}{-ik_1\rho}\right]^2 + 0\left|\frac{4\cot\theta_2}{ik_1\rho}\right|^3\right\}. \quad (7.69)$$

7.4b. Second limiting process.-- We now proceed to put $k_2 = 0$ in our previously established results which physically is equivalent to the approximation $k_2\rho << 1$ and, in fact, negligible but with $|k_1\rho| > 1$. Thus, noting that $k_2 = 0$ implies $n = 0$ and $(K/n) = \cot\theta_2$, we obtain from Eqs. (6.121), (6.125) and (6.133),

$$V_1^{(2)} \sim \frac{2}{\rho} e^{ik_1(h-z)}\left\{1 + \frac{1}{8}\left[\frac{4\cot\theta_2}{ik_1\rho}\right] + \frac{3}{128}\left[48\cot^2\theta_2 + \frac{25}{8}\right]\frac{1}{(ik_1\rho)^2}\right\} \quad (7.70)$$

to three terms. Comparing (7.70) term by term with (7.69) we see that the results begin to differ in the third term of the expansion,

$$V_1^{(2)} - \Lambda_1^{(2)} \sim \frac{75ik_1}{512(ik_1\rho)^3} e^{ik_1(h-z)} + \text{higher order terms}, \quad (7.71)$$

and thus we have established that the Lien approximation for $V_1^{(2)}$, namely his $\Lambda_1^{(2)}$, is in error by the amount given above for the region $k_2\rho << 1 < |k_1\rho|$.

Similarly, for $V_1^{(1)}$ in the limit $k_2 = 0$, we make use of Eq. (6.44) by substituting for $G(\theta_2)$ the expression deduced from the second of Eqs. (6.21) after placing $n = 0$; that is,

$$G(\theta_2) = -i \csc\theta_2 , \tag{7.72}$$

obtaining

$$V_1^{(1)} \sim -\frac{2i \cos\theta_2}{\rho} e^{ik_1R_2} \left\{ 1 + \frac{1}{8}\left[\frac{4\csc\theta_2}{ik_1\rho}\right] \right\} \tag{7.73}$$

which is seen to agree to two terms with $\Lambda_1^{(1)}$ as given in (7.68). Thus, again we find that the two limiting processes lead to results which begin to differ in the third term of their respective asymptotic expansions; and, therefore, conclude that Λ_1 is an adequate approximation to the physical situation when $k_2\rho << 1 < |k_1\rho|$.

Putting $k_2 = 0$ and $n = 0$ in Eqs. (7.26) and (7.27) we obtain the components of the Hertzian vector

$$\Pi_{x1} \sim -\frac{p}{2\pi\sigma k_1^2} \frac{1}{\rho^3} e^{-ik_1(z-h)} , \tag{7.74}$$

$$\Pi_{z1} \sim -\frac{ip}{2\pi\sigma k_1} \frac{\cos\phi}{\rho^2} e^{-ik_1(z-h)} , \tag{7.75}$$

which could have also been obtained from the Lien integral and which are suitable approximations when $k_2\rho << 1 < |k_1\rho|$. Similarly, from Eqs. (7.28) the electric field components become

$$\begin{aligned} E_{\rho 1} &\sim \frac{p}{2\pi\sigma} \frac{\cos\phi}{\rho^3} e^{-ik_1(z-h)} , \\ E_{\phi 1} &\sim \frac{p}{2\pi\sigma} 2\frac{\sin\phi}{\rho^3} e^{-ik_1(z-h)} , \\ E_{z1} &\sim -\frac{p}{2\pi\sigma} \frac{n(ik_2\rho)\cos\phi}{\rho^3} e^{-ik_1(z-h)} = 0 . \end{aligned} \tag{7.76}$$

The transverse components of the magnetic field for this limiting case, $k_2 = 0$, may be obtained from Eqs. (7.30) and (7.76); and all three components may be obtained from Eqs. (7.29) by setting $k_2 = 0$ and $n = 0$, yielding

$$\begin{aligned} H_{\rho 1} &\sim \frac{ip}{2\pi k_1} \frac{2\sin\phi}{\rho^3} e^{-ik_1(z-h)} , \\ H_{\phi 1} &\sim - \frac{ip}{2\pi k_1} \frac{\cos\phi}{\rho^3} e^{-ik_1(z-h)} , \\ H_{z1} &\sim - \frac{ip}{2\pi k_1} \left(\frac{3}{ik_1\rho}\right) \frac{\sin\phi}{\rho^3} e^{-ik_1(z-h)} . \end{aligned} \tag{7.77}$$

7.5 NUMERICAL EXAMPLE

By way of illustration of the results given in the present Chapter we now consider a numerical example. Before we can choose the appropriate expressions for the electric and magnetic field components, we must examine the magnitudes of the parameters involved; in particular, the three quantities: $\nu = \omega/2\pi$, the frequency; σ, the conductivity of the conducting medium; and p, the so-called "electric dipole moment" as defined by Eq. (2.8). Another parameter which must be specified but has little to do with the form of the equations is h, the depth of the source. For the present numerical example we choose the following values:

$$\begin{aligned} \nu &= 900 \text{ c.p.s.} ; \qquad & \sigma &= 5 \text{ mhos/meter} ; \\ p &= 500 \text{ amp x meter} ; & h &= 7.5 \text{ meters} \end{aligned} \tag{7.78}$$

from which we readily deduce

$$
\begin{aligned}
k_1 &\approx (i\omega\mu_0\sigma)^{\frac{1}{2}} = (1+i)/\delta \ , & k_2 &= \omega/c = 6\pi \times 10^{-6} \text{ meter}^{-1} \ , \\
\delta &= (2/\omega\mu_0\sigma)^{\frac{1}{2}} = 7.5 \text{ meters} \ , & \lambda_2 &= 2\pi/k_2 = 333 \text{ km} \ , \\
|k_1| &= 2^{\frac{1}{2}}/\delta = .189 \text{ meter}^{-1} \ , & |n|^2 &= \omega\epsilon_0/\sigma = 10^{-9} \ , \\
\lambda_1 &= 2\pi\delta = 47.5 \text{ meters} \ , & |n| &= 3.16 \times 10^{-5} \ .
\end{aligned}
\tag{7.79}
$$

The two quantities which must be considered in determining the proper form for the field components are $|k_1\rho|$ and $k_2\rho$. When $|k_1\rho| = 10$, we find $\rho = 53$ meters, and when $k_2\rho = 10^{-1}$, we find $\rho = 5300$ meters; so that the regions $|k_1\rho| < 10$ and $k_2\rho > 10^{-1}$ are to be excluded. Thus, for values of ρ satisfying the condition

$$
50 \text{ meters} \leq \rho \leq 5000 \text{ meters} \tag{7.80}
$$

we may use our results for the case $k_2\rho < 1 < |k_1\rho|$ as presented in Sections 7.1d and 7.2e where, from Eq. (7.24) and similar relations and from the value of $|n|$ in (7.79), we may neglect the contributions from the pole. Furthermore, from the discussion at the beginning of Section 7.1d it is apparent for the present case, $|n| = 3.16 \times 10^{-5}$, that we may neglect n and K altogether in comparison with unity especially for values of $\theta_2 \gtrsim \pi/4$ which are of practical interest. Thus, the expressions for the electric and magnetic field components that apply to the present numerical example are deduced from Eqs. (7.28) and (7.29) by neglecting the parts multiplied by n. It may be noted from the last of Eqs. (7.28) and the last of Eqs. (7.29) that the z components of the electric and magnetic fields are of less than $O(n)$ as compared with the transverse components and are therefore negligible in the present example.

The equations which are applicable to the present case are then given by

$$E_{\rho 1} \sim \frac{p}{2\pi\sigma}\frac{\cos\phi}{\rho^3}\left\{1 - \tfrac{1}{2}(k_2\rho)^2\right\} e^{ik_1(h-z)} ,$$
$$E_{\phi 1} \sim \frac{p}{2\pi\sigma}\frac{2\sin\phi}{\rho^3}\left\{1 + \tfrac{1}{2}(k_2\rho)^2\right\} e^{ik_1(h-z)} \qquad (7.81)$$

from which the magnetic field components, according to Eq. (7.30), are given by the vector equation

$$H_1 = \frac{i\sigma}{k_1}(E_1 \times e_z) \qquad (7.82)$$

where e_z is a unit vector in the positive z direction. Finally by neglecting $k_2\rho$ altogether, the case which corresponds to the asymptotic expansions of Lien's results, Eqs. (7.76), we have to an accuracy better than 0.5 percent for the whole range $k_2\rho \leqslant 10^{-1} < 1 < 10 \leqslant |k_1\rho|$, which for the present case limits the horizontal range from 50 to 5000 meters, the expressions

$$E_{\rho 1} \sim \frac{p}{2\pi\sigma}\frac{\cos\phi}{\rho^3} e^{(i-1)(h-z)/\delta} ,$$
$$E_{\phi 1} \sim \frac{p}{2\pi\sigma}\frac{2\sin\phi}{\rho^3} e^{(i-1)(h-z)/\delta} , \qquad (7.83)$$

where $\delta = (2/\omega\mu_0\sigma)^{\frac{1}{2}}$ is the so-called "skin depth".

The structure of the electric and magnetic field components is further elucidated by introducing the function

$$f(\rho, z) = (p/\pi\sigma\rho^3)\, e^{-(h-z)/\delta} \quad \text{volts/meter} \qquad (7.84)$$

in terms of which the magnitudes of the field components assume the simple forms

$$|E_{\rho 1}| = \tfrac{1}{2} f(\rho, z) \cos\phi , \qquad |E_{\phi 1}| = f(\rho, z) \sin\phi ;$$
$$|H_{\rho 1}| = \frac{\sigma\delta}{2^{\frac{1}{2}}} f(\rho, z) \sin\phi , \qquad |H_{\phi 1}| = \frac{\sigma\delta}{2(2)^{\frac{1}{2}}} f(\rho, z) \cos\phi . \qquad (7.85)$$

It is clear that the field components vary inversely as the cube of the horizontal range ρ and are exponentially attenuated with the aggregate depth of source and point of observation.

It is proper to point out in closing that the formulas employed in the present numerical example, Eqs. (7.84), were contained already in the papers by Lien and Ferris, as indicated in the references, and that our additional contribution in this respect has been to clearly point out the range of applicability of the results and the order of magnitude of the errors incurred in the corresponding approximate expressions.

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