Title
Test Particle Method in Kinetic Theory of a Plasma

Journal
Physics of Fluids, 7(4)

ISSN
00319171

Author
Rostoker, Norman

Publication Date
1964

DOI
10.1063/1.1711228

Peer reviewed

Test Particle Method in Kinetic Theory of a Plasma

NORMAN ROSTOKER*

John Jay Hopkins Laboratory for Pure and Applied Science
General Atomic Division of General Dynamics Corporation, San Diego, California
(Received 22 November 1963)

A test particle of coordinates $X = (\mathbf{x}, \mathbf{v})$ is surrounded by a shield cloud of field particles of coordinates X' characterized by a conditional probability function $P(X|X't)$. A relationship has been found between this function, the one-particle function $f(X, t)$ and the two-particle correlation function $G(X, X'; t)$. It is

$$G(X, X'; t) = f(Xt)P(X \mid X't) + f(X't)P(X' \mid Xt) + n\int dX'' f(X'', t)P(X'' \mid Xt)P(X'' \mid X't).$$

The first two terms indicate that each of the two particles involved is a test particle as well as part of the shield cloud of the other particle. The last term corresponds to the two particles shielding a third particle. This relation has been established without solving explicitly for anything and has none of the usual restrictions such as spatial homogeneity, adiabatic time behavior, etc., usually necessary for obtaining explicit solutions. It is useful because the problem of kinetic theory is reduced to determining P which involves only the Vlasov equation. In addition, superposition principles for fluctuations, etc., are apparent at the outset.

I. INTRODUCTION

We consider a gas of charged particles interacting only through Coulomb forces. The system may be described by the Liouville equation

$$\left\{\frac{\partial}{\partial t} + \sum_{i=1}^{N} \mathbf{v}_i\cdot\frac{\partial}{\partial \mathbf{x}_i} - \frac{e}{m}\left[\mathbf{F}_{\mathrm{ex}}(X_i t) + \sum_{j\neq i}^{N} \frac{\partial}{\partial \mathbf{x}_i}\frac{e}{|\mathbf{x}_i - \mathbf{x}_j|}\right]\cdot\frac{\partial}{\partial \mathbf{v}_i}\right\}D(Xt) = 0, \qquad (1)$$

$X_i = (\mathbf{x}_i, \mathbf{v}_i)$ the position and velocity of the ith particle, and $X = (X_1, X_2 \cdots X_N)$. $\mathbf{F}_{\mathrm{ex}}(X_i t) = \mathbf{E}_{\mathrm{ex}}(\mathbf{x}_i t) + (1/c)\, \mathbf{v}_i \times \mathbf{B}_{\mathrm{ex}}(\mathbf{x}_i t)$ where $\mathbf{E}_{\mathrm{ex}}$ and $\mathbf{B}_{\mathrm{ex}}$ are externally applied fields. Infinite mass randomly distributed ions are assumed leaving only electrons of charge $-e$ and mass m. This restriction is easily relaxed.

The system can also be described by the Bogoliubov–Born–Green–Kirkwood–Yvon heirarchy which is obtained by taking moments of Eq. (1)

$$\left\{\frac{\partial}{\partial t} + \sum_{i=1}^{s} \mathbf{v}_i\cdot\frac{\partial}{\partial \mathbf{x}_i} - \frac{e}{m}\left[\mathbf{F}_{\mathrm{ex}}(X_i t) + \sum_{j\neq i}^{s} \frac{\partial}{\partial \mathbf{x}_i}\frac{1}{|\mathbf{x}_i - \mathbf{x}_j|}\right]\cdot\frac{\partial}{\partial \mathbf{v}_i}\right\}f_s - \frac{ne^2}{m}\sum_{i=1}^{s}\int \frac{\partial}{\partial \mathbf{x}_i}\frac{1}{|\mathbf{x}_i - \mathbf{x}_{s+1}|}\cdot\frac{\partial f_{s+1}}{\partial \mathbf{v}_i}\,dX_{s+1} = 0, \qquad (2)$$

where

$$f_s(X_1 \cdots X_s; t) = V^s \int D(Xt)\, dX_{s+1} \cdots dX_N.$$

It has previously been shown that Eq. (2) can be solved[1] approximately by expanding in a parameter $g = 1/nL_{\mathrm{D}}^3$ where n is the density and L_{D} is the Debye length. Alternately one may expand in the discreteness parameters e, m, and $1/n$ considered as being of the same order. To first order one finds

$$f_s(X_1 \cdots X_s; t) = \prod_{i=1}^{s} f(X_i t) + \frac{1}{2}\sum_{j\neq k}^{N}\left[\prod_{i\neq j,k}^{N} f(X_i t)\right]G(X_j X_k; t), \qquad (3)$$

provided that $f(Xt)$ and $G(X, X'; t)$ satisfy the following equations

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} - \frac{e}{m}\,\mathbf{F}_M(Xt)\cdot\frac{\partial f}{\partial \mathbf{v}} = \frac{ne^2}{m}\int \frac{\partial}{\partial \mathbf{x}}\frac{1}{|\mathbf{x} - \mathbf{x}'|}\cdot\frac{\partial}{\partial \mathbf{v}}\,G(X, X'; t)\,dX', \qquad (4)$$

$$\frac{\partial}{\partial t}G(X, X'; t) + [O(Xt) + O(X't)]G(X, X'; t) = \frac{e^2}{m}\frac{\partial}{\partial \mathbf{x}}\frac{1}{|\mathbf{x} - \mathbf{x}'|}\cdot\left[f(X't)\frac{\partial f(Xt)}{\partial \mathbf{v}} - f(Xt)\frac{\partial f(X't)}{\partial \mathbf{v}'}\right] \qquad (5)$$

where

$$\mathbf{F}_M(Xt) = \mathbf{F}_{\mathrm{ex}}(Xt) + ne\int \frac{\partial}{\partial \mathbf{x}}\frac{1}{|\mathbf{x} - \mathbf{x}'|}\,f(X't)\,dX';$$

$O(Xt)$ is an operator that involves differentiation and integration, i.e.,

* Permanent Address: University of California, San Diego, La Jolla, California.

[1] N. Rostoker and M. N. Rosenbluth, Phys. Fluids **3**, 1 (1960); R. Balescu, Phys. Fluids **3**, 52 (1960); R. L. Guernsey, dissertation, University of Michigan (1960); A. Lenard, Ann. Phys. (N. Y.) **10**, 390 (1960).

$$O(Xt) = \mathbf{v}\cdot\frac{\partial}{\partial\mathbf{x}} - \frac{e}{m}\mathbf{F}_M(Xt)\cdot\frac{\partial}{\partial\mathbf{v}} - \frac{ne^2}{m}\frac{\partial}{\partial\mathbf{v}}f(Xt)\cdot\frac{\partial}{\partial\mathbf{x}}\int\frac{dX'}{|\mathbf{x}-\mathbf{x}'|}\{\cdots\}. \tag{6}$$

Our objective is to show the relationship between this problem and a test-particle problem. The test-particle problem can be formulated simply by assuming that there is an additional external electric field

$$\mathbf{E}_t(\mathbf{x}, t) = \frac{\partial}{\partial\mathbf{x}}\frac{e}{|\mathbf{x}-\mathbf{x}_0(t)|}$$

where $\mathbf{x}_0(t)$ is the test-particle orbit. Thus Eq. (1) becomes

$$\left\{\frac{\partial}{\partial t} + \sum_{i=1}^{N}\mathbf{v}_i\frac{\partial}{\partial\mathbf{x}_i} - \frac{e}{m}\left[\mathbf{F}_{ex}(X_it) + \sum_{j\neq i}^{N}\frac{\partial}{\partial\mathbf{x}_i}\frac{e}{|\mathbf{x}_i-\mathbf{x}_j|}\right]\cdot\frac{\partial}{\partial\mathbf{v}_i}\right\}\hat{D}(Xt) = \frac{e^2}{m}\sum_{i=1}^{N}\frac{\partial}{\partial\mathbf{x}_i}\frac{1}{|\mathbf{x}_i-\mathbf{x}_0(t)|}\cdot\frac{\partial\hat{D}}{\partial\mathbf{v}_i}. \tag{1'}$$

The previous procedure for producing a chain of equations and solving it by expansion results in the following modifications of Eqs. (4) and (5):

$$\frac{\partial\hat{f}}{\partial t} + \mathbf{v}\cdot\frac{\partial\hat{f}}{\partial\mathbf{x}} - \frac{e}{m}\mathbf{F}_M(Xt)\cdot\frac{\partial\hat{f}}{\partial\mathbf{v}} = \frac{ne^2}{m}\int\frac{\partial}{\partial\mathbf{x}}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\cdot\frac{\partial}{\partial\mathbf{v}}\hat{G}(X, X'; t)d\mathbf{x}' + \frac{e^2}{m}\frac{\partial\hat{f}(Xt)}{\partial\mathbf{v}}\cdot\frac{\partial}{\partial\mathbf{x}}\frac{1}{|\mathbf{x}-\mathbf{x}_0(t)|}, \tag{4'}$$

$$\frac{\partial}{\partial t}\hat{G}(X, X'; t) - \frac{e^2}{m}\frac{\partial}{\partial\mathbf{x}}\frac{1}{|\mathbf{x}-\mathbf{x}_0(t)|}\cdot\frac{\partial}{\partial\mathbf{v}}\hat{G}(X, X'; t) - \frac{e^2}{m}\frac{\partial}{\partial\mathbf{x}'}\frac{1}{|\mathbf{x}'-\mathbf{x}_0(t)|}\cdot\frac{\partial}{\partial\mathbf{v}'}\hat{G}(X, X't) + [\hat{O}(Xt) + \hat{O}(X't)]G(X, X't) = \frac{e^2}{m}\frac{\partial}{\partial\mathbf{x}}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\cdot\left[\hat{f}(X't)\frac{\partial\hat{f}(Xt)}{\partial v} - \hat{f}(Xt)\frac{\partial\hat{f}(X't)}{\partial\mathbf{v}'}\right]. \tag{5'}$$

Since we are only concerned with a first order calculation in $g = e$, m, or $1/n$, some simplifications are permitted. Since $\hat{G}$ is already first order, the second and third terms of Eq. (5′) are second order and may be omitted. Furthermore, Eq. (4′) differs from Eq. (4) only in the second term on the right which is clearly a first order quantity being due to only one test charge. Therefore we may assume

$$\hat{f}(Xt) = f(Xt) + \delta f(Xt)$$

where $\delta f(Xt)$ is a first-order quantity. Thus in Eq. (5′) $\hat{f}(Xt)$ can everywhere be replaced by $f(Xt)$. $\hat{O}(Xt)$ differs from $O(Xt)$ only in the fact that $f(Xt)$ in Eq. (6) is replaced by $\hat{f}(Xt)$. Therefore $\hat{O}(Xt) = O(Xt)$ in Eq. (5′) and finally we deduce that to first order $\hat{G}(X, X'; t) = G(X, X'; t)$. Equation (4′) to first order in g becomes

$$\frac{\partial}{\partial t}\delta f + \mathbf{v}\cdot\frac{\partial}{\partial\mathbf{x}}\delta f - \frac{e}{m}\mathbf{F}_M(X, t)\cdot\frac{\partial}{\partial\mathbf{v}}\delta f - \frac{e}{m}\delta\mathbf{F}_M(Xt)\cdot\frac{\partial}{\partial\mathbf{v}}f = \frac{e^2}{m}\frac{\partial f(Xt)}{\partial\mathbf{v}}\cdot\frac{\partial}{\partial\mathbf{x}}\frac{1}{|\mathbf{x}-\mathbf{x}_0(t)|},$$

where

$$\delta\mathbf{F}_M(X, t) = ne\int\frac{\partial}{\partial\mathbf{x}}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\delta f(X't)\,dX'$$

and we have made use of Eq. (4). The time dependence of $\delta f(Xt)$ can more conveniently be expressed by introducing $P(X_0(t) \mid X, t) = \delta f(Xt)$ in which case

$$\frac{\partial}{\partial t}\delta f(Xt) = \frac{\partial}{\partial t}P(X_0 \mid Xt) + v_0\cdot\frac{\partial}{\partial x_0}P + \dot{v}_0\cdot\frac{\partial P}{\partial v_0}.$$

The orbit $\mathbf{x}_0(t)$ has thus far not been specified. We are free to specify it as we please. Assume that it satisfies the equations

$$\frac{d\mathbf{x}_0}{dt} = \mathbf{v}_0, \qquad \dot{\mathbf{v}}_0 = \frac{d\mathbf{v}_0}{dt} = -\frac{e}{m}\mathbf{F}_M(X_0, t).$$

With these definitions the test particle problem can be stated as follows:

$$\frac{\partial}{\partial t}P(X_0 \mid Xt) + \left(\mathbf{v}_0\cdot\frac{\partial}{\partial\mathbf{x}_0} - \frac{e}{m}\mathbf{F}_M(X_0, t)\cdot\frac{\partial}{\partial\mathbf{v}_0}\right)P(X_0 \mid Xt) + O(Xt)P(X_0 \mid Xt) = \frac{e^2}{m}\frac{\partial f(Xt)}{\partial\mathbf{v}}\cdot\frac{\partial}{\partial\mathbf{x}}\frac{1}{|\mathbf{x}-\mathbf{x}_0|}. \tag{7}$$

We now proceed to show that a relationship exists between the solution of Eq. (7) and the solution of Eq. (5). It is

$$G(X, X'; t) = f(Xt)P(X \mid X't) + f(X't)P(X' \mid Xt) + n\int dX''\,f(X''t)P(X'' \mid Xt)P(X'' \mid X't), \tag{8}$$

which is valid to the same order of approximation that Eqs. (5) and (7) are valid. Equation (8) has the following physical interpretation. There are two particles involved in the correlation function $G(X, X'; t)$. The first term corresponds to X' being a field particle and X a test particle with the probability $f(Xt)$. In the second term, X' is the test particle and X the field particle. In the third term, X and X' are both field particles for any third particle X'' which is a test particle with the probability $f(X''t)$. Equation (8) is of practical significance because it replaces the problem of solving Eq. (5) by the problem of solving Eq. (7) which is simpler. It also makes the solution much easier to interpret. In Sec. II the proof of Eq. (8) is given. After generalizing the result to include finite mass ions we proceed to show the immediate consequences of Eq. (8) in terms of superposition principles.

II. RELATIONSHIP BETWEEN G(X, X′; t) AND P(X|X′t)

Equation (5) is an initial value problem. Assume that Eq. (8) is valid at $t = 0$. If we can show that Eq. (8) satisfies Eq. (5) for any t, it must be the unique solution. Therefore substitute Eq. (8) into Eq. (5) and the following assortment of terms results:

$$\begin{aligned}
&\frac{\partial f(Xt)}{\partial t} P(X \mid X't) + \frac{\partial f(X't)}{\partial t} P(X' \mid Xt) + n \int dX'' \frac{\partial f(X''t)}{\partial t} P(X'' \mid Xt) P(X'' \mid X't) + f(Xt) \frac{\partial}{\partial t} P(X \mid X't) \\
&+ f(X't) \frac{\partial}{\partial t} P(X' \mid Xt) + n \int dX'' f(X''t) \left[\frac{\partial P(X'' \mid Xt)}{\partial t} P(X'' \mid X't) + P(X'' \mid Xt) \frac{\partial P(X'' \mid X't)}{\partial t} \right] \\
&+ O(Xt)[f(Xt)P(X \mid X't)] + f(X't)O(Xt)P(X' \mid Xt) + O(X't)[f(X't)P(X' \mid Xt)] + f(Xt)O(X't)PX \mid X't) \\
&+ n \int dX'' f(X''t) \{ [O(Xt)P(X'' \mid Xt)]P(X'' \mid X't) + P(X'' \mid Xt)[O(X't)P(X'' \mid X't) \} \\
&= \frac{e^2}{m} \frac{\partial}{\partial \mathbf{x}} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \cdot \left[f(X't) \frac{\partial}{\partial \mathbf{v}} f(Xt) - f(Xt) \frac{\partial}{\partial \mathbf{v}'} f(X't) \right]. \qquad (9)
\end{aligned}$$

Now substitute Eq. (7) to eliminate all expressions of the type $(\partial/\partial t)\, P(X \mid X't) + O(X't)P(X \mid X't)$. This produces the following transformation of Eq. (1):

$$\begin{aligned}
&\frac{\partial f(Xt)}{\partial t} P(X \mid X't) + \frac{\partial f(X't)}{\partial t} P(X' \mid Xt) + n \int dX'' \frac{\partial f(X''t)}{\partial t} P(X'' \mid Xt) P(X'' \mid X't) \\
&\quad - f(Xt) \left[v \cdot \frac{\partial}{\partial x} P(X \mid X't) - \frac{e}{m} F_M(Xt) \cdot \frac{\partial}{\partial v} P(X \mid X't) \right] \\
&\quad - f(X't) \left[v' \cdot \frac{\partial}{\partial x'} P(X' \mid Xt) - \frac{e}{m} F_M(X't) \cdot \frac{\partial}{\partial v'} P(X' \mid Xt) \right] + O(Xt)[f(Xt)P(X \mid X't)] \\
&\quad + O(X't)[f(X't)P(X' \mid Xt)] + n \int dX'' f(X''t) P(X'' \mid X't) \left\{ -\mathbf{v}'' \cdot \frac{\partial}{\partial \mathbf{x}''} P(X'' \mid Xt) \right. \\
&\quad \left. + \frac{e}{m} F_M(X''t) \cdot \frac{\partial}{\partial \mathbf{v}''} P(X'' \mid Xt) + \frac{e^2}{m} \frac{\partial f(Xt)}{\partial \mathbf{v}} \cdot \frac{\partial}{\partial \mathbf{x}} \frac{1}{|\mathbf{x} - \mathbf{x}''|} \right\} + n \int dX'' f(X''t) P(X'' \mid Xt) \\
&\quad \cdot \left\{ -\mathbf{v}'' \cdot \frac{\partial}{\partial \mathbf{x}''} P(X'' \mid X't) + \frac{e}{m} F_M(X''t) \cdot \frac{\partial}{\partial \mathbf{v}''} P(X'' \mid X't) + \frac{e^2}{m} \frac{\partial f(X't)}{\partial \mathbf{v}'} \cdot \frac{\partial}{\partial \mathbf{x}'} \frac{1}{|\mathbf{x}' - \mathbf{x}''|} \right\} = 0.
\end{aligned}$$

Consider the last expression. The first two terms can be integrated by parts and the last term can be expressed in terms of $O(Xt)$ defined in Eq. (6). Thus it is transformed to

$$\begin{aligned}
&n \int dX'' P(X'' \mid Xt) \left[\mathbf{v}'' \cdot \frac{\partial}{\partial \mathbf{x}''} - \frac{e}{m} F_M(X''t) \cdot \frac{\partial}{\partial \mathbf{v}''} \right] \\
&\quad \cdot [f(X''t)P(X'' \mid Xt)] + \frac{ne^2}{m} \frac{\partial f(X't)}{\partial \mathbf{v}'} \cdot \frac{\partial}{\partial \mathbf{x}'} \\
&\quad \cdot \int \frac{dX''}{|\mathbf{x}' - \mathbf{x}''|} f(X''t) P(X'' \mid Xt)
\end{aligned}$$

$$\begin{aligned}
&= n \int dX'' f(X''t) P(X'' \mid Xt) \\
&\quad \cdot \left[\mathbf{v}'' \cdot \frac{\partial}{\partial \mathbf{x}''} - \frac{e}{m} F_M(X''t) \cdot \frac{\partial}{\partial \mathbf{v}''} \right] P(X'' \mid Xt) \\
&+ n \int dX'' P(X'' \mid Xt) P(X'' \mid X't) \\
&\quad \cdot \left[\mathbf{v}'' \cdot \frac{\partial}{\partial \mathbf{x}''} - \frac{e}{m} F_M(X''t) \cdot \frac{\partial}{\partial \mathbf{v}''} \right] f(X''t) \\
&- O(X't)[f(X't)P(X' \mid Xt)]
\end{aligned}$$

$$+\left[\mathbf{v}'\cdot\frac{\partial}{\partial x'}-\frac{e}{m}\mathbf{F}_M(X't)\cdot\frac{\partial}{\partial \mathbf{v}'}\right]f(X't)P(X'\mid Xt).$$

Combine this with the second last expression and the result is

$$n\int dX''\,P(X''\mid Xt)P(X''\mid X't)$$
$$\cdot\left(\mathbf{v}''\cdot\frac{\partial}{\partial \mathbf{x}''}-\frac{e}{m}F_M(X''t)\cdot\frac{\partial}{\partial \mathbf{v}''}\right)f(X''t)$$
$$-O(X't)[f(X't)P(X'\mid Xt)]-O(Xt)[f(Xt)P(X\mid X't)]$$
$$+\left[\mathbf{v}'\cdot\frac{\partial}{\partial x'}-\frac{e}{m}F_M(X't)\cdot\frac{\partial}{\partial \mathbf{v}'}\right][f(X't)P(X'\mid Xt)]$$
$$+\left[\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}}-\frac{e}{m}F_M(Xt)\cdot\frac{\partial}{\partial \mathbf{v}}\right]f(Xt)P(X\mid X't).$$

This now cancels many of the previous terms leaving the following result:

$$\left\{\left[\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}}-\frac{e}{m}\mathbf{F}_M(Xt)\cdot\frac{\partial}{\partial \mathbf{v}}\right]f(Xt)\right\}P(X\mid X't)$$
$$+\left\{\left[\frac{\partial}{\partial t}+\mathbf{v}'\cdot\frac{\partial}{\partial \mathbf{x}'}-\frac{e}{m}\mathbf{F}_M(X't)\cdot\frac{\partial}{\partial \mathbf{v}'}\right]f(X't)\right\}P(X'\mid Xt)$$
$$+n\int dX''\left\{\left[\frac{\partial}{\partial t}+\mathbf{v}''\frac{\partial}{\partial x''}-\frac{e}{m}\mathbf{F}_M(X''t)\right.\right.$$
$$\left.\left.\cdot\frac{\partial}{\partial \mathbf{v}''}\right]f(X''t)\right\}P(X''\mid Xt)P(X''\mid X't)=0. \quad (10)$$

Now referring back to Eq. (4) we see that

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}}-\frac{e}{m}\mathbf{F}_M(Xt)\cdot\frac{\partial}{\partial v}\right)f(Xt)$$
$$=\frac{ne^2}{m}\int\frac{\partial}{\partial \mathbf{x}}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\cdot\frac{\partial}{\partial \mathbf{v}}G(X,X't)\,dX',$$

where the right-hand side is a first order quantity. Since $P(X\mid X't)$ is also a first-order quantity all remaining terms in Eq. (10) are second order. Since second-order quantities have already been neglected in deriving Eq. (5), we conclude that Eq. (8) is the unique solution of Eq. (5) within the same limits of approximation that already apply to Eq. (5). It is apparent from the derivation that there are no limitations on the time- or space-dependence of $f(Xt)$.

III. GENERALIZATION TO INCLUDE IONS

If we use the label i to denote the species, q_i for the charge, and m_i for the mass, Eq. (4) becomes

$$\frac{\partial f_i(Xt)}{\partial t}+\mathbf{v}\cdot\frac{\partial f_i}{\partial \mathbf{x}}+\frac{q_i}{m_i}\mathbf{F}_M(Xt)\cdot\frac{\partial f_i}{\partial \mathbf{v}}$$
$$=\frac{nq_i}{m_i}\sum_j\int\frac{\partial}{\partial \mathbf{x}}\frac{q_j}{|\mathbf{x}-\mathbf{x}'|}\cdot\frac{\partial}{\partial \mathbf{v}}G_{ij}(X,X';t)\,dX'. \quad (11)$$

The macroscopic field is

$$F_M(Xt)=\mathbf{F}_{\text{ex}}(Xt)-n\sum_j\int\frac{\partial}{\partial \mathbf{x}}\frac{q_j}{|\mathbf{x}-\mathbf{x}'|}f_j(X't)\,dX'.$$

The pair correlation function $G_{ij}(X,X';t)$ has the symmetry property $G_{ij}(X,X';t)=G_{ji}(X',X;t)$ and satisfies the equation

$$\frac{\partial}{\partial t}G_{ij}(X,X';t)+\sum_l O_{il}(Xt)G_{lj}(X,X';t)$$
$$+O_{jl}(X't)G_{il}(X,X't)=\frac{q_iq_j}{m_j}\frac{\partial}{\partial \mathbf{x}}\frac{1}{|\mathbf{x}-\mathbf{x}'|}$$
$$\cdot\left[f_j(X't)\frac{\partial f_i(Xt)}{\partial \mathbf{v}}-f_i(Xt)\frac{\partial f_j(X't)}{\partial v'}\right], \quad (12)$$

which is the generalization of Eq. (5). The operator $O_{il}(Xt)$ is given by

$$O_{il}(Xt)=\delta_{il}\left[\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}}+\frac{q_i}{m_i}\mathbf{F}_M(Xt)\cdot\frac{\partial}{\partial \mathbf{v}}\right]$$
$$-n\frac{q_iq_l}{m_i}\frac{\partial f_i(Xt)}{\partial v}\cdot\int dX''\frac{\partial}{\partial \mathbf{x}}\frac{1}{|\mathbf{x}-\mathbf{x}''|}\{X''\},$$

where the function it operates on is to be placed in the curly brackets.

The generalization of Eq. (7) is

$$\frac{\partial}{\partial t}P_{ij}(X\mid X't)$$
$$+\left[\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}}+\frac{q_i}{m_i}\mathbf{F}_M(Xt)\cdot\frac{\partial}{\partial \mathbf{v}}\right]P_{ij}(X\mid X't)$$
$$+\sum_l O_{il}(X't)P_{il}(X\mid X't)=\frac{q_iq_j}{m_j}\frac{\partial f_j}{\partial \mathbf{v}'}\cdot\frac{\partial}{\partial \mathbf{x}'}\frac{1}{|\mathbf{x}'-\mathbf{x}|} \quad (13)$$

and

$$G_{ij}(X,X';t)$$
$$=f_i(Xt)P_{ij}(X\mid X't)+f_j(X't)P_{ji}(X'\mid Xt)$$
$$+n\int dX''\sum_l f_l(X'',t)P_{li}(X''\mid Xt)P_{lj}(X''\mid X't). \quad (14)$$

As an illustration of Eq. (14) consider the case of thermal equilibrium. Assume that

$$f_i(Xt)=(2\pi v_i^2)^{-\frac{3}{2}}e^{-v^2/2v_i^2},$$

where $m_iv_i^2=\Theta$ and $\mathbf{F}_{\text{ex}}(Xt)=0$. Since $f_i(\mathbf{v})$ is independent of $\mathbf{x}$, $\mathbf{F}_M(Xt) = 0$. For this case Eq. (13) simplifies to

$$\frac{\partial}{\partial t} P_{ij}(X \mid X't) + \left(\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} + \mathbf{v}'\cdot\frac{\partial}{\partial \mathbf{x}'}\right) P_{ij}(X \mid X't)$$

$$- n\,\frac{q_j}{m_j}\frac{\partial f_j}{\partial \mathbf{v}'}\cdot \sum_l q_l \int dX''\,\frac{\partial}{\partial \mathbf{x}}\frac{1}{|\mathbf{x}-\mathbf{x}''|} P_{il}(X \mid X''t)$$

$$= \frac{q_i q_j}{m_j}\frac{\partial f_j}{\partial \mathbf{v}'}\cdot\frac{\partial}{\partial \mathbf{x}'}\frac{1}{|\mathbf{x}'-\mathbf{x}|}. \tag{15}$$

This can be solved by Fourier and Laplace transformation,

$$P_{ij}(X \mid X't) = \int_{-i\infty+\gamma}^{i\infty+\gamma} \frac{dp}{2\pi i}\, e^{pt} \int \frac{d\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})} \hat{P}_{ij}(k;\mathbf{v} \mid \mathbf{v}'p),$$

where $\hat{P}_{ij}$ satisfies

$$[p + i\mathbf{k}\cdot(\mathbf{v}'-\mathbf{v})]\hat{P}_{ij} - 4\pi n \sum_l \frac{q_j q_l}{m_j}\frac{i\mathbf{k}}{k^2}\cdot\frac{\partial f_j}{\partial \mathbf{v}'}$$

$$\cdot \int d\mathbf{v}''\hat{P}_{il}(k;\mathbf{v} \mid \mathbf{v}'', p) = \frac{4\pi q_i q_j}{p m_j k^2}\, i\mathbf{k}\cdot\frac{\partial f_j}{\partial \mathbf{v}'}.$$

This can, of course, be solved by dividing by $p + i\mathbf{k}\cdot(\mathbf{v}'-\mathbf{v})$ and integrating. The solution is conveniently expressed in terms of the dielectric coefficient

$$\epsilon(k, p) = 1 - \sum_j \frac{\omega_{pj}^2}{k^2} \int \frac{i\mathbf{k}\cdot(\partial f_j/\partial \mathbf{v}')}{p + i\mathbf{k}\cdot\mathbf{v}'}\, d\mathbf{v}'$$

$$= 1 + \sum_j \omega_{pj}^2 \int_0^\infty dt\, t e^{-pt} e^{-\frac{1}{2}k^2 v_j^2 t^2}, \tag{16}$$

where $\omega_{pj}^2 = 4\pi n q^2/m_j$ is the plasma frequency of species j. Thus the asymptotic solution is

$$\hat{P}_{ij}(k;\mathbf{v} \mid v') = \lim_{p\to 0} p\hat{P}_{ij}(k;\mathbf{v} \mid v', p)$$

$$= -\frac{4\pi q_i q_j}{m_j v_j^2}\frac{(\mathbf{k}\cdot\mathbf{v}')f_j(\mathbf{v}')}{[\mathbf{k}\cdot(\mathbf{v}'-\mathbf{v}) - i\delta]}\frac{1}{k^2\epsilon(\mathbf{k}, -i\mathbf{k}\cdot\mathbf{v})}, \tag{17}$$

which describes the shielding of a particle in thermal equilibrium. Assuming

$$G_{ij}(X, X') = \int \frac{d\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})} \hat{G}_{ij}(k;\mathbf{v},\mathbf{v}'),$$

the Fourier transform of Eq. (14) is

$$\hat{G}_{ij}(k; v, v') = f_i(v)\hat{P}_{ij}(k;\mathbf{v} \mid \mathbf{v}') + f_j(v')\hat{P}^*_{ji}(k;\mathbf{v}' \mid \mathbf{v})$$

$$+ n\int dv'' \sum_l f_l(v'')\hat{P}^*_{li}(k;\mathbf{v}'' \mid \mathbf{v})P_{lj}(k; v'' \mid v').$$

After substituting Eq. (17) this becomes

$$\hat{G}_{ij}(k;\mathbf{v},\mathbf{v}') = -\frac{4\pi q_i q_j}{\Theta k^2} f_i(v) f_j(v')$$

$$\cdot\left\{\left[\frac{u'}{\epsilon(\mathbf{k}, -iku)} - \frac{u}{\epsilon^*(\mathbf{k}, -iku')}\right]\frac{1}{(u'-u-i\delta)}\right.$$

$$- \int du'' \sum_l \frac{4\pi n q_l^2}{\Theta k^2}\frac{F_l(u'')uu'}{(u-u''+i\delta)(u'-u''-i\delta)}$$

$$\left.\cdot\frac{1}{|\epsilon(\mathbf{k}, -iku'')|^2}\right\} \tag{16'}$$

where $u = \mathbf{k}\cdot\mathbf{v}/k$, $u' = \mathbf{k}\cdot\mathbf{v}'/k$, etc., and $F_l(u) = \int f_l(v)\,\delta[u - (\mathbf{k}\cdot\mathbf{v}/k)]\,d\mathbf{v}$. To evaluate the integral use the following information:

$$\frac{1}{(u-u''-i\delta)(u'-u''+i\delta)} = \frac{1}{u-u'+i\delta}$$

$$\cdot\left[\frac{1}{u''-u-i\delta} - \frac{1}{u''-u'+i\delta}\right],$$

$$\operatorname{Im}\epsilon(k, -iku) = \pi u \sum_l \frac{4\pi n q_l^2}{\Theta k^2} F_l(u),$$

$$\sum_l \frac{4\pi n q_l^2}{k^2\Theta}\frac{F_l(u)}{|\epsilon(k, -iku)|^2} = -\frac{1}{\pi}\operatorname{Im}\left[\frac{P}{u}\frac{1}{\epsilon(k, -iku)}\right],$$

where P means the principal part. (The imaginary part of the quantity in brackets is not singular at $u = 0$, but the real part is.) Dispersion relation can be derived from the following integral:

$$\int_{-\infty}^{\infty}\frac{du'}{u'-u+i\delta}\frac{1}{u'+i\delta}\frac{1}{\epsilon(k, -iku')} = 0.$$

This integral vanishes because $\epsilon(k, -iku')$ has no poles in the upper half of the u' plane, and $\lim_{|u'|\to\infty}\epsilon(k, -iku') = 1$ for $0 < \arg u' < \pi$. Writing out $1/(u'-u+i\delta) = P/(u'-u) - \pi i\delta(u'-u)$, etc., and taking real and imaginary parts leads to the following integral dispersion relations

$$\frac{1}{\pi}\int du'\,\frac{P}{u'-u}\operatorname{Re}\left[\frac{P}{u'}\frac{1}{\epsilon(k, -iku')}\right]$$

$$= -\operatorname{Im}\left[\frac{P}{u}\frac{1}{\epsilon(k, -iku)}\right] + \frac{\pi\delta(u)}{\epsilon(k, 0)},$$

$$\frac{1}{\pi}\int du'\,\frac{P}{u'-u}\operatorname{Im}\left[\frac{P}{u'}\frac{1}{\epsilon(k, -iku')}\right]$$

$$= \operatorname{Re}\left[\frac{P}{u}\frac{1}{\epsilon(k, -iku)}\right] - \frac{P}{u\epsilon(0)}. \tag{17'}$$

Returning now to Eq. (16′),

$$\hat{G}_{ij}(k, v, v') = -\frac{4\pi q_i q_j}{\Theta k^2} f_i(v) f_j(v')$$

$$\cdot\left\{\frac{1}{u'-u-i\delta}\left[\frac{u'}{\epsilon(k, -iku)} - \frac{u}{\epsilon(k, -iku)}\right]\right.$$

$$-\frac{uu'}{u'-u-i\delta}\int du''\left[\frac{1}{u''-u-i\delta}-\frac{1}{u''-u+i\delta}\right]\frac{1}{\pi}\,\mathrm{Im}\left[\frac{P}{u''}\frac{1}{\epsilon(k,-iku'')}\right].$$

Making use of Eq. (17′), the integral can be evaluated with the well known result

$$\hat{G}_{ij}(k;\mathbf{v},\mathbf{v}')=-\frac{4\pi q_i q_j}{\Theta k^2}\frac{f_i(v)f_j(v')}{\epsilon(\mathbf{k},0)}. \tag{18}$$

The point of this calculation is to show that verification of Eq. (14) or Eq. (8) directly from the explicit solutions is a complicated process even for the simplest case.[2] It appears to depend on analyticity properties and suggests unnecessary mathematical restrictions. This is the case for most of the mathematical manipulations necessary to get the results of kinetic theory into a form that can be interpreted physically. It is easier to carry them out before solving the problem as in the derivation of Eq. (8).

IV. FLUCTUATIONS

Consider for example the calculation of the ensemble average $\langle A(\mathbf{x}t)B(\mathbf{x}',t)\rangle$ where $A(\mathbf{x}t)$, $B(\mathbf{x}t)$ are any observables of the form

$$A(\mathbf{x}t)=\sum_{i=1}^{N}a(X_i\mid\mathbf{x}),\quad B(\mathbf{x},t)=\sum_{i=1}^{N}b(X_i\mid\mathbf{x}). \tag{19}$$

For example, if $a(X_i\mid\mathbf{x})=\partial/\partial x\,(e/|\mathbf{x}-\mathbf{x}_i|)\,A(\mathbf{x}t)=E_x(\mathbf{x}t)$, the x component of the electric field. $X_i=(\mathbf{x}_i,\mathbf{v}_i)$ are coordinates and velocities of the particles evaluated at time t and the sum is over all particles. The ensemble average is defined as

$$\langle A(\mathbf{x}t)B(\mathbf{x}',t)\rangle=\int dX\,D(Xt)\sum_{i,j=1}^{N}a(X_i\mid\mathbf{x})b(X_j\mid\mathbf{x}'). \tag{20}$$

Integrations over all coordinates but two can be carried out with the result

$$\langle A(\mathbf{x}t)B(\mathbf{x}'t)\rangle=n\int f_1(X_1t)a(X_1\mid\mathbf{x})b(X_1\mid\mathbf{x}')\,dX_1+n^2\int f_2(X_1,X_2;t)a(X_1\mid\mathbf{x})b(X_2\mid\mathbf{x}')\,dX_1\,dX_2.$$

The first term comes from $i=j$ in the sum and the second from $i\neq j$. Now substitute $f_2(X_1,X_2;t)=f(X_1t)f(X_2t)+G(X_1,X_2;t)$ and Eq. (8) for G.

$$\begin{aligned}\langle A(\mathbf{x},t)B(\mathbf{x}',t)\rangle&=\langle A(\mathbf{x}t)\rangle\langle B(\mathbf{x}',t)\rangle\\&+n\int f(X,t)a(X_1\mid\mathbf{x})b(X_1\mid\mathbf{x}')\,dX_1\\&+n^2\int f(X_1t)P(X_1\mid X_2t)a(X_1\mid\mathbf{x})b(X_2\mid\mathbf{x}')\,dX_1\,dX_2\\&+n^2\int f(X_2t)P(X_2\mid X_1t)a(X_1\mid\mathbf{x})b(X_2\mid\mathbf{x}')\,dX_1\,dX_2\\&+n^3\int f(X_3,t)P(X_3\mid X_1t)P(X_3\mid X_2t)\\&\quad\cdot a(X_1\mid\mathbf{x})b(X_2\mid\mathbf{x}')\,dX_1dX_2dX_3.\end{aligned} \tag{21}$$

Instead of the bare-particle quantities $a(X_1\mid x)$, $b(X_1\mid x)$, define the corresponding quantities for quasiparticles,

$$\hat{a}(X_1\mid\mathbf{x}t)=a(X_1\mid\mathbf{x})+n\int a(X_2\mid\mathbf{x})P(X_1\mid X_2t)\,dX_2, \tag{22}$$

etc. and substitute into Eq. (21). Thus

$$\begin{aligned}\langle A(\mathbf{x}t)B(\mathbf{x}'t)\rangle&=\langle A(\mathbf{x}t)\rangle\langle B(\mathbf{x}'t)\rangle\\&+n\int f(X_1t)a(X_1\mid\mathbf{x})b(X_1\mid\mathbf{x}')\,dX_1\\&+n\int dX_1\,f(X_1t)a(X_1\mid\mathbf{x})[\hat{b}(X_1\mid\mathbf{x}')t-b(X_1\mid\mathbf{x}')]\\&+n\int dX_2\,f(X_2t)[\hat{a}(X_2\mid\mathbf{x}t)-a(X_2\mid\mathbf{x})]b(X_2\mid\mathbf{x}')\\&+n\int dX_3\,f(X_3t)[\hat{a}(X_3\mid\mathbf{x}t)-a(X_3\mid\mathbf{x})]\\&\quad\cdot[\hat{b}(X_3\mid\mathbf{x}'t)-b(X_3\mid\mathbf{x}')]=\langle A(\mathbf{x}t)\rangle\langle B(\mathbf{x}'t)\rangle\\&+n\int f(X_1t)\hat{a}(X_1\mid\mathbf{x}t)\hat{b}(X_1\mid\mathbf{x}'t)\,dX_1,\end{aligned} \tag{23}$$

where

$$\langle A(xt)\rangle=n\int f(X_1t)a(X_1\mid\mathbf{x})\,dX_1,\quad\text{etc.}$$

Thus we have established the principle of superposition of statistically independent quasiparticles under very general circumstances, i.e., for any observable expressible by Eq. (19) and for any time and space variation of $f_1(Xt)$. Previously[3] such relationships have been derived by obtaining explicit solutions for specific cases and manipulating them into the form of Eq. (23) after a great deal of algebra.

V. TWO-TIME DISTRIBUTION FUNCTIONS

In order to calculate correlation functions, it is necessary to introduce two-time distribution func-

[2] Some further examples of direct verification are given in the previous paper [N. Rostoker, Phys. Fluids 7, 479 (1964)].

[3] N. Rostoker, Nucl. Fusion 1, 101 (1961).

tions. For example, $D_2(X, t; X't')\, dX\, dX'$ is the probability of finding the system in (X, dX) at t and in (X', dX') at t'. $D_2(Xt; X't')$ satisfies the Liouville equation in (X', t') and the initial condition

$$D_2(Xt, X't) = D(Xt)\delta(X - X').$$

We can obtain a BBKGY chain for D_2 and solve it approximately by expansion as in the case of $D(Xt)$. Since this has been done previously[3] we simply quote the results. For most calculations we require only two moments of D_2, namely

$$W_{11}(X_1t; X_1't') = V^2 \int D_2(Xt; X't')\, dX_2 \cdots dX_N\, dX_2' \cdots dX_N',$$

$$W_{12}(X_1t; X_2't') = V^2 \int D_2(Xt; X't')\, dX_2 \cdots dX_N\, dX_1'\, dX_3' \cdots dX_N'.$$

In terms of these moments the autocorrelation function $\langle A(\mathbf{x}t)B(\mathbf{x}'t')\rangle$ is

$$\begin{aligned}\langle A(\mathbf{x}t)B(\mathbf{x}', t')\rangle &= \int dX\, D_2(Xt; X't') \sum_{ij=1}^{N} a(X_i \mid \mathbf{x})b(X_j' \mid \mathbf{x}') \\ &= \frac{n}{V}\int dX_1\, dX_1' a(X_1 \mid \mathbf{x})b(X_1' \mid \mathbf{x}')W_{11}(X_1t; X_1't') \\ &\quad + n^2 \int dX_1\, dX_2'\, a(X_1 \mid \mathbf{x})b(X_2' \mid \mathbf{x}')W_{12}(X_1t; X_2't').\end{aligned} \tag{24}$$

If the parameter expansion ($g = e$, m, or $1/n$) is carried out as discussed in Sec. I, the equations for W_{11} and W_{12} are as follows:

$$\left\{\frac{\partial}{\partial t'} + \mathbf{v}_1'\cdot\frac{\partial}{\partial \mathbf{x}_1'} - \frac{e}{m}\mathbf{F}_M(X_1', t')\cdot\frac{\partial}{\partial \mathbf{v}_1'}\right\} \cdot W_{11}(X_1t, X_1't') = 0 \tag{25}$$

and

$$W_{11}(X_1t; X_1't) = Vf(X_1t)\delta(X_1' - X_1)$$

is the initial condition.

$$W_{12}(X_1t; X_2't') = f(X_1t)f(X_2't') + G_{12}(X_1t; X_2't'),$$

where

$$\left\{\frac{\partial}{\partial t'} + O(X_2't')\right\}G_{12}(X_1t; X_2't') = \frac{e^2}{m}\frac{\partial f(X_2't')}{\partial \mathbf{v}_2'}\cdot\frac{\partial}{\partial \mathbf{x}_2'}\frac{1}{V}\int \frac{W_{11}(X_1t; X_1't')\, dX_1'}{|\mathbf{x}_2' - \mathbf{x}_1'|} \tag{26}$$

and the initial condition is

$$G_{12}(X_1t; X_2't) = G(X_1, X_2'; t).$$

The formal solution of Eq. (26), as may be verified by direct substition, is

$$\begin{aligned}G_{12}(X_1t; X_2't') &= \frac{1}{V}\int dX_1' W_{11}(X_1t; X_1't')P(X_1' \mid X_2't') \\ &\quad + \frac{1}{V}\int dX_2\, W_{11}(X_2t; X_2't')P(X_2 \mid X_1t) \\ &\quad + \frac{n}{V}\int dX_3\, dX_3'\, W_{11}(X_3t; X_3't') \\ &\quad \cdot P(X_3 \mid X_1t)P(X_3' \mid X_2't').\end{aligned} \tag{27}$$

It is clear that this is a fairly obvious generalization of Eq. (8). Returning now to Eq. (24) and substituting Eq. (27), we obtain

$$\begin{aligned}\langle A(\mathbf{x}t)B(\mathbf{x}', t')\rangle &= \frac{n}{V}\int dX_1\, dX_1'\, W_{11}(X_1t; X_1't') \\ &\quad \cdot a(X_1 \mid x)b(X_1' \mid x') + \langle A(\mathbf{x}t)\rangle\langle B(\mathbf{x}', t')\rangle \\ &\quad + \frac{n^2}{V}\int dX_1\, dX_1'\, W_{11}(X_1t; X_1't') \\ &\quad \cdot a(X_1 \mid \mathbf{x})\int dX_2'\, b(X_2' \mid \mathbf{x}')P(X_1' \mid X_2't') \\ &\quad + \frac{n^2}{V}\int dX_2\, dX_2'\, W_{11}(X_2t; X_2't') \\ &\quad \cdot b(X_2' \mid \mathbf{x}')\int dX_1\, a(X_1 \mid \mathbf{x})P(X_2 \mid X_1t) \\ &\quad + \frac{n^3}{V}\int dX_3\, dX_3'\, W_{11}(X_3t; X_3't')\int dX_1 \\ &\quad \cdot a(X_1 \mid \mathbf{x})P(X_3 \mid X_1t)\int dX_2'\, b(X_2' \mid \mathbf{x}')P(X_3' \mid X_2't').\end{aligned}$$

After making use of the definitions of $\hat{a}(X_1 \mid \mathbf{x}t)$, etc., for quasiparticles given by Eq. (22), this reduces to

$$\begin{aligned}\langle A(\mathbf{x}t)B(\mathbf{x}'t')\rangle &= \frac{n}{V}\int dX_1\, dX_1'\, W_{11}(X_1t, X_1't') \\ &\quad \cdot \hat{a}(X_1 \mid \mathbf{x}t)\hat{b}(X_1' \mid \mathbf{x}'t') + \langle A(\mathbf{x}t)\rangle\langle B(\mathbf{x}'t')\rangle.\end{aligned} \tag{28}$$

The principle of superposition is thus obtained in a very general form.

In order to include finite-mass ions, Eq. (27) must be modified as follows

$$\begin{aligned}G_{ij}(X_1t; X_2't') &= \frac{1}{V}\int dX_1' W_{ii}(X_1t; X_1't')P_{ij}(X_1' \mid X_2't') \\ &\quad + \frac{1}{V}\int dX_2\, W_{jj}(X_2t; X_2't')P_{ji}(X_2 \mid X_1t) +\end{aligned}$$

$$+ \frac{n}{V} \int dX_3\, dX_3' \sum_l W_{ll}(X_3t; X_3't') \cdot P_{li}(X_3 \mid X_1t) P_{lj}(X_3' \mid X_2't'). \quad (29)$$

This is by now an obvious generalization of Eq. (14). Not all observables of interest are quite of the form of Eq. (19). For example, the electron density is

$$n_e(\mathbf{x}t) = \sum_{i=1}^{N} n_e(\mathbf{x}_i \mid \mathbf{x}), \quad (30)$$

where $n_e(\mathbf{x}_i \mid \mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_i)$ and the sum is only over electrons. The quantity $\langle n_e(\mathbf{x}t) n_e(\mathbf{x}'t')\rangle$ is of interest in connection with the scattering of electromagnetic waves.[4] It can be conveniently expressed in terms of electron densities for two kinds of quasiparticles,

$$\hat{n}_{ee}(X_1 \mid \mathbf{x}t) = n_e(\mathbf{x}_1 \mid \mathbf{x}) + n \int n_e(\mathbf{x}_2 \mid \mathbf{x}) P_{ee}(X_1 \mid X_2 t)\, dX_2, \quad (31)$$

$$\hat{n}_{Ie}(X_1 \mid \mathbf{x}t) = n \int n_e(\mathbf{x}_2 \mid \mathbf{x}) P_{Ie}(X_1 \mid X_2 t)\, dX_2. \quad (32)$$

$\hat{n}_{ee}(X_1 \mid \mathbf{x}t)$ means the electron density due to an electron quasiparticle at X_1; $\hat{n}_{Ie}(X_1 \mid \mathbf{x}t)$ is the electron density due to an ion quasiparticle at X_1. The subscripts i, j run over e, I to denote electrons and ions. The desired correlation function is

$$\langle n_e(\mathbf{x}t) n_e(\mathbf{x}'t')\rangle = \frac{n}{V} \int dX_1\, dX_1'\, n_e(\mathbf{x}_1 \mid \mathbf{x}) n_e(\mathbf{x}_1' \mid \mathbf{x}') W_{ee}(X_1 t; X_1't') + n^2 \int dX_1\, dX_2'\, n_e(\mathbf{x}_1 \mid \mathbf{x}) n_e(\mathbf{x}_2' \mid \mathbf{x}') W_{ee}(X_1 t; X_2't'). \quad (33)$$

After substituting from Eqs. (29), (31), and (32), this is easily brought to the form

$$\langle n_e(\mathbf{x}t) n_e(\mathbf{x}'t')\rangle = \langle n_e(\mathbf{x}t)\rangle \langle n_e(\mathbf{x}'t')\rangle + \frac{n}{V} \sum_{i=e,I} \int dX_1\, dX_1'\, W_{ii}(X_1 t, X_1't') \cdot \hat{n}_{ie}(X_1 \mid \mathbf{x}t) \hat{n}_{ie}(X_1' \mid \mathbf{x}'t'),$$

where

$$\langle n_e(\mathbf{x}t)\rangle = n \int f_e(X_1 t) n_e(\mathbf{x}_1 \mid \mathbf{x})\, dX_1.$$

This result was previously obtained from the explicit solution for particular cases after a very considerable amount of algebra.[4]

ACKNOWLEDGMENTS

This research was supported by the Advanced Research Projects Agency, Department of Defense, under Project Defender and was monitored by the Air Force Weapons Laboratory under Contract Number AF29(601)-5338.

[4] M. N. Rosenbluth and N. Rostoker, Phys. Fluids **5**, 726 (1962).