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A simple criterion that allows one to determine whether or not a given wave spectrum will generate zonal flows, is derived and analyzed. In the context of a coupled drift wave–zonal turbulence, the results are pertinent to the limit of small zonal flow damping, $\gamma_d \to 0$, in which previous analyses found that the turbulence vanishes. However, the practically important issue of the drift wave amplitude threshold for zonal flow excitation was not resolved. In its formal mathematical appearance, the criterion obtained is similar to the well-known Penrose criterion that is used for stability analysis of stellar distributions and particle distributions in plasmas. By contrast, the derived criterion, being applied to wave quanta rather than to particle distribution, shows that even ''normal'' (wave density decaying with wave number) distributions with an intensity above the threshold should generate zonal flows. This clearly points at the ubiquity of the latter. © *2001 American Institute of Physics.* [DOI: 10.1063/1.1330204]

I. INTRODUCTION

In many natural and laboratory environments, the equilibria of fluids or plasmas are characterized by stratification in one direction (say radial, frequently prescribed by gravity) while remaining homogeneous across it (say, azimuthally). In plasmas, the magnetic field determines the equilibrium configuration as the bulk rotation of fluids generally does. In such systems the gradient-specific modes are long known to be able to propagate in the direction of translational symmetry, i.e., perpendicularly to the gradient (e.g., Ref. 1). These waves generate turbulent transport in the radial direction, that compromises confinement in fusion devices (see, e.g., Refs. 2 and 3 for a review).

On the other hand, the drift-type turbulence is also capable of generating the so-called zonal flows which, in the most general case, are alternating, random jets streaming along the symmetry direction.[2–4] Their importance for confinement physics is mainly due to their ability to suppress the transport driving turbulence, essentially through shearing.[5–11] The mechanism of their generation is rather universal. Its key element is the turbulent Reynolds stress tensor whose divergence has a nonvanishing azimuthally averaged projection along the flow direction. One may also think of zonal flow generation as of a modulational instability of the Reynolds stress generated by the drift wave ''gas.''[12–15]

When the zonal flows are excited, they form an environment for the parent drift waves. A coupled system of ''predator–prey'' equations has been derived in Ref. 15 to self-consistently describe these two components of wave turbulence. This system has been analyzed in Refs. 15 and 16 under a condition in which the linear growth of drift waves was in a permanent, approximate balance with their nonlinear steepening. In other words, the system was assumed to be far from linear stability, rendering the generation of zonal

flows nonresonant. Two remarkable features have been revealed in this regime. First, the modulational instability develops in the presence of a ''normal'' ($\partial N_{\mathbf{k}} / \partial k < 0$) population of the drift wave quanta. Second, there is no amplitude threshold for the instability except one set by the zonal flow damping γ_d. The last aspect is particularly interesting in the limit of small γ_d when $N_{\mathbf{k}}$ turns out to scale as $N_{\mathbf{k}} \propto \gamma_d$, so that even in the case of vanishing $N_{\mathbf{k}}(\gamma_d \to 0)$ the zonal flow generation seems to be efficient enough to suppress the drift wave turbulence and transport.

The threshold-free generation of zonal flows could be at least tentatively attributed to the above-mentioned nonresonant regime that *a priori* requires a sufficiently strong drift wave nonlinearity to be able to balance their linear growth, so that the zonal flows develop on this background. One might argue then that in the limit $\gamma_d \to 0$, when $N_{\mathbf{k}}$ becomes vanishingly small and should behave linearly, such a balance would be no longer maintained. Then, two further possibilities would emerge. The first should occur if still there is an amplitude threshold N_{th} for zonal flow generation, so that when $N_{\mathbf{k}}$ drops below N_{th}, the drift wave turbulence is no longer suppressed, since zonal flows vanish. Thus, drift wave intensity must remain at some residual level $\sim N_{\mathrm{th}}$; even if $\gamma_d \to 0$. The second possibility should be expected if the zonal flows are indeed generated without the threshold. This means that there must be an essentially complete suppression of the drift wave turbulence when $\gamma_d \to 0$, as obtained in Refs. 15 and 16 for the case of nonresonant generation. Thus, in order to understand the dynamics of the coupled drift wave–zonal flow turbulence for small γ_d, it is necessary to study the purely ''*linear*'' problem of modulational stability of the drift waves. Namely, given a drift wave spectrum when both the linear growth rate and nonlinear self-interaction may be ignored, one derives a linear dispersion equation for the *resonant* zonal flow generation rate. Note

that the role of the distribution of the drift wave quanta $N_{\mathbf{k}}$ here is very much similar to the equilibrium particle distribution $f_0(\mathbf{v})$ in plasmas when the latter is subjected to stability analysis.

It should be noted that such studies do exist. For example, this approach was pursued recently in Refs. 17–19. However, these analyses were restricted to the case of a *monochromatic* (or quasimonochromatic[18]) spectra of the drift waves. They also revealed essentially no amplitude threshold in the nondissipative limit, which might be a result of the monochromatic treatment. Returning to the nonresonant regime of zonal flow generation addressed in Refs. 15 and 16, it should be emphasized that the instability does not require any degree of monochromaticity. Nor is the ''inverse population'' $\partial N_{\mathbf{k}}/\partial k>0$ of the drift wave quanta necessary, as mentioned before and which, at least in azimuthal wave number k_θ, is present in the monochromatic case. It is perhaps more important to mention here that the interaction of the drift waves with the zonal flow generally results in significant spreading of the drift wave spectrum in radial wave number k_r,[15,16,20] so that the quasimonochromatic approximation may (at best) be valid only during the initial stage of this interaction, provided that the drift wave linear growth rate is consistent with such an approximation. Thus, taking the above motivations and the importance of the case of weak collisionality in which the zonal flow mechanism of transport suppression appears to be particularly efficient, one needs to perform the stability analysis for an essentially arbitrary drift wave spectrum. The threshold issue ought to be the major concern of such an analysis. This will be our main goal in the present paper.

In the next section we give a brief derivation of the dispersion equation for the zonal flow generation in the most straightforward case of the planar geometry (keeping, however, r, θ notations for mainly historical reasons), very much in line with, e.g., Ref. 17. Next, we evaluate a general criterion providing the necessary and, under some obvious constraints also, the sufficient condition for the instability of drift wave spectra. This will be used for the further analysis of a narrow and a broad spectra, which set the limits for the quasimonochromatic theory and, more importantly, addresses the issue of the validity of the drift wave–zonal flow self-regulation model in the case of small collisionality.

II. BASIC EQUATIONS

As a model for the description of the zonal flow–drift wave turbulence, it is convenient to adopt one of the magnetohydrodynamics reduction schemes, e.g., one suggested in Ref. 21. The final evolution equation reads[17,21] as

$$\left(\frac{\partial}{\partial t}+V_0\frac{\partial}{\partial \theta}\right)\widetilde{\phi}+V_*\frac{\partial \widetilde{\phi}}{\partial \theta}-\rho_s^2\left(\frac{\partial}{\partial t}+\mathbf{V_E}\nabla\right)\nabla_\perp^2\phi=0, \quad (1)$$

where ϕ is the electrostatic potential, that consists of the poloidally symmetric zonal flow part Φ and the drift wave part $\widetilde{\phi}$, $\phi=\Phi+\widetilde{\phi}$. Accordingly, V_0 is the zonal flow part of the $\mathbf{E\times B}$ drift, $V_0=(c/B)\partial\Phi/\partial r$, while $\mathbf{V_E}=(c/B)\mathbf{e}_\parallel\times\nabla\phi$. The remaining notations are also standard, $V_*=-(cT_e/eB)d\ln n_0/dr$, $\rho_s^2=T_e/M\omega_{ci}^2=c_s^2/\omega_{ci}^2$. Averaging

this equation over poloidal and toroidal directions, which will be denoted by $\langle\cdot\rangle$, and taking into account $\langle\widetilde{\phi}\rangle=0$, one obtains then for the zonal flow generation,

$$\frac{\partial \Phi}{\partial t}=\frac{c}{B}\left\langle\frac{\partial \widetilde{\phi}}{\partial r}\frac{\partial \widetilde{\phi}}{\partial \theta}\right\rangle. \quad (2)$$

After the *Ansatz* $\Phi\propto\exp(-i\Omega t+iqr)$ and assuming that $q\ll\Delta k_r$, where Δk_r denotes the width of the drift wave spectrum in k_r, the last equation transforms to

$$-i\Omega\Phi_q=\frac{c}{B}\int k_r k_\theta|\widetilde{\phi}_{\mathbf{k}}|^2d\mathbf{k}. \quad (3)$$

As explained in the Introduction, our primary goal is to study the limit of small-amplitude drift waves and vanishing linear instability drive as pertinent to the threshold phenomena of the zonal flow generation. Therefore, the perturbation to the drift wave spectrum caused by the emerging zonal flows may be described in terms of the eikonal equation of the weak turbulence,

$$\frac{\partial N_{\mathbf{k}}}{\partial t}+\frac{\partial}{\partial \mathbf{k}}(k_\theta V_0+\omega_{\mathbf{k}})\cdot\frac{\partial N_{\mathbf{k}}}{\partial \mathbf{x}}-\frac{\partial}{\partial \mathbf{x}}(k_\theta V_0+\omega_{\mathbf{k}})\cdot\frac{\partial N_{\mathbf{k}}}{\partial \mathbf{k}}=0, \quad (4)$$

where $N_{\mathbf{k}}=(1+k_\perp^2\rho_s^2)^2|e\widetilde{\phi}_{\mathbf{k}}/T_e|^2$ is the density of the drift wave number of quanta and $\omega_{\mathbf{k}}=k_\theta V_*(1+k_\perp^2\rho_s^2)^{-1}$ is their frequency. In this equation the drift wave self-nonlinearity $\sim N_{\mathbf{k}}^2$ is neglected along with the linear instability term for the reasons mentioned above, whereas the zonal flow–drift wave nonlinear terms ($\sim V_0 N_{\mathbf{k}}$) are retained and are sufficient for a description of the finite-amplitude zonal flows. The latter is because the zonal flow self-nonlinearity vanishes due to the symmetry of the flow ($\partial/\partial\theta=0$) and the ''vector'' character of nonlinear term in the underlying equation (1). The importance of a finite-amplitude treatment of zonal flow perturbations, even in the case of a ''weakly turbulent'' drift wave spectrum, is due to the possibility of their secular growth via Eq. (2) or by an existence of residual flows. We will discuss this last issue in the concluding section. Now we assume that the drift wave spectrum consists of an equilibrium part $N_{\mathbf{k}}^0$, subject to the stability analysis and a perturbed part $\widetilde{N}_{\mathbf{k}}=N_{\mathbf{k}}-N_{\mathbf{k}}^0$, for which, from Eq. (4), one obtains

$$(-i\Omega+iqV_{gr})\widetilde{N}_{\mathbf{k}}=-\frac{c}{B}k_\theta q^2\frac{\partial N_{\mathbf{k}}^0}{\partial k_r}\Phi_q, \quad (5)$$

where $V_{gr}=\partial\omega_{\mathbf{k}}/\partial k_r$. Note, since $N_{\mathbf{k}}^0$ is an equilibrium spectrum, according to Eq. (3) it must satisfy

$$\int\frac{N_{\mathbf{k}}^0}{(1+k_\perp^2\rho_s^2)^2}k_r k_\theta d\mathbf{k}=\mathbf{0}. \quad (6)$$

Substituting then $\widetilde{N}_{\mathbf{k}}$ from Eq. (5) into Eq. (3) one obtains the following dispersion equation for Ω;[15,17]

$$\Omega=q^2 c_s^2\rho_s^2\int\frac{k_r k_\theta^2}{(1+k_\perp^2\rho_s^2)^2}\frac{\partial N_{\mathbf{k}}^0}{\partial k_r}\frac{d\mathbf{k}}{\Omega-qV_{gr}}. \quad (7)$$

As we mentioned before, this equation was used, e.g., in Ref. 17 to analyze the stability of a monochromatic drift wave
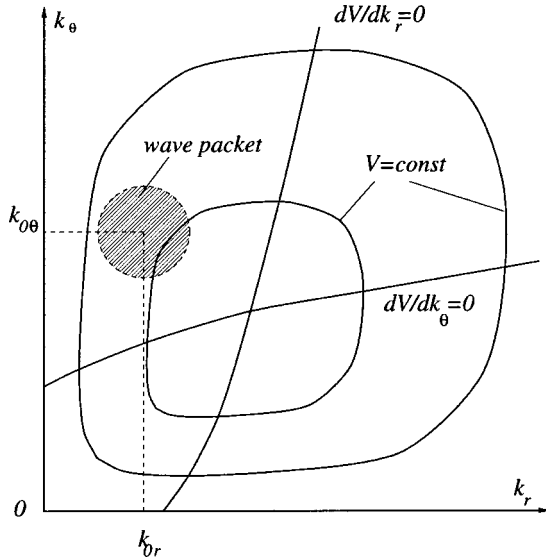
FIG. 1. The contours of the radial component of the group velocity V = const. The wave packet is located at $\mathbf{k}=\mathbf{k}_0$ on the k_r–k_θ plane. The special lines $\partial V/\partial k_r=0$ and $\partial V/\partial k_\theta=0$ cross at a point where $V(\mathbf{k})$ reaches its extremum $-V_*/4$.

$N_{\mathbf{k}}^0 \propto N_0 \delta(\mathbf{k}-\mathbf{k_0})$. In this case, the instability criterion was shown to be dependent only on $\mathbf{k}_0$ but not on N_0. To consider the case of a general spectrum $N_{\mathbf{k}}^0$, further evaluation of this dispersion relation is needed. This will be the subject of the next section.

III. INSTABILITY CRITERION

First, using the expression

$$V_{\mathrm{gr}}= -2\,\frac{V_* k_r k_\theta \rho_s^2}{(1+k_\perp^2 \rho_s^2)^2}, \tag{8}$$

we can re-write Eq. (7) to the following form.

$$1+\frac{qc_s^2}{2V_*}\int \frac{k_\theta d\mathbf{k}}{\Omega - qV_{\mathrm{gr}}}\frac{\partial N_{\mathbf{k}}^0}{\partial k_r}=0. \tag{9}$$

This clearly suggests the following variable transformation: $(k_r,k_\theta)\mapsto(V,k_\theta)$, where $V=V_{\mathrm{gr}}(\mathbf{k})$. In addition, we use the phase velocity $C=\Omega/q$ instead of Ω, which removes the parameter q from the dispersion equation. The latter then can be rewritten as

$$\Lambda(C)\equiv 1-\frac{c_s^2}{2V_*}\int_{-V_0}^{V_0}\frac{K'(V)dV}{V-C}=0, \tag{10}$$

where $V_0\equiv \max V_{\mathrm{gr}}(\mathbf{k})=|V_*|/4$ and

$$K(V)=\oint N^0(V,k_\theta)k_\theta dk_\theta. \tag{11}$$

The integral over k_θ is implied to be taken along the contours $V=$ const as shown in Fig. 1. The function $K(V)$ is a key to understanding the source of instability free energy. It constitutes the distribution of the azimuthal component of the drift wave momentum as a function of group velocity, which is in resonance with the phase velocity of the zonal flow perturbation. As we shall see, no net θ momentum of the drift

wave distribution is required for zonal flow generation. This is because no net momentum of the zonal flow is required i.e., no mean flow, only zonal shear needs to be generated.

The function $K(V)$ may be calculated in principle for any given distribution $N_{\mathbf{k}}^0$. Then, we need to find whether Eq. (10) has zeros in the upper half-plane of the complex variable C (with no loss of generality we may assume that $q>0$). This may be examined by using the standard Nyquist's method. First of all, the function $\Lambda(C)-1$ is a Cauchy-type integral which, for a ''reasonably good'' $K(V)$, is a holomorphic function of the degree -2 in the entire C plane cut along the interval $(-V_0,V_0)$ (it vanishes at $C\to\infty$ as $\sim C^{-2}$). It has a jump across this interval $\Delta\Lambda(C')=-\pi(c_s^2/V_*)K'(C')$, where $C=C'+iC''$. We consider a closed contour Γ in the upper half-plane of variable C consisting of a semicircle of infinite radius and of the real axis $C''\equiv\Im C=0+$ running in the positive direction. Within Γ, the function $\Lambda(C)$ has no singularities, and the number of zeros that $\Lambda(C)$ has there is equal to the number of rotations that its image $\hat{\Gamma}=\Lambda(\Gamma)$ on the Λ plane makes around the origin $\Lambda=0$. The contour $\hat{\Gamma}$ starts and ends at $\Lambda=1$ ($C=\infty$) and runs also in the positive direction. It makes no rotations around the origin if at every point $C'=C_r$ on the real axis of the C plane where $\Im\Lambda(C_r)=0$ and $\Im\Lambda'(C_r)<0$, the condition $\Re\Lambda(C_r)>0$ is met. Rewriting Eq. (10) on the real axis as

$$\Lambda(C')\equiv 1-\frac{c_s^2}{2V*}\mathcal{P}\int_{-V_0}^{V_0}\frac{K'(V)dV}{V-C'}-i\pi\frac{c_s^2}{2V_*}K'(C'), \tag{12}$$

we can formulate the sufficient stability condition as follows. If at all points where $K'(C'')=0$ and $K''(C'')>0$, the inequality

$$\frac{c_s^2}{2V*}\int_{-V_0}^{V_0}\frac{K'(V)dV}{V-C'}<1 \tag{13}$$

holds, then the distribution $N_0(\mathbf{k})$ in (11) is stable.

In contrast to a formally very similar condition for the stability of plasma velocity distributions (see, e.g., Ref. 22), the condition opposite to (13) does not lead automatically to the instability. This is because Eq. (10) does not contain any free parameter of the stability problem, like the wave number. Thus, the situation in which there is another point where the contour $\hat{\Gamma}$ crosses the real axis, $C'=C_{r2}$, $\Im\Lambda=0$ but $\Im\Lambda'(C_{r2})>0$ and $\Re\Lambda<0$ cannot be excluded by varying the free parameter. Clearly, in this situation the point $\Lambda=0$ may not be within $\hat{\Gamma}$. Nevertheless, the condition opposite to (13) can be shown to be sufficient for instability if the function $K(V)$ has not too many extrema (specified later) within the interval $(-V_0,V_0)$. Fortunately, this is the case for practically all interesting spectral shapes $N_{\mathbf{k}}^0$.

In order to turn the above necessary condition into a sufficient one, it is natural to start with the constraint (6). For practical purposes we confine our consideration to spectra that are symmetric in $k_r:N^0(k_r,k_\theta)=N^0(-k_r,k_\theta)$ or $k_\theta:N^0(k_r,k_\theta)=N^0(k_r,-k_\theta)$, or in both. Any of these conditions automatically satisfies Eq. (6), as well. In addi-
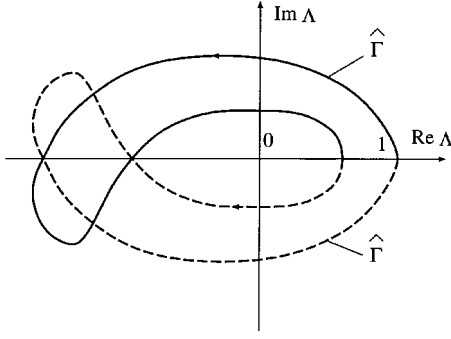
FIG. 2. The image of the contour Γ on the Λ plane. The solid curve maps the negative half-axis $C'<0$ while the complex conjugate part (dashed curve) maps the positive half-axis $C'>0$ on the Λ plane.

tion, the function $K(V)$ becomes symmetric, $K(-V)=K(V)$. According to Eq. (13), this means that the contour $\hat{\Gamma}$ is symmetric with respect to the real axis of the Λ plane since the $\hat{\Gamma}$ images of the half-axes $-\infty<C'<0$ and $0<C'<\infty$ are complex conjugates on the Λ plane, Fig. 2. Therefore, the situation in which the contour $\hat{\Gamma}$ crosses the real axis at $\Re\Lambda<0$ (necessary for the instability) and then recrosses it again at $\Re\Lambda<0$, so as to leave the origin outside it, may only occur when the total number of crossings [i.e., the number of extrema of $K(V)$ is five or more (note that this number must be odd by symmetry)]. In the case of only one or three extrema, the condition opposite to (13) guarantees instability. We can reformulate it as follows. If the symmetric function $K(V)$ has less than five extrema in the interval $(-V_0,V_0)$ and at least at one point C' where $K'(C')=0$,

$$\frac{c_s^2}{V*}\int_0^{V_0}\frac{VK'(V)dV}{V^2-C'^2}>1, \tag{14}$$

the distribution N^0 is unstable. Clearly, the extremum of $K(V)$ at $V=C'$ must be a minimum. In the case of larger number of extrema one can also formulate the sufficient instability conditions involving some obvious relations between integrals (14) taken at different points C' where the function K reaches its extrema. However, these are not very practical, so that in the cases of sufficiently complicated function $K(V)$, direct plotting of $\hat{\Gamma}$ using Eq. (12) should be more efficient. Moreover, in the next two sections, we consider the cases of a narrow and a broad wave packet for which the form of instability condition given by Eq. (14) is appropriate.

A. Instability of a narrow wave packet

To be specific, let us consider the case of a narrow Gaussian wave packet,

$$N^0(\mathbf{k})=\frac{N_0}{2\pi\Delta k^2}\left[\exp\left(-\frac{|\mathbf{k}-\mathbf{k}_0|^2}{\Delta k^2}\right)+\exp\left(-\frac{|\mathbf{k}-\mathbf{k}_1|^2}{\Delta k^2}\right)\right],$$

where $k_{1r}=-k_{0r}$ but $k_{1\theta}=k_{0\theta}$ to guarantee the stationarity condition (6). Here we normalize the wave number k to ρ_s^{-1}. The function $K(V)$ has two minima at $V=\pm V(\mathbf{k}_0)$ and a maximum at $V=0$ (by symmetry). Assuming $\Delta k \ll k_0$ we can expand $V(\mathbf{k})$ at $\mathbf{k}=\mathbf{k}_0$,

$$V(\mathbf{k})=V(\mathbf{k}_0)+\left.\frac{\partial V}{\partial \mathbf{k}}\right|_{\mathbf{k}=\mathbf{k}_0}\cdot(\mathbf{k}-\mathbf{k}_0). \tag{15}$$

Clearly, the wave packet should be separated from the caustic $\partial V/\partial k_r\equiv\partial^2\omega/\partial k_r^2=0$ by at least Δk. Otherwise, the quadratic term in $\mathbf{k}-\mathbf{k}_0$ needs to be included. Expressing k_r through V and k_θ and substituting into Eq. (11), we have

$$K(V)=\frac{V_r}{\sqrt{V_r^2+V_\theta^2}}\frac{k_{0\theta}N_0}{\sqrt{\pi}\Delta k}\exp\left(-\frac{[V-V(\mathbf{k}_0)]^2}{(V_r^2+V_\theta^2)\Delta k^2}\right). \tag{16}$$

We have denoted $V_{r,\theta}\equiv\partial V/\partial k_{r,\theta}$ at $\mathbf{k}=\mathbf{k}_0$. Substituting the last expression into the instability condition (14), we obtain

$$N_0\frac{(1+k_0^2)^3}{\Delta k^2}\frac{k_{0\theta}^2(1+k_{0\theta}^2-3k_{0r}^2)}{k_{0\theta}^2(1+k_{0\theta}^2-3k_{0r}^2)^2+k_{0r}^2(1+k_{0r}^2-3k_{0\theta}^2)^2}$$
$$>2\frac{\rho_s^2}{L_n^2}, \tag{17}$$

as a necessary and sufficient condition for the instability, where $L_n^{-1}=d\ln n_0/dr$.

In agreement with the monochromatic case analyzed in Ref. 17, the instability indeed requires wave localization for which k_r^2 is smaller than its value on the caustic $\partial V/\partial k_r=0$, i.e., $k_r^2<(1+k_\theta^2)/3$. However, there is also an amplitude threshold of the order of $N_{\rm th}\sim(\rho_s/L_n)^2\Delta k^2\rho_s^2$ for the onset of modulational instability of a packet that has a finite width Δk.

B. Instability of a broad wave packet

Let $\Delta k\gg1$ and normalize N as to keep the total number of quanta the same, as already considered for the case of a narrow packet. Then

$$K(0)=-2\int_0^\infty N(k_r=0,k_\theta)k_\theta dk_\theta\equiv-N_0.$$

In the case of a broad distribution $N^0(\mathbf{k})$ the function $K(V)$ increases monotonically from its minimum value $-N_0$ at $V=0$ to 0 at $V=V_0$. For the sake of simplicity we approximate $K(V)$ as $K(V)=K(0)(1-V^2/V_0^2)$. Substituting this into the instability condition (14) we finally obtain

$$N_0>\frac{\rho_s^2}{8L_n^2}. \tag{18}$$

Note that this instability condition does not formally depend on the width or the anisotropy of the distribution. However, an approximate isotropy of the spectrum has already been assumed by its derivation. According to Eq. (17) spectra elongated in k_r would generally require higher amplitudes to become unstable.

IV. CONCLUSIONS AND DISCUSSION

We have considered the modulational stability of a drift wave gas with respect to the zonal flow generation. The derived instability criterion indicates that narrow spectra $(\Delta k\rho_s\ll1)$, in order to be unstable must be localized within the caustics in k_r-k_θ plane, i.e., $k_r^2\rho_s^2<(1+k_\theta^2\rho_s^2)/3$, in

agreement with the monochromatic case.[17] Broad spectra, if not extended significantly in k_r, appear to be generically unstable as far as their spectral shape is concerned. However, in both cases there exists a significant threshold for the zonal flow excitation. The characteristic amplitude for drift wave threshold in terms of the number of quanta, is, in both cases, of the order of ρ_s^2/L_n^2, although narrow packets are excited more easily due to the additional threshold reduction by a factor of the order of $\Delta k^2 \rho_s^2 \ll 1$.

The back-reaction from the zonal flow on the parent drift waves is clearly stabilizing. In essence, it can be characterized by shearing[15,16] and thus by the diffusive spreading of the drift wave spectrum in k_r. A key for understanding of the stabilization is the function $K(V)$ that was identified as the azimuthal momentum distribution of the drift wave gas expressed in terms of its group velocity. Since shearing generally transports plasmons across the line $\partial V/\partial k_r = 0$ to larger k_r, the positive contribution to the $K(V)$ integral (11) from the larger k_r will increase, while the negative contribution from smaller k_r will decrease. This should result in the stabilization of modulational instability according to the criterion (14). Physically, this process implies a systematic decrease of the drift wave energy, since thereby the wave frequency decreases, subject to the constraint of the number of quanta held constant.

It should be remembered that the above consideration, as well as the treatment of the incoherent zonal flow generation in the accompanying paper,[16] focus mainly on stationary solutions and their stability. Nevertheless, our results may be used (with some caution) for considering time dependent evolution of coupled drift wave–zonal flow turbulence. The main merit of these analytical results is an explicit identification of critical parameters, their functional interrelations, and the capacity to scan the system behavior over a broad range in a multidimensional parameter space. What is particularly pertinent to the subject of the present paper is the case of the small and vanishing growth rate of the drift wave instability and collisional damping of the zonal flow. The importance of this regime is suggested by the recent result of Refs. 23 and 24, which demonstrated an asymptotic irrelevance of collisionless zonal flow damping. Let us consider first the case in which the latter is zero ($\gamma_d = 0$).

Based on the above discussion of the stabilization mechanism, one can expect that an initially unstable drift wave spectrum will relax to a quasiequilibrium state near the threshold of modulational instability due to diffusion in k_r by self-generated zonal flows. (This is quite similar to the quasilinear relaxation of unstable particle distributions, like ''beam'' or ''loss-cone'' instabilities.) Zonal flows will become thus marginally stable, but they still will maintain the diffusive flux (since they do not decay, $\gamma_d = 0$) of linearly unstable (e.g., due to the ion temperature gradient) drift waves to larger k_r, where they can be linearly absorbed by particles. For this steady state to exist, the linear growth rate of the drift waves γ_l must still be larger than some critical level, $\gamma_l \propto \nabla T - \nabla T_c > \gamma_{cr}$. Indeed, if we decrease γ_l below critical ($\gamma_l < \gamma_{cr}$, keeping $\gamma_d = 0$) the shearing damping rate of the drift waves by the undamped zonal flows will exceed their linear production rate, and ultimately eliminate them,

while the residual zonal flow will remain. This critical growth rate may be calculated from the quasilinear equation describing the evolution of the drift wave spectrum,[15]

$$\frac{\partial N_{\mathbf{k}}^0}{\partial t} - \frac{\partial}{\partial k_r} D_{\mathbf{k}} \frac{\partial N_{\mathbf{k}}^0}{\partial k_r} = \gamma_{\mathbf{k}} N_{\mathbf{k}}^0, \qquad (19)$$

where D is the diffusivity of drift waves by zonal flow:

$$D_{\mathbf{k}} = \sum_q k_\theta^2 q^4 \rho_s^4 c_s^2 \left(1 - \frac{q^2 \rho^2}{1 + k_\perp^2 \rho_s^2} \right)^2 \delta(\Omega_q - q V_{gr}) |\Phi_q|^2. \qquad (20)$$

This equation essentially implies that for a steady state $\gamma_{cr} \sim D_{\mathbf{k}}/\Delta k^2$ [see Eq. (17) of Ref. 16 for a more rigorous WKB result], which relates γ_l with the level of the residual zonal flow. In other words, if the actual $\gamma_l < \gamma_{cr}$, the steady state cannot be maintained and the drift waves are damped by shearing. Thus, if we now formally set $\gamma_l \to 0$, there should be extremely weak drift wave fluctuations with only the residual zonal flow. Such a situation may have at least qualitatively something to do with that observed in numerical simulations by Dimits et al.[7] and commonly referred to as the ''Dimits shifts'' regime. It was identified with an offset in γ_l by γ_{cr} on the $Q - \gamma$ diagram, where Q is the radial heat flux that is presumably proportional to the drift wave turbulence level N_k^0. For $0 < \gamma_l < \gamma_{cr}$, a state of zonal flows with negligible transport is observed, while for $\gamma_l > \gamma_{cr}$, heat flux Q increases. Note that in our simplified treatment the quantitative determination of γ_{cr} (i.e., residual zonal flow amplitude) would require a time dependent solution for the residual, undamped zonal flow specified during its generation phase, so that γ_{cr} will depend on the prehistory of the turbulence evolution to this steady state. In particular, the above scenario requires that the zonal flow is already generated in some way and then prevents the drift waves from growing above the threshold. This can be achieved, e.g., if an initially narrow, drift wave packet is spread in k_r upon generating zonal flows thus rising both the threshold [by the factor $(\Delta k \rho_s)^{-2} \gg 1$] and γ_{cr}. The state established in this way may now very well meet the above conditions, namely $\gamma_l < \gamma_{cr}$ and $N_{\mathbf{k}}^0 < N_{th}$ so that the drift waves will be damped via Eq. (19) while the zonal flow will sustain (if $\gamma_d = 0$). Otherwise (M. N. Rosenbluth, private communication), if we had started from a broad wave packet with a sufficiently high threshold of zonal flow generation, the latter could have never been excited due to the strong drift wave damping caused by their tree mode coupling.

It should be stressed that the above consideration is irrelevant to parameter regimes in which both the linear driver γ_l and collisional damping γ_d are sufficiently strong (well above the threshold) so that a robust steady state with the presence of both turbulence components is established. For yet higher γ_d, however, its uniqueness is examined in the accompanying paper,[16] where the possibility of a further bifurcation is demonstrated.

Another point to emphasize is that the order of the limits (first $\gamma_d \to 0$ then $\gamma_l \to 0$) is clearly important here. Indeed, if we first fix small $\gamma_d > \gamma_{dcr} > 0$ and let $\gamma_l \to 0$, then after the drift waves are sheared down to the threshold level N_{th}
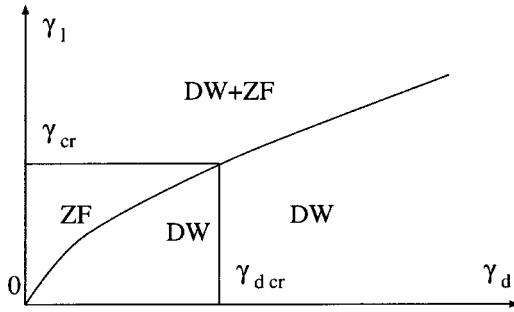
FIG. 3. The diagram, indicating the regions of residual drift wave, zonal flow turbulence as well as the region where they coexist on the $\gamma_d - \gamma_l$ plane.

$\sim \rho_s^2 / L_n^2$ and stop generating the zonal flows, the latter should be damped as well (γ_d-finite), producing no more shearing while residual drift wave turbulence may survive. To realize this scenario we need the zonal flows to be damped faster than the shearing rate $\gamma_d > \gamma_{dcr} \sim D/\Delta k^2$ since $D/\Delta k^2$ is the rate at which the drift waves themselves are indirectly damped by zonal flows. Notwithstanding the rather semiqualitative character of the above arguments, they clearly point at the intrinsic incompleteness of any two-dimensional (2D) (Q vs γ_l) study of the ''shift'' phenomena. The onset of the turbulent transport at small γ_l and γ_d (collisionality) can be fully understood only if Q is studied as a function of both (γ_d and $\nabla T - \nabla T_c$) parameters on an equal basis.

If we leave aside these rather peculiar limiting cases, the emerging steady-state picture does not differ in general from that considered in Refs. 15 and 16 (where the self-nonlinearity was also included into the drift wave balance), which was dictated by the consideration of reasonably large assumed γ_d. This resulted in a linear scaling of the drift wave turbulence level with the collisional damping of zonal flows, which implied the turbulence vanishes for the un-damped zonal flows and thus entered the regime where the self-nonlinearity of the drift waves can be neglected. Motivated by this, in the present paper we have realized that the behavior of the coupled drift wave–zonal flow turbulence, in the parameter region where the level of the both is small, may generally have a hysteretic character and the residual turbulence may remain in either component, depending on how we approach the origin $\gamma_l = \gamma_d = 0$, as described above. This is schematically depicted in Fig. 3.

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