Comment on C. W. Wong, Maxwell equations and the redundant gauge degree of freedom 2009 *Eur. J. Phys.* **30** 1401-1416

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(Dated: 23 October 2009)

Abstract

In the paper cited in the title, the author makes the claim that in classical electromagnetic theory the longitudinal electric field is instantaneous, corresponding to action at a distance, contrary to popular and correct belief. We point out that the determination of the speed of propagation of electromagnetic fields requires specification of the initial condition of the sources or equivalent. The Coulomb field of a stationary point charge proves nothing. We describe in detail a simple example to illustrate the universal onset of the static "instantaneous" regime throughout a region of space that expands with the speed of light.

In a recent paper in this journal[1] Chun Wa Wong argued that the longitudinal electric field E_{long}, whose curl vanishes, propagates instantaneously. In his introduction he states, "There is then always a longitudinal direction along which the magnetic induction vanishes, and the electric field component is instantaneous, acting at a distance." Later (In Sect. 5) he says, "Our conclusion in its simplest form is that the electrostatic potential of a stationary charge in vacuum is just the instantaneous Coulomb potential. So the electric field $\mathbf{E}$ generated from it is also instantaneous, acting at a distance."

In fact, the speed of propagation of the longitudinal electric field (or any other electromagnetic field) of a stationary distribution of charge or current cannot be deduced without considering the initial conditions. Perhaps his stationary charge was placed there a nanosecond ago. If he looks in his lab at the position of the charge from more than 30 cm away (forget about the "looking" involving light or other electromagnetic detection that has a finite speed of propagation) he will see nothing. But if he waits a split-second he will see his static longitudinal Coulomb field. If he had looked continuously he would have seen something more interesting - first, nothing; then complication; finally, simplicity.

For any localized distribution of sources created in vacuum at time $t = T$, the associated fields are confined to within a sphere of radius $r = c(t - T)$, where c is the speed of light and r is measured from the location of the sources. At early times the so-called static, induction, and radiation fields (and zones)[2] are mixed together, but as time goes on the component fields become recognizable through their different radial dependences. If the created sources are continuously time-varying, the three zones will continue to be populated by fields; if the sources are static for $t > T$, the induction and radiation fields move off to infinity at finite speed, leaving behind the static fields.

Consider the simple example of the creation of a subsequently static electric dipole at the origin at time $t = 0$. The dipole of strength p points in the z-direction; its dimensions are assumed infinitesimal on the length scale of interest. We could have considered the creation of charges $\pm q$, that moved away from each other a finite distance to final positions in a finite time, but the algebra is much messier and the conclusion no different.

We perform the calculation first by finding the retarded scalar and vector potentials. We use Gaussian units. We need the charge and current densities of the electric dipole:

$$\rho(\mathbf{r},t) \;=\; -\mathbf{p}\cdot\boldsymbol{\nabla}\delta(\mathbf{r})\Theta(t), \tag{1}$$

where $\Theta(t)$ is the Heaviside step function, and

$$\mathbf{J}(\mathbf{r},t) \;=\; \mathbf{p}\delta(\mathbf{r})\delta(t) \tag{2}$$

The formal expressions for the scalar and vector potentials are

$$\Phi(\mathbf{r},t) \;=\; \int d^3x' \frac{\rho(\mathbf{r}',\ t')}{R}|_{t'=t-R/c} \tag{3}$$

$$\mathbf{A}(\mathbf{r},t) \;=\; \int d^3x' \frac{\mathbf{J}(\mathbf{r}',t')}{c\ R}\ |_{t'=t-R/c} \tag{4}$$

The results of the integration with the sources of Eqs.(1) and (2) for arbitrary orientation of the dipole are

$$\Phi(\mathbf{r},t) \;=\; \frac{\mathbf{p}\cdot\mathbf{r}}{r^2}[\ \delta(ct-r)\ +\ \frac{1}{r}\Theta(ct-r)\ ] \tag{5}$$

$$\mathbf{A}(\mathbf{r},t) \;=\; \frac{\mathbf{p}}{r}\delta(ct-r) \tag{6}$$

Already we see that the potentials are confined to the interior or surface of the causality sphere, $r = ct$.

The fields are found from the familiar expressions in terms of the potentials to be

$$\mathbf{E}(\mathbf{r},t) \;=\; \frac{(3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p})}{r^3}[\ \Theta(ct-r)\ +\ r\ \delta(r-ct)\ ]+\frac{(\mathbf{p}-(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}})}{r}\ \delta'(r-ct) \tag{7}$$

$$\mathbf{B}(\mathbf{r},t) \;=\; \frac{\hat{\mathbf{r}}\times\mathbf{p}}{r^2}[\ -\delta(r-ct)+r\ \delta'(r-ct)] \tag{8}$$

Here the prime on a delta function means differentiation with respect to its argument.

The electric field exhibits the three components of the field, the static dipole field with its $1/r^3$ dependence occupying the interior of the causality sphere, the induction field falling off as $1/r^2$ and the transverse $1/r$ radiation field, both existing only on the surface of the causality sphere, $r = ct$. The magnetic induction has no static part, but only the induction and radiation terms, similarly existing only on the surface of the causality sphere.

Evidently, all components of the electric and magnetic field propagate with the speed of light. At any fixed distance $r = r_1$ from the dipole, after the time $t = t_1 = r_1/c$, all that will be observed is the static longitudinal field obtained solely from the Θ function term in the scalar potential. But this is no evidence for the instantaneous propagation of the longitudinal field or its action at a distance.

Acknowledgements

I thank Vladimir Hnizdo for drawing my attention to ref. 1. This work was supported by the Director, Office of Science, High Energy Physics, U.S. Department of Energy under Contract No. DE-AC02-05CH11231.

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References

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[2] Jackson J D 1998 *Classical Electrodynamics* 3rd edn (New York: Wiley), p.408.