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ANALYSIS OF STATISTICAL ERRORS FOR TRANSVERSE SECTION RECONSTRUCTION

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Analysis of Statistical Errors for Transverse Section Reconstruction

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ABSTRACT. The relationship between expected errors in the reconstruction of a transverse section from projections at multiple angles and the statistical errors of the projections is derived: $\sigma_\rho = \sigma_I \sqrt{(1.6D)/(nd^3)}$, where σ_ρ is the rms error of the reconstruction, σ_I is the rms error of the projections, n is the number of projection data points, D is the linear dimension of the reconstruction region and d is the linear dimension of the reconstructed cells into which the reconstruction region is subdivided (resolution length). The results are applicable to x-ray or nuclear particle transmission as well as radioisotope emission studies.

1. Introduction

Techniques for reconstruction of the 3-dimensional distribution of density in an object from projections at multiple angles have received extensive investigation and review, particularly for x-ray transmission (Gordon and Herman 1973) and photon emission from radioisotopes (Budinger and Gullberg 1974). However, the important problem of noise propagation and the analysis of the expected uncertainty in a reconstruction has only recently come under investigation (Chesler 1975, Barrett et. al. 1975). The present work provides a quantitative basis for estimating errors in a transverse section reconstructed from multiple projections. The results are applicable to x-ray or nuclear particle transmission as well as nuclear medicine image reconstruction.

The circular area to be reconstructed has diameter D, and is subdivided into small square cells of dimensions d x d. The area to be reconstructed consists of $(\pi/4)\ (D/d)^2$ cells, each of which is assumed to contain uniform density. The data consist of a collection of line integrals of the density over n coplanar paths through the transverse section. The paths traverse the reconstruction region at regularly spaced intervals and at regularly spaced angles. The geometry of the reconstruction region and an example of the paths for line integrals at one angle, θ, are shown in fig. 1. For the model chosen (uniform density in each cell) the relationship between the line integral I_i and the density, ρ_j, in the j^{th} cell is,

$$I_i = \sum_j \ell_{ij}\, \rho_j \qquad (1)$$

where ℓ_{ij} is the line length of the i^{th} path through the j^{th} cell, as shown in the inset of fig. 1. (ℓ_{ij} is zero if the i^{th} path does not intersect the j^{th} cell.)

2. Reconstruction and Error Analysis

The backprojection (simple superposition image) B_k, for the k^{th} cell is defined to be the line integral times the line length through that cell summed over the n paths,

$$B_k = \frac{D}{nd} \sum_i \frac{I_i}{d} \frac{\ell_{ik}}{d} = \frac{D}{nd^3} \sum_i I_i \ell_{ik} \tag{2}$$

The choice of the normalization factor $D/(nd^3)$ is explained below.

By substitution of eqn (1) into eqn (2), the backprojection can be written in terms of the density as,

$$B_k = \frac{D}{nd^3} \sum_{ij} \ell_{ik} \ell_{ij} \rho_j \tag{3}$$

Defining the matrix M by the expression,

$$M_{kj} = \frac{D}{nd^3} \sum_i \ell_{ik} \ell_{ij} \tag{4}$$

and substituting into eqn (3), gives,

$$B_k = \sum_j M_{kj} \rho_j \tag{5}$$

so that the backprojection vector is just the density vector multiplied by the matrix M. A diagonal element of M is given by,

$$M_{jj} = \frac{D}{nd^3} \sum_i \ell_{ij}^2 \tag{6}$$

but the fraction of line lengths which are non-zero is about d/D (only about nd/D of the line integrals intersect the j^{th} cell) and when non-zero ℓ_{ij} is about equal to d (the linear dimension of a cell) so that

$$M_{jj} \simeq \frac{D}{nd^3} \frac{nd}{D} d^2 = 1 \tag{7}$$

Thus the normalization factor $D/(nd^3)$ makes the matrix M roughly independent of the number of line integrals and the geometry of the reconstruction region.

The backprojection is just a matrix multiplication with the density vector, and if the problem is well posed, the density vector can be obtained after a matrix inversion. Since the matrix M may be very large, inversion is usually impractical. This problem is put aside for the moment and returned to below.

In order to investigate the uncertainty in the reconstruction, the density vector is expressed in terms of the inverse of the matrix M as,

$$\begin{aligned} \rho_j &= \sum_k M^{-1}_{jk} B_k = \sum_k M^{-1}_{jk} \frac{D}{nd^3} \sum_i I_i \ell_{ik} \\ &= \frac{D}{nd^3} \sum_i I_i \sum_k M^{-1}_{jk} \ell_{ik} \end{aligned} \tag{8}$$

Since the rms errors of different line integrals are uncorrelated, the rms error of ρ_j is the sum of the rms errors of the contributions from each I_i added in quadurature,

$$\begin{aligned} \sigma^2(\rho_j) &= \left(\frac{D}{nd^3}\right)^2 \sum_i \sigma^2(I_i) \left(\sum_k M^{-1}_{jk} \ell_{ik}\right)^2 \\ &= \left(\frac{D}{nd^3}\right)^2 \sum_i \sigma^2(I_i) \sum_{km} M^{-1}_{jk} M^{-1}_{jm} \ell_{ik} \ell_{im} \end{aligned} \tag{9}$$

where $\sigma(\rho_j)$ and $\sigma(I_i)$ are the rms errors of ρ_j and I_i respectively. If the rms errors of all I_i are equal to σ_I,

$$\sigma^2(\rho_j) = \left(\frac{D\sigma_I}{nd^3}\right)^2 \sum_{km} M^{-1}_{jk} M^{-1}_{jm} \sum_i \ell_{ik} \ell_{im} \tag{10}$$

and substitution from eqn (4) gives,

$$\sigma^2(\rho_j) = \frac{D\sigma_I^2}{nd^3} \sum_{km} M_{jk}^{-1} M_{jm}^{-1} M_{km} = \sigma_I^2 \frac{D}{nd^3} M_{jj}^{-1} \qquad (11)$$

It is well known that the operation of backprojection is simply a convolution with the function 1/r (Budinger and Gullberg 1974). Therefore the diagonal elements of the matrix M are equal, and the off-diagonal elements decrease proportional to the reciprocal of the distance between cells. That is, M_{jk} is proportional to the reciprocal of the distance between the k^{th} and j^{th} cells. Since the matrix is in practice very large, an attempt has been made to find an approximation to M^{-1} which is also a convolution but which is limited in extent. M_{jk}^{-1} has been set equal to zero when the j^{th} and k^{th} cells are greater than a specified distance apart, and the remaining M_{jk}^{-1} which best satisfy the relationship,

$$\sum_i M_{ji}^{-1} M_{ik} = \delta_{jk} \qquad (12)$$

in the least squares sense have been found, where δ_{jk} is the Kroneker delta. For all ranges of non-zero M_{jk}^{-1} tried, the diagonal element (central element of the convolution) has remained stable and is equal to 1.6.

A second approach to finding the convolution M^{-1} was also tried. In this approach the 2-dimensional Fourier convolution theorem was used. In our case the theorem states,

$$F_2(B) = F_2(M*\rho) = F_2(M)\ F_2(\rho) \qquad (13)$$

where F_2 indicates 2-dimensional Fourier transformation and * indicates convolution. Solving for $F_2(\rho)$,

$$F_2(\rho) = \frac{F_2(B)}{F_2(M)} = F_2(M^{-1}*B) = F_2(M^{-1})F_2(B) \qquad (14)$$

and the desired convolution is,

$$M^{-1} = F_2^{-1} \left(\frac{1}{F_2(M)} \right) \tag{15}$$

where F_2^{-1} is the inverse Fourier transformation. With this approach the central element of the convolution M^{-1} was also found to be equal to 1.6.

Setting M_{jj}^{-1} equal to 1.6. in eqn (11) gives,

$$\sigma^2(\rho_j) = \frac{1.6D}{nd^3} \sigma_I^2 \tag{16}$$

or,

$$\sigma_\rho = \sqrt{\frac{1.6D}{nd^3}} \sigma_I \tag{17}$$

where σ_ρ is the rms error of all reconstructed cell densities, since there is no j-dependence on the right-hand side of eqn (16).

This derivation assumes that the uncertainty of all measured line integrals are equal. This is not a serious drawback, since a practical estimate can be made using eqn (17) when this assumption is not true.

Although the logic of the derivation follows a particular reconstruction technique (convolution of the backprojection) the result is more general. The result is also applicable to the conventional convolution technique, as it has been shown that the two linear operators are equivalent (Budinger and Gullberg 1975).

Finally, it must be stressed that eqn (17) gives errors of the reconstruction due only to statistical errors of the line integrals. Systematic errors due to the particular reconstruction technique used are not treated here.

3. X-Ray Transmission

For an x-ray transmission device it is assumed that a water bath surrounds the reconstruction region and that the object to be reconstructed has approximately the same linear attenuation coefficient as that of water. This situation is depicted in fig. 2, and it can be seen that the total path length for the transmissions of x-rays along any of the n paths is approximately equal to a distance L of water.

In this case the function ρ to be reconstructed is the linear attenuation coefficient, usually denoted by μ. N_o x-rays are injected along the i^{th} path, and N_i, the number emerging from the opposite side of the water bath is counted. The expected value for N_i is given by

$$N_i = N_o \, e^{-\sum_j \ell_{ij} \rho_j} = N_o \, e^{-I_i} \qquad (18)$$

so that,

$$I_i = \log \left(\frac{N_o}{N_i} \right) \qquad (19)$$

There are several contributions to the rms error of I_i stemming from the uncertainties of both N_o and N_i. For the purposes of this example the uncertainty of N_o is neglected, as it is usually small. It is also assumed the uncertainty of N_i is purely statistical [$\sigma(N_i) = \sqrt{N_i}$] so that the rms error of I_i is,

$$\sigma(I_i) = \left| \frac{\partial I_i}{\partial N_i} \right| \sigma(N_i) = \frac{\sigma(N_i)}{N_i} = \frac{1}{\sqrt{N_i}} \qquad (20)$$

Since the n paths all traverse about an effective distance L of water, each N_i is approximately equal to $N_o \, e^{-\rho_o L}$, where ρ_o is the linear attenuation coefficient of water. Therefore the i-subscripts may be dropped and,

$$\sigma_I = \frac{1}{\sqrt{N}} \qquad (21)$$

Substituting eqn (21) into eqn (17) gives,

$$\sigma_\rho = \sqrt{\frac{1.6D}{Nnd^3}} \tag{22}$$

and the relative error is given by,

$$\frac{\sigma_\rho}{\rho_o} = \frac{1}{\rho_o}\sqrt{\frac{1.6D}{Nnd^3}} \tag{23}$$

A simplified version of eqn (23) emerges if the thickness of the water bath is equal to the diameter of the reconstruction region and the number of line integrals is equal to the square of the number of cells across the region, i.e.,

$$L = D \tag{24a}$$

$$n = \left(\frac{D}{d}\right)^2 \tag{24b}$$

For this special case, substitution of eqns (24) into eqn (23) gives,

$$\frac{\sigma_\rho}{\rho_o} = \frac{1}{\rho_o D}\sqrt{\frac{1.6D}{Nd}} \tag{25}$$

Using eqn (21) and the assumption that $I \simeq \rho_o L = \rho_o D$, eqn (25) can be further simplified to,

$$\frac{\sigma_\rho}{\rho_o} = \frac{\sigma_I}{I}\sqrt{\frac{1.6D}{d}} \tag{26}$$

which gives the relative error of the reconstruction of the relative error of the line integrals. For example assume that D = L = 27 cm., d = .15 cm (D/d = 180), $n = (180)^2$, $\rho_o = .19\ cm^{-1}$ and $N_i = N = 10^5$. Then,

$$\frac{\sigma_I}{I} = \frac{1}{\rho_o D\sqrt{N}} = \frac{1}{(.19)(27)\sqrt{10^5}} = 6\text{x}10^{-4} \tag{27}$$

and

$$\frac{\sigma_\rho}{\rho_o} = (6\text{x}10^{-4}) \sqrt{(1.6)\ (180)} = 10^{-2} \tag{28}$$

4. Gamma-Ray Emission

For the case of gamma emission, the assumption of equal line integral errors is not valid, however eqn (17) may be used to obtain a practical estimate of the reconstruction errors by making an approximation to σ_I. Here the function ρ to be reconstructed has the dimensions of counts per unit area.

If the total number of counts detected is Q, the average number used to evaluate each I_i is Q/n. (n = total number of line integrals) Since a line integral is proportional to the number of counts used to evaluate it, its relative rms error is equal to,

$$\frac{\sigma_I}{I} = \frac{1}{\sqrt{Q/n}} = \sqrt{\frac{n}{Q}} \tag{29}$$

so that,

$$\sigma_I = I \sqrt{\frac{n}{Q}} \tag{30}$$

Assuming that ρ is approximately uniform over the reconstruction region,

$$\rho \simeq \frac{Q}{\pi D^2/4} = \frac{4Q}{\pi D^2} \tag{31}$$

Since the average distance across a disk of diameter D is $\pi D/4$, the average line integral is equal to,

$$I \simeq \frac{\pi D \rho}{4} \simeq \frac{\pi D}{4} \frac{4Q}{\pi D^2} = \frac{Q}{D} \tag{32}$$

so that,

$$\sigma_I \simeq \frac{Q}{D}\sqrt{\frac{n}{Q}} = \frac{\sqrt{nQ}}{D} \tag{33}$$

Substituting eqn (33) into eqn (17) gives,

$$\sigma_\rho \simeq \sqrt{\frac{1.6D}{nd^3}}\;\frac{\sqrt{nQ}}{D} = \sqrt{\frac{1.6Q}{Dd^3}} \tag{34}$$

and using eqn (31) the relative rms error is given by,

$$\frac{\sigma_\rho}{\rho} \simeq \frac{\pi D^2}{4Q}\;\sigma_\rho = \frac{\pi}{4}\sqrt{\frac{1.6D^3}{Qd^3}} \tag{35}$$

Defining q as the number of counts per cell,

$$q = \rho d^2 = \frac{4Qd^2}{\pi D^2} \tag{36}$$

so that,

$$\frac{\sigma_\rho}{\rho} \simeq \frac{\pi}{4}\sqrt{\frac{1.6D^3}{d^3}\;\frac{4d^2}{\pi q D^2}} = \sqrt{\frac{1.6\pi D}{4qd}} \tag{37}$$

For example assume that $D = 27$ cm, $d = .15$ cm and $q = 10^3$ counts per cell. Eqn (37) gives

$$\frac{\sigma_\rho}{\rho} \simeq \sqrt{\frac{(1.6\pi)(27)}{(4)(10^3)(.15)}} = 0.15 \tag{38}$$

5. Conclusion

Given the size and number of cells in a region to be reconstructed and the number of events in the projection data, an estimate of the uncertainty of the reconstructed transverse section can be made in accordance with eqn (17),

$$\sigma_\rho = \sqrt{\frac{1.6D}{nd^3}}\;\sigma_I \tag{17}$$

After separating terms of eqn (17) and squaring,

$$\mathrm{var}(\rho) = \sigma_\rho^2 = 1.6\left(\frac{1}{d^2}\right)\left(\frac{D}{d}\right)\left(\frac{\sigma_I^2}{n}\right) \tag{39}$$

Thus the variance in the reconstructed function is related to the smallness in cell size and number of cells along a line integral.

The noise in a reconstructed element relative to noise in the projections is proportional to $\sqrt{D^3/(nd^3)}$ as ascertained by comparing the relative errors under the realistic conditions of eqns (27) and (28). Since the number of cells in the reconstruction region is given by $\nu = (\pi/4)(D/d)^2$, the noise amplification is proportional to,

$$\text{noise amp.} \ \alpha \ \sqrt{\frac{\nu}{n}\ \frac{D}{d}} \tag{40}$$

For practical estimates of relative rms errors of the reconstruction due to statistical uncertainty of the projections, the reader is referred to eqns (23) and (37) for transmission and emission respectively.

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Figure Captions

Figure 1. Geometry of the reconstruction region and an example of line integral paths at one angle, θ.

Figure 2. Physical set-up for x-ray transmission where a water bath is used to equalize total path length for line integrals.

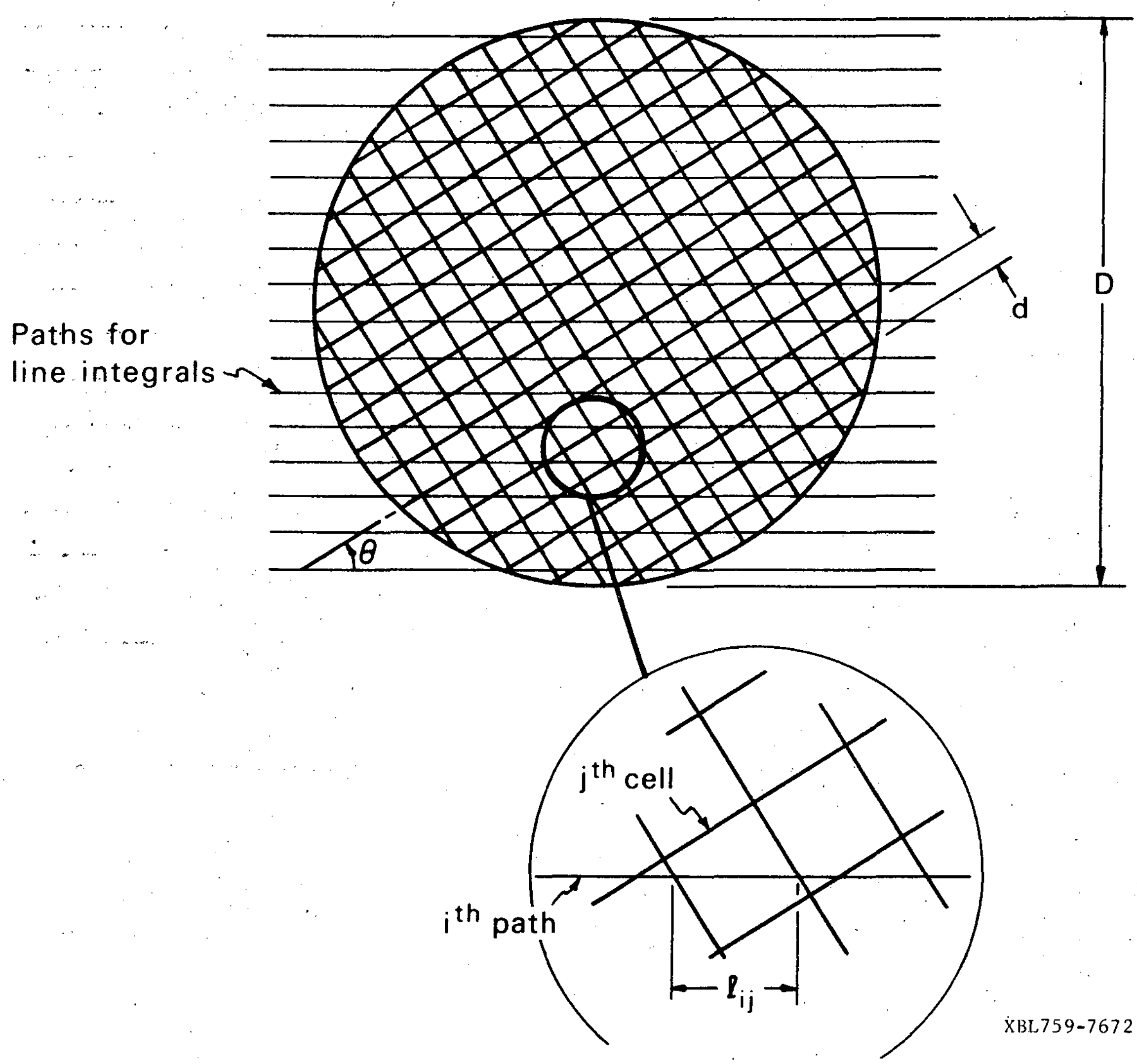
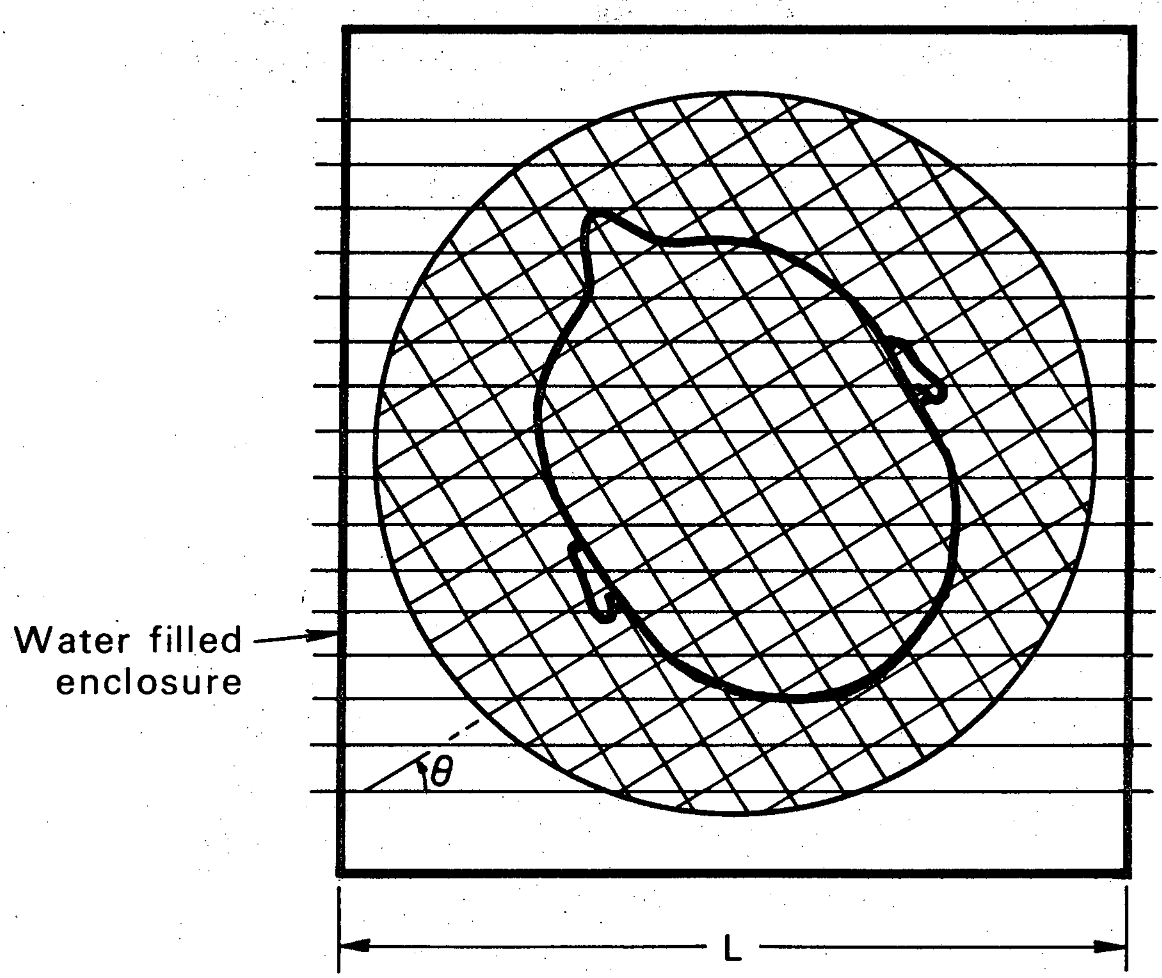

Paths for line integrals
θ
d
D
j^{th} cell
i^{th} path
ℓ_{ij}
XBL759-7672

Fig. 1



XBL759-7671

Fig. 2

0 0 0 0 4 4 0 2 6 7 2

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