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Positivity and vanishing theorems in complex and algebraic geometry

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Positivity and vanishing theorems in complex and algebraic geometry

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ABSTRACT OF THE DISSERTATION

Positivity and vanishing theorems in complex and algebraic geometry

by

Xiaokui Yang

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2012

Professor Kefeng Liu, Chair

In this thesis, we consider geometric properties of vector bundles arising from algebraic and Hermitian geometry.

On vector bundles in algebraic geometry, such as ample, nef and globally generated vector bundles, we are able to construct positive Hermitian metrics in different senses(e.g. Griffiths-positive, Nakano-positive and dual-Nakano-positive) by L^2-method and deduce many new vanishing theorems for them by analytic method instead of the Le Potier-Leray spectral sequence method.

On Hermitian manifolds, we find that the second Ricci curvature tensors of various metric connections are closely related to the geometry of Hermitian manifolds. We can derive various vanishing theorems for Hermitian manifolds and also for complex vector bundles over Hermitian manifolds by their second Ricci curvature tensors. We also introduce a natural geometric flow on Hermitian manifolds by using the second Ricci curvature tensor.

The dissertation of Xiaokui Yang is approved.

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Kefeng Liu, Committee Chair

University of California, Los Angeles

2012

To my family,

for their constant source of

love, concern, support and strength all these years.

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PUBLICATIONS

(1) K.-F. Liu and X.-K. Yang, Geometry of Hermitian manifolds. Internat. J. Math.**23** (2012), no. 6.

(2) K.-F. Liu; X.-F. Sun and X.-K. Yang, Positivity and vanishing theorems for ample vector bundles. *To appear in Journal of Algebraic Geometry*

(3) M. Dai and X.-K. Yang, Mappings of bounded distortion between complex manifolds. *To appear in Pure and Applied Math Quarterly*.

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CHAPTER 1

Positivity and vanishing theorems for vector bundles over Kähler manifolds

1.1 Introduction

Let E be a holomorphic vector bundle with a Hermitian metric h. In [Nakano55], Nakano introduced an analytic notion of positivity by using the curvature of (E, h), and now it is called Nakano positivity. Griffiths defined in [Griffiths69] Griffiths positivity of (E, h). On a Hermitian line bundle, these two concepts are the same. In general, Griffiths positivity is weaker than Nakano positivity. On the other hand, Hartshorne defined in [Hartshorne66] the ampleness of a vector bundle over a projective manifold. A vector bundle E is said to be ample if the tautological line bundle $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is ample over $\mathbb{P}(E^*)$. For a line bundle, it is well-known that the ampleness of the bundle is equivalent to its Griffiths positivity. In [Griffiths69], Griffiths conjectured that this equivalence is also valid for vector bundles, i.e. E is an ample vector bundle if and only if E carries a Griffiths-positive metric. As is well-known if E admits a Griffiths-positive metric, then $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ has a Griffiths-positive metric(see Proposition 1.2.11). Finding a Griffiths-positive metric on an ample vector bundle seems to be very difficult but is worth being investigated. In [Campana-Flenner90], Campana and Flenner gave an affirmative answer to the Griffiths conjecture when the base S is a projective curve, see also [Umemura73]. In [Siu-Yau80], Siu and Yau proved the Frankel conjecture that every compact Kähler manifold with positive holomorphic bisectional curvature is biholomorphic to the projective space. The positivity of holomorphic bisectional curvature is the same as Griffiths positivity of the holomorphic tangent bundle. On the other hand, Mori([Mori79]) proved the Hartshorne conjecture that any algebraic manifold with ample tangent vector bundle is biholomorphic to the projective space.

In this chapter, we consider the existence of positive Hermitian metrics on various vector bundles. It is well-known that metrics with good curvature properties are bridges between complex algebraic geometry and complex analytic geometry. We will construct Nakano-positive and dual-Nakano-positive metrics on various vector bundles associated to ample vector bundles.

Let E be a holomorphic vector bundle over a compact Kähler manifold S and F a line bundle over S. Let r be the rank of E and n be the complex dimension of S. In the following we briefly describe the main results in this chapter.

Theorem 1.1.1. *For any integer $k \geq 0$, if $S^{r+k}E \otimes \det E^* \otimes F$ is ample over S, then $S^k E \otimes F$ is both Nakano-positive and dual-Nakano-positive.*

Here we make no assumption on E and we allow E to be negative. For definitions about Nakano-positivity, dual-Nakano-positivity and ampleness, see Section 1.2. As pointed out by Berndtsson the Nakano positive part of Theorem 1.1.1 is a special case of [Berndtsson09a] where he proves it in the case of a general holomorphic fibration, but his method can not derive the dual-Nakano-positive part of Theorem 1.1.1. Note that Nakano-positive vector bundles are not necessarily dual-Nakano-positive and vice versa. For example, for any $n \geq 2$, the Fubini-Study metric h_{FS} on the holomorphic tangent bundle $T\mathbb{P}^n$ of $\mathbb{P}^n$ is semi-Nakano-positive and dual-Nakano-positive. It is well-known that $T\mathbb{P}^n$ does not admit a smooth Hermitian metric with Nakano-positive curvature for any $n \geq 2$. It is also easy to see that the holomorphic cotangent bundle of a complex hyperbolic space form is Nakano-positive and is not dual-Nakano-positive. On the other hand, by the dual Nakano-positivity, we can get various new vanishing theorems of type $H^{q,n}$. For more details, see Section 1.6.

As applications of Theorem 1.1.1, we get the following results:

Theorem 1.1.2. *Let E be an ample vector bundle over S.*

(1) If F is a nef line bundle, then there exists $k_0 = k_0(S, E)$ such that $S^k E \otimes F$ is Nakano-positive and dual-Nakano-positive for any $k \geq k_0$. In particular, $S^k E$ is Nakano-positive and dual-Nakano-positive for any $k \geq k_0$.

(2) If F is an arbitrary vector bundle, then there exists $k_0 = k_0(S, E, F)$ such that for any $k \geq k_0$, $S^kE \otimes F$ is Nakano-positive and dual-Nakano-positive.

Moreover, if the Hermitian vector bundle (E, h) is Griffiths-positive, then for large k, (S^kE, S^kh) is both Nakano-positive and dual-Nakano-positive.

The following results follow immediately from Theorem 1.1.1 and Theorem 1.1.2:

Corollary 1.1.3. *Let E be a holomorphic vector bundle over S.*

(1) If E is ample, $S^kE \otimes \det E$ is both Nakano-positive and dual-Nakano-positive for any $k \geq 0$.

(2) If E is ample and its rank r is greater than 1, then $S^mE^* \otimes (\det E)^t$ is both Nakano-positive and dual-Nakano-positive for any $t \geq r + m - 1$.

(3) If $S^{r+1}E \otimes \det E^*$ is ample, then E is both Nakano-positive and dual-Nakano-positive. In particular, E is Griffiths-positive.

If (E, h) is a Griffiths-positive vector bundle, Demailly-Skoda proved that $E \otimes \det E$ and $E^* \otimes (\det E)^r$ are Nakano-positive if $r > 1$([Demailly-Skoda80]). Berndtsson proved in [Berndtsson09a] that $S^kE \otimes \det E$ is Nakano-positive as soon as E is ample. For more related results, we refer the reader to recent works [Berndtsson09a], [Berndtsson09b], [Berndtsson], [Mourougane-Takayama07], [Mourougane-Takayama08] and [Schumacher] and references therein.

Let h_{FS} be the Fubini-Study metric on $T\mathbb{P}^n$ and S^kh_{FS} the induced metric on $S^kT\mathbb{P}^n$ by Veronese mapping. Let $n \geq 2$. It is easy to see that $T\mathbb{P}^n$ does not admit a Nakano-positive metric. In particular $(T\mathbb{P}^n, h_{FS})$ is not Nakano-positive. However, $(S^kT\mathbb{P}^n, S^kh_{FS})$ is Nakano-positive and dual-Nakano-positive for any $k \geq 2$ since $(S^{k+n}T\mathbb{P}^n \otimes K_{\mathbb{P}^n}, S^kh_{FS} \otimes \det(h_{FS})^{-1})$ is Griffiths-positive. This can be viewed as an evidence of positivity of some adjoint vector bundles, namely, vector bundles of type $S^kE \otimes (\det E)^\ell \otimes K_S$.

Theorem 1.1.4. *Let E be an ample vector bundle over S. Let r be the rank of E and n the dimension of S. If $r > 1$, then*

(1) $S^kE\otimes(\det E)^2\otimes K_S$ *is Nakano-positive and dual-Nakano-positive for any* $k \geq \max\{n-r, 0\}$. *Moreover, the lower bound is sharp.*

(2) $E\otimes(\det E)^k\otimes K_S$ *is Nakano-positive and dual-Nakano-positive for any* $k \geq \max\{n+1-r, 2\}$. *Moreover, the lower bound is sharp.*

In general, $\det E \otimes K_S$ is not an ample line bundle, for example, $(S, E) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(1))$. Similarly, in the case $n + 1 - r > 2$, i.e. $1 < r < n - 1$, the vector bundle $K_S \otimes (\det E)^{n-r}$ can be a negative line bundle, for example $(S, E) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1) \oplus \mathcal{O}_{\mathbb{P}^4}(1))$. So Theorem 1.1.4 is independent of the (dual-)Nakano-positivity of $S^kE \otimes \det E$.

The first application of Nakano-positivity and dual-Nakano-positivity for vector bundles are vanishing theorems. In this chapter, we obtain many vanishing theorems for various vector bundles which can also be viewed as generalizations of many classical vanishing theorems. In the literatures, many vanishing theorems have been obtained for the Dolbeault cohomology of ample and globally generated vector bundles on smooth projective manifolds, mainly due to the efforts of Le Potier, Schneider, Peternell, Sommese, Shiffman Demailly, Ein and Lazasfeld, Manivel, Layatini and Nahm([LePotier75], [Sommese78], [PLS87], [Demailly88], [Ein-Lazasfeld93], [Manivel97], [Laytimi-Nahm04], [Laytimi-Nahm05a], [Laytimi-Nahm05b]). The Le Potier vanishing theorem says that if E is an ample vector bundle over a smooth projective manifold X, then $H^{p,q}(X, E) = 0$ for any $p + q \geq n + r$ where $n = \dim_{\mathbb{C}} X$ and $r = \operatorname{rank}(E)$. When $r \leq n$, the vanishing pairs (p, q) are contained in a **triangle** enclosed by three lines $p+q = n+r$, $p = n$ and $q = n$. By using the Le Potier-Borel spectral sequence, many interesting generalizations are obtained for products of symmetric and skew-symmetric powers of an ample vector bundle, twisted by a suitable power of its determinant line bundle, see for examples, [Demailly88], [Manivel97], [Laytimi-Nahm04], [Laytimi-Nahm05a] and [Laytimi-Nahm05b]. The common feature of their results is that the vanishing theorems hold for (p, q) lying inside or on certain triangles.

As is well-known, except Nakano's vanishing theorem, few vanishing theorems for vector bundles are proved by analytic method. In this chapter, we use analytic method to prove vanishing theorems for certain Dolbeault cohomology groups of the bounded vector bundles. The new vanishing theorems have quite different feature and they hold for (p, q) lying inside or on certain

quadrilaterals. In order to describe the vanishing theorems much more effectively, we introduce

Definition 1.1.5. Let E be an arbitrary holomorphic vector bundle with rank r, L an ample line bundle and $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$. E is said to be $(\varepsilon_1, \varepsilon_2)$-bounded by L if there exists a Hermitian metric h on E and a positive Hermitian metric h^L on L such that the curvature of E is bounded by the curvatures of L^{ε_1} and L^{ε_2}, i.e.

$$\varepsilon_1 \omega_L \otimes Id_E \leq \Theta^{E,h} \leq \varepsilon_2 \omega_L \otimes Id_E \tag{1.1.1}$$

in the sense of Griffiths. E is called strictly $(\varepsilon_1, \varepsilon_2)$-bounded by L if at least one of $\Theta^{E,h} - \varepsilon_1 \omega_L \otimes Id_E$ and $\Theta^{E,h} - \varepsilon_2 \omega_L \otimes Id_E$ is not identically zero.

As is well-known, if $\det E$ is ample, we can choose $L = \det E$ as a natural bound. Hence, Definition 1.1.5 works naturally for many vector bundles in algebraic geometry. We list some examples as follows. See also Proposition 1.6.10.:

(1) If E is globally generated, E is $(0,1)$-bounded by $\det E$ and strictly $(0,1)$-bounded by $L \otimes \det E$ for any ample line bundle L;

(2) If E is an ample vector bundle with rank r, then E is strictly $(-1, r)$-bounded by $\det E$;

(3) If E is nef with rank r, then E is strictly $(-1, r)$-bounded by $L \otimes \det E$ for arbitrary ample line bundle L;

(4) If E is Griffiths-positive, E is strictly $(0,1)$-bounded by $\det E$.

Theorem 1.1.6. *If E is strictly $(\varepsilon_1, \varepsilon_2)$-bounded by L and $m + (r+k)\varepsilon_1 > 0$, then*

$$H^{p,q}(X, S^k E \otimes \det E \otimes L^m) = H^{q,p}(X, S^k E \otimes \det E \otimes L^m) = 0, \tag{1.1.2}$$

if $p \geq 1, q \geq 1$ satisfy

$$\frac{m + (r+k)\varepsilon_1}{m + (r+k)\varepsilon_2} \geq \min\left\{\frac{n-q}{p}, \frac{n-p}{q}\right\}. \tag{1.1.3}$$

In particular, $S^k E \otimes \det E \otimes L^m$ is Nakano-positive and dual-Nakano-positive and

$$H^{n,q}(X, S^k E \otimes \det E \otimes L^m) = H^{q,n}(X, S^k E \otimes \det E \otimes L^m) = 0$$

for $q \geq 1$.

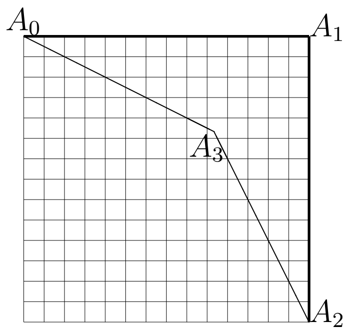

Figure 1

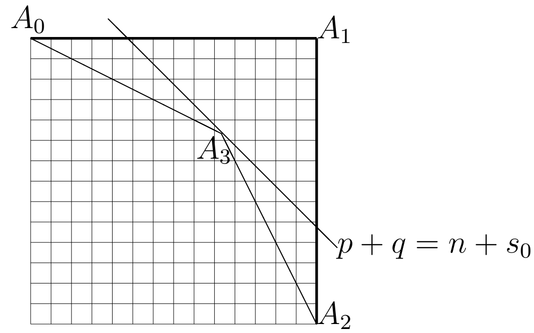

Figure 2

Remark 1.1.7. (1) (p, q) satisfies condition (1.1.3) if only if it lies inside or on the following quadrilateral $Q = A_0A_1A_2A_3$. See Figure 1 with A_0 and A_2 removed. Here

$$A_0 = (0, n), A_1 = (n, n), A_2 = (n, 0), A_3 = (c_0, c_0)$$

and

$$c_0 = \frac{n}{1 + \frac{m+(r+k)\varepsilon_1}{m+(r+k)\varepsilon_2}}, \tag{1.1.4}$$

It is obvious that Q is symmetric with respect to the line $p = q$.

(2) The condition $m + (r + k)\varepsilon_1 > 0$ is necessary, which guarantees that the vector bundle $S^kE \otimes \det E \otimes L^m$ is Griffiths-positive. In fact, in terms of Hermitian metrics,

$$S^kE \otimes \det E \otimes L^m = S^k(E \otimes L^{-\varepsilon_1}) \otimes \det(E \otimes L^{-\varepsilon_1}) \otimes L^{m+(r+k)\varepsilon_1} \geq L^{m+(r+k)\varepsilon_1}$$

and similarly $S^kE \otimes \det E \otimes L^m \leq L^{m+(r+k)\varepsilon_2}$. On the other hand, we will see that $S^kE \otimes \det E \otimes L^m$ has a nice metric h such that $(S^kE \otimes \det E \otimes L^m, h)$ behaves very similarly to a positive Hermitian "line bundle" $(\mathcal{L}, h_0)$. Moreover, $m + (r + k)\varepsilon_1$ and $m + (r + k)\varepsilon_2$ are the minimal and maximal eigenvalues of the curvature of $(\mathcal{L}, h_0)$ respectively. From these one can see that Theorem 1.1.6 is optimal.

(3) When ε_1 is close enough to ε_2, E is semi-stable with respect to L ([Kobayashi87]). Moreover, $H^{p,q}(X, S^kE \otimes \det E \otimes L^m) = 0$ for any $p + q \geq n + 1$.

(4) If $\varepsilon_1 \leq 0$, $\varepsilon_2 \geq 0$, and F is an arbitrary nef line bundle, Theorem 1.1.6 also holds for $S^kE \otimes \det E \otimes L^m \otimes F$.

As applications, we obtain

Theorem 1.1.8. *If E is globally generated and L is an ample line bundle, then for any $k \geq 1, m \geq 1$,*

$$H^{p,q}(X, S^k E \otimes (\det E)^m \otimes L) = H^{q,p}(X, S^k E \otimes (\det E)^m \otimes L) = 0$$

if $p \geq 1, q \geq 1$ satisfy

$$\frac{m-1}{m-1+(r+k)} \geq \min\left\{\frac{n-q}{p}, \frac{n-p}{q}\right\} \tag{1.1.5}$$

In particular, $S^k E \otimes (\det E)^m \otimes L$ is both Nakano-positive and dual-Nakano-positive and

$$H^{n,q}(X, S^k E \otimes (\det E)^m \otimes L) = H^{q,n}(X, S^k E \otimes (\det E)^m \otimes L) = 0$$

for any $q \geq 1$.

The right hand side of (1.1.5) depends only on the ratios and it makes Theorem 1.1.8 quite different from the results of [Demailly88], [Manivel97], [Laytimi-Nahm04] and [Laytimi-Nahm05a]. More precisely, for some specific vanishing pair (p, q), the power of $\det E$ may be independent on the dimension of X. For example, for $n = 3n_0 + 2$, by (1.1.5),

$$H^{2,n-1}(X, E \otimes (\det E)^{r+2} \otimes L) = 0 = H^{2n_0+2,2n_0+1}(X, E \otimes (\det E)^{r+2} \otimes L) \tag{1.1.6}$$

for any globally generated E and ample L. In general, we do not have $H^{p,q}(X, E \otimes (\det E)^{r+2} \otimes L) = 0$ **for all** $p+q \geq n+1$, if $1 < r \ll n$(cf. [Manivel97], Corollary B and [Laytimi-Nahm05a], Corollary 1.5). On the other hand, for fixed (k, m), the quadrilateral Q contains a triangle $p + q \geq n + s_0$ for some $s_0 \in (0, n]$. See Figure 2. Moreover, if the power m of $\det E$ is large enough, we obtain $H^{p,q}(X, S^k E \otimes (\det E)^m \otimes L) = 0$ for $p + q \geq n + 1$. Examples in [PLS87] and [Demailly88] indicate that a sufficient large power of $\det E$ is necessary in this case. For more details, see Corollary 1.6.14, Corollary 1.6.16 and Example 1.7.8.

Theorem 1.1.9. *If E is ample (resp. nef) and L is nef (resp. ample), then for any $k \geq 1$ and $m \geq k + r + 1$,*

$$H^{p,q}(X, S^k E \otimes (\det E)^m \otimes L) = H^{q,p}(X, S^k E \otimes (\det E)^m \otimes L) = 0,$$

if $p \geq 1, q \geq 1$ satisfy

$$\frac{(m-1)-(r+k)}{(m-1)+r(r+k)} \geq \min\left\{\frac{n-q}{p}, \frac{n-p}{q}\right\} \tag{1.1.7}$$

By a similar setting as (1.1.6), it is easy to see that the result in Theorem 1.1.9 is different from the results in [Demailly88], [Manivel97], [Laytimi-Nahm04] and [Laytimi-Nahm05a].

Remark: Our method is a generalization of the analytic proof of the Kodaira-Akizuki-Nakano vanishing Theorem for line bundles. We have obtained similar results for "partially" positive vector bundles.

1.2 Background materials

1.2.1 Various positivity and relations

Let E be a holomorphic vector bundle over a compact Kähler manifold S and h a Hermitian metric on E. There exists a unique connection ∇ which is compatible with the metric h and complex structure on E. It is called the Chern connection of (E, h). Let $\{z^i\}_{i=1}^n$ be local holomorphic coordinates on S and $\{e_\alpha\}_{\alpha=1}^r$ be a local frame of E. The curvature tensor $R^\nabla \in \Gamma(S, \Lambda^2 T^*S \otimes E^* \otimes E)$ has the form

$$R^\nabla = \frac{\sqrt{-1}}{2\pi} R^\gamma_{i\bar{j}\alpha} dz^i \wedge d\bar{z}^j \otimes e^\alpha \otimes e_\gamma \tag{1.2.1}$$

where $R^\gamma_{i\bar{j}\alpha} = h^{\gamma\overline{\beta}} R_{i\bar{j}\alpha\overline{\beta}}$ and

$$R_{i\bar{j}\alpha\overline{\beta}} = -\frac{\partial^2 h_{\alpha\overline{\beta}}}{\partial z^i \partial \overline{z}^j} + h^{\gamma\overline{\delta}} \frac{\partial h_{\alpha\overline{\delta}}}{\partial z^i} \frac{\partial h_{\gamma\overline{\beta}}}{\partial \overline{z}^j} \tag{1.2.2}$$

Here and henceforth we sometimes adopt the Einstein convention for summation.

Definition 1.2.1. A Hermitian vector bundle (E, h) is said to be *Griffiths-positive*, if for any nonzero vectors $u = u^i \frac{\partial}{\partial z^i}$ and $v = v^\alpha e_\alpha$,

$$\sum_{i,j,\alpha,\beta} R_{i\bar{j}\alpha\overline{\beta}} u^i \overline{u}^j v^\alpha \overline{v}^\beta > 0 \tag{1.2.3}$$

(E, h) is said to be *Nakano-positive*, if for any nonzero vector $u = u^{i\alpha} \frac{\partial}{\partial z^i} \otimes e_\alpha$,

$$\sum_{i,j,\alpha,\beta} R_{i\bar{j}\alpha\overline{\beta}} u^{i\alpha} \overline{u}^{j\beta} > 0 \tag{1.2.4}$$

(E, h) is said to be *dual-Nakano-positive*, if for any nonzero vector $u = u^{i\alpha} \frac{\partial}{\partial z^i} \otimes e_\alpha$,

$$\sum_{i,j,\alpha,\beta} R_{i\bar{j}\alpha\overline{\beta}} u^{i\beta} \overline{u}^{j\alpha} > 0 \tag{1.2.5}$$

It is easy to see that (E,h) is dual-Nakano-positive if and only if (E^*,h^*) is Nakano-negative. The notions of semi-positivity, negativity and semi-negativity can be defined similarly. We say E is Nakano-positive (resp. Griffiths-positive, dual-Nakano-positive, $\cdots$), if it admits a Nakano-positive(resp. Griffiths-positive, dual-Nakano-positive, $\cdots$) metric.

The following geometric definition of nefness is due to [DPS94].

Definition 1.2.2. Let (S,ω_0) be a compact Kähler manifold. A line bundle L over S is said to be nef, if for any $\varepsilon>0$, there exists a smooth Hermitian metric h_ε on L such that the curvature of (L,h_ε) satisfies

$$R=-\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h_\varepsilon\geq-\varepsilon\omega_0 \tag{1.2.6}$$

This means that the curvature of L can have an arbitrarily small negative part. Clearly a nef line bundle L satisfies

$$\int_C c_1(L)\geq 0$$

for any irreducible curve $C\subset S$. For projective algebraic S, both notions coincide.

By the Kodaira embedding theorem, we have the following geometric definition of ampleness.

Definition 1.2.3. Let (S,ω_0) be a compact Kähler manifold. A line bundle L over S is said to be ample, if there exists a smooth Hermitian metric h on L such that the curvature R of (L,h) satisfies

$$R=-\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h>0 \tag{1.2.7}$$

For comprehensive descriptions of positivity, nefness, ampleness and related topics, see [Demailly], [DPS94], [Lazasfeld04], [Griffiths69], [Shiffman-Sommese85] and [Umemura73].

In the seminal paper [Siu80], Siu introduced the following terminology:

Definition 1.2.4. Let (X,g) be a compact Kähler manifold. (X,g) has strongly negative curvature(resp. strongly positive) if

$$R_{i\overline{j}k\overline{\ell}}\left(A^i\overline{B}^j-C^i\overline{D}^j\right)\overline{\left(A^\ell\overline{B}^k-C^\ell\overline{D}^k\right)}\leq 0\ \text{ (resp. }\geq 0) \tag{1.2.8}$$

for any $A=A^i\frac{\partial}{\partial z^i}$, $B=B^j\frac{\partial}{\partial z^j}$, $C=C^i\frac{\partial}{\partial z^i}$, $D=D^j\frac{\partial}{\partial z^j}$ and the identity in the above inequality holds if and only if

$$A^i\overline{B}^j-C^i\overline{D}^j=0$$

for any i, j.

Remark 1.2.5. Note that if $\dim_{\mathbb{C}} X = 2$, the strong negativity in the sense of Siu is equivalent to the dual-Nakano negativity.

Example 1.2.6. The Hermitian holomorphic tangent bundle of $(\mathbb{P}^n, \omega_{FS})$ with $n > 1$ is dual-Nakano-positive and semi-Nakano-positive. In fact, in the normal coordinates of a given point on $\mathbb{P}^n$, the curvature tensor of $(T\mathbb{P}^n, \omega_{FS})$ is

$$R_{i\bar{j}k\bar{\ell}} = \delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj} \tag{1.2.9}$$

It is easy to verify the assertion from the following identities

$$R_{i\bar{j}k\bar{\ell}} u^{ik} \overline{u}^{j\ell} = \frac{1}{2} \sum_{i,j} |u^{ij} + u^{ji}|^2 \quad \text{and} \quad R_{i\bar{j}k\bar{\ell}} u^{i\ell} \overline{u}^{jk} = \Big| \sum_i u^{ii} \Big|^2 + \sum_{i,\ell} |u^{i\ell}|^2$$

Lemma 1.2.7. *Let $n > 1$.*

(1) $(T\mathbb{P}^n, \omega_{FS})$ is dual-Nakano-positive and semi-Nakano-positive.

(2) Let X be a hyperbolic space form with dimension n. If ω_B is the canonical metric on X, then (TX, ω_B) is dual-Nakano-negative and semi-Nakano-negative.

(3) For any n-dimensional compact Kähler manifold X, the holomorphic tangent bundle TX is neither Nakano-positive nor Nakano-negative.

Proof. The assertion (3) follows from Nakano-vanishing theorem and Serre duality. □

Lemma 1.2.8. *Let (E, h) be a Hermitian holomorphic vector bundle over a complex manifold X, S be a holomorphic subbudle of E and Q the corresponding quotient bundle.*

$$0 \to S \to E \to Q \to 0$$

(1) If E is (semi)-Nakano-negative, then S is also (semi)-Nakano negative.

(2) If E is (semi-)dual-Nakano-positive, then Q is also (semi-)dual-Nakano-positive.

Proof. This lemma is well-known(e.g. [Demailly]). For the sake of completeness, we include a proof here. It is obvious that (2) is the dual of (1). Let r be the rank of E and s the rank of S. Without loss of generality, we can assume, at a fixed point $p \in X$, there exists a local holomorphic frame $\{e_1, \cdots, e_r\}$ of E centered at point p such that $\{e_1, \cdots, e_s\}$ is a local holomorphic frame of S. Moreover, we can assume that

$$h(e_\alpha, e_\beta)(p) = \delta_{\alpha\beta}, \text{for } 1 \leq \alpha, \beta \leq r$$

Hence, the curvature tensor of S at point p is

$$R^S_{i\bar{j}\alpha\bar{\beta}} = -\frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z^i \partial \bar{z}^j} + \sum_{\gamma=1}^{s} \frac{h_{\alpha\bar{\gamma}}}{\partial z^i} \frac{h_{\gamma\bar{\beta}}}{\partial \bar{z}^j} \tag{1.2.10}$$

where $1 \leq \alpha, \beta \leq s$. The curvature tensor of E at point p is

$$R^E_{i\bar{j}\alpha\bar{\beta}} = -\frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z^i \partial \bar{z}^j} + \sum_{\gamma=1}^{r} \frac{h_{\alpha\bar{\gamma}}}{\partial z^i} \frac{h_{\gamma\bar{\beta}}}{\partial \bar{z}^j} \tag{1.2.11}$$

where $1 \leq \alpha, \beta \leq r$. By formula (1.2.4), it is easy to see that $R^E|_S - R^S$ is semi-Nakano-positive. Hence (1) follows. □

The following relations are well-known:

Lemma 1.2.9. *Let (X, g) be a Kähler manifold. We have the following relations between various curvature terminologies*

(1) dual-Nakano negativity implies strongly negativity in the sense of Siu;

(2) strongly negativity in the sense of Siu implies negativity of Riemannian sectional curvature;

(3) negativity of Riemannian sectional curvature implies negativity of holomorphic bisectional curvature.

1.2.2 Ampleness and Griffiths positivity for vector bundles

Let E be a Hermitian vector bundle of rank r over a compact Kähler manifold S, $L = \mathcal{O}_{\mathbb{P}(E^*)}(1)$ be the tautological line bundle of the projective bundle $\mathbb{P}(E^*)$ and π the canonical projection $\mathbb{P}(E^*) \to$

S. By definition([Hartshorne66]), E is an ample vector bundle over S if $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is an ample line bundle over $\mathbb{P}(E^*)$. E is said to be nef, if $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is nef. To simplify the notations we will denote $\mathbb{P}(E^*)$ by X and the fiber $\pi^{-1}(\{s\})$ by X_s.

Let $(e_1, \cdots, e_r)$ be the local holomorphic frame with respect to a given trivialization on E and the dual frame on E^* is denoted by $(e^1, \cdots, e^r)$. The corresponding holomorphic coordinates on E^* are denoted by $(W_1, \cdots, W_r)$. There is a local section e_{L^*} of L^* defined by

$$e_{L^*} = \sum_{\alpha=1}^{r} W_\alpha e^\alpha \tag{1.2.12}$$

Its dual section is denoted by e_L. Let h^E be a fixed Hermitian metric on E and h^L the induced quotient metric by the morphism $(\pi^* E, \pi^* h^E) \to L$.

If $\left(h_{\alpha\overline{\beta}}\right)$ is the matrix representation of h^E with respect to the basis $\{e_\alpha\}_{\alpha=1}^r$, then h^L can be written as

$$h^L = \frac{1}{h^{L^*}(e_{L^*}, e_{L^*})} = \frac{1}{\sum h^{\alpha\overline{\beta}} W_\alpha \overline{W}_\beta} \tag{1.2.13}$$

Proposition 1.2.10. *The curvature of (L, h^L) is*

$$R^{h^L} = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h^L = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\left(\sum h^{\alpha\overline{\beta}} W_\alpha \overline{W}_\beta\right) \tag{1.2.14}$$

where ∂ and $\overline{\partial}$ are operators on the total space $\mathbb{P}(E^*)$.

Although the following result is well-known([Demailly], [Griffiths69]), we include a proof here for the sake of completeness.

Proposition 1.2.11. *If (E, h^E) is a Griffiths-positive vector bundle, then E is ample.*

Proof. We will show that the induced metric h^L in (1.2.13) is positive. We fix a point $p \in \mathbb{P}(E^*)$, then there exist local holomorphic coordinates $(z^1, \cdots, z^n)$ centered at point $s = \pi(p)$ and local holomorphic basis $\{e_1, \cdots, e_r\}$ of E around s such that

$$h_{\alpha\overline{\beta}} = \delta_{\alpha\overline{\beta}} - R_{i\overline{j}\alpha\overline{\beta}} z^i \overline{z}^j + O(|z|^3) \tag{1.2.15}$$

Without loss of generality, we assume p is the point $(0, \cdots, 0, [a_1, \cdots, a_r])$ with $a_r = 1$. On the chart $U = \{W_r = 1\}$ of the fiber $\mathbb{P}^{r-1}$, we set $w^A = W_A$ for $A = 1, \cdots, r-1$. By formula

(1.2.14) and (1.2.15)

$$R^{h^L}(p) = \frac{\sqrt{-1}}{2\pi}\left(\sum R_{i\bar{j}\alpha\bar{\beta}}\frac{a_\beta \overline{a}_\alpha}{|a|^2}dz^i \wedge d\overline{z}^j + \sum_{A,B=1}^{r-1}\left(\delta_{AB} - \frac{a_B \overline{a}_A}{|a|^2}\right)dw^A \wedge d\overline{w}^B\right) \tag{1.2.16}$$

where $|a|^2 = \sum_{\alpha=1}^{r}|a_\alpha|^2$. If R^E is Griffith positive,

$$\left(\sum_{\alpha,\beta=1}^{r} R_{i\bar{j}\alpha\bar{\beta}}\frac{a_\beta \overline{a}_\alpha}{|a|^2}\right)$$

is a Hermitian positive $n \times n$ matrix. Consequently, $R^{h^L}(p)$ is a Hermitian positive $(1,1)$ form on $\mathbb{P}(E^*)$, i.e. h^L is a positive Hermitian metric. □

The following linear algebraic lemma will be used in Theorem 1.3.7.

Lemma 1.2.12. *If the matrix*

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is invertible and D is invertible, then $(A - BD^{-1}C)^{-1}$ exists and

$$T^{-1} = \begin{pmatrix} (A-BD^{-1}C)^{-1} & -(A-BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A-BD^{-1}C)^{-1} & D^{-1}C(A-BD^{-1}C)^{-1}BD^{-1} + D^{-1} \end{pmatrix}$$

Moreover, if T is positive definite, then $A - BD^{-1}C$ is positive definite.

1.2.3 Classical vanishing theorems

In the following, we will describe the idea of proving vanishing theorems by using an analytic method and the similar methods will be used in proving Theorem 1.1.6.

At first, we briefly describe the analytic proof of vanishing theorems for line bundles. Let $(\varphi_{i\bar{j}})_{n\times n}$ be a Hermitian positive matrix with eigenvalues

$$\lambda_1 \leq \cdots \leq \lambda_n \tag{1.2.17}$$

Let $u = \sum u_{I\overline{J}}dz^I \wedge d\overline{z}^J$ be a (p,q) form on $\mathbb{C}^n$ where $u_{I\overline{J}}$ is alternate in the indices $I = (i_1, \cdots, i_p)$ and $J = (j_1, \cdots, j_q)$. We define

$$T(u,u) = \langle [\varphi, \Lambda_\omega] u, u \rangle \tag{1.2.18}$$

where $\varphi = \sqrt{-1}\varphi_{i\bar{j}} dz^i \wedge d\bar{z}^j$ and Λ_ω is the contraction operator of the standard Kähler metric on $\mathbb{C}^n$. The following linear algebraic result is well-known([Demailly]):

Lemma 1.2.13. *We have the following estimate*

$$T(u,u) \geq \max\{p\lambda_1 - (n-q)\lambda_n, q\lambda_1 - (n-p)\lambda_n\}|u|^2 \tag{1.2.19}$$

Corollary 1.2.14. *Let (L,h) be a Hermitian line bundle over a compact Kähler manifold (X,ω_0). Let λ_1 and λ_n be the smallest and largest eigenvalue functions of R^L with respect to ω_0 respectively. Suppose $\lambda_n > 0$. If*

$$\max\{p\lambda_1 - (n-q)\lambda_n, q\lambda_1 - (n-p)\lambda_n\}$$

is positive everywhere, or equivalently

$$\frac{\lambda_1}{\lambda_n} > \max\left\{\frac{n-q}{p}, \frac{n-p}{q}\right\} \tag{1.2.20}$$

then

$$H^{p,q}(M,L) = H^{q,p}(M,L) = 0 \tag{1.2.21}$$

Proof. By a well-known Bochner formula for L,

$$\Delta'' = \Delta' + [R^L, \Lambda_\omega]$$

for any $u \in \Omega^{p,q}(M,L)$,

$$\langle \Delta'' u, u \rangle = \langle \Delta' u, u \rangle + T(u,u) \tag{1.2.22}$$

If $\Delta'' u = 0$, by the condition, we get $u = 0$. □

Remark 1.2.15. The condition in Corollary 1.2.14 can be satisfied if and only if (L,h) is Griffiths positive or Griffiths-negative. If (L,h) is a positive line bundle over a compact complex manifold X, we can define a Kähler metric on X

$$\omega_0 = R^L = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h \tag{1.2.23}$$

In this case, $\varphi = R^L$ in Lemma 1.2.14 and $\lambda_1 = \lambda_n = 1$. Hence, if $p+q \geq n+1$, $H^{p,q}(X,L) = 0$. This is the Kodaira-Akizuki-Nakano vanishing theorem.

Corollary 1.2.16 (Kodaira-Akizuki-Nakano)**.** *Let X be a compact complex manifold with complex dimension n. If L is an ample line bundle over X, then*

$$H^{p,q}(X, L) = 0 \quad \textit{for} \quad p + q \geq n + 1 \tag{1.2.24}$$

For ample vector bundles, Le Potier generalized Kodaira-Akizuki-Nakano vanishing theorem and obtained the famous Le Potier vanishing theorem

Theorem 1.2.17 (Le Potier)**.** *Let X be a compact complex manifold with complex dimension n and E be an ample vector bundle over X with rank r.*

$$H^{p,q}(X, E) = 0 \quad \textit{for} \quad p + q \geq n + r \tag{1.2.25}$$

However, when the rank r of E is very large, more precisely, when $r > n$, Le Potier's vanishing theorem can not provide any information. But the following result holds for any ample vector bundle

Proposition 1.2.18. *Let X be a compact complex manifold with complex dimension n and E be an ample vector bundle over X. Then*

$$H^{n,n}(X, E) = 0 \tag{1.2.26}$$

It is easy to see from the following example that Proposition 1.2.18 is optimal.

Example 1.2.19. It is well known that, for any $n \geq 2$, $E = T^{1,0}\mathbb{P}^n$ is ample, but

$$H^{n,n-1}(\mathbb{P}^n, E) = \mathbb{C} \neq 0 \tag{1.2.27}$$

The following vanishing theorem is due to Nakano([Nakano55])(see also ([Demailly])):

Lemma 1.2.20. *Let E be a holomorphic vector bundle over a compact Kähler manifold M. If E is Nakano-positive, then $H^{n,q}(M, E) = 0$ for any $q \geq 1$. If E is dual-Nakano-positive, then $H^{q,n}(M, E) = 0$ for any $q \geq 1$.*

The proof of Lemma 1.2.20 follows from formula (1.2.29) easily. Let (E,h) be a Hermitian holomorphic vector bundle with rank r over a compact Kähler manifold (X,ω_g). For any fixed point $p\in X$, there exists a local holomorphic coordinates system $\{z^i\}_{i=1}^n$ and local holomorphic frames $\{e_\alpha\}_{\alpha=1}^r$ such that

$$g_{i\bar{j}}(p)=\delta_{ij},\ \ h_{\alpha\overline{\beta}}(p)=\delta_{\alpha\overline{\beta}} \tag{1.2.28}$$

The curvature term

$$\begin{aligned} T(u,u) &= \langle [R^E,\Lambda_g]u,u\rangle \\ &= \sum R_{i\bar{j}\alpha\overline{\beta}}u_{I,\overline{iS},\alpha}\overline{u}_{I,\overline{jS}\beta}+\sum R_{i\bar{j}\alpha\overline{\beta}}u_{jR,\overline{J},\alpha}\overline{u}_{iR,\overline{J},\beta}-\sum R_{ii\alpha\overline{\beta}}u_{I\overline{J}\alpha}\overline{u}_{I\overline{J}\beta} \end{aligned} \tag{1.2.29}$$

for any $u=\sum u_{I\overline{J}\alpha}dz^I\wedge d\bar{z}^J\otimes e_\alpha$. For more details, see ([Demailly], p. 341). From formula (1.2.29), it is very difficult to obtain vanishing theorems for vector bundles. If the curvature R^E has a nice expression, for example

$$R_{i\bar{j}\alpha\overline{\beta}}=\varphi_{i\bar{j}}\tau_\alpha\overline{\tau}_\beta \tag{1.2.30}$$

then E behaviors as a line bundle with curvature $(\varphi_{i\bar{j}})$. Unfortunately, few examples with property (1.2.30) can be found. However, an integral version of (1.2.30) exists on vector bundles of type $E\otimes\det E$,

$$R^{E\otimes\det E}_{i\bar{j}\alpha\overline{\beta}}(s)=R_{i\bar{j}\alpha\overline{\beta}}(s)+\delta_{\alpha\beta}\cdot\sum_\gamma R_{i\bar{j}\gamma\overline{\gamma}}(s)=r!\cdot\int_{\mathbb{P}^{r-1}}\frac{\varphi_{i\bar{j}}W_\alpha\overline{W}_\beta}{|W|^2}\frac{\omega_{FS}^{r-1}}{(r-1)!} \tag{1.2.31}$$

where $[W_1,\cdots,W_r]$ are the homogeneous coordinates on $\mathbb{P}^{r-1}$, ω_{FS} is the Fubini-Study metric and

$$\varphi_{i\bar{j}}=(r+1)\sum_{\gamma,\delta}R_{i\bar{j}\gamma\bar{\delta}}(s)\frac{W_\delta\overline{W}_\gamma}{|W|^2} \tag{1.2.32}$$

It is obvious that if E is Griffiths-positive, then $E\otimes\det E$ is both Nakano-positive and dual-Nakano-positive. With the help of the nice formulation (1.2.31), we obtain vanishing theorem 1.1.8, Theorem 1.1.6 and Theorem 1.1.9 which are similar to Corollary 1.2.14.

1.3 Construction of Hermitian metrics on $S^kE \otimes F$

1.3.1 Curvature formulas

Let F be a holomorphic line bundle over S, $L = \mathcal{O}_{\mathbb{P}(E^*)}(1)$ and $\pi : \mathbb{P}(E^*) \to S$. For simplicity of notations, we set $\widetilde{L} = L^k \otimes \pi^*(F)$ for $k \geq 0$ and $X = \mathbb{P}(E^*)$. Let h_0 be a Hermitian metric on $\widetilde{L}$ and $\{\omega_s\}_{s\in S}$ a smooth family of Kähler metrics on the fibers $X_s = \mathbb{P}(E_s^*)$ of X which are induced by the curvature form of some metric on $\mathcal{O}_{\mathbb{P}(E^*)}(1)$. Let $\{w^A\}_{A=1}^{r-1}$ be the local holomorphic coordinates on the fiber X_s which are induced by the homogeneous coordinates $[W_1, \cdots, W_r]$ on a trivialization chart. Using these notations, we can write ω_s as

$$\omega_s = \frac{\sqrt{-1}}{2\pi} \sum_{A,B=1}^{r-1} g_{A\overline{B}}(s,w)dw^A \wedge d\overline{w}^B \tag{1.3.1}$$

It is well-known $H^0(\mathbb{P}^{r-1}, \mathcal{O}_{\mathbb{P}^{r-1}}(k))$ can be identified as the space of homogeneous polynomials of degree k in r variables. Therefore, the sections of $H^0(X_s, \widetilde{L}|_{X_s})$ are of the form $V_\alpha e_L^{\otimes k} \otimes \underline{e}$ where V_α are homogenous polynomials in $\{W_1, \cdots, W_r\}$ of degree k and $\underline{e}$ the base of $\pi^*(F)$ induced by a base e of F. For example, if $\alpha = (\alpha_1, \cdots, \alpha_r)$ with $\alpha_1 + \cdots + \alpha_r = k$ and α_j are nonnegative integers,

$$V_\alpha = W_1^{\alpha_1} \cdots W_r^{\alpha_r}. \tag{1.3.2}$$

Now we set

$$E_\alpha = e_1^{\otimes\alpha_1} \otimes \cdots \otimes e_r^{\otimes\alpha_r} \otimes e \quad \text{and} \quad e_{\widetilde{L}} = e_L^{\otimes k} \otimes \underline{e}$$

which are bases of $S^kE \otimes F$ and $\widetilde{L}$ respectively. We obtain a vector bundle whose fibers are $H^0(X_s, \widetilde{L}|_{X_s})$. In fact, this vector bundle is $\widetilde{E} = S^kE \otimes F$. Now we can define a smooth Hermitian metric f on $S^kE \otimes F$ by $(\widetilde{L}, h_0)$ and (X_s, ω_s), locally it is

$$\begin{aligned} f_{\alpha\overline{\beta}} := f(E_\alpha, E_\beta) &= \int_{X_s} \langle V_\alpha e_{\widetilde{L}}, V_\beta e_{\widetilde{L}} \rangle_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} \\ &= \int_{X_s} h_0 V_\alpha \overline{V}_\beta \frac{\omega_s^{r-1}}{(r-1)!} \end{aligned} \tag{1.3.3}$$

Here we regard h_0 locally as a positive function. In this general setting, the Hermitian metric h_0 on $\widetilde{L}$ and Kähler metrics ω_s on the fibers are independent.

Let $(z^1, \cdots, z^n)$ be local holomorphic coordinates on S. By definition, the curvature tensor of f is

$$R_{i\bar{j}\alpha\bar{\beta}} = -\frac{\partial^2 f_{\alpha\bar{\beta}}}{\partial z^i \partial \bar{z}^j} + \sum_{\gamma,\delta} f^{\gamma\bar{\delta}} \frac{\partial f_{\alpha\bar{\delta}}}{\partial z^i} \frac{\partial f_{\gamma\bar{\beta}}}{\partial \bar{z}^j} \tag{1.3.4}$$

In the following, we will compute the curvature of f. Let $T_{X/S}$ be the relative tangent bundle of the fibration $\mathbb{P}(E^*) \to S$, then $g_{A\bar{B}}$ is a metric on $T_{X/S}$ and $\det(g_{A\bar{B}})$ is a metric on $\det(T_{X/S})$. Let $\varphi = -\log(h_0 \det(g_{A\bar{B}}))$ be the local weight of induced Hermitian metric $h_0 \det(g_{A\bar{B}})$ on $\widetilde{L} \otimes \det(T_{X/S})$. In the sequel, we will use the following notations

$$\varphi_i = \frac{\partial \varphi}{\partial z^i}, \varphi_{i\bar{j}} = \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}, \varphi_{A\bar{B}} = \frac{\partial^2 \varphi}{\partial w^A \partial \bar{w}^B}, \varphi_{i\bar{B}} = \frac{\partial^2 \varphi}{\partial z^i \partial \bar{w}^B}, \varphi_{A\bar{j}} = \frac{\partial^2 \varphi}{\partial \bar{z}^j \partial w^A}$$

and $(\varphi^{A\bar{B}})$ is the transpose inverse of the $(r-1) \times (r-1)$ matrix $(\varphi_{A\bar{B}})$,

$$\sum_{B=1}^{r-1} \varphi^{A\bar{B}} \varphi_{C\bar{B}} = \delta_C^A$$

The following lemma can be deduced from the formulas in [Schumacher85], [Wolpert86] and [Siu86]. In the case of holomorphic fibration $\mathbb{P}(E^*) \to S$, it follows by straightforward computations.

Lemma 1.3.1. *The first order derivative of $f_{\alpha\bar{\beta}}$ is*

$$\frac{\partial f_{\alpha\bar{\beta}}}{\partial z^i} = -\int_{X_s} h_0 V_\alpha \overline{V}_\beta \varphi_i \frac{\omega_s^{r-1}}{(r-1)!} = \int_{X_s} \langle -V_\alpha \varphi_i e_{\widetilde{L}}, V_\beta e_{\widetilde{L}} \rangle_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} \tag{1.3.5}$$

Proof. By the local expression (1.3.1) of ω_s,

$$\frac{\omega_s^{r-1}}{(r-1)!} = \det(g_{A\bar{B}}) dV_{\mathbb{C}^{r-1}}$$

where $dV_{\mathbb{C}^{r-1}}$ is standard volume on $\mathbb{C}^{r-1}$. Therefore

$$f_{\alpha\bar{\beta}} = \int_{X_s} e^{-\varphi} V_\alpha \overline{V}_\beta dV_{\mathbb{C}^{r-1}}$$

and the first order derivative is

$$\begin{aligned}
\frac{\partial f_{\alpha\bar{\beta}}}{\partial z^i} &= \int_{X_s} \frac{\partial e^{-\varphi}}{\partial z^i} V_\alpha \overline{V}_\beta dV_{\mathbb{C}^{r-1}} \\
&= -\int_{X_s} \varphi_i e^{-\varphi} V_\alpha \overline{V}_\beta dV_{\mathbb{C}^{r-1}} \\
&= -\int_{X_s} h_0 V_\alpha \overline{V}_\beta \varphi_i \frac{\omega_s^{r-1}}{(r-1)!}
\end{aligned}$$

□

Theorem 1.3.2. *The curvature tensor of the Hermitian metric f on $S^kE \otimes F$ is*

$$R_{i\bar{j}\alpha\bar{\beta}} = \int_{X_s} h_0 V_\alpha \overline{V}_\beta \varphi_{i\bar{j}} \frac{\omega_s^{r-1}}{(r-1)!} - \int_{X_s} h_0 P_{i\alpha} \overline{P}_{j\beta} \frac{\omega_s^{r-1}}{(r-1)!} \tag{1.3.6}$$

where

$$P_{i\alpha} = -V_\alpha \varphi_i - \sum_\gamma V_\gamma \left(\sum_\delta f^{\gamma\bar{\delta}} \frac{\partial f_{\alpha\bar{\delta}}}{\partial z^i} \right) \tag{1.3.7}$$

Proof. The idea we use is due to Berndtsson([Berndtsson09a], Section 2). For simplicity of notations, we set $A_{i\alpha} = -V_\alpha \varphi_i$. The Hermitian metric (1.3.3) is also a norm on the smooth section space $\Gamma(X_s, \widetilde{L}|_{X_s})$, and it induces an orthogonal projection

$$\widetilde{\pi}_s : \Gamma(X_s, \widetilde{L}|_{X_s}) \to H^0(X_s, \widetilde{L}|_{X_s})$$

Using this projection, we can rewrite the first order derivative as

$$\begin{aligned}
\frac{\partial f_{\alpha\bar{\beta}}}{\partial z^i} &= \int_{X_s} \langle A_{i\alpha} e_{\widetilde{L}}, V_\beta e_{\widetilde{L}} \rangle_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} \\
&= \int_{X_s} \langle \widetilde{\pi}_s(A_{i\alpha} e_{\widetilde{L}}) + (A_{i\alpha} e_{\widetilde{L}} - \widetilde{\pi}_s(A_{i\alpha} e_{\widetilde{L}})), V_\beta e_{\widetilde{L}} \rangle_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} \\
&= \int_{X_s} \langle \widetilde{\pi}_s(A_{i\alpha} e_{\widetilde{L}}), V_\beta e_{\widetilde{L}} \rangle_{h_0} \frac{\omega_s^{r-1}}{(r-1)!}
\end{aligned}$$

since $(A_{i\alpha} e_{\widetilde{L}} - \widetilde{\pi}_s(A_{i\alpha} e_{\widetilde{L}}))$ is in the orthogonal complement of $H^0(X_s, \widetilde{L}|_{X_s})$. By this relation, we can write $\widetilde{\pi}_s(A_{i\alpha} e_{\widetilde{L}})$ in the basis $\{V_\alpha e_{\widetilde{L}}\}$ of $H^0(X_s, \widetilde{L}|_{X_s})$,

$$\widetilde{\pi}_s(A_{i\alpha} e_{\widetilde{L}}) = \sum_\gamma \left(\sum_\delta f^{\gamma\bar{\delta}} \frac{\partial f_{\alpha\bar{\delta}}}{\partial z^i} \right) (V_\gamma e_{\widetilde{L}}) \tag{1.3.8}$$

From this identity, we obtain

$$\int_{X_s} \left\langle \widetilde{\pi}_s(A_{i\alpha} e_{\widetilde{L}}), \widetilde{\pi}_s(A_{j\beta} e_{\widetilde{L}}) \right\rangle_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} = \sum_{\gamma,\delta} f^{\gamma\bar{\delta}} \frac{\partial f_{\alpha\bar{\delta}}}{\partial z^i} \frac{\partial f_{\gamma\bar{\beta}}}{\partial \bar{z}^j} \tag{1.3.9}$$

Suppose

$$P_{i\alpha} = A_{i\alpha} - \sum_\gamma V_\gamma \left(\sum_\delta f^{\gamma\bar{\delta}} \frac{\partial f_{\alpha\bar{\delta}}}{\partial z^i} \right) \tag{1.3.10}$$

then $A_{i\alpha} e_{\widetilde{L}} = \widetilde{\pi}_s(A_{i\alpha} e_{\widetilde{L}}) + P_{i\alpha} e_{\widetilde{L}}$, that is,

$$\widetilde{\pi}_s(P_{i\alpha} e_{\widetilde{L}}) = 0 \tag{1.3.11}$$

Similar to Lemma 1.3.1, we obtain the second order derivative

$$\begin{aligned}
\frac{\partial^2 f_{\alpha\overline{\beta}}}{\partial z^i \partial \overline{z}^j} &= -\int_{X_s} h_0 V_\alpha \overline{V}_\beta \varphi_{i\overline{j}} \frac{\omega_s^{r-1}}{(r-1)!} + \int_{X_s} \langle V_\alpha \varphi_i e_{\widetilde{L}}, V_\beta \varphi_j e_{\widetilde{L}} \rangle_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} \\
&= -\int_{X_s} h_0 V_\alpha \overline{V}_\beta \varphi_{i\overline{j}} \frac{\omega_s^{r-1}}{(r-1)!} + \int_{X_s} \langle A_{i\alpha} e_{\widetilde{L}}, A_{j\beta} e_{\widetilde{L}} \rangle_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} \\
&= -\int_{X_s} h_0 V_\alpha \overline{V}_\beta \varphi_{i\overline{j}} \frac{\omega_s^{r-1}}{(r-1)!} \\
&\quad + \int_{X_s} \left\langle P_{i\alpha} e_{\widetilde{L}} + \widetilde{\pi}_s (A_{i\alpha} e_{\widetilde{L}}), P_{j\beta} e_{\widetilde{L}} + \widetilde{\pi}_s (A_{j\beta} e_{\widetilde{L}}) \right\rangle_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} \\
&= -\int_{X_s} h_0 V_\alpha \overline{V}_\beta \varphi_{i\overline{j}} \frac{\omega_s^{r-1}}{(r-1)!} + \int_{X_s} h_0 P_{i\alpha} \overline{P}_{j\beta} \frac{\omega_s^{r-1}}{(r-1)!} + \\
&\quad \int_{X_s} \left\langle \widetilde{\pi}_s (A_{i\alpha} e_{\widetilde{L}}), \widetilde{\pi}_s (A_{j\beta} e_{\widetilde{L}}) \right\rangle_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} \\
&= -\int_{X_s} h_0 V_\alpha \overline{V}_\beta \varphi_{i\overline{j}} \frac{\omega_s^{r-1}}{(r-1)!} + \int_{X_s} h_0 P_{i\alpha} \overline{P}_{j\beta} \frac{\omega_s^{r-1}}{(r-1)!} \\
&\quad + f^{\gamma\overline{\delta}} \frac{\partial f_{\alpha\overline{\delta}}}{\partial z^i} \frac{\partial f_{\gamma\overline{\beta}}}{\partial \overline{z}^j}
\end{aligned}$$

By formula (1.3.4), we get the curvature formula (1.3.6). □

1.3.2 Positivity of Hermitian metrics on $S^k E \otimes F$

If (E, h) is a Griffiths-positive, Demailly-Skoda([Demailly-Skoda80]) showed that $(E \otimes \det E, h \otimes \det h)$ is Nakano-positive. They proved it by using a discrete Fourier transformation method. Here, we use a linear algebraic argument to show $(E \otimes \det E, h \otimes \det h)$ is both Nakano-positive and dual-Nakano-positive.

Let ω_{FS} be the standard Fubini-Study metric on $\mathbb{P}^{r-1}$ and $[W_1, \cdots W_r]$ the homogeneous coordinates on $\mathbb{P}^{r-1}$. If $A = (\alpha_1, \cdots, \alpha_k)$ and $B = (\beta_1, \beta_2, \cdots, \beta_k)$, we define the generalized Kronecker-δ for multi-index by the following formula

$$\delta_{AB} = \sum_{\sigma \in S_k} \prod_{j=1}^{k} \delta_{\alpha_{\sigma(j)} \beta_{\sigma(j)}} \tag{1.3.12}$$

where S_k is the permutation group in k symbols.

Lemma 1.3.3. *If $V_A = W_{\alpha_1} \cdots W_{\alpha_k}$ and $V_B = W_{\beta_1} \cdots W_{\beta_k}$, then*

$$\int_{\mathbb{P}^{r-1}} \frac{V_A \overline{V}_B}{|W|^{2k}} \frac{\omega_{FS}^{r-1}}{(r-1)!} = \frac{\delta_{AB}}{(r+k-1)!} \tag{1.3.13}$$

For simple-index notations,

$$\int_{\mathbb{P}^{r-1}} \frac{W_\alpha \overline{W}_\beta}{|W|^2} \frac{\omega_{FS}^{r-1}}{(r-1)!} = \frac{\delta_{\alpha\beta}}{r!}, \int_{\mathbb{P}^{r-1}} \frac{W_\alpha \overline{W_\beta} W_\gamma \overline{W_\delta}}{|W|^4} \frac{\omega_{FS}^{r-1}}{(r-1)!} = \frac{\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}}{(r+1)!} \tag{1.3.14}$$

Without loss of generality we can assume, at a fixed $s \in S$, $h_{\alpha\overline{\beta}}(s) = \delta_{\alpha\beta}$. The curvature of $(E \otimes \det E, h \otimes \det h)$ is

$$R^{E\otimes \det E}_{i\overline{j}\alpha\overline{\beta}}(s) = R_{i\overline{j}\alpha\overline{\beta}}(s) + \delta_{\alpha\beta} \cdot \sum_\gamma R_{i\overline{j}\gamma\overline{\gamma}}(s) \tag{1.3.15}$$

By Lemma 1.3.3, we obtain

$$R_{i\overline{j}\alpha\overline{\beta}}(s) + \delta_{\alpha\beta} \cdot \sum_\gamma R_{i\overline{j}\gamma\overline{\gamma}}(s) = r! \cdot \int_{\mathbb{P}^{r-1}} \frac{W_\alpha \overline{W}_\beta}{|W|^2} \varphi_{i\overline{j}} \frac{\omega_{FS}^{r-1}}{(r-1)!} \tag{1.3.16}$$

where

$$\varphi_{i\overline{j}} = (r+1) \sum_{\gamma,\delta} R_{i\overline{j}\gamma\overline{\delta}}(s) \frac{W_\delta \overline{W}_\gamma}{|W|^2} \tag{1.3.17}$$

If (E, h) is Griffiths-positive, then $(\varphi_{i\overline{j}})$ is Hermitian positive. For any nonzero $u = (u^{i\alpha})$

$$R^{E\otimes \det E}_{i\overline{j}\alpha\overline{\beta}} u^{i\beta} \overline{u}^{j\alpha} = (r+1) \int_{\mathbb{P}^{r-1}} \varphi_{i\overline{j}} \frac{(u^{i\beta}\overline{W}_\beta) \cdot \overline{(u^{j\alpha}\overline{W}_\alpha)}}{|W|^2} \frac{\omega_{FS}^{r-1}}{(r-1)!} > 0 \tag{1.3.18}$$

Therefore, $(E \otimes \det E, h \otimes \det h)$ is dual-Nakano-positive. By a similar formulation, we know $(E \otimes \det E, h \otimes \det h)$ is Nakano-positive.

In the following, we will prove similar results for ample vector bundles.

1.3.2.1 Nakano-positivity

In this subsection, we will use $\overline{\partial}$-estimate on a compact Kähler manifold to analyze the curvature formula in Theorem 1.3.2,

$$R_{i\overline{j}\alpha\overline{\beta}} = \int_{X_s} h_0 V_\alpha \overline{V}_\beta \varphi_{i\overline{j}} \frac{\omega_s^{r-1}}{(r-1)!} - \int_{X_s} h_0 P_{i\alpha} \overline{P}_{j\beta} \frac{\omega_s^{r-1}}{(r-1)!}$$

The first term on the right hand side involves the horizontal direction curvature $\varphi_{i\bar{j}}$ of the line bundle $\widetilde{L}\otimes\det(T_{X/S})$. If the line bundle $\widetilde{L}\otimes\det(T_{X/S})$ is positive in the horizontal direction, we can choose (h_0,ω_s) such that φ is positive in the horizontal direction, i.e. $(\varphi_{i\bar{j}})$ is Hermitian positive. We will get a lower bound of the second term by using Hörmander's L^2-estimate, following an idea of Berndtsson([Berndtsson09a]).

Lemma 1.3.4. *Let (M^n,ω_g) be a compact Kähler manifold and (L,h) a Hermitian line bundle over M. If there exists a positive constant c such that*

$$Ric(\omega_g)+R^h\geq c\omega_g \tag{1.3.19}$$

then for any $w\in\Gamma(M,T^{*0,1}M\otimes L)$ such that $\overline{\partial}w=0$, there exists a unique $u\in\Gamma(M,L)$ such that $\overline{\partial}u=w$ and $\widetilde{\pi}(u)=0$ where $\widetilde{\pi}:\Gamma(M,L)\to H^0(M,L)$ is the orthogonal projection. Moreover,

$$\int_M|u|_h^2\frac{\omega_g^n}{n!}\leq\frac{1}{c}\int_M|w|^2_{g^*\otimes h}\frac{\omega_g^n}{n!} \tag{1.3.20}$$

We refer the reader to [Demailly] and [Hormander66] for the proof of Lemma 1.3.4.

Now we apply Lemma 1.3.4 to each fiber (X_s,ω_s) and $(\widetilde{L}|_{X_s},h_0|_{X_s})$. At a fixed point $s\in S$, the fiber direction curvature of the induced metric on $\widetilde{L}\otimes\det(T_{X/S})$ is

$$-\frac{\sqrt{-1}}{2\pi}\partial_s\overline{\partial}_s\log(h_0\det(g_{A\overline{B}}))=R^{\widetilde{L}_s^{h_0}}+Ric_F(\omega_s) \tag{1.3.21}$$

On the other hand

$$-\frac{\sqrt{-1}}{2\pi}\partial_s\overline{\partial}_s\log(h_0\det(g_{A\overline{B}}))=\frac{\sqrt{-1}}{2\pi}\partial_s\overline{\partial}_s\varphi$$

where $\varphi=-\log(h_0\det(g_{A\overline{B}}))$. So condition (1.3.19) turns out to be

$$(\varphi_{A\overline{B}})\geq c_s(g_{A\overline{B}}) \tag{1.3.22}$$

for some positive constant $c_s=c(s)$.

Theorem 1.3.5. *If $(\varphi_{A\overline{B}})\geq c_s(g_{A\overline{B}})$ at point $s\in S$, then for any*

$$u=\sum_{i,\alpha}u^{i\alpha}\frac{\partial}{\partial z^i}\otimes E_\alpha\in\Gamma(S,T^{1,0}S\otimes\widetilde{E})$$

with $\widetilde{E}=S^kE\otimes F$, we have the following estimate at point s,

$$R_{i\bar{j}\alpha\overline{\beta}}u^{i\alpha}\overline{u^{j\beta}}\geq\int_{X_s}h_0(V_\alpha u^{i\alpha})\overline{(V_\beta u^{j\beta})}\left(\varphi_{i\bar{j}}-\frac{g^{A\overline{B}}\varphi_{i\overline{B}}\varphi_{A\bar{j}}}{c_s}\right)\frac{\omega_s^{r-1}}{(r-1)!} \tag{1.3.23}$$

Proof. At point $s \in S$, we set

$$P = \sum_{i,\alpha} P_{i\alpha} u^{i\alpha} e_{\widetilde{L}} \in \Gamma(X_s, \widetilde{L}_s), \quad K = -\sum_{i,\alpha} V_\alpha \varphi_i u^{i\alpha} e_{\widetilde{L}} \in \Gamma(X_s, \widetilde{L}_s)$$

It is obvious that $\overline{\partial}_s P = \overline{\partial}_s K$ where $\overline{\partial}_s$ is $\overline{\partial}$ on the fiber direction. On the other hand, by (1.3.11), $\widetilde{\pi}_s(P) = 0$. So we can apply Lemma 1.3.4 and get

$$\int_{X_s} |P|^2_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} \leq \frac{1}{c_s} \int_{X_s} |\overline{\partial}_s K|^2_{g_s^* \otimes h_0} \frac{\omega_s^{r-1}}{(r-1)!} \tag{1.3.24}$$

Since $\overline{\partial}_s K = -\sum\limits_{i,\alpha,B} V_\alpha \varphi_{i\overline{B}} u^{i\alpha} d\overline{z}^B \otimes e_{\widetilde{L}}$,

$$|\overline{\partial}_s K|^2_{g_s^* \otimes h_0} = \sum_{i,j} \sum_{\alpha,\beta} h_0 (V_\alpha u^{i\alpha}) \overline{(V_\beta u^{j\beta})} g^{A\overline{B}} \varphi_{i\overline{B}} \varphi_{A\overline{j}}$$

By inequality (1.3.24) and Theorem 1.3.2, we get the estimate (1.3.23). □

Before proving the main theorems, we need the following lemma:

Lemma 1.3.6. *If E is a holomorphic vector bundle with rank r over a compact Kähler manifold S and F is a line bundle over S such that $S^{k+r}E \otimes \det E^* \otimes F$ is ample over S, then there exists a positive Hermitian metric λ_0 on $\mathcal{O}_{\mathbb{P}(E^*)}(k) \otimes \pi^*(F) \otimes \det(T_{X/S})$.*

Proof. Let $\widehat{E}$ be $S^{k+r}E \otimes \det(E^*) \otimes F$. It is obvious that $\mathbb{P}(S^{k+r}E^*) = \mathbb{P}(\widehat{E}^*)$. The tautological line bundles of them are related by the following formula

$$\mathcal{O}_{\mathbb{P}(\widehat{E}^*)}(1) = \mathcal{O}_{\mathbb{P}(S^{k+r}E^*)}(1) \otimes \pi^*_{k+r}(\det E^*) \otimes \pi^*_{k+r}(F) \tag{1.3.25}$$

where $\pi_{k+r} : \mathbb{P}(S^{k+r}E^*) \to S$ is the canonical projection. Let $v_{k+r} : \mathbb{P}(E^*) \to \mathbb{P}(S^{k+r}E^*)$ be the standard Veronese embedding, then

$$\mathcal{O}_{\mathbb{P}(E^*)}(k+r) = v^*_{k+r}\left(\mathcal{O}_{\mathbb{P}(S^{k+r}E^*)}(1)\right) \tag{1.3.26}$$

Similarly, let μ_{k+r} be the induced mapping $\mu_{k+r} : \mathbb{P}(E^*) \to \mathbb{P}(\widehat{E}^*)$, then

$$\mu^*_{k+r}\left(\mathcal{O}_{\mathbb{P}(\widehat{E}^*)}(1)\right) = \mathcal{O}_{\mathbb{P}(E^*)}(k+r) \otimes \pi^*(F \otimes \det E^*) \tag{1.3.27}$$

By the identity

$$K_X = \pi^*(K_S) \otimes \mathcal{O}_{\mathbb{P}(E^*)}(-r) \otimes \pi^*(\det E), \tag{1.3.28}$$

we obtain

$$\mu_{k+r}^*\left(\mathcal{O}_{\mathbb{P}(\widehat{E}^*)}(1)\right)=\mathcal{O}_{\mathbb{P}(E^*)}(k)\otimes\pi^*(F)\otimes\det(T_{X/S})=\widetilde{L}\otimes\det(T_{X/S}) \tag{1.3.29}$$

If $\widehat{E}$ is ample, then $\mathcal{O}_{\mathbb{P}(\widehat{E}^*)}(1)$ is ample and so is $\widetilde{L}\otimes\det(T_{X/S})$. So there exists a positive Hermitian metric λ_0 on $\widetilde{L}\otimes\det(T_{X/S})$. □

Theorem 1.3.7. *Let E be a holomorphic vector bundle over a compact Kähler manifold S and F a line bundle over S. Let r be the rank of E and $k\geq 0$ an arbitrary integer. If $S^{k+r}E\otimes\det E^*\otimes F$ is ample over S, then there exists a smooth Hermitian metric f on $S^kE\otimes F$ such that $(S^kE\otimes F,f)$ is Nakano-positive.*

Proof. By Lemma 1.3.6, there exists a positive Hermitian metric λ_0 on the ample line bundle $\widetilde{L}\otimes\det(T_{X/S})$. We set

$$\omega_s=-\frac{\sqrt{-1}}{2\pi}\partial_s\overline{\partial}_s\log\lambda_0=\frac{\sqrt{-1}}{2\pi}\sum_{A,B=1}^{r-1}g_{A\overline{B}}(s,w)dw^A\wedge d\overline{w}^B$$

which is a smooth family of Kähler metrics on the fibers X_s. We get an induced Hermitian metric on $\widetilde{L}$, namely,

$$h_0=\frac{\lambda_0}{\det(g_{A\overline{B}})} \tag{1.3.30}$$

Let f be the Hermitian metric on the vector bundle $S^kE\otimes\det F$ induced by $(\widetilde{L},h_0)$ and (X_s,ω_s)(see (1.3.3)). In this setting, the weight φ of induced metric on $\widetilde{L}\otimes\det(T_{X/S})$ is

$$\varphi=-\log\left(h_0\det(g_{A\overline{B}})\right)=-\log\lambda_0$$

Hence

$$(\varphi_{A\overline{B}})=(g_{A\overline{B}}) \tag{1.3.31}$$

and in Theorem 1.3.5, $c_s=1$ for any $s\in S$. Therefore

$$\begin{aligned}R^{\widetilde{E}}(u,u) &= R_{i\overline{j}\alpha\overline{\beta}}u^{i\alpha}\overline{u^{j\beta}}\\ &\geq \int_{X_s}h_0(V_\alpha u^{i\alpha})\overline{(V_\beta u^{j\beta})}\left(\varphi_{i\overline{j}}-\sum_{A,B=1}^{r-1}g^{A\overline{B}}\varphi_{i\overline{B}}\varphi_{A\overline{j}}\right)\frac{\omega_s^{r-1}}{(r-1)!}\\ &= \int_{X_s}h_0(V_\alpha u^{i\alpha})\overline{(V_\beta u^{j\beta})}\left(\varphi_{i\overline{j}}-\sum_{A,B=1}^{r-1}\varphi^{A\overline{B}}\varphi_{i\overline{B}}\varphi_{A\overline{j}}\right)\frac{\omega_s^{r-1}}{(r-1)!}\end{aligned}$$

for any $u = \sum\limits_{i,\alpha} u^{i\alpha}\frac{\partial}{\partial z^i} \otimes E_\alpha \in \Gamma(S, T^{1,0}S \otimes \widetilde{E})$.

On the other hand λ_0 is a positive Hermitian metric on the line bundle $\widetilde{L} \otimes \det(T_{X/S})$. The curvature form of λ_0 can be represented by a Hermitian positive matrix, namely, the coefficients matrix of Hermitian positive $(1,1)$ form $\sqrt{-1}\partial\overline{\partial}\varphi$ on X. By Lemma 1.2.12,

$$\left(\varphi_{i\bar{j}} - \sum_{A,B=1}^{r-1} \varphi^{A\overline{B}}\varphi_{i\overline{B}}\varphi_{A\bar{j}}\right)$$

is a Hermitian positive $n \times n$ matrix. Since the integrand is nonnegative, $R^{\widetilde{E}}(u,u) = 0$ if and only if

$$\sum_{i,j}\sum_{\alpha,\beta} h_0(V_\alpha u^{i\alpha})\overline{(V_\beta u^{j\beta})}\left(\varphi_{i\bar{j}} - \sum_{A,B=1}^{r-1} \varphi^{A\overline{B}}\varphi_{i\overline{B}}\varphi_{A\bar{j}}\right) \equiv 0 \tag{1.3.32}$$

on X_s which means $(u^{i\alpha})$ is a zero matrix. In summary, we obtain

$$R^{\widetilde{E}}(u,u) > 0$$

for nonzero u, i.e. the induced metric f on $\widetilde{E} = S^kE \otimes F$ is Nakano-positive. $\square$

Corollary 1.3.8. *If E is ample, then for large k, S^kE is Griffiths positive, i.e. there exists a Hermitian metric h_k on S^kE such that h_k is Griffiths-positive.*

1.3.2.2 Dual-Nakano-positivity

By the curvature identity on $S^kE \otimes F$,

$$R_{i\bar{j}\alpha\overline{\beta}} = \int_{X_s} h_0 V_\alpha \overline{V}_\beta \varphi_{i\bar{j}} \frac{\omega_s^{r-1}}{(r-1)!} - \int_{X_s} h_0 P_{i\alpha}\overline{P}_{j\beta}\frac{\omega_s^{r-1}}{(r-1)!}$$

where φ is a weight of the line bundle $\mathcal{O}_{\mathbb{P}(E^*)}(k+r) \otimes \pi^*(\det E^*) \otimes \pi^*(F)$. Although this line bundle can not be negative, it is still possible that it is negative in the local horizontal direction, i.e. $(\varphi_{i\bar{j}})$ is a Hermitian negative matrix. For example, F is a " very negative" line bundle over S. If

$(\varphi_{i\bar{j}})$ is Hermitian negative, then for any nonzero $u = (u^{i\alpha})$,

$$\begin{aligned} R_{i\bar{j}\alpha\bar{\beta}} u^{i\alpha}\overline{u^{j\beta}} &= \int_{X_s} h_0 \varphi_{i\bar{j}} (V_\alpha u^{i\alpha}) \overline{(V_\beta u^{j\beta})} \frac{\omega_s^{r-1}}{(r-1)!} \\ &\quad - \int_{X_s} h_0 P_{i\alpha} \overline{P}_{j\beta} u^{i\alpha} \overline{u}^{j\beta} \frac{\omega_s^{r-1}}{(r-1)!} \\ &\leq \int_{X_s} h_0 \varphi_{i\bar{j}} (V_\alpha u^{i\alpha}) \overline{(V_\beta u^{j\beta})} \frac{\omega_s^{r-1}}{(r-1)!} \\ &< 0 \end{aligned}$$

Hence $S^k E \otimes F$ is Nakano-negative. In the following, we will prove that if $(S^{k+r}E \otimes \det E^* \otimes F)^*$ is ample, then $S^k E \otimes F$ is Nakano-negative which is equivalent to the statement: if $S^{k+r}E \otimes \det E^* \otimes F$ is ample, then $S^k E \otimes F$ is dual-Nakano-positive. Here we use a well-known fact ([Demailly]):

E is dual-Nakano-positive if and only if E^* is Nakano-negative.

For simplicity, we assume $k = 1$ and $F = \det E$. In the following we will show, if E^* is ample, then $E \otimes \det E$ is Nakano-negative.

As similar as the quotient metric on $\mathcal{O}_{\mathbb{P}(E^*)}(1)$(see Proposition 1.2.10) induced by the morphism $(\pi^* E, \pi^* h) \to \mathcal{O}_{\mathbb{P}(E^*)}(1)$, there is an induced metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$ by the morphism $(\pi^*(E^*), \pi^* h^*) \to \mathcal{O}_{\mathbb{P}(E)}(1)$. For a fixed point $s \in S$, we can choose a local coordinate system $(z^1, \cdots, z^n)$ and a local normal frame $(e_1, \cdots, e_r)$ of E centered at point s. With respect to this trivialization, we obtain:

Proposition 1.3.9. *If (E, h) is Griffiths-positive, then the quotient metric h^L on $L := \mathcal{O}_{\mathbb{P}(E)}(1)$ induced by $(\pi^* E^*, \pi^* h^*) \to \mathcal{O}_{\mathbb{P}(E)}(1)$ is negative in the local horizontal direction, i.e.*

$$\left(-\frac{\partial^2 \log h^L}{\partial z^i \partial \overline{z}^j} \right) \tag{1.3.33}$$

is Hermitian negative on the fiber $X_s = \pi^{-1}(s)$ where $\pi : \mathbb{P}(E) \to S$.

Proof. Let $h_{\alpha\bar{\beta}} = h(e_\alpha, e_\beta)$ and $R_{i\bar{j}\alpha\bar{\beta}}$ be the curvature components of h, then the quotient metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$ is,

$$h^L = \frac{1}{\sum h_{\alpha\bar{\beta}} W_\alpha \overline{W}_\beta} = \frac{1}{\sum (\delta_{\alpha\beta} - R_{i\bar{j}\alpha\bar{\beta}} z^i \overline{z}^j + O(|z|^3)) W_\alpha \overline{W}_\beta} \tag{1.3.34}$$

It is obvious that

$$-\frac{\partial^2 \log h^L}{\partial z^i \partial \overline{z}^j} = -\sum_{\alpha,\beta} R_{i\bar{j}\alpha\overline{\beta}}(s)\frac{W_\alpha \overline{W}_\beta}{|W|^2} \tag{1.3.35}$$

which is Hermitian negative on X_s if (E, h) is Griffiths-positive. $\square$

Let $v_k : E \to S^k E$ be the standard Veronese map which induces a map

$$\overline{v}_k : \mathbb{P}(E) \to \mathbb{P}(S^k E) \tag{1.3.36}$$

Let $\pi : \mathbb{P}(E) \to S$ and $\pi_k : \mathbb{P}(S^k E) \to S$, then $\pi_k \circ \overline{v}_k = \pi$. Now we fix a local holomorphic coordinate system $(z^1, \cdots, z^n)$ centered at point $s \in S$ and a local trivialization of E and $S^k E$. It is obivous that the map $\overline{v}_k$ sends (z, W) to $(z, S^k W)$ where $S^k W$ is the k-th symmetric power of homogeneous vector $W = [W_1, \cdots, W_r]$, and so the horizontal part of $\overline{v}_k$ is identity. With respect to this trivialization, we obtain

Theorem 1.3.10. *If E is ample, then there exists a Hermitian metric h^L on $L = \mathcal{O}_{\mathbb{P}(E)}(1)$ such that h^L is negative in the horizontal direction, i.e.*

$$\left(-\frac{\partial^2 \log h^L}{\partial z^i \partial \overline{z}^j}\right) \tag{1.3.37}$$

is Hermitian negative on the fiber $X_s = \pi^{-1}(s)$ where $\pi : \mathbb{P}(E) \to S$.

Proof. By Corollary 1.3.8, for large k, $S^k E$ is Griffiths-positive. By Proposition 1.3.9, there exists a Hermitian metric $\widehat{h}_k$ on $\mathcal{O}_{\mathbb{P}(S^k E)}(1)$, such that $\widehat{h}_k$ is Hermitian negative along the horizontal direction. By the relation

$$\mathcal{O}_{\mathbb{P}(E)}(k) = \overline{v}_k^* \left(\mathcal{O}_{\mathbb{P}(S^k E)}(1)\right) \tag{1.3.38}$$

there is an induced metric h^L on $\mathcal{O}_{\mathbb{P}(E)}(1)$

$$h^L := \left(\overline{v}_k^*(\widehat{h}_k)\right)^{\frac{1}{k}} \tag{1.3.39}$$

Hence, we obtain

$$-\frac{\partial^2 \log h^L}{\partial z^i \partial \overline{z}^j} = -\frac{1}{k}\frac{\partial^2 \log \widehat{h}_k}{\partial z^i \partial \overline{z}^j} \tag{1.3.40}$$

since the horizontal direction of $\overline{v}_k$ is identity with respect to that trivialization. $\square$

Theorem 1.3.11. *If E^* is ample, then there exists a Hermitian metric on $E \otimes \det E$ which is Nakano-negative.*

Proof. By Theorem 1.3.10, if E^* is ample, then there exists a Hermitian metric h^L on $L := \mathcal{O}_{\mathbb{P}(E^*)}(1)$ such that

$$\left(-\frac{\partial^2 \log h^L}{\partial z^i \partial \overline{z}^j}\right) \tag{1.3.41}$$

is Hermitian negative. Let $\{\omega_s\}_{s\in S}$ be a smooth family of Hermitian metric of the fiber X_s. We can set

$$h_0 = \frac{(h^L)^{r+1}}{\det(\omega_s)}$$

and let

$$\varphi = -\log(h_0 \det(\omega_s)) = -(r+1)\log h^L \tag{1.3.42}$$

Hence, we obtain

$$\varphi_{i\overline{j}} = -(r+1)\frac{\partial^2 \log h^L}{\partial z^i \partial \overline{z}^j} \tag{1.3.43}$$

Therefore $(\varphi_{i\overline{j}})$ is Hermitian negative. On the other hand, the metric induced by h_0 and $\{\omega_s\}_{s\in S}$ on $E \otimes \det E$ has curvature components

$$R_{i\overline{j}\alpha\overline{\beta}} = \int_{X_s} h_0 W_\alpha \overline{W}_\beta \varphi_{i\overline{j}} \frac{\omega_s^{r-1}}{(r-1)!} - \int_{X_s} h_0 P_{i\alpha} \overline{P}_{j\beta} \frac{\omega_s^{r-1}}{(r-1)!} \tag{1.3.44}$$

Therefore, for any nonzero $u = (u^{i\alpha})$,

$$\begin{aligned} R_{i\overline{j}\alpha\overline{\beta}} u^{i\alpha}\overline{u^{j\beta}} &\leq \int_{X_s} h_0 \varphi_{i\overline{j}} (W_\alpha u^{i\alpha}) \overline{(W_\beta u^{j\beta})} \frac{\omega_s^{r-1}}{(r-1)!} \\ &< 0 \end{aligned}$$

The proof of Nakano-negativity of $E \otimes \det E$ is completed. □

Combined with Theorem 1.3.7, Lemma 1.3.6 and Theorem 1.3.11 we obtain,

Theorem 1.3.12. *Let E be a holomorphic vector bundle over a compact Kähler manifold S and F a line bundle over S. Let r be the rank of E and $k \geq 0$ an arbitrary integer. If $S^{k+r}E \otimes \det E^* \otimes F$ is ample over S, then $S^k E \otimes F$ is both Nakano-positive and dual-Nakano-positive.*

1.3.2.3 Applications

Corollary 1.3.13. *If E is an ample vector bundle and F is a nef line bundle, then there exists $k_0 = k_0(S, E)$ such that $S^k E \otimes F$ is Nakano-positive and dual-Nakano-positive for any $k \geq k_0$. In particular, $S^k E$ is Nakano-positive and dual-Nakano-positive for $k \geq k_0$.*

Proof. It is easy to see that there exists $k_0 = k_0(S, E)$ such that for any $k \geq k_0$, $S^{k+r} E \otimes \det E^*$ is ample, and so is $S^{k+r} E \otimes \det E^* \otimes F$. By Theorem 1.3.12, $S^k E \otimes F$ is Nakano-positive and dual-Nakano-positive. In particular, $S^k E$ is Nakano-positive and dual-Nakano-positive for $k \geq k_0$. □

Corollary 1.3.14. *If E is an ample vector bundle and F is a nef line bundle, or E is a nef vector bundle and F is an ample line bundle,*

(1) $S^k E \otimes \det E \otimes F$ is Nakano-positive and dual-Nakano-positive for any $k \geq 0$.

(2) If the rank r of E is greater than 1, then $S^m E^* \otimes (\det E)^t \otimes F$ is Nakano-positive and dual-Nakano-positive if $t \geq r + m - 1$.

Proof. (1) It follows by the ampleness of $S^{k+r} E \otimes F = S^{k+r} E \otimes \det E^* \otimes (\det E \otimes F)$.

(2) If $r > 1$, it is easy to see $E^* \otimes \det E = \wedge^{r-1} E$. By the relation

$$S^{r+m}(E^* \otimes \det E) \otimes (\det E)^{t-r-m+1} \otimes F = S^{r+m} E^* \otimes \det E \otimes (\det E)^t \otimes F$$

we can apply Theorem 1.3.12 to the pair $(E^*, (\det E)^t \otimes F)$ and obtain the Nakano-positivity and dual-Nakano-positivity of $S^m E^* \otimes (\det E)^t \otimes F$ when $t \geq r + m - 1$. Let $E = T\mathbb{P}^2$, then $E = E^* \otimes \det E$ is Griffiths-positive but not Nakano-positive. So we can not remove the restriction $t \geq r + m - 1$. □

Corollary 1.3.15. *If $S^{r+1} E \otimes \det E^*$ is ample, then E is Nakano-positive and dual-Nakano-positive and so E is Griffiths-positive.*

Remark 1.3.16. By Corollary 1.3.15, the ampleness of $\mathcal{O}_{\mathbb{P}(E^*)}(r+1) \otimes \pi^*(\det E^*)$ implies the ampleness of $\mathcal{O}_{\mathbb{P}(E^*)}(1)$. But in general, the ampleness of $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ can not imply the ampleness of $\mathcal{O}_{\mathbb{P}(E^*)}(r+1) \otimes \pi^*(\det E^*)$.

1.4 Nakano-positivity and dual-Nakano-positivity of adjoint vector bundles

The following lemma is due to Fujita ([Fujita91]) and [Ye-Zhang90].

Lemma 1.4.1. *Let E be an ample vector bundle over S. Let r be the rank of E and n the dimension of S. If $r \geq n+1$, then $\det E \otimes K_S$ is ample except $(S, E) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1})$.*

Theorem 1.4.2. *Let E be an ample vector bundle over S. Let r be the rank of E and n the dimension of S.*

(1) If $r > 1$, then $S^k E \otimes (\det E)^2 \otimes K_S$ is Nakano-positive and dual-Nakano-positive for any $k \geq \max\{n-r, 0\}$.

(2) If $r = 1$, then the line bundle $E^{\otimes(n+2)} \otimes K_S$ is Nakano-positive.

Moreover, the lower bound on k is sharp.

Proof. **(1)** If $r > 1$, then $X = \mathbb{P}(E^*)$ is a $\mathbb{P}^{r-1}$ bundle which is not isomorphic to any projective space. By Lemma 1.4.1, $\mathcal{O}_{\mathbb{P}(E^*)}(n+r) \otimes K_X$ is ample. So

$$\mathcal{O}_{\mathbb{P}(E^*)}(n) \otimes \pi^* (K_S \otimes \det E)$$

is ample and it is equivalent to the ampleness of $S^n E \otimes (\det E^*) \otimes (\det E)^2 \otimes K_S$. If $k \geq \max\{n-r, 0\}$, $S^{r+k} E \otimes \det E^* \otimes (\det E)^2 \otimes K_S$ is also ample, hence by Theorem 1.3.12, $S^k E \otimes (\det E)^2 \otimes K_S$ is Nakano-positive and dual-Nakano-positive.

(2) It follows from Lemma 1.4.1. In fact, the vector bundle $\widetilde{E} = E^{\oplus(n+2)}$ is an ample vector bundle of rank $n+2$ and $\det \widetilde{E} = E^{\otimes(n+2)}$. By Lemma 1.4.1, $\det \widetilde{E} \otimes K_S = E^{\otimes(n+2)} \otimes K_S$ is ample.

Here the lower bound $n-r$ is sharp. For any integer $k_0 < n-r$, there exists some ample vector E such that $E \otimes (\det E)^{k_0} \otimes K_S$ is not Nakano-positive, for example $(S, E) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1) \oplus \mathcal{O}_{\mathbb{P}^4}(1))$. □

Theorem 1.4.3. *Let E be an ample vector bundle over S. Let r be the rank of E and n the dimension of S. If $r > 1$, then $E \otimes (\det E)^k \otimes K_S$ is Nakano-positive and dual-Nakano-positive for any $k \geq \max\{n+1-r, 2\}$. Moreover, the lower bound is sharp.*

Proof. If $r \geq n-1$, by Theorem 1.4.2, $E \otimes (\det E)^2 \otimes K_S$ is Nakano-positive and dual-Nakano-positive. Now we consider $1 < r < n-1$. By ([Ishihara01, Theorem 2.5]), $K_S \otimes (\det E)^{n-r}$ is nef except the case $(S, E) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1) \oplus \mathcal{O}_{\mathbb{P}^4}(1))$. It is easy to check

$$S^{r+1}E \otimes K_S \otimes (\det E)^{n-r}$$

is also ample in that case. By Theorem 1.3.12, $E \otimes (\det E)^{n+1-r} \otimes K_S$ is Nakano-positive and dual-Nakano-positive. Here the lower bound $n+1-r$ is sharp. For any integer $k_0 < n+1-r$, there exists an ample vector bundle E such that $E \otimes (\det E)^{k_0} \otimes K_S$ is not Nakano-positive, for example $(S, E) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1) \oplus \mathcal{O}_{\mathbb{P}^4}(1))$. $\square$

Remark 1.4.4. In Theorem 1.4.2 and 1.4.3, if $r \geq n$, $E \otimes (\det E)^2 \otimes K_S$ is Nakano-positive and dual-Nakano-positive. If $E = T\mathbb{P}^n$, then $S^2E \otimes \det E \otimes K_{\mathbb{P}^n}$ is Nakano-positive and dual-Nakano-positive.

Problem: Is $S^2E \otimes \det E \otimes K_S$ Nakano-positive and dual-Nakano-positive when E is ample and $r \geq n$? If one can show $S^{n+2}E \otimes K_S$ is ample, or equivalently, $\mathcal{O}_{\mathbb{P}(E^*)}(n+2) \otimes \pi^*(K_S)$ is ample, by Theorem 1.3.12, $S^2E \otimes \det E \otimes K_S$ is Nakano-positive and dual-Nakano-positive.

1.5 Comparison of Griffiths-positive and Nakano-positive metrics

Let (E, h) be a Hermitian vector bundle. In general, it is not so easy to write down the exact curvature formula of (S^kE, S^kh). In this section, we give an algorithm to compute the curvature of (S^kE, S^kh). As applications, we can disprove the Griffiths-positivity and Nakano-positivity of a given metric on $\mathbb{P}^n$.

Let h be a Hermitian metric on E, h^L be the induced metric in (1.2.13) on $L = \mathcal{O}_{\mathbb{P}(E^*)}(1)$. Let F be a line bundle with Hermitian metric h^F. Naturally, there is an induced metric $S^kh \otimes h^F$ on the vector bundle $S^kE \otimes F$. On the other hand, we can construct a new metric f on $S^kE \otimes F$ by formula (1.3.3). There is a **canonical way** to do it. Let $\widetilde{L} = L^k \otimes \pi^*(F)$. The induced metric on $\widetilde{L}$ is $h_0 = (h^L)^k \otimes \pi^*(h^F)$ and the induced metric on $\det(T_{X/S}) = L^r \otimes \pi^*(\det E^*)$ is $(h^L)^r \otimes \pi^*(\det(h)^{-1})$. These two metrics induce a metric $\lambda_0 = (h^L)^{k+r} \otimes \pi^*\left(h^F \cdot \det(h)^{-1}\right)$ on

$\widetilde{L} \otimes \det(T_{X/S})$. Now we can polarize each fiber X_s by the curvature of λ_0. By formula (1.2.14),

$$\omega_s = -\frac{\sqrt{-1}}{2\pi}\partial_s\overline{\partial}_s \log \lambda_0 = \frac{(k+r)\sqrt{-1}}{2\pi}\partial_s\overline{\partial}_s \log\left(\sum h^{\alpha\overline{\beta}} W_\alpha \overline{W}_\beta\right) = (k+r)\omega_{FS} \qquad (1.5.1)$$

By a simple linear algebraic argument, we obtain

$$\frac{\lambda_0}{\det(\omega_s)} = \frac{(h^L)^k \otimes \pi^*(h^F)}{(k+r)^{r-1}} = \frac{h_0}{(k+r)^{r-1}} \qquad (1.5.2)$$

Now we can use $(\widetilde{L}, h_0)$ and (X_s, ω_s) to construct a "new" metric f on $S^kE \otimes F$ by formula (1.3.3).

Theorem 1.5.1. *The metric f has the form*

$$f = \frac{(r+k)^{r-1}}{(r+k-1)!} \cdot S^k h \otimes h^F \qquad (1.5.3)$$

Moreover, f is a constant multiple of the metric constructed in Theorem 1.3.12.

Proof. Without loss of generality, we can choose normal coordinates for the metric h at a fix point $s \in S$. By formula (1.2.14), the metric $h_0 = (h^L)^k \otimes h^F$ on $L^k \otimes F$ induced by (E, h) and (F, h^F) can be written as $\frac{h^F}{|W|^{2k}}$ locally on the fiber $X_s \cong \mathbb{P}^{r-1}$. By formula (1.5.1), the metric f defined by (1.3.3) has the following form

$$f_{\alpha\overline{\beta}} = \int_{X_s} h_0 V_\alpha \overline{V}_\beta \frac{\omega_s^{r-1}}{(r-1)!} = (k+r)^{r-1} h^F \int_{\mathbb{P}^{r-1}} \frac{V_\alpha \overline{V}_\beta}{|W|^{2k}} \frac{\omega_{FS}^{r-1}}{(r-1)!}$$

Here V_α, V_β are homogeneous monomials of degree k in $W_1, \cdots, W_r$. By Lemma 1.3.3,

$$f_{\alpha\beta} = \frac{(r+k)^{r-1}}{(r+k-1)!}\delta_{\alpha\beta} h^F$$

that is $f = \frac{(r+k)^{r-1}}{(r+k-1)!} \cdot S^k h \otimes h^F$. By formulas (1.5.2) and (1.3.30), f is a constant multiple of the metric constructed in Theorem 1.3.12. □

Theorem 1.5.2. *If (E, h) is a Griffiths-positive vector bundle, then*

(1) $(S^kE \otimes (\det E)^\ell, S^k h \otimes (\det h)^\ell)$ is Nakano-positive and dual-Nakano-positive for any $k \geq 0$ and $\ell \geq 1$.

(2) There exists $k_0 = k_0(M, E)$ such that $(S^kE, S^k h)$ is Nakano-positive and dual-Nakano-positive for any $k \geq k_0$.

Proof. These follow by Theorem 1.3.12 and Theorem 1.5.1. □

Proposition 1.5.3. *(1) (E, h) is Griffiths-positive if and only if $(S^k E, S^k h)$ is Griffiths-positive for some $k \geq 1$.*

(2) If (E, h) is (dual-)Nakano-positive, then $(S^k E, S^k h)$ is (dual-)Nakano-positive for any $k \geq 1$.

Proof. By Theorem 1.5.1, $S^k h$ is a constant multiple of the metric constructed by formula (1.3.3). So by Theorem 1.3.2, we can write down the curvature formula of $S^k h$ explicitly. In a normal coordinates of h at a fixed point, the curvature formula (1.3.6) can be simplified by Lemma 1.3.3. We obtain curvature formulas (1.5.4) and (1.5.6).

For the convenience of the reader, we assume $k = 2$ at first. We can choose normal coordinates at a fixed point. Let $\{e_1, \cdots, e_r\}$ be the local basis at that point. The ordered basis of $S^2 E$ at that point are $\{e_1 \otimes e_1, e_1 \otimes e_2, \cdots, e_r \otimes e_{r-1}, e_r \otimes e_r\}$. We denote them by $e_{(\alpha,\beta)} = e_\alpha \otimes e_\beta$ with $\alpha \leq \beta$. The curvature tensor $S^2 h$ is

$$R_{i\bar{j}(\alpha,\gamma)\overline{(\beta,\delta)}} = R_{i\bar{j}\alpha\bar{\beta}}\delta_{\gamma\delta} + R_{i\bar{j}\gamma\bar{\delta}}\delta_{\alpha\beta} + R_{i\bar{j}\gamma\bar{\beta}}\delta_{\alpha\delta} + R_{i\bar{j}\alpha\bar{\delta}}\delta_{\gamma\beta} \tag{1.5.4}$$

where $R_{i\bar{j}\alpha\bar{\beta}}$ is the curvature tensor of E. Let $u = \sum_i \sum_{\alpha \leq \gamma} u_{i(\alpha,\gamma)} e_{(\alpha,\gamma)} \in \Gamma(M, T^{1,0}M \otimes S^2 E)$. For simplicity of notations, we extend the values of $u_{i(\alpha,\gamma)}$ to all indices (α, γ) by setting $u_{i(\alpha,\gamma)} = 0$ if $\gamma < \alpha$. Therefore

$$\begin{aligned}
& \sum_{i,j} \sum_{\substack{\alpha \leq \gamma \\ \beta \leq \delta}} R_{i\bar{j}(\alpha,\gamma)\overline{(\beta,\delta)}} u_{i(\alpha,\gamma)} \overline{u}_{j(\beta,\delta)} \\
= & \sum_{i,j} \sum_{\alpha,\gamma,\beta,\delta} R_{i\bar{j}(\alpha,\gamma)\overline{(\beta,\delta)}} u_{i(\alpha,\gamma)} \overline{u}_{j(\beta,\delta)} \\
= & \sum_{i,j,\alpha,\beta,\gamma,\delta} \big(R_{i\bar{j}\alpha\bar{\beta}} u_{i(\alpha,\gamma)} \overline{u}_{j(\beta,\gamma)} + R_{i\bar{j}\gamma\bar{\delta}} u_{i(\alpha,\gamma)} \overline{u}_{j(\alpha,\delta)} \\
& + R_{i\bar{j}\gamma\bar{\beta}} u_{i(\alpha,\gamma)} \overline{u}_{j(\beta,\alpha)} + R_{i\bar{j}\alpha\bar{\delta}} u_{i(\alpha,\gamma)} \overline{u}_{j(\gamma,\delta)} \big) \qquad (1.5.5) \\
= & \sum_{\gamma} \sum_{i,j,\alpha,\beta} R_{i\bar{j}\alpha\bar{\beta}} \left(u_{i(\alpha,\gamma)} + u_{i(\gamma,\alpha)} \right) \overline{\left(u_{j(\beta,\gamma)} + u_{j(\gamma,\beta)} \right)}
\end{aligned}$$

Hence $(S^2 E, S^2 h)$ is Nakano-positive if (E, h) is Nakano-positive. For the general case, we set $A = (\alpha_1, \cdots, \alpha_k)$ and $B = (\beta_1, \cdots, \beta_k)$ with $\alpha_1 \leq \cdots \leq \alpha_k$ and $\beta_1 \leq \cdots \leq \beta_k$. The basis of

S^kE are $\{e_A = e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k}\}$. The curvature tensor of (S^kE, S^kh) is

$$R_{i\bar{j}A\overline{B}} = \sum_{\alpha,\beta=1}^{r} \sum_{s,t=1}^{k} R_{i\bar{j}\alpha\overline{\beta}} \delta_{\alpha\alpha_s} \delta_{\beta\beta_t} \delta_{A_sB_t} \tag{1.5.6}$$

where $A_s = (\alpha_1, \cdots, \alpha_{s-1}, \alpha_{s+1}, \cdots, \alpha_k)$, $B_t = (\beta_1, \cdots, \beta_{t-1}, \beta_{t+1}, \cdots, \beta_k)$ and $\delta_{A_sB_t}$ is the multi-index delta function(see formula (1.3.12)). We have the curvature formula,

$$\begin{aligned} &\sum_{i,j,A,B} R_{i\bar{j}A\overline{B}} u_{iA} \overline{u}_{jB} \\ = &\sum_{\alpha_1,\cdots,\alpha_{k-1}} \sum_{\sigma\in S_{k-1}} \sum_{i,j,\alpha,\beta} R_{i\bar{j}\alpha\overline{\beta}} V_{i\alpha\alpha_{\sigma(1)}\cdots\alpha_{\sigma(k-1)}} \overline{V}_{j\beta\alpha_{\sigma(1)}\cdots\alpha_{\sigma(k-1)}} \end{aligned} \tag{1.5.7}$$

where S_{k-1} is the permutation group in $(k-1)$ symbols and

$$V_{i\alpha\alpha_1\cdots\alpha_{k-1}} = \sum_{s=1}^{k} u_{iA^s}, \quad A^s = (\alpha_1, \cdots, \alpha_{s-1}, \alpha, \alpha_{s+1}, \cdots, \alpha_k)$$

The Nakano-positivity of (S^kE, S^kh) follows immediately from the Nakano-positivity of (E, h) by formula (1.5.7). With the help of curvature formula (1.5.6), we can prove Griffiths-positivity and dual-Nakano-positivity of S^kE in a similar way. Here, we use another way to show it. S^kE can be viewed as a quotient bundle of $E^{\otimes k}$. If (E, h) is Griffiths-positive(resp. dual-Nakano-positive), $(E^{\otimes k}, h^{\otimes k})$ is Griffiths-positive(resp. dual-Nakano-positive) and so the quotient bundle S^kE is Griffiths-positive(resp. dual-Nakano-positive)([Demailly]). The induced metrics on quotient bundles are exactly the given ones. $\square$

Remark 1.5.4. Part (1) is an analogue of ampleness: E is ample if and only if S^kE is ample for some $k \geq 1$. The converse of part (2) is not valid in general. We know $(S^2T\mathbb{P}^n, S^2h_{FS})$ is Nakano-positive, but $(T\mathbb{P}^n, h_{FS})$ is not Nakano-positive as shown in the following.

1.6 Vanishing theorems

1.6.1 Vanishing theorems for adjoint vector bundles

Theorem 1.6.1. *Let $E, E_1, \cdots, E_\ell$ be vector bundles over an n-dimensional compact Kähler manifold M. Their ranks are $r, r_1, \cdots, r_\ell$ respectively. Let L be a line bundle on M.*

(1) If E is ample, L is nef and $r > 1$, then

$$H^{n,q}(M, S^kE \otimes (\det E)^2 \otimes K_M \otimes L) = H^{q,n}(M, S^kE \otimes (\det E)^2 \otimes K_M \otimes L) = 0$$

for any $q \geq 1$ and $k \geq \max\{n - r, 0\}$.

(2) If E is ample, L is nef and $r > 1$, then

$$H^{n,q}(M, E \otimes (\det E)^k \otimes K_M \otimes L) = H^{q,n}(M, E \otimes (\det E)^k \otimes K_M \otimes L) = 0$$

for any $q \geq 1$ and $k \geq \max\{n + 1 - r, 2\}$.

(3) Let $r > 1$. If E is ample and L is nef, or E is nef and L is ample, then

$$H^{n,q}(M, S^mE^* \otimes (\det E)^t \otimes L) = H^{q,n}(M, S^mE^* \otimes (\det E)^t \otimes L) = 0$$

for any $q \geq 1$ and $t \geq r + m - 1$.

(4) If all E_i are ample and L is nef, or, all E_i are nef and L is ample, then for any $k_1 \geq 0, \cdots, k_\ell \geq 0$,

$$\begin{aligned} & H^{n,q}(M, S^{k_1}E_1 \otimes \cdots \otimes S^{k_\ell}E_\ell \otimes \det E_1 \otimes \cdots \otimes \det E_\ell \otimes L) \\ = \; & H^{q,n}(M, S^{k_1}E_1 \otimes \cdots \otimes S^{k_\ell}E_\ell \otimes \det E_1 \otimes \cdots \otimes \det E_\ell \otimes L) = 0 \end{aligned}$$

for $q \geq 1$.

Proof. By Theorem 1.4.2, Theorem 1.4.3 and Corollary 1.3.14, the vector bundles in consideration are all Nakano-positive and dual-Nakano-positive. The results follow from Lemma 1.2.20. □

Remark 1.6.2. Part (4) can be regarded as a generalization of Griffiths ([Griffiths69], Theorem G) and Demailly([Demailly88], Theorem 0.2).

The following results generalize Griffiths' vanishing theorem(see also [Laytimi-Nahm05a], Corollary 1.5):

Proposition 1.6.3. *Let r be the rank of E and $k \geq 1$. For any $t \geq 0$, if $S^{t+kr}E \otimes L$ is ample,*

$$H^{n,q}(M, S^tE \otimes (\det E)^k \otimes L) = H^{q,n}(M, S^tE \otimes (\det E)^k \otimes L) = 0$$

for any $q \geq 1$.

Proof. By Theorem 1.3.12, $S^t E \otimes (\det E)^k \otimes L$ is Nakano-positive and dual-Nakano-positive. The results follow by Nakano's vanishing theorem. □

Remark 1.6.4. Theorem 1.1.1 allows us to do induction to deduce more positivity results. For example, if $S^m E \otimes L$ is ample, then $S^{m-r} E \otimes \det E \otimes L$ is (dual-)Nakano-positive and so it is ample. Using Theorem 1.1.1 again, we get $S^{m-2r} \otimes (\det E)^2 \otimes L$ is Nakano-positive and dual-Nakano-positive. Finally, we get $S^t E \otimes (\det E)^k \otimes L$ is Nakano-positive and dual-Nakano-positive, if $m = t + kr$ for some $0 \leq t < r$. It is obvious that the (dual-)Nakano-positivity turns stronger and stronger under induction. This explains why a lot of vanishing theorems involve a power of $\det E$.

If L is an ample line bundle over a compact Kähler manifold M and F is an arbitrary line bundle over M. By comparing the Chern classes, there exists a constant m_0 such that $L^{m_0} \otimes F$ is ample and so it is positive. If E is an ample vector bundle and F is an arbitrary vector bundle, it is easy to see $S^k E \otimes F$ is ample for large k. But, in general, we don't know whether an ample vector bundle carries a Griffiths-positive or Nakano-positive metric. In the following, we will construct Nakano-positive and dual-Nakano-positive metrics on various ample vector bundles.

Lemma 1.6.5. *If L is an ample line bundle over M and F is an arbitrary vector bundle. There exists an integer m_0 such that $L^{m_0} \otimes F$ is Nakano-positive and dual-Nakano-positive.*

Proof. Let h_0 be a positive metric on L and ω be the curvature of h_0 which is also the Kähler metric fixed on M. For any metric g on F, the curvature R^g has a lower bound in the sense

$$\min_{x \in M} \inf_{u \neq 0} \frac{R^g(u(x), u(x))}{|u(x)|^2} \geq -(m_0 - 1) \tag{1.6.1}$$

where $u \in \Gamma(M, T^{1,0}M \otimes F)$. The curvature of metric $h^{m_0} \otimes g$ on $L^{m_0} \otimes F$ is given by

$$\widehat{R} = m_0 \omega \cdot g + h_0^m \cdot R^g \tag{1.6.2}$$

Therefore

$$\widehat{R}(v \otimes u, v \otimes u) \geq |u|^2 h_0^{m_0}(v, v)$$

for any $v \in \Gamma(M, L^{m_0})$ and $u \in \Gamma(M, T^{1,0}M \otimes F)$. □

Lemma 1.6.6. *If E is (dual-)Nakano-positive and F is a nef* line *bundle, then $E \otimes F$ is (dual-)Nakano-positive.*

Proof. Fix a Kähler metric on M. Let g be a Nakano-positive metric on E, then there exists $2\varepsilon > 0$ such that

$$R^g(u(x), u(x)) \geq 2\varepsilon |u(x)|^2$$

for any $u \in \Gamma(M, T^{1,0}M \otimes E)$. On the other hand, by a result of [DPS94], there exists a smooth metric h_0 on the nef line bundle F such that

$$R^{h_0} \geq -\varepsilon \omega h_0 \tag{1.6.3}$$

The curvature of $g \otimes h_0$ on $E \otimes F$ is

$$\widehat{R} = R^g \cdot h_0 + g \cdot R^{h_0}$$

For any $u \in \Gamma(M, T^{1,0}M \otimes E)$ and $v \in \Gamma(M, F)$

$$\widehat{R}(u \otimes v, u \otimes v) \geq \left(R^g(u, u) - \varepsilon |u|^2\right) h_0(v, v) \geq \varepsilon |u|^2 h_0(v, v) \tag{1.6.4}$$

For dual-Nakano-positivity, the proof is similar. □

Theorem 1.6.7. *If E is an ample vector bundle and F is an arbitrary vector bundle over M, then there exists $k_0 = k_0(M, E, F)$ such that $S^k E \otimes F$ is Nakano-positive and dual-Nakano-positive for any $k \geq k_0$.*

Proof. By Lemma 1.6.5, there exists m_0 such that $(\det E)^{m_0} \otimes F$ is Nakano-positive and dual-Nakano-positive. On the other hand, there exists $k_0 = k_0(E, m_0, M)$ such that $\mathcal{O}_{\mathbb{P}(E^*)}(r + k) \otimes \pi^*(\det E^*)^{m_0+1}$ is ample for $k \geq k_0$. It is equivalent to the ampleness of vector bundle $S^{r+k}E \otimes (\det E^*)^{m_0+1}$. By Theorem 1.3.12, $S^k E \otimes (\det E^*)^{m_0}$ is Nakano-positive and dual-Nakano-positive. Since the tensor product of two (dual-)Nakano-positive vector bundles is (dual-)Nakano-positive, $S^k E \otimes F = (S^k E \otimes (\det E^*)^{m_0}) \otimes ((\det E)^{m_0} \otimes F)$ is Nakano-positive and dual-Nakano-positive for $k \geq k_0$. □

The following results are well-known in algebraic geometry, but merit a proof in our setting.

Corollary 1.6.8. *If E is ample over M, L is a nef line bundle and F is an arbitrary vector bundle,*

(1) there exists $k_0 = k_0(M, E, F)$ such that for any $k \geq k_0$.

$$H^{p,q}(M, S^k E \otimes F) = 0$$

for $q \geq 1$ and $p \geq 0$.

(2) there exists $k_0 = k_0(M, E)$ such that for any $k \geq k_0$,

$$H^{p,q}(M, S^k E \otimes L) = 0$$

for any $q \geq 1$ and $p \geq 0$.

Proof. **(1)** By Theorem 1.6.7, there exists $k_0 = k_0(M, E, F)$ such that $S^k E \otimes F \otimes \Lambda^{n-p} T^{1,0} M$ is Nakano-positive for any p. On the other hand

$$H^{p,q}(M, S^k E \otimes F) = H^{n,q}(M, S^k E \otimes F \otimes \Lambda^{n-p} T^{1,0} M)$$

By Nakano vanishing theorem, $H^{p,q}(M, S^k E \otimes F) = 0$ for $q \geq 1$ and $p \geq 0$ if $k \geq k_0$. The proof of part **(2)** is similar. □

1.6.2 Vanishing theorems for bounded vector bundles

Firstly, we would like to introduce the following

Definition 1.6.9. Let E be an arbitrary holomorphic vector bundle with rank r, L an ample line bundle and $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$. E is said to be $(\varepsilon_1, \varepsilon_2)$-bounded by L if there exists a Hermitian metric h on E and a positive Hermitian metric h^L on L such that the curvature of E is bounded by the curvatures of L^{ε_1} and L^{ε_2}, i.e.

$$\varepsilon_1 \omega_L \otimes Id_E \leq \Theta^{E,h} \leq \varepsilon_2 \omega_L \otimes Id_E \tag{1.6.5}$$

in the sense of Griffiths. E is called strictly $(\varepsilon_1, \varepsilon_2)$-bounded by L if at least one of $\Theta^{E,h} - \varepsilon_1 \omega_L \otimes Id_E$ and $\Theta^{E,h} - \varepsilon_2 \omega_L \otimes Id_E$ is not identically zero.

It is easy to see that E is $(\varepsilon_1, \varepsilon_2)$-bounded by L if and only if $E \otimes L^{-\varepsilon_1}$ and $E^* \otimes L^{\varepsilon_2}$ are semi-Griffiths positive. Similarly, if E is strictly $(\varepsilon_1, \varepsilon_2)$-bounded by L, then at least one of the semi-Griffiths positive vector bundles $E \otimes L^{-\varepsilon_1}$ and $E^* \otimes L^{\varepsilon_2}$ is not trivial.

Proposition 1.6.10. *Let E be a holomorphic vector bundle with rank r over a projective manifold.*

(1) If E is globally generated, E is strictly $(0,1)$-bounded by $L \otimes \det E$ for any ample line bundle L;

(2) If E ample, E is strictly $(-1, r)$-bounded by $\det E$;

(3) If E is nef, E is strictly $(-1, r)$-bounded by $L \otimes \det E$ for any ample line bundle L;

(4) If E is Griffiths-positive, E is strictly $(0,1)$-bounded by $\det E$.

Proof. (1) As is well-known, if E is globally generated, there exists a Hermitian metric h on E such that $\Theta^{E,h}$ is semi-Griffiths-positive and $E \otimes \det E^* = \Lambda^{r-1}E^*$ is semi-Griffiths-negative. If L is an ample line bundle, $E \otimes \det E^* \otimes L^*$ is Griffiths-negative and

$$\Theta^{E,h} < \omega_{L\otimes \det E} \otimes Id_E$$

Hence, E is strictly $(0,1)$-bounded by $L \otimes \det E$.

(2) We assume $r > 1$. By a result of [Berndtsson09a] and [Mourougane-Takayama07], if E is ample, $E \otimes \det E$ is Griffiths-positive. On the other hand, $E^* \otimes \det E = \Lambda^{r-1}E$ is ample and so is $S^{r+1}(E^* \otimes \det E)$. By a result of [Liu-Sun-Yang], $(E^* \otimes \det E) \otimes \det(E^* \otimes \det E) = E^* \otimes (\det E)^r$ is Griffiths-positive.

(3) If E is nef, $S^{r+1}E \otimes L$ is ample and by a result of [Liu-Sun-Yang], $E \otimes \det E \otimes L$ is Griffiths-positive. Similarly, we know $S^{r+1}(E^* \otimes \det E) \otimes L$ is ample and so $E^* \otimes (\det E)^r \otimes L$ is Griffiths-positive.

(4) It is obvious. □

Remark 1.6.11. In general, if E is $(-1, r)$-bounded by $\det E$, E is not necessarily ample. For example, $E = L^3 \oplus L^{-1}$ for some ample line bundle L.

Let h be a Hermitian metric on the vector bundle E. At a fixed point $p \in X$, if we assume $h_{\alpha\overline{\beta}} = \delta_{\alpha\overline{\beta}}$, then the naturally induced bundle $(E \otimes (\det E)^m, h \otimes (\det h)^m)$ has curvature component

$$R^{E\otimes(\det E)^m}_{i\bar{j}\alpha\overline{\beta}} = R_{i\bar{j}\alpha\overline{\beta}} + m\sum_{\delta} R_{i\bar{j}\delta\bar{\delta}} \tag{1.6.6}$$

where $R_{i\bar{j}\alpha\overline{\beta}}$ is the curvature component of (E, h). It is obvious that S^kE has basis

$$\{e_A = e_1^{\alpha_1} \otimes \cdots \otimes e_r^{\alpha_r}\} \tag{1.6.7}$$

if $A = (\alpha_1, \cdots, \alpha_r)$ with $\alpha_1 + \cdots + \alpha_r = k$ and α_j are nonnegative integers. The naturally induced bundle $(S^kE \otimes (\det E)^m, S^kh \otimes (\det h)^m)$ has curvature components

$$R^{S^kE\otimes(\det E)^m}_{i\bar{j}A\overline{B}} = R_{i\bar{j}A\overline{B}} + m\sum_{\delta} R_{i\bar{j}\delta\bar{\delta}}. \tag{1.6.8}$$

Lemma 1.6.12. *If (E, h) is a Hermitian vector bundle, the curvature of $(S^kE \otimes (\det E)^m, S^kh \otimes (\det h)^m$ can be written as*

$$R^{S^kE\otimes(\det E)^m}_{i\bar{j}A\overline{B}}(p) = (r+k-1)! \cdot \int_{\mathbb{P}^{r-1}} \frac{V_A\overline{V}_B}{|W|^{2k}}\varphi_{i\bar{j}}\frac{\omega_{FS}^{r-1}}{(r-1)!} \tag{1.6.9}$$

where

$$\varphi_{i\bar{j}} = (r+k)\sum_{\gamma,\delta} R_{i\bar{j}\gamma\bar{\delta}}(p)\frac{W_\delta\overline{W}_\gamma}{|W|^2} + (m-1)\sum_{\delta} R_{i\bar{j}\delta\bar{\delta}}. \tag{1.6.10}$$

Proof. This follows from Lemma 1.3.3. □

Theorem 1.6.13. *If E is strictly $(\varepsilon_1, \varepsilon_2)$-bounded by L and $m + (r+k)\varepsilon_1 > 0$, then*

$$H^{p,q}(X, S^kE \otimes \det E \otimes L^m) = H^{q,p}(X, S^kE \otimes \det E \otimes L^m) = 0 \tag{1.6.11}$$

if $p \geq 1, q \geq 1$ satisfy

$$\frac{m+(r+k)\varepsilon_1}{m+(r+k)\varepsilon_2} \geq \min\left\{\frac{n-q}{p}, \frac{n-p}{q}\right\}. \tag{1.6.12}$$

In particular, if $m+(r+k)\varepsilon_1 > 0$, $S^kE\otimes\det E\otimes L^m$ is Nakano-positive and dual-Nakano-positive and

$$H^{n,q}(X, S^kE \otimes \det E \otimes L^m) = H^{q,n}(X, S^kE \otimes \det E \otimes L^m) = 0$$

for $q \geq 1$.

Proof. Let h be a Hermitian metric on E and h^L a positive Hermitian metric on L such that

$$\varepsilon_1 \omega_L \otimes Id_E \leq \Theta^{E,h} \leq \varepsilon_2 \omega_L \otimes Id_E.$$

We can polarize X by

$$\omega_g = \omega_L = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h^L. \tag{1.6.13}$$

At the fixed point $p \in X$, we can assume

$$g_{i\bar{j}}(p) = \delta_{i\bar{j}} \quad \text{and} \quad h_{\alpha\overline{\beta}}(p) = \delta_{\alpha\overline{\beta}}.$$

Therefore,

$$g_{i\bar{j}}(p) = R^{h^L}_{i\bar{j}}(p) = \delta_{i\bar{j}}. \tag{1.6.14}$$

Let

$$\varphi_{i\bar{j}} = (r+k)\left(\sum_{\gamma,\delta} R^h_{i\bar{j}\gamma\bar{\delta}}(p)\frac{W_\delta \overline{W}_\gamma}{|W|^2}\right) + mR^{h_L}_{i\bar{j}}. \tag{1.6.15}$$

then the curvature of $S^kE \otimes \det E \otimes L^m$ is

$$R^{S^kE\otimes\det E\otimes L^m}_{i\bar{j}A\overline{B}}(p) = (r+k-1)! \cdot \int_{\mathbb{P}^{r-1}} \frac{V_A\overline{V}_B}{|W|^{2k}}\varphi_{i\bar{j}}\frac{\omega_{FS}^{r-1}}{(r-1)!}. \tag{1.6.16}$$

By formula (1.6.15), it is easy to see that, at point p, for any $v = (v^1, \cdots, v^n) \in \mathbb{C}^n \setminus \{0\}$,

$$(m + (r+k)\varepsilon_1)\,|v|^2 \leq \varphi_{i\bar{j}}v^i\overline{v}^j \leq (m + (r+k)\varepsilon_2)\,|v|^2 \tag{1.6.17}$$

Since $m + (r+k)\varepsilon_1 > 0$, it is obvious that $S^kE \otimes \det E \otimes L^m$ is both Nakano-positive and dual-Nakano-positive by (1.6.16). Let λ_1 be the smallest eigenvalue of $(\varphi_{i\bar{j}})$ and λ_n the largest one, then

$$m + (r+k)\varepsilon_1 \leq \lambda_1 \leq \lambda_n \leq m + (r+k)\varepsilon_2 \tag{1.6.18}$$

Let $\varphi = \frac{\sqrt{-1}}{2\pi}\varphi_{i\bar{j}}dz^i \wedge d\bar{z}^j$, we obtain the curvature term of $S^kE \otimes \det E \otimes L^m$

$$\begin{aligned}
T(u,u) &= \langle [R, \Lambda_g]u, u\rangle \\
&= (r+k-1)! \int_{\mathbb{P}^{r-1}} \langle [\varphi, \Lambda_g]U, U\rangle \cdot \frac{1}{|W|^{2k}} \cdot \frac{\omega_{FS}^{r-1}}{(r-1)!} \\
&\geq (r+k-1)! \int_{\mathbb{P}^{r-1}} \max\{p\lambda_1 - (n-q)\lambda_n, q\lambda_1 - (n-p)\lambda_n\}|U|^2 \cdot \frac{1}{|W|^{2k}} \cdot \frac{\omega_{FS}^{r-1}}{(r-1)!} \\
&= \max\{pK_1 - (n-q)K_n, qK_1 - (n-p)K_n\}
\end{aligned}$$

for any nonzero $u = u_{I\bar{J}A} dz^I \wedge d\bar{z}^J \otimes e_A \in \Omega^{p,q}(X, S^k E \otimes \det E \otimes L^m)$ where

$$U = \sum_A u_{I\bar{J}A} V_A dz^I \wedge d\bar{z}^J, \quad \text{and} \qquad K_i = (r+k-1)! \cdot \int_{\mathbb{P}^{r-1}} \frac{|U|^2}{|W|^{2k}} \lambda_i \frac{\omega_{FS}^{r-1}}{(r-1)!}, i = 1, n,$$

By (1.6.18), if $m + (r+k)\varepsilon_1 > 0$,

$$\frac{K_1}{K_n} > \frac{m + (r+k)\varepsilon_1}{m + (r+k)\varepsilon_2} \tag{1.6.19}$$

since E is *strictly* $(\varepsilon_1, \varepsilon_2)$-bounded by L. If $p \geq 1, q \geq 1$ satisfy

$$\frac{m + (r+k)\varepsilon_1}{m + (r+k)\varepsilon_2} \geq \min\left\{\frac{n-q}{p}, \frac{n-p}{q}\right\} \tag{1.6.20}$$

we obtain

$$\frac{K_1}{K_n} > \min\left\{\frac{n-q}{p}, \frac{n-p}{q}\right\} \tag{1.6.21}$$

By standard Bochner formulas, we deduce that $H^{p,q}(X, S^k E \otimes (\det E)^m \otimes L) = H^{p,q}(X, S^k E \otimes (\det E)^m \otimes L) = 0$. $\square$

Theorem 1.1.8 and Theorem 1.1.9 follow immediately from Theorem 1.6.13 and Proposition 1.6.10.

Now we want to analyze the condition

$$\lambda_0 \geq \min\left\{\frac{n-q}{p}, \frac{n-p}{q}\right\} \tag{1.6.22}$$

for some $\lambda_0 \in [0, 1)$. Without loss of generality, we assume $p \geq q \geq 1$, then that is a linear cindition

$$p + \lambda_0 q \geq n. \tag{1.6.23}$$

When $p = q$, we obtain

$$c_0 = \frac{n}{1 + \lambda_0}, \tag{1.6.24}$$

(p, q) satisfies (1.6.22) if and only if (p, q) lies in the quadrilateral $Q = A_0 A_1 A_2 A_3$ where

$$A_0 = (0, n), A_1 = (n, n), A_2 = (n, 0), A_3 = (c_0, c_0) \tag{1.6.25}$$

Corollary 1.6.14. *Let E be globally generated and L be ample.*

(1) If the pair (k,m,s) satisfies

$$m \geq \frac{1}{s}\left[\frac{n-s}{2}\right](r+k)+1 \tag{1.6.26}$$

where $[\bullet]$ is the integer part of $\bullet$, then

$$H^{p,q}(X, S^kE \otimes (\det E)^m \otimes L) = 0$$

for any $p+q \geq n+s$.

(2) For fixed (k,m), we have

$$H^{p,q}(X, S^kE \otimes (\det E)^m \otimes L) = 0$$

for any (p,q) satisfies

$$p+q \geq n + \left(\frac{2n}{1+\frac{m-1}{r+k+m-1}} - n\right). \tag{1.6.27}$$

Proof. If $m \geq \frac{1}{s}\left[\frac{n-s}{2}\right](r+k)+1$, we get

$$\frac{m-1}{r+k+m-1} \geq \frac{\left[\frac{n-s}{2}\right]}{\left[\frac{n-s}{2}\right]+s}. \tag{1.6.28}$$

If $p+q \geq n+s$,

$$\max_{p+q\geq n+s} \min\left\{\frac{n-q}{p}, \frac{n-p}{q}\right\} = \frac{\left[\frac{n-s}{2}\right]}{\left[\frac{n-s}{2}\right]+s}. \tag{1.6.29}$$

Part (1) is proved. For part (2), if

$$p+q \geq n + \left(\frac{2n}{1+\frac{m-1}{r+k+m-1}} - n\right) = \frac{2n}{1+\frac{m-1}{r+k+m-1}}$$

then

$$\max\{p,q\} \geq \frac{n}{1+\frac{m-1}{r+k+m-1}}. \tag{1.6.30}$$

That is

$$\frac{m-1}{r+k+m-1} \geq \min\left\{\frac{n-q}{p}, \frac{n-p}{q}\right\}.$$

So the vanishing result holds. □

Remark 1.6.15. Theorem 1.1.8 and Corollary 1.6.14 are also valid for semi-Griffiths positive E. Consider the example $E = T\mathbb{P}^2 \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ with the canonical metric. Since $r = n = 2$, by Corollary 1.6.14, we obtain

$$H^{p,q}(X, E \otimes (\det E)^m \otimes L) = 0 \tag{1.6.31}$$

for any $p + q \geq n + 1$ if $m \geq 1$. It is obvious that the lower bound 1 is sharp since

$$H^{n,n-1}(X, E \otimes L) \cong H^{1,1}(\mathbb{P}^n, \mathbb{C}) = \mathbb{C} \tag{1.6.32}$$

if we choose $L = \mathcal{O}_{\mathbb{P}^n}(1)$ and $m = 1$. So the lower bound

$$\frac{1}{s}\left[\frac{n-s}{2}\right](r+k) + 1$$

can not be improved by a universal constant, i.e., a constant independent of r, s, n, k. Hence the lower bound is optimal in that sense.

Similarly, we obtain

Corollary 1.6.16. *Let E be ample (resp.) and L be nef (resp. ample). Suppose $k \geq 1$ and $m \geq r + k + 1$.*

(1) If the pair (k, m, s) satisfies

$$m \geq \frac{1}{s}\left[\frac{n-s}{2}\right](r+k)(r+1) + (r+1) + k,$$

then

$$H^{p,q}(X, S^k E \otimes (\det E)^m) = 0$$

for any $p + q \geq n + s$.

(2) For fixed (k, m), we have

$$H^{p,q}(X, S^k E \otimes (\det E)^m \otimes L) = 0$$

for any (p, q) satisfies

$$p + q \geq n + \left(\frac{2n}{1 + \frac{(m-1)-(r+k)}{(m-1)+r(r+k)}} - n\right).$$

1.7 Examples

1.7.1 Positivity of $T\mathbb{P}^n$

It is well-known that the holomorphic tangent bundle $T\mathbb{P}^n$ of $\mathbb{P}^n$ is ample and also Griffiths positive.

Corollary 1.7.1. *Let h_{FS} be the Fubini-Study metric on $T\mathbb{P}^n$ with $n \geq 2$, then*

(1) $(S^{n+1}T\mathbb{P}^n \otimes K_{\mathbb{P}^n}, S^{n+1}h_{FS} \otimes \det(h_{FS})^{-1})$ *is semi-Griffiths-positive. Moreover,* $S^{n+1}T\mathbb{P}^n \otimes K_{\mathbb{P}^n}$ *can not admit a Griffiths-positive metric.*

(2) $(T\mathbb{P}^n, h_{FS})$ *is dual-Nakano-positive and semi-Nakano-positive.*

(3) $(S^kT\mathbb{P}^n \otimes K_{\mathbb{P}^n}, S^kh_{FS} \otimes \det(h_{FS})^{-1})$ *is Griffiths-positive for any* $k \geq n+2$.

(4) $(S^kT\mathbb{P}^n, S^kh_{FS})$ *is Nakano-positive and dual-Nakano-positive for any* $k \geq 2$.

Proof. **(1)** By the Euler sequence

$$0 \to \mathbb{C} \to T\mathbb{P}^n \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \to 0 \tag{1.7.1}$$

we know $T\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(-1)$ is the quotient bundle of trivial bundle $\mathbb{C}^{\oplus(n+1)}$. Hence

$$S^{n+1}T\mathbb{P}^n \otimes K_{\mathbb{P}^n} = S^{n+1}\left(T\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(-1)\right)$$

with the canonical metric is semi-Griffiths-positive. However, if $S^{n+1}T\mathbb{P}^n \otimes K_{\mathbb{P}^n}$ admits a Griffiths-positive metric, by Corollary 1.3.15, $T\mathbb{P}^n$ is Nakano-positive which is impossible for $n \geq 2$. **(2)** The curvature of $E = T\mathbb{P}^n$ with respect to the standard Fubini-Study metric h_{FS} is

$$R_{i\bar{j}k\bar{\ell}} = h_{i\bar{j}}h_{k\bar{\ell}} + h_{i\bar{\ell}}h_{k\bar{j}} \tag{1.7.2}$$

Without loss of generality, we assume $h_{i\bar{j}} = \delta_{ij}$ at a fixed point, then

$$R_{i\bar{j}k\bar{\ell}}u^{ik}\overline{u}^{j\ell} = \frac{1}{2}\sum_{j,k}|u^{jk} + u^{kj}|^2 \tag{1.7.3}$$

which means that (E, h_{FS}) is semi-Nakano-positive but not Nakano-positive. For the dual-Nakano-positivity of $(T\mathbb{P}^n, h_{FS})$ we can check that by definition. We can also show it by the monotone property of dual-Nakano-positivity of quotient bundles. By the Euler sequence (1.7.1), $T\mathbb{P}^n$ is the quotient bundle of dual-Nakano-positive bundle $\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)}$ and so $T\mathbb{P}^n$ is dual-Nakano-positive.

(3) It follows by the identity

$$S^kT\mathbb{P}^n \otimes K_{\mathbb{P}^n} = S^k(T\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(-1)) \otimes \mathcal{O}_{\mathbb{P}^n}(k-n-1)$$

and semi-Griffiths positivity of $T\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(-1)$.

(4) By Theorem 1.3.12, the canonically induced metric f is Nakano-positive and dual-Nakano-positive. On the other hand, by Theorem 1.5.1, f is a constant multiple of S^kh_{FS}. The lower bound of k follows from **(1)** and **(2)**. $\square$

Example 1.7.2. In this example, we will show the Nakano-positivity of $(S^2T\mathbb{P}^2, S^2h_{FS})$ in local coordinates. At a fixed point, we choose normal coordinates of $T\mathbb{P}^2$. Let $\{e_1, e_2\}$ be the ordered basis of $T\mathbb{P}^2$ at that point. The ordered basis of $S^2T\mathbb{P}^2$ are $e_{(1,1)} = e_1 \otimes e_1, e_{(1,2)} = e_1 \otimes e_2$ and $e_{(2,2)} = e_2 \otimes e_2$. Using the same notation as Proposition 1.5.3, we set $V_{i\alpha\gamma} = u_{i(\alpha,\gamma)} + u_{i(\gamma,\alpha)}$ where $u = \sum_i \sum_{\alpha\leq\gamma} u_{i(\alpha,\gamma)} \frac{\partial}{\partial z^i} \otimes e_{(\alpha,\gamma)} \in \Gamma(\mathbb{P}^2, T^{1,0}\mathbb{P}^2 \otimes S^2T\mathbb{P}^2)$. For $\gamma = 1$, the 2×2 matrix $(V_{i\alpha 1})$ has the form

$$T_1 = \begin{pmatrix} 2u_{1(1,1)} & u_{1(1,2)} \\ 2u_{2(1,1)} & u_{2(1,2)} \end{pmatrix}$$

For $\gamma = 2$, the 2×2 matrix $(V_{i\alpha 2})$ is

$$T_2 = \begin{pmatrix} u_{1(1,2)} & 2u_{1(2,2)} \\ u_{2(1,2)} & 2u_{2(2,2)} \end{pmatrix}$$

The total 2×3 matrix $\left(u_{i(\alpha,\beta)}\right)$ is

$$T = \begin{pmatrix} u_{1(1,1)} & u_{1(1,2)} & u_{1(2,2)} \\ u_{2(1,1)} & u_{2(1,2)} & u_{2(2,2)} \end{pmatrix}$$

By formulas (1.5.5) and (1.7.3),

$$\begin{aligned} \sum_{i,j,\alpha,\gamma,\beta,\delta} R_{i\bar{j}(\alpha,\gamma)\overline{(\beta,\delta)}} u_{i(\alpha,\gamma)} \overline{u}_{j(\beta,\delta)} &= \sum_{i,j,\alpha,\beta} \left(R_{i\bar{j}\alpha\bar{\beta}} V_{i\alpha 1}\overline{V}_{j\beta 1} + R_{i\bar{j}\alpha\bar{\beta}} V_{i\alpha 2}\overline{V}_{j\beta 2}\right) \\ &= \frac{1}{2}\sum_{i,\alpha} |V_{i\alpha 1} + V_{\alpha i 1}|^2 + \frac{1}{2}\sum_{i,\alpha} |V_{i\alpha 2} + V_{\alpha i 2}|^2 \end{aligned}$$

It equals zero if and only if T_1 and T_2 are skew-symmetric which means $T \equiv 0$. The Nakano-positivity of $(S^2T\mathbb{P}^2, S^2h_{FS})$ is proved.

1.7.2 Curvature properties of Kodaira sufaces

In this subsection we will investigate various curvature properties on Kodaira surfaces. By definition, a Kodaira surface X corresponds to a non-trivial deformation of complex structures of Riemann surfaces of genus ≥ 2. Naturally, we can identify a point $x \in X$ on the fiber X_t with the punctured Riemann surface $X_t \setminus \{x\}$, and we get a map from X to the moduli space $\mathcal{M}_{g,1}$. This map lifts to local immersions to the Teichmuller space $\mathcal{T}_{g,1}$. So there is a naturally induced metric on the Kodaira surface.

Let $f : \mathcal{T}_{g,n} \to \mathcal{M}_{g,n}$ be the universal curve. The Poincaré metric on each fiber of $\mathcal{T}_{g,n}$, which is a complete metric on the n-punctured Riemann surface with constant curvature -1, patches together to give a smooth metric on the relative cotangent bundle $K_{\mathcal{T}_{g,n}/\mathcal{M}_{g,n}}$. It is well-known that $f_*(K^{\otimes 2}_{\mathcal{T}_{g,n}/\mathcal{M}_{g,n}})$ is isomorphic to the cotangent bundle $T^{*1,0}\mathcal{M}_{g,n}$ of the moduli space $\mathcal{M}_{g,n}$. For a given point $z \in \mathcal{M}_{g,n}$, the fiber of the bundle $f_*(K^{\otimes 2}_{\mathcal{T}_{g,n}/\mathcal{M}_{g,n}})$ is

$$H^0(C_z, K^{\otimes 2}_{C_z}) \cong \big(H^1(C_z, TC_z)\big)^*$$

where $C_z = f^{-1}(z)$ is a Riemann surface of genus g with n punctures. The space $H^1(C_z, TC_z)$ can be identified as the space $\mathbb{H}^{0,1}_{(2)}(C_z, TC_z)$ of $L^2(dA_z)$-integrable harmonic Beltrami differentials where dA_z is the Poincaré metric on C_z. Let μ_1, μ_2 be two such Beltrami differentials, the *Weil-Petersson metric* on the holomorphic tangent bundle of $\mathcal{M}_{g,n}$ is defined by

$$(\mu_1, \mu_2)_z = \int_{C_z} \mu_1 \cdot \overline{\mu}_2 dA_z \tag{1.7.4}$$

The following result can be deduced from the similar methods as in [Liu-Sun-Yau08]. For more details, we refer to [Liu-Yang2].

Proposition 1.7.3. *The curvature of Weil-Petersson metric $g_{WP,g,n}$ on the Teichmüller space $\mathcal{T}_{g,n}$ of Riemann surfaces of genus $g \geq 2$ is dual-Nakano-negative and semi-Nakano-negative.*

Lemma 1.7.4 ([To-Yeung11]). *Let X be a Kodaira surface. There exists a holomorphic map Φ from X to $\mathcal{T}_{g,1}$ such that Φ is a local holomorphic immersion.*

Proposition 1.7.5. *Let X be a Kodaira surface. Let h be the naturally induced metric from the Teichmüller space $(\mathcal{T}_{g,1}, \omega_{WP,g,1})$ with $g \geq 2$.*

(1) (X, h) *is Griffiths-negative, i.e.* (X, h) *has negative holomorphic bisectional curvature;*

(2) (X, h) *is semi-Nakano-negative but it can not be quasi-Nakano-negative;*

(3) X *can not admit a Kähler metric with non-positive Riemannian sectional curvature;*

(4) X *can not admit a Kähler metric with semi-dual-Nakano negative curvature.*

Proof. By Lemma 1.7.4, $T^{1,0}X$ is a holomorphic subbundle of the tangent bundle $\mathcal{T}_{g,1}$. Hence, (1) follows from the decreasing property of subbundles (i.e. Lemma 1.2.8) and Proposition 1.3.2.

For (2), as similar as (1), $(T^{1,0}X, h)$ is semi-Nakano-negative. On the other hand,

$$H^0(X, End(T^{1,0}X)) \cong H^0(X, T^{*1,0}X \otimes T^{1,0}X) \cong H^{1,0}(X, T^{1,0}X)$$

Hence, if $(T^{1,0}X, h)$ is quasi-Nakano-negative, then $H^{1,0}(X, T^{1,0}X) = 0$ by Nakano vanishing theorem which is a contradiction.

(3) Suppose X admits a Kähler metric with non-positive Riemannian sectional curvature. It is well-known that every Kodaira surface is algebraic and of general type. Moreover, $c_1^2 > 2c_2$ and so by [Zheng95, Proposition 3], the Kodaira surface X is strongly rigid which is a contradiction.

(4) As an analog of Lemma 1.2.9, we know that if X has a Kähler metric with semi-dual-Nakano negative curvature, that metric has non-positive Riemannian sectional curvature. However, in virtue of part (3), it is impossible. □

Remark 1.7.6. The property (3) in Proposition 1.7.5 answers a question of [To-Yeung11] in a negative way. That is, Kodaira surface can not carry a Kähler metric with non-positive Riemannian sectional curvature, although it admits a Kähler metric with negative holomorphic bisectional curvature.

1.7.3 Examples of bounded vector bundles

It is well-known that globally generated vector bundles are semi-Griffiths positive. On the other hand, any globally generated vector bundle has a quotient metric induced from the trivial vector bundle and so it is semi-dual-Nakano-positive([Demailly]).

Corollary 1.7.7. *Let E be a globally generated vector bundle and L an ample line bundle over a projective manifold X, then $S^kE \otimes L$ is dual-Nakano-positive for any $k \geq 1$. Moreover,*

$$H^{p,n}(X, S^kE \otimes L) = 0 \tag{1.7.5}$$

for any $p \geq 1$.

We can not obtain a vanishing quadrilateral for $S^kE \otimes L$ as Figure 1. It is easy to see that the result in Corollary 1.7.7 is a vertical line of the quadrilateral in Figure 1. In [PLS87], the authors found more vanishing elements close to that vertical line. More precisely, they proved that

$$H^{p,n-1}(X, S^kE \otimes L) = 0, \qquad \text{for any} \quad p \geq r+1 \tag{1.7.6}$$

But in general, there exists some $1 \leq q \leq n$ such that $H^{n,q}(X, S^kE \otimes L) \neq 0$. In particular, $S^kE \otimes L$ is not necessarily Nakano-positive. For example, $E = T\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(-1)$ and $L = \mathcal{O}_{\mathbb{P}^n}(1)$. It is obvious E is globally generated. When $n \geq 2$, $E \otimes L = T\mathbb{P}^n$ is dual-Nakano-positive but not Nakano-positive. More generally, we have

Example 1.7.8 ([Demailly88]). Let $X = G(r, V)$ be the Grassmannian of subspaces of codimension r of a vector space V, $\dim_{\mathbb{C}} V = d$, and E the tautological quotient vector bundle of rank r over X. Then E is globally generated and $L = \det E$ is very ample.

$$H^{n,q}(X, S^kE \otimes \det E) = \begin{cases} 0, & q \neq (r-1)(d-r); \\ S^{k+r-d}V \otimes \det V, & q = (r-1)(d-r) \end{cases} \tag{1.7.7}$$

where $n = \dim_{\mathbb{C}} X = r(d-r)$. If $r = d-1$, then $X = \mathbb{P}^n = \mathbb{P}^{d-1}$ and $E = T\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(-1)$, $\det E = \mathcal{O}_{\mathbb{P}^n}(1)$. That is

$$H^{n,q}(\mathbb{P}^n, S^kT\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(1-k)) = \begin{cases} 0, & q \neq n-1; \\ S^{k-1}V \otimes \det V, & q = n-1 \end{cases} \tag{1.7.8}$$

Therefore, if $n \geq 2$, $S^kT\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(1-k)$ can not be Nakano-positive. However, we will see that for any $\ell \geq 2-k$, $S^kT\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(\ell)$ is both Nakano-positive and dual-Nakano-positive. Moreover, we can obtain more vanishing results about it.

Let h_{FS} be the Fubini-Study metric on $\mathbb{P}^n$ and it also induces a metric on $L = \mathcal{O}_{\mathbb{P}^n}(1)$. It is easy to see that

$$\omega_L \otimes Id \leq \Theta^{T\mathbb{P}^n} \leq 2\omega_L \otimes Id. \tag{1.7.9}$$

So $T\mathbb{P}^n$ is strictly $(1,2)$-bounded by L. Similarly, $H = T\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(-1)$ is strictly $(0,1)$-bounded by L.

Proposition 1.7.9. *If $\ell \geq 2-k$, $S^kT\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(\ell)$ is Nakano-positive and dual-Nakano-positive and*

$$H^{p,q}(\mathbb{P}^n, S^kT\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(\ell)) = H^{q,p}(\mathbb{P}^n, S^kT\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(\ell)) = 0 \tag{1.7.10}$$

for any $p \geq 1, q \geq 1$ satisfy

$$\frac{\ell+k-1}{\ell+n+2k-1} \geq \min\left\{\frac{n-p}{q}, \frac{n-q}{p}\right\}. \tag{1.7.11}$$

Proof. It follows from the relation

$$S^kH \otimes \det H \otimes \mathcal{O}_{\mathbb{P}^n}(\ell+k-1) = S^kT\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(\ell) \tag{1.7.12}$$

and Theorem 1.1.6. Here $\ell+k-1 \geq 1$, i.e., $\ell \geq 2-k$ is necessary by Example 1.7.8. $\square$

Remark 1.7.10. Although $T\mathbb{P}^n$ is not Nakano-positive when $n \geq 2$, $S^kT\mathbb{P}^n$ is both Nakano-positive and dual-Nakano-positive for any $k \geq 2$.

CHAPTER 2

Geometry of Hermitian manifolds

2.1 Introduction

It is well-known([Bochner46]) that on a compact Kähler manifold, if the Ricci curvature is positive, then the first Betti number is zero; if the Ricci curvature is negative, then there is no holomorphic vector field. The key ingredient for the proofs of such results is the Kähler symmetry. On the other hand, on a Hermitian manifold, we don't have such symmetry and there are several different Ricci curvatures. While on a Kähler manifold, all these Ricci curvatures coincide, since the Chern curvature on a Kähler manifold coincides with the curvature of the (complexified) Levi-Civita connection. We can see this more clearly on an abstract Hermitian holomorphic bundle (E, h). The Chern connection ∇^{CH} on E is the unique connection which is compatible with the holomorphic structure and the Hermitian metric h on E. Hence, the Chern curvature $\Theta^E \in \Gamma(M, \Lambda^{1,1}T^*M \otimes E^* \otimes E)$. There are two ways to take trace of Θ^E. If we take trace of Θ^E with respect to the Hermitian metric h on E, we get a $(1,1)$-form $Tr_h\Theta^E \in \Gamma(M, \Lambda^{1,1}T^*M)$ on M which is called the first Ricci-Chern curvature of (E, h). It is well-known that the first Ricci-Chern curvature represents the first Chern class of the bundle. On the other hand, if we take trace on the $(1,1)$-part by using the metric of the manifold, we obtain an endomorphism of E, $Tr_\omega\Theta^E \in \Gamma(M, E^* \otimes E)$. It is called the second Ricci-Chern curvature of (E, h). The first and second Ricci-Chern curvatures have different geometric meanings, which were not clearly studied in some earlier literatures. We should point out that the nonexistence of holomorphic sections of a Hermitian holomorphic vector bundle E is characterized by the second Ricci-Chern curvature of E. Let E be the holomorphic tangent bundle $T^{1,0}M$. If M is Kähler, the first and second Ricci-Chern curvatures are the same by the Kähler symmetry. Unfortunately, on a Hermitian manifold, the Chern curvature is not symmetric, i.e., the first and second Ricci-Chern curvatures are different. Moreover, in general they can not be compared. An interesting example is the Hopf manifold $\mathbb{S}^{2n+1} \times \mathbb{S}^1$. As is well-known the Hopf manifold is non-Kähler and has vanishing first Chern class. However, the canonical metric on it has strictly positive second Ricci-Chern curvature! Moreover, the first Ricci-Chern curvature is nonnegative and not identically zero, whereas it represents the zero first Chern class! For more details, see Proposition 2.6.4.

In this chapter, we study the nonexistence of holomorphic and harmonic sections of an abstract vector bundle over a compact Hermitian manifold. Let E be a holomorphic vector bundle over a compact Hermitian manifold (M, ω). Since the holomorphic section space $H^0(M, E)$ is independent on the connections on E, we can choose any connection on E to detect $H^0(M, E)$. As mentioned above, the key part, is the second Ricci curvature of that given connection. For example, on the holomorphic tangent bundle $T^{1,0}M$ of a Hermitian manifold M, there are three typical connections

(1) the complexified Levi-Civita connection ∇ on $T^{1,0}M$;

(2) the Chern connection ∇^{CH} on $T^{1,0}M$;

(3) the Bismut connection ∇^{B} on $T^{1,0}M$.

It is well-known that if M is Kähler, all three connections are the same. However, in general, the relations among them are somewhat mysterious. In this chapter, we derive certain relations about their curvatures on certain Hermitian manifolds.

Let E be a Hermitian *complex* (possibly *non-holomorphic*) vector bundle or a Riemannian *real* vector bundle over a compact Hermitian manifold (M, ω). Let ∇^E be an arbitrary connection on E and $\partial_E, \overline{\partial}_E$ the $(1,0), (0,1)$ part of ∇^E respectively. The $(1,1)$-curvature of ∇^E is denoted by $R^E \in \Gamma(M, \Lambda^{1,1}T^*M \otimes E^* \otimes E)$. It can be viewed as a representation of the operator $\partial_E\overline{\partial}_E + \overline{\partial}_E\partial_E$. We can define harmonic section spaces associated to (E, ∇^E) by

$$\mathcal{H}^{p,q}_{\overline{\partial}_E}(M, E) = \{\varphi \in \Omega^{p,q}(M, E) \mid \overline{\partial}_E\varphi = \overline{\partial}^*_E\varphi = 0\} \tag{2.1.1}$$

In general, on a complex vector bundle E, there is no terminology such as "holomorphic section of E". However, if the vector bundle E is holomorphic and ∇^E is the Chern connection on E, i.e., $\overline{\partial}_E = \overline{\partial}$, then $\mathcal{H}^{p,q}_{\overline{\partial}_E}(M, E)$ is isomorphic to the Dolbeault cohomology group $H^{p,q}_{\overline{\partial}}(M, E)$ and $H^0_{\overline{\partial}}(M, E)$ is the holomorphic section space $H^0(M, E)$ of E.

If (E, h, ∇^E) is a Hermitian complex vector bundle with a fixed connection ∇^E over a compact Hermitian manifold (M, ω), we will call $Tr_h R^E \in \Gamma(M, \Lambda^{1,1}T^*M)$ the first Hermitian-Ricci curvature of (E, h, ∇^E) and $Tr_\omega R^E \in \Gamma(M, E^* \otimes E)$ the second Hermitian-Ricci curvature. If ∇^E is the Chern connection of a Hermitian holomorphic vector bundle (E, h), they are called the first and second Ricci-Chern curvatures of (E, h) respectively.

Theorem 2.1.1. *Let E be a Hermitian complex vector bundle or a Riemannian real vector bundle over a compact Hermitian manifold (M, ω) and ∇^E be any metric connection on E.*

(1) If the second Hermitian-Ricci curvature $Tr_\omega R^E$ is nonpositive everywhere, then every $\overline{\partial}_E$-closed section of E is parallel, i.e. $\nabla^E s = 0$;

(2) If the second Hermitian-Ricci curvature $Tr_\omega R^E$ is nonpositive everywhere and negative at some point, then $\mathcal{H}^0_{\overline{\partial}_E}(M, E) = 0$;

(3) If the second Hermitian-Ricci curvature $Tr_\omega R^E$ is p-nonpositive everywhere and p-negative at some point, then $\mathcal{H}^0_{\overline{\partial}_E}(M, \Lambda^q E) = 0$ for any $p \leq q \leq rank(E)$.

The proof of this theorem is based on generalized Bochner-Kodaira identities on vector bundles over Hermitian manifolds (Theorem 2.4.5). We prove that (Theorem 2.4.8) the torsion integral of the Hermitian manifold can be killed if the background Hermitian metric ω on M is Gauduchon, i.e. $\partial\overline{\partial}\omega^{n-1} = 0$. On the other hand, in the conformal class of any Hermitian metric, the Gauduchon metric always exists ([Gauduchon84]). So we can change the background metric in the conformal way. It is obvious that the positivity of the second Hermitian-Ricci curvature is preserved under conformal transformations. This method is very useful on Hermitian manifolds. Kobayashi-Wu([Kobayashi-Wu70]) and Gauduchon([Gauduchon77b]) obtained similar result in the special case when ∇^E is the Chern connection of the Hermitian *holomorphic* vector bundle E. Now we go back to the Hermitian manifold (M, ω).

Corollary 2.1.2. *Let (M, ω) be a compact Hermitian manifold and Θ is the Chern curvature of $(T^{1,0}M, \omega)$.*

(1) if the second Ricci-Chern curvature $Tr_\omega\Theta$ is nonnegative everywhere and positive at some point, then $H^{p,0}_{\overline{\partial}}(M) = 0$ for any $1 \leq p \leq n$. In particular, the arithmetic genus $\chi(M, \mathcal{O}) = 1$;

(2) if the second Ricci-Chern curvature $Tr_\omega\Theta$ is nonpositive everywhere and negative at some point, then the holomorphic vector bundle $\Lambda^p T^{1,0}M$ has no holomorphic vector field for any $1 \leq p \leq n$.

As is well-known, if a Hermitian manifold has positive first Ricci-Chern curvature, it must be Kähler. However, we can not draw the same conclusion if the second Ricci-Chern curvature is positive, since the first and second Ricci-Chern curvatures of a Hermitian manifold can not be compared. In fact, the first Ricci-Chern curvature is d-closed, but in general the second Ricci-Chern curvature is not d-closed and they are in the different $(d, \overline{\partial}, \partial)$-cohomology classes. For example, the Hopf manifold $\mathbb{S}^{2n+1} \times \mathbb{S}^1$ with standard Hermitian metric has strictly positive second Ricci-Chern curvature and nonnegative first Ricci-Chern curvature, but it is non-Kähler. For more details, see Proposition 2.6.4.

Now we consider several special Hermitian manifolds. An interesting class of Hermitian manifolds is the balanced Hermitian manifolds, i.e., Hermitian manifolds with coclosed Kähler forms. It is well-known that every Kähler manifold is balanced. In some literatures, they are also called semi-Kähler manifolds. In complex dimension 1 and 2, every balanced Hermitian manifold is Kähler. However, in higher dimensions, there exist non-Kähler manifolds which admit balanced Hermitian metrics. Such examples were constructed by E. Calabi([Calabi-Eckmann53]), see also [Gray66] and [Michelson83]. There are also some other important classes of non-Kähler balanced manifolds, such as: complex solvmanifolds, 1-dimensional families of Kähler manifolds ([Michelson83]) and compact complex parallelizable manifolds (except complex torus) ([Urakawa81]). On the other hand, Alessandrini-Bassanelli([Alessandrini-Bassanelli93]) proved that every Moishezon manifold is balanced and so balanced manifolds can be constructed from Kähler manifolds by modification. For more examples, we refer the reader to [Alessandrini-Bassanelli04], [Michelson83], [Ganchev-Ivanov01], [Ganchev-Ivanov00], [Fu-Yau08], [Fu-Li-Yau], [Fu-Wang-Wu] and references therein.

Every balanced metric ω is Gauduchon. In fact, $d^*\omega = 0$ is equivalent to $d\omega^{n-1} = 0$ and so

$\partial\overline{\partial}\omega^{n-1} = 0$. By [Gauduchon84], every Hermitian manifold has a Gauduchon metric. However, there are many manifolds which can not support balanced metrics. For example, the Hopf surface $\mathbb{S}^3 \times \mathbb{S}^1$ is non-Kähler, so it has no balanced metric. For more discussions, we refer the reader to [Calabi-Eckmann53], [Michelson83],[Tosatti-Weinkove10b], [Alessandrini-Bassanelli93] and references therein.

On a compact balanced Hermitian manifold M, we can also detect the holomorphic section spaces $H^0(M, T^{1,0}M)$ and $H^{p,0}_{\overline{\partial}}(M)$ by the Levi-Civita connection on (M, ω_h). Let ∇ be the complexified Levi-Civita connection on M and R the complexified Riemannian curvature. It is easy to see that $R(X, Y, Z, W) = R(Z, W, X, Y)$ for any $X, Y, Z, W \in \Gamma(M, T_{\mathbb{C}}M)$. In the local holomorphic coodinates $(z^1, \cdots, z^n)$ of M, we set

$$R_{i\bar{j}k\bar{\ell}} = R\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{z}^\ell}\right), \quad R_{i\bar{j}} = h^{k\bar{\ell}} R_{i\bar{j}k\bar{\ell}} \left(= h^{k\bar{\ell}} R_{k\bar{\ell}i\bar{j}}\right)$$

and call $\left(R_{i\bar{j}}\right)$ the Hermitian-Ricci curvature of (M, h). Since ∇ is a connection on the complex vector bundle $T_{\mathbb{C}}M$, there is an induced connection on the Hermitian holomorphic vector bundle $(T^{1,0}M, h)$ and we denote it still by ∇. The curvature of $(T^{1,0}M, h, \nabla)$ is denoted by $\widehat{R}$. In general, the first and second Hermitian-Ricci curvatures of $\widehat{R}$ are different. Moreover, $\widehat{R}$ and R are different but they can be compared(see Proposition 2.2.12). This property can be viewed as a connection between Riemannian geometry and Hermitian geometry(or Symplectic geometry). For example, we can use it to study the non-existence of certain complex structures on complete Riemannian manifolds. In particular,

Corollary 2.1.3. *Let (M, h) be a compact Hermitian manifold. If the Hermitian-Ricci curvature $(R_{i\bar{j}})$ is quasi-positive, then $H^2_{dR}(M, \mathbb{C}) \neq 0$.*

As applications, I can deduce that $\mathbb{S}^{2n+1} \times \mathbb{S}^1$ can not admit a Hermitian metric with quasi-positive Hermitian-Ricci curvature and also Lebrun's result that there is no complex structure on $\mathbb{S}^6$ which is compatible with the round metric.

Theorem 2.1.4. *Let (M, ω) be a compact balanced Hermitian manifold. Suppose the Hermitian-Ricci curvature $(R_{i\bar{j}})$ of M is nonnegative everywhere.*

(1) If φ is a holomorphic p-form, then $\Delta_{\partial}\varphi = 0$ and so $\dim_{\mathbb{C}} H^{p,0}_{\overline{\partial}}(M) \leq \dim_{\mathbb{C}} H^{0,p}_{\overline{\partial}}(M)$ for any $1 \leq p \leq n$;

(2) If the Hermitian-Ricci curvature $(R_{i\bar{j}})$ is positive at some point, then $H^{p,0}_{\overline{\partial}}(M) = 0$ for any $1 \leq p \leq n$. In particular, the arithmetic genus $\chi(M, \mathcal{O}) = 1$.

Let $\widehat{R}^{(2)}_{i\bar{j}}$ be the components of the second Hermitian-Ricci curvature of $\widehat{R}$. The dual of Theorem 2.1.4 is

Theorem 2.1.5. *Let (M, ω) be a compact balanced Hermitian manifold. If $2\widehat{R}^{(2)}_{i\bar{j}} - R_{i\bar{j}}$ is nonpositive everywhere and negative at some point, there is no holomorphic vector field on M.*

Remark 2.1.6. It is easy to see that the Hermitian-Ricci curvature tensor $(R_{i\bar{j}})$ and second Ricci-Chern curvature tensor $\Theta^{(2)} := Tr_{\omega}\Theta$ can not be compared. Therefore, Theorem 2.1.4 and Corollary 2.1.2 are independent of each other. For the same reason, Theorem 2.1.5 and Corollary 2.1.2 are independent. Balanced Hermitian manifolds with nonnegative Hermitian-Ricci curvatures are discussed in Proposition 2.3.5.

As we discuss in the above, on Hermitian manifolds, the second Hermitian-Ricci curvature tensors of various metric connections are closely related to the geometry of Hermitian manifolds. A natural idea is to define a flow by using second Hermitian-Ricci curvature tensors of various metric connections. For example,

$$\frac{\partial h}{\partial t} = -\Theta^{(2)} + \mu h, \quad \mu \in \mathbb{R} \tag{2.1.2}$$

on a general Hermitian manifold (M, h) by using the second Ricci-Chern curvature. This flow preserves the Kähler and the Hermitian structures and has short time solution on any compact Hermitian manifold. It is very similar to and closely related to the Hermitian Yang-Mills flow, the Kähler-Ricci flow and the harmonic map heat flow. It may be a bridge to connect them.

2.2 Various connections and curvatures on Hermitian manifolds

2.2.1 Complexified Riemannian curvature

Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇, the curvature R of (M, g, ∇) is defined as

$$R(X, Y, Z, W) = g\left(\left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\right) Z, W\right) \tag{2.2.1}$$

On a Hermitian manifold (M, h), let ∇ be the complexified Levi-Civita connection and g the background Riemannian metric. Two metrics g and h are related by

$$ds_h^2 = ds_g^2 - \sqrt{-1}\omega_h \tag{2.2.2}$$

where ω_h is the fundamental $(1,1)$-form (or Kähler form) associated to h. For any two holomorphic vector fields $X, Y \in \Gamma(M, T^{1,0}M)$,

$$h(X, Y) = 2g(X, \overline{Y}) \tag{2.2.3}$$

This formula will be used in several definitions. In the local holomorphic coordinates $\{z^1, \cdots, z^n\}$ on M, the complexified Christoffel symbols are given by

$$\Gamma_{AB}^C = \sum_E \frac{1}{2} g^{CE}\left(\frac{\partial g_{AE}}{\partial z^B} + \frac{\partial g_{BE}}{\partial z^A} - \frac{\partial g_{AB}}{\partial z^E}\right) = \sum_E \frac{1}{2} h^{CE}\left(\frac{\partial h_{AE}}{\partial z^B} + \frac{\partial h_{BE}}{\partial z^A} - \frac{\partial h_{AB}}{\partial z^E}\right) \tag{2.2.4}$$

where $A, B, C, E \in \{1, \cdots, n, \overline{1}, \cdots, \overline{n}\}$ and $z^A = z^i$ if $A = i$, $z^A = \overline{z}^i$ if $A = \overline{i}$. For example

$$\Gamma_{ij}^k = \frac{1}{2} h^{k\bar{\ell}}\left(\frac{\partial h_{j\bar{\ell}}}{\partial z^i} + \frac{\partial h_{i\bar{\ell}}}{\partial z^j}\right), \ \Gamma_{\bar{i}j}^k = \frac{1}{2} h^{k\bar{\ell}}\left(\frac{\partial h_{j\bar{\ell}}}{\partial \overline{z}^i} - \frac{\partial h_{j\bar{i}}}{\partial \overline{z}^\ell}\right) \tag{2.2.5}$$

The complexified curvature components are

$$\begin{aligned} R_{ABCD} :\ &= 2\mathbf{g}\left(\left(\nabla_{\frac{\partial}{\partial z^A}} \nabla_{\frac{\partial}{\partial z^B}} - \nabla_{\frac{\partial}{\partial z^B}} \nabla_{\frac{\partial}{\partial z^A}}\right) \frac{\partial}{\partial z^C}, \frac{\partial}{\partial z^D}\right) \\ &= \mathbf{h}\left(\left(\nabla_{\frac{\partial}{\partial z^A}} \nabla_{\frac{\partial}{\partial z^B}} - \nabla_{\frac{\partial}{\partial z^B}} \nabla_{\frac{\partial}{\partial z^A}}\right) \frac{\partial}{\partial z^C}, \frac{\partial}{\partial z^{\overline{D}}}\right) \end{aligned} \tag{2.2.6}$$

Hence

$$R_{ABC}^D = \sum_E R_{ABCE} h^{ED} = -\left(\frac{\partial \Gamma_{AC}^D}{\partial z^B} - \frac{\partial \Gamma_{BC}^D}{\partial z^A} + \Gamma_{AC}^F \Gamma_{FB}^D - \Gamma_{BC}^F \Gamma_{AF}^D\right) \tag{2.2.7}$$

By the Hermitian property, we have, for example

$$R_{i\bar{j}k}^{l} = -\left(\frac{\partial \Gamma_{ik}^{l}}{\partial \bar{z}^{j}} - \frac{\partial \Gamma_{\bar{j}k}^{l}}{\partial z^{i}} + \Gamma_{ik}^{s}\Gamma_{\bar{j}s}^{l} - \Gamma_{\bar{j}k}^{s}\Gamma_{is}^{l} - \Gamma_{\bar{j}k}^{\bar{s}}\Gamma_{i\bar{s}}^{l}\right) \tag{2.2.8}$$

Remark 2.2.1. We have $R_{ABCD} = R_{CDAB}$. In particular,

$$R_{i\bar{j}k\bar{\ell}} = R_{k\bar{\ell}i\bar{j}} \tag{2.2.9}$$

Unlike the Kähler case, we can define several Ricci curvatures:

Definition 2.2.2. (1) The *complexified Ricci curvature* on (M, h) is defined by

$$\mathfrak{R}_{k\bar{\ell}} := h^{i\bar{j}}\left(R_{k\bar{j}i\bar{\ell}} + R_{ki\bar{j}\bar{\ell}}\right) \tag{2.2.10}$$

The *complexified scalar curvature* of h is defined as

$$s_h := h^{k\bar{\ell}}\mathfrak{R}_{k\bar{\ell}} \tag{2.2.11}$$

(2) The *Hermitian-Ricci curvature* is

$$R_{k\bar{\ell}} := h^{i\bar{j}}R_{i\bar{j}k\bar{\ell}} \tag{2.2.12}$$

The *Hermitian-scalar curvature* of h is given by

$$S := h^{k\bar{\ell}}R_{k\bar{\ell}} \tag{2.2.13}$$

Lemma 2.2.3. *On a Hermitian manifold,*

$$\overline{R_{ABCD}} = R_{\overline{ABCD}}, \quad \overline{\mathfrak{R}_{k\bar{\ell}}} = \mathfrak{R}_{\ell\bar{k}}, \quad \overline{R_{k\bar{\ell}}} = R_{\ell\bar{k}} \tag{2.2.14}$$

and

$$\mathfrak{R}_{k\bar{\ell}} = h^{i\bar{j}}\left(2R_{k\bar{j}i\bar{\ell}} - R_{k\bar{\ell}i\bar{j}}\right) \tag{2.2.15}$$

Proof. The Hermitian property of curvature tensors is obvious. By first Bianchi identity, we have

$$R_{ki\bar{j}\bar{\ell}} + R_{k\bar{j}\bar{\ell}i} + R_{k\bar{\ell}i\bar{j}} = 0$$

That is $R_{ki\bar{j}\bar{\ell}} = R_{k\bar{j}i\bar{\ell}} - R_{k\bar{\ell}i\bar{j}}$. The curvature formula (2.2.10) turns out to be

$$\mathfrak{R}_{k\bar{\ell}} = h^{i\bar{j}}\left(2R_{k\bar{j}i\bar{\ell}} - R_{k\bar{\ell}i\bar{j}}\right) \tag{2.2.16}$$

$\square$

Definition 2.2.4. The Ricci curvatures are called *positive* (resp. *nonnegative, negative, non-positive*) if the corresponding Hermitian matrices are positive (resp. nonnegative, negative, non-positive).

The following three formulas are used frequently in the sequel.

Lemma 2.2.5. *Assume $h_{i\bar{j}} = \delta_{ij}$ at a fixed point $p \in M$, we have the following formula*

$$\begin{aligned} R_{i\bar{j}k\bar{\ell}} \;=\; & -\frac{1}{2}\left(\frac{\partial^2 h_{i\bar{\ell}}}{\partial z^k \partial \bar{z}^j} + \frac{\partial^2 h_{k\bar{j}}}{\partial z^i \partial \bar{z}^\ell}\right) \\ & +\frac{1}{4}\left(\frac{\partial h_{k\bar{q}}}{\partial z^i}\frac{\partial h_{q\bar{\ell}}}{\partial \bar{z}^j} + \frac{\partial h_{i\bar{q}}}{\partial z^k}\frac{\partial h_{q\bar{j}}}{\partial \bar{z}^\ell}\right) + \frac{1}{4}\left(\frac{\partial h_{i\bar{q}}}{\partial z^k}\frac{\partial h_{q\bar{\ell}}}{\partial \bar{z}^j} + \frac{\partial h_{k\bar{q}}}{\partial z^i}\frac{\partial h_{q\bar{j}}}{\partial \bar{z}^\ell}\right) \\ & +\frac{1}{4}\left(\frac{\partial h_{q\bar{\ell}}}{\partial z^i}\frac{\partial h_{k\bar{j}}}{\partial \bar{z}^q} + \frac{\partial h_{q\bar{j}}}{\partial z^k}\frac{\partial h_{i\bar{\ell}}}{\partial \bar{z}^q}\right) + \frac{1}{4}\left(\frac{\partial h_{i\bar{\ell}}}{\partial z^q}\frac{\partial h_{k\bar{q}}}{\partial \bar{z}^j} + \frac{\partial h_{k\bar{j}}}{\partial z^q}\frac{\partial h_{i\bar{q}}}{\partial \bar{z}^\ell}\right) \\ & -\frac{1}{4}\left(\frac{\partial h_{q\bar{\ell}}}{\partial z^i}\frac{\partial h_{k\bar{q}}}{\partial \bar{z}^j} + \frac{\partial h_{q\bar{j}}}{\partial z^k}\frac{\partial h_{i\bar{q}}}{\partial \bar{z}^\ell}\right) - \frac{1}{4}\left(\frac{\partial h_{i\bar{\ell}}}{\partial z^q}\frac{\partial h_{k\bar{j}}}{\partial \bar{z}^q} + \frac{\partial h_{k\bar{j}}}{\partial z^q}\frac{\partial h_{i\bar{\ell}}}{\partial \bar{z}^q}\right) \end{aligned} \tag{2.2.17}$$

By a linear transformation on the local holomorphic coordinates, one can get the following Lemma. For more details, we refer the reader to [Street-Tian2].

Lemma 2.2.6. *Let (M, h, ω) be a Hermitian manifold. For any $p \in M$, there exist local holomorphic coordinates $\{z^i\}$ centered at a point p such that*

$$h_{i\bar{j}}(p) = \delta_{ij} \quad \textit{and} \quad \Gamma_{ij}^k(p) = 0 \tag{2.2.18}$$

By Lemma 2.2.6, we have a simplified version of curvatures:

Lemma 2.2.7. *Assume $h_{i\bar{j}}(p) = \delta_{ij}$ and $\Gamma_{ij}^k(p) = 0$ at a fixed point $p \in M$,*

$$R_{i\bar{j}k\bar{\ell}} = -\frac{1}{2}\left(\frac{\partial^2 h_{i\bar{\ell}}}{\partial z^k \partial \bar{z}^j} + \frac{\partial^2 h_{k\bar{j}}}{\partial z^i \partial \bar{z}^\ell}\right) - \sum_q \left(\frac{\partial h_{q\bar{\ell}}}{\partial z^i}\frac{\partial h_{k\bar{q}}}{\partial \bar{z}^j} + \frac{\partial h_{q\bar{j}}}{\partial z^k}\frac{\partial h_{i\bar{q}}}{\partial \bar{z}^\ell}\right) \tag{2.2.19}$$

For Hermitian-Ricci curvatures

$$R_{k\bar{\ell}} = h^{i\bar{j}} R_{i\bar{j}k\bar{\ell}} = -\frac{1}{2}\sum_s \left(\frac{\partial^2 h_{s\bar{\ell}}}{\partial z^k \partial \bar{z}^s} + \frac{\partial^2 h_{k\bar{s}}}{\partial z^s \partial \bar{z}^\ell}\right) - \sum_{q,s}\left(\frac{\partial h_{q\bar{\ell}}}{\partial z^s}\frac{\partial h_{k\bar{q}}}{\partial \bar{z}^s} + \frac{\partial h_{k\bar{q}}}{\partial z^s}\frac{\partial h_{q\bar{\ell}}}{\partial \bar{z}^s}\right) \tag{2.2.20}$$

and

$$h^{i\bar{j}} R_{k\bar{j}i\bar{\ell}} = h^{i\bar{j}} R_{i\bar{\ell}k\bar{j}} = -\frac{1}{2}\sum_s \left(\frac{\partial^2 h_{k\bar{\ell}}}{\partial z^s \partial \bar{z}^s} + \frac{\partial^2 h_{s\bar{s}}}{\partial z^k \partial \bar{z}^\ell}\right) - \sum_{q,s}\left(\frac{\partial h_{q\bar{\ell}}}{\partial z^k}\frac{\partial h_{s\bar{q}}}{\partial \bar{z}^s} + \frac{\partial h_{q\bar{s}}}{\partial z^s}\frac{\partial h_{k\bar{q}}}{\partial \bar{z}^\ell}\right) \tag{2.2.21}$$

For complexified Ricci curvature,

$$\begin{aligned}\mathfrak{R}_{k\bar{\ell}} &= \frac{1}{2}\sum_{s}\left(\frac{\partial^2 h_{s\bar{\ell}}}{\partial z^k \partial \bar{z}^s} + \frac{\partial^2 h_{k\bar{s}}}{\partial z^s \partial \bar{z}^\ell}\right) - \sum_{s}\left(\frac{\partial^2 h_{k\bar{\ell}}}{\partial z^s \partial \bar{z}^s} + \frac{\partial^2 h_{s\bar{s}}}{\partial z^k \partial \bar{z}^\ell}\right) \\ &+ \sum_{q,s}\left(\frac{\partial h_{q\bar{\ell}}}{\partial z^s}\frac{\partial h_{k\bar{q}}}{\partial \bar{z}^s} + \frac{\partial h_{k\bar{q}}}{\partial z^s}\frac{\partial h_{q\bar{\ell}}}{\partial \bar{z}^s}\right) - 2\sum_{q,s}\left(\frac{\partial h_{q\bar{\ell}}}{\partial z^k}\frac{\partial h_{s\bar{q}}}{\partial \bar{z}^s} + \frac{\partial h_{q\bar{s}}}{\partial z^s}\frac{\partial h_{k\bar{q}}}{\partial \bar{z}^\ell}\right) \quad (2.2.22)\end{aligned}$$

2.2.2 Curvature of complexified Levi-Civita connection on $T^{1,0}M$

Since $T^{1,0}M$ is a subbundle of $T_{\mathbb{C}}M$, there is an induced connection $\widehat{\nabla}$ on $T^{1,0}M$ given by

$$\widehat{\nabla} = \pi \circ \nabla : T^{1,0}M \xrightarrow{\nabla} \Gamma(M, T_{\mathbb{C}}M \otimes T_{\mathbb{C}}M) \xrightarrow{\pi} \Gamma(M, T_{\mathbb{C}}M \otimes T^{1,0}M) \quad (2.2.23)$$

The curvature $\widehat{R} \in \Gamma(M, \Lambda^2 T_{\mathbb{C}}M \otimes T^{*1,0}M \otimes T^{1,0}M)$ of $\widehat{\nabla}$ is given by

$$\widehat{R}(X,Y)s = \widehat{\nabla}_X \widehat{\nabla}_Y s - \widehat{\nabla}_Y \widehat{\nabla}_X s - \widehat{\nabla}_{[X,Y]} s \quad (2.2.24)$$

for any $X, Y \in T_{\mathbb{C}}M$ and $s \in T^{1,0}M$. It has components

$$\widehat{R}^l_{ABk} = \frac{\partial \Gamma^l_{Bk}}{\partial z^A} - \frac{\partial \Gamma^l_{Ak}}{\partial z^B} - \Gamma^s_{Ak}\Gamma^l_{Bs} + \Gamma^s_{Bk}\Gamma^l_{As} \quad (2.2.25)$$

where

$$\widehat{R}\left(\frac{\partial}{\partial z^A}, \frac{\partial}{\partial z^B}\right)\frac{\partial}{\partial z^k} = \sum_l \widehat{R}^l_{ABk}\frac{\partial}{\partial z^\ell} \quad (2.2.26)$$

For example,

$$\widehat{R}^l_{i\bar{j}k} = -\left(\frac{\partial \Gamma^l_{ik}}{\partial \bar{z}^j} - \frac{\partial \Gamma^l_{\bar{j}k}}{\partial z^i} + \Gamma^s_{ik}\Gamma^l_{\bar{j}s} - \Gamma^s_{\bar{j}k}\Gamma^l_{si}\right) \quad (2.2.27)$$

With respect to the Hermitian metric h on $T^{1,0}M$, we can define

$$\widehat{R}_{ABk\bar{l}} = \sum_{s=1}^{n} \widehat{R}^s_{ABk} h_{s\bar{\ell}} \quad (2.2.28)$$

Definition 2.2.8. The *first Hermitian-Ricci curvature* of the Hermitian vector bundle $\left(T^{1,0}M, \widehat{\nabla}\right)$ is defined by

$$\widehat{R}^{(1)}_{i\bar{j}} = h^{k\bar{\ell}}\widehat{R}_{i\bar{j}k\bar{\ell}} \quad (2.2.29)$$

The *second Hermitian-Ricci curvature* of it is

$$\widehat{R}^{(2)}_{k\bar{\ell}} = h^{i\bar{j}}\widehat{R}_{i\bar{j}k\bar{\ell}} \quad (2.2.30)$$

The *scalar curvature* of $\widehat{\nabla}$ on $T^{1,0}M$ is denoted by

$$S^{LC} = h^{i\bar{j}} h^{k\bar{\ell}} \widehat{R}_{i\bar{j}k\bar{\ell}} \tag{2.2.31}$$

By Lemma 2.2.6, we have the following formulas

Lemma 2.2.9. *On a Hermitian manifold (M, h), on a point p with $h_{i\bar{j}}(p) = \delta_{ij}$ and $\Gamma_{ij}^k(p) = 0$,*

$$\widehat{R}_{i\bar{j}k\bar{\ell}} = -\frac{1}{2}\left(\frac{\partial^2 h_{i\bar{\ell}}}{\partial z^k \partial \bar{z}^j} + \frac{\partial^2 h_{k\bar{j}}}{\partial z^i \partial \bar{z}^\ell}\right) - \sum_q \frac{\partial h_{q\bar{\ell}}}{\partial z^i}\frac{\partial h_{k\bar{q}}}{\partial \bar{z}^j} \tag{2.2.32}$$

For the first and second Hermitian-Ricci curvatures,

$$\widehat{R}^{(1)}_{i\bar{j}} = -\frac{1}{2}\sum_k \left(\frac{\partial^2 h_{i\bar{k}}}{\partial z^k \partial \bar{z}^j} + \frac{\partial^2 h_{k\bar{j}}}{\partial z^i \partial \bar{z}^k}\right) - \sum_{k,q} \frac{\partial h_{q\bar{k}}}{\partial z^i}\frac{\partial h_{k\bar{q}}}{\partial \bar{z}^j} \tag{2.2.33}$$

and

$$\widehat{R}^{(2)}_{i\bar{j}} = -\frac{1}{2}\sum_k \left(\frac{\partial^2 h_{i\bar{k}}}{\partial z^k \partial \bar{z}^j} + \frac{\partial^2 h_{k\bar{j}}}{\partial z^i \partial \bar{z}^k}\right) - \sum_{k,q} \frac{\partial h_{i\bar{q}}}{\partial \bar{z}^k}\frac{\partial h_{q\bar{j}}}{\partial z^k} \tag{2.2.34}$$

Moreover,

$$\widehat{R}^{(1)}_{i\bar{j}} - \widehat{R}^{(2)}_{i\bar{j}} = h_{m\bar{j}} h^{\ell\bar{k}} \Gamma^{\bar{q}}_{\bar{k}i}\Gamma^m_{\ell\bar{q}} - \Gamma^{\bar{q}}_{k\bar{j}}\Gamma^k_{i\bar{q}} = \sum_{k,q}\left(\frac{\partial h_{i\bar{q}}}{\partial \bar{z}^k}\frac{\partial h_{q\bar{j}}}{\partial z^k} - \frac{\partial h_{i\bar{q}}}{\partial z^k}\frac{\partial h_{q\bar{j}}}{\partial \bar{z}^k}\right) \tag{2.2.35}$$

2.2.3 Curvature of Chern connection on $T^{1,0}M$

On the Hermitian holomorphic vector bundle $(T^{1,0}M, h)$, the Chern connection ∇^{CH} is the unique connection which is compatible with the complex structure and the Hermitian metric. Its curvature components are

$$\Theta_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \bar{z}^j} + h^{p\bar{q}}\frac{\partial h_{p\bar{\ell}}}{\partial \bar{z}^j}\frac{\partial h_{k\bar{q}}}{\partial z^i} \tag{2.2.36}$$

It is well-known that the *first Ricci-Chern curvature*

$$\Theta^{(1)} := \frac{\sqrt{-1}}{2\pi}\Theta^{(1)}_{i\bar{j}} dz^i \wedge d\bar{z}^j \tag{2.2.37}$$

represents the first Chern class of M where

$$\Theta^{(1)}_{i\bar{j}} = h^{k\bar{\ell}}\Theta_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 \log\det(h_{k\bar{\ell}})}{\partial z^i \partial \bar{z}^j} \tag{2.2.38}$$

The *second Ricci-Chern curvature* components are

$$\Theta^{(2)}_{i\bar{j}} = h^{k\bar{\ell}}\Theta_{k\bar{\ell}i\bar{j}} \tag{2.2.39}$$

The *scalar curvature* of the Chern connection is defined by

$$S^{CH} = h^{i\bar{j}}h^{k\bar{\ell}}\Theta_{i\bar{j}k\bar{\ell}} \tag{2.2.40}$$

2.2.4 Curvature of Bismut connection on $T^{1,0}M$

In [Bismut89], Bismut defined a class of connections on Hermitian manifolds. In this subsection, we choose one of them (see [Ma-Marinescu07], p. 21). The *Bismut connection* ∇^B on the holomorphic tangent bundle $(T^{1,0}M, h)$ is characterized by

$$\nabla^B = \nabla + S^B \tag{2.2.41}$$

where ∇ is the Levi-Civita connection and S^B is a 1-form with values in $End(T^{1,0}M)$ defined by

$$\mathbf{h}(S^B(X)Y, Z) = 2\mathbf{g}(S^B(X)Y, \overline{Z}) = \sqrt{-1}(\partial - \overline{\partial})\omega_h(X, Y, \overline{Z}) \tag{2.2.42}$$

for any $Y, Z \in T^{1,0}M$ and $X \in T_{\mathbb{C}}M$. Let $\widetilde{\Gamma}^{\beta}_{i\alpha}$ and $\widetilde{\Gamma}^{\beta}_{\bar{j}\alpha}$ be the Christoffel symbols of the Bismut connection where $i, j, \alpha, \beta \in \{1, \cdots, n\}$. We use different types of letters since the Bismut connection is not torsion free.

Lemma 2.2.10. *We have the following relations between $\widetilde{\Gamma}$ and Γ,*

$$\widetilde{\Gamma}_{i\alpha\overline{\beta}}(:= h_{\beta\overline{\gamma}}\widetilde{\Gamma}^{\overline{\gamma}}_{i\alpha}) = \Gamma_{i\alpha\overline{\beta}} + \Gamma_{\alpha\overline{\beta}i} = \frac{\partial h_{i\overline{\beta}}}{\partial z^{\alpha}}, \quad \widetilde{\Gamma}_{\bar{j}\alpha\overline{\beta}} = 2\Gamma_{\bar{j}\alpha\overline{\beta}} \tag{2.2.43}$$

Proof. Let $X = \frac{\partial}{\partial z^i}, Y = \frac{\partial}{\partial z^j}, Z = \frac{\partial}{\partial z^k}$. Since $\omega_h = \frac{\sqrt{-1}}{2}h_{m\overline{n}}dz^m \wedge d\overline{z}^n$, we obtain

$$\begin{aligned}
\sqrt{-1}(\partial - \overline{\partial})\omega_h(X, Y, \overline{Z}) &= -\frac{1}{2}\frac{\partial h_{m\overline{n}}}{\partial z^p}dz^p dz^m d\overline{z}^n\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}, \frac{\partial}{\partial \overline{z}^k}\right) \\
&= \frac{1}{2}\left(\frac{\partial h_{i\overline{k}}}{\partial z^j} - \frac{\partial h_{j\overline{k}}}{\partial z^i}\right) \\
&= \Gamma^{\overline{s}}_{j\overline{k}}h_{i\overline{s}} = \Gamma_{j\overline{k}i}
\end{aligned}$$

On the other hand

$$h\left(\nabla^B_{\frac{\partial}{\partial z^i}}\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}\right) = \widetilde{\Gamma}_{ij\overline{k}} \tag{2.2.44}$$

By the definition (2.2.41) of Bismut connection, we get

$$\begin{aligned}\widetilde{\Gamma}_{i\alpha\overline{\beta}} = h\left(\nabla^B_{\frac{\partial}{\partial z^i}}\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta}\right) &= h\left(\nabla_{\frac{\partial}{\partial z^i}}\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta}\right) + h\left(S^B\left(\frac{\partial}{\partial z^i}\right)\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta}\right)\\ &= \Gamma_{i\alpha\overline{\beta}} + \Gamma_{\alpha\overline{\beta}i} = \frac{\partial h_{i\overline{\beta}}}{\partial z^\alpha}\end{aligned}$$

The proof of the other one is similar. $\square$

The Bismut curvature $B \in \Gamma\left(M, \Lambda^{1,1}T^*M \otimes End(T^{1,0}M)\right)$ is given by

$$B^{\beta}_{i\overline{j}\alpha} = -\frac{\partial \widetilde{\Gamma}^{\beta}_{i\alpha}}{\partial \overline{z}^j} + \frac{\partial \widetilde{\Gamma}^{\beta}_{\overline{j}\alpha}}{\partial z^i} - \widetilde{\Gamma}^{\gamma}_{i\alpha}\widetilde{\Gamma}^{\beta}_{\overline{j}\gamma} + \widetilde{\Gamma}^{\gamma}_{\overline{j}\alpha}\widetilde{\Gamma}^{\beta}_{i\gamma} \tag{2.2.45}$$

Lemma 2.2.11. *Assume $h_{i\overline{j}}(p) = \delta_{ij}$ and $\Gamma^k_{ij}(p) = 0$ at a fixed point $p \in M$,*

$$B_{i\overline{j}\alpha\overline{\beta}} = -\left(\frac{\partial^2 h_{i\overline{\beta}}}{\partial \overline{z}^j \partial z^\alpha} + \frac{\partial^2 h_{\alpha\overline{j}}}{\partial z^i \partial \overline{z}^\beta} - \frac{\partial^2 h_{\alpha\overline{\beta}}}{\partial z^i \partial \overline{z}^j}\right) + \sum_\gamma \frac{\partial h_{\alpha\overline{\gamma}}}{\partial z^i}\frac{\partial h_{\gamma\overline{\beta}}}{\partial \overline{z}^j} - 4\sum_\gamma \frac{\partial h_{\alpha\overline{\gamma}}}{\partial \overline{z}^j}\frac{\partial h_{\gamma\overline{\beta}}}{\partial z^i} \tag{2.2.46}$$

Proof. It follows by (2.2.43) and (2.2.45). $\square$

We can define the first Ricci-Bismut curvature $B^{(1)}_{i\overline{j}}$, the second Ricci-Bismut curvature $B^{(2)}_{i\overline{j}}$ and scalar curvature S^{BM} similarly.

2.2.5 Relations among four curvatures on Hermitian manifolds

Proposition 2.2.12. *On a Hermitian manifold (M, h), we have*

$$R_{ijk\overline{l}} = \widehat{R}_{ijk\overline{\ell}}, \quad R_{\overline{i}\overline{j}k\overline{\ell}} = \widehat{R}_{\overline{i}\overline{j}k\overline{\ell}} \tag{2.2.47}$$

and for any $u, v \in \mathbb{C}^n$,

$$\left(R_{i\overline{j}k\overline{\ell}} - \widehat{R}_{i\overline{j}k\overline{\ell}}\right)u^i\overline{w}^j v^k \overline{v}^\ell \leq 0 \tag{2.2.48}$$

In particular, $R_{i\overline{j}} \leq \widehat{R}^{(1)}_{i\overline{j}}$ and $R_{i\overline{j}} \leq \widehat{R}^{(2)}_{i\overline{j}}$ in the sense of Hermitian matrices.

Proof. By formulas (2.2.8) and (2.2.27), we can set

$$T_{i\overline{j}k\overline{\ell}} := R_{i\overline{j}k\overline{\ell}} - \widehat{R}_{i\overline{j}k\overline{\ell}} = \Gamma^{\overline{s}}_{\overline{j}k}\Gamma^{t}_{i\overline{s}}h_{t\overline{\ell}} \tag{2.2.49}$$

Without loss generality, we assume $h_{i\bar{j}} = \delta_{ij}$ at a fixed point, then

$$T_{i\bar{j}k\bar{\ell}} = \sum_s \Gamma_{\bar{j}ks}\Gamma_{i\bar{s}\bar{\ell}} = -\sum_s \Gamma_{i\bar{s}\bar{\ell}}\overline{\Gamma_{j\bar{s}\bar{k}}} \tag{2.2.50}$$

where

$$\Gamma_{i\bar{s}\bar{\ell}} = \frac{1}{2}\left(\frac{\partial h_{i\bar{\ell}}}{\partial \bar{z}^s} - \frac{\partial h_{i\bar{s}}}{\partial \bar{z}^\ell}\right) = -\Gamma_{i\bar{\ell}\bar{s}} \tag{2.2.51}$$

and so

$$T_{i\bar{j}k\bar{\ell}}u^i\overline{u}^j v^k \overline{v}^\ell = -\sum_s \left(\sum_{i,\ell} \Gamma_{i\bar{s}\bar{\ell}} u^i \overline{v}^\ell\right)\overline{\left(\sum_{k,j}\Gamma_{j\bar{s}\bar{k}}u^j\overline{v}^k\right)} \leq 0$$

□

Remark 2.2.13. (1) Because of the second order terms in R, $\widehat{R}$, Θ and B, we can not compare $R, \widehat{R}$ with Θ, B.

(2) Since the third order terms of $\partial\Theta^{(2)}$ are not zero in general, it is possible that $\Theta^{(1)}$ and $\Theta^{(2)}$ are not in the same $(d, \partial, \overline{\partial})$-cohomology class. For the same reason $B^{(1)}$ and $B^{(2)}$ are not in the same $(d, \partial, \overline{\partial})$-cohomology class.

(3) If the manifold (M, h) is Kähler, all curvatures are the same.

2.3 Curvature relations on special Hermitian manifolds

2.3.1 Curvatures relations on balanced Hermitian manifolds

The following lemma is well-known(for example [Gauduchon77b]), and we include a proof here in our setting.

Lemma 2.3.1. *Let (M, ω) be a compact Hermitian manifold. The following conditions are equivalent:*

(1) $d^*\omega = 0$*;*

(2) $d\omega^{n-1} = 0$*;*

(3) For any smooth function $f \in C^\infty(M)$,

$$\frac{1}{2}\Delta_d f = \Delta_{\overline{\partial}} f = \Delta_\partial f = -h^{i\overline{j}} \frac{\partial^2 f}{\partial z^i \partial \overline{z}^j} \tag{2.3.1}$$

(4) $\Gamma^{\ell}_{\overline{i}\ell} = 0$ for any $1 \leq i \leq n$.

Proof. On a compact Hermitian manifold, $d^*\omega = - * d * \omega = -c_n * d\omega^{n-1}$ where c_n is a constant depending only on the complex dimension n of M. On the other hand, the Hodge $*$ is an isomorphism, and so (1) and (2) are equivalent. If f is a smooth function on M,

$$\begin{cases} \Delta_{\overline{\partial}} f = -h^{i\overline{j}} \frac{\partial^2 f}{\partial z^i \partial \overline{z}^j} + 2h^{i\overline{j}} \Gamma^{\overline{\ell}}_{i\overline{j}} \frac{\partial f}{\partial \overline{z}^\ell} \\ \Delta_\partial f = -h^{i\overline{j}} \frac{\partial^2 f}{\partial z^i \partial \overline{z}^j} + 2h^{i\overline{j}} \Gamma^{k}_{\overline{j}i} \frac{\partial f}{\partial z^k} \end{cases} \tag{2.3.2}$$

On the other hand,

$$h^{i\overline{j}}\Gamma^{\overline{\ell}}_{i\overline{j}} = -\Gamma^{\overline{j}}_{k\overline{j}} h^{k\overline{\ell}} \quad \text{and} \quad h^{i\overline{j}}\Gamma^{k}_{\overline{j}i} = -\Gamma^{i}_{\overline{\ell}i} h^{k\overline{\ell}} \tag{2.3.3}$$

Therefore (3) and (4) are equivalent. For the equivalence of (1) and (4), see Lemma 2.9.8. □

Definition 2.3.2. A Hermitian manifold (M, ω) is called *balanced* if it satisfies one of the conditions in Lemma 2.3.1.

On a balanced Hermitian manifold, there are more symmetries on the second derivatives of the metric.

Lemma 2.3.3. *Let (M, h) be a balanced Hermitian manifold. On a point p with $h_{i\overline{j}}(p) = \delta_{ij}$ and $\Gamma^k_{ij}(p) = 0$, we have*

$$\sum_s \frac{\partial h_{s\overline{i}}}{\partial \overline{z}^s} = \sum_s \frac{\partial h_{s\overline{s}}}{\partial \overline{z}^i} = 0 \tag{2.3.4}$$

and

$$\sum_i \frac{\partial^2 h_{i\overline{\ell}}}{\partial z^k \partial \overline{z}^i} = \sum_i \frac{\partial^2 h_{k\overline{i}}}{\partial z^i \partial \overline{z}^\ell} = \sum_i \frac{\partial^2 h_{i\overline{i}}}{\partial z^k \partial \overline{z}^\ell} - 2\sum_{i,q} \frac{\partial h_{q\overline{\ell}}}{\partial \overline{z}^i} \frac{\partial h_{k\overline{q}}}{\partial z^i} \tag{2.3.5}$$

Proof. At a fixed point p, if $h_{i\overline{j}} = 0$ and $\Gamma^k_{ij} = 0$, then

$$\frac{\partial h_{i\overline{j}}}{\partial \overline{z}^k} = -\frac{\partial h_{i\overline{k}}}{\partial \overline{z}^j} \tag{2.3.6}$$

The balanced condition $\sum_s \Gamma_{\bar{i}s}^s = 0$ is reduced to

$$\sum_s \frac{\partial h_{s\bar{s}}}{\partial \bar{z}^i} = \sum_s \frac{\partial h_{s\bar{i}}}{\partial \bar{z}^s} = 0$$

by formula (2.3.6). By the balanced condition, we get

$$\begin{aligned} 0 = \frac{\partial \Gamma_{\bar{\ell}i}^i}{\partial z^k} &= \frac{\partial}{\partial z^k}\left(\frac{1}{2}h^{i\bar{q}}\left(\frac{\partial h_{i\bar{q}}}{\partial \bar{z}^\ell} - \frac{\partial h_{i\bar{\ell}}}{\partial \bar{z}^q}\right)\right) \\ &= \frac{1}{2}\sum_i \left(\frac{\partial^2 h_{i\bar{i}}}{\partial z^k \partial \bar{z}^\ell} - \frac{\partial^2 h_{i\bar{\ell}}}{\partial z^k \partial \bar{z}^i}\right) - \sum_{i,q} \frac{\partial h_{q\bar{\ell}}}{\partial \bar{z}^i}\frac{\partial h_{k\bar{q}}}{\partial z^i} \end{aligned}$$

Hence, we obtain formula (2.3.5). □

Proposition 2.3.4. *Let (M, h) be a balanced Hermitian manifold. At a point p with $h_{i\bar{j}}(p) = \delta_{ij}$ and $\Gamma_{ij}^k(p) = 0$, we have following formulas about various Ricci curvatures:*

$$\Theta_{k\bar{\ell}}^{(1)} = \widehat{R}_{k\bar{\ell}}^{(1)} = B_{k\bar{\ell}}^{(1)} = -\sum_i \frac{\partial^2 h_{i\bar{i}}}{\partial z^k \partial \bar{z}^\ell} + \sum_{q,i} \frac{\partial h_{q\bar{\ell}}}{\partial \bar{z}^i}\frac{\partial h_{k\bar{q}}}{\partial z^i} \tag{2.3.7}$$

$$\Theta_{k\bar{\ell}}^{(2)} = -\sum_i \frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \bar{z}^i} + \sum_{i,q} \frac{\partial h_{q\bar{\ell}}}{\partial \bar{z}^i}\frac{\partial h_{k\bar{q}}}{\partial z^i} \tag{2.3.8}$$

$$\widehat{R}_{k\bar{\ell}}^{(2)} = -\sum_i \frac{\partial^2 h_{i\bar{i}}}{\partial z^k \partial \bar{z}^\ell} + \sum_{i,q}\left(2\frac{\partial h_{q\bar{\ell}}}{\partial \bar{z}^i}\frac{\partial h_{k\bar{q}}}{\partial z^i} - \frac{\partial h_{k\bar{q}}}{\partial \bar{z}^i}\frac{\partial h_{q\bar{\ell}}}{\partial z^i}\right) \tag{2.3.9}$$

$$B_{k\bar{\ell}}^{(2)} = -\sum_i \frac{\partial^2 h_{i\bar{i}}}{\partial z^k \partial \bar{z}^\ell} + \sum_{i,q}\left(5\frac{\partial h_{q\bar{\ell}}}{\partial \bar{z}^i}\frac{\partial h_{k\bar{q}}}{\partial z^i} - 4\frac{\partial h_{k\bar{q}}}{\partial \bar{z}^i}\frac{\partial h_{q\bar{\ell}}}{\partial z^i}\right) \tag{2.3.10}$$

$$R_{k\bar{\ell}} = -\sum_i \frac{\partial^2 h_{i\bar{i}}}{\partial z^k \partial \bar{z}^\ell} + \sum_{i,q}\left(\frac{\partial h_{q\bar{\ell}}}{\partial \bar{z}^i}\frac{\partial h_{k\bar{q}}}{\partial z^i} - \frac{\partial h_{k\bar{q}}}{\partial \bar{z}^i}\frac{\partial h_{q\bar{\ell}}}{\partial z^i}\right) \tag{2.3.11}$$

$$\mathfrak{R}_{k\bar{\ell}} = -\sum_i \frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \bar{z}^i} - \sum_{i,q}\left(\frac{\partial h_{q\bar{\ell}}}{\partial \bar{z}^i}\frac{\partial h_{k\bar{q}}}{\partial z^i} - \frac{\partial h_{k\bar{q}}}{\partial \bar{z}^i}\frac{\partial h_{q\bar{\ell}}}{\partial z^i}\right) \tag{2.3.12}$$

Proof. In (2.2.33), (2.2.34), (2.2.38), (2.2.39), (2.2.20), (2.2.22), we get expressions for all Ricci curvatures on Hermitian manifolds. By balanced relations (2.3.4) and (2.3.5), we get simplified versions of all Ricci curvatures. □

Proposition 2.3.5. *(1) A balanced Hermitian manifold with positive Hermitian-Ricci curvature $\left(R_{i\bar{j}}\right)$ is Kähler.*

(2) Let (M, h) be a compact balanced Hermitian manifold. If the Hermitian-Ricci curvature $\left(R_{i\bar{j}}\right)$ is nonnegative everywhere and positive at some point, then M is Moishezon.

Proof. **(1)** On a balanced Hermitian manifold, we have

$$\left(\Theta^{(1)}_{i\bar{j}}\right) = \left(\widehat{R}^{(1)}_{i\bar{j}}\right) \geq \left(R_{i\bar{j}}\right) \tag{2.3.13}$$

by Proposition 2.2.12 and Proposition 2.3.4. If $\left(R_{i\bar{j}}\right)$ is Hermitian positive, then $\Theta^{(1)}_{i\bar{j}}$ is Hermitian positive, and so

$$\Omega = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\det(h_{k\bar{\ell}}) \tag{2.3.14}$$

is a Kähler metric.

(2) If the Hermitian-Ricci curvature is nonnegative everywhere and positive at some point, so is $\left(\Theta^{(1)}_{i\bar{j}}\right)$. The Hermitian line bundle $L = \det(T^{1,0}M)$ satisfies

$$\int_M c_1(L)^n > 0 \tag{2.3.15}$$

By Siu-Demailly's solution of Grauert-Riemenschneider conjecture ([Siu84], [Demailly87]), M is Moishezon. □

2.3.2 Curvature relations on Hermitian manifolds with $\Lambda(\partial\overline{\partial}\omega) = 0$

Now we consider a compact Hermitian manifold (M,ω) with $\Lambda(\partial\overline{\partial}\omega) = 0$. The condition $\Lambda(\partial\overline{\partial}\omega) = 0$ is equivalent to

$$\sum_k \left(\frac{\partial h_{i\bar{j}}}{\partial z^k \partial \overline{z}^k} + \frac{\partial h_{k\bar{k}}}{\partial z^i \partial \overline{z}^j}\right) = \sum_k \left(\frac{\partial h_{i\bar{k}}}{\partial z^k \partial \overline{z}^j} + \frac{\partial^2 h_{k\bar{j}}}{\partial z^i \partial \overline{z}^k}\right) \tag{2.3.16}$$

for any i,j. Similar to Proposition 2.3.4, we can use (2.3.16) to simplify Ricci curvatures and get relations among them.

Proposition 2.3.6. *Let (M,h) be a compact Hermitian manifold with $\Lambda(\partial\overline{\partial}\omega) = 0$. At a point p*

with $h_{i\bar{j}}(p)=\delta_{ij}$ and $\Gamma_{ij}^k(p)=0$, the following identities about Ricci curvatures hold:

$$\Theta^{(1)}_{k\bar{\ell}} = -\sum_i \frac{\partial^2 h_{i\bar{i}}}{\partial z^k \partial \overline{z}^\ell} + \sum_{q,i} \frac{\partial h_{q\bar{\ell}}}{\partial \overline{z}^i} \frac{\partial h_{k\bar{q}}}{\partial z^i} \tag{2.3.17}$$

$$\Theta^{(2)}_{k\bar{\ell}} = -\sum_i \frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \overline{z}^i} + \sum_{i,q} \frac{\partial h_{q\bar{\ell}}}{\partial \overline{z}^i} \frac{\partial h_{k\bar{q}}}{\partial z^i} \tag{2.3.18}$$

$$\widehat{R}^{(1)}_{k\bar{\ell}} = -\frac{1}{2}\sum_i \left(\frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \overline{z}^i} + \frac{\partial^2 h_{i\bar{i}}}{\partial z^k \partial \overline{z}^\ell}\right) - \sum_{i,q} \frac{\partial h_{q\bar{\ell}}}{\partial \overline{z}^i} \frac{\partial h_{k\bar{q}}}{\partial z^i} \tag{2.3.19}$$

$$\widehat{R}^{(2)}_{k\bar{\ell}} = -\frac{1}{2}\sum_i \left(\frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \overline{z}^i} + \frac{\partial^2 h_{i\bar{i}}}{\partial z^k \partial \overline{z}^\ell}\right) - \sum_{i,q} \frac{\partial h_{k\bar{q}}}{\partial \overline{z}^i} \frac{\partial h_{q\bar{\ell}}}{\partial z^i} \tag{2.3.20}$$

$$B^{(1)}_{k\bar{\ell}} = -\sum_i \frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \overline{z}^i} + \sum_{i,q} \left(\frac{\partial h_{q\bar{\ell}}}{\partial \overline{z}^i} \frac{\partial h_{k\bar{q}}}{\partial z^i} - 4\frac{\partial h_{q\bar{\ell}}}{\partial z^i} \frac{\partial h_{k\bar{q}}}{\partial \overline{z}^i}\right) \tag{2.3.21}$$

$$B^{(2)}_{k\bar{\ell}} = -\sum_i \frac{\partial^2 h_{i\bar{i}}}{\partial z^k \partial \overline{z}^\ell} + \sum_{i,q} \left(\frac{\partial h_{q\bar{\ell}}}{\partial \overline{z}^i} \frac{\partial h_{k\bar{q}}}{\partial z^i} - 4\frac{\partial h_{k\bar{q}}}{\partial \overline{z}^i} \frac{\partial h_{q\bar{\ell}}}{\partial z^i}\right) \tag{2.3.22}$$

$$R_{k\bar{\ell}} = -\frac{1}{2}\sum_i \left(\frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \overline{z}^i} + \frac{\partial^2 h_{i\bar{i}}}{\partial z^k \partial \overline{z}^\ell}\right) - \sum_{i,q} \left(\frac{\partial h_{q\bar{\ell}}}{\partial \overline{z}^i} \frac{\partial h_{k\bar{q}}}{\partial z^i} + \frac{\partial h_{k\bar{q}}}{\partial \overline{z}^i} \frac{\partial h_{q\bar{\ell}}}{\partial z^i}\right) \tag{2.3.23}$$

$$\begin{aligned}\mathfrak{R}_{k\bar{\ell}} &= -\frac{1}{2}\sum_i \left(\frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \overline{z}^i} + \frac{\partial^2 h_{i\bar{i}}}{\partial z^k \partial \overline{z}^\ell}\right) + \sum_{i,q} \left(\frac{\partial h_{q\bar{\ell}}}{\partial \overline{z}^i} \frac{\partial h_{k\bar{q}}}{\partial z^i} + \frac{\partial h_{k\bar{q}}}{\partial \overline{z}^i} \frac{\partial h_{q\bar{\ell}}}{\partial z^i}\right) \\ &\quad -2\sum_{q,i} \left(\frac{\partial h_{q\bar{\ell}}}{\partial z^k} \frac{\partial h_{i\bar{q}}}{\partial \overline{z}^i} + \frac{\partial h_{q\bar{i}}}{\partial z^i} \frac{\partial h_{k\bar{q}}}{\partial \overline{z}^\ell}\right)\end{aligned} \tag{2.3.24}$$

Proposition 2.3.7. *If (M,ω) is a compact Hermitian manifold with $\Lambda(\partial\overline{\partial}\omega)=0$, then*

$$B^{(2)} \leq \Theta^{(1)} \quad \textit{and} \quad B^{(1)} \leq \Theta^{(2)} \tag{2.3.25}$$

in the sense of Hermitian matrices and identities hold if and only if (M,ω) is Kähler. Moreover,

$$\Theta^{(2)} + B^{(2)} = \Theta^{(1)} + R^{(1)} \tag{2.3.26}$$

Finally, we would like to discuss the relations of special metrics on Hermitian manifolds. By [Alessandrini-Bassanelli93], every Moishezon manifold is balanced, i.e., there exists a smooth Hermitian metric ω such that $d^*\omega = 0$. On the other hand, by [Demailly-Paun04](see also [Ji-Shiffman93]), on each Moishezon manifold, there exists a singular Hermitian metric ω such that $\partial\overline{\partial}\omega = 0$ in the sense of current. However, these two conditions can not be satisfied simultaneously in the smooth sense on a Hermitian non-Kähler manifold.

Proposition 2.3.8. *Let (M, ω) be a compact Hermitian manifold. If $d^*\omega = 0$ and $\Lambda(\partial\overline{\partial}\omega) = 0$, then $d\omega = 0$, i.e. (M, ω) is Kähler. In particular, if a compact Hermitian manifold admits a smooth metric ω such that $d^*\omega = 0$ and $\partial\overline{\partial}\omega = 0$, then it is Kähler.*

Proof. Let (M, ω) be a balanced Hermitian manifold with $\Lambda(\partial\overline{\partial}\omega) = 0$. The condition $\Lambda(\partial\overline{\partial}\omega) = 0$ is equivalent to

$$\sum_i \frac{\partial h_{i\bar{i}}}{\partial z^k \partial \overline{z}^\ell} + \sum_i \frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \overline{z}^i} = \sum_i \frac{\partial h_{i\bar{\ell}}}{\partial z^k \partial \overline{z}^i} + \sum_i \frac{\partial^2 h_{k\bar{i}}}{\partial z^i \partial \overline{z}^\ell} \tag{2.3.27}$$

By formula 2.3.5, at a point p with $h_{i\bar{j}} = \delta_{ij}$ and $\Gamma_{ij}^k(p) = 0$, we have

$$\begin{aligned} \sum_i \frac{\partial h_{i\bar{i}}}{\partial z^k \partial \overline{z}^\ell} + \sum_i \frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \overline{z}^i} &= \sum_i \frac{\partial h_{i\bar{\ell}}}{\partial z^k \partial \overline{z}^i} + \sum_i \frac{\partial^2 h_{k\bar{i}}}{\partial z^i \partial \overline{z}^\ell} \\ &= 2\sum_i \frac{\partial h_{i\bar{i}}}{\partial z^k \partial \overline{z}^\ell} - 4\sum_{q,i} \frac{\partial h_{q\bar{\ell}}}{\partial \overline{z}^i} \frac{\partial h_{k\bar{q}}}{\partial z^i} \end{aligned}$$

That is

$$\sum_i \frac{\partial h_{i\bar{i}}}{\partial z^k \partial \overline{z}^\ell} = \sum_i \frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \overline{z}^i} + 4\sum_{q,i} \frac{\partial h_{q\bar{\ell}}}{\partial \overline{z}^i} \frac{\partial h_{k\bar{q}}}{\partial z^i} \tag{2.3.28}$$

By taking trace of it, we obtain

$$4\sum_{q,i,k} \frac{\partial h_{q\bar{k}}}{\partial \overline{z}^i} \frac{\partial h_{k\bar{q}}}{\partial z^i} = 0 \Longleftrightarrow \frac{\partial h_{k\bar{q}}}{\partial z^i} = 0 \tag{2.3.29}$$

at point p. Since p is arbitrary, we have $d\omega \equiv 0$, therefore, (M, ω) is Kähler. $\square$

Remark 2.3.9. This result is known in [Alexandrov-Ivanov01] and also [Fino-Parton-Salamon04] in the conditions of $d^*\omega = 0$ and $\partial\overline{\partial}\omega = 0$. By carefully computations, we find that their method works also for $d^*\omega = 0$ and $\Lambda\left(\partial\overline{\partial}\omega\right) = 0$. Our method is quite different from theirs.

2.4 Bochner formulas on Hermitian complex and Riemannian real vector bundles over compact Hermitian manifolds

Let (M, h, ω) be a compact Hermitian manifold. The complexified Levi-Civita connection ∇ on $T_{\mathbb{C}}M$ induces a linear connection on $\Omega^{p,q}(M)$:

$$\nabla : \Omega^{p,q}(M) \to \Omega^1(M) \otimes \left(\Omega^{p,q}(M) \oplus \Omega^{p-1,q+1}(M) \oplus \Omega^{p+1,q-1}(M)\right) \tag{2.4.1}$$

We consider the following two canonical components of ∇,

$$\begin{cases} \nabla' : \Omega^{p,q}(M) \to \Omega^{1,0}(M) \otimes \Omega^{p,q}(M) \\ \nabla'' : \Omega^{p,q}(M) \to \Omega^{0,1}(M) \otimes \Omega^{p,q}(M) \end{cases} \tag{2.4.2}$$

Note that $\nabla \neq \nabla' + \nabla''$ if (M, h, ω) is not Kähler. The following calculation rule follows immediately

$$\nabla'(\varphi \wedge \psi) = (\nabla'\varphi) \wedge \psi + \varphi \wedge \nabla'\psi \tag{2.4.3}$$

for any $\varphi, \psi \in \Omega^\bullet(M)$.

Lemma 2.4.1. *On a Hermitian manifold* (M, h)*, we have*

$$\begin{cases} \partial h(\varphi, \psi) = h(\nabla'\varphi, \psi) + h(\varphi, \nabla''\psi) \\ \overline{\partial} h(\varphi, \psi) = h(\nabla''\varphi, \psi) + h(\varphi, \nabla'\psi) \end{cases} \iff \begin{cases} \frac{\partial}{\partial z^i} h(\varphi, \psi) = h(\nabla'_i\varphi, \psi) + h(\varphi, \nabla''_{\bar{i}}\psi) \\ \frac{\partial}{\partial \bar{z}^j} h(\varphi, \psi) = h(\nabla''_{\bar{j}}\varphi, \psi) + h(\varphi, \nabla'_j\psi) \end{cases}$$

for any $\varphi, \psi \in \Omega^{p,q}(M)$.

Remark 2.4.2. (1) Here we use the compact notations

$$\nabla'_i = \nabla'_{\frac{\partial}{\partial z^i}}, \quad \nabla''_{\bar{j}} = \nabla''_{\frac{\partial}{\partial \bar{z}^j}}$$

Note that $\nabla'_{\bar{j}} = \nabla''_i = 0$ and $\nabla_i \neq \nabla'_i$, $\nabla_{\bar{j}} \neq \nabla'_{\bar{j}}$.

(2) If we regard $\Lambda^{p,q}T^*M$ as an abstract vector bundle E, the above lemma says that ∇' and ∇'' are compatible with the Hermitian metric on E.

Now we go to an abstract setting. Let (E, h) be a Hermitian *complex* (possibly *non-holomorphic*) vector bundle or a *Riemannian* real vector bundle over a compact Hermitian manifold (M, ω). Let ∇^E be an arbitrary *metric connection* on (E, h), i.e.,

$$dh(s, t) = h(\nabla^E s, t) + h(s, \nabla^E t) \tag{2.4.4}$$

for any $s, t \in \Gamma(M, E)$. There is a natural decomposition

$$\nabla^E = \nabla'^E + \nabla''^E \tag{2.4.5}$$

where

$$\begin{cases} \nabla'^E : \Gamma(M, E) \to \Omega^{1,0}(M, E) \\ \nabla''^E : \Gamma(M, E) \to \Omega^{0,1}(M, E) \end{cases} \tag{2.4.6}$$

∇'^E and ∇''^E induce two differential operators. The first one is $\partial_E : \Omega^{p,q}(M, E) \to \Omega^{p+1,q}(M, E)$ defined by

$$\partial_E(\varphi \otimes s) = (\partial\varphi) \otimes s + (-1)^{p+q}\varphi \wedge \nabla'^E s \tag{2.4.7}$$

for any $\varphi \in \Omega^{p,q}(M)$ and $s \in \Gamma(M, E)$. The other one is $\overline{\partial}_E : \Omega^{p,q}(M, E) \to \Omega^{p,q+1}(M, E)$ defined by

$$\overline{\partial}_E(\varphi \otimes s) = \left(\overline{\partial}\varphi\right) \otimes s + (-1)^{p+q}\varphi \wedge \nabla''^E s \tag{2.4.8}$$

for any $\varphi \in \Omega^{p,q}(M)$ and $s \in \Gamma(M, E)$. The following formula is well-known

$$\left(\partial_E\overline{\partial}_E + \overline{\partial}_E\partial_E\right)(\varphi \otimes s) = \varphi \wedge \left(\partial_E\overline{\partial}_E + \overline{\partial}_E\partial_E\right) s \tag{2.4.9}$$

for any $\varphi \in \Omega^{p,q}(M)$ and $s \in \Gamma(M, E)$. The operator $\partial_E\overline{\partial}_E + \overline{\partial}_E\partial_E$ is represented by its $(1, 1)$ curvature tensor $R^E \in \Gamma(M, \Lambda^{1,1}T^*M \otimes E)$. For any $\varphi, \psi \in \Omega^{\bullet,\bullet}(M, E)$, there is a *sesquilinear pairing*

$$\{\varphi, \psi\} = \varphi^\alpha \wedge \overline{\psi^\beta}\langle e_\alpha, e_\beta\rangle \tag{2.4.10}$$

if $\varphi = \varphi^\alpha e_\alpha$ and $\psi = \psi^\beta e_\beta$ in the local frame $\{e_\alpha\}$ of E. By the metric compatible property of ∇^E,

$$\partial\{\varphi, \psi\} = \{\partial_E\varphi, \psi\} + (-1)^{p+q}\{\varphi, \overline{\partial}_E\psi\} \tag{2.4.11}$$

if $\varphi \in \Omega^{p,q}(M, E)$.

Let ω be the Kähler form of the Hermitian metric h, i.e.,

$$\omega = \frac{\sqrt{-1}}{2} h_{i\overline{j}} dz^i \wedge d\overline{z}^j \tag{2.4.12}$$

On the Hermitian manifold (M, h, ω), the norm on $\Omega^{p,q}(M)$ is defined by

$$(\varphi, \psi) = \int_M \langle\varphi, \psi\rangle \frac{\omega^n}{n!} = \frac{2^n}{(p+q)!}\int_M h(\varphi, \psi)\frac{\omega^n}{n!} = \int_M \varphi \wedge *\overline{\psi} \tag{2.4.13}$$

The norm on $\Omega^{p,q}(M, E)$ is defined by

$$(\varphi, \psi) = \int_M \{\varphi, *\psi\} = \int_M \left(\varphi^\alpha \wedge *\overline{\psi^\beta}\right)\langle e_\alpha, e_\beta\rangle \tag{2.4.14}$$

for $\varphi, \psi \in \Omega^{p,q}(M, E)$. The dual operators of $\partial, \overline{\partial}, \partial_E$ and $\overline{\partial}_E^*$ are denoted by $\partial^*, \overline{\partial}^*, \partial_E$ and $\overline{\partial}_E^*$ respectively.

The following lemma was firstly shown by Demailly using Taylor expansion method(e.g. [Demailly]). For the convenience of the reader, we will take another approach which seems to be useful in local computations.

Lemma 2.4.3. *Let (M, h, ω) be a compact Hermitian manifold. If τ is the operator of type $(1,0)$ defined by $\tau = [\Lambda, 2\partial\omega]$ on $\Omega^\bullet(M, E)$,*

$$\begin{cases} [\Lambda, \partial] = \sqrt{-1}\left(\overline{\partial}^* + \overline{\tau}^*\right) \\ \left[\Lambda, \overline{\partial}\right] = -\sqrt{-1}(\partial^* + \tau^*) \end{cases} \tag{2.4.15}$$

For the dual equation, it is

$$\begin{cases} [\overline{\partial}^*, L] = \sqrt{-1}(\partial + \tau) \\ [\partial^*, L] = -\sqrt{-1}(\overline{\partial} + \overline{\tau}) \end{cases} \tag{2.4.16}$$

where L is the operator $L\varphi = 2\omega \wedge \varphi$ and Λ is the adjoint operator of L.

Proof. See Lemma 2.9.7 of the Appendix. □

In the rest of this section E is assumed to be a Hermitian complex vector bundle or a Riemannian real vector bundle over a compact Hermitian manifold M.

Lemma 2.4.4. *Let ∇^E be a metric connection on E over a compact Hermitian manifold (M, ω). If τ is the operator of type $(1,0)$ defined by $\tau = [\Lambda, 2\partial\omega]$ on $\Omega^\bullet(M, E)$, then*

(1) $[\overline{\partial}_E^*, L] = \sqrt{-1}(\partial_E + \tau)$;

(2) $[\partial_E^*, L] = -\sqrt{-1}(\overline{\partial}_E + \overline{\tau})$;

(3) $[\Lambda, \partial_E] = \sqrt{-1}(\overline{\partial}_E^* + \overline{\tau}^*)$;

(4) $[\Lambda, \overline{\partial}_E] = -\sqrt{-1}(\partial_E^* + \tau^*)$.

Proof. See Lemma 2.9.10 of the Appendix. □

Theorem 2.4.5. *Let ∇^E be a metric connection E over a compact Hermitian manifold (M,ω).*

$$\Delta_{\overline{\partial}_E}=\Delta_{\partial_E}+\sqrt{-1}\left[\partial_E\overline{\partial}_E+\overline{\partial}_E\partial_E,\Lambda\right]+(\partial_E\tau^*+\tau^*\partial_E)-(\overline{\partial}_E\overline{\tau}^*+\overline{\tau}^*\overline{\partial}_E)\tag{2.4.17}$$

where

$$\begin{cases}\Delta_{\overline{\partial}_E}=\overline{\partial}_E\overline{\partial}_E^*+\overline{\partial}_E^*\overline{\partial}_E\\ \Delta_{\partial_E}=\partial_E\partial_E^*+\partial_E^*\partial_E\end{cases}\tag{2.4.18}$$

Proof. It follows from Lemma 2.4.4. □

We make a useful observation on the torsion τ:

Lemma 2.4.6. *For any $s\in\Gamma(M,E)$, we have*

$$\tau(s)=-2\sqrt{-1}\left(\overline{\partial}^*\omega\right)\cdot s,\qquad \overline{\tau}(s)=2\sqrt{-1}\left(\partial^*\omega\right)\cdot s\tag{2.4.19}$$

Proof. By definition

$$\begin{aligned}([\Lambda,2\partial\omega])\,s &= 2\Lambda\left((\partial\omega)\cdot s\right)\\ &= 2\left(\Lambda(\partial\omega)\right)\cdot s\\ &= -2\sqrt{-1}\left(\overline{\partial}^*\omega\right)\cdot s\end{aligned}$$

Here we use the identity

$$\overline{\partial}^*\omega=\sqrt{-1}\Lambda(\partial\omega)\tag{2.4.20}$$

where the proof of it is contained in Lemma 2.9.8 of the Appendix. □

Corollary 2.4.7. *If (M,ω) is a compact balanced Hermitian manifold, and ∇^E a metric connection on E over M, then*

$$\|\overline{\partial}_Es\|^2=\|\partial_Es\|^2+\left(\sqrt{-1}\left[\partial_E\overline{\partial}_E+\overline{\partial}_E\partial_E,\Lambda\right]s,s\right)\tag{2.4.21}$$

for any $s\in\Gamma(M,E)$.

Proof. Since for any $s\in\Gamma(M,E)$, $\tau s=\overline{\tau}s=0$ and $\tau^*s=\overline{\tau}^*s=0$ on a balanced Hermitian manifold, the result follows from formula (2.4.17). □

Theorem 2.4.8. *Let (M, ω) be a compact Hermitian manifold with $\partial\overline{\partial}\omega^{n-1} = 0$. If ∇^E is a metric connection on E over M, then*

$$0 = \|\overline{\partial}_E s\|^2 = \|\partial_E s\|^2 + \left(\sqrt{-1}\left[\partial_E\overline{\partial}_E + \overline{\partial}_E\partial_E, \Lambda\right] s, s\right) \tag{2.4.22}$$

for any $s \in \Gamma(M, E)$ with $\overline{\partial}_E s = 0$.

Proof. By formula (2.4.17), we only have to prove that

$$\left((\partial_E\tau^* + \tau^*\partial_E)s - (\overline{\partial}_E\overline{\tau}^* + \overline{\tau}^*\overline{\partial}_E)s, s\right) = 0 \tag{2.4.23}$$

It is equivalent to $(\partial_E s, \tau s) = 0$ since $\tau^* s = \overline{\tau}^* s = \overline{\partial}_E s = 0$. By formula (2.4.19) and Stokes' Theorem,

$$\begin{aligned}
(\tau^*\partial_E s, s) &= (\partial_E s, \tau s) = \int_M \{\partial_E s, *(\tau s)\} \\
&= 2\sqrt{-1}\int_M \left\{\partial_E s, *\left(\overline{\partial}^*\omega \cdot s\right)\right\} \\
&= 2\sqrt{-1}\int_M \left\{\partial_E s, \left(*\overline{\partial}^*\omega\right) \cdot s\right\} \\
&= -2\sqrt{-1}\int_M \left\{s, \overline{\partial}_E\left(\left(*\overline{\partial}^*\omega\right) \cdot s\right)\right\} \\
&= -2\sqrt{-1}\int_M \left\{s, \left(\overline{\partial} * \overline{\partial}^*\omega\right) \cdot s - \left(*\overline{\partial}^*\omega\right) \wedge \overline{\partial}_E s\right\}
\end{aligned}$$

It is easy to see that

$$\overline{\partial} * \overline{\partial}^*\omega = -\overline{\partial} * *\partial * \omega = c_n\overline{\partial}\partial\omega^{n-1} = 0 \tag{2.4.24}$$

since $*\omega = c_n\omega^{n-1}$ where c_n is a constant depending only on the complex dimension of M. Hence

$$(\partial_E s, \tau s) = 2\sqrt{-1}\int_M \left\{s, \left(*\overline{\partial}^*\omega\right) \wedge \overline{\partial}_E s\right\} = 0 \tag{2.4.25}$$

since $\overline{\partial}_E s = 0$. ∎

Remark 2.4.9. By these formulas, we can obtain classical vanishing theorems on Kähler manifolds and rigidity of harmonic maps between compact Hermitian and compact Riemannian manifolds.

2.5 Vanishing theorems on Hermitian manifolds

2.5.1 Vanishing theorems on compact Hermitian manifolds

Let E be a Hermitian *complex* (possibly *non-holomorphic*) vector bundle or a Riemannian *real* vector bundle over a compact Hermitian manifold (M, ω). Let ∇^E be an arbitrary connection on E and $\partial_E, \overline{\partial}_E$ the $(1,0),(0,1)$ part of ∇^E respectively. The $(1,1)$-curvature of ∇^E is denoted by $R^E \in \Gamma(M, \Lambda^{1,1}T^*M \otimes E^* \otimes E)$. It can be viewed as a representation of the operator $\partial_E\overline{\partial}_E + \overline{\partial}_E\partial_E$. We can define harmonic section spaces associated to (E, ∇^E) by

$$\mathcal{H}^{p,q}_{\overline{\partial}_E}(M,E) = \{\varphi \in \Omega^{p,q}(M,E) \mid \overline{\partial}_E\varphi = \overline{\partial}^*_E\varphi = 0\} \tag{2.5.1}$$

In general, on a complex vector bundle E, there is no terminology such as "holomorphic section of E". However, if the vector bundle E is holomorphic and ∇^E is the Chern connection on E, i.e., $\overline{\partial}_E = \overline{\partial}$, then $\mathcal{H}^{p,q}_{\overline{\partial}_E}(M,E)$ is isomorphic to the Dolbeault cohomology group $H^{p,q}_{\overline{\partial}}(M,E)$ and $H^0_{\overline{\partial}}(M,E)$ is the holomorphic section space $H^0(M,E)$ of E.

Definition 2.5.1. Let A be an $r \times r$ Hermitian matrix and $\lambda_1 \leq \cdots \leq \lambda_r$ be eigenvalues of A. A is said to be *p-nonnegative* (resp. *positive, negative, nonpositive*) for $1 \leq p \leq r$ if

$$\lambda_{i_1} + \cdots + \lambda_{i_p} \geq 0 (\quad \text{resp.} \quad > 0, < 0, \leq 0) \quad \text{for any} \quad 1 \leq i_1 < i_2 < \cdots < i_p \leq n \tag{2.5.2}$$

Theorem 2.5.2. *Let E be a Hermitian complex vector bundle or a Riemannian real vector bundle over a compact Hermitian manifold (M, ω) and ∇^E be any metric connection on E.*

(1) If the second Hermitian-Ricci curvature $Tr_\omega R^E$ is nonpositive everywhere, then every $\overline{\partial}_E$-closed section of E is parallel, i.e. $\nabla^E s = 0$;

(2) If the second Hermitian-Ricci curvature $Tr_\omega R^E$ is nonpositive everywhere and negative at some point, then $\mathcal{H}^0_{\overline{\partial}_E}(M,E) = 0$;

(3) If the second Hermitian-Ricci curvature $Tr_\omega R^E$ is p-nonpositive everywhere and p-negative at some point, then $\mathcal{H}^0_{\overline{\partial}_E}(M, \Lambda^q E) = 0$ for any $p \leq q \leq rank(E)$.

Proof. By [Gauduchon84], there exists a smooth function $u : M \to \mathbb{R}$ such that $\omega_G = e^u \omega$ is a Gauduchon metric, i.e. $\partial\overline{\partial}\omega_G^{n-1} = 0$. Now we replace the metric ω on M by the Gauduchon metric ω_G. By the relation $\omega_G = e^u\omega$, we get

$$Tr_{\omega_G} R^E = e^{-u} Tr_\omega R^E \tag{2.5.3}$$

Therefore, the positivity conditions in the Theorem are preserved. Let $s \in \Gamma(M, E)$ with $\overline{\partial}_E s = 0$, by formula (2.4.22), we obtain

$$0 = \|\partial_E s\|^2 + \left(\sqrt{-1}\left[\partial_E\overline{\partial}_E + \overline{\partial}_E\partial_E, \Lambda_G\right] s, s\right) = \|\partial_E s\|^2 - \left(Tr_{\omega_G} R^E s, s\right) \tag{2.5.4}$$

where

$$R^E = \partial_E\overline{\partial}_E + \overline{\partial}_E\partial_E = R_{i\bar{j}\alpha}^{\beta} dz^i \wedge d\overline{z}^j \otimes e^\alpha \otimes e_\beta \tag{2.5.5}$$

Since the second Hermitian-Ricci curvature $Tr_{\omega_G} R^E$ has components

$$R_{\alpha\overline{\beta}} = h_G^{i\bar{j}} R_{i\bar{j}\alpha\overline{\beta}} \tag{2.5.6}$$

formula (2.5.4) can be written as

$$0 = \|\partial_E s\|^2 - \int_M R_{\alpha\overline{\beta}} s^\alpha \overline{s}^\beta \tag{2.5.7}$$

Now **(1)** and **(2)** follow by identity (2.5.7) with the curvature conditions immediately. For **(3)**, we set $F = \Lambda^q E$ with $p \leq q \leq r = rank(E)$. Let $\lambda_1 \leq \cdots \leq \lambda_r$ be the eigenvalues of $-Tr_{\omega_G} R^E$, then we know

$$\lambda_1 + \cdots + \lambda_p \geq 0 \tag{2.5.8}$$

and it is strictly positive at some point. If $p \leq q \leq r$, the smallest eigenvalue of $-Tr_{\omega_G} R^F$ is $\lambda_1 + \cdots + \lambda_q \geq 0$ and it is strictly positive at some point. By **(2)**, we know $\mathcal{H}^0_{\overline{\partial}_E}(M, F) = 0$. □

If ∇^E is the Chern connection of the Hermitian holomorphic vector bundle E, we know

$$\mathcal{H}^0_{\overline{\partial}_E}(M, E) \cong H^0(M, E)$$

since $\overline{\partial}_E = \nabla''^E = \overline{\partial}$ for the Chern connection.

Corollary 2.5.3 (Kobayashi-Wu[Kobayashi-Wu70], Gauduchon [Gauduchon77b])**.** *Let ∇^E be the Chern connection of a Hermitian holomorphic vector bundle E over a compact Hermitian manifold (M, h, ω).*

(1) If the second Ricci-Chern curvature $Tr_\omega R^E$ is nonpositive everywhere, then every holomorphic section of E is parallel, i.e. $\nabla^E s = 0$;

(2) If the second Ricci-Chern curvature $Tr_\omega R^E$ is nonpositive everywhere and negative at some point, then E has no holomorphic section, i.e. $H^0(M, E) = 0$;

(3) If the second Ricci-Chern curvature $Tr_\omega R^E$ is p-nonpositive everywhere and p-negative at some point, then $\Lambda^q E$ has no holomorphic section for any $p \leq p \leq rank(E)$.

Now we can apply it to the tangent and cotangent bundles of compact Hermitian manifolds.

Corollary 2.5.4. *Let (M, ω) be a compact Hermitian manifold and Θ is the Chern curvature of the Chern connection ∇^{CH} on the holomorphic tangent bundle $T^{1,0}M$.*

(1) If the second Ricci-Chern curvature $\Theta^{(2)}$ is nonpositive everywhere and negative at some point, then M has no holomorphic vector field, i.e. $H^0(M, T^{1,0}M) = 0$;

(2) If the second Ricci-Chern curvature $\Theta^{(2)}$ is nonnegative everywhere and positive at some point, then M has no holomorphic p-form for any $1 \leq p \leq n$, i.e. $H^{p,0}_{\overline{\partial}}(M) = 0$; In particular, the arithmetic genus

$$\chi(M, \mathcal{O}) = \sum (-1)^p h^{p,0}(M) = 1 \tag{2.5.9}$$

(3) If the second Ricci-Chern curvature $\Theta^{(2)}$ is p-nonnegative everywhere and p-positive at some point, then M has no holomorphic q-form for any $p \leq q \leq n$, i.e. $H^{q,0}_{\overline{\partial}}(M) = 0$. In particular, if the scalar curvature S^{CH} is nonnegative everywhere and positive at some point, then $H^0(M, mK_M) = 0$ for all $m \geq 1$ where K_M is the canonical line bundle of M.

Proof. Let $E = T^{1,0}M$ and h be a Hermitian metric on E such that the second Ricci-Chern curvature $Tr_{\omega_h}\Theta$ of (E, h) satisfies the assumption. It is obvious that all section spaces in consideration are independent of the choice of the metrics and connections.

The metric on the vector bundle E is fixed. Now we choose a Gauduchon metric $\omega_G = e^u \omega_h$ on M. Then the second Ricci-Chern curvature $\widetilde{\Theta}^{(2)} = Tr_{\omega_G}\Theta = e^{-u} Tr_{\omega_h}\Theta$ shares the semi-definite property with $\Theta^{(2)} = Tr_{\omega_h}\Theta$. For the safety, we repeat the arguments in Theorem 2.5.2 briefly. If s is a holomorphic section of E, i.e., $\overline{\partial}_E s = \overline{\partial} s = 0$, by formula (2.4.22), we obtain

$$0 = \|\partial_E s\|^2 + \left(\sqrt{-1}\left[\partial_E \overline{\partial}_E + \overline{\partial}_E \partial_E, \Lambda_G\right] s, s\right) = \|\partial_E s\|^2 - (Tr_{\omega_G}\Theta s, s) \tag{2.5.10}$$

If $Tr_\omega \Theta$ is nonpositive everywhere, then $\partial_E s = 0$ and so $\nabla^E s = 0$. If $Tr_\omega\Theta$ is nonpositive everywhere and negative at some point, we get $s = 0$, therefore $H^0(M, T^{1,0}M) = 0$. The proofs of (2) and (3) are similar. $\square$

Remark 2.5.5. It is well-known that the first Ricci-Chern curvature $\Theta^{(1)}$ represents the first Chern class of M. But on a Hermitian manifold, it is possible that the second Ricci-Chern curvature $\Theta^{(2)}$ is not in the same $(d, \partial, \overline{\partial})$-cohomology class as $\Theta^{(1)}$. For example, $\mathbb{S}^3 \times \mathbb{S}^1$ with canonical metric has strictly positive second Ricci-Chern curvature but it is well-known that it has vanishing first Chern number c_1^2. For more details see Proposition 2.6.4. Therefore, $\Theta^{(2)}$ in Proposition 2.5.4 can NOT be replaced by $\Theta^{(1)}$. It seems to be an interesting question: if (M, ω) is a compact Hermitian manifold and its first Ricci-Chern curvature is nonnegative everywhere and positive at some point, is the first Betti number of M zero? In particular, is it Kähler in dimension 2?

As special cases of our results, the following results for Kähler manifolds are well-known, and we list them here for the convenience of the reader. Let (M, h, ω) be a compact Kähler manifold.

(1) If the Ricci curvature is nonnegative everywhere, then any holomorphic $(p, 0)$ form is parallel;

(2) If the Ricci curvature is nonnegative everywhere and positive at some point, then $h^{p,0} = 0$ for $p = 1, \cdots, n$. In particular, the arithmetic genus $\chi(M, \mathcal{O}) = 1$ and $b_1(M) = 0$;

(3) If the scalar curvature is nonnegative everywhere and positive at some point, then $h^{n,0} = 0$.

(A) If the Ricci curvature is nonpositive everywhere, then any holomorphic vector field is parallel;

(B) If the Ricci curvature is nonpositive everywhere and negative at some point, there is no holomorphic vector field.

2.5.2 Vanishing theorems on special Hermitian manifolds

Let (M, h, ω) be a compact Hermitian manifold and ∇ be the Levi-Civita connection.

Lemma 2.5.6. *Let (M, ω) be a compact balanced Hermitian manifold. For any $(p, 0)$-form φ on M,*

(1) If φ is holomorphic, then $\partial^*\varphi = 0$;

(2) If $\nabla'\varphi = 0$, then $\partial\varphi = 0$.

Proof. For simplicity, we assume $p = 1$. For the general case, the proof is the same. By Lemma 2.9.5, we know, for any $(1, 0)$-form $\varphi = \varphi_i dz^i$,

$$\partial^*\varphi = -h^{i\bar{j}}\frac{\partial\varphi_i}{\partial\bar{z}^j} \tag{2.5.11}$$

where we use the balanced condition $h^{i\bar{j}}\Gamma^s_{i\bar{j}} = 0$. If φ is holomorphic, then $\frac{\partial\varphi_i}{\partial\bar{z}^j} = 0$, hence $\partial^*\varphi = 0$. On the other hand,

$$\nabla'\varphi = \left(\frac{\partial\varphi_i}{\partial z^j} - \Gamma^m_{ji}\varphi_m\right) dz^j \otimes dz^i \tag{2.5.12}$$

If $\nabla'\varphi = 0$, we obtain

$$\partial\varphi = \frac{\partial\varphi_i}{\partial z^j}dz^j \wedge dz^i = \Gamma^m_{ji}\varphi_m dz^j \wedge dz^i = 0 \tag{2.5.13}$$

□

Theorem 2.5.7. *Let (M, ω) be a compact balanced Hermitian manifold with Levi-Civita connection ∇.*

(1) If the Hermitian-Ricci curvature $(R_{i\bar{j}})$ is p-nonnegative everywhere, then any holomorphic $(q, 0)$-form ($p \leq q \leq n$) is ∂-harmonic; in particular, $\dim_{\mathbb{C}} H^{q,0}_{\bar{\partial}}(M) \leq \dim_{\mathbb{C}} H^{0,q}_{\bar{\partial}}(M)$ for any $p \leq q \leq n$;

(2) If the Hermitian-Ricci curvature $(R_{i\bar{j}})$ is p-nonnegative everywhere and p-positive at some point, $H^{q,0}_{\bar{\partial}}(M) = 0$ for any $p \leq q \leq n$;

In particular,

(3) if the Hermitian-Ricci curvature $(R_{i\bar{j}})$ is nonnegative everywhere and positive at some point, then $H^{p,0}_{\bar{\partial}}(M) = 0$, for $p = 1, \cdots, n$ and so the arithmetic genus $\chi(M, \mathcal{O}) = 1$ and $b_1(M) \leq h^{0,1}(M)$.

(4) if the Hermitian-scalar curvature S is nonnegative everywhere and positive at some point, then

$$H^0(M, mK_M) = 0 \quad \textit{for any} \quad m \geq 1$$

where $K_M = \det T^{*1,0}M$.

Proof. At first, we assume $p = 1$ for **(1)** and **(2)**. Now we consider $E = T^{*1,0}M$ with the induced metric connection $\nabla^E = \widehat{\nabla}$ for h (see (2.2.23)). By formula (2.4.7), we have

$$\|\overline{\partial}_E s\|^2 = \|\partial_E s\|^2 + \sqrt{-1}\left(\left[R^E, \Lambda\right] s, s\right) \tag{2.5.14}$$

where R^E is the $(1,1)$-part curvature of E with respect to the connection ∇^E. More precisely,

$$R^E = \partial_E \overline{\partial}_E + \overline{\partial}_E \partial_E = -\widehat{R}_{i\bar{j}k}^{\ell} dz^i \wedge d\bar{z}^j \otimes \frac{\partial}{\partial z^\ell} \otimes dz^k \tag{2.5.15}$$

since E is the dual vector bundle of $T^{1,0}M$ and the $(1,1)$-part of the curvature of $T^{1,0}M$ is

$$\widehat{R}_{i\bar{j}k}^{\ell} dz^i \wedge d\bar{z}^j \otimes dz^k \otimes \frac{\partial}{\partial z^\ell} \tag{2.5.16}$$

If $s = f_i dz^i$ is a holomorphic 1-form, i.e.

$$\overline{\partial} s = \frac{\partial f_i}{\partial \bar{z}^j} d\bar{z}^j \wedge dz^i = 0 \tag{2.5.17}$$

then

$$\overline{\partial}_E s = \left(\frac{\partial f_i}{\partial \bar{z}^j} - f_k \Gamma_{\bar{j}i}^k\right) d\bar{z}^j \otimes dz^i = -f_k \Gamma_{\bar{j}i}^k d\bar{z}^j \otimes dz^i \tag{2.5.18}$$

Without loss of generality, we assume $h_{i\bar{j}} = \delta_{ij}$ at a given point. By Proposition 2.2.12, the quantity

$$|\overline{\partial}_E s|^2 = \sum_{i,j,t,n} f_i \overline{f}_n \Gamma_{\bar{j}t\bar{i}} \overline{\Gamma_{\bar{j}t\bar{n}}} = \sum_{i,n} \left(\widehat{R}_{n\bar{i}}^{(2)} - R_{n\bar{i}}\right) f_i \overline{f}_n \tag{2.5.19}$$

On the other hand

$$\sqrt{-1}\left\langle \left[R^E,\Lambda\right] s,s\right\rangle=\sum_{i,n}\widehat{R}^{(2)}_{n\bar{i}}f_i\overline{f}_n \tag{2.5.20}$$

That is

$$|\overline{\partial}_E s|^2-\sqrt{-1}\left\langle \left[R^E,\Lambda\right] s,s\right\rangle=-\sum_{i,n}R_{n\bar{i}}f_i\overline{f}_n\leq 0 \tag{2.5.21}$$

if the Hermitian-Ricci curvature $(R_{n\bar{i}})$ of (M,h,ω) is nonnegative everywhere. Then we get

$$0\leq\|\partial_E s\|^2=\|\overline{\partial}_E s\|^2-\sqrt{-1}\left(\left[R^E,\Lambda\right] s,s\right)\leq 0 \tag{2.5.22}$$

That is $\partial_E s=0$. Since

$$\partial_E s=\nabla'^E s=\widehat{\nabla}' s=\nabla' s=\left(\frac{\partial f_i}{\partial z^j}-f_\ell\Gamma^\ell_{ij}\right)dz^j\otimes dz^i$$

we obtain $\nabla' s=0$. By Lemma 2.5.6, we know $\Delta_\partial s=0$. In summary, we get

$$H^{1,0}_{\overline{\partial}}(M)\subset H^{1,0}_{\partial}(M)\cong H^{0,1}_{\overline{\partial}}(M) \tag{2.5.23}$$

If the Hermitian-Ricci curvature $(R_{n\bar{i}})$ is nonnegative everywhere and positive at some point, then $f_i=0$ for each i, that is $s=0$. Now we obtain $H^{1,0}_{\overline{\partial}}(M)=0$. The general cases follow by the same arguments as Theorem 2.5.2 and Theorem 2.5.4. In part (3), $b_1(M)\leq\dim_{\mathbb{C}}H^{0,1}_{\overline{\partial}}(M)$ follows form the Frölicher relation $b_1(M)\leq h^{1,0}(M)+h^{0,1}(M)$. $\square$

The dual of Theorem 2.5.7 is

Theorem 2.5.8. *Let (M,h,ω) be a compact balanced Hermitian manifold.*

(1) If $2\widehat{R}^{(2)}_{i\bar{j}}-R_{i\bar{j}}$ is nonpositive everywhere, then any holomorphic vector field is ∇'-closed;

(2) If $2\widehat{R}^{(2)}_{i\bar{j}}-R_{i\bar{j}}$ is nonpositive everywhere and negative at some point, there is no holomorphic vector field.

Proof. Let $E=T^{1,0}M$ and $\widehat{\nabla}$ the induced connection on it. If $s=f^i\frac{\partial}{\partial z^i}$ is a holomorphic section, then

$$\overline{\partial}_E s=f^i\Gamma^\ell_{\bar{j}i}d\bar{z}^j\otimes\frac{\partial}{\partial z^\ell}\in\Gamma(M,\Lambda^{0,1}T^*M\otimes E) \tag{2.5.24}$$

Without loss generality, we assume $h_{i\bar{j}} = \delta_{ij}$ at a given point. By Proposition 2.2.12,

$$\begin{aligned} |\overline{\partial}_E s|^2 - \sqrt{-1}\left\langle \left[\widehat{R}^{1,1}, \Lambda\right] s, s\right\rangle &= \left(\widehat{R}^{(2)}_{i\bar{j}} - R_{i\bar{j}}\right) f^i \overline{f}^j + \widehat{R}^{(2)}_{i\bar{j}} f^i \overline{f}^j \\ &= \left(2\widehat{R}^{(2)}_{i\bar{j}} - R_{i\bar{j}}\right) f^i \overline{f}^j \end{aligned}$$

By formula (2.4.17),

$$0 \leq \|\partial_E s\|^2 = \|\overline{\partial}_E s\|^2 - \sqrt{-1}\left(\left[\widehat{R}^{1,1}, \Lambda\right] s, s\right) \tag{2.5.25}$$

So if $2\widehat{R}^{(2)}_{i\bar{j}} - R_{i\bar{j}}$ is nonpositive everywhere, $\partial_E s = \nabla' s = 0$. If $2\widehat{R}^{(2)}_{i\bar{j}} - R_{i\bar{j}}$ is nonpositive everywhere and negative at some point, there is no holomorphic vector field. $\square$

Remark 2.5.9. (1) It is obvious that the second Ricci-Chern curvature $\Theta^{(2)}_{k\bar{\ell}}$ and Hermitian-Ricci curvature $R_{k\bar{\ell}}$ can not be compared. Therefore, Theorem 2.5.4 and Theorem 2.5.7 are independent of each other. For the same reason, Theorem 2.5.4 and Theorem 2.5.8 are independent.

(2) For a special case in Theorem 2.5.7, if the Hermitian-Ricci curvature $R_{k\bar{\ell}}$ is nonnegative everywhere and positive at some point, by Proposition 2.3.5, the manifold (M, ω) is Moishezon. It is well-known that every 2-dimensional Moishezon/balanced manifold is Kähler, but there are many Moishezon non-Kähler manifolds in higher dimension(See [Michelson83]).

The following result was firstly obtained in [Ivanov-Papadopoulos01]:

Corollary 2.5.10. *Let (M, ω) be a compact Hermitian manifold with $\Lambda(\partial\overline{\partial}\omega) = 0$. Let ∇^B be the Bismut connection on $T^{1,0}M$.*

(1) If the first Ricci-Bismut curvature $B^{(1)}$ is nonnegative everywhere, then every holomorphic $(p, 0)$-form is parallel with respect to the Chern connection ∇^{CH};

(2) If the first Ricci-Bismut curvature $B^{(1)}$ is nonnegative everywhere and positive at some point, then M has no holomorphic $(p, 0)$-form for any $1 \leq p \leq n$, i.e. $H^{p,0}_{\overline{\partial}}(M) = 0$; in particular, the arithmetic genus $\chi(M, \mathcal{O}) = 1$.

(3) If the first Ricci-Bismut curvature $B^{(1)}$ is p-nonnegative everywhere and p-positive at some point then M has no holomorphic $(q, 0)$-form for any $p \leq q \leq n$, i.e. $H^{q,0}_{\overline{\partial}}(M) = 0$. In

particular, if the scalar curvature S^{BM} of the Bismut connection is nonnegative everywhere and positive at some point, then $H^0(M, mK_M) = 0$ for any $m \geq 1$.

Proof. By Proposition 2.3.7, if $\Lambda(\partial\overline{\partial}\omega) = 0$, then

$$B^{(1)} \leq \Theta^{(2)} \tag{2.5.26}$$

Now we can apply Corollary 2.5.4 to get (1), (2) and (3). $\square$

Remark 2.5.11. For more vanishing theorems on special Hermitian manifolds, one can consult [Alexandrov-Ivanov01], [Ivanov-Papadopoulos01], [Ganchev-Ivanov01], [Ganchev-Ivanov00] and references therein.

2.6 Examples of non-Kähler manifolds with nonnegative curvatures

Let $M = \mathbb{S}^{2n-1} \times \mathbb{S}^1$ be the standard n-dimensional ($n \geq 2$) Hopf manifold. It is diffeomorphic to $\mathbb{C}^n - \{0\}/G$ where G is cyclic group generated by the transformation $z \longrightarrow \frac{1}{2}z$. It has an induced complex structure of $\mathbb{C}^n - \{0\}$. For more details about such manifolds, we refer the reader to [Kobayashi-Nomizu69]. On M, there is a natural metric

$$h = \sum_{i=1}^{n} \frac{4}{|z|^2} dz^i \otimes d\overline{z}^i \tag{2.6.1}$$

The following identities follow immediately

$$\frac{\partial h_{k\overline{\ell}}}{\partial z^i} = -\frac{4\delta_{k\ell}\overline{z}^i}{|z|^4}, \quad \frac{\partial h_{k\overline{\ell}}}{\partial \overline{z}^j} = -\frac{4\delta_{k\ell}z^j}{|z|^4} \tag{2.6.2}$$

and

$$\frac{\partial^2 h_{k\overline{\ell}}}{\partial z^i \partial \overline{z}^j} = -4\delta_{k\ell}\frac{\delta_{i\overline{j}}|z|^2 - 2\overline{z}^i z^j}{|z|^6} \tag{2.6.3}$$

Example 2.6.1 (Curvatures of Chern connection)**.** Straightforward computations show that, the Chen curvature components are

$$\Theta_{i\overline{j}k\overline{\ell}} = -\frac{\partial^2 h_{k\overline{\ell}}}{\partial z^i \partial \overline{z}^j} + h^{p\overline{q}} \frac{\partial h_{k\overline{q}}}{\partial z^i} \frac{\partial h_{p\overline{\ell}}}{\partial \overline{z}^j} = \frac{4\delta_{kl}(\delta_{ij}|z|^2 - z^j \overline{z}^i)}{|z|^6} \tag{2.6.4}$$

and the first and second Ricci-Chern curvatures are

$$\Theta^{(1)}_{k\bar{\ell}} = \frac{n\left(\delta_{k\ell}|z|^2 - z^\ell \bar{z}^k\right)}{|z|^4}, \quad \Theta^{(2)}_{k\bar{\ell}} = \frac{(n-1)\delta_{k\ell}}{|z|^2} \tag{2.6.5}$$

It is easy to see that the eigenvalues of $\Theta^{(1)}$ are

$$\lambda_1 = 0, \lambda_2 = \cdots = \lambda_n = \frac{n}{|z|^2} \tag{2.6.6}$$

Hence, $\Theta^{(1)}$ is nonnegative and 2-positive everywhere.

Example 2.6.2 (Curvatures of Levi-Civita connection)**.** Similarly, we have

$$\Gamma^\ell_{ik} = -\frac{\delta_{i\ell}\bar{z}^k + \delta_{k\ell}\bar{z}^i}{2|z|^2}, \quad \Gamma^\ell_{\bar{j}k} = \frac{\delta_{jk}z^\ell - \delta_{k\ell}z^j}{2|z|^2} \tag{2.6.7}$$

and

$$\frac{\partial \Gamma^\ell_{ik}}{\partial \bar{z}^j} = -\frac{\delta_{k\ell}\delta_{ij} + \delta_{i\ell}\delta_{jk}}{2|z|^2} + \frac{\delta_{i\ell}z^j\bar{z}^k + \delta_{k\ell}z^j\bar{z}^i}{2|z|^4} \tag{2.6.8}$$

$$\frac{\partial \Gamma^\ell_{\bar{j}k}}{\partial z^i} = \frac{\delta_{jk}\delta_{i\ell} - \delta_{k\ell}\delta_{ij}}{2|z|^2} - \frac{(\delta_{jk}z^\ell - \delta_{k\ell}z^j)\bar{z}^i}{2|z|^4} \tag{2.6.9}$$

The complexified Riemannian curvature components are

$$R^\ell_{i\bar{j}k} = -\left(\frac{\partial \Gamma^\ell_{ik}}{\partial \bar{z}^j} - \frac{\partial \Gamma^\ell_{\bar{j}k}}{\partial z^i} + \Gamma^s_{ik}\Gamma^\ell_{\bar{j}s} - \Gamma^s_{\bar{j}k}\Gamma^\ell_{is} - \Gamma^{\bar{s}}_{\bar{j}k}\Gamma^\ell_{i\bar{s}}\right) = \frac{\delta_{i\ell}\delta_{jk}}{2|z|^2} - \frac{\delta_{i\ell}z^j\bar{z}^k + \delta_{jk}z^\ell\bar{z}^i}{4|z|^4} \tag{2.6.10}$$

and

$$R_{i\bar{j}k\bar{\ell}} = \frac{2\delta_{i\ell}\delta_{jk}}{|z|^4} - \frac{\delta_{i\ell}z^j\bar{z}^k + \delta_{jk}z^\ell\bar{z}^i}{|z|^6}, \quad R_{k\bar{\ell}} = \frac{\delta_{k\ell}|z|^2 - z^\ell\bar{z}^k}{2|z|^4} \tag{2.6.11}$$

Example 2.6.3 (Curvatures of Bismut connection)**.** By definition (2.2.45) and Lemma 2.2.10, we obtain

$$B^\ell_{i\bar{j}k} = \frac{\delta_{jk}\delta_{i\ell} - \delta_{k\ell}\delta_{ij}}{|z|^2} + \frac{\delta_{ij}\bar{z}^k z^\ell + \delta_{k\ell}\bar{z}^i z^j - \delta_{i\ell}\bar{z}^k z^j - \delta_{jk}\bar{z}^i z^\ell}{|z|^4} \tag{2.6.12}$$

Two Ricci curvatures are

$$B^{(1)}_{i\bar{j}} = B^{(2)}_{i\bar{j}} = \frac{(2-n)(\delta_{ij}|z|^2 - \bar{z}^i z^j)}{4|z|^2} \tag{2.6.13}$$

On the other hand, by formula (2.6.3), it is easy to see $\partial\bar{\partial}\omega = 0$ and $B^{(1)} = 0$ for $n = 2$.

Proposition 2.6.4. *Let $M = \mathbb{S}^{2n-1} \times \mathbb{S}^1$ be the standard n-dimensional ($n \geq 2$) Hopf manifold with canonical metric h,* *(1)* (M,h) *has positive second Ricci-Chern curvature* $\Theta^{(2)}$*;*

(2) (M,h) *has vanishing first Chern class but has nonnegative first Ricci-Chern curvature* $\Theta^{(1)}$*. Moreover,*

$$\int_M \left(\Theta^{(1)}\right)^n = 0 \tag{2.6.14}$$

(3) (M,h) *is semi-positive in the sense of Griffiths, i.e.*

$$\Theta_{i\bar{j}k\bar{\ell}} u^i \overline{u}^j v^k \overline{v}^\ell \geq 0 \tag{2.6.15}$$

for any $u, v \in \mathbb{C}^n$*;*

(4) The Hermitian-Ricci curvature $(R_{k\bar{\ell}})$ *is nonnegative and* 2*-positive everywhere;*

(5) (M,h) *has nonpositive and* 2*-negative first Ricci-Bismut curvature. In particular,* $(\mathbb{S}^3 \times \mathbb{S}^1, \omega)$ *satisfies* $\partial\overline{\partial}\omega = 0$ *and has vanishing first Ricci-Bismut curvature* $B^{(1)}$*.*

Although we know all Betti numbers of Hopf manifold $\mathbb{S}^{2n-1} \times \mathbb{S}^1$, $h^{p,0}$ is not so obvious.

Corollary 2.6.5. *Let* (M,h) *be* n*-dimensional Hopf manifold with* $n \geq 2$*,*

(1) $h^{p,0}(M) = 0$ *for* $p \geq 1$ *and* $\chi(M, \mathcal{O}) = 1$*.*

(2) $\dim_{\mathbb{C}} H^0(M, mK) = 0$ *for any* $m \geq 1$ *where* $K = \det(T^{*1,0}M)$*.*

Remark 2.6.6. By Leray-Borel spectral sequence, one can compute all Hodge numbers of all Hopf manifolds. For more details, one can see [Hofer93].

2.7 Non-existence of complex structures on Riemannian manifolds

Let ∇^E be a connection on the complex vector bundle E. Let r be the rank of E, then there is a naturally induced connection $\nabla^{\det(E)}$ on the determine line bundle $\det(E) = \Lambda^r E$,

$$\nabla^{\det(E)}(s_1 \wedge \cdots \wedge s_r) = \sum_{i=1}^r s_1 \wedge \cdots \wedge \nabla^E s_i \wedge \cdots \wedge s_r$$

The curvature tensor of (E, ∇^E) is denoted by

$$R^E \in \Gamma\left(X, \Lambda^2 T^* X \otimes End(E)\right)$$

and the curvature tensor of $(\det E, \nabla^{\det(E)})$ is denoted by

$$R^{\det(E)} \in \Gamma\left(X, \Lambda^2 T^* X\right)$$

Note that the trace operator is well-defined without using metric.

Lemma 2.7.1. *We have the relation that*

$$tr R^E = R^{\det E} \in \Gamma(X, \Lambda^2 T^* X)$$

Note that the trace operator is well-defined without using metrics on the vector bundle E.

Proof. Let $\{e_1, \cdots, e_r\}$ be a local frame of the vector bundle E.

$$\begin{aligned} \left(\nabla^{\det(E)}\right)^2 (e_1 \wedge \cdots \wedge e_r) &= \sum_{i=1}^r e_1 \wedge \cdots \wedge (\nabla^E)^2 e_i \wedge \cdots \wedge e_r \\ &+ \sum_{i \neq j} e_1 \wedge \cdots \wedge (\nabla^E e_i) \wedge \cdots \wedge (\nabla^E e_j) \wedge \cdots \wedge e_r \end{aligned}$$

It is obvious that the second term on the right hand side is zero. Hence, we obtain

$$\left(\nabla^{\det(E)}\right)^2 (e_1 \wedge \cdots \wedge e_r) = (tr R^E)(e_1 \wedge \cdots \wedge e_r)$$

which finishes the proof of the Lemma. □

Corollary 2.7.2. *$tr R^E$ is a d-closed 2-form.*

Proof. By Bianchi identity, we know, for any vector bundle (F, ∇^F)

$$\nabla^{F \otimes F^*} R^F = 0$$

In particular, if F is a line bundle, $F \otimes F^* = \underline{\mathbb{C}}$ and $\nabla^{F \otimes F^*} = d$. Hence $d\left(R^{\det E}\right) = 0$. □

Theorem 2.7.3. *Let (M, h) be a compact Hermitian manifold. If the Hermitian-Ricci curvature $(R_{i\bar{j}})$ is quasi-positive, then $H^2(M) \neq 0$.*

Proof. If the Hermitian-Ricci curvature $(R_{i\bar{j}})$ is quasi-positive, by Proposition 2.2.12, so is the first Hermitian-Ricci curvature $(\widehat{R}^{(1)}_{i\bar{j}})$. Let $R^{\det}$ be the curvature of $(\det T^{1,0}M, \widehat{\nabla}^{\det T^{1,0}M})$ induced by the Hermitian vector bundle $(T^{1,0}M, \widehat{\nabla})$. By Lemma 2.7.1 and Corollary 2.7.2, $R^{\det}$ is a d-closed 2-form on M and it has a natural decomposition

$$R^{\det} = \omega^{2,0} + \omega^{0,2} + \omega^{1,1}$$

It is obvious that

$$\omega^{1,1} = \widehat{R}^{(1)}_{i\bar{j}} dz^i \wedge d\bar{z}^j$$

On the other hand, since the connection is metric compatible, we get

$$\omega^{2,0} = -\overline{\omega}^{0,2}$$

Hence

$$(\sqrt{-1})^n \int \left(R^{\det}\right)^n = (\sqrt{-1})^n \sum_{\ell=0}^{\left[\frac{n}{2}\right]} \binom{n}{2\ell}\binom{2\ell}{\ell} \int (-\omega^{2,0} \wedge \overline{\omega}^{2,0})^{\ell} \wedge \left(\omega^{1,1}\right)^{n-2\ell} \tag{2.7.1}$$

It is obvious that, if $\omega^{1,1}$ is quasi-positive,

$$(\sqrt{-1})^n \int (-\omega^{2,0} \wedge \overline{\omega}^{2,0})^{\ell} \wedge \left(\omega^{1,1}\right)^{n-2\ell} \geq 0$$

for $1 \leq \ell \leq \left[\frac{n}{2}\right]$ and $(\sqrt{-1})^n \int \left(\omega^{1,1}\right)^n > 0$. That is

$$(\sqrt{-1})^n \int \left(R^{\det}\right)^n > 0$$

So $R^{\det}$ is a d-closed but not d-exact 2-form on M, which implies $H^2(M) \neq 0$. □

Remark 2.7.4. (1) Similar results were also obtained in [Tang06] and [Bol-Hernadez-Lamoneda99].

(2) It is obvious that, the Hermitian Ricci curvature $(R_{i\bar{j}})$ defined in (2.2.12) exists on any Riemannian manifold, i.e., we do not need a complex structure or a compatible Hermitian metric on M. So it is very natural to ask the following question

Question 2.7.5. On a Riemannian manifold (M, g), which kinds of Riemannian curvature conditions on g can imply the quasi-positivity of the Hermitian-Ricci curvature?

The first sufficient curvature condition is the "strictly $\frac{1}{4}$-pinched Riemannian sectional curvature". In fact, Yau and Zheng proved in [Yau-Zheng91] that, if (M, g) has strictly $\frac{1}{4}$-pinched Riemannian sectional curvature, the complexified curvature operator is positive. In particular, the Hermitian-Ricci curvature is positive. On the other hand, by the celebrated Brendle-Schoen-Hopf differential sphere theorem ([Brendle-Schoen09]), we know that if (M, g) has strictly $\frac{1}{4}$-pinched Riemannian sectional curvature, M must be a sphere. In particular, we obtain a generalization of Lebrun's result

Corollary 2.7.6. *There is no integrable complex structure which is compatible with a strictly $\frac{1}{4}$-pinched Riemannian metric on $\mathbb{S}^6$.*

Definition 2.7.7. Let (M, g) be a Riemannian manifold. (M, g) has weakly positive constant sectional curvature, if there exists a quasi-positive smooth function λ on M such that

$$R(X, Y, Z, W) = \lambda(g(X, W)g(Y, Z) - g(X, Z)g(Y, W)) \tag{2.7.2}$$

for any real vector fields $X, Y, Z, W \in \Gamma(M, TM)$.

Corollary 2.7.8. *Let (M, g) be a Riemannian manifold with weakly positive constant sectional curvature.*

(1) (M, g) has quasi-positive Hermitian-Ricci curvature $(R_{i\bar{j}})$.

(2) If the Riemannian metric g is the background metric a Hermitian manifold (M, h), then M must be Kähler. In particular, $H^2(M) \neq 0$.

Proof. Let $\{x^1, \cdots, x^n, x^{n+1}, \cdots, x^{2n}\}$ be a real local coordinate system on M centered at a point p and $\{z^i = x^i + \sqrt{-1}y^i\}_{i=1}^n$ be the complex coordinate system where $y^i = x^{n+i}, i = 1, \cdots, n$. If (M, g) has weakly positive constant sectional curvature, the complexified curvature tensor

$$R_{ijk\bar{\ell}} = R\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^k}, \frac{\partial}{\partial z^\ell}\right) = 0$$

In fact, we can verify it by using formula (2.7.2) and the relation

$$\frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial y^i}\right)$$

Similarly, we can verify $R_{\overline{i}\overline{j}k\overline{\ell}} = 0$. Without loss of generality, we can assume at point p,

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \delta_{ij}, \quad \text{for} \quad i, j = 1, \cdots, 2n$$

By formula (2.2.6) and condition (2.7.2),

$$R_{i\overline{j}k\overline{\ell}} = R\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}, \frac{\partial}{\partial \overline{z}^k}, \frac{\partial}{\partial z^\ell}\right) = \frac{\lambda}{2}\delta_{i\ell}\delta_{jk}$$

Now we obtain the complexified Ricci curvature at point p,

$$R_{i\overline{j}} = \frac{\lambda}{2}\delta_{ij}$$

Hence the Hermitian-Ricci curvature is quasi-positive. If (M, h) is non-Kähler, by Proposition 2.2.12, Lemma 2.7.1 and Corollary 2.7.2, we obtain that

$$\widehat{R}^{(1)} = \widehat{R}^{(1)}_{i\overline{j}} dz^i \wedge d\overline{z}^j > Ric = R_{i\overline{j}} dz^i \wedge d\overline{z}^j = \frac{\lambda}{2}\delta_{ij} dz^i \wedge d\overline{z}^j$$

That is, the first Hermitian-Ricci curvature is a strictly positive closed $(1, 1)$ form. Since it is the curvature of the line bundle $K_M^* = \det(T^{1,0}M)$, it can be viewed as a Kähler metric on M. $\square$

Corollary 2.7.9. *Let $(\mathbb{S}^{2n-1} \times \mathbb{S}^1, h)$ be the Hermitian manifold defined in the last section. The Hermitian-Ricci curvature $(R_{k\overline{\ell}})$ is nonnegative everywhere and it can not be strictly positive at any point.*

Since $H^2(\mathbb{S}^{2n-1} \times \mathbb{S}^1) = 0$, we know the quasi-positive curvature condition in Theorem 2.7.3 can not be replaced by nonnegative curvature condition. Moreover,

(1) by Theorem 2.7.3, $\mathbb{S}^{2n-1} \times \mathbb{S}^1$ can not admit a Hermitian metric with quasi-positive Hermitian-Ricci curvature.

(2) by Corollary 2.7.8, $\mathbb{S}^{2n-1} \times \mathbb{S}^1$ can not admit a Hermitian metric with positive constant Riemannain sectional curvature.

2.8 A natural geometric flow on Hermitian manifolds

As we discussed in the above sections, on Hermitian manifolds, the second Ricci curvature tensors of various metric connections are closely related to the geometry of Hermitian manifolds. A natural idea is to define a flow by using second Ricci curvature tensors of various metric connections. We describe it in the following.

Let (M, h) be a compact Hermitian manifold. Let ∇ be an *arbitrary metric connection* on the holomorphic tangent bundle $(E, h) = (T^{1,0}M, h)$.

$$\nabla : E \to \Omega^1(E) \tag{2.8.1}$$

It has two components ∇' and ∇'',

$$\nabla = \nabla' + \nabla'' \tag{2.8.2}$$

∇' and ∇'' induce two differential operators

$$\partial_E : \Omega^{p,q}(E) \to \Omega^{p+1,q}(E) \tag{2.8.3}$$

$$\overline{\partial}_E : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E) \tag{2.8.4}$$

Let R^E be the $(1,1)$ curvature of the metric connection ∇. More precisely R^E is a representation of $\partial_E\overline{\partial}_E + \overline{\partial}_E\partial_E$. It is easy to see that

$$R^E \in \Gamma(M, \Lambda^{1,1}T^*M \otimes End(E)) \tag{2.8.5}$$

and locally, we can write it as

$$R^E = R_{i\overline{j}A}^B dz^i \wedge dz^j \otimes e^A \otimes e_B \tag{2.8.6}$$

Here we set $e_A = \frac{\partial}{\partial z^A}, e^B = dz^B$ where $A, B = 1, \cdots, n$, since the geometric meanings of j and A are different. It is well-known that a metric connection ∇ is determined by its Christoffel symbols

$$\nabla_{\frac{\partial}{\partial z^i}} e_A = \Gamma_{iA}^B e_B, \quad \nabla_{\frac{\partial}{\partial \overline{z}^j}} e_A = \Gamma_{\overline{j}A}^B e_B \tag{2.8.7}$$

In particular, we don't have notations such as Γ_{Ai}^B. It is obvious that

$$R_{i\overline{j}B}^A = -\frac{\partial \Gamma_{iA}^B}{\partial \overline{z}^j} + \frac{\partial \Gamma_{\overline{j}A}^B}{\partial z^i} - \Gamma_{iA}^C \Gamma_{\overline{j}C}^B + \Gamma_{\overline{j}A}^C \Gamma_{iC}^B \tag{2.8.8}$$

We set the second Hermitian-Ricci curvature tensor of (∇, h) as

$$R^{(2)} = h^{i\overline{j}} R_{i\overline{j}A\overline{B}} e^A \otimes \overline{e}^B \in \Gamma(M, E^* \otimes \overline{E}^*) \tag{2.8.9}$$

In general we can study a new class of flows on Hermitian manifolds

$$\begin{cases} \frac{\partial h}{\partial t} = \mathcal{F}(h) + \mu h \\ h(0) = h_0 \end{cases} \tag{2.8.10}$$

where $\mathcal{F}$ can be a linear combination of the first and the second Hermitian-Ricci curvature tensors of different metric connections on $(T^{1,0}M, h)$. For examples, $\mathcal{F}(h) = -\Theta^{(2)}$, the second Ricci-Chern curvature tensor of the Chern connection, and $\mathcal{F}(h) = -\widehat{R}^{(2)}$, the second Hermitian-Ricci curvature tensor of the complexified Levi-Civita connection, or the second Ricci curvature of any other Hermitian connection. Quite interesting is to take $\mathcal{F}(h) = s\Theta^{(1)} + (1-s)\Theta^{(2)}$ as the mixed Ricci-Chern curvature, or $\mathcal{F}(h) = B^{(2)} - 2\widehat{R}^{(2)}$ where $B^{(2)}$ is the second Ricci curvature of the Bismut connection. More generally, we can set $\mathcal{F}(h)$ to be certain suitable functions on the metric h.

The following result holds for quite general $\mathcal{F}(h)$, but here for simplicity we will only take $\mathcal{F}(h) = -\Theta^{(2)}$ as an example.

$$\begin{cases} \frac{\partial h}{\partial t} = -\Theta^{(2)} + \mu h \\ h(0) = h_0 \end{cases} \tag{2.8.11}$$

where μ is a real parameter. By formula (2.2.39), the second Ricci-Chern curvature tensor has components

$$\Theta^{(2)}_{k\bar{\ell}} = h^{i\bar{j}}\Theta_{i\bar{j}k\bar{\ell}} = -h^{i\bar{j}}\frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \bar{z}^j} + h^{i\bar{j}}h^{p\bar{q}}\frac{\partial h_{k\bar{q}}}{\partial z^i}\frac{\partial h_{p\bar{\ell}}}{\partial \bar{z}^j} \tag{2.8.12}$$

Theorem 2.8.1. *Let (M, h_0) be a compact Hermitian manifold.*

(1) There exists small ε such that, the solution of flow (2.8.11) exists for $|t| < \varepsilon$, and it preserves the Hermitian structure;

(2) The flow (2.8.11) preserves the Kähler structure, i.e., if the initial metric h_0 is Kähler, then $h(t)$ are also Kähler.

Proof. **(1).** Let Δ_c be the canonical Laplacian operator on the Hermitian manifold (M, h) defined by

$$\Delta_c = h^{p\bar{q}}\frac{\partial^2}{\partial z^p \partial \bar{z}^q}. \tag{2.8.13}$$

Therefore, the second Ricci-Chern curvature $-\Theta^{(2)}_{i\bar{j}}$ has leading term $\Delta_c h_{i\bar{j}}$ which is strictly elliptic. The local existence of the flow (2.8.11) follows by general theory of parabolic PDE, and the solution is a Hermitian metric on M.

(2). The coefficients of the tensor $\partial\omega$ are given by

$$f_{i\bar{j}k} = \frac{\partial h_{i\bar{j}}}{\partial z^k} - \frac{\partial h_{k\bar{j}}}{\partial z^i} \tag{2.8.14}$$

Under the flow (2.8.11), we have

$$\begin{cases} \frac{\partial f_{i\bar{j}k}}{\partial t} = \frac{\partial \Theta^{(2)}_{k\bar{j}}}{\partial z^i} - \frac{\partial \Theta^{(2)}_{i\bar{j}}}{\partial z^k} + \mu f_{i\bar{j}k} \\ f_{i\bar{j}k}(0) = 0 \end{cases} \tag{2.8.15}$$

At first, we observe that $f_{i\bar{j}k}(t) \equiv 0$ is a solution of (2.8.15). In fact, if $f_{i\bar{j}k}(t) \equiv 0$, then $h_{i\bar{j}}(t)$ are Kähler metrics, and so

$$\Theta^{(2)}_{i\bar{j}} = \Theta^{(1)}_{i\bar{j}} = -\frac{\partial^2 \log\det(h_{m\bar{n}})}{\partial z^i \partial \bar{z}^j}$$

Therefore,

$$\frac{\partial \Theta^{(2)}_{k\bar{j}}}{\partial z^i} - \frac{\partial \Theta^{(2)}_{i\bar{j}}}{\partial z^k} = -\frac{\partial^3 \log\det(h_{m\bar{n}})}{\partial z^i \partial z^k \partial \bar{z} j} + \frac{\partial^3 \log\det(h_{m\bar{n}})}{\partial z^i \partial z^k \partial \bar{z} j} = 0 \tag{2.8.16}$$

On the other hand,

$$\frac{\partial \Theta^{(2)}_{k\bar{j}}}{\partial z^i} - \frac{\partial \Theta^{(2)}_{i\bar{j}}}{\partial z^k} = \Delta_c\left(f_{i\bar{j}k}\right) + \quad \text{lower order terms} \tag{2.8.17}$$

Hence the solution of (2.8.15) is unique. □

Remark 2.8.2. Theorem 2.8.1 holds also for quite general $\mathcal{F}(h)$.

The flow (2.8.11) has close connections to several important geometric flows:

1. It is very similar to the Hermitian Yang-Mills flow on holomorphic vector bundles. More precisely, if the flow (2.8.11) has long time solution and it converges to a Hermitian metric h_∞ such that

$$\Theta^{(2)}_{i\bar{j}} = \mu h_{i\bar{j}} \tag{2.8.18}$$

   The Hermitian metric h_∞ is Hermitian-Einstein. So, by [Li-Yau87], the holomorphic tangent bundle $T^{1,0}M$ is stable. As shown in Example 2.6.1, the Hopf manifold $\mathbb{S}^{2n+1} \times \mathbb{S}^1$ is stable for any $n \geq 1$. In fact, in the definition of $\Theta^{(2)}_{i\bar{j}}$, if we take trace by using the initial metric h_0, then we get the original Hermitian-Yang-Mills flow equation.

2. If the initial metric is Kähler, then this flow is reduced to the usual Kähler-Ricci flow([Cao85]).

3. The flow (2.8.11) is similar to the harmonic map flow equation as shown in Theorem 2.8.1. It is strictly parabolic, and so the long time existence depends on certain curvature condition of the target manifold as discussed in the pioneering work of Eells-Sampson in [Eells-Sampson64]. The long time existence of this flow and other geometric properties of our new flow will be studied in our subsequent work.

Certain geometric flows and related results have been considered on Hermitian manifolds recently, we refer the reader to [Street-Tian1], [Street-Tian2], [Street-Tian3] and [Gill].

2.9 Appendix: The proof of the refined Bochner formulas

Lemma 2.9.1. *On a compact Hermitian manifold (M, h, ω), we have*

$$[\Lambda, 2\partial\omega] = A + B + C \tag{2.9.1}$$

where

$$\begin{cases} A = -h^{k\bar{\ell}} h_{i\overline{m}} \Gamma^{\overline{m}}_{s\bar{\ell}} dz^s \wedge dz^i I_k \\ \overline{A}^* = -h^{s\bar{t}} \Gamma^{\bar{i}}_{s\bar{k}} d\bar{z}^k I_{\bar{i}} I_{\bar{t}} \end{cases} \tag{2.9.2}$$

$$\begin{cases} B = -2\Gamma^{\bar{\ell}}_{i\bar{j}} dz^i \wedge d\bar{z}^j I_{\bar{\ell}} \\ \overline{B}^* = 2h^{p\bar{j}} \Gamma^{\bar{s}}_{\ell\bar{j}} dz^\ell I_p I_{\bar{s}} \end{cases} \tag{2.9.3}$$

$$\begin{cases} C = \Lambda(2\partial\omega) = 2\Gamma^{\bar{\ell}}_{j\bar{\ell}} dz^j \\ \overline{C}^* = 2h^{j\bar{\ell}} \Gamma^{\bar{s}}_{j\bar{s}} I_{\bar{\ell}} = -2h^{j\bar{i}} \Gamma^{\bar{\ell}}_{j\bar{i}} I_{\bar{\ell}} \end{cases} \tag{2.9.4}$$

Moreover,

(1) $[\Lambda, A] = -\sqrt{-1}\overline{B}^*$;

(2) $[\Lambda, B] = -\sqrt{-1}(2\overline{A}^* + \overline{B}^* + \overline{C}^*)$;

(3) $[\Lambda, C] = -\sqrt{-1}\overline{C}^*$.

Proof. All formulas follow by straightforward computations. $\square$

Definition 2.9.2. With respect to ∇' and ∇'', we define

$$\begin{cases} D' := dz^i \wedge \nabla'_i \\ D'' := d\bar{z}^j \wedge \nabla''_{\bar{j}} \end{cases} \tag{2.9.5}$$

The dual operators of $\partial, \overline{\partial}, D', D''$ with respect to the norm in (2.4.13) are denoted by $\partial^*, \overline{\partial}^*, \delta', \delta''$ and define

$$\begin{cases} \delta'_0 := -h^{i\bar{j}} I_i \nabla''_{\bar{j}} \\ \delta''_0 := -h^{j\bar{i}} I_{\bar{i}} \nabla'_j \end{cases} \tag{2.9.6}$$

where I the contraction operator and $I_i = I_{\frac{\partial}{\partial z^i}}$ and $I_{\bar{i}} = I_{\frac{\partial}{\partial \bar{z}^i}}$.

Remark 2.9.3. It is obvious that these first order differential operators D', D'', δ'_0 and δ''_0 are well-defined and they don't depend on the choices of holomorphic frames. If (M, h) is Kähler, $D' = \partial$, $D'' = \overline{\partial}$, $\delta'_0 = \delta' = \partial^*$ and $\delta''_0 = \delta'' = \overline{\partial}^*$.

Lemma 2.9.4. *In the local holomorphic coordinates,*

$$\partial = D' - \frac{B}{2} \quad \text{and} \quad \overline{\partial} = D'' - \frac{\overline{B}}{2} \tag{2.9.7}$$

Proof. We only have to check them on functions and 1-forms. $\square$

Lemma 2.9.5. *On a compact Hermitian manifold (M, h), we have*

$$\begin{cases} \delta'' = \delta''_0 - \frac{\overline{C}^*}{2} \\ \delta' = \delta'_0 - \frac{C^*}{2} \end{cases} \tag{2.9.8}$$

For ∂ and $\overline{\partial}$, we have

$$\begin{cases} \partial^* = \delta'_0 - \frac{B^* + C^*}{2} \\ \overline{\partial}^* = \delta''_0 - \frac{\overline{B}^* + \overline{C}^*}{2} \end{cases} \tag{2.9.9}$$

Proof. For any $\varphi \in \Omega^{p,q-1}(M)$ and $\psi \in \Omega^{p,q}(M)$, by Stokes' theorem

$$\begin{aligned}
0 &= \int_M \overline{\partial}(\varphi \wedge *\overline{\psi}) \\
&= \int_M \frac{\partial}{\partial \overline{z}^j}\left(d\overline{z}^j \wedge \varphi \wedge *\overline{\psi}\right) \\
&= \int_M \frac{\partial}{\partial \overline{z}^j}\left(\langle d\overline{z}^j \wedge \varphi, \psi\rangle \frac{\omega^n}{n!}\right) \\
&= \int_M \frac{\partial}{\partial \overline{z}^j}\left(\left\langle \varphi, h^{j\overline{i}} I_{\overline{i}}\psi\right\rangle \frac{\omega^n}{n!}\right) \\
&= \int_M \left(\left\langle \nabla''_{\overline{j}}\varphi, h^{j\overline{i}} I_{\overline{i}}\psi\right\rangle + \left\langle \varphi, \nabla'_j h^{j\overline{i}} I_{\overline{i}}\psi\right\rangle + \left\langle \varphi, h^{j\overline{i}} I_{\overline{i}}\psi\right\rangle \frac{\partial \log\det(h_{m\overline{n}})}{\partial \overline{z}^j}\right)\frac{\omega^n}{n!} \\
&= \int_M \left(\left\langle d\overline{z}^j \wedge \nabla''_{\overline{j}}\varphi, \psi\right\rangle + \left\langle \varphi, h^{j\overline{i}} \nabla'_j I_{\overline{i}}\psi\right\rangle + \left\langle \varphi, \frac{\partial h^{j\overline{i}}}{\partial z^j} I_{\overline{i}}\psi\right\rangle + \left\langle \varphi, h^{j\overline{i}} I_{\overline{i}}\psi\right\rangle \frac{\partial \log\det(h_{m\overline{n}})}{\partial \overline{z}^j}\right)\frac{\omega^n}{n!}
\end{aligned}$$

That is

$$(D''\varphi, \psi) = \left(d\overline{z}^j \wedge \nabla''_{\overline{j}}\varphi, \psi\right) = -\left(\varphi, h^{j\overline{i}} \nabla'_j I_{\overline{i}}\psi\right) - \left(\varphi, \left(\frac{\partial h^{j\overline{i}}}{\partial z^j} + h^{j\overline{i}} \frac{\partial \log\det(h_{m\overline{n}})}{\partial z^j}\right) I_{\overline{i}}\psi\right) \tag{2.9.10}$$

Now we will compute the second and third terms on the right hand side.

$$\frac{\partial h^{j\overline{i}}}{\partial z^j} + h^{j\overline{i}} \frac{\partial \log\det(h_{m\overline{n}})}{\partial z^j} = h^{j\overline{i}} h^{s\overline{t}}\left(\frac{\partial h_{s\overline{t}}}{\partial z^j} - \frac{\partial h_{j\overline{t}}}{\partial z^s}\right) = 2h^{j\overline{i}}\Gamma^{\overline{t}}_{j\overline{t}} = -2h^{j\overline{\ell}}\Gamma^{\overline{i}}_{j\overline{\ell}} \tag{2.9.11}$$

On the other hand

$$\begin{aligned}
-h^{j\overline{i}}\nabla'_j I_{\overline{i}} &= -h^{j\overline{i}} I_{\overline{i}} \nabla'_j - h^{j\overline{i}} I\left(\nabla'_j \frac{\partial}{\partial \overline{z}^i}\right) \\
&= \delta''_0 - h^{j\overline{i}}\Gamma^{\overline{\ell}}_{j\overline{i}} I_{\overline{\ell}}
\end{aligned} \tag{2.9.12}$$

In summary, by formulas (2.9.10), (2.9.11) and (2.9.12), the adjoint operator δ'' of D'' is

$$\delta'' = \left(\delta''_0 - h^{j\overline{i}}\Gamma^{\overline{\ell}}_{j\overline{i}} I_{\overline{\ell}}\right) + 2h^{j\overline{i}}\Gamma^{\overline{\ell}}_{j\overline{i}} I_{\overline{\ell}} = \delta''_0 - \frac{\overline{C}^*}{2}$$

Since $\overline{\partial} = D'' - \frac{\overline{B}}{2}$, we get

$$\overline{\partial}^* = \delta'' - \frac{\overline{B}^*}{2} = \delta''_0 - \frac{\overline{B}^* + \overline{C}^*}{2}$$

$\square$

Lemma 2.9.6. *On a compact Hermitian manifold* (M, h)*, we have*

$$\begin{cases} [\Lambda, D'] = \sqrt{-1}\left(\delta'' + \frac{\overline{C}^*}{2}\right) \\ [\Lambda, D''] = -\sqrt{-1}(\delta' + \frac{C^*}{2}) \end{cases} \quad \textit{and} \quad \begin{cases} [\delta'', L] = \sqrt{-1}(D' + \frac{C}{2}) \\ [\delta', L] = -\sqrt{-1}(D'' + \frac{\overline{C}}{2}) \end{cases} \tag{2.9.13}$$

Proof. By definition

$$\begin{aligned} (\Lambda D')\varphi &= \left(\sqrt{-1}h^{i\overline{j}}I_iI_{\overline{j}}\right)(dz^k \wedge \nabla'_k\varphi) \\ &= -\sqrt{-1}h^{i\overline{j}}I_i\left(dz^k \wedge I_{\overline{j}}\nabla'_k\varphi\right) \\ &= -\sqrt{-1}h^{i\overline{j}}I_{\overline{j}}\nabla'_i\varphi + \sqrt{-1}h^{i\overline{j}}dz^kI_iI_{\overline{j}}\nabla'_k\varphi \\ &= \sqrt{-1}\delta''_0 + dz^k \wedge \nabla'_k\left(\sqrt{-1}h^{i\overline{j}}I_iI_{\overline{j}}\varphi\right) \\ &= \sqrt{-1}\delta''_0 + D'\Lambda\varphi \end{aligned}$$

where we use the metric compatible condition

$$\nabla'\omega = 0 \Longrightarrow \nabla'_k(\Lambda\varphi) = \Lambda(\nabla'_k\varphi) \tag{2.9.14}$$

□

Lemma 2.9.7. *On a compact Hermitian manifold* (M, h)*, we have*

$$\begin{cases} [\Lambda, \partial] = \sqrt{-1}\left(\overline{\partial}^* + \overline{\tau}^*\right) \\ \left[\Lambda, \overline{\partial}\right] = -\sqrt{-1}(\partial^* + \tau^*) \end{cases} \tag{2.9.15}$$

For the dual case, it is

$$\begin{cases} [\overline{\partial}^*, L] = \sqrt{-1}(\partial + \tau) \\ [\partial^*, L] = -\sqrt{-1}(\overline{\partial} + \overline{\tau}) \end{cases} \tag{2.9.16}$$

Proof. By Lemma 2.9.6, 2.9.4 and 2.9.1,

$$\begin{aligned} [\Lambda, \partial] &= [\Lambda, D'] - \left[\Lambda, \frac{B}{2}\right] \\ &= \sqrt{-1}\left(\delta''_0 + \frac{2\overline{A}^* + \overline{B}^* + \overline{C}^*}{2}\right) \\ &= \sqrt{-1}\left(\delta'' + \frac{\overline{C}^*}{2} + \frac{2\overline{A}^* + \overline{B}^* + \overline{C}^*}{2}\right) \\ &= \sqrt{-1}(\overline{\partial}^* + \overline{\tau}^*) \end{aligned}$$

The other relations follow by complex conjugate and adjoint operations. □

Lemma 2.9.8. *On a Hermitian manifold* (M, h, ω)*,*

$$\overline{\partial}^*\omega = \sqrt{-1}\Lambda(\partial\omega) = \sqrt{-1}\Gamma_{\ell\overline{j}}^{\overline{j}} dz^\ell \tag{2.9.17}$$

Proof. We have

$$\frac{C}{2} = \Lambda(\partial\omega) = \Gamma_{j\overline{\ell}}^{\overline{\ell}} dz^j$$

On the other hand, by Lemma 2.9.5 and $\delta_0''\omega = 0$

$$\begin{aligned}
\overline{\partial}^*\omega &= \left(\delta_0'' - \frac{\overline{B}^* + \overline{C}^*}{2}\right)\omega = -\frac{\overline{B}^*\omega}{2} - \frac{\overline{C}^*}{2}\omega \\
&= \left(h_{\ell\overline{k}} h^{p\overline{j}} h^{i\overline{s}} \Gamma_{i\overline{j}}^{\overline{k}} dz^\ell I_p I_{\overline{s}}\right)\left(\frac{\sqrt{-1}}{2} h_{m\overline{n}} dz^m \wedge d\overline{z}^n\right) - \frac{\overline{C}^*}{2}\omega \\
&= -\frac{\sqrt{-1}}{2} h_{\ell\overline{k}} h^{i\overline{j}} \Gamma_{i\overline{j}}^{\overline{k}} dz^\ell - \frac{\overline{C}^*}{2}\omega \\
&= \frac{\sqrt{-1}}{2} \Gamma_{\ell\overline{j}}^{\overline{j}} dz^\ell - \frac{\overline{C}^*}{2}\omega \\
&= \sqrt{-1}\Gamma_{\ell\overline{j}}^{\overline{j}} dz^\ell \\
&= \sqrt{-1}\Lambda(\partial\omega)
\end{aligned}$$

□

Now we assume E is a Hermitian complex vector bundle or a Riemannian vector bundle over a compact Hermitian manifold (M, h, ω) and ∇^E is a metric connection on E.

Lemma 2.9.9. *We have the following formula:*

$$\overline{\partial}_E^*(\varphi \otimes s) = (\overline{\partial}^*\varphi) \otimes s - h^{i\overline{j}}\left(I_{\overline{j}}\varphi\right) \wedge \nabla_i^E s \tag{2.9.18}$$

for any $\varphi \in \Omega^{p,q}(M)$ *and* $s \in \Gamma(M, E)$.

Proof. The proof of it is the same as Lemma 2.9.5. □

Lemma 2.9.10. *If* τ *is the operator of type* $(1, 0)$ *defined by* $\tau = [\Lambda, 2\partial\omega]$ *on* $\Omega^\bullet(M, E)$*, then*

(1) $[\overline{\partial}_E^*, L] = \sqrt{-1}(\partial_E + \tau)$*;*

(2) $[\partial_E^*, L] = -\sqrt{-1}(\overline{\partial}_E + \overline{\tau})$;

(3) $[\Lambda, \partial_E] = \sqrt{-1}(\overline{\partial}_E^* + \overline{\tau}^*)$;

(4) $[\Lambda, \overline{\partial}_E] = -\sqrt{-1}(\partial_E^* + \tau^*)$.

Proof. We only have to prove **(3)**. For any $\varphi \in \Omega^\bullet(M)$ and $s \in \Gamma(M, E)$,

$$\begin{aligned}
(\Lambda\partial_E)(\varphi \otimes s) &= \Lambda\left(\partial\varphi \otimes s + (-1)^{|\varphi|}\varphi \wedge \partial_E s\right) \\
&= (\Lambda\partial\varphi) \otimes s + (-1)^{|\varphi|}\sqrt{-1}h^{k\overline{\ell}} I_k I_{\overline{\ell}}\left(\varphi \wedge \partial_E s\right) \\
&= (\Lambda\partial\varphi) \otimes s + (-1)^{|\varphi|}\sqrt{-1}h^{k\overline{\ell}} I_k\left((I_{\overline{\ell}}\varphi) \wedge \partial_E s\right) \\
&= (\Lambda\partial\varphi) \otimes s + (-1)^{|\varphi|}\sqrt{-1}h^{k\overline{\ell}}\left(I_k\left(I_{\overline{\ell}}\varphi\right)\right) \wedge \partial_E s - \sqrt{-1}h^{k\overline{\ell}} I_{\overline{\ell}}(\varphi) \wedge I_k\partial_E s \\
&= (\Lambda\partial\varphi) \otimes s + (-1)^{|\varphi|}(\Lambda\varphi) \wedge \partial_E s - \sqrt{-1}h^{k\overline{\ell}} I_{\overline{\ell}}(\varphi) \wedge \nabla_k^E s
\end{aligned}$$

On the other hand

$$\begin{aligned}
(\partial_E\Lambda)(\varphi \otimes s) &= \partial_E\left((\Lambda\varphi) \otimes s\right) \\
&= (\partial\Lambda\varphi) \otimes s + (-1)^{|\varphi|}(\Lambda\varphi) \wedge \partial_E s
\end{aligned}$$

Therefore

$$\begin{aligned}
[\Lambda, \partial_E](\varphi \otimes s) &= ([\Lambda, \partial]\varphi) \otimes s - \sqrt{-1}h^{k\overline{\ell}} I_{\overline{\ell}}(\varphi) \wedge \nabla_k^E s \\
&= \sqrt{-1}\left(\left(\overline{\partial}^* + \overline{\tau}^*\right)\varphi\right) \otimes s - \sqrt{-1}h^{k\overline{\ell}} I_{\overline{\ell}}(\varphi) \wedge \nabla_k^E s \\
&= \sqrt{-1}\left(\overline{\partial}_E^* + \overline{\tau}^*\right)(\varphi \otimes s)
\end{aligned}$$

where the last step follows by formula (2.9.18). $\square$

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