Title

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Numerical Simulation of Richards Equation: Current Approaches and an Alternate Perspective

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NUMERICAL SIMULATION OF RICHARDS EQUATION: CURRENT APPROACHES AND AN ALTERNATE PERSPECTIVE

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ABSTRACT. The transient flow of water in saturated-unsaturated media is described by a non-linear parabolic partial differential equation, familiarly known as Richards equation. Numerical solution of Richards equation is often beset with difficulties related to stability, convergence, and verification, particularly when water saturations are low and when material heterogeneities exist. It is suggested that these difficulties arise largely due to the fact that conventional numerical techniques based on finite differences and finite elements do not take into account the nature of the local flow geometry in estimating fluxes. Nor do they recognize that the Darcy-Buckingham equation, in the presence of gravity, heterogeneities or nonuniform flow geometry, is an implicit statement relating flux to the potential distribution between two surfaces of equal potential. Moreover, for an elemental volume in a transient nonlinear system, capacitance has to be defined in an operational sense, being specifically associated with a chosen location of observation within the elemental volume. Finally, in order to compute fluxes accurately, the time-averaging factor has to be made a function of space and of time. Theoretical discussions are provided to demonstrate how these ideas may be synthesized to solve the problem of transient flow in a flow tube of non uniform cross sectional area.

1. INTRODUCTION

1.1. Motivation

The transient flow of water in an isothermal porous medium under conditions of partial saturation is often expressed in the form of a partial differential equation. Originally proposed by L.A. Richards in 1931, this governing equation is subject to the important assumption that the air phase is at a constant pressure within the zone of partial saturation. Richards equation is extremely non-linear in nature due to the strong dependences of material properties on the dependent variable, water phase pressure. As a consequence, closed form solutions to Richards

equation are extremely difficult to obtain, especially when one is interested in multidimensional heterogeneous systems with complex geometries. Therefore, for applying Richards equation to any realistic field problem, the preferred approach among researchers at the present time is the use of numerical models.

Within the past thirty years a variety of numerical models have appeared in the literature for solving Richards equation (e.g., Brutsaert, 1971; Cooley, 1971; Freeze, 1971; Narasimhan and Witherspoon, 1978; Neuman, 1973; Rubin et al., 1964; and many others). Despite the availability of many such algorithms, practical difficulties do exist in the credible implementation of these models. These difficulties relate not only to the task of merely obtaining a solution (stability; convergence) but also to the verification of the solutions that are so obtained.

The present work is motivated by a desire to identify the causes of these difficulties and to explore rational ways of overcoming them.

1.2. Scope

The transient transport process in the vadose zone is one that involves multiple fluid phases and heat. Yet, Richards equation idealizes the system purely in terms of single phase water transport. Some researchers (Morel-Seytoux, 1987) have attempted to minimize the effect of this constraint by treating the vadose zone as a two-fluid system involving water and air or as a multi-component system involving heat as well (Philip and de Vries, 1957; Sophocleous, 1979). In the present work we will not be concerned with these more general approaches and we shall restrict ourselves to the single-phase isothermal idealization of Richards equation.

It has been recognized in the literature that the strong non-linearity of Richards equation could be eased by simply casting the equations using water content rather than pressure head as the dependent variable. Because, in heterogeneous media water content is discontinuous at material interfaces, this formulation has to be supplemented by continuity criteria on capillary pressure head when applied to heterogeneous media. Thus, the ultimate solution of Richards' equation has to take into account the variation of fluid pressure. Because most realistic field problems in the earth sciences involve heterogeneous media, we will devote our attention in the present work exclusively to the pressure-head based formulation.

A majority of the numerical models proposed for solving Richards equation involve the discretization of the flow domain as well as time into finite subsets and integrating the equation in terms of discrete sums. These methods fall into two general categories, the Integral Finite Difference Methods IFDM [*] and the Finite Element Methods (FEM). A relatively new technique known as the Boundary Element Method (BEM) is used by some researchers to solve the Richards equation. This method consists generally in discretizing the boundary surface of each material within the system and numerically integrating the Green's functions over these segments. However, the Green's Functions are primarily well-suited for linear problems and are not well-defined for non-linear equations. In the present work we shall restrict ourselves to the IFDM and the FEM.

The scope of this work is a modest one of recalling certain well-accepted numerical modeling approaches and to look for rational ways of extending beyond these approaches.

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[*] We include the classical Finite Difference Methods (FDM) as subsets of the IFDM in the present work.

This work does not include a detailed review of all the relevant literature on numerically modeling Richards' equation.

2. THE GOVERNING EQUATIONS

In developing Richards' equation using fluid pressure as the dependent variable, two conventions are usually followed in the literature. In one, *suction head (or, in short, suction)*, denoted by the symbol h is used. Suction, defined as the difference between atmospheric pressure head and water pressure head is always positive and is merely the capillary pressure expressed in units of water head. In the other convention, one chooses to use a *gauge pressure head (or, simply pressure head)*, in which the gauge reads zero at atmospheric pressure. Denoted by the symbol ψ, the pressure head is always negative in the unsaturated zone. Also, in order to take into account gravity, one may choose to express the vertical axis either positive upwards, as is frequently done in the hydrogeology literature (elevation) or positive downwards (depth). One could use any of these conventions as long as one is careful to be consistent. In the present work we shall use suction head h in conjunction with the vertical axis z positive upwards.

Note that suction and pressure head are simply related by $h = -\psi$. The *potential (or potentiometric head)*, usually denoted by the symbol ϕ, is given by, $\phi = z - h$ or, equivalently by, $\phi = z + \psi$.

In its essence Richards equation expresses the evolution in time of some state variables such as pressure head and water saturation in an elemental volume within a variably saturated porous medium. For practical purposes, it is convenient to represent this evolution over a small interval of time Δt. Consider a small elemental volume of index j as shown in Figure 1, bounded by a closed surface Γ. At the initial time t_o, the mass of water contained in the element and the average suction head over the element are, respectively, $M^o_{w_j}$ and h^o_j.

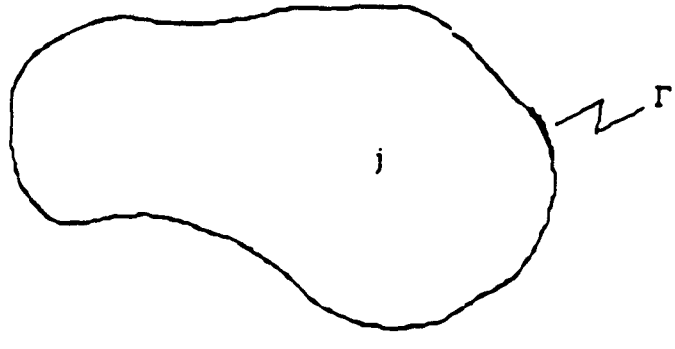

Figure 1: An elemental volume j enclosed by the surface Γ.

State at t_o:

$$M_{w,j}(t_o) = M^o_{w,j} \tag{1a}$$

$$h_j(t_o) = h^o_j \tag{1b}$$

Change of State t_o $t_o + \Delta t$:

$$-\left\{\sum_m \rho_w \vec{q}\cdot\vec{n}_{jm}\ \Delta\Gamma_{jm} + \sum_b \rho_w \vec{q}\cdot\vec{n}_{jb}\ \Delta\Gamma_{jb} + \rho_w G_j\right\}\Delta t$$

$$= \Delta M_{w,j} = -M_{c,j}\Delta h_j \tag{2}$$

where ρ_w is the mass density of water, $\vec{q}$ is flux density or darcy velocity, $\vec{n}_{jm}$ is the unit outer normal to the m th surface segment of volume element j that is interior to the flow domain, $\vec{n}_{jb}$ is the outer normal to the b th surface segment of volume element j that coincides with the external surface of the flow domain, $\Delta\Gamma_{jm}$ and $\Delta\Gamma_{jb}$ are surface segments that enclose volume element j, G_j volumetric rate of fluid generation (positive when fluid is injected or negative when fluid is withdrawn) from elemental volume j, $\Delta M_{w,j}$ is the change in mass of water over j during the time interval Δt, $M_{c,j}$ is the *capacitance* of element j (synonymous with *fluid mass capacity* [Narasimhan and Witherspoon, 1977]) and, Δh_j is the change in average suction head over j during Δt. Defined by the relation,

$$M_{c,j} = -\frac{\partial M_w}{\partial h},$$

the *capacitance* is defined as the change in the mass of water stored in the elemental volume j associated with a unit change in the average suction head h over the element, with the external stresses held constant.

The capacitance, $M_{c,j}$ includes the effects of three independent processes; changes in pore volume, saturation and water density. All these processes can be expressed in terms of of water-phase pressure. Thus,

$$M_{c,j} = -V_{s_j}\rho_w\left[e_j S_j \gamma_w c_w + S_j \gamma_w \chi' a_{v,j} + e_j \frac{dS_j}{dh}\right] \tag{3}$$

where V_{s_j} denotes the volume of solids, e is void ratio, S is water saturation, c_w is compressibility of water, γ_w is unit weight of water, χ' is a parameter used to convert changes in fluid pressure to changes in effective stress, and, a_v is the coefficient of compressibility.

We now use the Darcy-Buckingham equation to express flux density by the relation,

$$\vec{q} = - K_s k_r(h) \vec{\nabla}(z - h) \tag{4}$$

In view (4), the conservation equation (3) becomes,

$$\Delta t \left\{ \sum_m \rho_w K_s k_r(h) \vec{\nabla}(z - h) \cdot \vec{n}_{jm} \ \Delta\Gamma_{jm} + \sum_b \rho_w K_s k_r(h) \vec{\nabla}(z - h) \cdot \vec{n}_{jb} \ \Delta\Gamma_{jb} + \rho_w G_j \right\}$$

$$= - M_{c,j} \Delta h_j \tag{5}$$

Note that because of the way we have defined $M_{c,j}$ in (3), the volume element is fixed in the solid phase, and as such is deformable like the solid skeleton. In this context, Darcy velocity is understood to be the velocity of the fluid relative to an observer fixed in the solid phase.

In (5) the summation over b includes all Dirichlet boundaries, including seepage faces and evaporation/evapotranspiration boundaries. The source term G_j includes conventional source-sink terms as well as Neumann boundaries.

By letting the elemental volume j become infinitesimally small, one could readily derive the parabolic partial differential equation form (5). Nevertheless, it is not essential for our purposes to resort to the differential equation. Numerical models of interest to us can be directly related to the discretized equation (5) without requiring the PDE as an intermediary.

State at $t_o + \Delta t$

$$M_{w,j}(t_o + \Delta t) = M^o_{w,j} + \Delta M_{w,j} \tag{6a}$$

$$h_j(t_o + \Delta t) = h^o_j + \Delta h_j \tag{6b}$$

The sequence, initial state, change of state, final state, forms the framework within which Richards equation is constructed.

3. THE NUMERICAL APPROACH

We now examine how (5) gets translated into a set of discretized numerical equations in the context of the IFDM or the FEM. These numerical schemes are founded on the notion that the dependent variable h_j as well as other intensive variables are known at discrete locations within the flow domain of interest. These locations are frequently known as nodes or nodal points. The intensive quantities are physically taken to be volume averages over the elemental volume of interest. In the present work we shall assume very small deformations of the porous medium and hence treat the elevations of the nodal points to be invariant in time. This notion

of discrete locations immediately dictates the character of numerical models.

Consider the left hand side of (5). The gradient of potential included within the summations need now to be expressed in terms of the magnitudes of potential at discrete locations. Accordingly, in the context of numerical modeling (5) translates to,

$$\Delta t\left\{\sum_m U_{jm}[(z_m - z_j) - (h_m - h_j)] + \sum_b U_{jb}[(z_b - z_j) - (h_b - h_j)] + \rho_w G_j\right\}$$

$$= - M_{c_j}\Delta h_j \qquad (7)$$

where, U_{jm} and U_{jb} are *conductances* defined as the time rate of flux between adjoining regions per unit difference in potential.

Also, because of the dynamic nature of the system, h_m and h_j continually change during Δt. Therefore, on the left hand side of (7) one has to use time-averaged values of h. Thus, if $\bar{h}_m$ and $\bar{h}_j$ are time averages over Δt, then,

$$\Delta t\left\{\sum_m U_{jm}[(z_m - z_j) - (\bar{h}_m - \bar{h}_j)] + \sum_b U_{jb}[(z_b - z_j) - (\bar{h}_b - \bar{h}_j)] + \rho_w G_j\right\}$$

$$= - M_{c_j}\Delta h_j \qquad (8)$$

Looking at (8) it is clear that the goal is to solve for Δh_j, using the known quantities U and M_c and the known forcing functions on the boundaries as well as the source terms. It follows therefore that whatever difficulties that arise in numerical simulations must be related to a large degree to errors inherent in estimating the the conductances, the capacitances and the time-averaged values of h. Therefore, the following three questions are critical to numerical modeling.

1. How best to calculate the conductances U_{jm} and U_{jb} *?*
2. How best to define the time-averages $\bar{h}$ *?, and,*
3. How best to calculate the capacitances M_c *?*

In partially saturated systems, both the conductances and the capacitances continually vary in time.

Indeed, if we look at the final set of linear equations that arise in either in the IFDM or the FEM, we find that they have essentially the same form as (8). Therefore equation 8, in conjunction with the three questions raised above constitutes a basis to analyze the difficulties encountered in the numerical simulation of Richards equation.

4. CURRENT APPROACHES

At the present time, the typical practice to set up numerical equations to solve Richards equation is to start with the partial differential equation and integrate it. Using different techniques for discretization, the goal of the integration process is to assemble a set of linear algebraic equations that are ultimately solved by algebraic methods. Integration is carried out in space as well as in time. We will now discuss how the integration procedures influence the estimation of conductances, capacitances and the time averages in (8).

4.1. The Conductance

By definition *conductance* is the time rate of transfer of mass of water between two adjoining elemental volumes per unit difference in potentiometric head. The magnitude of conductance in the context of saturated unsaturated flow is a function of several factors including relative hydraulic conductivity (a material property) and local flow geometry. In fact, conductance is merely the reciprocal of the resistance to flow. Thus, as shown by Narasimhan (1985),

$$\vec{Q}_{jm} = U_{jm}[(z_m - z_j) - (h_m - h_j)]$$

$$= \frac{1}{R_{jm}}[(z_m - z_j) - (h_m - h_j)]$$

$$= \frac{[(z_m - h_m) - (z_j + h_j)]}{\frac{1}{K_s}\int_{x_j}^{x_m} \frac{dy}{k_r(h(y))A(y)}} \qquad (9)$$

where,

$$R_{jm} = \frac{1}{U_{jm}}$$

is the *resistance*, K_s is the saturated hydraulic conductivity and k_r is the relative permeability to water. In (9) we consider unidimensional flow in a flow tube of nonuniform cross sectional area bounded by isopotential surfaces at x_j and x_m. Flux is defined as positive if it is directed towards j. For simplicity, we assume in (9) that the flow tube in (9) is composed of a single material under unsaturated conditions. Note that in (9) both k_r and the spatially dependent A occur within the integral sign. Now, since h is known only at the end points x_j and x_m but not within the interval $x_j < x < x_m$, (9) is in general an implicit statement when z_m is not equal to z_j (that is when gravity is present) or when A or K_s is a function of space. This implicit nature of the flux law has to be duly recognized in setting up the numerical equations. Nevertheless, in current numerical practices the flux law is invariably treated as an explicit statement, using

ad hoc assumptions.

In the IFDM and in the conventional FDM, one typically has a situation shown in Figure 2. In this situation, flux explicitly written in the form,

$$\vec{Q}_{jm} = K_s \bar{k}_{r,jm} [(z_m - h_m) - (z_j - h_j)] \frac{A_{jm}}{d_{jm} + d_{mj}} \qquad (10)$$

where $\bar{k}_{r,jm}$ is a spatial average of the relative hydraulic conductivity for the region between j and m and, A_{jm} , d_{jm} and d_{mj} are as shown in Figure 2. For simplicity we assume in (10) that both j and m consist of the same material. It is common practice in the literature to use a variety of predetermined mean values for $K_{r,jm}$ (e.g., harmonic mean, geometric mean, upstream weighting). All these a priori chosen mean values have errors inherent in them. For, as we have already seen, these do not take into account either the local flow geometry nor do they recognize the implicit nature of the flux law.

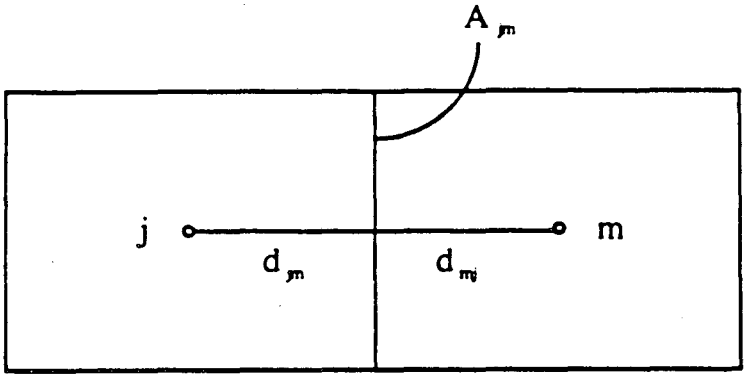

Figure 2: Two interacting volume elements, j and m, in the IFDM.

In the FEM, one usually has a situation such as that depicted in Figure 3. Here, the triangular region is the finite element and k_r and h are known at the locations of the corner points of the finite element. The conductance U_{jm} is the sum of two components,

$$U_{jm} = U_{jm,e_1} + U_{jm,e_2} \tag{11}$$

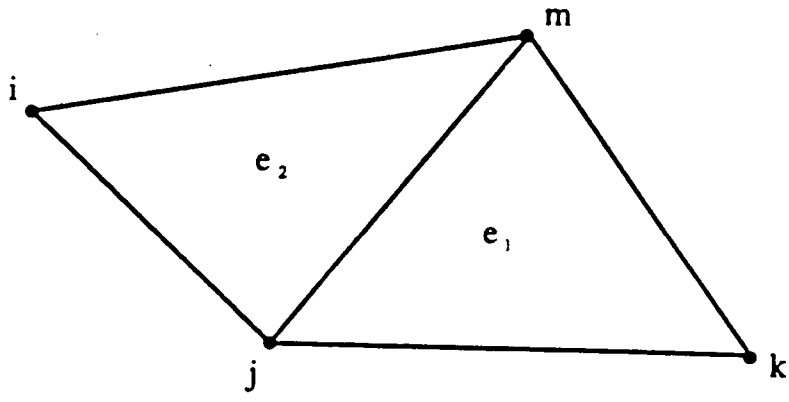

Figure 3: Finite elements e_1 and e_2 defined by nodal points i, j, k, and m.

A basic task in the FEM is to calculate the components of U_{jm} from each of the two finite elements. This is commonly accomplished by using a weighted integration logic with the Galerkin method (also known as Rayleigh-Ritz method or the Method of Weighted Residuals). In using the weighted volume integration logic, one simply uses a pre selected functional form for the variation of k_r over the finite element and explicitly arrives at the component of U_{jm} for that element. Here again, one neither gives consideration to the local flow geometry over the finite element nor does one recognize the implicit nature of the flux law. Thus, for essentially the same reasons both the IFDM and the FEM have errors inherent in their logic that they rely on to calculate conductance.

4.2. The Time-Average

It is widely known that if Δt is large, then, in order to assure stability as well as accuracy of solution, one has to use time-averaged values of h in evaluating the fluxes included within the summations in (8). If we assume that we have the ability to evaluate U_{jm} and U_{jb} accurately, it is easy to see that the mass of water transferred from m to j during Δt is given by,

$$\text{Mass transferred} = \int_{t_o}^{t_o+\Delta t} U_{jm}[(z_m - z_{j)} - (h_m - h_j)]dt \tag{12}$$

For purposes of setting up the linear equations in the numerical model, we wish to replace (12) by,

$$\int_{t_o}^{t_o+\Delta t} U_{jm}[(z_m - z_j) - (h_m - h_j)]dt$$

$$= U_{jm}(t_o + \lambda_{jm}\Delta t)\left\{(z_m - z_j) - (h_m(t_o + \lambda_{jm}\Delta t) - h_j(t_o + \lambda_{jm}\Delta t)\right\}$$

$$= \bar{U}_{jm}\left[(z_m - z_j) - (\bar{h}_m - \bar{h}_j)\right] \tag{13}$$

where $\bar{U}_{jm}$, $\bar{h}_m$ and $\bar{h}_j$ are time-averaged values evaluated at $\lambda_{jm}\Delta t$ where, $0 < \lambda_{jm} < 1.0$. To assure maximum accuracy in evaluating the fluxes, therefore, we must choose the time-weighting factor λ_{jm} in such a way that (13) is satisfied as accurately as possible. Because of the dependence of conductance on k_r as well as the local flow geometry, it is to be expected that λ_{jm} will in general be a function of space and time. In other words, for assuring maximum accuracy in evaluating fluxes in the numerical model, λ_{jm} must be allowed to vary from elemental interface to elemental interface and from one time step to the next. Nevertheless, it is the general custom in numerical modeling practices (either the IFDM or the FEM) to use a single global value of λ for the entire flow domain. The special cases of $\lambda = 0.0$, 0.5 and 1.0 are respectively known as forward differencing *(explicit)*, central differencing *(Crank-Nicolson)* and backward differencing *(implicit)* schemes. Very few workers (e.g., Edwards, 1972) have provided for the flexibility to vary this global λ in time. The common practice is to set λ equal to 0.5 or 1.0. The latter value is often preferred to assure unconditional stability in strongly non-linear problems.

4.3. The Capacitance

Capacitance is an essential parameter in transient systems because without it the system would be a steady state one. The classical notion of capacitance can be conveniently illustrated with the help of the heat conduction analogy. The *Heat Capacity* of a mass of material is defined by,

$$C = \frac{\Delta H}{\Delta T} \tag{14}$$

where C is the heat capacity of the material, ΔH is the change in heat content and ΔT is the change in temperature. Although this definition is simple, one has to recognize its constraints. Note that C is uniquely defined only if ΔT is a constant over the mass of material. This will indeed be the case if the mass of material is well stirred or if it occupies an infinitesimally small volume. However, if the mass occupies a finite volume and it is not well-stirred, as is the case when we deal with an elemental volume within a transient system, then the denominator ΔT in (14) is not unique. In principle, therefore, C is poorly defined, unless one specifies the particular location at which ΔT is measured. If so, C is in fact defined only in an operational

sense and it is a function of the property of the material that occupies the elemental volume and also the *location of measurement*. In the use of the IFDM for solving the Richards equation, the capacitance is purely treated as material property and is evaluated using (3). In the FEM too the logic is similar, except that it is treated as a sum of several capacitance components generated from each finite element of which the nodal point is part of. As suggested by Narasimhan (1985), *the location of average* or the location at which ΔT is measured, depends, especially in the case of nonlinear problems, both on local flow geometry and on the nature of the constituent material. Because local flow geometry is seldom considered in evaluating the capacitance term, IFDM and FEM methods, as they are currently implemented, have inherent errors.

4.4. Summary

The difficulties relating to stability, convergence, accuracy and verification of numerical solutions of Richards equation using conventional IFDM and FEM approaches are attributable to errors inherent in the evaluation of conductances and capacitances as well as in the time-averaging of fluid potentials. It stands to reason that by developing improved logic to the accurate estimation of these three quantities, one can greatly increase the power and credibility of numerical models that are used to solve Richards equation.

5. AN ALTERNATE APPROACH

As we have seen, the accurate determination of conductances and capacitances involve spatial as well as temporal considerations. In the space domain, accuracy depends both on a knowledge of local flow geometry and on the dependence of material properties on fluid potential. Now, if we restrict ourselves to systems involving laminar flow, they can always be visualized as a collection of flow tubes. By definition, flow occurs only along flow tubes. In isotropic materials, flow lines lie aligned perpendicular to surfaces of equal potential. Therefore, any multidimensional, laminar flow system can be analyzed as a composite of many one dimensional problems. Accordingly, it is of interest for us to choose a single flow tube of nonuniform cross sectional area as a fundamental unit of interest and investigate how conductances, capacitances and time-averages may be calculated in respect of a single such flow tube. It is true that in a transient unsaturated system the geometric disposition of the flow tubes themselves will change within the time interval Δt. For purposes of our discussion here we will assume an average disposition of the flow tube that is appropriate for the time interval. This logic is similar to that frequently used in linearizing a nonlinear problem over small intervals of time.

5.1. Richards Equation for a Single Flow Tube

In Figure 4 we depict a segment of a flow tube of non uniform cross sectional area. This segment is discretized into three elemental volumes, 1, j and 2. The three elemental volumes are separated by interfaces which are isopotential surfaces shown by bold lines. For purposes

of reference, we use an appropriate flow line as a curvilinear x axis. The FInodal pointsFR of the elemental volumes are located at $\bar{x}_1$, $\bar{x}_j$, and $\bar{x}_2$.

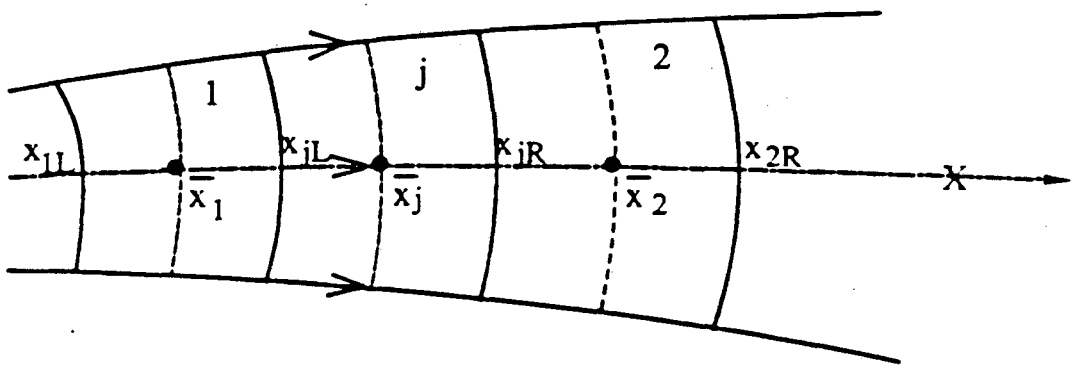

Figure 4: The segment of a flow tube divided into 3 volume elements 1, j and 2.

If we recognize that from an empirical point of view Darcy-Buckingham law pertains to macroscopic flux between two surfaces of equal potential, it follows that the nodal points can be located anywhere along the isopotential surfaces passing through $\bar{x}_1$, $\bar{x}_j$, and $\bar{x}_2$.

In the context of Figure 4, we may write the equation of mass conservation for elemental volume j as follows.

$$\Delta t \left\{ \bar{U}_{j1}[(z_1 - z_j) - (\bar{h}_1 - \bar{h}_j)] + \bar{U}_{j2}[(z_2 - z_j) - (\bar{h}_2 - \bar{h}_j)] \right\}$$

$$= - M_{c,j} \Delta h_j \qquad (15)$$

For simplicity we have neglected the source term in (15). We now proceed to analyze how the $\bar{U}$'s , the $\bar{h}$'s and M_c in (15) can be accurately evaluated.

5.2. Evaluating Conductance

Consider the time-averaged conductance $\bar{U}_{jm}$, where m = 1,2. In view of (12), $\bar{U}_{jm} = U(\bar{h}_j , \bar{h}_m)$, where, $\bar{h} = h(t_o + \lambda_{jm}\Delta t)$. Suppose, based on a knowledge of the past behavior of the system we have accurately estimated $\bar{h}_m$ and $\bar{h}_j$. Then, according to (9),

$$\bar{Q}_{jm}^{est} = \frac{[(z_m - \bar{h}_m^{est}) - (\bar{z}_j - \bar{h}_j^{est})]}{\frac{1}{K_s} \int_{x_j}^{x_m} \frac{dy}{k_r[h(y)]A(y)}} \qquad (16)$$

where, $h(\bar{x}_j) = \bar{h}_j$ and $h(\bar{x}_m) = \bar{h}_m$. Again, for simplicity, we consider that elements j and m are composed of the same material so the K_s is a constant. In principle, (16) can be iteratively solved to obtain $\bar{Q}_{jm}^{est}$. Then, in view of (12),

$$\bar{Q}^{est} = \bar{U}_{jm}^{est}[(z_m - z_j) - (\bar{h}_m^{est} - \bar{h}_j^{est})] \tag{17}$$

and,

$$\bar{U}_{jm}^{est} = \frac{\bar{Q}^{est}}{[(z_m - \bar{h}^{est}) - (z_j - \bar{h}^{est})]} \tag{18}$$

5.3. The Time Integration Factor, λ_{jm}.

Equation (12) is an integral expression for the mass of water transferred from element m to element j during Δt. As indicated in (12), we wish to approximate this integral by using time average values for conductance and fluid potential. Suppose, based on past behavior of the system we algebraically express the temporal variations of U_{jm} , h_m and h_j as convenient algebraic expressions. Then,

$$\int_{t_o}^{t_o + \Delta t} U_{jm}(t)[(z_m - z_j) - (h_m - h_j)]dt$$

$$= - U(\lambda_{jm})[h_m(\lambda_{jm}) - h_j(\lambda_{jm})]\Delta t \tag{19}$$

The left hand side of (19) can be explicitly evaluated using the convenient algebraic expressions. Also, because the algebraic expressions are known, U_{jm} , h_m andh_j on the right hand side are all functions only of λ_{jm}. Therefore, (18) could be solved to get an accurate value for λ_{jm}.

5.4. The Capacitance

As we have already discussed, the capacitance $M_{c,j}$ is the ratio of change in mass of water divided by the change in suction head. We now consider how capacitance may be estimated accurately.

The variation of h over elemental volume j is depicted in Figure 5 at the initial time t_o and at the end of the time interval Δt. The suction heads at the left and right extremities of the elemental volume are known from the initial condition to be $h_{jL}^{o\ est}$ and $h_{jR}^{o\ est}$. In keeping with (9), the profile of h over j is shown in the Figure by the curve labeled $h(t_o)$. Let h_{jL}^{est} andh_{jR}^{est} be the estimated values at $t_o + \Delta t$. The profile of h corresponding to these

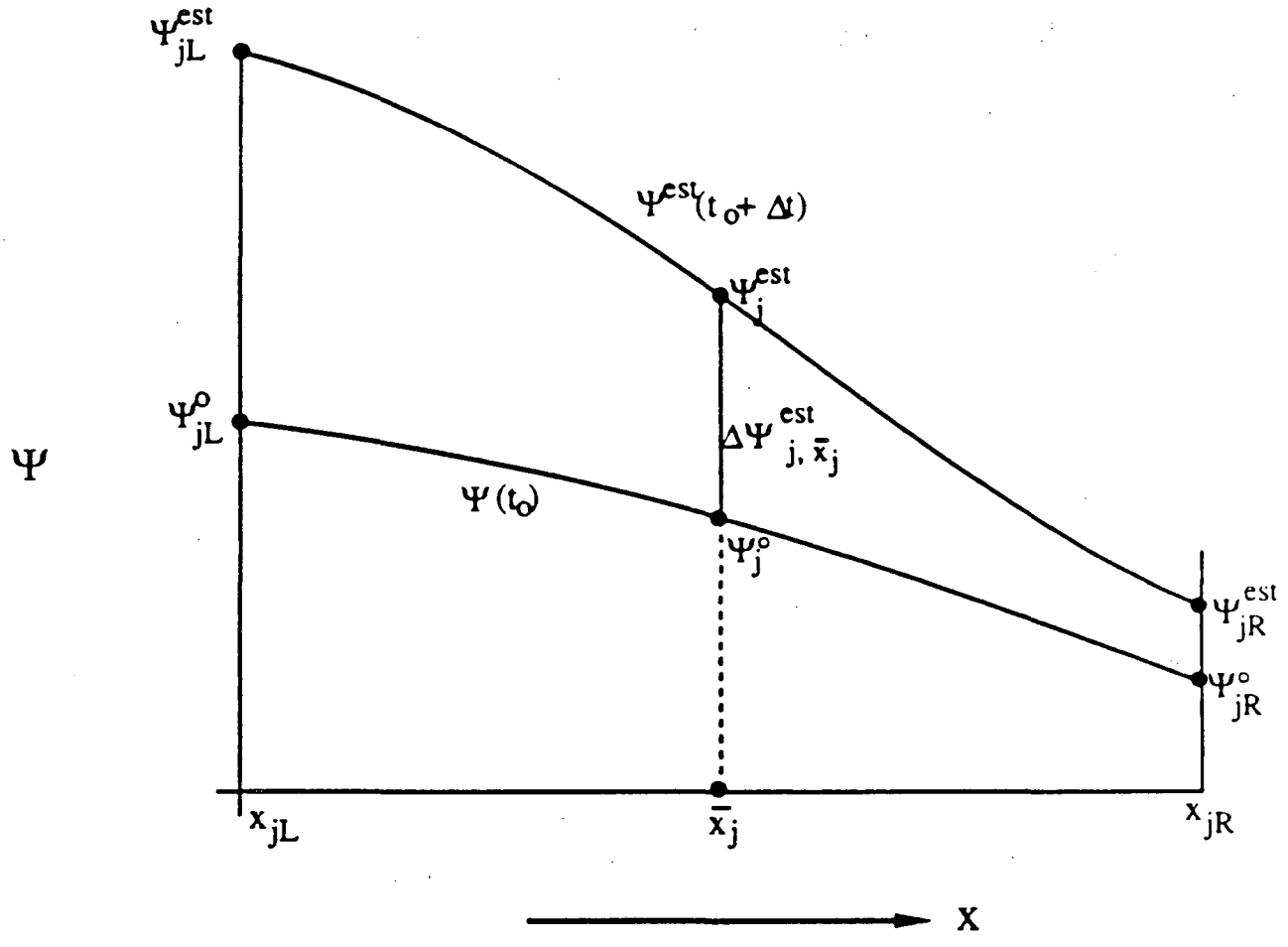

Figure 5: Schematic representation of profile of Ψover volume element j at t_o and at $(t_o + \Delta t)$.

estimated values is shown in Figure 5 by the curve labeled $h_j^{est}(t_o + \Delta t)$. Then, the mass of water contained in j at t_o and $t_o + \Delta t$ are, respectively,

$$M_{w,j}^{est}(t_o + \Delta t) = V_{s_j}\rho_w \int_{x_{jL}}^{x_{jR}} e_j^{est}(x)S_j^{est}(x)dx \tag{20a}$$

$$M_{w,j}(t_o) = V_{s_j}\rho_w \int_{x_{jL}}^{x_{jR}} e^o(x)S^o(x)dx \tag{20b}$$

and,

$$\Delta M_{w,j} = M_{w,j}^{est}(t_o + \Delta t) - M_{w,j}(t_o) \tag{20c}$$

Note from the profiles given in Figure 5 that the change in suction head Δh_j^{est} is a function of position within j. However, we have *a priori* chosen the nodal point location to be $\bar{x}_j$. Therefore, by dividing the change in mass of water over Δt by the change in suction head at the nodal point location, we obtain an *operational capacitance* for j, which pertains to the particular choice of nodal point location. Thus,

$$M_{c,j}(\bar{x}_j) = -\frac{\Delta M_{w,j}^{est}}{\Delta h_j^{est}} \tag{21}$$

5.5. The Numerical Equation

In view of the foregoing we may now write the governing discretized equation for elemental volume j.

$$\Delta t \left\{ \bar{U}_{j1}^{est}[(z_1 - z_j) - (\bar{h}_1 - \bar{h}_{j)1}] + \bar{U}_{j2}^{est}[(z_2 - z_{j)} - (h_2 - h_{j)1}] \right\}$$

$$= -M_{c,j}\Delta h_j \tag{22}$$

5.6. Corrections for Estimates

Inherent to the approach of solving the linear equations is the need to use *a priori* knowledge of conductances and capacitances. In nonlinear problems these values vary continuously in time and we are constrained to using estimated values for these to implement the

solution process. Therefore, in order to render the solution as accurately as possible, one may correct for the estimation errors by using predictor-corrector schemes or by using the Newton-Raphson iteration method.

6. EXTENSION TO MULTIDIMENSIONS

In the beginning of section 5 we noted that a multidimensional laminar flow system may be idealized as a collection of flow tubes. In a transient, partially saturated system, the flow geometry must in general be expected to vary with time. Thus, the disposition of the flow geometry is *a priori* unknown. Yet, the theory developed above merely describes how the problem may be accurately solved *if* the flow geometry is known. How then is the theory pertinent to multidimensional systems ?

In order to answer this question, one must address a related question of basic importance. In a heterogeneous system, resistances to flow depend on flow geometry. What cause dictates the particular flow geometry preferred by the flowing water in response to the particular combination of forcing functions ? Addressing this question recently, Narasimhan (1988) postulated that the flow geometry will adjust itself in such a fashion that the rate at which the moving fluid dissipates energy over the system (as it moves down slope in the potential field) is minimized. If this postulate is valid, the overall problem cannot be credibly solved without identifying the particular flow geometry that minimizes energy dissipation. The theory presented above is useful in solving for fluxes and potential drops provided geometry is known. These fluxes and potentials are indeed the components that are needed to quantify the minimization process. Thus, it is reasonable to state that for a satisfactory solution of the multidimensional problem one has to start with an estimated flow geometry and calculate the energy dissipation using using the ideas presented above to calculate fluxes and potential drops. One has to progressively adjust the flow geometry until the the energy dissipation is globally minimized.

The conventional wisdom that h is the primarily dependent variable in Richards equation is correct only for systems with known flow geometry. In systems with unknown flow geometry, h and flow geometry are *both* dependent variables. For very much these same reasons, but in the context of two-phase flow theory, Morel-Seytoux (1987) suggests to write equations in terms of total flux, water flux and water content in a curvilinear coordinate system which is essentially set of stream lines for the total velocity field.

7. CONCLUDING REMARKS

The difficulties inherent in the numerical simulation of Richards equation stem from neglecting the role of local flow geometry in determining the conductances and capacitances, as well as from a failure to recognize that in the general nonlinear case the flux law is an implicit statement. These difficulties could be effectively countered by developing appropriate computational logic to imbed geometry into the estimation of fluxes and capacitances.

On the face of it it may appear as though the task of solving for flow geometry may render the problem too difficult to solve. However, the integral nature of the ideas presented in this work may actually render the solution of the problem much easier than one may suspect a

priori.

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