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Authors
O'Neil, Thomas
Rostoker, Norman

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Triplet Correlation for a Plasma

THOMAS O'NEIL AND NORMAN ROSTOKER
Department of Physics, University of California at San Diego, La Jolla, California
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The three-body and two-body electron-correlation functions are calculated for a plasma in thermal equilibrium. The method involves convergent kinetic equations developed from the hierarchy by an expansion in $\epsilon = 1/4\pi n L_D^3$ which is carried to second order. The free energy determined by this method agrees with the previous result of Abé.

I. INTRODUCTION

MANY authors have treated the statistical mechanics of a plasma by using the Bogoliubov–Born–Green–Kirkwood–Yvon chain of equations.[1,2] These authors terminated this chain by expanding the s-body functions in terms of $\epsilon = 1/4\pi n L^3{}_D$ and by neglecting terms of higher order than ϵ. One would expect the next term in the expansion to be of order ϵ^2 or $\epsilon^2 \ln \epsilon$. A recent field theoretic calculation of the mean free path by Misawa[3] indicates that the second-order term may be even larger than the first-order term. This rather surprising result motivated us to begin an investigation of the next term in plasma expansions.

In this paper, we consider only the thermal equilibrium plasma. The calculation of the mean free path considered by Misawa will be discussed in a subsequent paper. We calculate the second-order pair correlation function and the second-order triple correlation function. We then use the pair correlation function to calculate the free energy of the plasma. The purpose of the calculation is to investigate the nature of the next order in the plasma expansion and, for as simple a case as possible, to see if there are any divergence difficulties.

II. BASIC EQUATIONS

Consider a plasma of N electrons and N infinite mass randomly distributed ions. Let the plasma be contained in a volume V, and let the position and velocity coordinates of the ith electron be given by $X_i = (\mathbf{x}_i, \mathbf{v}_i)$. For an ensemble of such plasmas, the density in phase space $D(X_1, X_2, \ldots X_N, t)$ satisfies the Liouville equation,

$$\left\{\frac{\partial}{\partial t} + \sum_{i=1}^{N} \mathbf{v}_i \cdot \frac{\partial}{\partial \mathbf{x}_i} - \frac{e}{m} \sum_{i=1}^{N} \left[\mathbf{E}(\mathbf{x}_i) + \frac{\mathbf{v}_i}{c} \times \mathbf{B}(\mathbf{x}_i)\right] \cdot \frac{\partial}{\partial \mathbf{v}_i}\right\} D = 0. \quad (1)$$

We consider only Coulomb forces of interaction between the electrons and assume all external fields are zero; hence,

$$\mathbf{E}(\mathbf{x}_i) = \sum_{j=1, j\neq i}^{N} \frac{\partial}{\partial \mathbf{x}_i} \frac{e}{|\mathbf{x}_i - \mathbf{x}_j|}, \qquad \mathbf{B}(\mathbf{x}_i) = 0.$$

The s-body function is defined as

$$f_s(X_1, \cdots X_s, t) = V^s \int D(X_1, \ldots X_N, t)\, dX_{s+1} \cdots dX_N.$$

By taking moments of the Liouville equation, we generate the Bogoliubov–Born–Green–Kirkwood–Yvon chain of equations,

$$\left\{\frac{\partial}{\partial t} + \sum_{i=1}^{s} \mathbf{v}_i \cdot \frac{\partial}{\partial \mathbf{x}_i} - \frac{e^2}{m} \sum_{i,j=1}^{s}{}' \frac{\partial}{\partial \mathbf{x}_i} \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|} \cdot \frac{\partial}{\partial \mathbf{v}_i}\right\} f_s - \frac{ne^2}{m} \sum_{i=1}^{s} \int dX_{s+1} \frac{\partial}{\partial \mathbf{x}_i} \frac{1}{|\mathbf{x}_i - \mathbf{x}_{s+1}|} \cdot \frac{\partial f_{s+1}}{\partial \mathbf{v}_i} = 0. \quad (2)$$

We can rewrite this chain of equations by expressing f_s in terms of the Mayer cluster expansion,

$$f_s = \prod_{i=1}^{s} f(X_i, t) + \sum_{P} [\pi f(X_i, t)] P(X_j, X_k, t) + \sum_{P,P} [\pi f(X_i, t)] P(X_j, X_k, t) P(X_l, X_m, t) + \sum_{t} [\pi f(X_i, t)] T(X_j, X_k, X_l, t) + \cdots, \quad (3)$$

where the second term is summed over pairs, the third over pairs of pairs and the fourth over triplets. $P(X_i, X_j, t)$ is called the pair correlation function and $T(X_i, X_j, X_k, t)$ is called the triple correlation function. The quadruple- and higher-order correlation functions have not been explicitly shown, because they make negligibly small contributions in all our calculations and are dropped at the outset.

[1] N. Rostoker and M. N. Rosenbluth, Phys. Fluids **3**, 1 (1960).
[2] E. A. Frieman and D. L. Book, Phys. Fluids **6**, 1700 (1963).
[3] S. Misawa, Phys. Rev. Letters **13**, 337 (1964).

Using this expansion, the equation for the one-body function becomes

$$\left\{\frac{\partial}{\partial t} + \mathbf{v}_1\cdot\frac{\partial}{\partial \mathbf{x}_1} - \frac{e}{m}\mathbf{E}_M(\mathbf{x}_1)\cdot\frac{\partial}{\partial \mathbf{v}_1}\right\} f(X_1, t) = \frac{ne^2}{m}\int \frac{\partial}{\partial \mathbf{x}_1}\frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|}\cdot\frac{\partial P}{\partial \mathbf{v}_1}(X_1, X_2)\, dX_2, \tag{4}$$

where

$$\mathbf{E}_M(\mathbf{x}_1) = \int \frac{\partial}{\partial \mathbf{x}_1}\frac{e}{|\mathbf{x}_1 - \mathbf{x}_2|} f(X_2)\, dX_2.$$

The equation for the two-body function becomes

$$\begin{aligned}
&\left\{\frac{\partial}{\partial t} + \mathbf{v}_1\cdot\frac{\partial}{\partial \mathbf{x}_1} + \mathbf{v}_2\cdot\frac{\partial}{\partial \mathbf{x}_2} - \frac{e}{m}\mathbf{E}_M(\mathbf{x}_1)\cdot\frac{\partial}{\partial \mathbf{v}_1} - \frac{e}{m}\mathbf{E}_M(\mathbf{x}_2)\cdot\frac{\partial}{\partial \mathbf{v}_2}\right\} P(X_1, X_2, t) \\
&\quad - \frac{ne^2}{m}\frac{\partial f}{\partial \mathbf{v}_1}\cdot\int \frac{\partial}{\partial \mathbf{x}_1}\frac{1}{|\mathbf{x}_1 - \mathbf{x}_3|} P(X_2, X_3)\, dX_3 - \frac{ne^2}{m}\frac{\partial f}{\partial \mathbf{v}_2}\cdot\int \frac{\partial}{\partial \mathbf{x}_2}\frac{1}{|\mathbf{x}_2 - \mathbf{x}_3|} P(X_1, X_3)\, dX_3 \\
&\quad - \frac{ne^2}{m}\int \frac{\partial}{\partial \mathbf{x}_1}\frac{1}{|\mathbf{x}_1 - \mathbf{x}_3|}\cdot\frac{\partial T(1, 2, 3)}{\partial \mathbf{v}_1}\, dX_3 - \frac{ne^2}{m}\int \frac{\partial}{\partial \mathbf{x}_2}\frac{1}{|\mathbf{x}_2 - \mathbf{x}_3|}\cdot\frac{\partial T(1, 2, 3)}{\partial \mathbf{v}_2}\, dX_3 \\
&\quad = \frac{e^2}{m}\frac{\partial}{\partial \mathbf{x}_1}\frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|}\cdot\left(\frac{\partial}{\partial \mathbf{v}_1} - \frac{\partial}{\partial \mathbf{v}_2}\right)[f(X_1)f(X_2) + P(X_1, X_2)].
\end{aligned} \tag{5}$$

Dropping the quadruple correlation function, the equation for the three-body function becomes

$$\begin{aligned}
&\left\{\frac{\partial}{\partial t} + \mathbf{v}_1\cdot\frac{\partial}{\partial \mathbf{x}_1} + \mathbf{v}_2\cdot\frac{\partial}{\partial \mathbf{x}_2} + \mathbf{v}_3\cdot\frac{\partial}{\partial \mathbf{x}_3} - \frac{e}{m}\mathbf{E}_M(\mathbf{x}_1)\cdot\frac{\partial}{\partial \mathbf{v}_1} - \frac{e}{m}\mathbf{E}_M(\mathbf{x}_2)\cdot\frac{\partial}{\partial \mathbf{v}_2}\right. \\
&\quad \left. - \frac{e}{m}\mathbf{E}_M(\mathbf{x}_3)\cdot\frac{\partial}{\partial \mathbf{v}_3}\right\} T(1, 2, 3) - \frac{ne^2}{m}\frac{\partial f}{\partial \mathbf{v}_1}\cdot\int \frac{\partial}{\partial \mathbf{x}_1}\frac{1}{|\mathbf{x}_1 - \mathbf{x}_4|} T(2, 3, 4)\, dX_4 \\
&\quad - \frac{ne^2}{m}\frac{\partial f}{\partial \mathbf{v}_2}\cdot\int \frac{\partial}{\partial \mathbf{x}_2}\frac{1}{|\mathbf{x}_2 - \mathbf{x}_4|} T(3, 4, 1)\, dX_4 - \frac{ne^2}{m}\frac{\partial f}{\partial \mathbf{v}_3}\cdot\int \frac{\partial}{\partial \mathbf{x}_3}\frac{1}{|\mathbf{x}_3 - \mathbf{x}_4|} T(4, 1, 2)\, dX_4 \\
&\quad - \frac{ne^2}{m}\int \frac{\partial}{\partial \mathbf{x}_1}\frac{1}{|\mathbf{x}_1 - \mathbf{x}_4|}\cdot\frac{\partial P(1, 2)}{\partial \mathbf{v}_1} P(3, 4)\, dX_4 - \frac{ne^2}{m}\int \frac{\partial}{\partial \mathbf{x}_2}\frac{1}{|\mathbf{x}_2 - \mathbf{x}_4|}\cdot\frac{\partial P(1, 2)}{\partial \mathbf{v}_2} P(3, 4)\, dX_4 \\
&\quad - \frac{ne^2}{m}\int \frac{\partial}{\partial \mathbf{x}_3}\frac{1}{|\mathbf{x}_3 - \mathbf{x}_4|}\cdot\frac{\partial P(2, 3)}{\partial \mathbf{v}_3} P(1, 4)\, dX_4 - \frac{ne^2}{m}\int \frac{\partial}{\partial \mathbf{x}_2}\frac{1}{|\mathbf{x}_2 - \mathbf{x}_4|}\cdot\frac{\partial P(2, 3)}{\partial \mathbf{v}_2} P(1, 4)\, dX_4 \\
&\quad - \frac{ne^2}{m}\int \frac{\partial}{\partial \mathbf{x}_1}\frac{1}{|\mathbf{x}_1 - \mathbf{x}_4|}\cdot\frac{\partial P(1, 3)}{\partial \mathbf{v}_1} P(2, 4)\, dX_4 - \frac{ne^2}{m}\int \frac{\partial}{\partial \mathbf{x}_3}\frac{1}{|\mathbf{x}_3 - \mathbf{x}_4|}\cdot\frac{\partial P(1, 3)}{\partial \mathbf{v}_3} P(2, 4)\, dX_4 \\
&= \frac{e^2}{m}\left\{\frac{\partial}{\partial \mathbf{x}_1}\frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|}\cdot\left(\frac{\partial}{\partial \mathbf{v}_1} - \frac{\partial}{\partial \mathbf{v}_2}\right) T(1, 2, 3) + \frac{\partial}{\partial \mathbf{x}_1}\frac{1}{|\mathbf{x}_1 - \mathbf{x}_3|}\cdot\left(\frac{\partial}{\partial \mathbf{v}_1} - \frac{\partial}{\partial \mathbf{v}_3}\right) T(1, 2, 3)\right. \\
&\quad + \frac{\partial}{\partial \mathbf{x}_2}\frac{1}{|\mathbf{x}_2 - \mathbf{x}_3|}\cdot\left(\frac{\partial}{\partial \mathbf{v}_2} - \frac{\partial}{\partial \mathbf{v}_3}\right) T(1, 2, 3) + P(2, 3)\frac{\partial f}{\partial \mathbf{v}_1}\cdot\frac{\partial}{\partial \mathbf{x}_1}\left[\frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|} + \frac{1}{|\mathbf{x}_1 - \mathbf{x}_3|}\right] \\
&\quad + P(3, 1)\frac{\partial f}{\partial \mathbf{v}_2}\cdot\frac{\partial}{\partial \mathbf{x}_2}\left[\frac{1}{|\mathbf{x}_2 - \mathbf{x}_3|} + \frac{1}{|\mathbf{x}_2 - \mathbf{x}_1|}\right] + P(1, 2)\frac{\partial f}{\partial \mathbf{v}_3}\cdot\frac{\partial}{\partial \mathbf{x}_3}\left[\frac{1}{|\mathbf{x}_3 - \mathbf{x}_1|} + \frac{1}{|\mathbf{x}_3 - \mathbf{x}_2|}\right] \\
&\quad + f(1)\left[\frac{\partial}{\partial \mathbf{x}_2}\frac{1}{|\mathbf{x}_2 - \mathbf{x}_1|}\cdot\frac{\partial}{\partial \mathbf{v}_2} + \frac{\partial}{\partial \mathbf{x}_3}\frac{1}{|\mathbf{x}_3 - \mathbf{x}_1|}\cdot\frac{\partial}{\partial \mathbf{v}_3}\right] P(2, 3) \\
&\quad + f(2)\left[\frac{\partial}{\partial \mathbf{x}_3}\frac{1}{|\mathbf{x}_3 - \mathbf{x}_2|}\cdot\frac{\partial}{\partial \mathbf{x}_3} + \frac{\partial}{\partial \mathbf{x}_1}\frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|}\cdot\frac{\partial}{\partial \mathbf{v}_1}\right] P(3, 1) \\
&\quad \left. + f(3)\left[\frac{\partial}{\partial \mathbf{x}_1}\frac{1}{|\mathbf{x}_1 - \mathbf{x}_3|}\cdot\frac{\partial}{\partial \mathbf{v}_1} + \frac{\partial}{\partial \mathbf{x}_2}\frac{1}{|\mathbf{x}_2 - \mathbf{x}_3|}\cdot\frac{\partial}{\partial \mathbf{v}_2}\right] P(1, 2)\right\}.
\end{aligned} \tag{6}$$

III. THERMAL EQUILIBRIUM CORRELATION FUNCTIONS

The Gibbs distribution dictates the following forms for the thermal equilibrium correlation functions:

$$P(X_1, X_2) = f(v_1)f(v_2)\phi(r_{12}),$$

$$T(X_1, X_2, X_3) = f(v_1)f(v_2)f(v_3)\psi(r_{12}, r_{13}, r_{23}),$$

where

$$f(v) = \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \exp\left(-\frac{mv^2}{2kT}\right), \quad r_{12} = |\mathbf{x}_1 - \mathbf{x}_2|.$$

Substituting these functions into Eqs. (5) and (6) and using the fact that ϕ and ψ must be invariant with respect to interchange of particle indices, we obtain the following two equations:

$$\frac{\partial\phi}{\partial r_{12}} - \frac{1}{4\pi L_D^2}\int \frac{(\mathbf{x}_1 - \mathbf{x}_3)\cdot\hat{x}_{12}}{|\mathbf{x}_1 - \mathbf{x}_3|^3} \cdot[\phi(r_{23}) + \psi(r_{12}, r_{13}, r_{23})]\, d\mathbf{x}_3 = \frac{b}{r_{12}^2}[1 + \phi] \qquad (7)$$

$$\frac{\partial\psi}{\partial r_{12}} + \frac{1}{4\pi L_D^2}\frac{\partial}{\partial r_{12}}\int \frac{\psi(r_{23}, r_{24}, r_{34})}{r_{14}}\, d\mathbf{x}_4 + \frac{\phi(r_{13})}{4\pi L_D^2}\frac{\partial}{\partial r_{12}}\int \frac{\phi(r_{24})}{r_{14}}\, d\mathbf{x}_4 = \frac{b}{r_{12}^2}[\phi(r_{23}) + \phi(r_{13}) + \psi], \qquad (8)$$

where

$$L_D^2 = \frac{kT}{4\pi ne^2}, \qquad b = \frac{e^2}{kT}, \qquad \hat{x}_{12} = \frac{(\mathbf{x}_1 - \mathbf{x}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|}.$$

To solve these equations, we use the standard procedure of expanding in terms of the parameter $\epsilon = 1/4\pi nL_D^3 = b/L_D$. We will first find expansions for ϕ and ψ that satisfy Eqs. (7) and (8) in the region where particle separations are greater than $r_0 = 1/n^{\frac{1}{3}}$ and that satisfy boundary conditions which demand $\phi(r_{12})$ to approach zero when r_{12} approaches infinity and $\psi(r_{12}, r_{13}, r_{23})$ to approach zero when any one of the particles approaches infinity. Then we will find expansions for ϕ and ψ that satisfy Eqs. (7) and (8) in the region where particle separations are less than r_0 and that match the long range solutions when particle separations are equal to r_0. To find the long range solutions, we scale Eqs. (7) and (8) to $r'_{12}L_D = r_{12}$, $r'_{13}L_D = r_{13}$, and $r'_{23}L_D = r_{23}$; and we neglect terms of higher order than ϵ^2. Writing the resulting equations in terms of the unscaled coordinates, gives

$$\frac{\partial\phi_I^{(1)}}{\partial r_{12}} + \frac{1}{4\pi L_D^2}\frac{\partial}{\partial r_{12}}\int \frac{\phi_I^{(1)}(r_{23})}{r_{13}}\, d\mathbf{x}_3 = \frac{b}{r_{12}^2}, \qquad (9)$$

$$\frac{\partial\phi_I^{(2)}}{\partial r_{12}} - \frac{1}{4\pi L_D^2}\int \frac{(\mathbf{x}_1 - \mathbf{x}_3)\cdot\hat{x}_{12}}{|\mathbf{x}_1 - \mathbf{x}_3|^3}[\phi_I^{(2)}(r_{23}) + \psi_I(r_{12}, r_{13}, r_{23})]\, d\mathbf{x}_3 = \frac{b}{r_{12}^2}\phi_I^{(1)}(r_{12}), \qquad (10)$$

$$\frac{\partial\psi_I}{\partial r_{12}} + \frac{\phi_I^{(1)}(r_{12})}{4\pi L_D^2}\frac{\partial}{\partial r_{12}}\int \frac{\phi_I^{(1)}(r_{24})}{r_{14}}\, d\mathbf{x}_4 + \frac{1}{4\pi L_D^2}\frac{\partial}{\partial r_{12}}\int \frac{\psi_I(r_{23}, r_{24}, r_{34})}{r_{14}}\, d\mathbf{x}_4 = \frac{b}{r_{12}^2}[\phi_I^{(1)}(r_{13}) + \phi_I^{(1)}(r_{23})], \qquad (11)$$

where the subscript I indicates our specialization to the long range region. We note that the above equations contain integrals of ϕ_I and ψ_I over all space and that ϕ_I and ψ_I are valid only in the long range region. However, the error so incurred is less than ϵ^2 and, thus, is tolerable.

The solution of Eq. (9) is the well-known screened potential

$$\phi_I^{(1)}(r_{12}) = -\frac{b}{r_{12}}\exp\left(-\frac{r_{12}}{L_D}\right).$$

Using

$$\int \frac{\phi_I^{(1)}(r_{23})}{r_{13}}\, d\mathbf{x}_3 = -4\pi L_D^2\left[\frac{b}{r_{12}} + \phi_I^{(1)}(r_{12})\right],$$

one can easily verify that the solution of Eq. (11) is

$$\psi_I = \phi_I^{(1)}(r_{12})\phi_I^{(1)}(r_{13}) + \phi_I^{(1)}(r_{12})\phi_I^{(1)}(r_{23}) + \phi_I^{(1)}(r_{13})\phi_I^{(1)}(r_{23}) + n\int d\mathbf{x}_4\, \phi_I^{(1)}(r_{14})\phi_I^{(1)}(r_{24})\phi_I^{(1)}(r_{34}).$$

In Appendix A, we Fourier analyze Eq. (10), and we show that

$$\phi_I^{(2)}(k) = \frac{-1}{nL_D^3}\frac{1}{8\pi n}\frac{(kL_D)^3}{[1 + (kL_D)^2]^2}\frac{1}{2i}\ln\left[\frac{2i + kL_D}{2i - kL_D}\right].$$

We also show that the inverse transformation is

$$\phi_I^{(2)}(r) = \frac{\epsilon^2}{2}\frac{L_D}{r}\left\{\frac{L_D}{r}e^{-2r/L_D} - \tfrac{3}{4}e^{-r/L_D}\,\mathrm{Ei}\left(\frac{-r}{L_D}\right) + \tfrac{3}{4}e^{r/L_D}\,\mathrm{Ei}\left(\frac{-3r}{L_D}\right) + \tfrac{1}{2}e^{-2r/L_D} + \frac{r}{4L_D}e^{-r/L_D}\,\mathrm{Ei}\left(\frac{-r}{L_D}\right) + \frac{r}{4L_D}e^{r/L_D}\,\mathrm{Ei}\left(\frac{-3r}{L_D}\right) + e^{-r/L_D}\left[\frac{1}{4}\frac{r}{L_D}\ln(3) - \tfrac{1}{3} - \tfrac{3}{4}\ln(3)\right]\right\}.$$

For the purpose of matching the short range solutions, we note here that in the region around $r = r_0$ the dominant term of $\phi_I^{(2)}$ is

$$\phi_I^{(2)}(r) = \epsilon^2 e^{-2r/L_D}\left[\frac{L_D^2}{2r^2} + O(\ln \epsilon)\right].$$

To find the short range solutions, we scale Eqs. (7) and (8) to $r'_{12}b = r_{12}$, $r'_{13}b = r_{13}$, and $r'_{23}b = r_{23}$. We may neglect terms of order $\epsilon^2 \ln \epsilon$, since our free energy calculation will permit a larger error in the short range correlation function than in the long range correlation function and since the remaining lower order terms will be sufficient to match the dominant terms of the long range solutions obtained above. Writing the resulting equations in terms of unscaled coordinates, gives

$$\frac{\partial \phi_{II}}{\partial r_{12}} + \frac{1}{4\pi L_D^2}\frac{\partial}{\partial r_{12}}\int \frac{\phi_{II}(r_{23})\, d\mathbf{x}_3}{r_{13}} = \frac{b}{r_{12}^2}(1 + \phi_{II}), \quad (12)$$

$$\frac{\partial \psi_{II}}{\partial r_{12}} = \frac{b}{r_{12}^2}[\phi_{II}(r_{13}) + \phi_{II}(r_{23}) + \psi_{II}], \quad (13)$$

where the subscript II indicates our specialization to the short range region.

Using

$$\int \frac{\phi_I^{(1)}(r_{23})\, d\mathbf{x}_3}{r_{13}} = -4\pi L_D^2\left[\frac{b}{r_{12}} + \phi_I^{(1)}(r_{12})\right],$$

Eq. (12) becomes

$$\frac{\partial \phi_{II}}{\partial r_{12}} + \frac{\partial}{\partial r_{12}}\left(\frac{b}{r_{12}}\right)\phi_{II} = -\frac{\partial}{\partial r_{12}}\left(\frac{b}{r_{12}} e^{-r_{12}/L_D}\right). \quad (14)$$

Since

$$\frac{\partial}{\partial r_{12}}\left(\frac{b}{r_{12}}\right) = \frac{\partial}{\partial r_{12}}\left(\frac{b}{r_{12}} e^{-r_{12}/L_D}\right) + O\left(\frac{b}{L_D^2}\right),$$

we can rewrite Eq. (14) as

$$\frac{\partial \phi_{II}}{\partial r_{12}} + \frac{\partial}{\partial r_{12}}\left(\frac{b}{r_{12}} e^{-r_{12}/L_D}\right)\phi_{II} = -\frac{\partial}{\partial r_{12}}\left(\frac{b}{r_{12}} e^{-r_{12}/L_D}\right).$$

The general solution of this equation is

$$\phi_{II} = -1 + A \exp\left[-\frac{b}{r_{12}} e^{-r_{12}/L_D}\right].$$

By setting A equal to 1, we can make ϕ_{II} match $\phi_I = \phi_I^{(1)} + \phi_I^{(2)}$ to all orders less than $\epsilon^2 \ln \epsilon$ at $r_{12} = r_0$. In fact,

$$\phi_{II} = -1 + \exp\left[-\frac{b}{r_{12}} e^{-r_{12}/L_D}\right]$$

is accurate to all orders less than $\epsilon^2 \ln \epsilon$ over the whole range of r_{12}.

If we replace b/r_{12}^2 by $-\partial[(b/r_{12}) \exp(-r_{12}/L_D)]/\partial r_{12}$ in Eq. (13), then this equation becomes

$$\frac{\partial \psi_{II}}{\partial r_{12}} = -\frac{\partial}{\partial r_{12}}\left[\frac{b}{r_{12}} e^{-r_{12}/L_D}\right][\phi_{II}(r_{13}) + \phi_{II}(r_{23}) + \psi_{II}].$$

The general solution of this equation is

$$\psi_{II} = A' \exp\left[-\frac{b}{r_{12}} e^{-r_{12}/L_D} - \frac{b}{r_{13}} e^{-r_{13}/L_D} - \frac{b}{r_{23}} e^{-r_{23}/L_D}\right] + 2 - \exp\left[-\frac{b}{r_{12}} e^{-r_{12}/L_D}\right] - \exp\left[-\frac{b}{r_{13}} e^{-r_{13}/L_D}\right] - \exp\left[-\frac{b}{r_{23}} e^{-r_{23}/L_D}\right].$$

By setting A' equal to 1, we can make ψ_{II} match ψ_I to all orders less than $\epsilon^2 \ln \epsilon$ at $r_{12} = r_{13} = r_{23} = r_0$.

We can use ψ_{II} to obtain an approximate expression for ψ that is valid in the region where one particle is far from two other particles which are close together (i.e., $r_{12} \simeq b$, $r_{13} \simeq L_D$ and $r_{23} \simeq L_D$). Making the appropriate expansions, ψ_{II} becomes

$$\psi_{III} = \left[-1 + \exp\left(-\frac{b}{r_{12}} e^{-r_{12}/L_D}\right)\right] \cdot \left[-\frac{b}{r_{13}} e^{-r_{13}/L_D} - \frac{b}{r_{23}} e^{-r_{23}/L_D}\right],$$

where the subscript III indicates our specialization to this mixed range region.

IV. FREE ENERGY

The internal energy of the plasma is given by

$$E = \tfrac{3}{2}NkT + \frac{N^2e^2}{2V}\int_0^\infty 4\pi r\phi(r)\, dr.$$

We can divide the integral into the following two parts:

$$\int_0^\infty 4\pi r\phi(r)\, dr = \int_0^{r_0} 4\pi r\phi_{II}(r)\, dr + \int_{r_0}^\infty 4\pi r\phi_I(r)\, dr$$

Note that we may neglect terms of order $\epsilon^2 \ln \epsilon$ in ϕ_{II} but that we must know ϕ_I to order ϵ^2. In Appendix B, we evaluate these integrals and show that

$$E = \tfrac{3}{2}NkT - \tfrac{1}{2}NkT\{\epsilon + \epsilon^2[\gamma - \tfrac{2}{3} + \tfrac{1}{2}\ln(3\epsilon)]\}.$$

Using $\partial/\partial T(F/T) = -E/T^2$, gives

$$F = -NkT \ln\left[\frac{eV}{N}\left(\frac{mkT}{2\pi\hbar^2}\right)^{\frac{3}{2}}\right] - \tfrac{1}{6}NkT\{2\epsilon + \epsilon^2[\gamma - \tfrac{11}{12} + \tfrac{1}{2}\ln(3\epsilon)]\}.$$

The first term is just the free energy of an ideal gas and the second term is the contribution to the free energy from the coulomb interaction of the electrons.

This result was obtained earlier by Abé,[4] who used the thermal equilibrium Mayer cluster expansion. Shure[5] has also derived this result up to order $\epsilon^2 \ln \epsilon$ by the methods discussed in this paper.

V. CONCLUSION

We have calculated the long range pair correlation and the long range triple correlation to order ϵ^2. Also, we have calculated the short range pair correlation, short range triple correlation and mixed range triple correlation to all orders less than $\epsilon^2 \ln \epsilon$. These results are tabulated below.

$$\phi_{\mathrm{I}} = -\frac{b}{r}e^{-r/L_{\mathrm{D}}} + \frac{\epsilon^2}{2}\frac{L_{\mathrm{D}}}{r}\left\{\frac{L_{\mathrm{D}}}{r}e^{-2r/L_{\mathrm{D}}} - \tfrac{3}{4}e^{-r/L_{\mathrm{D}}}\,\mathrm{Ei}\left(\frac{-r}{L_{\mathrm{D}}}\right) + \tfrac{3}{4}e^{r/L_{\mathrm{D}}}\,\mathrm{Ei}\left(\frac{-3r}{L_{\mathrm{D}}}\right) + \tfrac{1}{2}e^{-2r/L_{\mathrm{D}}}\right.$$
$$\left. + \frac{r}{4L_{\mathrm{D}}}e^{-r/L_{\mathrm{D}}}\,\mathrm{Ei}\left(\frac{-r}{L_{\mathrm{D}}}\right) + \frac{r}{4L_{\mathrm{D}}}e^{r/L_{\mathrm{D}}}\,\mathrm{Ei}\left(\frac{-3r}{L_{\mathrm{D}}}\right) + e^{-r/L_{\mathrm{D}}}\left[\frac{1}{4}\frac{r}{L_{\mathrm{D}}}\ln(3) - \tfrac{1}{3} - \tfrac{3}{4}\ln(3)\right]\right\},$$

$$\psi_{\mathrm{I}} = \frac{b^2}{r_{12}r_{13}}\exp\left[-\left(\frac{r_{12}+r_{13}}{L_{\mathrm{D}}}\right)\right] + \frac{b^2}{r_{12}r_{23}}\exp\left[-\left(\frac{r_{12}+r_{23}}{L_{\mathrm{D}}}\right)\right] + \frac{b^2}{r_{13}r_{23}}\exp\left[-\left(\frac{r_{13}+r_{23}}{L_{\mathrm{D}}}\right)\right]$$
$$- n\int d\mathbf{x}_4\,\frac{b^3}{r_{14}r_{24}r_{34}}\exp\left[-\left(\frac{r_{14}+r_{24}+r_{34}}{L_{\mathrm{D}}}\right)\right],$$

$$\phi_{\mathrm{II}} = -1 + \exp\left[\frac{-b}{r}e^{-r/L_{\mathrm{D}}}\right],$$

$$\psi_{\mathrm{II}} = \exp\left[-\frac{b}{r_{12}}e^{-r_{12}/L_{\mathrm{D}}} - \frac{b}{r_{13}}e^{-r_{13}/L_{\mathrm{D}}} - \frac{b}{r_{23}}e^{-r_{23}/L_{\mathrm{D}}}\right] + 2$$
$$- \exp\left[-\frac{b}{r_{12}}e^{-r_{12}/L_{\mathrm{D}}}\right] - \exp\left[-\frac{b}{r_{13}}e^{-r_{13}/L_{\mathrm{D}}}\right] - \exp\left[-\frac{b}{r_{23}}e^{-r_{23}/L_{\mathrm{D}}}\right],$$

$$\psi_{\mathrm{III}} = \left[-1 + \exp\left(-\frac{b}{r_{12}}e^{-r_{12}/L_{\mathrm{D}}}\right)\right]\left[-\frac{b}{r_{13}}e^{-r_{13}/L_{\mathrm{D}}} - \frac{b}{r_{23}}e^{-r_{23}/L_{\mathrm{D}}}\right].$$

The next order in the plasma expansion is evidently straight-forward and gives only a small correction to the first order. The result of Misawa and the earlier discussion of Sandri[6] indicate that one might expect a very large correction or a divergence. It has been established that this is not the case in any thermal equilibrium calculation.

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APPENDIX A

If we put $\phi_{\mathrm{I}}^{(1)}$ and ψ_{I} into Eq. (10) and Fourier analyze, we obtain

$$\phi_{\mathrm{I}}^{(2)}(\mathbf{k}) = \frac{-1}{n^2}\frac{(kL_{\mathrm{D}})^4}{[1+(kL_{\mathrm{D}})^2]^2(2\pi)^2k}\int_0^\infty \frac{k'^3\,dk'}{[1+(k'L_{\mathrm{D}})^2]}\cdot\int_{-1}^{1}\frac{d\mu\,\mu}{[1+L_{\mathrm{D}}^2(k^2+k'^2+2kk'\mu)]}.$$

Carrying out the μ integration and replacing $k'L_{\mathrm{D}}$ by x', gives

$$\phi_{\mathrm{I}}^{(2)}(\mathbf{k}) = \frac{-1}{n^2L_{\mathrm{D}}^3}\frac{(kL_{\mathrm{D}})^2}{[1+(kL_{\mathrm{D}})^2]^2}\frac{1}{(2\pi)^2}\int_0^\infty\frac{dx'\,x'^2}{[1+x'^2]}\cdot\left\{1 - \frac{[1+(kL_{\mathrm{D}})^2+x'^2]}{4kL_{\mathrm{D}}x'}\ln\left[\frac{1+(kL_{\mathrm{D}}+x')^2}{1+(kL_{\mathrm{D}}-x')^2}\right]\right\}.$$

Using the even symmetry of the integrand and dividing the logarithm into two parts, gives

$$\phi_{\mathrm{I}}^{(2)}(\mathbf{k}) = \frac{-1}{n^2L_{\mathrm{D}}^3}\frac{(kL_{\mathrm{D}})^2}{[1+(kL_{\mathrm{D}})^2]^2(2\pi)^2}\,\mathrm{Re}\int_{-\infty}^{+\infty}\frac{dx'\,x'^2}{1+x'^2}\cdot\left\{\frac{1}{2} - \frac{[1+(kL_{\mathrm{D}})^2+x'^2]}{4kL_{\mathrm{D}}x'}\ln\left[\frac{x'+kL_{\mathrm{D}}+i}{x'-kL_{\mathrm{D}}+i}\right]\right\},$$

where Re indicates that we must take the real part of the integral. Since the branch cut is in the lower half x'-plane, we may evaluate the integral by using a contour that runs along the real axis and then closes in the upper half plane with an infinite semicircle. Picking up the pole at $x' = i$ and the contribution

[4] R. Abé, Progr. Theoret. Phys. (Kyoto) **21**, 475 (1959)
[5] F. Shure, Phys. Rev. Letters **12**, 353 (1964).
[6] G. Sandri, Ann. Phys. (N. Y.) **24**, 332 (1963).

from $|x'| = \infty$, gives

$$\phi_{\mathrm{I}-}^{(2)}(\mathbf{k}) = \frac{-1}{nL_{\mathrm{D}}^3}\frac{1}{8\pi n}\frac{(kL_{\mathrm{D}})^3}{[1+(kL_{\mathrm{D}})^2]^2}\frac{1}{2i}\ln\left[\frac{2i+kL_{\mathrm{D}}}{2i-kL_{\mathrm{D}}}\right].$$

The inverse transform can easily be put into the form

$$\phi_{\mathrm{I}}^{(2)}(r) = -\epsilon^2\frac{L_{\mathrm{D}}}{r}\frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{dx\sin(xr/L_{\mathrm{D}})}{[1+x^2]}\frac{x^4}{i}\cdot\ln\left[\frac{2i+x}{2i-x}\right].$$

In the upper half x-plane, there is a double pole at $x = i$ and a branch cut running from $x = 2i$ to $x = i\,\infty$. To evaluate the integral, we replace sin (xr/L_{D}) by Im [exp (ixr/L_{D})] and use a contour that runs down the real axis and then closes in the upper half plane with an infinite semicircle which is indented to pass down around the branch cut. The result is

$$\begin{aligned}\phi_{\mathrm{I}}^{(2)}(r) = \tfrac{1}{2}\epsilon^2\frac{L_{\mathrm{D}}}{r}\Big\{&\frac{L_{\mathrm{D}}}{r}e^{-2r/L_{\mathrm{D}}} - \tfrac{3}{4}e^{-r/L_{\mathrm{D}}}\,\mathrm{Ei}\left(\frac{-r}{L_{\mathrm{D}}}\right)\\ &+ \tfrac{3}{4}e^{r/L_{\mathrm{D}}}\,\mathrm{Ei}\left(\frac{-3r}{L_{\mathrm{D}}}\right) + \tfrac{1}{2}e^{-2r/L_{\mathrm{D}}}\\ &+ \frac{r}{4L_{\mathrm{D}}}e^{-r/L_{\mathrm{D}}}\,\mathrm{Ei}\left(\frac{-r}{L_{\mathrm{D}}}\right) + \frac{r}{4L_{\mathrm{D}}}e^{r/L_{\mathrm{D}}}\,\mathrm{Ei}\left(\frac{-3r}{L_{\mathrm{D}}}\right)\\ &+ e^{-r/L_{\mathrm{D}}}\left[\frac{1}{4}\frac{r}{L_{\mathrm{D}}}\ln(3) - \tfrac{1}{3} - \tfrac{3}{4}\ln(3)\right]\Big\}.\end{aligned}$$

APPENDIX B

In Sec. IV, we pointed out that the internal energy of the plasma is

$$E = \tfrac{3}{2}NkT + \frac{N^2e^2}{2V}\int_0^{r_0}4\pi r\phi_{\mathrm{II}}(r)\,dr + \frac{N^2e^2}{2V}\int_{r_0}^{\infty}4\pi r\phi_{\mathrm{I}}(r)\,dr.$$

The first integral is equal to

$$\begin{aligned}\int_0^{r_0}4\pi r\left[-1+\exp\left(\frac{-b}{r}e^{-r/L_{\mathrm{D}}}\right)\right]dr &= -2\pi r_0^2 + 4\pi e^{b/L_{\mathrm{D}}}\int_0^{r_0}re^{-b/r}\left(1-\frac{br}{2L_{\mathrm{D}}^2}\right)dr\\ &\simeq \frac{1}{nb}\Big[\tfrac{1}{2}(4\pi)^{2/3}\epsilon^{5/3} - (4\pi)^{1/3}\epsilon^{4/3}\\ &\quad + \tfrac{3}{4}\epsilon^2 - \tfrac{1}{2}\gamma\epsilon^2 - \tfrac{1}{2}\epsilon^2\ln\left(\frac{\epsilon^{\frac{3}{2}}}{(4\pi)^{\frac{1}{2}}}\right) - \frac{4\pi}{6}\epsilon^2\Big].\end{aligned}$$

The second integral may be written in the following form:

$$\int_{r_0}^{\infty}4\pi r\phi_{\mathrm{I}}(r)\,dr = \frac{2}{\pi}\int_0^{\infty}\phi_{\mathrm{I}}(k)\cos(kr_0)\,dk.$$

Using the even symmetry of $\phi(k)$ and replacing kL_{D} by x, we find

$$\int_{r_0}^{\infty}4\pi r\phi_{\mathrm{I}}(r)\,dr = \frac{-1}{\pi L_{\mathrm{D}}}\int_{-\infty}^{+\infty}dx\cos\left(\frac{xr_0}{L_{\mathrm{D}}}\right)\frac{1}{n}\frac{1}{(1+x^2)}\cdot\left\{1+\frac{1}{8\pi nL_{\mathrm{D}}^3}\frac{x^3}{1+x^2}\frac{1}{2i}\ln\left[\frac{2i+x}{2i-x}\right]\right\}.$$

In the upper half x-plane the first term of the integrand has a pole at $x = i$, and the second term has a double pole at $x = i$ and a branch cut running from $x = 2i$ to $x = i\,\infty$. To evaluate the integral, we replace cos (xr_0/L_{D}) by Re $(e^{ixr_0}L_{\mathrm{D}})$ and use a contour that runs down the real axis and then closes in the upper half plane with an infinite semicircle which is indented to pass down around the branch cut. The result is

$$\begin{aligned}\int_{r_0}^{\infty}4\pi r\phi_{\mathrm{I}}(r)\,dr &= \frac{1}{bn}\{-\epsilon + (4\pi)^{1/3}\epsilon^{4/3} - \tfrac{1}{2}(4\pi)^{2/3}\epsilon^{5/3} + \tfrac{4}{6}\pi\epsilon^2\}\\ &\quad - \tfrac{1}{2}\epsilon^2\{\gamma + \tfrac{1}{6} + \ln(3) + \ln[(4\pi)^{\frac{1}{2}}\epsilon^{\frac{3}{2}}]\}.\end{aligned}$$

Thus, we find

$$E = \tfrac{3}{2}NkT - \tfrac{1}{2}NkT\{\epsilon + \epsilon^2[\gamma - \tfrac{2}{3} + \tfrac{1}{2}\ln(3\epsilon)]\}.$$