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DISTRIBUTIONS OF SELF-INTERACTIONS AND VOIDS IN (1+1)-d DIRECTED PERCOLATION[1]

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Distributions of Self-Interactions and Voids

in (1+1)-d Directed Percolation

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We investigate the scaling of self-interactions and voids in $(1 + 1)$-d directed percolation clusters and backbones. We verify that the meandering of the backbone scales like the directed cluster. A geometric relation between the size distribution and the fractal dimensions of a set of objects is applied to find the scaling properties of self-interactions in directed percolation. Lastly we connect the geometric properties of the backbone with the avalanche distribution generated by interface dynamics at the depinning transition.

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Directed percolation (DP) is the common name for branching processes with an absorbing (here, "dead") state. Denoting the active sites as "live," directed percolation can then be defined as a time-directed process for the propagation of live sites. For a given lattice, each site can come alive with a probability p, if and only if one of its neighbors was live at the previous timestep [1]. Directed percolation has a wide applicability, and was proposed to be closely related to Reggeon field theory [2] which describes the evolution of a density ψ of live sites for a large class of branching processes:

$$\frac{d\psi}{dt} = \psi + \frac{d^2\psi}{dx^2} - \lambda\psi^2 + \eta(x,t) \qquad (1)$$

This is a nonlinear Langevin equation where the noise term obeys $\langle \eta(x,t)\eta(x',t')\rangle \propto \psi(x,t)\delta(x-x')\delta(t-t')$ and thus vanishes when $\psi \to 0$. Directed percolation has been related to spatio-temporal intermittency [3], self-organized critical models [4–6] and the propagation of interfaces at depinning transitions [7, 8, 4, 9–11], as well.

Despite its usefulness, DP remains unsolved. Only in four (spatial) dimensions or more is it tractable, since there the active branches effectively do not meet [12]. In that limit, where we effectively have a random neighbour updating, the critical branching process has a size distribution for extinction at time t that reflects the first-return scaling for a random walk in the number of active sites, $p(t) \propto t^{-3/2}$ [13]. In three dimensions or less, the overlapping of different branches leads to analytical difficulties, and this effect is most pronounced in $(1 + 1)$-dimensional DP, which we make focus of this work.

The DP *backbone* is the time-reversal invariant subset of the (time-) directed percolation process. A picture of the $(1 + 1)$-d DP network consists of branches that die ("dangling ends") and branches that continue, on all scales; within the branches there are voids, of all sizes, enclosed by live branches (see Fig. 1). The

backbone, however, has no dangling ends – it has the geometry of a badly tangled fishnet, where stringy lengths separate the multiply-connected blobs [14]. Recently much discussion has centered on the backbone in connection with models of interface pinning in disordered media [7, 4]. In this letter we study the distribution of self-interactions and voids in DP, in the backbone network, and associate voids with the avalanches expected in interface motion at the depinning threshold [15, 9–11]. Our first step is to introduce a general formula that relates the fractal dimensions of a set of objects to the size distribution of the objects.

Size distributions and fractal dimensions. Consider a set of self-similar objects, each with fractal dimension D whose union is a fractal of dimension D_{tot}. Define a subset of this total set, that consists of one point from each object. If this point set has dimension D_{num}, then the number of objects between sizes s and $s + ds$ contained within a box of length L is:

$$n_L(s)\, ds = L^{D_{\text{num}}}\, s^{-\tau}\, f(s/L^D)\, ds \tag{2}$$

where the scaling function $f(x)$ approaches 1 for $x \ll 1$, and 0 for $x \gg 1$. Since $\int s n_L(s) ds \propto L^{D_{\text{tot}}}$, by matching powers of L it follows that the number of objects of size s (for large, fixed L) is distributed as $n(s) \propto s^{-\tau}$ with:

$$\tau - 1 = \frac{D_{\text{num}} - D_{\text{tot}} + D}{D}. \tag{3}$$

In the case of $D_{\text{num}} = D_{\text{tot}}$, then an uncovering argument (like in [16]) yields

$$\tau - 1 = \frac{D_{\text{num}}}{D}. \tag{4}$$

We call these respectively the *triplex* and *duplex* formulae. The formulae apply to self-affine objects as long as the dimensions are measured along a common axis, and the objects are disjoint.

One simple application is the distribution of intervals separated by a fractal dust of dimension D_{ust} in one dimension. Each interval can be associated with a point of the dust, so that $D_{\text{num}} = D_{\text{ust}}$, and is self-similar with the full line, so that $D = D_{\text{tot}} = 1$. Therefore the interval lengths created by the fractal dust are distributed like

$$n(\ell) \propto \ell^{-1-D_{\text{num}}} \tag{5}$$

This result is discussed in [17] and was recently used [18] for the scaling of the first-return time of activity at a given site.

Directed-percolation exponents. If the percolation parameter p lies below a critical threshold p_c, the propagation of live sites has a finite lifetime. If p lies above p_c, the propagation of live sites can continue forever. For p just below p_c, the time-like correlation length (lifetime) diverges $\propto (p_c - p)^{-\nu_{\parallel}}$ and the space-like correlation length (width) $\propto (p_c - p)^{-\nu_{\perp}}$, where the $(1 + 1)$-d exponents are $\nu_{\parallel} = 1.733$ and $\nu_{\perp} = 1.097$ [12]. The *order-parameter* exponent β is defined as the scaling of the density of the infinite cluster (above threshold) with distance to threshold $\epsilon(= |p - p_c|)$: $\rho \propto \epsilon^{\beta}$. For $(1 + 1)$-d DP, $\beta = 0.277$. These three exponents are believed necessary and sufficient to completely characterize DP structures and correlations.

For the backbone network, $\nu_{\perp}$ and $\nu_{\parallel}$ are the same as for the full cluster. This was checked by direct simulation of the meandering of the backbone $\langle x^2 \rangle \propto t^{1.26 \pm 0.01}$, i.e. $2\nu_{\perp}/\nu_{\parallel} = 1.26$, and by the scaling of the number of singly-connected bonds ("red bonds" [19]) along the backbone, which leads to $1/\nu_{\parallel} = 0.58$ (see later discussion). By contrast, the order-parameter scaling for the backbone is given by $\tilde{\beta} = 2\beta$, because a point belongs to the backbone precisely when it belongs *both* to the forward and the backward DP network [20,21]. (Hereafter, a *tilde* denotes a backbone exponent.) Other exponents are easily deduced, for example, $\chi = \nu_{\perp}/\nu_{\parallel} = 0.633$ the exponent

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for the time development of the average width of living sites. The scaling of the mass m of the infinite cluster up to a correlation length $\ell_\parallel = \epsilon^{-\nu_\parallel}$, $m \propto \epsilon^{\beta - \nu_\parallel - \nu_\perp}$, leads to dimensions for one-dimensional transverse and longitudinal cuts: $D_\perp = 1 - \beta/\nu_\perp$, $D_\parallel = 1 - \beta/\nu_\parallel$. In this notation, the full dimension measured longitudinally is then $D_\parallel + \chi$. Measuring how the cluster mass scales with its length for both DP and BDP, see Fig. 2b,c we get $m(t) \propto t^{1.47 \pm 0.02}$ and $\widetilde{m}(t) \propto t^{1.30 \pm 0.03}$ we deduce the respective dimensions of longitudinal cuts $D_\parallel = 0.84 \pm 0.02$ and $\widetilde{D}_\parallel = 0.67 \pm 0.03$, consistent with $\widetilde{\beta} = 2\beta = 0.55$. Note that the distance between cluster sites in the longitudinal cross-section is power-law distributed with exponent $1 + D_\parallel$.

Now if clusters are initiated everywhere in space and time, their size distribution follows from the duplex formula (Eq. 4) with $D = D_\parallel + \chi$, $D_{\text{num}} = D_{\text{tot}} = 1 + \chi$, the latter dimensions coming from the scaling of our affine box. Therefore,

$$\tau_{\text{all}} - 1 = \frac{1 + \chi}{D_\parallel + \chi} = \frac{\nu_\parallel + \nu_\perp}{\nu_\parallel + \nu_\perp - \beta} \tag{6}$$

The distribution of clusters initiated at a single point is then $\tau_{\text{1pt}} = \tau_{\text{all}} - 1$.

Self-interactions in directed percolation. To calculate when two branches in $(1+1)$-d directed percolation eventually interfere with each other, we examine voids. A void is a region completely enclosed by the DP network. The size of a cluster void is the number non-cluster sites enclosed by two merging branches; backbone voids are defined similarly. In both cases, the scaling of the void-size distribution can be obtained from the triplex formula. The voids themselves scale like the area of the affine region, $D = 1 + \chi$. As the voids are dense on the DP network, $D_{\text{num}} = D_\parallel + \chi$ and $D_{\text{tot}} = D = 1 + \chi$. The void areas s are then power-law distributed $n(s) \propto s^{-\tau_{\text{void}}}$ with

$$\tau_{\text{void}} - 1 = \frac{D_\parallel + \chi}{1 + \chi} = 1 - \frac{\beta}{\nu_\parallel + \nu_\perp}. \tag{7}$$

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The above argument applied to the backbone voids (replacing $D_\parallel$ with $\widetilde{D}_\parallel$) gives $\tilde{\tau}_{\text{void}} = (\nu_\parallel + \nu_\perp - 2\beta)/(\nu_\parallel + \nu_\perp)$. These are in agreement with the numerical values $\tau_{\text{void}} = 1.93 \pm 0.02$ and $\tilde{\tau}_{\text{void}} = 1.82 \pm 0.02$ obtained from simulations, see Fig. 2a. There is an obvious duality reflected in Eqs. 6) and 7): the distribution of all clusters in space is related to the distribution of voids in the cluster by $(\tau_{\text{all}} - 1) = (\tau_{\text{void}} - 1)^{-1}$. Simply related to τ_{void} is the τ-exponent for void length ($\parallel$): $\tau_l - 1 = (\tau_{\text{void}} - 1)(1 + \chi)$. This confirms the measured distribution of times between subsequent self-interactions $P(t) \propto t^{-2.55 \pm 0.02}$. The time between self-interactions differs from the time between subsequent activity at a given site, the latter having a τ-exponent of $D_\parallel + 1$ (c.f. fractal dust result).

As an application, consider the distribution of voids touching an interface which forms the 1-dimensional boundary along the backbone. The distribution of border voids will differ from the overall distribution of voids because larger voids will have a larger probability to touch the 1-d interface. Along a 1-d path on the backbone, small voids sit densely, implying $D_{\text{num}} = 1$; as before, $D = D_{\text{tot}} = 1 + \chi$. Thus the distribution of backbone voids along the interface scales with exponent

$$\tilde{\tau}_{1d} - 1 = \frac{1}{1 + \chi} \tag{8}$$

These border voids play a special role in the dynamics of interfaces driven through a quenched-disordered media. Examples of such models are the depinning dynamics at an externally-tuned critical threshold [7] or when driven similar to invasion percolation [4, 8]. In the model of ref. [4] an interface with small slopes advances at the point of globally-minimal pinning, followed by neighboring advances which keep slopes small. This mimics the quasistatic dynamics of the KPZ equation with quenched noise [10]. This intermittently jumping interface gets pinned along a DP backbone of high pinning strengths [9]. The advance of the interface occurs in bursts

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associated with punctuations of voids in the underlying backbone [22, 15, 9–11]. We now associate the distribution of these bursts with the distribution of voids on the outer surface (hull) of the DP backbone.

A void of length ℓ that borders the backbone hull, has a probability to be punctuated proportional to the number of singly-connected sites $n_{\text{red}}(\ell)$ on the length ℓ of the hull. These are the famous *red* bonds introduced by Stanley [19]. To calculate $n_{\text{red}}(\ell)$ consider a segment of length ℓ on the backbone hull at the critical point $p = p_c$. If p is decreased by an amount ϵ below the critical point, the cluster may be disconnected on length ℓ because each site on the cluster has probability ϵ to be removed. The average number of bonds that disconnect on length ℓ is then $N(\ell, \epsilon) = n_{\text{red}}(\ell)\epsilon + O(\epsilon^2)$. Ignoring the higher-order terms, the correlation length of the cluster becomes equal to ℓ at the ϵ value where $N(\ell, \epsilon)$ is equal to 1. From the known correlation length $\ell = \epsilon^{-\nu_\|}$ we get $n_{\text{red}}(\ell) = \ell^{1/\nu_\|}$. This result was first obtained for isotropic percolation by Coniglio [23]. This is confirmed by direct numerical simulation of the number of singly-connected bonds on length ℓ: $n_{\text{red}} \propto \ell^{0.58 \pm 0.02}$. Weighting the void-size distribution by the chance a given void is punctuated ($\propto n_{\text{red}}(\ell)$) and noting that the void area is $s = \ell^{1+\chi}$, one obtains the distribution of the areas of the interface avalanches is a power law with exponent

$$\tilde{\tau}_{\text{wtd}} - 1 = \frac{1}{1+\chi} - \frac{1}{\nu_\|}\frac{1}{1+\chi} = \frac{\nu_\| - 1}{\nu_\| + \nu_\perp} \tag{9}$$

which is identical to the earlier formula of Maslov and Paczuski [11], and also discussed for other interface model by [24]. The above value $\tau = 1.259$ agrees both with large-scale simulations of the interface model [4, 25], as well as with direct simulations of eroded void areas resulting from the removal of one backbone hull site [22]. A punctuation event typically leads to the elimination of an entire cascade of backbone voids, as indicated in Fig. 1b. (probability to eliminate N voids with one punctuation

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is in fact $\propto N^{-1.35\pm0.05}$ [26]). Thus the agreement between numerics and the above formula, using $\nu_\perp = 1.10$ and $\nu_{||} = 1.73$ from directed percolation, could be accidental. In fact, large-scale simulations [25] for the scaling of average avalanche size indicate that for the self-organized depinning model [4] $\nu_\perp(\text{model}) = 1.05 \pm 0.01$ which is smaller than $\nu_\perp = 1.10$ for directed percolation. The measured $\tau = 1.255 \pm 0.005$ [25] and Eq. 9 then dictate a corresponding change in the value of $\nu_{||}(\text{model})$ and therefore a slight change in the value of the roughness exponent χ [27] for the interface generated by the model.

Finally, we note that the above derivation of the avalanche exponent can be repeated for standard invasion percolation [28, 29] using the triplex formula. Identify D_{num} as the dimension of the region of the growing cluster where new bursts can initiate ($D_{\text{num}} = D_{\text{perimeter}}$ which depends on the invasion rule) and D as the dimension of a single avalanche (cluster). The ratio between these gives the exponent for potential clusters that have at least one connection in common with the invading cluster, just as in Eq. 5. To obtain the exponent for the generated avalanches one assigns weights to larger clusters because they typically can be invaded through a number of connections equal to the number of red bonds on the scale defined by the cluster size. This gives the second term in the exponent for invasion clusters, so we finally obtain

$$\tau_{INV} - 1 = \frac{D_{\text{num}}}{D} - \frac{1/\nu}{D} \qquad (10)$$

This exponent for the invaded regions associated with punctuations of the threshold for invasion percolation was first derived in [30].

Summary: We have investigated self-interactions and voids in both DP clusters and backbones, and have demonstrated how their scaling depends on the DP exponents. It is noteworthy that the self-interaction time for branches has finite

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mean, in contrast to the time between subsequent activity at a given site. We have also discussed avalanches occurring in models of interface motion near the depinning threshold in terms of voids of the DP backbone. Further we presented a general formula for relating dimensions to size distributions which opens up for a study of scaling properties of many other self-affine and self-similar structures.

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[26] The eroded area resulting from a single punctuation consists of a cascade of backbone voids. The number of voids in this cascade is distributed $n(N_{\text{void}}) \sim N^{-1.35 \pm 0.05}$ (numerics). As voids sit densely on the backbone, the number of voids is proportional to the backbone mass $\widetilde{m}$ on the scale of the eroded area s: $N_{\text{void}} \sim \widetilde{m} \sim s^{(\widetilde{D}_\parallel + x)/(1+x)}$. Transforming the void area distribution (i.e. $\widetilde{\tau}_{\text{wtd}}$) to void number N_{void} distribution: $n(N_{\text{void}}) \sim N_{\text{void}}^{-\widetilde{\tau}_{\text{casc}}}$, where $\widetilde{\tau}_{\text{casc}} - 1 = \frac{1 - D_{\text{red}}}{\widetilde{D}_\parallel + x} = \frac{\nu_\parallel - 1}{\nu_\parallel + \nu_\perp - 2\beta}$ is in reasonable agreement with the value 1.35.

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FIG. 1. (a) Directed percolation cluster (thin lines) and backbone (heavy lines).

(b) Punctuation of backbone hull. The crossed area indicates the erosion due to one punctuation event.

(c) The backbone where the eroded area is removed.

FIG. 2. (a) The dashed lines are the distribution of times between self interactions. Full lines are the distributions of void areas. In both cases then lower curve: DP; upper curve: Backbone. Crosses is distribution of erroded areas, caused by a single punctuation event. The distributions are all measured at critical $p = 0.7055$ using [?] and generating clusters of length $\ell = 100000$ for DP and $\ell = 100000$ for the backbone simulations.

(b) Solid line is the scaling of void area/length with void length ℓ. Solid circles is the distribution of the rms meandring of the directed percolation network as a function of "time" t. In both cases we observe scaling $\propto \ell^{0.63\pm0.01}$. Open circles show scaling of the mass of the network at a cross section at given time t.

(c) As in b), but for backbone instead. In addition we show with dotted line the accumulated number of red bonds as function of t.
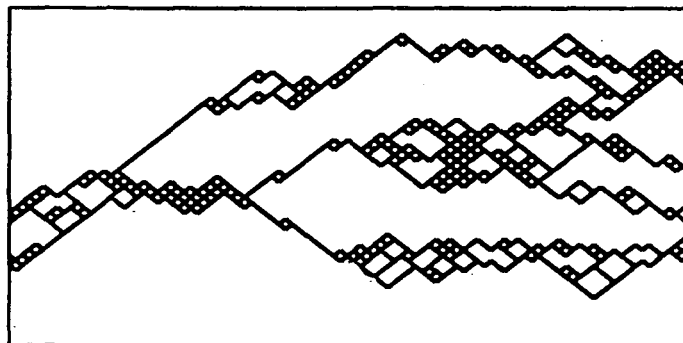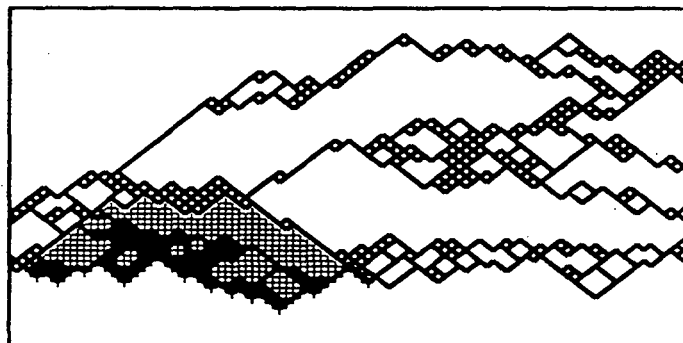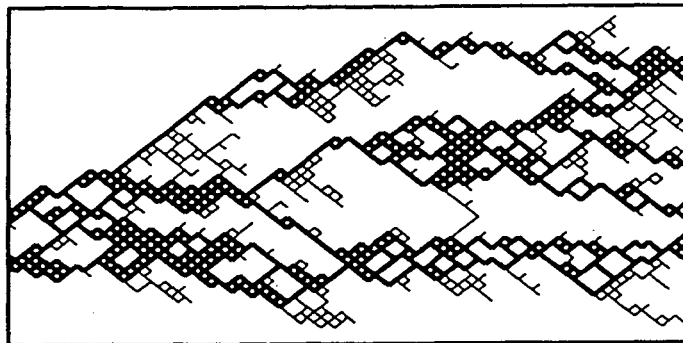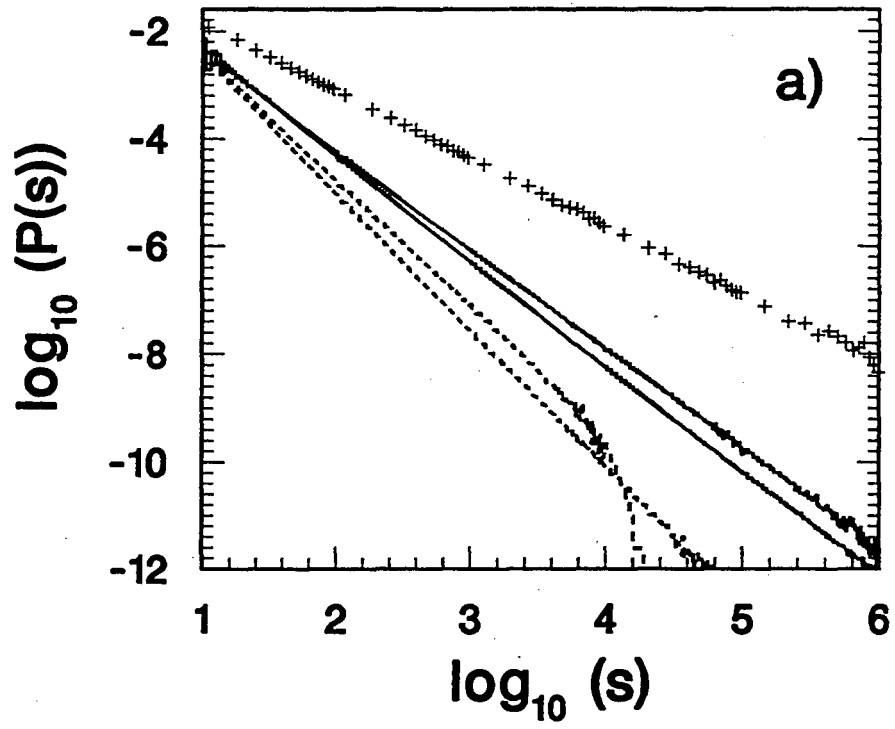
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Fig. 1 abc

Fig. 2a

Fig. 2b