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IN DEEP HOMOGENEOUS HYDROSOLS

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On the Structure of the Light Field at Shallow Depths

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ABSTRACT

Recent experimental determinations of the up and downwelling irradiances in Lake Pend Oreille are studied with the purpose of explaining certain observed nonlinear trends in the semilog plots of these irradiances at shallow depths. A mathematical model which describes these irradiances is derived from the basic equations of radiative transfer. The model explains the observed phenomena in terms of the inherent optical properties of the medium and its external lighting conditions. On the basis of the cited experimental evidence and supporting theory the following hypothesis about light fields in all homogeneous natural hydrosols are proposed: (a) The ratio of the upwelling irradiance to the downwelling irradiance is invariably monotonic increasing or decreasing with increasing depth (depending on the medium) and approaches a limit which is independent of the external lighting conditions and which depends only on the inherent optical properties of the medium. (b) The logarithmic derivatives of the up and downwelling irradiance are monotonic increasing or decreasing with increasing depth (depending on the medium) and approach a common limit which is independent of the external lighting conditions

and which depends only on the inherent optical properties of the medium. In this way we arrive at a fairly detailed understanding of the light field at extreme depths (shallow and deep) in all homogeneous natural hydrosols.

INTRODUCTION

For many practical purposes in applied hydrological optics, the downwelling irradiance $H(z,-)$ at a depth z in a natural hydrosol may be represented by the following simple formula

$$H(z,-) = H(o,-) \, e^{-Kz},\tag{1}$$

where K is a fixed number which characterizes the overall flux transmitting properties of the hydrosol. A similar formula may be used to determine the upwelling irradiance $H(z,+)$ at any depth z :

$$H(z,+) = H(o,+) \, e^{-Kz},\tag{2}$$

where—again for many practical purposes— K is a fixed number, and in fact identical to the one appearing in equation (1).

Still another practical formula is the one which describes the depth dependence of scalar irradiance $h(z)$ at each depth z :

$$h(z) = h(o) \, e^{-Kz},\tag{3}$$

where K is the same number as that appearing in (1) and (2).

In practice $H(z, +)$ and $H(z, -)$ are measured by suitably designed horizontal flat plate collectors exposed to the appropriate hemisphere, and $h(z)$ is measured by a suitably designed spherical collector. In view of (1), (2), and (3), quick estimates of a K for a particular natural body of water can be obtained by measuring any one of these three radiometric quantities at two distinct depths, and using the formula:

$$K = \frac{1}{(z_2 - z_1)} \ln \frac{A(z_1)}{A(z_2)}, \tag{4}$$

where $A(z)$ stands for any one of the three quantities

$H(z, +)$, $H(z, -)$ or $h(z)$ at depth z.

The practical procedures of hydrological optics summarized in formulas (1) - (4) are quite analogous to the following well known procedure used in applied heat conduction studies to estimate the temperature $T(t)$ of a cooling spherical body at time t immersed in a bath of zero temperature:

$$T(t) = T(0) e^{-kt}, \tag{5}$$

where k is a known fixed number which characterizes the overall heat conducting properties of the material comprising the spherical body. Conversely, Equation (5) may be used to estimate k by measuring $T(t)$ at two distinct times and using a formula exactly analogous to (4).

The specialists who use Equation (5) are aware of the fact that it is a useful approximate formula which becomes an exact formula for $T(t)$ in the limit as $t \longrightarrow \infty$. They also realize that Equation (5) becomes quite inadequate for relatively accurate estimates of $T(t)$ whenever t is small, and must resort in such estimates to more general forms representing $T(t)$. These more general forms, of which Equation (5) is a special limiting case, are well-known and are solidly founded in the general theory of heat conduction[1], and experimental fact.

Equations (1) - (3) are regarded by the specialists in hydrological optics in much the same way as Equation (5) is regarded in its own discipline: they are useful approximate relations which can be shown[2] to become exact formulas for $H(z, \pm)$ and $h(z)$ in the limit as $z \longrightarrow \infty$ in deep homogeneous plane parallel optical media. Perhaps what is not well known — or at any rate not fully realized — is that, like Equation (5), these equations do not exactly represent $H(z, \pm)$ or $h(z)$ for small values of z, even in homogeneous hydrosols with uniform external lighting conditions and perfectly calm air-water surfaces. Thus, for relatively accurate estimates of $H(z, \pm)$ and $h(z)$ such as those required in basic scientific studies of the light fields in natural hydrosols, Equations (1) - (3) are quite inadequate. They do not represent the small but experimentally demonstrable departures from linearity of the semilog plots of $H(z, \pm)$ and $h(z)$.

What is required at present and which appears to be absent in the
discipline of hydrological optics is a set of more general formulas
which can accurately represent the quantities $H(z, \pm)$ and
$h(z)$ in the small- z ranges and which reduce to these
simpler formulas in the limit as $z \longrightarrow \infty$.

One of the two purposes of this paper is to present a set of
formulas for $H(z, \pm)$ which yield a closer approximation
to reality than (1) and (2). These formulas are motivated by the re-
sults of recently performed measurements in the light field in real
natural hydrosols, and their derivations are founded on the tenets of
general radiative transfer theory.

The second and perhaps more important purpose of the paper is to
examine the resulting formulas for indications of possible general
qualitative rules that may be hypothesized about the fine structure
of the light field and to put the hypotheses into forms which will be
amenable to further theoretical study or experimental verification.
On the basis of the present model, it was possible to formulate three
such hypotheses about the quantities:

$$K(z, \pm) = \frac{-1}{H(z, \pm)} \frac{dH(z, \pm)}{dz} \tag{6}$$

and

$$R(z, -) = \frac{H(z, +)}{H(z, -)} . \tag{7}$$

These hypotheses are presented in detail below.

The quantities $K(z,+)$ and $K(z,-)$ are simply the slopes of the semilog plots of $H(z,+)$ and $H(z,-)$. According to the simple formulas (1) and (2), these slopes do not change with depth and in fact are of the form:

$$K(z,+) = K(z,-) = K \; ,$$

where K is defined in (1) and (2). Careful experiments show, however, that $K(z,+)$ and $K(z,-)$ are distinct numbers and do change with depth. Furthermore, it is known[2] on theoretical grounds that, in homogeneous media,

$$\lim_{z \to \infty} K(z,+) = \lim_{z \to \infty} K(z,-) = k_\infty$$

where k_∞ is a number which depends only on the inherent optical properties of the medium and is completely independent of the external lighting conditions on the upper boundary of the medium. The present goal is to find out something about the nonlinear behavior of $K(z,\pm)$ at relatively <u>small</u> depths.

The quantity $R(z,-)$ summarizes the flux transmitting and reflecting properties of the medium both above and below the hypothetical plane at depth z . According to the simple formulas (1) and (2),

$$R(z,-) = \frac{H(0,+)}{H(0,-)} \; ,$$

a fixed number for all z. Careful experiments show, however, that $R(z,-)$ changes with depth; and in all homogeneous media it can be shown[2] to approach a well-defined limit as $z \to \infty$:

$$\lim_{z \to \infty} R(z,-) = R_\infty \; ,$$

where R_∞ is a number which depends only on the inherent optical properties of the hydrosol and is completely independent of the external lighting conditions on the upper boundary of the medium. Our present goal is to understand the nonlinear behavior of $R(z, -)$ for relatively small values of z .

To prepare the groundwork leading to these objectives we now consider some experimental data which supplies graphic evidence of the nonlinear depth behavior of $K(z, \pm)$ and $R(z, -)$ in near-surface regions of a specific hydrosol. The experimental evidence presented below has been computed from the data obtained in Lake Pend Oreille, Idaho by J. E. Tyler.[3]

<div align="center">EXPERIMENTAL DATA</div>

Figure 1 depicts the semilog plots of $H(z, +)$ and $H(z, -)$ over the range of depths $5 \leqslant z \leqslant 55$ meters. $H(z, \pm)$ are associated with a wavelength interval of width 64 mμ centered at 480 mμ. This depth range corresponds to a range of about 20 optical depths, so that the light field in the vicinity of 50 meters should have practically attained the asymptotic limit—assuming complete homogeneity of the medium.

Just how close is the present hydrosol to being homogeneous? To answer this, we must know the values of the volume attenuation

function α within the medium. Figure 2 shows a plot of α vs depth for the present medium. An optical medium is by definition homogeneous if α is a constant function within that medium. It is clear from the α -plot of Figure 2 that the hydrosol was not strictly homogeneous at the time of the experiment. There is about a 12% variation in the values of α over the indicated depth range. In several places, in particular the bracketed range, there is a relatively abrupt change of the order of 5% in the values of α . In addition to $\alpha(z)$, the values $a(z)$ of the volume absorption function are plotted for several depths. Observe how the values $a(z)$ tend to follow the changes in $\alpha(z)$. This feature will be noted again later in the study when the mathematical model is being discussed. However, with this background information in mind we may now turn to a detailed examination of the plots of $H(z,\pm)$.

Observe that each of the plots in Figure 1 exhibits a small but noticeable nonlinearity. The curves are slightly concave upward indicating relatively steep slope at shallow depths and less steep slopes at greater depths. To facilitate the examination of these logarithmic slopes, they have been plotted as functions of depth in Figure 3. Both $K(z,+)$ and $K(z,-)$ exhibit a uniform downward trend toward a common asymptote defined by the horizontal line across the figure at ordinate $k_\infty = 0.178/meter$. This value was obtained using the fact cited earlier that $K(z,\pm)$ have a common limit, and then performing a suitable extrapolation based on this fact.[4]

The uniform approach of $K(z, \pm)$ to this common limit appears to be interrupted in the neighborhood of 40 meters. There appears to be some optical disturbance in the medium within the bracketed depth range that results in a marked deviation of these curves from their expected paths. We can explain this anomalous behavior on the basis of our observations of the depth dependence of $\alpha(z)$ in Figure 2. The abrupt change in the values of α in the same depth range appear to hold the key to the explanation when the following formulas are examined:

$$K(z,-) = \alpha(z,-) - \frac{\int_{\Xi_-} N_*(z, \xi)\, d\Omega}{H(z,-)} , \quad (8)$$

$$K(z,+) = -\alpha(z,+) + \frac{\int_{\Xi_+} N_*(z, \xi)\, d\Omega}{H(z,+)} . \quad (9)$$

These are exact formulas relating $K(z, \pm)$ to the values of α and the angular distribution of the light field at depth z . These formulas need not be derived here; they are discussed in complete detail elsewhere.[5] However, we must explain the definitions of the quantities $\alpha(z, \pm)$:

$$\alpha(z, \pm) = \alpha(z)\, D(z, \pm) .$$

Here $D(z, \pm)$ are the values of the distribution functions for the downwelling $(-)$ and upwelling $(+)$ streams of radiant flux at depth z . $D(z, \pm)$ are defined as:

$$D(z, \pm) = \frac{h(z, \pm)}{H(z, \pm)} ,$$

and are measures of the shape of the radiance distributions in their respective hemispheres. Figure 2 shows a plot of $D(z_1, +)$ and $D(z, -)$ for the present hydrosol. It is evident that these quantities are nearly constant over the entire depth range under study.

On the basis of Equation (8), whenever there is an abrupt change in the values $\alpha(z)$ over some small depth interval, and whenever $D(z, -)$ is relatively fixed over this depth interval, (so that the integral term is relatively constant) we predict that $K(z, -)$ must exhibit a change in the <u>same direction</u> as that of $\alpha(z)$. Thus if $\alpha(z)$ abruptly decreases, $K(z, -)$ is expected to exhibit a decrease in value.

On the basis of (9), on the other hand, under the same conditions, there is simultaneous change of $K(z, +)$, but in the <u>opposite direction</u> as that of $\alpha(z)$. Thus if $\alpha(z)$ abruptly decreases, $K(z, +)$ is expected to exhibit an increase in value.

Returning now to Figures 3 and 2, these predictions are apparently borne out by the portions of the α , and K-curves in the bracketed depth range. Therefore the abrupt inhomogeneity in the structure of the hydrosol in this depth range appears to induce the observed interruption of the orderly trend of the K-curves toward their limit.

One might inquire why the comparable change in α in the vicinity of 20 meters does not produce a similar marked effect on the K-curves.

The answer lies in the fact that the light field in the vicinity of
5-30 meters (2-12 optical depths) is still in the process of settling
down and attaining a spatial steady state configuration, so that changes
in K-values are naturally relatively great in this region; changes in
α - values thus have relatively little influence on the fine structure
of the K-functions, and their effects are obscured by the settling-down
changes taking place. However, at around 40 meters the light field has
begun to assume its asymptotic angular structure. Any abrupt change in
$\alpha(z)$ would now cause the entire smoothing process to recommence;
in particular the K-values are now very close to their limit, and the
effect of any inhomogeneity would be relatively magnified.

As a final step in the examination of the experimental evidence we
turn to Figure 4 which exhibits a plot of $R(z, -)$ versus depth.
In this case there is a uniform upward trend, as depth increases, of
the values of $R(z, -)$ toward the limiting value $R_\infty =$
0.0278. This limiting value may be found from the formula[2]

$$R_\infty = \frac{k_\infty - a(-)}{k_\infty + a(+)} , \qquad (10)$$

where

$$a(\pm) = a\, D(\pm) .$$

The quantities $D(\pm)$ are the limits of $D(z, \pm)$
as $z \rightarrow \infty$, and are found from the plots of $D(z, \pm)$
in Figure 2. The values used were $D(+) = 2.77$,

$D(-) = 1.33$ $. k_\infty$ is as defined in Figure 3,

and α was taken as the value of the volume absorption function at

depth 42 meters: $\alpha(42) = 0.123/meter$.

As in the case of the plots of $K(z, \pm)$, the plot of

$R(z, -)$ exhibits a change in trend in the bracketed depth range

discussed earlier. From the exact representation:[5]

$$R(z,-) = \frac{K(z,-) - \alpha(z,-)}{K(z,+) + \alpha(z,+)} \qquad (11)$$

of the function $R(z, -)$, and the observed changes in

$K(z, +)$ and $K(z, -)$ we see that the observed anomaly

(downward trend) exhibited by $R(z, -)$ in the vicinity of

40 meters is traceable directly to the abrupt downward change of

$\alpha(z)$ in this vicinity.

Summary of the Experimental Evidence

We may now summarize the preceding observations:

(a) Over the depth ranges where the hydrosol is practically

homogeneous, the magnitudes of the K-functions exhibit a

monotonic decrease with increasing depth, with

$K(z,-) > K(z, +)$. It appears that if the medium

were homogeneous and infinitely optically deep, the monotonic

decrease would continue indefinitely toward a common

limit k_∞ .

(b) Under the same conditions as in (a), the values $R(z, -)$

appear to exhibit a <u>monotonic</u> increase toward a well-defined

limit R_∞ .

(c) The distribution functions $D(z, \pm)$ are

<u>practically constant</u> with depth.

(d) The ratio of $\alpha(z)$ to $a(z)$ and hence the

ratio $A(z)/\alpha(z)$ $\left(\alpha(z) = a(z) + A(z) \right)$ appears to be

<u>practically constant</u> with depth.

FORMULATION OF THE MODEL

On the basis of the experimental evidence summarized above, in

particular statements (c) and (d), we may adopt the two-D model of the

light field. The equations forming the basis of this model have been

explored in detail in a previous work.[6] Therefore it simply remains

to solve the equations of this model for the particular context at hand.

Specifically, we assume that,

The optical medium is

(i) Optically infinitely deep.

(ii) Separable. (The ratio of $A(z)/\alpha(z)$

is invariant with depth - see experimental statement (d).)

(iii) Irradiated by a collimated radiance distribution of magnitude

N^0 incident at an angle θ_0 from the normal to its

upper boundary.

Formulas for $H(z, \pm)$

It follows from the two-D theory that the expressions for

$H(z, \pm)$ are:

$$H(z, -) = N^0 \left[C(\mu_0, -) \, e^{-k_\infty z} + (\mu_0 - C(\mu_0, -)) \, e^{-\alpha z / \mu_0} \right] \quad (12)$$

$$H(z, +) = N^0 \left[C(\mu_0, -) \, R_\infty \, e^{-k_\infty z} + C(\mu_0, +) \, e^{-\alpha z / \mu_0} \right] . \quad (13)$$

The quantities $C(\mu_0, \pm)$, k_∞ , and R_∞ and their

component parts are defined in detail in reference 6. For convenience

we repeat their basic definitions here:

$$C(\mu_0, \pm) = \frac{A \, b^*(\mp) + \sigma_{\mp}(\mu_0) \left[a^*(\mp) \mp \left(\frac{\alpha}{\mu_0} \right) \right]}{\left(k_+ + \frac{\alpha}{\mu_0} \right) \left(k_- + \frac{\alpha}{\mu_0} \right)} , \quad (14)$$

$$\left(\mu_0 = \cos \theta_0 \right) ,$$

$$k_\pm = \frac{1}{2} \left\{ \left[a^*(+) + b^*(+) - a^*(-) - b^*(-) \right] \pm \right.$$
$$\left. \left[(a^*(+) + b^*(+) + a^*(-) + b^*(-))^2 - 4 \, b^*(-) \, b^*(+) \right]^{1/2} \right\} , \quad (15)$$

$$k_\infty = -k_- , \quad k_\infty \leq \alpha , \quad (16)$$

$$R_\infty = \frac{k_\infty - a^*(-)}{k_\infty + a^*(+)} . \quad (17)$$

Because of assumption (iii), the $H(z, \pm)$ depend implicitly on the quantity μ_0 and to explicitly note this we should write $H(z, \pm ; \mu_0)$. If the medium is irradiated by an arbitrary radiance distribution $N^0(\mu, \phi)$ then the associated irradiances are found by an appropriate integration of the normalized forms of (12) and (13):

$$H(z, \pm) = \frac{1}{\mu_0} \int_{\phi=0}^{2\pi} \int_{\mu=0}^{1} H(z, \pm ; \mu) N^0(\mu, \phi) \, d\mu \, d\phi \qquad (18)$$

However, the present results can be deduced by a direct examination of (12) and (13) for an arbitrary μ_0 without having to consider the general μ_0 -effects as summarized in Equation (18).

<div align="center">Formulas for $K(z, \pm)$</div>

Using the definitions in (6) and the formulas for $H(z, \pm)$ given in (12) and (13), we have,

$$K(z, \div) = \frac{k_\infty C(\mu_0, -) e^{-k_\infty z} + \frac{\alpha}{\mu_0}(\mu_0 - C(\mu_0, -)) e^{-\alpha z/\mu_0}}{C(\mu_0, -) e^{-k_\infty z} + (\mu_0 - C(\mu_0, -)) e^{-\alpha z/\mu_0}} , \qquad (19)$$

or

$$K(z, -) = \frac{k_\infty - \frac{\alpha}{\mu_0} A(\mu_0, -) e^{-\left(\frac{\alpha}{\mu_0} - k_\infty\right) z}}{1 - A(\mu_0, -) e^{-\left(\frac{\alpha}{\mu_0} - k_\infty\right) z}} , \qquad (20)$$

where
$$A(\rho_0,-) = \frac{C(\rho_0,-) - \rho_0}{C(\rho_0,-)} \, . \tag{21}$$

Furthermore,
$$K(z,+) = \frac{k_\omega \, C(\rho_0,-) \, R_\omega \, e^{-k_\omega z} - \frac{\alpha}{\rho_0} C(\rho_0,+) \, e^{-\alpha z/\rho_0}}{C(\rho_0,-) \, R_\omega \, e^{-k_\omega z} - C(\rho_0,+) e^{-\alpha z/\rho_0}} \, , \tag{22}$$

or

$$K(z,+) = \frac{k_\omega - \frac{\alpha}{\rho_0} A(\rho_0,+) \, e^{-\left(\frac{\alpha}{\rho_0} - k_\omega\right) z}}{1 - A(\rho_0,+) \, e^{-\left(\frac{\alpha}{\rho_0} - k_\omega\right) z}} \, , \tag{23}$$

where
$$A(\rho_0,+) = \frac{1}{R_\omega} \frac{C(\rho_0,+)}{C(\rho_0,-)} \, . \tag{24}$$

Formulas (20) and (23) may be used to predict the depth dependence of

$K(z, \pm)$. We deduce immediately from these equations the general fact that:

$$\lim_{z \to \infty} K(z, \pm) = k_\omega \, . \tag{25}$$

Furthermore the K-functions approach this limit in a monotonic manner, as can be seen by taking their derivatives with respect to z :

$$\frac{dK(z,\pm)}{dz} = \frac{\left(\frac{\alpha}{\rho_0} - k_\omega\right)^2 A(\rho_0,\pm) \, e^{-\left(\frac{\alpha}{\rho_0} - k_\omega\right) z}}{\left[1 - A(\rho_0,\pm) \, e^{-\left(\frac{\alpha}{\rho_0} - k_\omega\right) z}\right]} \, . \tag{26}$$

The monotonic behavior of $K(z, \pm)$ follows from the observation that $dK(z, \pm)/dz$ are of <u>constant sign</u> for all z, and for arbitrary choice of μ_o. In particular, the model predicts that

$$\frac{dK(z, \pm)}{dz} > 0 \quad \text{if } A(\mu_o, \pm) > 0 \quad \left.\begin{array}{l} K \text{ increasing} \\ H \text{ concave downward} \end{array}\right\} \quad (27)$$

and

$$\frac{dK(z, \pm)}{dz} = 0 \quad \text{if } A(\mu_o, \pm) = 0 \quad \left.\begin{array}{l} K \text{ constant} \\ H \text{ linear} \end{array}\right\} \quad (28)$$

and

$$\frac{dK(z, \pm)}{dz} < 0 \quad \text{if } A(\mu_o, \pm) < 0 \quad \left.\begin{array}{l} K \text{ decreasing} \\ H \text{ concave upward} \end{array}\right\} \quad (29)$$

It thus appears that the model qualitatively reproduces the experimentally observed depth-behavior of $K(z, \pm)$ as summarized in (a) above. The question of whether $K(z, +)$ and $K(z, -)$ increase or decrease with increasing depth is settled by evaluating the quantities $A(\mu_o, +)$ and $A(\mu_o, -)$, respectively, and applying the criteria (27) - (29). Clearly the increase or decrease of the K-functions is governed, according to the present model, by the nature of the external lighting conditions (summarized by the parameter μ_o) and the inherent optical properties of the medium (summarized by α, k_∞, $A(\mu_o, \pm)$).

Some specific illustrations of this fact are given below. For the present, we turn to the consideration of $R(z,-)$.

Formula for $R(z,-)$

Using the definition (7) and the formulas for $H(z,\pm)$ given in (12) and (13), we have

$$R(z,-) = \frac{C(\mu_0,-)\,R_\infty\,e^{-k_\omega z} - C(\mu_0,+)\,e^{-\alpha z/\mu_0}}{C(\mu_0,-)\,e^{-k_\omega z} + (\mu_0 - C(\mu_0,-))\,e^{-\alpha z/\mu_0}}\,, \qquad (30)$$

or

$$R(z,-) = R_\infty\,\frac{1 - A(\mu_0,+)\,e^{-\left(\frac{\alpha}{\mu_0} - k_\infty\right)z}}{1 - A(\mu_0,-)\,e^{-\left(\frac{\alpha}{\mu_0} - k_\infty\right)z}}\,. \qquad (31)$$

Formula (31) may be used to predict the depth dependence of $R(z,-)$. We deduce immediately that, in general,

$$\lim_{z\to\infty} R(z,-) = R_\omega\,,$$

and that $R(z,-)$ approaches this limit in a monotonic manner, as can be seen by taking the derivative of (31) with respect to z :

$$\frac{dR(z,-)}{dz} = \frac{\left(\frac{\alpha}{\mu_0} - k_\omega\right)\left(R_\omega - R(0,-)\right)\mu_0\,C(\mu_0,-)\,e^{-\left(\frac{\alpha}{\mu_0} - k_\omega\right)z}}{\left[C(\mu_0,-) + (\mu_0 - C(\mu_0,-))\,e^{-\left(\frac{\alpha}{\mu_0} - k_\omega\right)z}\right]^2} \qquad (32)$$

or

$$\frac{d R(z,-)}{dz} = \frac{R_\infty \left(\frac{\alpha}{\mu_0} - k_\infty\right) \left[A(\mu_0,+) - A(\mu_0,-) \right] e^{-\left(\frac{\alpha}{\mu_0} - k_\infty\right)z}}{\left[1 - A(\mu_0,-) e^{-\left(\frac{\alpha}{\mu_0} - k_\infty\right)z} \right]^2} \qquad (33)$$

The monotonic behavior of $R(z,-)$ follows from the obser-
vation that $dR(z,-)/dz$ is of _constant sign_ for
all z , and for arbitrary choice of μ_0 . It appears that the model
can qualitatively reproduce the experimentally observed depth-behavior
of $R(z,-)$ as summarized in (b) above.

In particular, the model predicts that:

$$\frac{d R(z,-)}{dz} > 0 \quad \text{if} \quad A(\mu_0,+) > A(\mu_0,-) \; ; \; R \text{ increasing} \qquad (34)$$

$$\frac{d R(z,-)}{dz} = 0 \quad \text{if} \quad A(\mu_0,+) = A(\mu_0,-) \; ; \; R \text{ constant} \qquad (35)$$

$$\frac{d R(z,-)}{dz} < 0 \quad \text{if} \quad A(\mu_0,+) < A(\mu_0,-) \; ; \; R \text{ decreasing} \qquad (36)$$

The increase or decrease of $R(z,-)$ with increasing depth is
therefore governed by the relative magnitudes of $A(\mu_0,\pm)$,
according to the criteria (34) - (36).

COMPARISON OF EXPERIMENTAL DATA WITH CALCULATIONS BASED ON THE MODEL

Figure 5 shows a graphical comparison of the experimentally

determined K-function values (the crosses) with the calculated values

of these functions (solid curves) using the formulas (20) and (23)

deduced from the model. Table I gives a numerical comparison of the

values. The agreement between experimental data and theory appears to

be good.

Figure 6 shows a graphical comparison of the experimentally de-

termined R-function values (crosses) with the calculated values of these

functions (solid curve) using the formula (31) deduced from the model.

Table I includes a numerical comparison of the values. The agreement

between the computed and measured values is excellent in this case.

A word about the calculation procedure may be in order. The fol-

lowing values of the inherent optical properties were used:

k_ω = 0.178/meter, α = 0.430/meter. The quantities $A(\mu_0, \pm)$,

for curve-fitting purposes, may be considered as constants of integration.

Their values were therefore determined in the present by using the

following boundary conditions:

$$K(12.2, -) = 0.216/\text{meter},$$

$$K(12.2, +) = 0.206/\text{meter}.$$

The corresponding values of $A(\mu_0, \pm)$ were then found to be:

$$A(\mu_0, +) = -1.337,$$

$$A(\mu_0, -) = -2.141.$$

A value μ_0 = 1.583 was found by computing the slope at z = 12.2

meters of the experimental $K(z, -)$ curve. This is an

effective μ_0 in the sense that it simulates the non-collimated

external lighting conditions and interreflection effects at the boundary.

Recall that assumption (iii) of the model makes it strictly applicable

only to media with nonreflecting boundaries irradiated by a collimated

radiance distribution. In this way the lengthy integration process of

the kind described in Equation (18) (which is strictly necessary) was

conveniently bypassed.

When the values of the constants $A(\mu_0, \pm)$ and μ_0,

so found at the single depth z = 12.2 meters, were substituted in

(20), (23) and (31), these formulas predicted a set of values of

$K(z, \pm)$ and $R(z, -)$ for all other depths.

These predicted values are shown in Table I.

HYPOTHESES ON THE FINE STRUCTURE OF LIGHT FIELDS IN NATURAL HYDROSOLS

We have seen that there is experimental evidence of a regular non-linear trend in the logarithmic derivatives and the ratios of the up and downwelling irradiances in near-homogeneous natural hydrosols. On the basis of this evidence, and the ability of the present mathematical model of the light field for homogeneous natural hydrosols to quantitatively reproduce these effects, we conclude that these nonlinearities are effects which may be expected to be observed in all homogeneous natural hydrosols. We are, thus, led to tentatively propose the following hypotheses about the fine structure of the light field in all homogeneous source-free natural hydrosols. The part of the hypotheses concerned with the limiting behavior of the K and R functions has been proved on general grounds elsewhere (reference 2), but is included here for completeness.

I. The ratio of the upwelling to the downwelling irradiance, $R(z,-)$, is monotonic increasing or decreasing with increasing depth. $R(z,-)$ always approaches a limit $R\infty$ which depends only on the inherent optical properties of the hydrosol and which is independent of the external lighting conditions at the upper boundary of the medium.

II. The logarithmic derivatives $K(z, \pm)$ of the up and downwelling irradiances are monotonic increasing or decreasing with increasing depth. $K(z, \pm)$ always approach a common limit $k\infty$ which depends only on the inherent optical

properties of the hydrosol and which is independent of the ex-
ternal lighting conditions at the upper boundary of the medium.

On the basis of the experimental evidence cited above, and on an

examination of the mathematical model of the observed phenomena, we can

propose an additional hypothesis which goes on to state more specifically

the depth behavior of the K- and R-functions:

III. **In all homogeneous source-free natural hydrosols,**

(a) $K(z,-) > K(z,+)$ for all $z \geqq 0$.

(b) $dK(z,\pm)/dz < 0$ for all $z \geqq 0$.

We immediately deduce that

$$\frac{dR(z,-)}{dz} > 0 , \tag{37}$$

which follows from (a) and the general relation:[5]

$$\frac{dR(z,-)}{dz} = R(z,-)\left[K(z,-) - K(z,+) \right] . \tag{38}$$

Hypothesis III is more specific than hypotheses I and II: (b) implies

hypothesis II, and Equation (37) asserts that the reflectance function

$R(z,-)$ monotonically increases with increasing depth, thus (a)

implies I.

The hypothesis cited in III is actually but one of a score of pos-

sibilities. It has, however, a relatively high probability of occurring.

The sense of this "probability" will be made clear below. In order to facilitate further theoretical study of hypothesis III and anticipate the other possibilities, we append a catalog of all possible configurations of the K-functions, as predicted by the present mathematical model.

CATALOG OF K-CONFIGURATIONS

The catalog of K-configurations in Figs. 7 - 12 is a graphical listing of all ways in which $K(z,-)$ and $K(z,+)$ may approach their common limit k_∞ in various homogeneous source-free natural hydrosols. It would be of interest to try to reproduce each of the possible configurations under controlled laboratory conditions.

A K-configuration is defined as an ordered quadruple of the four quantities: $K(0,\pm)$, k_∞, and O. The catalog consists of 25 configurations. These configurations are divided into three classes:

1. Nondegenerate Configurations   (9 members)

2. Degenerate Configurations

    (a) First Kind   (8 members)

    (b) Second Kind   (3 members)

3. Forbidden Configurations   (5 members)

The nondegenerate configuration is defined as one in which

$$O \neq k_\infty \neq K(0,-) \neq K(0,+) \neq k_\infty . \tag{39}$$

A degenerate configuration is defined as one in which at least one of the inequalities in (39) is replaced by an equality.

A forbidden configuration is one in which the following basic inequality of the general theory is violated:

$$K(0,+) R(0,-) \leq K(0,-) . \tag{40}$$

Observe that the K-configurations are defined in terms of the values of the K-functions for $\not{z} = 0$. This is possible because the monotonic behavior of the K and R functions fixes their relative behavior at all depths once their initial values are known. For example, in C1

$$K(0,-) > K(0,+) > R_\infty .$$

Then since $K(\not{z}, \pm)$ must always decrease or always increase toward its limit we must have, in the present example, a _decrease_ of both $K(\not{z}, +)$ and $K(\not{z}, -)$ toward R_∞ . Furthermore, since $R(\not{z}, -)$ also exhibits a fixed monotonic behavior for all $\not{z}$, C1 must have -- by virtue of (38) -- $K(\not{z}, -) > K(\not{z}, +)$ for all $\not{z}$. Similar arguments show that all the configurations are well-defined in terms of the initial values of the K-functions. Knowing the initial magnitudes of $K(0, +)$ and $K(0, -)$ therefore fixes each configuration for all $\not{z}$. The general relation (38) may be used to determine whether $R(\not{z}, -)$ increases or decreases for a particular K-configuration.

We now give evidence that a configuration in which $K(\not{z}, -) > K(\not{z}, +)$ is preferred to one with $K(z, -) < K(\not{z}, +)$. The most unlikely configuration is $D_2 3$ which is associated with non-absorbing media with

$\sigma_+ (\mu_0) \neq 0$ for all μ_0 .

In fact, in such media, if they are infinitely deep, homogeneous and source-free, it follows from (12) and (13) that

$$H(\not{z}, -) = N^0 \left[C(\mu_0, -) + (\mu_0 - C(\mu_0, -)) e^{-\Delta \not{z}/\mu_0} \right],$$

$$H(\not{z}, +) = N^0 \left[C(\mu_0, -) + C(\mu_0, +) e^{-\Delta \not{z}/\mu_0} \right],$$

Here we have, on the assumption that $a = 0:$ $C(\mu_0, +) = \mu_0 - C(\mu_0, -)$, and $k_\omega = 0$,

and that $R(z, -) = 1$ for all z.

Furthermore, from (14),

$$C(\mu_0, -) = \mu_0 \left(\mu_0 + \frac{\sigma_+(\mu_0)}{\Delta} \right)$$

so that

$$\lim_{z \to \infty} H(z, +) = \lim_{z \to \infty} H(z, -) = N^0 \mu_0 \left[\mu_0 + \frac{\sigma_+(\mu_0)}{\Delta} \right] = H(\infty, \pm).$$

If we choose $\mu_0 = 1$, for example, then

$$1 + \frac{\sigma_+(1)}{\Delta} > 0$$

and

$$H(\infty, \pm) > H(0, \pm)$$

so that

$$K(0, +) = K(0, -) < 0.$$

Now if the absorption coefficient is very slightly increased from

O to ϵ, a small positive number, then the resulting effect is:

$$k_\omega > 0,$$
$$R_\omega < 1.$$

Furthermore $K(0, -)$ and $K(0, +)$ now are distinct and

$K(0, -)$ is greater than $K(0, +)$. That is:

$$k_\omega > 0 > K(0, -) > K(0, +),$$

which is represented by configuration C6 (see also F3 to see that the

reverse inequality on the K-functions is impossible). As the value of

a increases, $K(0, -)$ and $K(0, +)$ move upward

on the vertical axis, maintaining the above inequality as they approach
and assume configuration C5. At this point, as a is further increased,
the configuration assumed depends on the external lighting conditions and
the volume scattering function σ . If σ is highly anisotropic
with high forward scattering values and small backward scattering values,
as is the case in most natural hydrosols, then the configurations C3, C2,
C1 are most likely to be assumed on the basis of various simple models
obtained by assuming the appropriate forms for σ . Thus the term,
"configuration X is more probable than configuration Y " means that the
values of the optical parameters associated with configuration X are
more likely to be observed in nature than those associated with Y .
The ordering of the likelihood of occurrence of optical parameters is
based on known experimental evidence. The configurations are ordered
roughly in the order of decreasing probability of occurrence.

The discussion will not go into further detail on these configurations.
We merely mention that various special models can be obtained by assuming
specific, but simple, forms of σ . These easily yield most of the
20 possible configurations. These forms for σ are:

(i) $\sigma(\xi; \xi') = \sigma_+ \, \delta(\xi - \xi') + \sigma_- \, \delta(\xi + \xi')$,

where σ_+, σ_- are fixed constants, and δ is the
Dirac delta function (Stick model).

(ii) $\sigma(\xi; \xi') = \alpha/4\pi$ where α is
the total scattering coefficient (Ball model).

SOME SPECIAL FINE STRUCTURE RELATIONS

The model developed above may be used to answer the following questions.

1. What estimates can be made of the differences:

$$K(0,-) - K(0,+)$$
$$K(0,-) - k_\omega$$
$$K(0,+) - k_\omega \quad ?$$

2. What estimates can be made of the difference

$$R(0,-) - R_\infty$$

knowing either $\quad R_\infty \quad$ or $\quad R(0,-) \quad ?$

3. What can be said about the relative magnitudes of

$$K(z,+), \quad K(z,-), \quad \text{and} \quad k(z) \quad ?$$

4. If $\quad K(0,\pm) < 0, \quad$ this implies that

$$H(z,\pm) \quad$$ (respectively) has a maximum at some

depth z_{max} . What estimates can be made of z_{max} ?

The numerical examples following the general answers to the first

three questions show that the variations of $\quad K(z,\pm)$

and $\quad R(z,-) \quad$ in the present hydrosol are not less

than the expected errors of the observed data. Therefore, detailed precise

measurements of the light field in natural hydrosols should be generally

expected to exhibit the nonlinearity in the H , K and R

plots discussed in this paper; the presence of these nonlinearities and

their classification (by means of the appropriate catalog number of the

observed $(K$ -configuration) should form part of the detailed des-
cription of the hydrosols.

Answer to Question 1. From (20) and (23) after simplification, we have

$$K(0,-) - K(0,+) = \frac{\left(\frac{\alpha}{\mu_o} - k_\omega\right)\left[A(\mu_o,+) - A(\mu_o,-)\right]}{\left[1 - A(\mu_o,-)\right]\left[1 - A(\mu_o,+)\right]} \cdot (41)$$

As an example of the use of this formula we use the values of
$A(\mu_o, \pm)$ obtained in the computations above, whence:

$$K(0,-) - K(0,+) = 0.010/meter .$$

In addition to (41), we have

$$K(0, \pm) - k_\omega = \frac{-\left(\frac{\alpha}{\mu_o} - k_\omega\right) A(\mu_o, \pm)}{1 - A(\mu_o, \pm)} \qquad (42)$$

For the case of the present medium, we estimate that

$$K(0,-) - k_\omega = 0.064/meter$$

$$K(0,+) - k_\omega = 0.054/meter$$

Answer to Question 2.   From (31) we have

$$R(0,-) - R_\infty = \frac{R_\infty \left[A(\mu_0,-) - A(\mu_0,+) \right]}{1 - A(\mu_0,-)}$$

In the present case, we know R_∞ and $A(\mu_0,\pm)$.

This leads to the estimate

$$R(0,-) - R_\infty = -0.007$$

for the present medium. Therefore the spread $R(0,-) - R_\infty$

of $R(z,-)$ values in the present hydrosol is on the

order of 30% of $R(0,-)$.

Answer to Question 3. The definition of $k(z)$ is

exactly analogous to the definition of $K(z,\pm)$

given in (6):

$$k(z) = \frac{-1}{h(z)} \frac{dh(z)}{dz}.$$

To get an expression involving $K(z,\pm)$

and $k(z)$ we use the notion of the dis-

tribution function which links $H(z,\pm)$

and the corresponding components $h(z,\pm)$

of $h(z)$:

$$h(z) = h(z,-) + h(z,+),$$

where $h(z, \pm)$ is the scalar irradiance associated with the downwelling (-) or upwelling (+) stream of radiant energy. From the definitions of $D(z, \pm)$,

$$D(z, \pm) = \frac{h(z, \pm)}{H(z, \pm)} \text{ ,}$$

and that of $k(z)$, we have:

$$k(z) = -\frac{1}{h(z)} \frac{d}{dz} \left[D(z, -) H(z, -) + D(z, +) H(z, +) \right]$$

$$= \frac{1}{h(z)} \left[D(z, -) H(z, -) K(z, -) + D(z, +) H(z, +) K(z, +) \right]$$

$$\therefore -\frac{1}{h(z)} \left[\frac{d D(z, -)}{dz} H(z, -) + \frac{d D(z, +)}{dz} H(z, +) \right]$$

This is the exact representation of $k(z)$ in terms of the D and K functions.

According to the present model, however,

$$\frac{d D(z, \pm)}{dz} = 0$$

This assumption is in good agreement with experimental fact. For the present medium,

$$\frac{d D(z, \pm)}{dz} \sim 0.001 / meter \text{ ,}$$

as may be verified from Figure 2. The number 0.001 is an upper limit of

the derivative values over the indicated depth range. Since

$K(z,\pm)$ are usually determined to about

10^{-3}/meter, the contribution to $\hat{k}(z)$ by the

terms containing the derivatives of $D(z,\pm)$

is not significant. Hence we may write

$$\hat{k}(z) = \gamma(z)\, K(z,-) + \left[1 - \gamma(z)\right] K(z,+) ,$$

where

$$\gamma(z) = \frac{h(z,-)}{h(z)} < 1.$$

This representation of $\hat{k}(z)$ shows that

$\hat{k}(z)$ is expected to be between $K(z,-)$

and $K(z,+)$, regardless of the algebraic signs of

$K(z,+)$ and $K(z,-)$. As an example, let $z = 30$

meters. Computations from the data yield the value $\hat{k}(30) = 0.187$/meter.

Therefore we have, as expected,

$$0.188/\text{meter} = K(30,-) > 0.187 = \hat{k}(30) > K(30,+) = 0.185.$$

Answer to Question 4. Configurations C3, C5, C6, $D_1 8$, $D_2 3$ exhibit all

possible ways in which either $K(0,+)$ or

$K(0,-)$ may be negative. All except the last

configuration exhibits a finite depth Z_{max} at which

$$K(Z_{max}, +) = 0 \qquad \text{or} \qquad K(Z_{max}, -) = 0.$$

In these cases, Z_{max} is the abscissa of the maximum point on the corresponding H-curve (observe that Z_{max} may differ for $K(Z, +)$ and $K(Z, -)$).

An estimate of Z_{max} for the upwelling (+) or downwelling (-) stream can be made directly from (20) or (23) by setting

$$K(Z_{max}, \pm) = 0,$$

and solving for Z_{max}. Thus, from (20) and (23):

$$0 = K(Z_{max}, \pm) = \frac{k_\omega - \frac{\alpha}{\mu_0} A(\mu_0, \pm) e^{-\left(\frac{\alpha}{\mu_0} - k_\omega\right) Z_{max}}}{1 - A(\mu_0, \pm) e^{-\left(\frac{\alpha}{\mu_0} - k_\omega\right) Z_{max}}},$$

from which

$$0 = k_\omega - \frac{\alpha}{\mu_0} A(\mu_0, \pm) e^{-\left(\frac{\alpha}{\mu_0} - k_\omega\right) Z_{max}}.$$

Solving for Z_{max}:

$$Z_{max}(\pm) = \frac{\ln\left[\frac{\alpha}{k_\omega \mu_0} A(\mu_0, \pm)\right]}{\left[\frac{\alpha}{\mu_0} - k_\omega\right]},$$

where the plus sign refers to the upwelling stream and the minus sign refers to the downwelling stream.

The criterion for the existence of a positive Z_{max} value is evidently:

$$\frac{\alpha}{k_\omega \mu_0} A(\mu_0, \pm) > 1 .$$

If in particular, the argument of the natural logarithm is positive but less than unity then Z_{max} has a negative sign, which means that $H(Z, +)$ $\left(\text{or} \ H(Z, -) \right)$ has no maximum value. In this case the irradiance simply decreases monotonically for all depths $Z \gtrless 0$.

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TABLE 1

z meters	$K(z, -)$		$K(z, +)$		$R(z, -)$	
	Data	Calculated	Data	Calculated	Data	Calculated
6.10					0.0221	0.0221
12.20	0.216	0.216	0.206	0.206		
18.30	0.206	0.204	0.198	0.196	0.0250	0.0249
24.41	0.196	0.195	0.191	0.189		
30.52	0.189	0.188	0.185	0.185	0.0266	0.0266
36.64	0.183	0.184	0.179	0.182		
42.76	0.180	0.182	0.182	0.180	0.0279	0.0274
48.88	0.178	0.180	0.184	0.179		
54.99					0.0258	0.0277

$\mu_0 = 1.583$ $A(\mu_0, +) = -1.337$ $A(\mu_0, -) = -2.141$

$R_\infty = 0.0278$ $k_\infty = 0.178/\text{meter}$ $\alpha = 0.430/\text{meter}$

$\lambda = 480 \pm 64 \; m\mu$

PLOTS OF
UPWELLING IRRADIANCE H(Z,+)
DOWNWELLING IRRADIANCE H(Z,-)
AS FUNCTIONS OF DEPTH Z

H(Z,-)

H(Z,+)

Relative Magnitudes

100,000

10,000

1000

100

10

Depth Z Meters

0   10   20   30   40   50   60
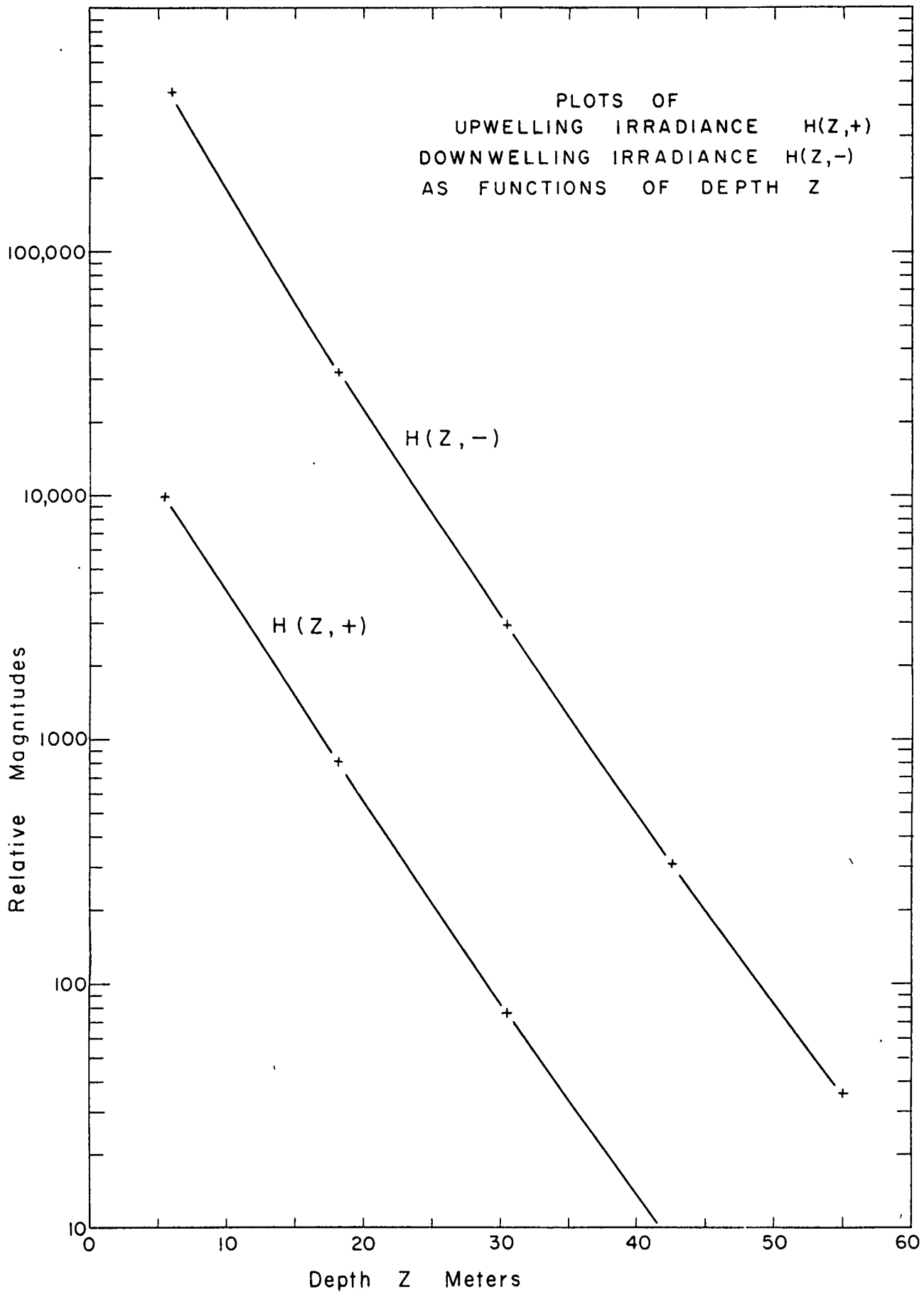
Figure 1

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DISTRIBUTION FUNCTIONS D(Z, ±)

VOLUME ATTENUATION FUNCTION α(Z) (+)
VOLUME ABSORPTION FUNCTION a(Z) (⊕)

DEPTH Z METERS

Figure 2

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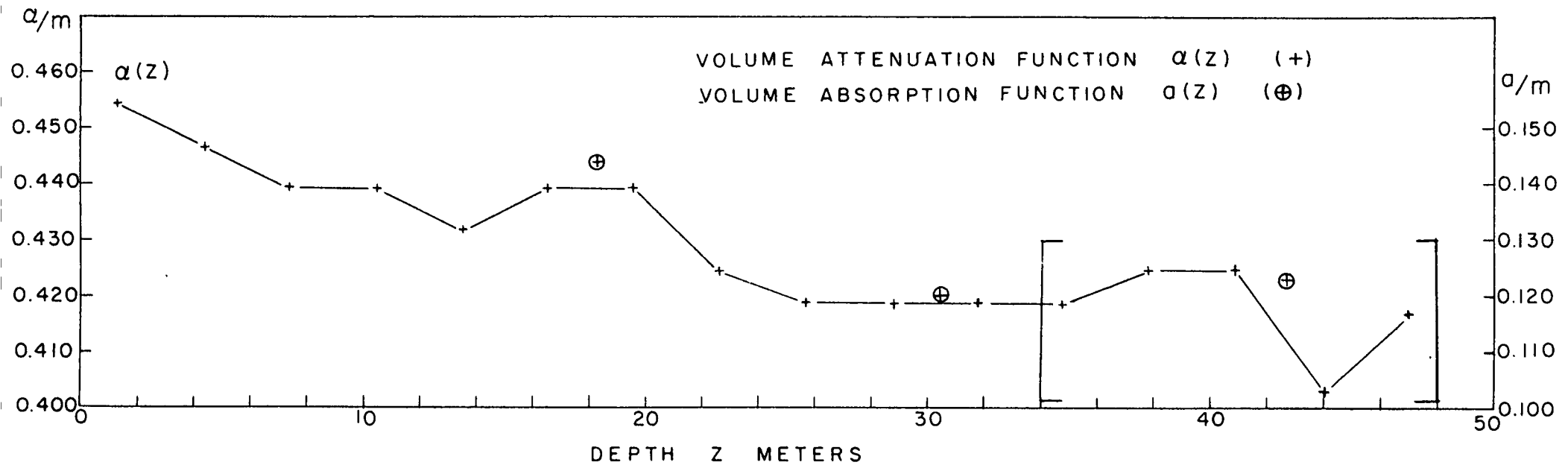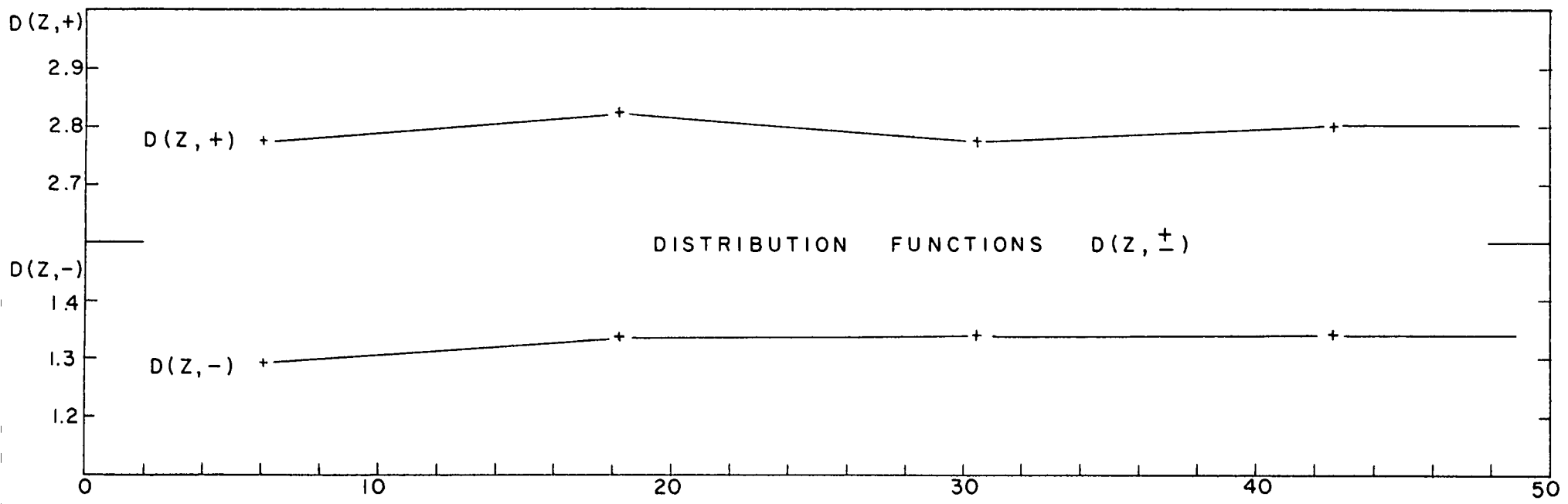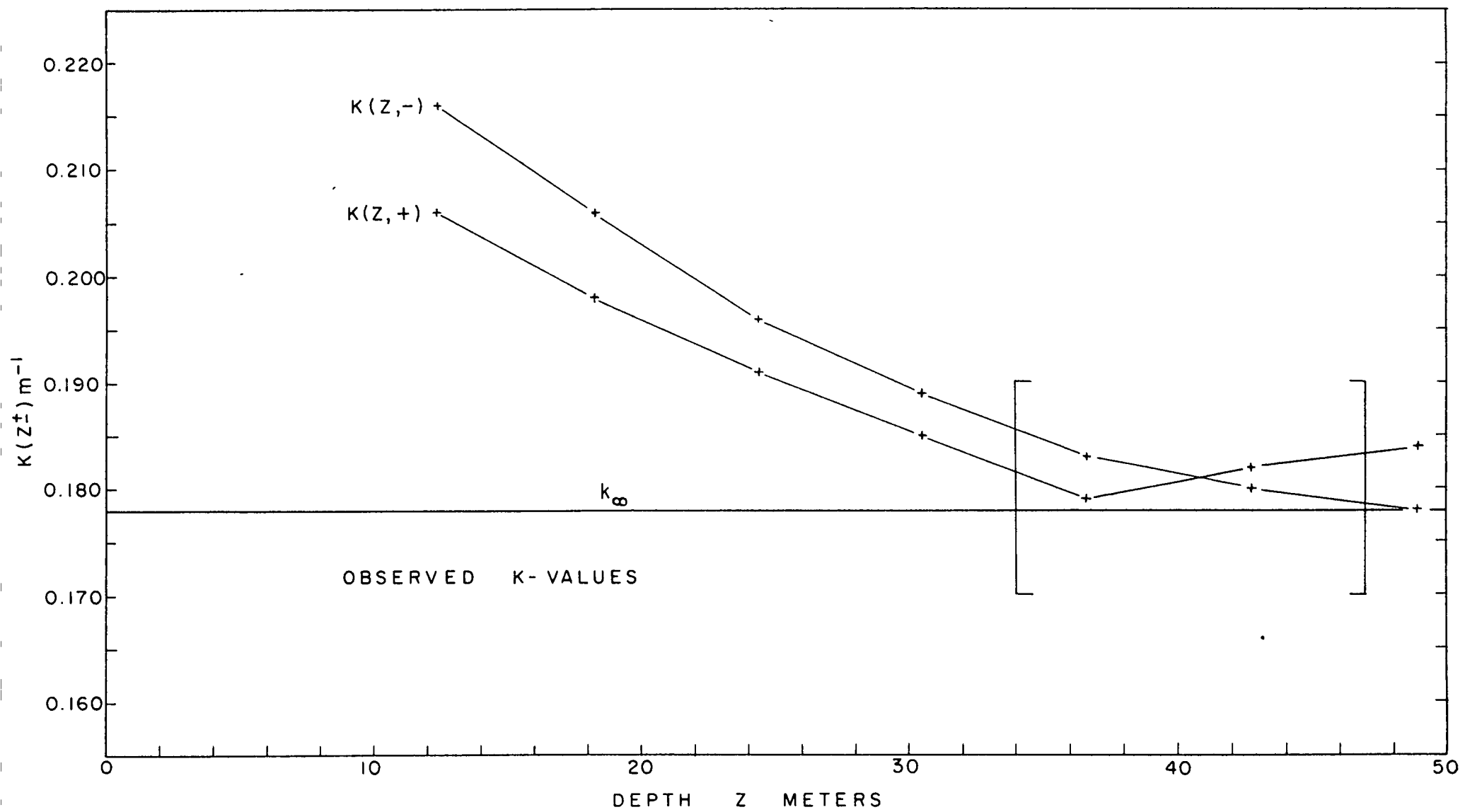
Rudolph W. Preisendorfer
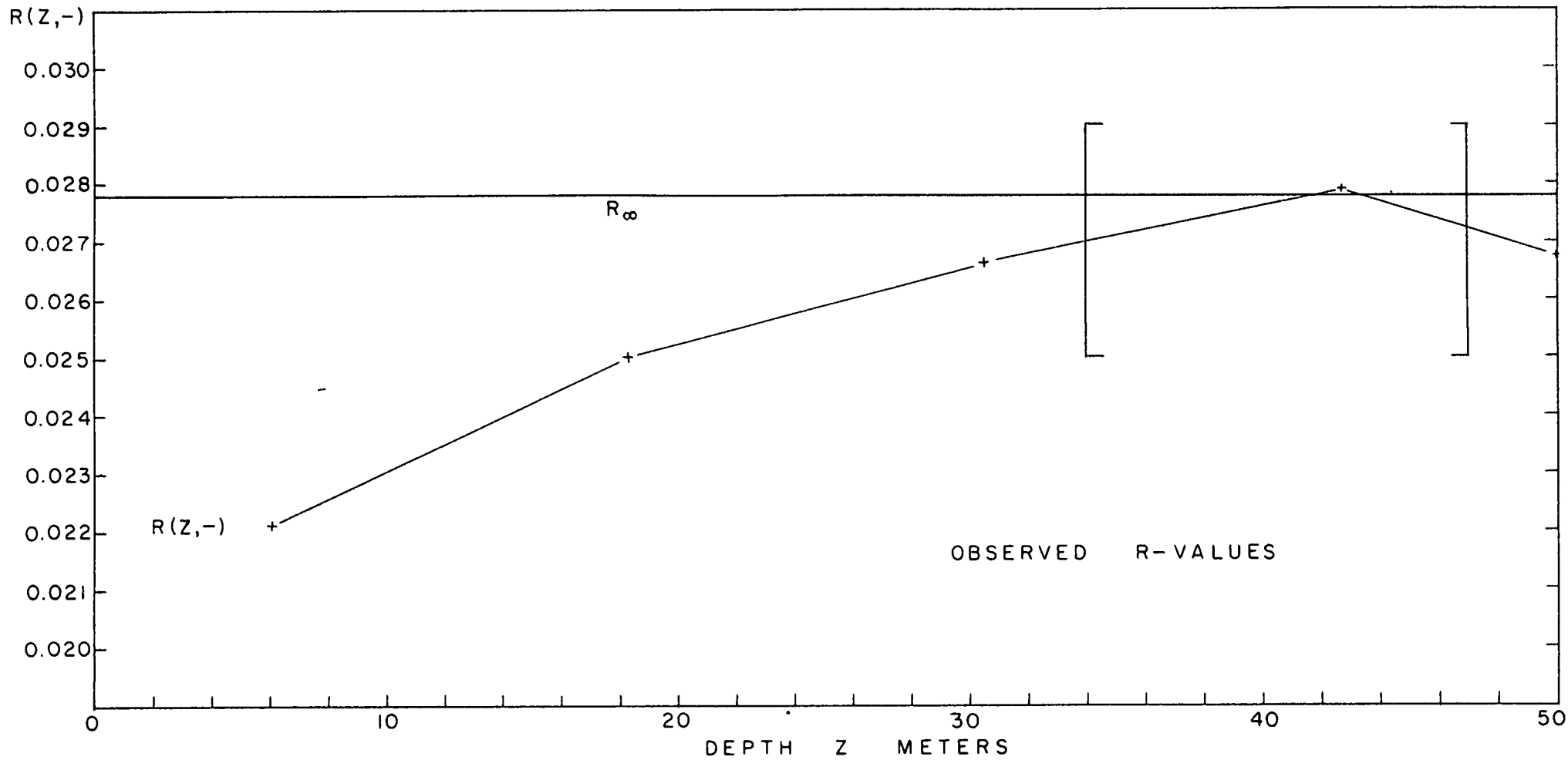
Figure 3

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Figure 4
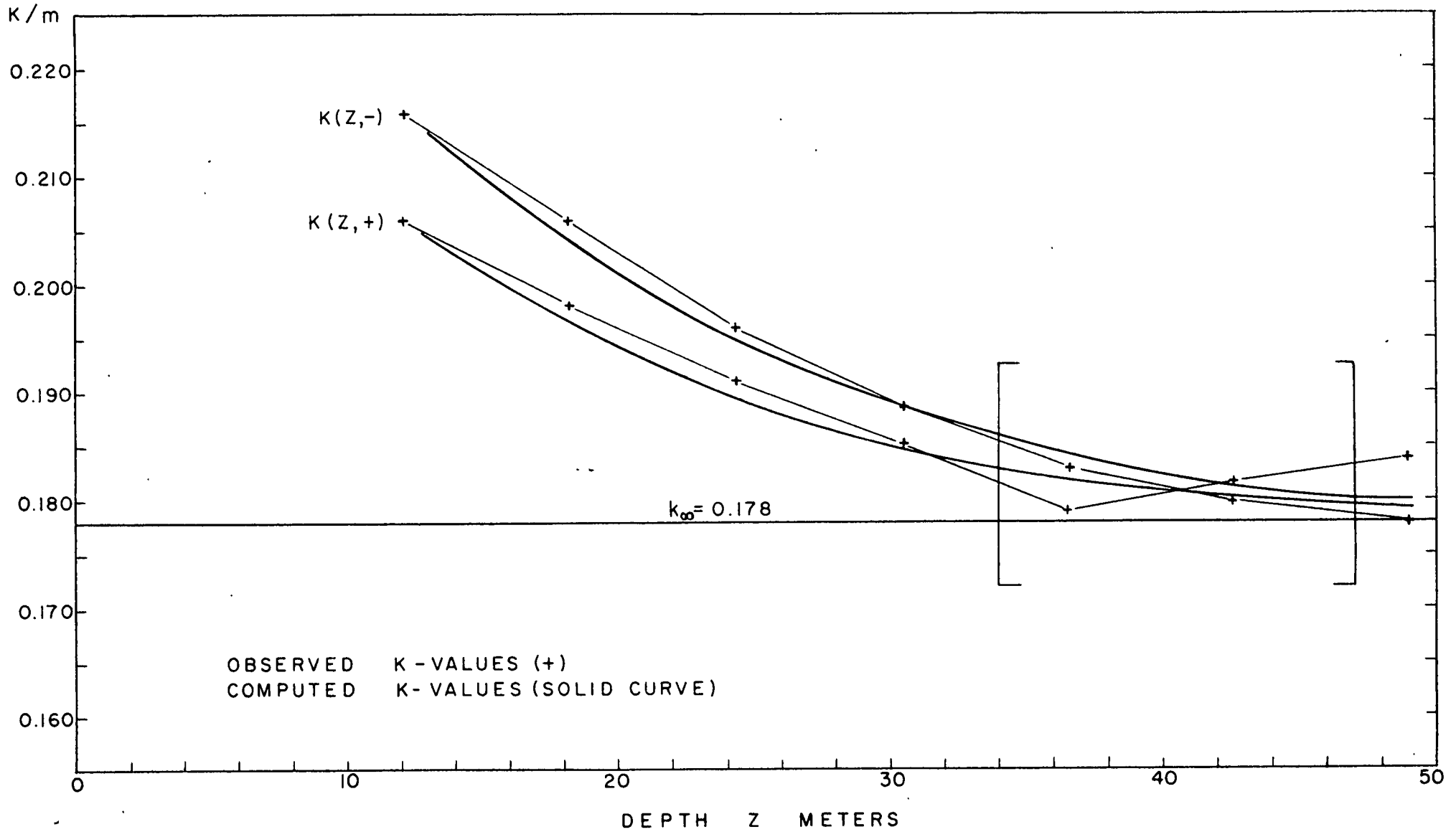
Rudolph W. Preisendorfer

Figure 5
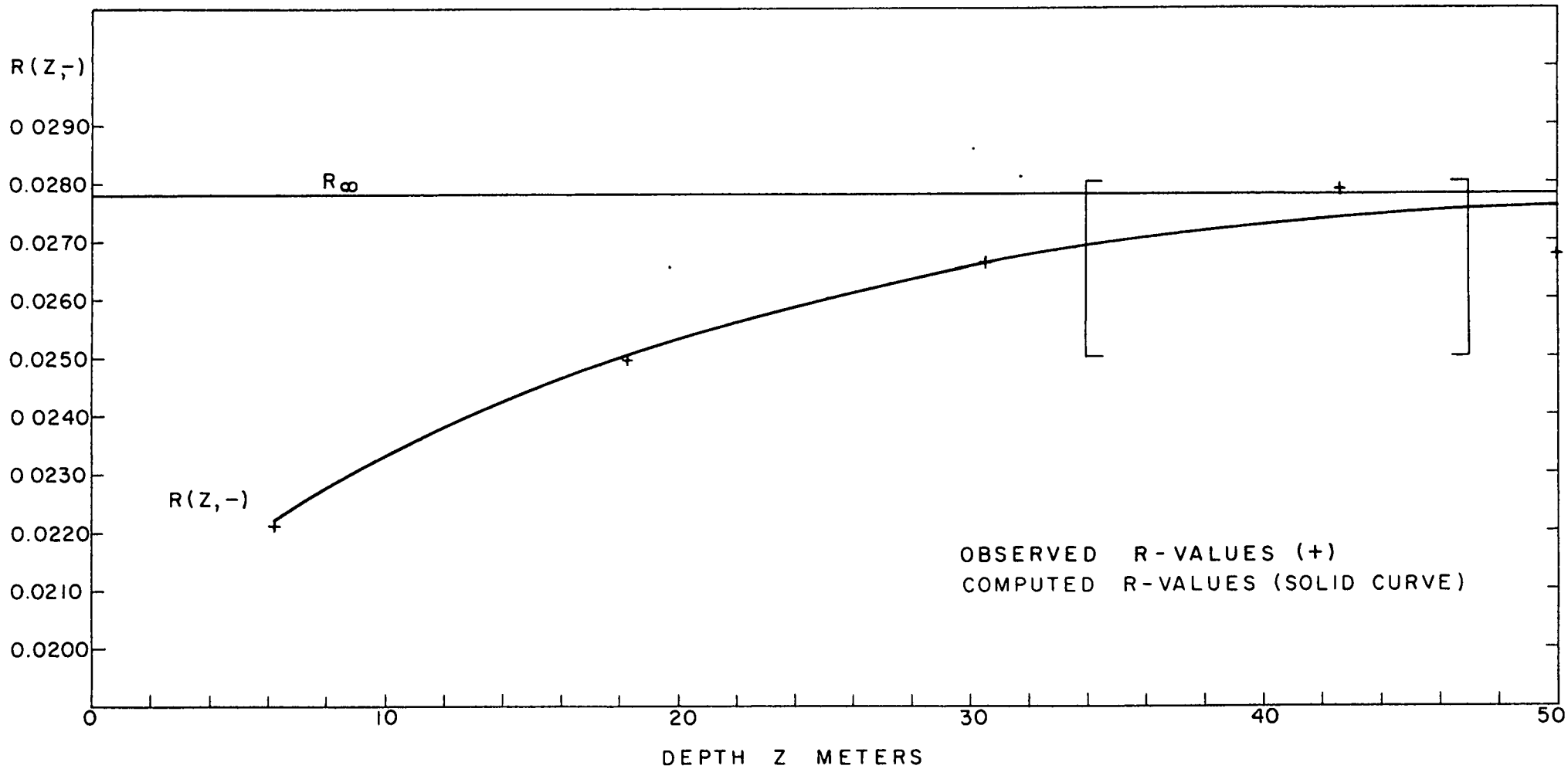
Rudolph W. Preisendorfer

Figure 6
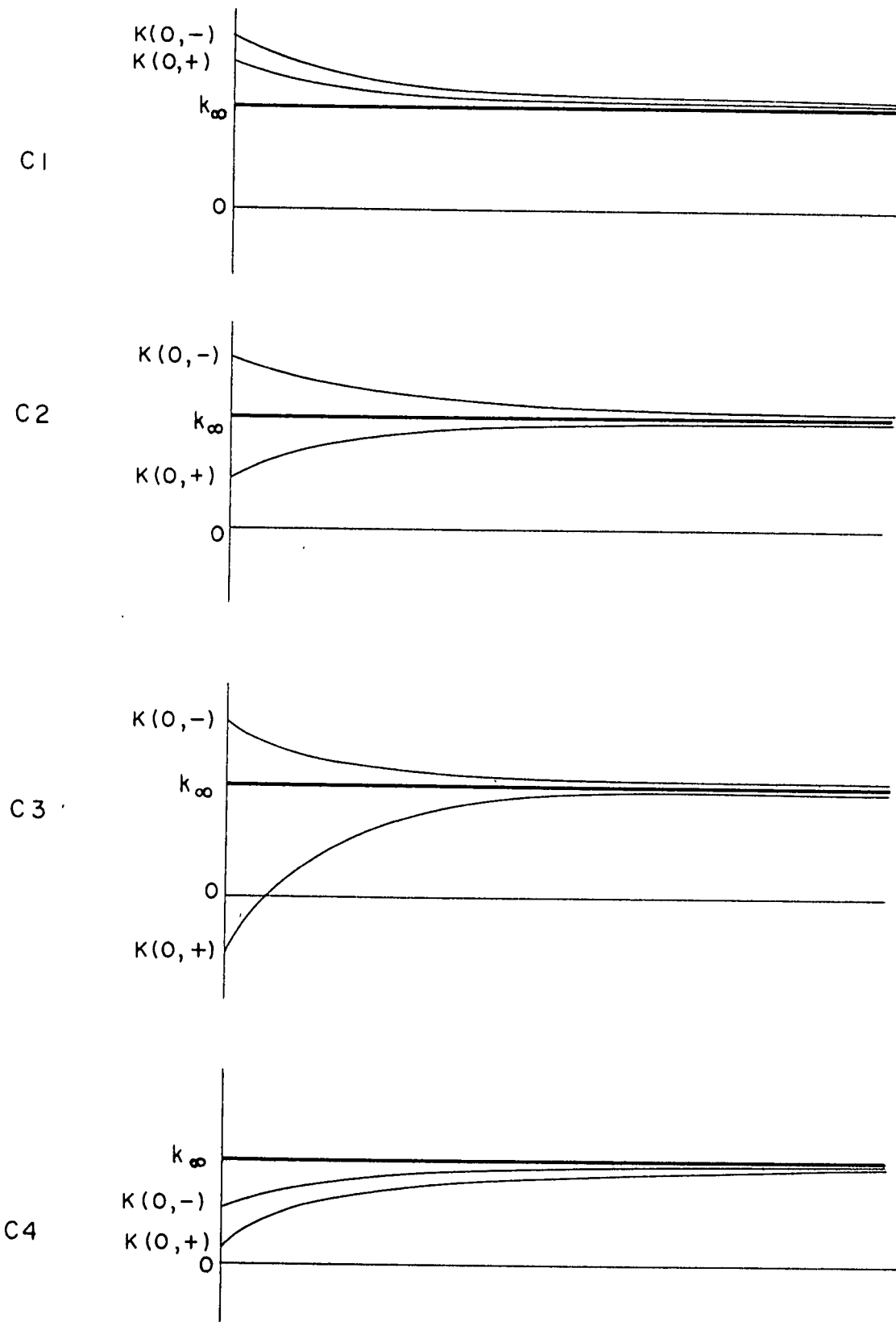
C I

C 2

C 3 ·

C 4

Figure 7

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Figure 8

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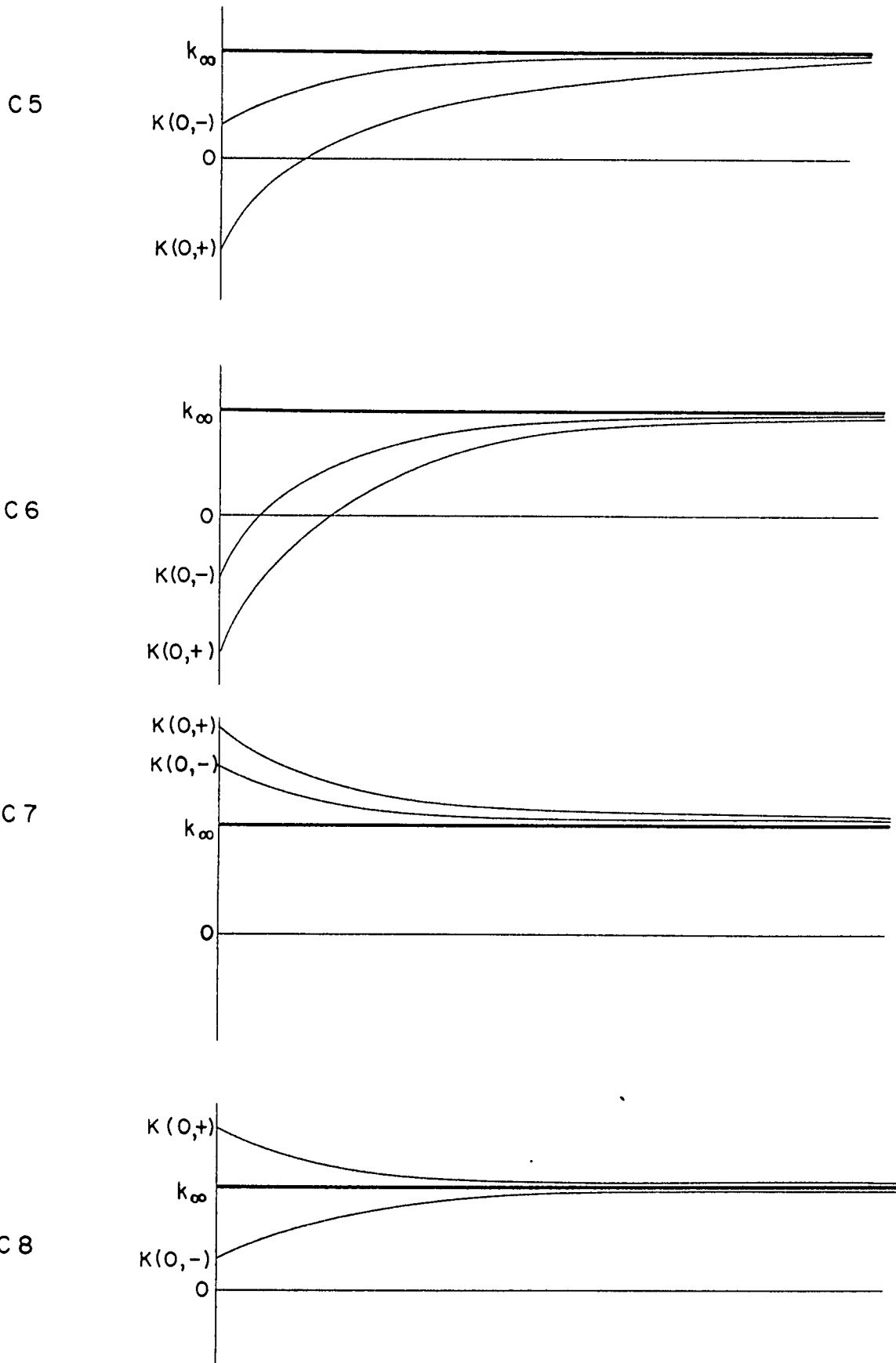
NONDEGENERATE CONFIGURATIONS, CONCLUDED
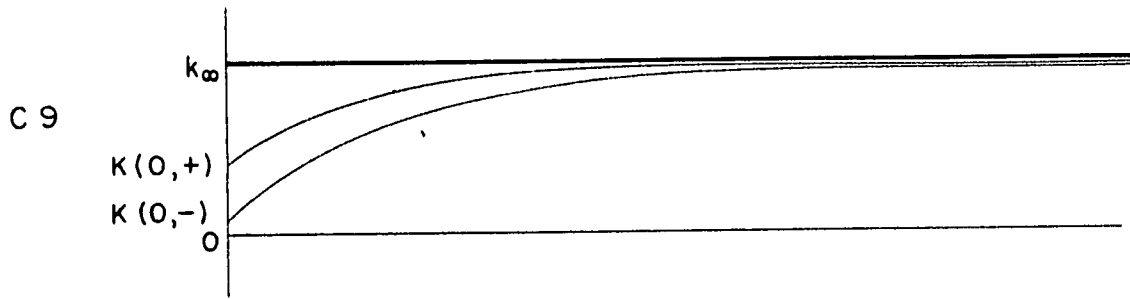
C 9

k_∞

$K(0,+)$

$K(0,-)$

0

DEGENERATE CONFIGURATIONS, FIRST KIND $(k_\infty > 0)$

$K(0,-)$

$K(0,+) = k_\infty$

$D_I 1$

0

$K(0,-) =$
$K(0,+) = k_\infty$

$D_I 2$

0

$K(0,+) = k_\infty$

$D_I 3$ $K(0,-)$

0

Figure 9

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$D_1 4$

$K(0,-) = K(0,+)$

k_∞

O

$D_1 5$

k_∞

$K(0,-) = K(0,+)$

O

$D_1 6$

$K(0,-) = k_\infty$

$K(0,+)$

O

$D_1 7$

$K(0,+)$

$K(0,-) = k_\infty$

O

Figure 10

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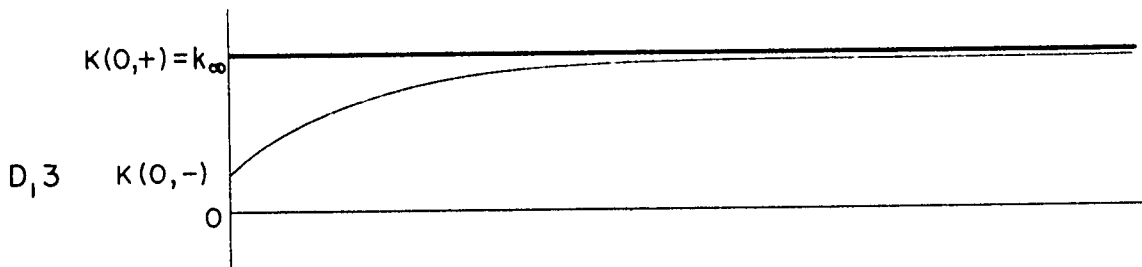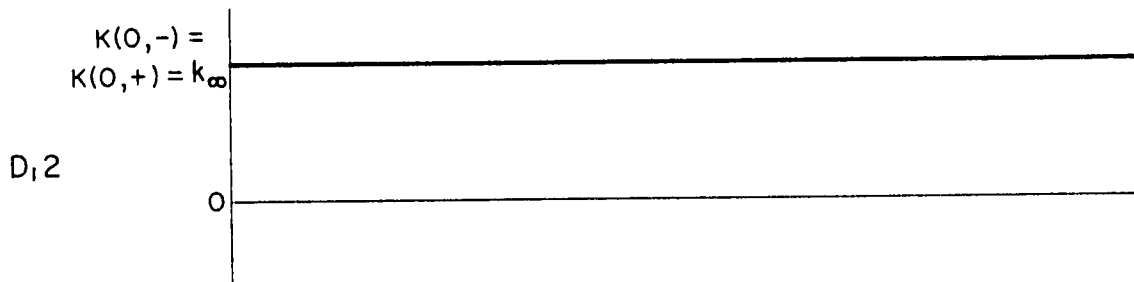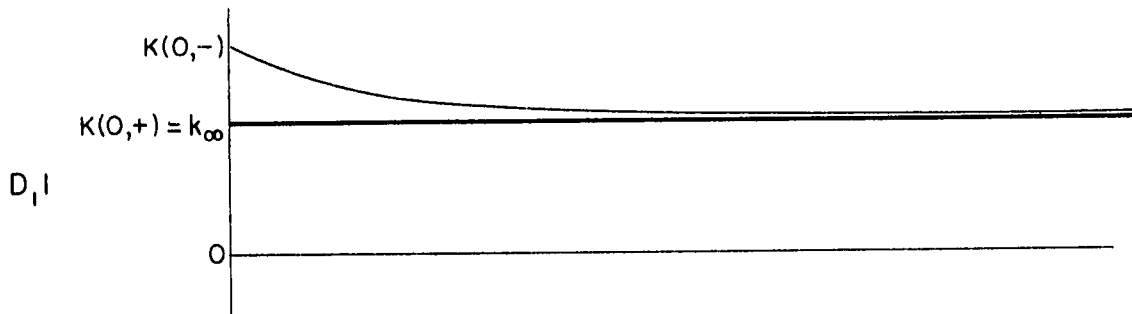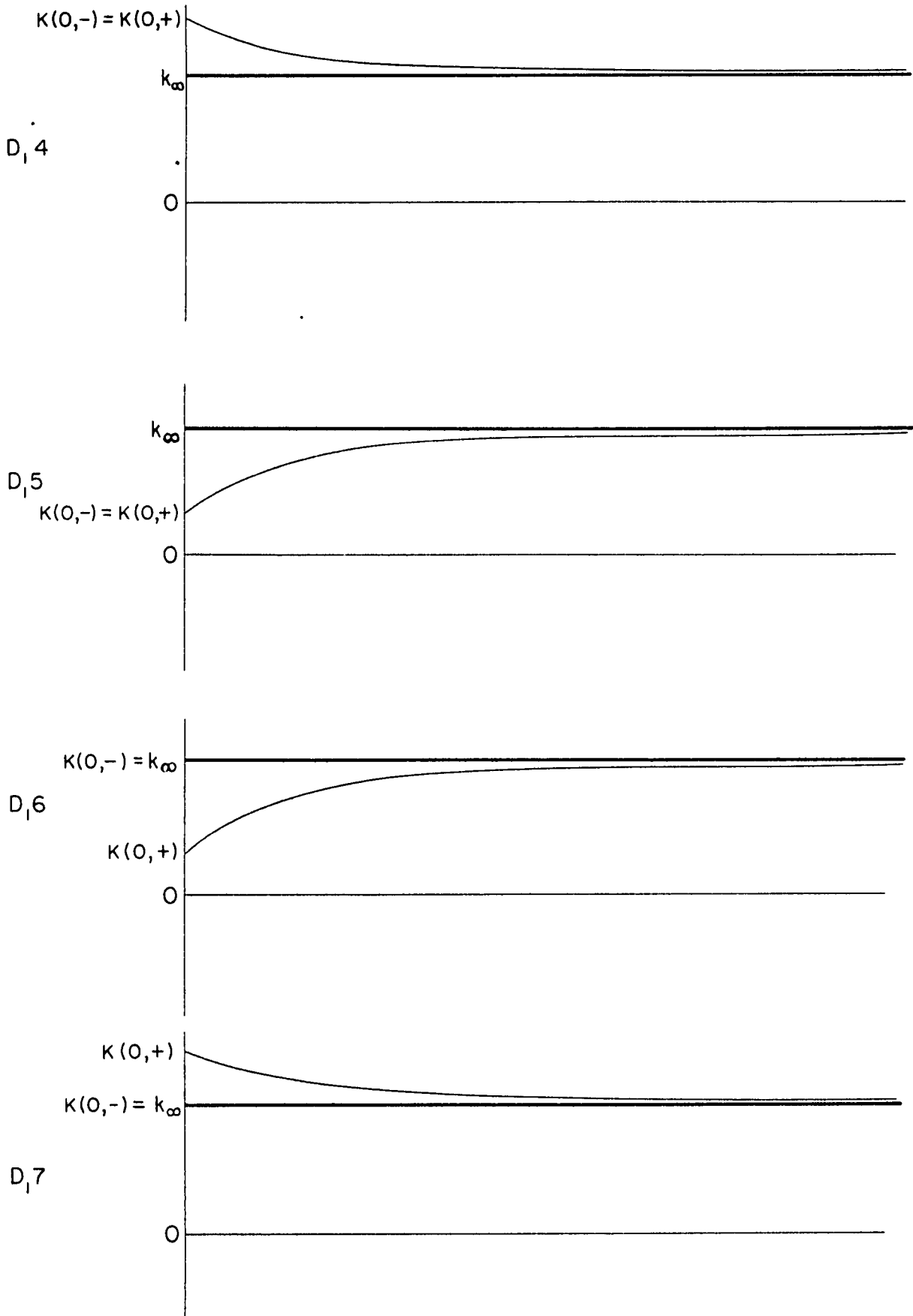
DEGENERATE CONFIGURATIONS, FIRST KIND ($k_\infty > 0$) CONCLUDED

$D_1 8$

$K(0,-) = k_\infty$

0

$K(0,+)$
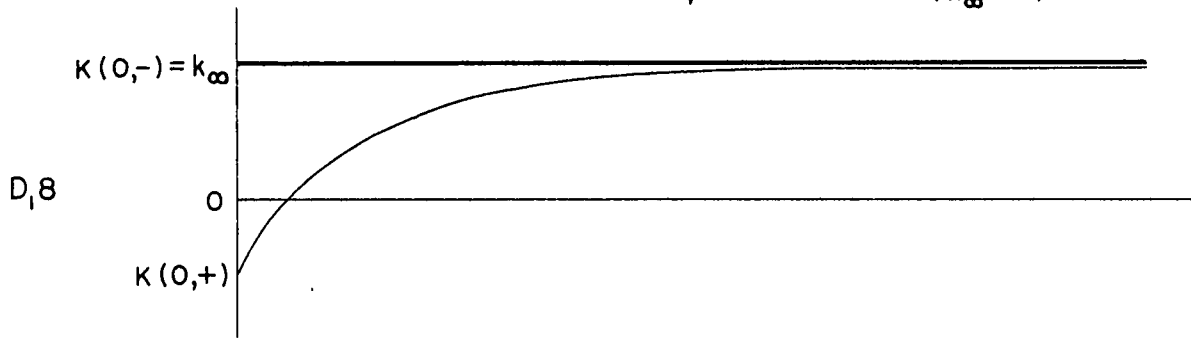
DEGENERATE CONFIGURATIONS, SECOND KIND ($k_\infty = 0$)

$D_2 1$

$K(0,-) = K(0,+)$

$k_\infty = 0$

$D_2 2$ $\begin{aligned} K(0,-) &= \\ K(0,+) &= \\ k_\infty &= 0 \end{aligned}$
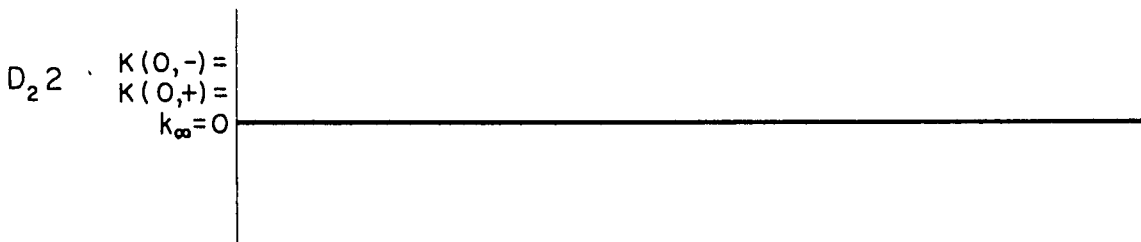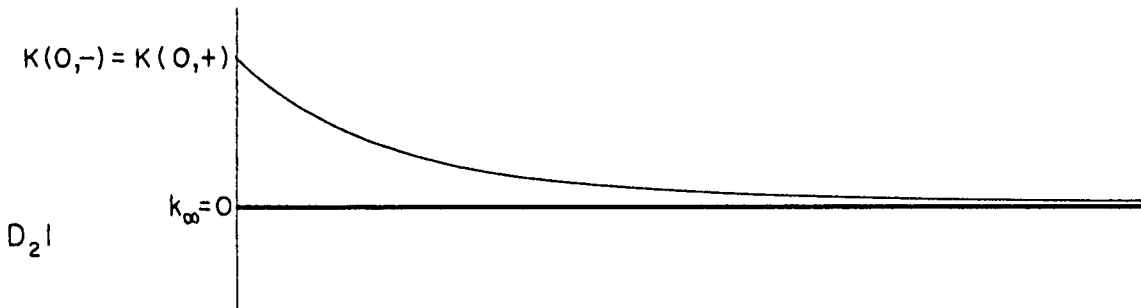
$D_2 3$

$k_\infty = 0$

$K(0,-) = K(0,+)$

Figure II

Rudolph W. Preisendorfer
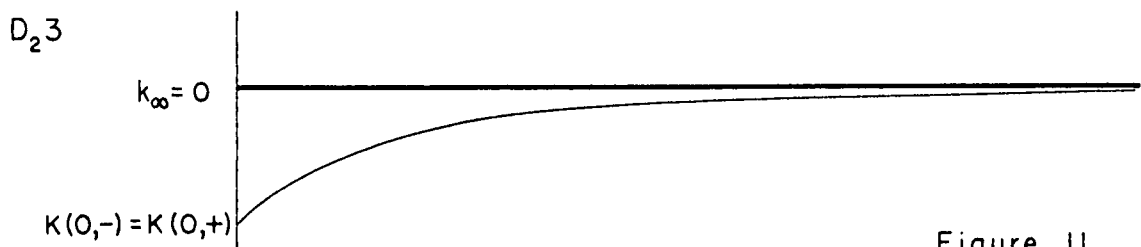
FORBIDDEN CONFIGURATIONS

Figure 12

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