Title
Boundary Characterization of Iterated Automorphism Orbits on Bounded Domains

Author
Strong, Joshua Alexander

Publication Date
2016

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA
RIVERSIDE

Boundary Characterization of Iterated Automorphism Orbits on Bounded Domains

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

by

Joshua Alexander Strong

June 2016

Dissertation Committee:

Dr. Bun Wong, Chairperson
Dr. Yat Sun Poon
Dr. Stefano Vidussi

The Dissertation of Joshua Alexander Strong is approved:

Committee Chairperson

University of California, Riverside

Acknowledgments

Thank you to all who have supported me over these five years, namely my fellow grads, friends, and family. Especially thank you to my advisor, Dr. Bun Wong, for all the help and guidance.

To my parents.

ABSTRACT OF THE DISSERTATION

Boundary Characterization of Iterated Automorphism Orbits on Bounded Domains

by

Joshua Alexander Strong

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, June 2016
Dr. Bun Wong, Chairperson

The problem of characterizing bounded domains in $\mathbb{C}^n$ can be related to the automorphism group and the geometry of the boundary. It is a conjecture of Greene and Krantz that if a smoothly bounded domain has a noncompact automorphism group, then the boundary is of finite type at any automorphism accumulation point. While there have been numerous supporting results, the conjecture is as yet unsolved. The purpose of this dissertation is to provide another result in support of the Greene-Krantz conjecture. Specifically, if the boundary of a smoothly bounded convex domain admits an iterated automorphism orbit nontangentially, then it is of finite type.

Contents

Chapter 1

Introduction

A major consideration in many subjects of mathematics is that one would like to know which objects are "the same." That is, under some equivalence relation, how many different types of equivalence classes are there and what are these types of equivalences. When studying domains in $\mathbb{C}^n$, we care about equivalence under biholomorphism. That is, two domains in $\mathbb{C}^n$ are equivalent if there is a biholomorphism between them. This equivalence is especially useful when our domains are endowed with the Kobayashi or Carathéodory metrics, for under theses metrics, any biholomorhpism preserves the distance between between any two points. So no matter how much their Euclidean distances may differ, they are still the same distance apart in the Kobayashi metric. The Kobayashi metric will be an essential tool in whats follows. Some other useful tools for bounded domains are the automorphism group (biholomorphic self mappings) of the domain and the type (order of contact with a variety) of the boundary. It is a conjecture of Greene and Krantz that a smoothly bounded domain with a noncompact automorphism group is of finite type at any boundary orbit accumulation point. If this conjecture is true, it would classify all smoothly bounded domains in $\mathbb{C}^2$ with a noncompact automorphism group, for they would be, up to biholomorphism, the ball or a complex ellipsoid.

In Chapter 2, we give some background on the problem of classifying domains as well as a series of definitions and theorems about the analytic and geometric properties that will be used throughout. Chapter 3 will cover the Kobayashi metric and Gromov hyperbolicity. We discuss the relationship between the two and important properties. In chapter 4 we provide the specific details that lead up top the proof of the main theorem.

Chapter 2

Background

2.1 Preliminaries

Firstly, let us provide some definitions and basic notation to be used throughout.

Definition 2.1.1 *Let Ω be an open subset of $\mathbb{C}^n$. A function $f : \Omega \to \mathbb{C}$ is said to be holomorphic (or analytic) if at each $p \in \Omega$ there is som open neighborhood U of p such that f has a power series expansion*

$$f(z) = \sum_{j_1,\ldots,j_n=0}^{\infty} a_{j_1\ldots j_n}(z_1 - p_1)^{j_1} \ldots (z_n - p_n)^{j_n}$$

which converges for all $z \in U$. If instead $f : \Omega \to \mathbb{C}^m$ then we say f is holomorphic if the component functions

$$f_j(z_1, ..., z_n)$$

are holomorphic for $j = 1, ..., m$.

It is a nontrivial fact that a function $f : \Omega \subset \mathbb{C}^n \to \mathbb{C}$ is holomorphic if and only if f is holomorphic in each variable separately. That is the mapping

$$\zeta \mapsto f(z_1, ..., z_{j-1}, \zeta, z_{j+1}, ..., z_n)$$

is a holomorphic function of one complex variable for all $j = 1, ..., n$.

Definition 2.1.2 *For two open subsets $W, V \subset \mathbb{C}^n$, a function $f : W \to V$ is said to be a biholomorphism if f is holomorphic and admits a holomorphic inverse $f^{-1} : V \to W$.*

Note that, contrary to the real case, if f is a one-to-one holomorphic map, then it is a biholomorphism onto its range. We will denote by $\mathrm{Hol}(U, V)$ the collection of holomorphic maps from U to V. The unit disk in $\mathbb{C}$ is given by

$$\Delta = \{z \in \mathbb{C} : |z| < 1\},$$

the upper half plane in $\mathbb{C}$ by

$$\mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\},$$

and the unit polydisk in $\mathbb{C}^n$ by

$$\Delta^n = \Delta \times \cdots \times \Delta = \{(z_1, ..., z_n) \in \mathbb{C}^n : |z_j| < 1 \text{ for all } j = 1, ..., n\},$$

Finally the unit ball in $\mathbb{C}^n$ is

$$B^n = \{(z_1, ..., z_n) \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_n|^2 < 1\}.$$

2.2 The Complex Plane and $\mathbb{C}^n$

We begin our consideration of the classification of complex domains with the plane, $\mathbb{C}$. In this case, the Riemann mapping theorem classifies all simply connected domains.

Definition 2.2.1 *We say that a domain $\Omega \subset \mathbb{C}$ is simply connected if it is connected and any closed curve in Ω can be continuously shrunk to a point in Ω.*

Originally, the Riemann mapping theorem was written as a statement about the existence of a bijective holomorphism from domains to the unit disk. For our purposes, we will use an equivalent statement.

Theorem 2.2.2 (Riemann Mapping Theorem) *Let $\Omega \subset \mathbb{C}$ be a simply connected domain such that $\Omega \neq \mathbb{C}$. Then Ω is biholomorphic to Δ.*

This powerful theorem tells us that for simply connected domains in $\mathbb{C}$, there are only two equivalence classes: (1) the entire plane and (2) any other simply connected domain. Now one might hope that the Riemann mapping theorem could be extended to higher dimensions. However, once $n = 2$ the theorem fails. Not only that, but the number of equivalence classes increases dramatically in higher dimensions.

Theorem 2.2.3 (Burns/Schneider/Wells [4]) *If $n \geq 2$, there exists an infinite dimensional family of holomorphically distinct bounded strictly pseudoconvex domains in $\mathbb{C}^n$ obtained by C^∞ perturbations of the unit ball.*

2.3 Domains of Holomorphy

The differences between $\mathbb{C}$ and $\mathbb{C}^n$ do not end with the ball and the bidisk. For example, if $\Omega \subset \mathbb{C}$ is a bounded domain then a function $f : \Omega \to \mathbb{C}$ may be meromorhic. That is f has isolated singularities in Ω. If these singularities are removable, then f can be extended to a holomorphic function Ω. However, if these are poles or essential singularities, then there is no hope of extending the function f. Of course this also does not extend to higher dimensions.

Theorem 2.3.1 (Hartogs's Phenomenon) *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with $n \geq 2$. Let $K \subset \Omega$ be compact such that $\Omega \setminus K$ is connected. If $f : \Omega \setminus K \to \mathbb{C}^n$ is holomorphic, then there is a holomorphic function $F : \Omega \to \mathbb{C}^n$ where $F|_{\Omega \setminus K} = f$.*

One can infer from this theorem that there can be no isolated poles or essential singularities of a holomorphic function on a bounded domain in $\mathbb{C}^n$. For if there were such a singularity, then we could extend the function to include such points giving us a contradiction.

Theorem 2.3.1 does not actually take care of all cases when a holomorphic function can be extended to a larger domain.

Definition 2.3.2 *A domain $\Omega \subset \mathbb{C}^n$ is called a domain of holomorphy if there does not exist any nonempty open sets U, V, where V is connected and not contained in Ω, $U \subset V \cap \Omega$, and for any holomorphic function f on Ω there is a holomorphic function g on V such that $f = g$ on U.*

Simply put, Ω is a domain of holomorphy if there is at least one holomorphic function on Ω which cannot be holomorphically extended past $\partial\Omega$. On the complex plane, every domain is a domain of holomorphy. We give a short proof that the unit disk is such a domain.

Proposition 2.3.3 *The unit disk $\Delta \subset \mathbb{C}$ is a domain of holomorphy.*

Proof. Define $f : \Delta \to \mathbb{C}$ by

$$f(z) = \sum_{k=0}^{\infty} 2^{-k} z^{2^k}.$$

By the Weierstrass M-test, we get that f is analytic on Δ and continuous on $\bar{\Delta}$. However, on $\partial\Delta$, the mapping

$$\theta \mapsto f(e^{i\theta}) = \sum_{k=0}^{\infty} 2^{-k} e^{i\theta 2^k}$$

is a nowhere differentiable function. Thus, we cannot extend f holomorphically past $\partial\Delta$ and so Δ is a domain of holomorphy. ■

An important fact about open subsets is that an open subset $\Omega \subset \mathbb{C}^n$ is that Ω is a domain of holomorphy if and only if it is peudoconvex. We discuss characteristics of psedoconvexity in the next section.

2.4 Pseudoconvexity

Psudoconvexity is the complex analog of convexity in the real sense. When describing domains, it is convenient to use the notion of a defining function.

Definition 2.4.1 *Let $\Omega \subset \mathbb{R}^n$ be an open set with C^k boundary. A function $\rho : \mathbb{R}^n \to \mathbb{R}$ is said to be a defining function for Ω if ρ is C^k and*

1. *$\rho(x) < 0$ for all $x \in \Omega$,*

2. *$\rho(x) > 0$ for all $x \notin \Omega$, and*

3. *$\nabla\rho(x) \neq 0$ for all $x \in \partial\Omega$.*

Example 2.4.2 *The unit ball B^n.*

The unit ball in $\mathbb{C}^n$ can be described by the function $\rho : \mathbb{C}^n \to \mathbb{R}$ defined by

$$\rho(z) = ||z||^2 - 1.$$

That is

$$B^n = \{z \in \mathbb{C}^n : \rho(z) < 0\}.$$

Now for our purposes, we will only consider domains with C^2 boundary. Our notion of convexity will then depend on tangent vectors of the boundary.

Definition 2.4.3 *Let $\Omega \subset \mathbb{R}^n$ have a C^1 defining function ρ. Let $p \in \partial\Omega$. Then $w = (w_1, ..., w_n)$ is a tangent vector to $\partial\Omega$ at p if*

$$\sum_{k=1}^{n} \left.\frac{\partial\rho}{\partial x_k}\right|_p w_k = 0.$$

In this case we write $w \in T_p(\partial\Omega)$.

Definition 2.4.4 *Let $\Omega \subset \mathbb{R}^n$ be a domain with C^2 boundary, $p \in \partial\Omega$, and ρ be a defining function for Ω. We say that $\partial\Omega$ is convex at p if*

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial x_j \partial x_k}\bigg|_p w_j w_k \geq 0$$

for all $w = (w_1, ..., w_2) \in T_p(\partial\Omega)$. If, instead, we have a strict inequality for all nonzero w satisfying the second equation, we say that q is a point of strict convexity.

For domains in $\mathbb{C}^n$ with C^2 boundary, the notion of Levi pseudoconvexity uses complex tangent vectors rather that real tangent vectors.

Definition 2.4.5 *Let $\Omega \subset \mathbb{C}^n$ be a doimain with C^2 boundary, $p \in \partial\Omega$, and ρ be a defining function for Ω. We say that p is a point of Levi pseudoconvexity if*

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}\bigg|_p w_j \bar{w}_k \geq 0$$

for all $w \in \mathbb{C}^n$ such that

$$\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}\bigg|_p w_j = 0.$$

If instead, we have a strict inequality for all nonzero w satisfying the second equation, we say that x is a point of strict (Levi) peudoconvexity. In general, when we say a boundary point is pseudoconvex we mean that it is weakly pseudoconvex.

The vectors satisfying the second equation in the above definition are called complex tangent vectors. We shall denote the complex tangent space by $T_p^{(1,0)}(\partial\Omega)$. Note that $T_p^{(1,0)}(\partial\Omega) \subset T_p(\partial\Omega)$. In fact, the complex tangent space at a point $p \in \partial\Omega$ is the largest complex subspace of the real tangent space to $\partial\Omega$ at p. We call $\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}\Big|_p w_j \bar{w}_k$ the Levi form of ρ at p. So $p \in \partial\Omega$ is a point of weak (respectively strong) psuedoconvexity if its Levi form is positive semidefinite (respectively positive definite).

Proposition 2.4.6 *Let $\Omega \subset \mathbb{C}^2$ be a domain with C^2 boundary and $p \in \partial\Omega$. If $\partial\Omega$ is convex at p, then it is also pseudoconvex at p.*

Proof. Let ρ be a defining function for Ω. Let $w = (w_1, ..., w_n) \in T_p^{(1,0)}(\partial\Omega)$. Write $w_j = \xi_j + i\eta_j$ for $j = 1, ..., n$. Then $(\xi_1, \eta_1, \xi_2, \eta_2, ..., \xi_n, \eta_n) \in T_p(\partial\Omega)$. Since $\partial\Omega$ is convex at p then

$$\begin{aligned}
0 &\leq \sum_{j,k=1}^{n} \left.\frac{\partial^2\rho}{\partial x_j \partial x_k}\right|_p \xi_j\xi_k + 2\sum_{j,k=1}^{n} \left.\frac{\partial^2\rho}{\partial x_j \partial x_k}\right|_p \xi_j\eta_k + \sum_{j,k=1}^{n} \left.\frac{\partial^2\rho}{\partial x_j \partial x_k}\right|_p \eta_j\eta_k \\
&= \frac{1}{4}\sum_{j,k=1}^{n} \left(\frac{\partial}{\partial z_j} + \frac{\partial}{\partial \bar{z}_j}\right)\left(\frac{\partial}{\partial z_k} + \frac{\partial}{\partial \bar{z}_k}\right)\rho\Big|_p (w_j + \bar{w}_j)(w_k + \bar{w}_k) \\
&\quad + \frac{1}{2}\sum_{j,k=1}^{n} \left(\frac{\partial}{\partial z_j} + \frac{\partial}{\partial \bar{z}_j}\right)\left[i\left(\frac{\partial}{\partial z_k} - \frac{\partial}{\partial \bar{z}_k}\right)\right]\rho\Big|_p (w_j + \bar{w}_j)[-i(w_k - \bar{w}_k)] \\
&\quad + \frac{1}{4}\sum_{j,k=1}^{n} \left[i\left(\frac{\partial}{\partial z_j} - \frac{\partial}{\partial \bar{z}_j}\right)\right]\left[i\left(\frac{\partial}{\partial z_k} - \frac{\partial}{\partial \bar{z}_k}\right)\right]\rho\Big|_p [-i(w_j - \bar{w}_j)][-i(w_k - \bar{w}_k)] \\
&= \sum_{j,k=1}^{n} \left.\frac{\partial^2\rho}{\partial z_j \partial z_k}\right|_p w_jw_k + \sum_{j,k=1}^{n} \left.\frac{\partial^2\rho}{\partial \bar{z}_j \partial \bar{z}_k}\right|_p \bar{w}_j\bar{w}_k + 2\sum_{j,k=1}^{n} \left.\frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k}\right|_p w_j\bar{w}_k \\
&= 2\mathrm{Re}\left[\sum_{j,k=1}^{n} \left.\frac{\partial^2\rho}{\partial z_j \partial z_k}\right|_p w_jw_k\right] + 2\sum_{j,k=1}^{n} \left.\frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k}\right|_p w_j\bar{w}_k.
\end{aligned}$$

Now since $w \in T_p^{(1,0)}(\partial\Omega)$ then so is iw. So we get

$$0 \leq -2\mathrm{Re}\left[\sum_{j,k=1}^{n} \left.\frac{\partial^2\rho}{\partial z_j \partial z_k}\right|_p w_jw_k\right] + 2\sum_{j,k=1}^{n} \left.\frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k}\right|_p w_j\bar{w}_k.$$

Combining these two inequalities then yields

$$0 \leq 4\sum_{j,k=1}^{n} \left.\frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k}\right|_p w_j\bar{w}_k.$$

Thus, p is a point of pseudoconvexity. ■

Note that a pseudoconvex domain need not be convex. We can see this by considering domains in $\mathbb{C}$. Recall that the complex tangent space for a boundary point p

of any domain $\Omega \subset \mathbb{C}$ is $T_p^{(1,0)}(\partial\Omega) = \{0\}$. Therefore, every domain in $\mathbb{C}$ is vacuously pseudoconvex.

Our definition of pseudoconvexity seems dependent on the defining function chosen, but we shall see in the next proposition that it is in fact independent of the given defining function.

Proposition 2.4.7 *Pseudoconvexity is independent of the chosen defining function.*

Proof. Let $\Omega = \{\rho < 0\} \subset \mathbb{C}^n$ be a domain with C^2 boundary and $p \in \partial\Omega$ be a point of pseudoconvexity. Let $\hat{\rho}$ be another defining function for Ω in a neighborhood, U, of p. Then there is a C^2 function, $h : U \to \mathbb{R}$, that is nonvanishing on U (shrinking U if necessary), such that $\hat{\rho} = h\rho$. Now

$$\frac{\partial^2 \hat{\rho}}{\partial z_j \partial \bar{z}_k} = \frac{\partial^2 (h\rho)}{\partial z_j \partial \bar{z}_k} = h\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} + \frac{\partial \rho}{\partial \bar{z}_k}\frac{\partial h}{\partial z_j} + \frac{\partial \rho}{\partial z_j}\frac{\partial h}{\partial \bar{z}_k} + \rho\frac{\partial^2 h}{\partial z_j \partial \bar{z}_k}.$$

Therefore, evaluating at p yields

$$\left.\frac{\partial^2 \hat{\rho}}{\partial z_j \partial \bar{z}_k}\right|_p = \left[h\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} + \frac{\partial \rho}{\partial \bar{z}_k}\frac{\partial h}{\partial z_j} + \frac{\partial \rho}{\partial z_j}\frac{\partial h}{\partial \bar{z}_k}\right]_p$$

since $\rho(p) = 0$. So for $w \in T^{(1,0)}(\partial\Omega)$,

$$\begin{aligned}
\sum_{j,k=1}^{n} \left.\frac{\partial^2 \hat{\rho}}{\partial z_j \partial \bar{z}_k}\right|_p w_j \bar{w}_k &= \sum_{j,k=1}^{n} \left[h\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} + \frac{\partial \rho}{\partial \bar{z}_k}\frac{\partial h}{\partial z_j} + \frac{\partial \rho}{\partial z_j}\frac{\partial h}{\partial \bar{z}_k}\right]_p w_j \bar{w}_k \\
&= h(p) \sum_{j,k=1}^{n} \left.\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}\right|_p w_j \bar{w}_k + 2\mathrm{Re} \sum_{j,k=1}^{n} \left.\frac{\partial \rho}{\partial z_j}\frac{\partial h}{\partial \bar{z}_k}\right|_p w_j \bar{w}_k \\
&= h(p) \sum_{j,k=1}^{n} \left.\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}\right|_p w_j \bar{w}_k.
\end{aligned}$$

And finally, since $h(p) \neq 0$, we get that

$$\sum_{j,k=1}^{n} \left.\frac{\partial^2 \hat{\rho}}{\partial z_j \partial \bar{z}_k}\right|_p w_j \bar{w}_k = 0$$

if and only if

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}\bigg|_p w_j \bar{w}_k = 0.$$

Thus pseudoconvexity does not rely on the defining function. ■

Now there is also a notion of pseudoconvexity for domains which do not have a C^2 boundary. First we will need a few definitions.

Definition 2.4.8 *Let $\Omega \subset \mathbb{C}$ be an open set. A function $u : \Omega \to [-\infty, \infty)$ is called upper semicontinuous if*

$$u(a) \geq \limsup_{z \to a} u(z).$$

Definition 2.4.9 *Let $\Omega \subset \mathbb{C}$ be an open set. A function $u : \Omega \to [-\infty, \infty)$ is called subharmonic if u is upper semicontinuous and for each $a \in \Omega$ there is a neighborhood U of a such that*

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + Re^{i\theta})\, d\theta$$

whenever the closed disk $\{a + re^{i\theta} : 0 \leq r \leq R, 0 \leq \theta \leq 2\pi\} \subset U$.

There is, of course, an analog of subharmonic functions for several complex variables, called plurisubharmonic functions.

Definition 2.4.10 *For $p, v \in \mathbb{C}^n$, a complex line in $\mathbb{C}^n$ passing through p in the direction of v is given by $\{p + zv : z \in \mathbb{C}\}$*

Definition 2.4.11 *Let $\Omega \subset \mathbb{C}^n$ be an open set. A function $u : \Omega \to [-\infty, \infty)$ is called plurisubharmonic if u is upper semicontinuous and $u|_L : L \cap \Omega \to [-\infty, \infty)$ is subharmonic where L is any complex line passing through some point in Ω.*

Definition 2.4.12 *Let $\Omega \subset \mathbb{C}^n$ be an open set. A continuous function $u : \Omega \to \mathbb{R}$ is called an exhaustion function for Ω if for any $c \in \mathbb{R}$ the set*

$$\{z \in \Omega : u(z) \leq c\}$$

is compact in Ω.

Now we can define a notion of pseudoconvexity that does not require any smoothness of the boundary.

Definition 2.4.13 *Let $\Omega \subset \mathbb{C}^n$ be an open set. We say that Ω is pseudoconvex if it admits a continuous plurisubharmonic exhaustion function.*

For domains in $\mathbb{C}^n$ with C^2 boundary, the definition above and the definition of Levi pseudoconvexity are equivalent, see Gunning [12].

Now we show that pseudoconvexity is preserved under biholomorphisms.

Proposition 2.4.14 *Let $\Omega_1, \Omega_2 \subset \mathbb{C}^n$ and $\varphi : \Omega_1 \to \Omega_2$ a biholomorphism. Suppose Ω_1 is a pseudoconvex domain. Then Ω_2 is also a psuedoconvex domain.*

Proof. If Ω_1 is pseudoconvex then there is a continuous plurisubharmonic exhaustion function $u : \Omega_1 \to [-\infty, \infty)$. Then the composite $u \circ \varphi^{-1} : \Omega_2 \to [-\infty, \infty)$ is also a continuous plurisubharmonic function, see Gunning [12]. Furthermore, we can see that $u \circ \varphi^{-1}$ is also an exhaustion function for W_2. Thus, W_2 is pseudoconvex. ∎

We now discuss a couple examples of pseudoconvex domains.

Example 2.4.15 *The unit ball $B^2 \subset \mathbb{C}^2$.*

Consider the unit ball in $\mathbb{C}^2$. That is

$$B^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 - 1 < 0\}.$$

Then

$$\sum_{j,k=1}^{2} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k = |w_1|^2 + |w_2|^2,$$

which vanishes only when $w = (w_1, w_2) = (0, 0)$. Thus, every boundary point is strictly pseudoconvex.

Example 2.4.16 *A complex ellipsoid E_2.*

Let

$$E_2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^4 - 1 < 0\}.$$

Then

$$\sum_{j,k=1}^{2} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k = |w_1|^2 + 4|z_2|^2 |w_2|^2.$$

So for any boundary point of the form $p = (e^{i\theta}, 0) \in \partial E_2$, we must have $w_1 = 0$ and

$$\sum_{j,k=1}^{2} \left. \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} \right|_p w_j \bar{w}_k = 0.$$

These are then the points of weak pseudoconvexity. Every other boundary point is strictly pseudoconvex.

2.5 Variety Type

From our discussions in the previous sections, one might infer that, for $n \geq 2$, a domain $\Omega \subset \mathbb{C}^n$ may be characterized by boundary properties. Therefore, we will develop the notion of variety type in the sense of D'Angelo.

Definition 2.5.1 *Let $U \subset \mathbb{C}^n$. A subset $V \subset U$ is called a holomorphic variety if it is composed of the roots of a finite number of holomorphic functions. That is*

$$V = \{z \in U : f_1(z) = f_2(z) = \cdots = f_k(z) = 0\}$$

where f_i are holomorphic functions on U.

When a variety, V, is one (complex) dimensional, then it can be parameterized. See Gunning [13] for a precise statement of the local parameterization theorem. We state only what is necessary for our purposes.

Proposition 2.5.2 *If $V \subset \mathbb{C}^n$ is a one dimensional holomorphic variety and $p \in V$, then there is a neighborhood U of p and a holomorphic function, $f : \Delta \to \mathbb{C}^n$ with $f(0) = p$ and $f(\Delta) \subset U \cap V$.*

We often refer to a one dimensional holomorphic variety as a holomorphic disk or curve. When appropriate, we will refer to the image, $f(\Delta)$, as the holomorphic disk.

Given a smooth function $f : \mathbb{C} \to \mathbb{C}$ with $f(0) = 0$, we let $\nu(f)$ denote the order of vanishing of f at 0. If $g : \mathbb{C} \to \mathbb{C}^n$ is a smooth function with $g(0) = 0$ we let $\nu(g) = \min_i \nu(g_i)$, where $g = (g_1, ..., g_d)$.

Definition 2.5.3 *Let Ω be a smooth domain in $\mathbb{C}^n$ and $q \in \partial\Omega$. Let $\rho(z)$ be a defining function for Ω in a neighborhood of q. We say that $\partial\Omega$ is of finite type C in the sense of D'Angelo if*

$$\sup_f \left\{ \frac{\nu(\rho \circ f)}{\nu(f)} \right\} = C < \infty,$$

where f ranges through nonconstant holomorphic parameterizations of one dimensional holomorphic subvarieties of $\mathbb{C}^n$ with $f(0) = q$.

We say that $\partial\Omega$ is of finite line type L if

$$\sup_\ell \{\nu(\rho \circ \ell)\} = L < \infty,$$

where ℓ ranges through complex lines in $\mathbb{C}^n$ with $\ell(0) = q$.

Note that $\nu(\rho \circ \ell) \geq 2$ if and only if the image of ℓ is tangent to $\partial\Omega$ at q. So if we have a domain $\Omega \subset \mathbb{C}^n$ and a point $q \in \partial\Omega$ such that there is a holomorphic disk V passing through q, the D'Angelo (or variety) type of q is essentially a measurement of "how close" is V to actually lying in $\partial\Omega$. Now if $V \subset \partial\Omega$ then q would be a point of infinite type. However, this is not a necessary condition as we shall see in an example. When working with geometrically convex domains, one need only consider the line type rather than the more general variety type. This is due to McNeal, who showed the following proposition.

Proposition 2.5.4 (McNeal [21]) *Let $\Omega \subset \mathbb{C}^n$ be a convex domain with $q \in \partial\Omega$. Then q is a point of finite variety type if and only if it is of finite line type.*

The following examples are all convex sets, so we only consider the line type.

Example 2.5.5 *The unit ball B^n.*

The unit ball in $\mathbb{C}^2$ is given by

$$B^2 = \{(z_1, z_2) \in \mathbb{C}^2 : \rho(z_1, z_2) = |z_1|^2 + |z_2|^2 - 1 < 0\}.$$

Consider the point $(1, 0) \in \partial B^2$. The complex line tangent to ∂B^2 at $(1, 0)$ is given by the function $\ell : \mathbb{C} \to \mathbb{C}^n$ defined by

$$\ell(\zeta) = (1, \zeta).$$

So we get that

$$\rho \circ \ell(\zeta) = |\zeta|^2$$

and so we see that

$$\nu(\rho \circ \ell) = 2.$$

Thus, $(1, 0)$ is a point of finite type. Furthermore, since we can rotate the unit ball so that any boundary point is sent to $(1, 0)$, we have that every boundary point is of finite type.

Example 2.5.6 *The complex ellipsoid E_m.*

The complex ellipsoid (or egg domain) in $\mathbb{C}^2$ is given by

$$E_m = \{(z_1, z_2) \in \mathbb{C}^2 : \rho(z_1, z_2) = |z_1|^2 + |z_2|^{2m} - 1 < 0\},$$

where m is a positive integer. The point $(1, 0)$ lies on ∂E_m and the complex line tangent to ∂E_m at $(1, 0)$ is again the function

$$\ell(\zeta) = (1, \zeta).$$

Here we have

$$\rho \circ \ell(\zeta) = |\zeta|^{2m}$$

and so

$$\nu(\rho \circ \ell) = 2m.$$

Thus, $(1, 0)$ is a point of finite type.

Example 2.5.7 *The bidisk* Δ^2*.*

The bidisk in $\mathbb{C}^2$ is given by

$$\Delta^2 = \{(z_1, z_2) \in \mathbb{C}^2 : \rho_1(z_1, z_2) = |z_1|^2 - 1 < 0 \text{ and } \rho_2(z_1, z_2) = |z_2|^2 - 1 < 0\}.$$

Again $(1, 0) \in \partial\Delta^2$. In this case, a neighborhood of the complex line

$$\ell(\zeta) = (1, \zeta)$$

is contained in $\partial\Delta^2$ and

$$\rho_1 \circ \ell(\zeta) = 0.$$

Thus,

$$\nu(\rho \circ \ell) = \infty.$$

So we say $\partial\Delta^2$ is of infinite type at $(1, 0)$.

Now one might hope that a boundary point of a convex domain is infinite type only when there is a complex line passing through such a point which is contained in the boundary. However, this is not the case.

Example 2.5.8 *An exponentially flat domain, E_∞.*

Consider the domain

$$E_\infty = \{(z_1, z_2) \in \mathbb{C}^2 : \rho(z_1, z_2) = |z_1|^2 + 2\exp\left(-|z_2|^{-2}\right) - 1 < 0\}.$$

$(1, 0) \in \partial E_\infty$ and the complex line tangent to ∂E_∞ at $(1, 0)$ is again given by

$$\ell(\zeta) = (1, \zeta).$$

Then

$$\rho_1 \circ \ell(\zeta) = 2\exp\left(-|\zeta|^{-2}\right)$$

and so

$$\frac{2\exp\left(-|\zeta|^{-2}\right)}{|\zeta|^n} \longrightarrow 0$$

as $\zeta \to 0$ for all $n \geq 0$. So

$$\nu(\rho \circ \ell) = \infty.$$

Thus, $(1, 0)$ is a point of infinite type. Here, the complex line, ℓ, intersects ∂E_∞ only at $(1, 0)$, but $(1, 0)$ is still of infinite type.

2.6 Automorphism Orbits

For a domain $\Omega \subset \mathbb{C}^n$, the group of automorphisms will be denoted by $\mathrm{Aut}(\Omega)$. That is, $\mathrm{Aut}(\Omega)$ is the collection of biholomorphic self mappings of Ω. As the name suggests, $\mathrm{Aut}(\Omega)$ is a group under composition of functions. $\mathrm{Aut}(\Omega)$ is also a topological space under the compact-open topology. The automorphism group can yield information about the domain in question. One very important property is whether $\mathrm{Aut}(\Omega)$ is compact or not.

Definition 2.6.1 *A mapping $f : \Omega_1 \to \Omega_2$ between topological spaces is called proper if $f^{-1}(K)$ is compact in Ω_1 whenever K is compact in Ω_2. For bounded domains, Ω_1 and Ω_2, if $\{z_k\} \subset \Omega$ is a sequence such that $z_k \to q \in \partial\Omega_1$, then $f(z_k) \to p \in \partial\Omega_2$.*

If $f \in \mathrm{Aut}(\Omega)$, then f is a proper mapping since f^{-1} is continuous.

Definition 2.6.2 *Let G be a group and X a topological space. We say that G acts on X if there is a mapping $\sigma : G \times X \to X : (g, x) \mapsto gx$, with the property that if $e \in G$ is the identity element, then $\sigma(e, x) = x$ and if $g, h \in G$, then $\sigma(gh, x) = g(hx)$.*

Definition 2.6.3 *Let G be a group and X a topological space. Lex $x \in X$. The orbit of x under the action σ is the set*

$$\{y \in X : \sigma(g, x) = y \text{ for some } g \in G\}.$$

For domains $\Omega \subset \mathbb{C}^n$, $\mathrm{Aut}(\Omega)$ acts on Ω by the mapping $(\varphi, z) \mapsto \varphi(z)$. We can use the action of $\mathrm{Aut}(\Omega)$ on Ω to determine the compactness of $\mathrm{Aut}(\Omega)$.

Definition 2.6.4 *Let $\Omega \subset \mathbb{C}^n$ be a domain and $q \in \Omega$. We say p is an orbit accumulation point of $\mathrm{Aut}(\Omega)$ if there is a sequence $\{\varphi_k\} \subset \mathrm{Aut}(\Omega)$ such that $\varphi_k(q) \to p$. If $p \in \partial\Omega$ then we say p is a boundary orbit accumulation point for $\{\varphi_k\}$.*

H. Cartan showed that for a bounded domain, Ω, $\mathrm{Aut}(\Omega)$ is a Lie group which acts properly on Ω, see Narasimhan [22]. We can determine the compactness of $\mathrm{Aut}(\Omega)$ by examining the orbits.

Proposition 2.6.5 *Suppose $\Omega \subset \mathbb{C}^n$ is a bounded domain.* $\mathrm{Aut}(\Omega)$ *admits a boundary orbit accumulation point if and only if* $\mathrm{Aut}(\Omega)$ *is noncompact.*

Proof. Suppose there is some $q \in \Omega$ and a sequence $\{\varphi_k\} \subset \mathrm{Aut}(\Omega)$ such that

$$\varphi(q) \to p \in \partial\Omega.$$

Suppose, for a contradiction, that $\mathrm{Aut}(\Omega)$ is compact. Then there is a subsequence $\{\varphi_{k_j}\} \subset \{\varphi_k\}$ such that

$$\varphi_{k_j} \to \varphi \in \mathrm{Aut}(\Omega).$$

Now

$$\varphi(q) = \lim_{j\to\infty} \varphi_{k_j}(q) = p \in \partial\Omega.$$

This is a contradiction since φ is an automorphism. Thus, $\mathrm{Aut}(\Omega)$ is noncompact. Conversely, suppose that $\mathrm{Aut}(\Omega)$ is noncompact. Then there is a sequence $\{\varphi_k\} \subset \mathrm{Aut}(\Omega)$ such that $\varphi_k \to \varphi \notin \mathrm{Aut}(\Omega)$ as $k \to \infty$. Now it is a theorem of H. Cartan that either $\varphi \in \mathrm{Aut}(\Omega)$ or $\varphi(\Omega) \subset \partial\Omega$, see Narasimhan [22]. Therefore, for any $z \in \Omega$,

$$\lim_{k\to\infty} \varphi_k(z) = \varphi(z) \in \partial\Omega$$

and so $\mathrm{Aut}(\Omega)$ admits a boundary orbit accumulation point. ∎

We shall now discuss some examples of domains with noncompact automorphism groups.

Definition 2.6.6 *A domain $\Omega \subset \mathbb{C}^n$ is said to be homogeneous if it possesses a transitive automorphism group. That is, for all $z, w \in \Omega$, there is some $\varphi \in \mathrm{Aut}(\Omega)$ such that $\varphi(z) = w$.*

Example 2.6.7 *The unit disk Δ.*

By the Schwarz lemma we see that any automorphism of the unit is given by

$$\mathrm{Aut}(\Delta) = \left\{ \varphi(z) = e^{i\theta} \frac{z-a}{1-\bar{a}z} : |a| < 1, 0 \leq \theta < 2\pi \right\}.$$

Let $a, b \in \Delta$ and consider the automorphisms

$$\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$$

and

$$\varphi_{-b}(z) = \frac{z+b}{1+\bar{b}z}.$$

Then

$$\varphi_{-b} \circ \varphi_a(a) = \varphi_{-b}(0) = b,$$

Thus, Δ is homogeneous.

Example 2.6.8 *The polydisk $\Delta^n, n \geq 2$.*

The automorphism group of the polydisk is given by

$$\mathrm{Aut}(\Delta^n) = \left\{ \varphi(z_1, ..., z_n) = \left(e^{i\theta_1} \frac{z_{\sigma(1)} - a_1}{1 - \bar{a}_1 z_{\sigma(1)}}, ..., e^{i\theta_n} \frac{z_{\sigma(n)} - a_n}{1 - \bar{a}_n z_{\sigma(n)}} \right) \right\}$$

where σ is a permutation of $\{1, ..., n\}$, $a = (a_1, ..., a_n) \in \Delta^n$, and $\theta = (\theta_1, ..., \theta_n) \in [0, 2\pi)^n$. Note that the coordinate functions are simply the automorphisms of the unit disk in each coordinate after a permutation. Thus, one might assume that Δ^n is homogeneous, which is indeed true. Let $a = (a_1, ..., a_n), b = (b_1, ..., b_n) \in \Delta^n$ and consider the automorphisms

$$\varphi_a(z) = \left(\frac{z_1 - a_1}{1 - \bar{a}_1 z_1}, ..., \frac{z_n - a_n}{1 - \bar{a}_n z_n} \right)$$

and

$$\varphi_{-b}(z) = \left(\frac{z_1 + b_1}{1 + \bar{b}_1 z_1}, ..., \frac{z_n + b_n}{1 + \bar{b}_n z_n} \right).$$

Then

$$\varphi_{-b} \circ \varphi_a(a) = \varphi_{-b}(0) = b,$$

Thus, Δ^n is homogeneous.

Example 2.6.9 *The unit ball B^n.*

Firstly, a complex rotation is clearly an automorphism of the unit ball. Complex rotations are the linear maps that make up the group of unitary transformations, U_n, of $\mathbb{C}^n$. That is,

$$U_n = \{A \in \mathrm{Mat}_{n \times n}(\mathbb{C}) : A\bar{A}^t = I = \bar{A}^t A\}.$$

Also, for $a \in \Delta$, the mappings

$$\psi_a(z_1, ..., z_n) = \left(\frac{z_1 - a}{1 - \bar{a} z_1}, \frac{\sqrt{1 - |a|^2} z_2}{1 - \bar{a} z_1}, ..., \frac{\sqrt{1 - |a|^2} z_n}{1 - \bar{a} z_1} \right)$$

are automorphisms of B^n. Now the group of automorphisms of B^n is the group generated by these two types of mappings. That is,

$$\mathrm{Aut}(B^n) = \langle U_n \cup \{\psi_a : a \in \Delta\} \rangle .$$

Let $a = (a_1, ..., a_n), b = (b_1, ...b_n) \in B^n$. There are points $a' = (a'_1, 0, ..., 0), b' = (b'_1, 0, ..., 0) \in B^n$ such that $|a| = |a'|$ and $|b| = |b'|$. That is a' and b' are just the image of a and b under a rotation that sends each to lie on the z_1-axis. Let $\Phi, \Psi \in U_n$ be the maps such that

$$\Phi(a) = a'$$

and

$$\Psi(b') = b.$$

Recall from our example of the unit disk, Δ, that we have automorphisms $\varphi_{a'_1} : \Delta \to \Delta$ and $\varphi_{-b'_1} : \Delta \to \Delta$ such that

$$\varphi_{a'_1}(a'_1) = 0$$

and

$$\varphi_{-b'_1}(0) = b'_1.$$

Furthermore, we can write

$$\psi_{a'_1} = \left(\varphi_{a'_1}, \frac{\sqrt{1 - |a'_1|^2} z_2}{1 - \bar{a}'_1 z_1}, ..., \frac{\sqrt{1 - |a'_1|^2} z_n}{1 - \bar{a}'_1 z_1} \right)$$

and

$$\psi_{-b'_1} = \left(\varphi_{-b'_1}, \frac{\sqrt{1 - |-b'_1|^2} z_2}{1 + \bar{b}'_1 z_1}, ..., \frac{\sqrt{1 - |-b'_1|^2} z_n}{1 + \bar{b}'_1 z_1} \right).$$

Therefore,

$$\Psi \circ \psi_{-b'_1} \circ \psi_{a'_1} \circ \Phi(a) = \Psi \circ \psi_{-b'_1} \circ \psi_{a'_1}(a') = \Psi \circ \psi_{-b'_1}(0) = \Psi(b') = b.$$

Thus, B^n is homogeneous.

In the last three examples, we saw that Δ, Δ^n, and B^n all posses a transitive automorphism group. So why then are their automorphism groups noncompact? To answer this, we give the following proposition.

Proposition 2.6.10 *Let $\Omega \subset \mathbb{C}^n$ be a homogeneous domain. Then* $\operatorname{Aut}(\Omega)$ *is noncompact.*

Proof. Let $\{z_k\} \subset \Omega$ be a sequence so that $z_k \to p \in \partial\Omega$. Then since Ω is homogeneous, there is $\varphi_k \in \operatorname{Aut}(\Omega)$ such that $\varphi_k(z_k) = z_{k+1}$ for all $k \in \mathbb{N}$. Now let

$$\psi_k = \varphi_k \circ \cdots \circ \varphi_1 \in \operatorname{Aut}(\Omega).$$

Then

$$\lim_{k\to\infty} \psi_k(z_1) = p.$$

Thus, $\operatorname{Aut}(\Omega)$ admits a boundary orbit accumulation point and so, by proposition 2.6.5, $\operatorname{Aut}(\Omega)$ is noncompact. ■

Example 2.6.11 *The egg domain E_m.*

Recall

$$E_m = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2m} - 1 < 0\}.$$

The group of automorphisms of E_m is given by

$$\operatorname{Aut}(E_m) = \left\{ \psi_a(z_1, z_2) = \left(\frac{z_1 - a}{1 - \bar{a}z_1}, \left(\frac{\sqrt{1-|a|^2}}{1-\bar{a}z_1} \right)^{\frac{1}{m}} z_2 \right) : |a| < 1 \right\}.$$

The automorphism group of E_m is not transitive as in the previous examples. However $\operatorname{Aut}(E_m)$ is still noncompact. To see this, one just needs to take any sequence $\{a_j\} \subset \Delta$ such that $a_j \to -1$ as $j \to \infty$ and consider $\psi_{a_j} \in \operatorname{Aut}(E_m)$. Then for any $z \in E_m$,

$$\psi_{a_j}(z) = \left(\frac{z_1 - a_j}{1 - \bar{a}_j z_1}, \left(\frac{\sqrt{1-|a_j|^2}}{1-\bar{a}_j z_1} \right)^{\frac{1}{m}} z_2 \right) \longrightarrow (1, 0) \in \partial E_m.$$

So $(1, 0)$ is a boundary orbit accumulation point. Thus, $\operatorname{Aut}(E_m)$ is noncompact.

2.7 The Greene-Krantz Conjecture

We now have all the necessary machinery to state the Greene-Krantz conjecture.

Conjecture 2.7.1 (Greene/Krantz) *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with smooth C^∞ boundary. If $p \in \partial\Omega$ is a boundary orbit accumulation point for $\mathrm{Aut}(\Omega)$, then $\partial\Omega$ is of finite type at p.*

There are numerous results that support this conjecture. We list just a few.

Theorem 2.7.2 (Wong [26]) *Let $\Omega \subset \mathbb{C}^2$ be a bounded domain and $\{\varphi_j\} \subset \mathrm{Aut}(\Omega)$ such that*

1. *$\varphi_j(\Omega) \to W$ as $j \to \infty$ where $W \subset \partial\Omega$ is a one dimensional complex subvariety of $\mathbb{C}^2$,*
2. *W is contained in an open subset $U \subset \partial\Omega$ where $\partial\Omega$ is C^1 at U and there is an open set $V \subset \mathbb{C}^2$ such that $V \cap \partial\Omega = U$ and $V \cap \Omega$ is convex, and*
3. *There is a point $p \in \Omega$ such that $\varphi_j(p) \to q \in W$ nontangentially.*

Then Ω is biholomorphic to Δ^2.

In the above theorem, no global smoothness is assumed. One can see that the boundary could not be globally smooth for that would imply a contradiction that Ω is also biholomorphic to the ball B^2, see Wong [25].

Theorem 2.7.3 (Kim [16]) *Suppose that $\Omega \subset \mathbb{C}^2$ is a bounded convex domain with piecewise C^∞ smooth Levi flat boundary. If $\mathrm{Aut}(\Omega)$ is noncompact then Ω is biholomorphic to Δ^2.*

Notice that, to be bounded, convex, and piecewise smooth Levi flat, $\partial\Omega$ cannot be globally smooth. So while the above theorems of Wong and Kim do not imply the conjecture, we see that the boundary could have orbit accumulation points of infinite type when $\partial\Omega$ is not smooth.

Theorem 2.7.4 *Let M be a real analytic subvariety of $\mathbb{C}^n$. Then $p \in M$ is a point of finite type if and only if there does not exist a nontrivial holomorphic variety passing through p and lying in M.*

See D'Angelo [7] for a proof of the above theorem. The boundary of an arbitrary smoothly bounded domain $\Omega \subset \mathbb{C}^n$ could contain a point $p \in \partial\Omega$ of infinite type even though there is no holomorphic variety passing through p and lying in $\partial\Omega$. In this case $\partial\Omega$ would not be real analytic. Now it is a theorem of Diederich and Fornæss, see [8], that any compact real analytic subvariety of $\mathbb{C}^n$ contains no nontrivial complex analytic subvariety. Combining this with the above theorem yields that if a smoothly bounded domain, $\Omega \subset \mathbb{C}^n$, has a real analytic boundary, then $\partial\Omega$ must be of finite type.

Theorem 2.7.5 (Lee/Thomas/Wong [20]) *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded convex domain. Suppose there is a sequence $\{\varphi_j\} \subset \mathrm{Aut}(\Omega)$ such that $\varphi(z)$ converges nontangentially to some boundary point for all $z \in \Omega$. If $p \in \partial\Omega$ is an orbit accumulation point, then there does not exist any non trivial complex analytic variety passing through p and lying in $\partial\Omega$.*

In [14], Hamann shows that we can remove the nontangential requirement in the above theorem. Again, the nonexistence of a holomorphic variety contained in the boundary is not enough to show that a boundary point is of finite type, in general. Though, finite type would imply that there is no holomorphic variety in the boundary.

Finally, we state the well-known ball characterization theorem of Wong, which classifies strongly pseudoconvex domains with noncompact automorphism group.

Theorem 2.7.6 (Wong [25]) *If $\Omega \subset \mathbb{C}^n$ is a strongly pseudoconvex bounded domain with a noncompact automorphism group, then Ω is biholomorphic to the unit ball B^n.*

Chapter 3

The Kobayashi Pseudometric and Gromov Hyperbolicity

3.1 The Kobayashi Pseudometric

An important biholomorphic invariant is the Koboayashi pseudometric. We start with a chain of holomorphic disks.

Definition 3.1.1 *Let $z, w \in \mathbb{C}^n$. We say there is a chain of holomorphic disks from z to w if there exist $z = x_1, x_2, ..., x_{k+1} = w \in \Omega$ and analytic maps $\varphi_1, ..., \varphi_k : \Delta \to \Omega$ such that $x_i, x_{i+1} \in \varphi_i(\Delta)$ for $i = 1, ..., k$.*

In order to define the length of such a chain, we will use the Poincaré metric on the unit disk.

Definition 3.1.2 *Let $z \in \Delta$ and $v \in \mathbb{C}$. The Poincaré metric for the unit disk is defined* *by*

$$K_\Delta(z,v) = \frac{|v|}{1-|z|^2}.$$

The Poincaré metric is a complete metric on Δ. It yields a pseudodistance function given by

$$d_\Delta(z,\omega) = \tanh^{-1}\left|\frac{z-\omega}{1-z\bar{\omega}}\right| = \frac{1}{2}\log\left(\frac{|1-z\bar{w}|+|z-w|}{|1-z\bar{w}|-|z-w|}\right).$$

Definition 3.1.3 *For a chain, α, of holomorphic disks, the length of α is given by*

$$\ell(\alpha) = \sum_{i=1}^{k} d_\Delta(\varphi^{-1}(x_i), \varphi^{-1}(x_{i+1})).$$

Of course, we want to take a minimizing chain as our distance definition.

Definition 3.1.4 *The Kobayashi pseudodistance between z and w is then given by*

$$d_\Omega(z,w) = \inf_\alpha \ell(\alpha),$$

where α ranges over all chains of holomorphic disks from z to w.

Note that, using this definition, on the unit disk the Kobayashi distance coincides with the Poincaré distance. For our purposes, we will use an integral formula for the Kobayashi pseudodistance.

Definition 3.1.5 *Given $\Omega \subset \mathbb{C}^n$, $p \in \Omega$, and $v \in \mathbb{C}^n$, the Kobayashi pseudometric is given by*

$$K_\Omega(p,v) = \inf\{|\zeta| : f \in \mathrm{Hol}(\Delta,\Omega), f(0) = p, f'(\zeta) = v\}.$$

Again, the Poincaré metric coincides with this metric on Δ. Now we can also define an integrated pseudodistance using this infinitesimal pseudometric. Furthermore, Royden showed that the pseudodistances are equivalent.

Theorem 3.1.6 (Royden [23]) *The Kobayashi pseudodistance is given by*

$$d_\Omega(z,w) = \inf_\gamma \int_0^1 K_\Omega(\gamma(t), \gamma'(t))\ dt.$$

where $z, w \in \Omega$ and a curve $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = z, \gamma(1) = w$.

For convex domains that do not contain any complex lines, Barth [2] showed that the Kobayashi pseudodistance is an actual distance in the sense that $d_\Omega(z,w) > 0$ if $z \neq w$. The following example may allude to the fact that containing no complex lines is necessary.

Example 3.1.7 $d_{\mathbb{C}} \equiv 0$

For any $z, w \in \mathbb{C}$ there is a holomorphic map, $f : \Delta \to \mathbb{C}$, such that

$$f(0) = z$$

and

$$f(\epsilon) = w$$

where ϵ is arbitrarily small. To see this, one just needs to take a rotation and dilation of Δ so that $\epsilon \mapsto w - z$ and then translate 0 to z. Thus,

$$d_{\mathbb{C}}(z,w) = 0.$$

We now discuss some properties of the Kobayashi metric.

Definition 3.1.8 *We say that a subset $\Omega \subset \mathbb{C}^n$ is $\mathbb{C}$-proper if Ω does not contain any nontrivial complex affine lines.*

Definition 3.1.9 *Let $\Omega \subset \mathbb{C}^n$ be a $\mathbb{C}$-proper open set. For $z \in \Omega$ and $v \in \mathbb{C}^n$, let $L(z, v) \subset \mathbb{C}^n$ be the complex line passing through z in the direction of v. We set*

$$\delta_\Omega(z, v) = d_{Euc}(z, \partial\Omega \cap L(z, v))$$

and

$$\delta_\Omega(z) = d_{Euc}(z, \partial\Omega).$$

That is, $\delta_\Omega(z, v)$ is the Euclidean distance from z to $\partial\Omega$ in the complex direction of v and $\delta_\Omega(z)$ is the overall Euclidean distance from z to $\partial\Omega$.

Proposition 3.1.10 *Let U, V be domains in $\mathbb{C}^n$ and $f : U \to V$ be a holomorphic map. Then*

$$K_V(f(p), f'(v)) \leq K_U(p, v)$$

and

$$d_V(f(z), f(w)) \leq d_U(z, w).$$

Proof. Put $q = f(p)$ and $w = f'(v)$ and let $g \in \mathrm{Hol}(U, V)$ such that $g(0) = p$ and $g'(\zeta) = v$. Then

$$f \circ g(0) = f(p) = q$$

and

$$f' \circ g'(\zeta) = f'(v) = w.$$

Thus

$$\{|\zeta| : f \in Hol(\Delta, U), f(0) = p, f'(\zeta) = v\} \subset \{|\zeta| : f \in Hol(\Delta, V), f(0) = p, f'(\zeta) = v\}$$

and so

$$K_V(f(p), f'(v)) \leq K_U(p, v).$$

Then the second inequality is now clear since

$$\begin{aligned} d_V(f(z), f(w)) &= \inf_{\gamma} \int_0^1 K_V(f \circ \gamma(t), f' \circ \gamma'(t)) \ dt \\ &\leq \inf_{\gamma} \int_0^1 K_U(\gamma(t), \gamma'(t)) \ dt \\ &= d_U(z, w). \end{aligned}$$

■

Corollary 3.1.11 *If U, V are domains in $\mathbb{C}^n$ and $f : U \to V$ is a biholomorphism, then*

$$K_V(f(p), f'(v)) = K_U(p, v)$$

and

$$d_V(f(z), f(w)) = d_U(z, w).$$

Proof. Apply f^{-1} to the previous proposition. ■

We call this the *distance decreasing* property of holomorphic maps for the Kobayashi metric. A more precise term would be distance "nonincreasing" since equality may still hold. However "decreasing" is the standard term used in the literature. Another very useful property of the Kobayashi metric is upper and lower estimates. For general domains, we have an upper estimate.

Proposition 3.1.12 *Let $\Omega \subset \mathbb{C}^n$ be a domain, $z \in \Omega$, and $v \in \mathbb{C}^n$. Then*

$$K_\Omega(z, v) \leq \frac{||v||}{\delta_\Omega(z, v)}.$$

Proof. Let D be the largest open disk contained in $\{z + \mathbb{C}v\} \cap \Omega$. Then $\delta_\Omega(z,v) = \delta_D(z)$. Let r be the radius of D. Since translations, dialations, and rotations are biholomorphisms, we may assume $z = 0$, $v = (v_1, 0, ...0)$, and $D = \Delta$. Thus,

$$K_\Omega(z,v) \leq K_D(0,v) = \frac{|v_1|}{1} = \frac{|v_1|}{\delta_D(0)} = \frac{||v||}{\delta_\Omega(z,v)}.$$

■

Now when Ω is convex, we see that the Kobayashi metric also has a lower estimate. However, we will need some information about the Poincaré metric on the upper half plane $\mathcal{H} \subset \mathbb{C}$.

Definition 3.1.13 *For $z \in \mathcal{H}$ and $v \in \mathbb{C}$, the Poincaré metric for the upper half plane is defined by*

$$K_\mathcal{H}(z,v) = \frac{|v|}{2\text{Im}(z)}.$$

Since the Poincaré models for the unit disk and half plane are equivalent, we see that on $\mathcal{H}$ the Kobayashi and Poincaré metrics coincide. We also have an explicit distance function on $\mathcal{H}$ given by

$$d_\mathcal{H}(z,\omega) = \frac{1}{2}\cosh^{-1}\left(1 + \frac{|z-w|^2}{2\text{Im}(z)\text{Im}(w)}\right).$$

Proposition 3.1.14 *Let $\Omega \subset \mathbb{C}^n$ be a conxex domain, $z \in \Omega$, and $v \in \mathbb{C}^n$. Then*

$$K_\Omega(z,v) \geq \frac{||v||}{2\delta_\Omega(z,v)}.$$

Proof. Put $x \in \partial\Omega$ so that $\delta_\Omega(z,v) = d_{Euc}(z,x)$. By rotating and translating, we may assume $x = 0$, $z = z_1, 0, ..., 0)$, $v = (v_1, 0, ..., 0)$, and $\Omega \subset \{z \in \mathbb{C}^n : \text{Im}(z_1) > 0\}$. Let

$\pi : \mathbb{C}^n \to \mathbb{C}$ be the projection onto the first coordinate. Then

$$\begin{aligned} K_\Omega(z,v) &\geq K_{\pi(\Omega)}(z_1,v_1) \geq K_{\mathcal{H}}(z_1,v_1) \\ &= \frac{|v_1|}{2\mathrm{Im}(z_1)} \geq \frac{|v_1|}{2|z_1|} = \frac{||v||}{\delta_\Omega(z,v)}. \end{aligned}$$

■

On product domains, the Kobayashi metric has a well know nice property, see Kobayashi [17].

Proposition 3.1.15 *Let $U, V \subset \mathbb{C}^n$ be domains. Then for any $(u,v),(u',v') \in U \times V$,*

$$d_{U\times V}((u,v),(u',v')) = \max\{d_U(u,u'), d_V(v,v')\}.$$

3.2 Gromov Hyperbolocity

For bounded domains, the notion of Gromov hyperbolicity is related to the variety type of the boundary. Gromov hyperbolicity concerns geodesic triangles. Geodesics are just generalizations of straight lines in Euclidean space.

Definition 3.2.1 *Let (X, d) be a metric space. We say $\sigma : [a, b] \to X$ is a geodesic if*

$$d(\sigma(t), \sigma(s)) = |t - s|$$

for all $t, s \in [a, b]$. For $A \geq 1, B \geq 0$, σ is an (A, B)-quasigeodesic if

$$\frac{1}{A}|t - s| - B \leq d(\sigma(t), \sigma(s)) \leq A|t - s| + B$$

for all $t, s \in [a, b]$.

Just as with holomorphic disks, we sometimes refer to the image of σ as the geodesic or quasigeodesic.

Definition 3.2.2 *Let (X, d) be a proper geodesic metric space (that is, any closed ball under the metric d is compact and every two points is connected by a minimizing geodesic). Let $x, y, z \in X$, σ_{xy} (similarly σ_{yz} and σ_{zx}) be the geodesic segment from x to y (similarly y to z and z to x). We say that the geodesic triangle formed by these three points is δ thin if there is some $\delta > 0$ such that the δ neighborhood of $\sigma_{xy} \cup \sigma_{yz}$ conatains σ_{zx}. If there is some fixed $\delta > 0$ such that every geodesic triangle in X is δ thin, then we say that (X, d) is a Gromov hyperbolic metric space.*

We will also consider the Gromov product.

Definition 3.2.3 *Let (X,d) be a metric space and $x,y,z \in X$. The Gromov product of x and y at z, denoted $(x|y)_z$, is given by*

$$(x|y)_z = \frac{1}{2}(d(x,z) + d(z,y) - d(x,y)).$$

Due to the triangle inequality we see that the Gromov product is always nonnegative. The Gromov product can be used to determine Gromov hyperbolicity as well, see Buyalo and Schroeder [5].

Theorem 3.2.4 *A proper geodesic metric space (X,d) is Gromov hyperbolic if and only if there is a $\delta > 0$ such that for all $x,y,z,p \in X$,*

$$(x|y)_p \geq \min\{(x|z)_p, (z|y)_p\} - \delta.$$

Now in an arbitrary domain, $\Omega \subset \mathbb{C}^n$, under the Kobaysahi metric, geodesics are difficult to find in general. However, there are certain quasigeodesics that can be found relatively easier. A nice property of Gromov hyperbolic metric spaces is that every quasigeodesic is close to an actual geodesic, see Buyalo and Schroeder [5].

Proposition 3.2.5 *Let (X,d) be a Gromov hyperbolic metric space for some δ and $x,y \in X$. Fix some $A \geq 1, B \geq 0$ and let σ_{xy} be a geodesic and γ_{xy} be an (A,B)-quasigeodesic such that $\sigma_{xy}(s) = x = \gamma_{xy}(s')$ and $\sigma_{xy}(t) = y = \gamma_{xy}(t')$ for some $s,t,s',t' \in \mathbb{R}$. Then there is some constant $H = H(A,B,\delta) \geq 0$ such that σ_{xy} is contained in the H-neighborhood of γ_{xy}.*

Using the above proposition, we see that for any (A, B)-quasigeodesic segment, σ between two points, x and y, in a Gromov hyperbolic metric space (X, d), there is an actual geodesic segment between x and y which has a maximum distance H from σ where H is dependent on A, B, and δ. So if every geodesic triangle is δ thin then we can find an $M > 0$ such that every (A, B) quasigeodesic is M thin.

For bounded domains in $\mathbb{C}^n$, Gromov hyperbolicity in the Kobayashi metric is related to the variety type $\partial\Omega$. Of course, every bounded domain is trivially Gromov hyperbolic under the Euclidean metric. However, we have seen that for bounded convex domains, Ω, the distance from any point $x \in \Omega$ to $\partial\Omega$ is infinite. So it is feasible to assume there could be bounded domains which are not Gromov hyperbolic under the Kobayashi metric.

Example 3.2.6 *The bidisk, Δ^2, is not Gromov hyperbolic under the Kobayashi metric.*

In fact, for any bounded convex domain, $\Omega \subset \mathbb{C}^n$, such that $\partial\Omega$ contains a complex affine disk, Ω is not Gromov hyperbolic under the Kobayashi metric. The key to the proof of this fact is that we can find a sequence of (A, B)-quasigeodesics which are parallel to the affine disk which is contained in the boundary. Furthermore, as we shall see in the sequel, we can parameterize certain line segments, that end at the boundary, to be (A, B)-quasigeodesics as well. Now this forms a sequence of geodesic triangles where one side converges uniformly to $\partial\Omega$ and since we have seen that $\partial\Omega$ is infinitely far away from any other point in Ω, under the Kobayashi metric, then this essentially gives us that we can find a quasigeodesic triangle that is not M-thin for any $M > 0$.

For smoothly bounded convex sets, there is an open neighborhood of the boundary, where the real normal lines to the boundary can be parameterized as quasigeodesics. First, we need a global estimation of the Kobayashi distance in terms of supporting hyperplanes for convex subset of $\mathbb{C}^n$.

Lemma 3.2.7 *Suppose $\Omega \subset \mathbb{C}^n$ is a convex open set and $H \subset \mathbb{C}^n$ is a complex hyperplane such that $H \cap \Omega = \emptyset$. Then for all $z, w \in \Omega$*

$$d_\Omega(z, w) \geq \frac{1}{2} \left| \log \frac{d_{Euc}(H, z)}{d_{Euc}(H, w)} \right| .$$

Proof. Since Ω is convex there is a real hyperplane $H_\mathbb{R}$ such that $H \subset H_\mathbb{R}$ and $H_\mathbb{R} \cap \Omega = \emptyset$. We may assume

$$H_\mathbb{R} = \{(z_1, ..., z_n) \in \mathbb{C}^n : \mathrm{Im}(z_1) = 0\},$$

$$\Omega \subset \{(z_1, ..., z_n) \in \mathbb{C}^n : \mathrm{Im}(z_1) > 0\},$$

$$H = \{(0, z_2, ..., z_n) \in \mathbb{C}^n\}.$$

Let $P : \mathbb{C}^n \to \mathbb{C}$ be the projection onto the first coordinate. Then $P(\Omega) \subset \mathcal{H}$ which implies

$$d_\Omega(w.z) \geq d_{P(\Omega)}(P(w), P(z)) \geq d_\mathcal{H}(P(w), P(z)).$$

Also

$$\begin{aligned} d_\mathcal{H}(w, z) &= \frac{1}{2} \cosh^{-1} \left(1 + \frac{|w - z|^2}{2|w||z|} \right) \\ &\geq \frac{1}{2} \cosh^{-1} \left(\frac{|w|}{2|z|} + \frac{|z|}{2|w|} \right) \\ &= \frac{1}{2} \log \left(\frac{|w|}{|z|} \right) \end{aligned}$$

and the fact that $|P(z)| = d_{Euc}(H, z)$ gives us

$$d_\Omega(w, z) \geq \frac{1}{2} \left| \log \frac{d_{Euc}(H, z)}{d_{Euc}(H, w)} \right|.$$

■

Definition 3.2.8 *For a bounded domain $\Omega \subset \mathbb{C}^n$ with C^1 boundary and a point $x \in \partial\Omega$, we denote by n_x the inward pointing unit normal vector to $\partial\Omega$ at x.*

Proposition 3.2.9 *Suppose $\Omega \subset \mathbb{C}^n$ is a bounded convex open set with C^∞ boundary. Then there exists $\epsilon, K > 0$ such that if $x \in \partial\Omega$ then the curve $\sigma_x : \mathbb{R}_{\geq 0} \to \Omega$ given by*

$$\sigma_x(t) = x + e^{-2t}\epsilon n_x$$

is a $(1, K)$ quasigeodesic with respect to the Kobayashi metric.

Proof. Since $\partial\Omega$ is smooth, there is a disk, $D \subset \mathbb{C}$, centered at some $p \in \mathbb{C}$ on the positive real axis with radius p such that the image of the function $\varphi_x : D \to \mathbb{C}^n$ defined by $\varphi_x(z) = x + zn_x$ is contained in Ω. Define $\psi : D \to \Delta$ by

$$\psi(z) = \frac{p - z}{p}.$$

Then ψ is a biholomorphism with $\psi(\mathbb{R} \cap D) = \mathbb{R} \cap \Delta$ and

$$0 \leq 1 - ct = \psi(t) \leq 1 - \frac{t}{c}$$

where $c = p^{-1}$ and $t \in [0, p]$. Now $d_\Delta(z, w) = \frac{1}{2}\log\left(\frac{|1-z\bar{w}|+|z-w|}{|1-z\bar{w}|-|z-w|}\right)$, so for $0 < a < b \leq p$

$$\begin{aligned} d_D(a,b) &= d_\Delta(\psi(a), \psi(b)) \\ &= \frac{1}{2}\log\left(\frac{(1+\psi(a))(1-\psi(b))}{(1-\psi(a))(1+\psi(b))}\right) \\ &\leq \frac{1}{2}\log\left(\frac{(1+\psi(a))(1-\psi(b))}{(1-\psi(a))}\right) \\ &\leq \log\sqrt{2}c + \log\frac{b}{a}. \end{aligned}$$

Then since $d_D(e^{-2t}\epsilon, e^{-2s}\epsilon) = d_\Delta(\psi(e^{-2t}\epsilon), \psi(e^{-2s}\epsilon))$ we have

$$\begin{aligned} d_\Omega(\sigma_x(t), \sigma_x(s)) &= d_\Omega(\varphi_x(e^{-2t}\epsilon), \varphi_x(e^{-2s}\epsilon)) \\ &\leq d_D(e^{-2t}\epsilon, e^{-2s}\epsilon) \\ &\leq \log\sqrt{2}c + \left|\log\frac{e^{-2t}\epsilon}{e^{-2s}\epsilon}\right| \\ &= \log\sqrt{2}c + |t-s|. \end{aligned}$$

Also,

$$\begin{aligned} d_\Omega(\sigma_x(t), \sigma_x(s)) &\geq \frac{1}{2}\left|\log\frac{d_{Euc}(H_x, \sigma_x(t))}{d_{Euc}(H_x, \sigma_x(s))}\right| \\ &= \frac{1}{2}\left|\frac{e^{-2t}\epsilon}{e^{-2s}\epsilon}\right| \\ &= |t-s| \end{aligned}$$

where H_x is the complex tangent hyperplane to $\partial\Omega$ at x. Thus, σ_x is a $(1, \log\sqrt{2}c)$ quasi-geodesic. ■

Chapter 4

Iterated Orbit Accumulation

4.1 Nontangential Convergence

The direction of travel of an automorphism orbit can yield certain conclusions. Nontangential convergence provides us with useful properties.

Definition 4.1.1 *For a domain $\Omega \subset \mathbb{C}^n$ with C^1 boundary, a sequence $\{q_j\} \subset \Omega$, and a point $q \in \partial\Omega$, we say that $q_j \to q$* nontangentially *if for all j large enough*

$$q_j \in \Gamma_\alpha(q) = \{z \in \Omega : ||z - q|| \leq \alpha\delta_\Omega(z)\}$$

for some $\alpha > 1$. We say that $q_j \to q$ normally if the q_j's approach q along the real normal line to $\partial\Omega$ at q.

Lemma 4.1.2 *Let $\Omega \subset \mathbb{C}^n$ be a convex domain with C^1 boundary. Let $z \in \Omega$ and $q' = q + tn_q$ for some $t > 0$. Then*

$$\Gamma_\alpha(q) \subset \left\{z \in \Omega : 0 \leq \angle zqq' \leq \arccos\left(\frac{1}{\alpha}\right)\right\}.$$

Proof. Put $H = \{z \in \mathbb{C}^n : \mathrm{Im}(z_1) > 0\}$. We may assume $q = 0$, $n_q = (i, 0, ..., 0)$, and $\Omega \subset H$. Then $\delta_\Omega(z) \leq \delta_H(z) = \mathrm{Im}(z_1)$ which implies that $||z - q|| \leq \alpha \mathrm{Im}(z_1) = \alpha ||(\mathrm{Im}(z_1), 0, ..., 0)||$. Then since

$$\cos(\angle zqq') = \frac{||(\mathrm{Im}(z_1), 0, ..., 0)||}{||z - q||},$$

we have $\angle zqq' \leq \arccos(1/\alpha)$. ■

When $\partial\Omega$ admits a nontangential orbit accumulation point, Lee, Thomas, and Wong [20] showed that there is a sequence of points $\{p_j\} \subset \Omega$, within some fixed Kobayashi distance from $p \in \Omega$, such that the action of the sequence of automorphisms $\{\varphi_j\} \subset \mathrm{Aut}(\Omega)$ on the respective p_j's approaches the accumulation point $q \in \partial\Omega$ along the real normal line to the boundary at q. To be precise:

Lemma 4.1.3 *Let $\Omega \subset \mathbb{C}^n$ be a convex domain with C^1 boundary. Suppose $\{\varphi_j\} \subset Aut(\Omega)$ and $\varphi_j(p) \to q \in \partial\Omega$ nontangentially for some $p \in \Omega$. Then there exists $\{p_j\} \subset \Omega$ such that $\varphi_j(p_j) \to q$ normally and $d_\Omega(p, p_j) \leq r$ for some $r > 0$.*

Proof. Let $\ell_q = \{q + tn_q : t \in \mathbb{R}\}$ and define $\pi : \mathbb{C}^n \to \ell_q$ as the projection mapping onto ℓ_q. Put $q_j = \varphi_j(p)$, $\tilde{q}_j = \pi(q_j)$, and $p_j = \varphi^{-1}(\tilde{q}_j)$. Then $\tilde{q}_j \to q$ normally and $||\tilde{q}_j - q_j|| \leq ||q_j - q||$. Now by lemma 4.1.2

$$\frac{1}{\alpha} \leq \cos(\angle zqq') = \frac{||\tilde{q}_j - q||}{||q_j - q||}.$$

Let $\gamma(t) = (1-t)q_j + t\tilde{q}_j$. Then

$$\begin{aligned}
d_\Omega(p, p_j) &= d_\Omega(q_j, \tilde{q}_j) \\
&\leq \int_0^1 K_\Omega(\gamma(t), \gamma'(t))\ dt \\
&\leq \int_0^1 \frac{||\gamma'(t)||}{\delta_\Omega(\gamma(t), \gamma'(t))}\ dt \\
&\leq \int_0^1 \frac{||\gamma'(t)||}{\delta_\Omega(\gamma(t))}\ dt \\
&\leq \int_0^1 \frac{||\gamma'(t)||\alpha}{||\gamma(t) - q||}\ dt \\
&\leq \frac{||\tilde{q}_j - q_j||\alpha}{||\tilde{q}_j - q||} \\
&\leq \frac{||q_j - q||\alpha}{||\tilde{q}_j - q||} \\
&\leq \alpha^2.
\end{aligned}$$

Finally, we let $r = \alpha^2$. ∎

Essentially, this gives us that the Kobayashi distance from each $\varphi_j(p)$ to the real normal line of the boundary at $q \in \partial\Omega$ remains bounded by a fixed constant.

4.2 A Class of Holomorphic Mappings

We will need to know that a certain class of holomorphic mappings from the bi-disk to smoothly bounded convex subsets of $\mathbb{C}^n$ cannot exist. Much of this section is a modification of an argument of Zimmer, see [28], which was originally under the assumption that Ω had a $C^{1,\alpha}$ boundary. However, we will only consider domains with C^∞ boundary.

For convex sets, we also have an estimation of the Kobayashi distance, see Abate [1], for two points sufficiently away from each other. In this section, we denote by H_x to be the complex hyperplane in $\mathbb{C}^n$ that is tangent to $\partial\Omega$ at $x \in \partial\Omega$.

Lemma 4.2.1 *Suppose $\Omega \in \mathbb{C}^n$ is a bounded convex set with C^2 boundary and $x, y \in \partial\Omega$ with $H_x \neq H_y$. Then there are $\epsilon > 0$ and $C \in \mathbb{R}$ such that*

$$d_\Omega(p,q) \geq \frac{1}{2}\log\frac{1}{\delta_\Omega(p)} + \frac{1}{2}\log\frac{1}{\delta_\Omega(q)} - C$$

for all $p, q \in \Omega$ with $d_{Euc}(p, H_x), d_{Euc}(q, H_y) \leq \epsilon$.

Lemma 4.2.2 *Suppose $\Omega \subset \mathbb{C}^n$ is a bounded convex open set with C^∞ boundary, $o \in \Omega$, and $\{p_j\}, \{q_k\} \subset \Omega$ are sequences such that $p_j \to x \in \partial\Omega$ and $q_k \to y \in \partial\Omega$. If*

$$\limsup_{j,k\to\infty} (p_j|q_k)_o = \infty,$$

then $H_x = H_y$.

Proof. By proposition 3.2.9 there exists $1 \geq \epsilon > 0$ and $K > 0$ so that the function $\sigma_z : \mathbb{R}_{\geq 0} \to \Omega$ given by $\sigma_z(t) = z + e^{-2t}\epsilon n_z$ is a $(1, K)$ quasigeodesic for all $z \in \partial\Omega$. We can pick $x_j, y_j \in \partial\Omega$ and $t_j, s_j \in \mathbb{R}$ such that $p_j = \sigma_{x_j}(t_j)$ and $q_j = \sigma_{y_j}(s_j)$ where $t_j, s_j \to \infty$ as

$j \to \infty$. Also there is some $R \geq 0$ such that $d_\Omega(\sigma_z(0), o) \leq R$ for all $z \in \partial\Omega$. Now suppose, by way of contradiction, that

$$H_x \neq H_y.$$

Notice that

$$t_j = \frac{1}{2} \log \frac{\epsilon}{\delta_\Omega(p_j)} \leq \frac{1}{2} \log \frac{1}{\delta_\Omega(p_j)}.$$

Thus

$$\begin{aligned} d_\Omega(o, p_j) &\leq d_\Omega(o, \sigma_{x_j}(0)) + d_\Omega(\sigma_{x_j}(0), p_j) \\ &\leq R + t_j + K \\ &\leq R + K + \frac{1}{2} \log \frac{1}{\delta_\Omega(p_j)}. \end{aligned}$$

Similarly,

$$d_\Omega(o, q_j) \leq R + K + \frac{1}{2} \log \frac{1}{\delta_\Omega(q_j)}.$$

Also, by lemma 4.2.1, there is a $C \in \mathbb{R}$ such that for n large enough

$$d_\Omega(p_j, q_j) \geq \frac{1}{2} \log \frac{1}{\delta_\Omega(p_j)} + \frac{1}{2} \log \frac{1}{\delta_\Omega(q_j)} - C.$$

Let $C' = \max\{R + K, C\}$. Then

$$2(p_j|q_j)_o \leq C' + \frac{1}{2} \log \frac{1}{\delta_\Omega(p_j)} + C' + \frac{1}{2} \log \frac{1}{\delta_\Omega(q_j)} - \frac{1}{2} \log \frac{1}{\delta_\Omega(p_j)} - \frac{1}{2} \log \frac{1}{\delta_\Omega(q_j)} + C' = 3C'$$

which implies that

$$\limsup_{n \to \infty} (p_j|q_j)_o \leq \frac{3}{2} C'$$

and hence, a contradiction. Therefore we must have $H_x = H_y$. ■

Definition 4.2.3 *For a set $A \subset \mathbb{C}^n$, let $\mathcal{N}_\epsilon(A)$ be the ϵ-neighborhood of A under the standard Euclidean distance. The Hausdorff distance between bounded sets is given by*

$$d_H(A,B) = \inf\{\epsilon > 0 : A \subset \mathcal{N}_\epsilon(B) \text{ and } B \subset \mathcal{N}_\epsilon(A)\}.$$

We say that a sequence $\{A_j\} \subset 2^{\mathbb{C}^n}$ converges to $A \subset \mathbb{C}^n$ in the local Hausdorff topology if

$$\lim_{j\to\infty} d_H(A_j \cap B_R(0), A \cap B_R(0)) = 0$$

for all $R > 0$.

For a sequence $\{A_j\} \subset 2^{\mathbb{C}^n}$ with $u_j \in A_j$ and a set $A \subset \mathbb{C}^n$ with $u \in A$ we denote by

$$(A_j, u_j) \to (A, u)$$

to mean that $A_j \to A$ in the local Hausdorff topology and $u_j \to u$.

The following result of Frankel will be used in the main propositions of this section. It allows us to work in lower dimensions when concerning applications of affine transformations. $\mathrm{Aff}(V)$ denotes the set of complex affine transformations on some $V \subset \mathbb{C}^n$.

Lemma 4.2.4 (Frankel [9]) *Suppose $\Omega \subset \mathbb{C}^n$ is a $\mathbb{C}$-proper convex open set. If $V \subset \mathbb{C}^n$ is a complex affine subspace intersecting Ω and $\{A_j\} \subset \mathrm{Aff}(V)$ is a sequence of affine transformations such that $A_j(\Omega \cap V)$ converges in the local Hausdorff topology to a $\mathbb{C}$-proper convex open set $\hat{\Omega}_V \subset V$, then there exists affine maps $B_j \in \mathrm{Aff}(\mathbb{C}^n)$ such that $B_j\Omega$ converges in the local Hausdorff topology to a $\mathbb{C}$-proper convex open set $\hat{\Omega}$ with $\hat{\Omega} \cap V = \Omega \cap V$.*

We can, under a sequence of affine transformations, send a domain with a boundary point of infinite type to a $\mathbb{C}$-proper domain (not necessarily bounded) such that the boundary contains a holomorphic disk. Note that proposition 4.2.5 below only concerns domains in $\mathbb{C}^2$. We will apply lemma 4.2.4 for domains in $\mathbb{C}^n$.

Proposition 4.2.5 *Suppose $\Omega \subset \mathbb{C}^2$ is a $\mathbb{C}$-proper convex open set with $0 \in \partial\Omega$ and*

$$\Omega \cap \mathcal{O} = \{(x+iy, z) \in \mathcal{O} : y > f(x,z)\}$$

where $\mathcal{O}$ is a neighborhood of 0 and $f : \mathbb{R} \times \mathbb{C} \to \mathbb{R}$ is a convex nonnegative function such that $t \mapsto f(t,0)$ is C^1 at $t = 0$. Suppose further that there is no nontrivial holomorphic disk contained in the boundary and $0 \in \partial\Omega$ is a point of infinite type. Then there exists $t_j \to \infty$ and complex affine maps A_j such that

$$A_j(\Omega, (ie^{-t_j}, 0)) \to (\hat{\Omega}, (i, 0)),$$

where

$$\hat{\Omega} \cap (\mathbb{C} \times \{1\}) = \varnothing,$$

$$\mathcal{H} \times \Delta \subset \hat{\Omega} \subset \{(z,w) \in \mathbb{C}^2 : \mathrm{Im}(z) > 0\},$$

$\hat{\Omega}$ is $\mathbb{C}$-proper and $\partial\hat{\Omega}$ contains a nontrivial holomorphic disk.

Proof. We may assume $\mathcal{O} = (V + iW) \times U$ where $V, W \subset \mathbb{R}$ and $U \subset \mathbb{C}$ are neighborhoods of 0 and by rescaling $B_1(0,0) \subset U$. Since $0 \in \partial\Omega$ is of infinite type, then for all $j > 0$

$$\lim_{z \to 0} \frac{f(0,z)}{|z|^j} = 0.$$

So we can find some $a_j \to 0$ and $z_j \in \mathbb{C}$ with $|z_j| < 1$ such that $f(0, z_j) = a_j |z_j|^j$ and for any $w \in \mathbb{C}$ with $|w| < |z_j|$ we have $f(0,w) \leq a_j |w|^j$. Since there is no nontrivial holomorphic disk contained in the boundary, we may assume that $f(0,z_j)\neq 0$ and $z_j, f(0,z_j)\to 0$. Define

$$A_j = \begin{pmatrix} f(0,z_j)^{-1} & 0 \\ 0 & z_j^{-1} \end{pmatrix}$$

and let $\Omega_j = A_j\Omega$. By possibly passing to a subsequence, we see that there is a $t_j\to\infty$ such that $e^{-t_j} = f(0,z_j)$ which gives us

$$A_j(ie^{-t_j},0)\to(i,0).$$

Now there are $\epsilon_1,\epsilon_2>0$ such that $B_{\epsilon_1}(\epsilon_2 i,0)\subset\Omega_j$ for all j and since, for any $R>0$, $\{\Omega' : \Omega' \text{ is open, convex, and } B_{\epsilon_1}(\epsilon_2 i,0)\subset\Omega'\subset B_R(0,0)\}$ is compact in the local Hausdorff topology, then we may pass to a subsequence such that $\overline{\Omega_j}$ converges to a closed convex set $C\subset\{(z,w)\in\mathbb{C} : \mathrm{Im}(z)\geq 0\}$. Since $t\mapsto f(t,0)$ is C^1 at $t=0$ then we must have

$$\mathcal{H}\times\{0\}\subset C.$$

Put $\mathcal{O}_j = A_j\mathcal{O}$. Then

$$\Omega_j\cap\mathcal{O}_j = \{(x+iy,z) : x\in V_j, z\in U_j, y>f_j(x,z)\}$$

where $V_j = f(0,z_j)^{-1}V$, $U_j = z_j^{-1}U$, and $f_j(x,z) = f(0,z_j)^{-1}f(f(0,z_j)x, z_jz)$. Now for $|w|<1$

$$\begin{aligned} f_j(0,w) &= \frac{f(0,z_jw)}{f(0,z_j)} \\ &\leq \frac{a_j|z_j|^j|w|^j}{f(0,z_j)} \\ &= |w|^j\to 0 \end{aligned}$$

as $j \to \infty$, and so

$$\{0\} \times \Delta \subset \partial C.$$

Since C is convex, $\mathcal{H} \times \Delta \subset C$. Let $\hat{\Omega}$ be the interior of C. Then Ω_j converges to $\hat{\Omega}$, where there is a holomorphic disk contained in $\partial\hat{\Omega}$, and

$$\mathcal{H} \times \Delta \subset \hat{\Omega} \subset \{(z, w) \in \mathbb{C}^2 : \mathrm{Im}(z) > 0\}.$$

Notice that $f_j(0, 1) = 1$ for all j which implies $(i, 1) \in \partial\hat{\Omega}$. So $(i, 1), (0, 1) \in \partial\hat{\Omega}$ which gives us that $\mathcal{H} \times \{1\} \subset \partial\hat{\Omega}$ and so

$$\hat{\Omega} \cap (\mathbb{C} \times \{1\}) = \varnothing.$$

Therefore any affine map $z \mapsto (a_1, a_2)z + b$ with its image in $\hat{\Omega}$ must have $a_1 = a_2 = 0$. Thus, $\hat{\Omega}$ is $\mathbb{C}$-proper. ■

Proposition 4.2.6 *Suppose $\hat{\Omega} \subset \mathbb{C}^n$, $n \geq 2$, is a $\mathbb{C}$-proper convex open set such that $\mathcal{H} \times \Delta \times \{(0, ..., 0)\} \subset \hat{\Omega} \subset \{z \in \mathbb{C}^n : \mathrm{Im}(z_1) > 0\}$ and $\mathbb{C} \times \{(1, 0, ..., 0)\} \cap \hat{\Omega} = \varnothing$. Then there exist a holomorphic map $f : \Delta \times \Delta \to \hat{\Omega}$ such that*

1. *for all $z, w \in \Delta$*

$$d_{\hat{\Omega}}(f(z, 0), f(w, 0)) = d_{\Delta}(z, w),$$

2. *for all $s, t \geq 0$*

$$|t - s| - \log\sqrt{2} \leq d_{\Omega}(f(0, \tanh(t)), f(0, \tanh(s))) \leq |t - s|$$

Proof. Firstly, we claim that the map $g : \mathcal{H} \to \hat{\Omega}$ given by $g(z) = (z, 0, ..., 0)$ induces an isometric embedding $(\mathcal{H}, d_{\mathcal{H}}) \to (\hat{\Omega}, d_{\hat{\Omega}})$. We have $d_{\hat{\Omega}}(g(z), g(w)) \leq d_{\mathcal{H}}(z, w)$ by the distance decreasing property of holomorphic maps. Let $P : \mathbb{C}^n \to \mathbb{C}$ be the projection onto the first coordinate. Since

$$\hat{\Omega} \subset \{(z_1, ..., z_n) \in \mathbb{C}^n : \mathrm{Im}(z_1) > 0\}$$

and

$$\mathcal{H} \times \{(0, ..., 0)\} = \hat{\Omega} \cap (\mathbb{C} \times \{(0, ..., 0)\})$$

then $P(\Omega) = \mathcal{H}$ and $P \circ g = \mathrm{Id}$, then $d_{\mathcal{H}}(z, w) = d_{\mathcal{H}}(P(g(z)), P(g(w))) \leq d_{\hat{\Omega}}(g(z), g(w))$ and so

$$d_{\hat{\Omega}}(g(z), g(w)) = d_{\mathcal{H}}(z, w).$$

Now define $f : \Delta \times \Delta \to \hat{\Omega}$ by

$$f(z, w) = \left(i\frac{1+z}{1-z}, w, 0, ..., 0 \right).$$

Then for all $z, w \in \Delta$

$$\begin{aligned} d_{\Delta}(z, w) &= d_{\mathcal{H}} \left(i\frac{1+z}{1-z}, i\frac{1+w}{1-w} \right) \\ &= d_{\hat{\Omega}} \left(\left(i\frac{1+z}{1-z}, 0, ..., 0 \right), \left(i\frac{1+w}{1-w}, 0, ..., 0 \right) \right) \\ &= d_{\hat{\Omega}}(f(z, 0), f(w, 0)). \end{aligned}$$

Now put $x_t = f(0, \tanh(t)) = (i, \tanh(t), 0, ..., 0)$. Then

$$\begin{aligned} d_{\hat{\Omega}}(x_t, x_s) &\leq d_{\Delta \times \Delta}((0, \tanh(t)), (0, \tanh(s)) \\ &\leq d_{\Delta}(\tanh(t), \tanh(s)) \\ &= |t - s|. \end{aligned}$$

Consider the complex line $L = \mathbb{C} \times \{(1, 0, ..., 0)\}$. Then $L \cap \hat{\Omega} = \varnothing$ so there is a complex hyperplane, H, with $L \subset H$ and $H \cap \hat{\Omega} = \varnothing$. So

$$\begin{aligned} d_{\hat{\Omega}}(x_t, x_s) &\geq \frac{1}{2}\left|\log \frac{d_{Euc}(H, x_t)}{d_{Euc}(H, x_s)}\right| \\ &= \frac{1}{2}\left|\log \frac{d_{Euc}(L, x_t)}{d_{Euc}(L, x_s)}\right| \\ &= \frac{1}{2}\left|\log \frac{1 - \tanh(t)}{1 - \tanh(s)}\right|. \end{aligned}$$

Finally, using the fact that $\tanh(x) = 1 - \frac{2}{e^{2x}+1}$ for $x \in \mathbb{R}$ we get

$$\begin{aligned} d_{\hat{\Omega}}(x_t, x_s) &\geq \frac{1}{2}|\log(e^{2t} + 1) - \log(e^{2s} + 1)| \\ &\geq \frac{1}{2}(|\log(e^{2t}) - \log(e^{2s})| - \log(1 + e^{-2s})) \\ &\geq |t - s| - \log \sqrt{2}. \end{aligned}$$

■

Now for a smoothly bounded domain, Ω, we see that the mappings above cannot have Ω as a codomain.

Proposition 4.2.7 *Suppose $\Omega \subset \mathbb{C}^n$ is a bounded convex open set with C^∞ boundary. Then there does not exist a holomorphic map $f : \Delta \times \Delta \to \Omega$ and $c \geq 0$ such that*

1. *for all $z, w \in \Delta$*

$$d_\Delta(z, w) - c \leq d_\Omega(f(z, 0), f(w, 0)) \leq d_\Delta(z, w) + c,$$

2. *for all $s, t \geq 0$*

$$|t - s| - c \leq d_\Omega(f(0, \tanh(t)), f(0, \tanh(s))) \leq |t - s| + c$$

Note that if $f : \Delta^2 \to \Omega$ is holomorphic and induces an isometric embedding of (Δ^2, d_{Δ^2}) into (Ω, d_Ω) then

$$d_\Omega(f(0, \tanh(t)), f(0, \tanh(s))) = |t - s|.$$

Proof. Suppose there is a function $f : \Delta \times \Delta \to \Omega$ with the properties above. Then

$$\lim_{t\to\infty} d_\Omega(f(\tanh(t)e^{i\theta}, 0), f(0, 0)) \geq \lim_{t\to\infty} d_\Delta(\tanh(t)e^{i\theta}, 0) - c = \lim_{t\to\infty} t - c = \infty.$$

Also

$$\lim_{t\to\infty} d_\Omega(f(0, \tanh(t)), f(0, 0)) \geq \lim_{t\to\infty} t - c = \infty.$$

Now put

$$\sigma_{\theta,j} = f(\tanh(j)e^{i\theta}, 0)$$

and

$$\sigma_k = f(0, \tanh(k)).$$

By passing to a subsequence, we may assume that there is an $x \in \partial\Omega$ and for every $e^{i\theta} \in \partial\Delta$ there is some $x_\theta \in \partial\Omega$ such that $\sigma_{\theta,j} \to x_\theta$ and $\sigma_k \to x$. Put $o = f(0, 0)$. Then

$$\begin{aligned} 2(\sigma_{\theta,j}|\sigma_k)_o &\geq j + k - 2c - d_\Omega(\sigma_{\theta,j}, \sigma_k) \\ &\geq j + k - 2c - d_{\Delta\times\Delta}((\tanh(j)e^{i\theta}, 0), (0, \tanh(k)) \\ &= j + k - 2c - \max\{d_\Delta((\tanh(j)e^{i\theta}, 0), d_\Delta(0, \tanh(k))\} \\ &= j + k - 2c - \max\{j, k\}. \end{aligned}$$

Thus

$$\lim_{j,k\to\infty} (\sigma_{\theta,j}|\sigma_k)_o = \infty$$

and so, by lemma 4.2.2,

$$H_{x_\theta} = H_x$$

for all $\theta \in \mathbb{R}$. We may assume that

$$H_x = \{(z_1, ...z_n) \in \mathbb{C}^n : z_1 = 0\}$$

and

$$\Omega \subset \{(z_1, ...z_n) \in \mathbb{C}^n : \mathrm{Im}(z_1) > 0\}.$$

Let $\pi : \mathbb{C}^n \to \mathbb{C}$ be the projection onto the first coordinate and define $g : \Delta \to \mathbb{C}$ by

$$g(z) = \pi(f(z, 0)).$$

Then $\mathrm{Im}(g(z)) > 0$ for all $z \in \Delta$, g is bounded, and for all $\theta \in \mathbb{R}$,

$$\lim_{r \to 1^-} g(re^{i\theta}) = 0.$$

So for any $w \in \Delta$,

$$g(w) = \int_{|z|=r} \frac{g(z)}{z-w}\, dz \longrightarrow \int_{|z|=1} \frac{g(z)}{z-w}\, dz = 0,$$

where the convergence is given by the dominated convergence theorem as $r \to 1^-$. Thus, we have a contradiction. ■

4.3 Finite Type

We will now be able to showcase a condition that guarantees finite type for some boundary point of a smoothly bounded convex domain. We will use another result of Frankel that gives us a sufficient condition for a sequence of affine transformations to send a domain, Ω, to another domain, $\hat{\Omega}$, which are biholomorphic to each other.

Theorem 4.3.1 (Frankel [10]) *Suppose Ω is a $\mathbb{C}$-proper convex set, $X \subset \Omega$ is a compact subset, and $\{\varphi_j\} \subset \mathrm{Aut}(\Omega)$. If there exists $x_j \in X$ and complex affine maps A_j such that*

$$A_j(\Omega, \varphi_j x_j) \to (\hat{\Omega}, p)$$

where $\hat{\Omega}$ is a $\mathbb{C}$-proper convex set, then Ω is biholomorphic to $\hat{\Omega}$.

Definition 4.3.2 *For a domain $\Omega \subset \mathbb{C}^n$ denote by $B_\Omega(o, M)$ the closed ball centered at $o \in \Omega$ with Kobayashi radius M. That is*

$$B_\Omega(o, M) = \{z \in \Omega : d_\Omega(o, z) \leq M\}$$

The key to our main result is the ability to cover a quasigeodesic, with an endpoint that is a boundary orbit accumulation point for some sequence $\{\varphi_j\} \subset \mathrm{Aut}(\Omega)$, with the image under the action of each φ_j on a compact subset of Ω. In this case, Zimmer showed that the boundary point is indeed of finite type. Essentially, the sequence of affine transformations, $\{A_j\}$, from proposition 4.2.5 will send a bounded convex domain, Ω, that admits a boundary point of infinite type to an unbounded, yet still $\mathbb{C}$-proper, convex domain, $\hat{\Omega}$, in which the boundary contains a nontrivial affine disk. Now the affine transformations are constructed using the defining function for a neighborhood of the infinite type boundary point, x, and a specific sequence of points which converge to x. This new domain then admits a function which "almost" induces an isometric embedding of the bidisk into $\hat{\Omega}$, which we saw was impossible for smoothly bounded domains. At this juncture, we just need Ω to be biholomorphic to $\hat{\Omega}$, which would imply a contradiction. Based on the formula for the affine transformations, there is a specific sequence of points that lie on the real normal line to $\partial\Omega$ at x, p_j, which converge to x while each $A_j p_j = \zeta \in A_j(\Omega)$, where ζ is fixed.

Ultimately, $\zeta \in \hat{\Omega}$. According to Frankel, if each p_j was the image of some automorphism acting on points contained in a compact subset of Ω, then Ω is indeed biholomorphic to $\hat{\Omega}$. If we could construct the affine maps based on the automorphism orbit rather than the defining function of Ω, we may be able to remove the covering condition. However, it is unclear whether that can be done. Now if every point sufficiently close to x and on the real normal line to x is the image of some automorphism $\varphi(q)$ with q in a compact subset of Ω, then we have our biholomorphic equivalence. So once we cover the real normal line segment up to the boundary with the action of the automorphism group on a compact subset, we have our contradiction.

Theorem 4.3.3 (Zimmer [28]) *Suppose $\Omega \subset \mathbb{C}^n$ is a bounded convex open set with C^∞ boundary. If there exists $o \in \Omega, x \in \partial\Omega, M \geq 0$, and $T \in \mathbb{R}$ so that*

$$\{x + e^{-t} n_x : t > T\} \subset Aut(\Omega) B_\Omega(o, M)$$

then x is of finite type in the sense of D'Angelo.

Proof. Suppose x is of infinite type. By assumption, for any sequence $\{t_j\} \subset \mathbb{R}$ with $t_j > T$ and $t_j \to \infty$, we can find $x_j \in B_\Omega(o, M)$ and $\varphi_j \in \mathrm{Aut}(\Omega)$ such that $\varphi_j(x_j) = x + e^{-t_j} n_x$. Now combining lemma 4.2.4 and proposition 4.2.5 gives us that there are a sequence of affine maps, A_j, such that $A_j \Omega \to \hat{\Omega}$ and $A_j \varphi_j(x_j) \to (i, 0, ..., 0) \in \hat{\Omega}$, where $\hat{\Omega}$ is $\mathbb{C}$-proper, $\partial\hat{\Omega}$ contains a nontrivial holomorphic disk, and there is a holomorphic function $f : \Delta \times \Delta \to \hat{\Omega}$, with the same properties as in proposition 4.2.6. Also by theorem 4.3.1 we must have that Ω and $\hat{\Omega}$ are biholomorphic. But, there can be no such function into Ω by proposition 4.2.7. Thus we have a contradiction, and so x must be of finite type. ■

4.4 Main Result

We can now present our main result. While it is still unclear whether or not the nontangential assumption is necessary, it does provide us some sufficient tools.

Theorem 4.4.1 *Suppose $\Omega \subset \mathbb{C}^n$ is a bounded convex domain with C^∞ boundary. Suppose there exists $\varphi \in \mathrm{Aut}(\Omega)$ and $p \in \Omega$ such that for the sequence of iterates $\{\varphi^j\} \subset \mathrm{Aut}(\Omega)$ we have $\varphi^j(p) \to x \in \partial\Omega$ nontangentially. Then x is of finite type.*

Note that smoothness of the boundary is indeed a necessary assumption. By theorem 2.7.5, a boundary orbit accumulation point for $\mathrm{Aut}(\Omega)$ cannot be in any holomorphic disk in $\partial\Omega$. Moreover, the estimate of lemma 4.2.1 can fail if there is a singularity of the boundary. In the specific case of the bidisk, Δ^2, just take $(t,s),(t',s) \in \Delta^2$, where $t,t',s \in \mathbb{R}$ are sufficiently close to $(1,1) \in \partial\Delta^2$. In this case, $H_1 = \mathbb{C} \times \{1\}$ and $H_2 = \{1\} \times \mathbb{C}$ are complex hyperplanes that are not equal and do not intersect Δ^2. Then (t,s) and (t',s) are sufficiently close to both H_1 and H_2. If we fix t,t' and send $s \to 1$, then $d_{\Delta^2}((t,s),(t',s)) = d_\Delta(t,t')$ which remains constant. Now this would imply that lemma 4.2.2 (involving the Gromov product) may fail for domains without a smooth boundary. Lemma 4.2.2 also provides us with the conditions that guarantee the nonexistence of our bidisk function from proposition 4.2.7. Also, we are not guaranteed a tubular neighborhood of the boundary where the real normal lines to the boundary are quasigeodesics, which is key to the proof of our main theorem. Finally, in $\mathbb{C}^2$, singularities of the boundary could yield the hypothesis of theorem 2.7.2, which would also be contrary to our conclusion.

Before we begin the proof, let us first provide an example that satisfies the hypothesis and conclusion of the main result.

Example 4.4.2 *The egg domain, E_m.*

Recall that $E_m = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^{2m} - 1 < 0\}$ and has automorphism group

$$\mathrm{Aut}(E_m) = \left\{ \varphi_a(z,w) = \left(\frac{z-a}{1-\bar{a}z}, \left(\frac{\sqrt{1-|a|^2}}{1-\bar{a}z} \right)^{\frac{1}{m}} w \right) : |a| < 1 \right\}.$$

Let $a = -\frac{1}{2}$ and put

$$\varphi(z,w) = \varphi_{-\frac{1}{2}}(z,w) = \left(\frac{z+\frac{1}{2}}{1+\frac{1}{2}z}, \left(\frac{\sqrt{1-|\frac{1}{2}|^2}}{1+\frac{1}{2}z} \right)^{\frac{1}{m}} w \right).$$

Then $\varphi(0,0) = (\frac{1}{2}, 0)$. Also, for any iteration of φ, $\varphi^j(0,0) = (\varphi_1^j(0,0), \varphi_2^j(0,0)) = (\varphi_1^j(0,0), 0)$. Since the first coordinate function does not depend on w, we will only concern ourselves with the mapping

$$f(z) = \varphi_1(z,0) = \frac{z+\frac{1}{2}}{1+\frac{1}{2}z}.$$

Then we see that for the sequence given by $\{f^j(0)\}$,

$$f^j(0) = \frac{3^j - 1}{3^j + 1} \longrightarrow 1$$

as $j \to \infty$. Then since each $f^j(0) \in \mathbb{R}$, we see that $\varphi^j(0,0)$ converges normally to $(1,0) \in \partial E_m$. And as we have seen previously, the point $(1,0)$ is of finite type.

Proof of theorem 4.4.1. Firstly, we may assume that there is no nontrivial holomorphic disk contained in $\partial\Omega$ passing through x due to theorem 2.7.5. Since $\varphi \in \mathrm{Aut}(\Omega)$ and the composition of any two automorphisms remains an automorphism, we have that $\varphi^j \in$

$\mathrm{Aut}(\Omega)$ for all $j \in \mathbb{N}$ and $\varphi^{-1} \in \mathrm{Aut}(\Omega)$. Put $M = d_\Omega(p, \varphi(p))$. We may assume $M > 0$, since otherwise, φ would fix p. Then for every consecutive pair of iterates we have

$$\begin{aligned}
d_\Omega(\varphi^j(p), \varphi^{j+1}(p)) &= d_\Omega(\varphi^{-1}(\varphi^j(p)), \varphi^{-1}(\varphi^{j+1}(p))) \\
&= d_\Omega(\varphi^{j-1}(p), \varphi^j(p)) \\
&\vdots \\
&= d_\Omega(p, \varphi(p)) \\
&= M.
\end{aligned}$$

By lemma 4.1.3, there exists $\{p_j\} \subset \Omega$ such that $\varphi^j(p_j) \to x$ normally and $d_\Omega(p, p_j) \leq r$ for some $r > 0$. So we have

$$\begin{aligned}
d_\Omega(\varphi^j(p_j), \varphi^{j+1}(p_{j+1})) &\leq d_\Omega(\varphi^j(p_j), \varphi^j(p)) + d_\Omega(\varphi^j(p), \varphi^{j+1}(p)) + d_\Omega(\varphi^{j+1}(p), \varphi^{j+1}(p_{j+1})) \\
&\leq r + d_\Omega(\varphi^j(p)), \varphi^{j+1}(p)) + r \\
&= 2r + d_\Omega(p, \varphi(p)) \\
&= 2r + M,
\end{aligned}$$

for all $j \in \mathbb{N}$. By convexity, we may assume $x = 0$ and $n_x = (i, 0, ..., 0)$. Furthermore, we may assume that $|\varphi^j(p_j)| > |\varphi^{j+1}(p_{j+1})|$ since $\varphi^j(p_j) \to 0$ as $j \to \infty$. Now consider some $z, y, w \in \Omega$ that lie on the real normal line to $\partial\Omega$ at x such that $|w| < |y| < |z|$. We claim that, for sufficiently small $|z|$, if $w \in B_\Omega(z, R)$ for some $R > 0$, then $y \in B_\Omega(z, R)$ as well. If z is sufficiently small, then there is a one (complex) dimensional affine disk, D, centered at z, such that $D \subset \Omega \cap \{\zeta \in \Omega : \mathrm{Im}(\zeta_1) > 0\}$, $0 \in \partial D$, and ∂D is tangent to $\partial\Omega$ at 0. Note that D is essentially a copy of the unit disk under a translation and dialation and so D is biholomorphic to Δ. Now any geodesic under the Poincaré (equivalently Kobayashi) metric passing through z in D is a straight line. Thus

$$d_D(z,w) = d_D(z,y) + d_D(y,w).$$

Let $\pi : \mathbb{C}^n \to \mathbb{C}$ be the projection onto the first coordinate so $\pi(\Omega) \subset \mathcal{H}$. Then

$$\begin{aligned}
d_\Omega(z,w) &\geq d_{\pi(\Omega)}(\pi(z),\pi(w)) \\
&\geq d_{\mathcal{H}}(\pi(z),\pi(w)) \\
&= d_D(z,w) \\
&= d_D(z,y) + d_D(y,w) \\
&\geq d_D(z,y) \\
&\geq d_\Omega(z,y)
\end{aligned}$$

where the last inequality is given by the inclusion map from D into Ω. Therefore, if $w \in B_\Omega(z,R)$ for $z,w \in \{x + e^{-t}n_x : t > T\}$ with T sufficiently large, then $y \in B_\Omega(z,R)$ for all $y \in \{x + e^{-t}n_x : t > T\}$ with $|w| < |y| < |z|$. Note that we can derive this fact using the estimates of the Kobayashi metric as well. Finally, since $d_\Omega(\varphi^j(p_j), \varphi^{j+1}(p_{j+1})) \leq 2r + M$, then there is a $T > 0$ such that

$$\{x + e^{-t}n_x : t > T\} \subset \cup_{j\in\mathbb{N}} B_\Omega(\varphi^j p_j, 2r + M),$$

and so

$$\{x + e^{-t}n_x : t > T\} \subset \mathrm{Aut}(\Omega) B_\Omega(p, 3r + M).$$

Thus, by theorem 4.3.3, $x \in \partial\Omega$ is of finite type. ■

Now, in $\mathbb{C}^2$, our main theorem gives will us a classification of domains with the properties of the hypothesis. Recall that our example of a nontangential iterated automorphism accumulation point was given by the egg domain E_m. Berteloot and Cœuré [3] showed that if a smoothly bounded domain $\Omega \subset \mathbb{C}^2$ admits an automorphism accumulation point which is of finite type, then Ω is biholomorphic to E_m for some m. Thus, we have a stronger conclusion for domains in $\mathbb{C}^2$ for our main theorem, which we state as a corollary.

Corollary 4.4.3 *Suppose $\Omega \subset \mathbb{C}^2$ is a bounded convex domain with C^∞ boundary. Suppose there exists $\varphi \in \mathrm{Aut}(\Omega)$ and $p \in \Omega$ such that for the sequence of iterates $\{\varphi^j(p)\} \subset \mathrm{Aut}(\Omega)$ we have $\varphi^j(p) \to x \in \partial\Omega$ nontangentially. Then Ω is biholomorphic to an egg domain, E_m.*

Chapter 5

Conclusions

Our result is a special case of the Greene-Krantz conjecture under the additional hypothesis that the domain is convex, the automorphism orbit comes from the iterations of a single automorphism, and the convergence is nontangential. Of course, we would like to remove theses additional hypotheses to prove the conjecture. One might note that in the proof of the main theorem, we just need the Kobayashi distance of each consecutive point in the automorphism orbit sequence to be bounded by some fixed number. So if there was a boundary automorphism orbit accumulation point in which a sequence converges nontangentially so that the distances of each consecutive terms were bounded by a fixed constant, then we would have the result of our main theorem. Of course, we would like to solve the Greene-Krantz conjecture completely, but the next logical step would be solving the conjecture in the case of convex domains.

Bibliography

[1] M. Abate. *Iteration Theory of Holomorphic Maps on Taut Manifolds.* Mediterranean Press, Italy, 1989.

[2] Theodore J. Barth. Convex domains and kobayashi hyperbolicity. *Proceedings of the American Mathematical Society*, 79(4):556–558, August 1980.

[3] F. Berteloot and Gérard Cœuré. Domaines de $\mathbb{C}^2$, pseudoconvexes et de type fini ayant un groupe non compact d'automorphismes. *Annales de l'institut Fourier*, 41(1):77–86, 1991.

[4] D. Burns, S. Shnider, and R. O. Wells. Deformations of strictly pseudoconvex domains. *Inventiones mathematcae*, 46:237–253, 1978.

[5] Sergei Buyalo and Viktor Schroeder. *Elements of Asymptotic Geometry.* European Mathematical Society, Germany, 2007.

[6] John B. Conway. *Functions of One Complex Variable I.* Springer-Verlag, New York, 2nd edition, 1978.

[7] John P. D'Angelo. *Several Complex Variables and the Geometry of Real Hypersurfaces.* CRC Press, Florida, 1993.

[8] Klas Diederich and John E. Fornæss. Pseudoconvex domains with real-analytic boundary. *Annals of Matematics*, 107:371–384, 1978.

[9] Sidney Frankel. Complex goemetry of convex domains that cover varieties. *Acta Mathematica*, 163:109–149, 1989.

[10] Sidney Frankel. Applications of Affine Geometry to Geometric Function Theory in Several Complex Variables: Part I. Convergent Rescalings and Intrinsic Quasi-Isometric Structure. In *Proceedings of Symposia in Pure Mathematics, Part 2*, volume 52, pages 183–208, 1991.

[11] Ian Graham. Distortion theorems for holomorphic maps between convex domains in $\mathbb{C}^n$. *Complex Variables, Theory and Application: An International Journal*, 15(1):37–42, 1990.

[12] Robert C. Gunning. *Introduction to Holomorphic Functions of Several Complex Variables Volume I: Function Theory*. Wadsworth & Brooks/Cole, Pacific Grove, California, 1990.

[13] Robert C. Gunning. *Introduction to Holomorphic Functions of Several Complex Variables Volume II: Local Theory*. Wadsworth & Brooks/Cole, Pacific Grove, California, 1990.

[14] Kaylee Joy Hamann. *On the Characterization of Convex Domains with Non-compact Automorphism Group*. PhD thesis, UCR, 2016.

[15] Thomas W. Hungerford. *Algebra*. Springer-Verlag, New York, 1974.

[16] Kang-Tae Kim. Geometry of bounded domains and the scaling techniques in several complex variables. *Lecture Notes Series Seoul National University*, (13), 1993.

[17] Shoshichi Kobayashi. *Hyperbolic Complex Spaces*. Springer-Verlag, Berlin, 1998.

[18] Steven G. Krantz. *Function Theory of Several Complex Variables*. American Mathematical Society, Providence, 2nd edition, 2001.

[19] Serge Lang. *Introduction to Complex Hyperbolic Spaces*. Springer-Verlag, New York, 1987.

[20] Lina Lee, Bradley Thomas, and Bun Wong. On boundary accumulation points of a convex domain in $\mathbb{C}^n$. *Methods and Applications of Analysis*, 21(4):427–440, December 2014.

[21] Jeffery D. McNeal. Convex domains of finite type. *Journal of Functional Analysis*, 108:361–373, 1992.

[22] Raghavan Narasimhan. *Several Complex Variables*. The University of Chicago Press, Chicago, 1971.

[23] H. L. Royden. Remarks on the Kobayashi metric. In *Several Complex Variables II Maryland 1970*, volume 185 of *Lecture Notes in Mathematics*, pages 125–137, 1971.

[24] Bradley Gray Thomas. *Boundary Characterization of a Smooth Domain with Non-Compact Automorphism Group*. PhD thesis, UCR, 2012.

[25] B. Wong. Characterization of the unit ball in $\mathbb{C}^n$ by its automorphism group,. *Inventiones mathematcae*, 41:253–257, 1977.

[26] B. Wong. Characterization of the bidisc by its automorphism group,. *American Journal of Mathematics*, 117(2):279–288, 1995.

[27] Andrew M. Zimmer. Gromov hyperbolicity and the kobayashi metric on convex domains of finite type. ArXiv e-prints, June 2014.

[28] Andrew M. Zimmer. Characterizing polynomial domains by their automorphism group. ArXiv e-prints, June 2015.