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The Brown Measure of Non-Hermitian Sums of Atomic Operators

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The Brown Measure of Non-Hermitian Sums of Atomic Operators

A dissertation submitted in partial satisfaction
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by

Max Sun Zhou

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ABSTRACT OF THE DISSERTATION

The Brown Measure of Non-Hermitian Sums
of Atomic Operators

by

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Doctor of Philosophy in Mathematics
University of California, Los Angeles, 2024
Professor Dimitri Y. Shlyakhtenko, Chair

Let (M, τ) be a tracial von Neumann algebra. For $X \in (M, \tau)$, the Brown measure of X is a complex probability measure supported on the spectrum of X. It is the spectral measure when X is normal and the empirical spectral distribution when X is a random matrix. We consider operators of the form $X = p + iq$, where $p, q \in (M, \tau)$ are Hermitian, freely independent, and the spectral distributions of p and q consist of finitely many atoms. There is an associated random matrix model $X_n = P_n + iQ_n$, where $P_n, Q_n \in M_n(\mathbb{C})$ are independently Haar-rotated Hermitian matrices.

Using a Hermitization technique, we will compute the Brown measure of $X = p + iq$ when the spectral distributions of p and q are 2 atoms and prove the convergence of the empirical spectral distribution of X_n to the Brown measure of X when the law of P_n converges to the law of p and the law of Q_n converges to the law of q.

For the general case, we will relate the operator-valued Cauchy transform from the mathematical literature to the Quaternionic analogue of the Cauchy transform in the physics literature. When $X = p + iq$ for $p, q \in (M, \tau)$ Hermitian and freely independent, the Quaternionic Cauchy transform provides heuristics for the boundary and support of the Brown measure of X. We verify these heuristics when the spectral distributions of p and q

are 2 atoms, and show that in general, the heuristic implies that the boundary of $X = p + iq$ is an algebraic curve. We conclude by discussing the atoms of the Brown measure.

The dissertation of Max Sun Zhou is approved.

Sorin Popa

Terence Chi-Shen Tao

Jun Yin

Dimitri Y. Shlyakhtenko, Committee Chair

University of California, Los Angeles

2024

To my parents,

who have always supported me.

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VITA

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CHAPTER 1

Background material

In this chapter, we will review the background material needed to define the Brown measure and prove our results about it. The three major topics to be discussed are (tracial) von Neumann algebras, non-commutative laws/random matrices, and free probability.

1.1 von Neumann algebras

In this section, we introduce the definition of von Neumann algebras and discuss traces. This material is summarized from [SZ79] and [AP].

Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. We will assume that the inner product is conjugate linear in the second variable.

Let x be a linear operator from H to H. Then, let $\|x\|$ be the usual operator norm of a linear operator between normed vector spaces,

$$\|x\| = \sup_{\substack{\xi \in H \\ \|\xi\|=1}} \|x\xi\| . \tag{1.1}$$

In general, $\|x\|$ may be infinite.

Let $B(H)$ be the set of bounded linear operators from H to itself, where the operator norm is finite. For $x \in B(H)$, there exists a unique $x^* \in B(H)$, defined by the equation:

$$\langle x\xi, \eta \rangle = \langle \xi, x^*\eta \rangle, \qquad \xi, \eta \in H. \tag{1.2}$$

The operator x^* is called the ***adjoint*** of x.

Then, $B(H)$ is a Banach $*$-algebra with the following operations:

- The sum $x + y$ is the usual sum of linear operators.
- The product xy is the usual composition of linear operators.
- The norm $\|x\|$ is the usual operator norm (which is finite).
- The adjoint x^* is the usual adjoint operator.

We define an element $p \in B(H)$ to be a ***projection*** if $p^2 = p$ and $p = p^*$. We highlight that all of the projections we consider are orthogonal projections.

Consider the following topologies on $B(H)$:

- The ***norm topology*** (or ***strong topology***) on $B(H)$ is the topology induced by the metric
$$d(x, y) = \|x - y\| . \tag{1.3}$$
Hence, a net $(x_i)_{i\in I}$ converges to x if and only if $(\|x_i - x\|)_{i\in I}$ converges to 0.
- The ***strong operator topology*** (or ***s.o. topology*** or ***s.o.t***) on $B(H)$ is the locally convex topology induced by the seminorms:
$$p_\xi(x) = \|x\xi\| , \qquad \xi \in H. \tag{1.4}$$
Hence, a net $(x_i)_{i\in I}$ converges to x if and only if $(x_i\xi)_{i\in I}$ converges to $x\xi$ for every $\xi \in H$.
- The ***weak operator topology*** (or ***w.o. topology*** or ***w.o.t***) on $B(H)$ is the locally convex topology induced by the seminorms:
$$\omega_{\xi,\eta}(x) = |\langle x\xi, \eta\rangle| , \qquad \xi, \eta \in H. \tag{1.5}$$
Hence, a net $(x_i)_{i\in I}$ converges to x if and only if $(\langle x_i\xi, \eta\rangle)_{i\in I}$ converges to $\langle x_i\xi, \eta\rangle$ for every $\xi, \eta \in H$.

We state some well-known facts about these topologies:

Proposition 1.1. *Let $B(H)$ be the set of bounded linear operators on a Hilbert space H.*

- *In general, the norm topology is strictly finer than the strong operator topology and the strong operator topology is strictly finer than the weak operator topology.*
- *A linear functional on $B(H)$ is w.o. continuous if and only if it is s.o. continuous.*
- *A convex subset of $B(H)$ is w.o. closed if and only if it is s.o. closed.*
- *For a subspace $M \subset B(H)$, a linear functional on M is w.o. continuous if and only if it is s.o. continuous.*
- *The unit ball in $B(H)$ is compact with respect to the w.o. topology.*

Define $M \subset B(H)$ to be a $*$-subalgebra if M is a subalgebra of $B(H)$ (i.e. a subset closed under multiplication and addition that contains 0) that is closed under the $*$ operation. Now, we proceed to define C*-algebras and von Neumann algebras:

Definition 1.2.

- *$M \subset B(H)$ is a **C*-algebra** if M is a $*$-subalgebra that is closed in the norm topology of $B(H)$.*
- *$M \subset B(H)$ is a **von Neumann algebra** (or **W*-algebra**) if M is a $*$-subalgebra that contains 1 and is closed in the strong operator topology of $B(H)$.*

Note that since the norm topology is finer than the s.o. topology, then every von Neumann algebra is a C*-algebra. There are examples that disprove the converse (ex. the set of compact operators on an infinite dimensional Hilbert space H). Also note that since M is convex, then it is equivalent to define von Neumann algebras as those *-subalgebras that are closed in the w.o topology.

For C*-algebras, there is a completely abstract definition (i.e. with no mention of $B(H)$):

Definition 1.3. *An **abstract C*-algebra** is a Banach algebra (i.e. a complete normed vector space with a norm that satisfies $\|xy\| \leq \|x\| \, \|y\|$) that has an involution $*$ that is* *conjugate linear, $(xy)^* = y^*x^*$, and satisfies the C^* identity:*

$$\|x^*x\| = \|x\|^2 . \tag{1.6}$$

The Gelfand-Naimark theorem states that an abstract C*-algebra is isometrically isomorphic to a concrete C*-algebra (i.e. a norm-closed *-subalgebra of $B(H)$). The heart of the proof of this theorem is the Gelfand-Naimark-Siegel (GNS) construction.

There is a similar abstract definition of a von Neumann algebra (originally proved in [Sak56]):

Definition 1.4. *An* ***abstract von Neumann algebra*** *is a C*-algebra M where M is the dual space of some separable Banach space M_*.*

We will not use this characterization of von Neumann algebras and instead always view them as subalgebras of $B(H)$.

A practical difficulty in determining whether or not a $*$-subalgebra of $B(H)$ is actually a von Neumann algebra is determining its s.o.-closure (or equivalently, w.o.-closure). The von Neumann bicommutant theorem makes the following connection between the topology and algebraic structure of $B(H)$:

Theorem 1.5. *Let M be a *-subalgebra of $B(H)$. The closure of M in the s.o. topology is equal to the closure of M in the w.o. topology, and these are equal to the bicommutant M'' of M.*

Now, we examine the abelian C*-algebras and von Neumann algebras to give a heuristic for the difference between C*-algebras and von Neumann algebras.

Let X be a locally compact Hausdorff space. Let $C_0(X)$ be the space of continuous complex-valued functions on X that vanish at infinity. $C_0(X)$ is a C*-algebra with the usual functional addition/multiplication, $f^* = \overline{f}$, and supremum norm. Note that $C_0(X)$ is norm closed. Let $C_c(X)$ be the space of compactly supported continuous complex-valued functions on X. $C_c(X)$ is a C*-subalgebra of $C_0(X)$.

The Gelfand representation gives an (isometric) isomorphism between any abelian C*-algebra and $C_0(X)$ for some locally compact Hausdorff space X. Further, the C*-algebra is unital if and only if X is compact, in which case $C_0(X) = C_c(X)$.

For the analogous theorem for abelian von Neumann algebras, consider (X, μ) a σ-finite measure space. Then, $L^\infty(X, \mu)$ acts on the Hilbert space $L^2(X, \mu)$ by left multiplication. $L^\infty(X, \mu)$ is a von Neumann algebra with the usual functional addition/multiplication, $f^* = \overline{f}$ and the L^∞ norm. The fact that $L^\infty(X, \mu)$ is closed in the s.o. topology can be verified by hand or follows from von Neumann's bicommutant theorem and the fact that the commutant of $L^\infty(X, \mu)$ is itself.

Every abelian von Neumann algebra on a separable Hilbert space is isometrically isomorphic to $L^\infty(X, \mu)$, where (X, μ) is a standard measure space.

This gives the heuristic that C*-algebras are analogous to continuous functions, while von Neumann algebras are analogous to bounded measurable functions.

If von Neumann algebras are analogous to bounded measurable functions, then there should be a way to recover the analogue of the measure. In the case of $L^\infty(X, \mu)$, we recover the measure through the integration of characteristic functions χ_E, where E is a measurable subset of X. Note that the χ_E are exactly the Hermitian projections in $L^\infty(X, \mu)$. The analogy to this is the trace on certain von Neumann algebras.

First, we define some types of linear functionals on a von Neumann algebra M:

Definition 1.6. *Let φ be a linear functional on $M \subset B(H)$.*

- *φ is **positive** if $\varphi(x^*x) \geq 0$ for $x \in M$.*
- *φ is a **state** if φ is positive and $\varphi(1) = 1$.*
- *φ is **faithful** if $\varphi(x^*x) = 0$ implies $x = 0$.*
- *φ is a **trace** if φ is positive and $\varphi(xy) = \varphi(yx)$ for all $x, y \in M$.*
- *If φ is a trace and a state, we call φ a **tracial state**.*

Recall that a linear functional φ on M is positive if and only if φ is bounded with $\|\varphi\| = \varphi(1)$. Hence, a positive linear functional φ is a state if and only if $\|\varphi\| = 1$.

Note that the integral on $L^\infty(X, \mu)$ is a state when μ is a positive measure and is a state when μ is a probability measure. The analogue of the countable additivity axiom for measures is countable additivity:

Definition 1.7. *Let $M \subset B(H)$ be a von Neumann algebra. A positive linear functional φ is **completely additive** if for every family $\{p_i : i \in I\}$ of mutually orthogonal projections in M, $\varphi(\sum_i p_i) = \sum_i \varphi(p_i)$.*

There is a more typical continuity condition that is the analogue of the monotone convergence theorem:

Definition 1.8. *Let $M \subset B(H)$ be a von Neumann algebra. A positive linear functional φ is **normal** if for every bounded increasing net $(x_i)_{i\in I}$ of positive elements in M, $\varphi(\sup_i x_i) = \sup_i \varphi(x_i)$.*

Recall that the element $\sup_i x_i \in M$ since it is the s.o. limit of x_i.

It turns out that normality and complete additivity are equivalent and can also be viewed as a continuity condition:

Proposition 1.9. *Let φ be a positive linear functional on a von Neumann algebra $M \subset B(H)$. The following conditions are equivalent:*

- *φ is normal.*
- *φ is completely additive.*
- *The restriction of φ to the unit ball of M is w.o. continuous.*
- *The restriction of φ to the unit ball of M is s.o. continuous.*

Thus, as a generalization of measure spaces, we will work with tracial von Neumann algebras:

Definition 1.10. *Let $M \subset B(H)$ be a von Neumann algebra with a normal, faithful, tracial state τ. Then, we call the pairing (M, τ) a* ***tracial von Neumann algebra*** *and we just refer to τ as the* ***trace****.*

On (M, τ), τ also acts as a generalization of the dimension function for finite vector spaces. For instance, for projections $p, q \in M \subset B(H)$, let $p \vee q$ be the projection onto $\overline{pH + qH}$ and let $p \wedge q$ be the projection onto $pH \cap qH$. Then, $p \wedge q, p \vee q \in M$ and we have the following formula:

$$\tau(p \vee q) = \tau(p) + \tau(q) - \tau(p \wedge q)\,. \tag{1.7}$$

This is a generalization of the following formula for finite-dimensional vector subspaces V, W:

$$\dim(V + W) = \dim(V) + \dim(W) - \dim(V \cap W)\,. \tag{1.8}$$

Similarly, there is a generalization of the rank-nullity theorem: Let $x \in M \subset B(H)$. If $n(x)$ is the projection onto $\ker(x)$ and $l(x)$ is the projection onto $\overline{\mathrm{im}(x)}$, then

$$\tau(n(x)) + \tau(l(x)) = 1\,. \tag{1.9}$$

A von Neumann algebra M is ***finite*** if every isometry is unitary, i.e. for $v \in M$ if $v^*v = 1$, then $vv^* = 1$. It is clear that vv^* is a projection. When M has a tracial state, then M is finite: $\tau(1 - vv^*) = 1 - \tau(vv^*) = 1 - \tau(v^*v) = 1 - \tau(1) = 0$ and $1 - vv^* \geq 0$, so $1 - vv^* = 0$, i.e. $vv^* = 1$.

For the converse, a von Neumann algebra M is a ***factor*** if the center of M is only $\mathbb{C}$. Then, a finite factor has a unique normal, faithful, tracial state. Finite factors that have a minimal non-zero projection are just $M_n(\mathbb{C})$ (with the normalized trace $\tau = \mathrm{tr}/n$ being the unique trace). Those finite factors that do not have a minimal projection are called $\mathrm{II}_\mathbf{1}$ ***factors***.

Finally, we consider the GNS construction applied to the trace τ on (M, τ). Then, the sesquilinear form $\langle a, b\rangle = \tau(b^*a)$ is actually an inner product, so we may complete M with respect to this inner product to form the Hilbert space L^2M. M is a dense subset of L^2M. Further, M acts faithfully on L^2M by left multiplication, and left multiplication is a bounded operator on L^2M. Hence, M can be thought of as a subset of $B(L^2M)$. Moreover, in this situation, this subset of M in $B(L^2M)$ is s.o. closed, and hence is a von Neumann algebra on $B(L^2M)$.

1.2 Functional calculus and the spectral theorem

In this section, we describe various forms of the functional calculus and the spectral theorem. These will be used extensively in the rest of the paper. This material is summarized from [Con90], [SZ79], [AP], and [Shl].

First, we recall the definition of the spectrum of an operator and some basic facts:

Definition 1.11. *Let H be a Hilbert space. The* ***spectrum*** *of an operator $x \in B(H)$ is the set*

$$\sigma(x) = \{z \in \mathbb{C} : z - x \text{ is not invertible}\}. \tag{1.10}$$

The following Proposition lists some basic facts about $\sigma(x)$:

Proposition 1.12. *Let H be a Hilbert space and let $x \in B(H)$.*

- *$\sigma(x)$ is a non-empty, compact subset of $\mathbb{C}$.*
- *$\lim\limits_{n\to\infty} \|x^n\|^{1/n} = \sup_{\lambda\in\sigma(x)} |\lambda|$. This quantity is called the* ***spectral radius*** *of x and will be denoted $|\sigma(x)|$.*
- *If x is normal, then $|\sigma(x)| = \|x\|$.*
- *If x is Hermitian, then $\sigma(x) \subset \mathbb{R}$.*
- *If x is positive, then $\sigma(x) \subset [0, \infty)$.*

The most basic functional calculus is the continuous functional calculus for Hermitian elements. It can be stated on the level of C*-algebras. This is another instance of the heuristic that C*-algebras are analogous to continuous functions.

For any $X \subset \mathbb{C}$, let $C(X)$ be the set of complex-valued continuous functions on X. Recall that $C(X)$ is a C*-algebra with the usual functional addition/multiplication, $f^* = \overline{f}$, and supremum norm.

The continuous functional calculus follows from the observation that $\sigma(p(x)) = p(\sigma(x))$ for any polynomial p:

Theorem 1.13. *Let $x \in B(H)$ be a Hermitian operator. There exists a unique $*$-algebra homomorphism taking $f \in C(\sigma(x))$ to $f(x) \in B(H)$, such that:*

1. *If f is a polynomial, $f(\lambda) = a_n\lambda^n + \cdots + a_0$, then $f(x) = a_n x^n + \cdots + a_0$.*

2. *$\|f(x)\| = \|f\|$.*

This map is an isometric $*$-isomorphism between $C(\sigma(x))$ and the C-algebra generated by x and 1 in $B(H)$, $C^*(\{x, 1\})$.*

Let $x \in (M, \tau)$ where x is Hermitian. From the continuous functional calculus, the map taking $f \in C(\sigma(x))$ to $\tau(f(x))$ is a continuous linear functional. It is also easy to check that this linear functional is positive and takes 1 to 1. Hence, it corresponds to a Borel probability measure on $\sigma(x) \subset \mathbb{C}$, μ_x. This measure is called the ***spectral measure*** of x. For $f \in C(\sigma(x))$,

$$\tau(f(x)) = \int_{\sigma(x)} f(t)\, d\mu_x(t)\,. \tag{1.11}$$

Next, we proceed to examine the Borel functional calculus. For $X \subset \mathbb{C}$, let $B(X)$ be the set of bounded Borel functions $f : X \to \mathbb{C}$. The continuous functional calculus can be extended to the Borel functional calculus by considering the following continuous linear functionals on $C(\sigma(x))$:

$$C(\sigma(x)) \ni f \mapsto \langle f(x)\xi, \eta\rangle, \qquad \xi, \eta \in H\,. \tag{1.12}$$

From the Riesz-Markov representation theorem, this map corresponds to integration by a Borel measure $\mu_{\xi,\eta}$ on $C(\sigma(x))$. From the continuous functional calculus, $\|\mu_{\xi,\eta}\| \leq \|\xi\|\,\|\eta\|$.

Hence, for any $f \in B(\sigma(x))$, we may define $\langle f(x)\xi, \eta\rangle$ by integration with respect to $\mu_{\xi,\eta}$. For fixed f, this is a bounded sesquilinear form on H, and hence $f(x)$ is given by a unique operator in $B(H)$. The Borel functional calculus satisfies the following properties:

Theorem 1.14. *Let $x \in B(H)$ be a Hermitian operator. There exists a unique $*$-algebra homomorphism taking $f \in B(\sigma(x))$ to $f(x) \in B(H)$ such that:*

1. *This map extends the continuous functional calculus.*
2. *If $f, f_n \in B(\sigma(x))$, $\sup_n \|f_n\| < \infty$ and $f_n \to f$ pointwise, then $f_n(x) \to f(x)$ in the s.o. topology on $B(H)$.*

This map takes the C-algebra $B(\sigma(x))$ into the von Neumann algebra generated by $\{x\}$ in $B(H)$.*

Recall that any $f \in B(\sigma(x))$ is a pointwise limit of a sequence of uniformly bounded continuous functions. Also, recall that τ is continuous with respect to the s.o. topology on bounded subsets of M. Then, by taking limits, we see that the following formula holds for all $f \in B(\sigma(x))$:

$$\tau(f(x)) = \int_{\sigma(x)} f(t)\, d\mu_x(t)\,. \tag{1.13}$$

In the case of the continuous functional calculus, there is a more general functional calculus for normal elements. It follows from the fact that if $x \in B(H)$ is normal, then $C^*(\{x, 1\})$ is an abelian C*-algebra, and hence is isomorphic to $C(X)$, where X is a compact Hausdorff space. Then, the spectrum of any $f \in C(X)$ is just $f(X)$ and the continuous functional calculus just corresponds to the composition of functions. Thus, we have the continuous functional calculus for normal operators:

Theorem 1.15. *Let $x \in B(H)$ be a normal operator. There exists a unique $*$-algebra homomorphism taking $f \in C(\sigma(x))$ to $f(x) \in B(H)$ such that:*

1. *If f is a polynomial in z and $\overline{z}$, $f(z, \overline{z}) = a_{00} + a_{10}z + a_{01}\overline{z} + \cdots + a_{nm}z^n\overline{z}^m$, then $f(x, x^*) = a_{00} + a_{10}x + a_{01}x^* + \cdots + a_{nm}x^n(x^*)^m$.*

2. $\|f(x, x^*)\| = \|f\|$.

This map is an isometric $*$-isomorphism between $C(\sigma(x))$ and the C^*-algebra generated by x and 1 in $B(H)$, $C^*(\{x, 1\})$.

We can extend the continuous functional calculus to the analogous Borel functional calculus that is valid on the von Neumann algebra generated by x. We also have the similar spectral measure μ_x that is now a complex probability measure supported on $\sigma(x) \subset \mathbb{C}$, where for all $f \in B(\sigma(x))$,

$$\tau(f(x)) = \int_{\sigma(x)} f(t)\, d\mu_x(t)\,. \tag{1.14}$$

Our final functional calculus is the analytic functional calculus. Let $x \in B(H)$ and let $A(\sigma(x))$ be the set of all complex analytic functions defined on a neighborhood of $\sigma(x)$. For $f \in A(\sigma(x))$, let $\gamma_1, \ldots, \gamma_k$ be closed, rectifiable Jordan curves such that the interiors of these curves are disjoint, the unions of the interiors of these curves contain $\sigma(x)$, and the closure of this region is in the domain of f. Let $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$. Then, it is possible to define the following operator-valued integral as the limit of norm-convergent Riemann sums:

$$f(x) = \frac{1}{2\pi i} \int_{\Gamma} f(z)(\zeta - x)^{-1}\, dz\,. \tag{1.15}$$

From an operator-valued version of Cauchy's theorem, $f(x)$ is independent of the curve γ. Then, we have the following analytic functional calculus:

Theorem 1.16. *Let $x \in B(H)$. There exists a unique algebra homomorphism taking $f \in A(\sigma(x))$ to $f(x) \in B(H)$ such that:*

1. *If f is a polynomial, $f(z) = a_n z^n + \cdots + a_0$, then $f(x) = a_n x^n + \cdots + a_0$.*

2. *If $f_n, f \in A(\sigma(x))$ and $f_n \to f$ locally uniformly, then $f_n(x) \to f(x)$ in norm.*

Note that the uniqueness statement follows from Runge's approximation theorem. When x is Hermitian, then the analytic functional calculus is just the restriction of the continuous functional calculus to $A(\sigma(x))$.

Since the operator-valued integral is norm-convergent, then in (M, τ),

$$\tau(f(x)) = \tau \left[\frac{1}{2\pi i} \int_\Gamma f(z)(z-x)^{-1}\, dz \right] = \frac{1}{2\pi i} \int_\Gamma f(z)\tau[(z-x)^{-1}]\, dz\,. \tag{1.16}$$

Now, we briefly mention the spectral theorem. There is a notion of a ***projection-valued measure*** where the measurable sets in a measure space correspond to projections in $B(H)$, for a Hilbert space H. Similar axioms hold for these projections as for measurable sets in a measure space. Then, we have the following spectral theorem:

Theorem 1.17. *Let $x \in B(H)$ where x is Hermitian. Then, there exists a projection-valued measure E_t for which:*

$$x = \int_{\sigma(x)} t\, dE_t\,. \tag{1.17}$$

The integration in the spectral theorem is the operator-valued analogue to the Riemann-Stieltjes integral. It is also possible to make sense of the following identity for $f \in C(\sigma(x))$:

$$f(x) = \int_{\sigma(x)} f(t)\, dE_t\,. \tag{1.18}$$

In particular, if $x \in (M, \tau)$, then $\tau(E_t)$ is just a Borel measure, and then we recover the spectral measure:

$$\tau(f(x)) = \int_{\sigma(x)} f(t)\, d\tau(E_t)\,. \tag{1.19}$$

1.3 Non-commutative probability spaces and random matrices

In this section, we define non-commutative probability spaces, the law of elements in a non-commutative probability space, and what it means to converge in law. We will introduce and discuss random matrices in each of these contexts. This material is summarized from [AGZ10], [Tao12], and [Shl].

First, we define a non-commutative probability space:

Definition 1.18. *A **non-commutative probability space** is a pair (A, ϕ), where A is a* *unital *-algebra and $\phi : A \to \mathbb{C}$ is a linear functional where $\phi(1) = 1$. The elements of the non-commutative probability space (A, ϕ) are called* ***non-commutative random variables****.*

This definition is very general: it does not impose any positivity/continuity conditions on ϕ or any closure conditions on A (i.e. that A is norm closed or s.o. closed). It does not even impose any boundedness conditions on A.

As in the case of tracial von Neumann algebras, we will highlight some of the aspects of non-commutative probability spaces that are a generalization of concepts for the commutative probability space $L^\infty(X, \mu)$.

Consider the following examples of non-commutative probability spaces:

- Let (X, μ) be a probability space. Then, taking $A = L^\infty(X, \mu)$ and ϕ to be integration with respect to μ makes $(L^\infty(X, \mu), d\mu)$ a non-commutative probability space. Then, the non-commutative probability space $(L^\infty(X, \mu), d\mu)$ consists of bounded random variables on the probability space (X, μ). Recall in this situation that $(L^\infty(X, \mu), d\mu)$ is also a tracial von Neumann algebra acting on the Hilbert space $L^2(X, \mu)$.

- Instead of taking $A = L^\infty(X, \mu)$ in the previous example, we can take $A = L^{\infty-}(X, \mu) = \bigcap_{p \geq 1} L^p(X, \mu)$. A is an algebra and taking ϕ to be integration with respect to μ makes (A, ϕ) a non-commutative probability space. This space consists of more general unbounded random variables on (X, μ) where integration still makes sense. In particular, if we consider A to act on $L^2(X, \mu)$, these are potentially unbounded operators and hence do not form a von Neumann algebra. In our situation, all of our operators are bounded, so we will not deal with these issues.

- In general, for a tracial von Neumann algebra (M, τ), $A = M$ and $\phi = \tau$ is a non-commutative probability space.

- Analogous to the example of bounded random variables, we can consider ***random matrices***: Let (X, μ) be a probability space. Then, consider $n \times n$ matrices with entries in $L^\infty(X, \mu)$, $M_n(L^\infty(X, \mu))$. This is a von Neumann algebra acting on $H = L^2(X, \mu)^{\oplus n}$

and it is tracial with $\tau = \mathbb{E}_\mu[\frac{1}{n}\mathrm{tr}]$. Hence, $n \times n$ random matrices on (X, μ) is a non-commutative probability space with $\phi = \tau$.

- Note that instead of taking bounded matrices, we can consider matrices with entries in $L^{\infty-}(X, \mu)$. $\phi = \mathbb{E}_\mu[\frac{1}{n}\mathrm{tr}]$ is still well-defined and makes $A = M_n(L^{\infty-}(X, \mu))$ a non-commutative probability space. Again, for our thesis, we will only deal with bounded random matrices.

Next, we consider the law of a family of non-commutative random variables. Let J be a subset of $\mathbb{N}$ and let $\mathbb{C}\langle X_i, \overline{X_i} : i \in J\rangle$ denote the set of polynomials in non-commutative variables $X_i, \overline{X_i}$, $i \in J$.

Definition 1.19. *Let (A, ϕ) be a non-commutative probability space. Let $\{a_i\}_{i\in J} \subset A$. The* ***law of*** *$\{a_i\}_{i\in J}$ is the map $\mu_{\{a_i\}_{i\in J}} : \mathbb{C}\langle X_i, \overline{X_i} : i \in J\rangle \to \mathbb{C}$ given by:*

$$\mu_{\{a_i\}_{i\in J}}(P) = \phi(P(\{a_i, a_i^*\}_{i\in J}))\,. \tag{1.20}$$

Consider the following examples of non-commutative laws:

- Consider a probability space (X, μ) and the non-commutative probability space $(L^\infty(X, \mu), d\mu)$. Let $f \in L^\infty(X, \mu)$. Then, for a polynomial $P = P(z, \overline{z})$,

$$\mu_f(P) = \int_X P(f, f^*)\, d\mu = \int_{f(X)} P(z, \overline{z})\, df_*(\mu)\,, \tag{1.21}$$

  where $f_*(\mu)$ is the pushforward measure of μ under f. As $f \in L^\infty(X, \mu)$, then $f_*(\mu)$ is a compactly supported probability measure. From the Stone-Weierstrass theorem, polynomials in z and $\overline{z}$ are dense in $C(f(X))$, and hence $\mu_f(P)$ uniquely determines the measure $df_*(\mu)$. Hence, the non-commutative law of f corresponds to the classical law of f.

- Consider $(M_n(\mathbb{C}), \frac{1}{n}\mathrm{tr})$ and $X_n \in M_n(\mathbb{C})$ where X_n is Hermitian. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of X_n. Then, computing the (normalized) traces in an orthogonal eigenbasis of X_n, it is clear that

$$\mu_{X_n}(x^k) = \tau(X_n^k) = \frac{1}{n}\sum_{i=1}^{n} \lambda_i^k \,. \tag{1.22}$$

Letting

$$\nu = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i} \,, \tag{1.23}$$

then in general, for a polynomial $P(t)$,

$$\mu_{X_n}(P) = \frac{1}{n}\mathrm{tr}(P(X_n)) = \frac{1}{n}\sum_{i=1}^{n} P(\lambda_i) = \int_{\mathbb{R}} P(t)\, d\nu \,. \tag{1.24}$$

The law of X_n corresponds with ν, where ν is the spectral measure of X_n in $(M_n(\mathbb{C}), \frac{1}{n}\mathrm{tr})$. For a normal $X_n \in M_n(\mathbb{C})$, the exact same formula for ν uniquely satisfies the following equation for polynomials in $z, \overline{z}$:

$$\mu_{X_n}(P) == \frac{1}{n}\mathrm{tr}(P(X_n, X_n^*)) = \frac{1}{n}\sum_{i=1}^{n} P(\lambda_i, \overline{\lambda_i}) = \int_{\mathbb{C}} P(z, \overline{z})\, d\nu \,, \tag{1.25}$$

so again the law of X_n is ν, the spectral measure of X_n in $(M_n(\mathbb{C}), \frac{1}{n}\mathrm{tr})$.

- For a normal random matrix $X_n \in M_n(L^\infty(X, \mu))$, we can consider the random eigenvalue distribution $\nu : (X, \mu) \to \mathcal{P}(\mathbb{C})$ given by:

$$\nu = \sum_{i=1}^{n} \delta_{\lambda_i} \,. \tag{1.26}$$

Since X_n is uniformly bounded, then ν is almost surely supported on a compact set, K. Then, the map taking $f(z, \overline{z}) \in C(K)$ to $\mathbb{E}_\mu\left[\int_K f(z, \overline{z})\, d\nu\right]$ is a bounded linear functional, and hence is represented by a compactly supported measure $\overline{\nu}$. By integrating the formula for deterministic matrices with respect to μ,

$$\begin{aligned}\mu_{X_n}(P) &= \tau(P(X_n, X_n^*)) \\ &= \mathbb{E}_\mu\left[\frac{1}{n}\mathrm{tr}(P(X_n, X_n^*))\right] \\ &= \mathbb{E}_\mu\left[\int_{\mathbb{C}} P(z,\overline{z})\, d\nu\right] \\ &= \int_{\mathbb{C}} P(z,\overline{z})\, d\overline{\nu}\,.\end{aligned} \tag{1.27}$$

Thus, the law of X_n is $\overline{\nu}$, the spectral measure of the random matrix X_n with respect to $\mathbb{E}_\mu[\frac{1}{n}\mathrm{tr}]$.

Motivated by the examples, we define the empirical spectral distribution of a random matrix:

Definition 1.20. *Let $X_n \in M_n(L^{\infty-}(X,\mu))$ be a random matrix. Let $\lambda_1, \ldots, \lambda_n$ be the random eigenvalues of X_n (repeated with algebraic multiplicity). Then, the* ***empirical spectral distribution (ESD)*** *of X_n is the random probability measure $\mu_{X_n} : (X,\mu) \to \mathcal{P}(\mathbb{C})$ given by:*

$$\mu_{X_n} = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i}\,. \tag{1.28}$$

When $X_n \in M_n(L^{\infty}(X,\mu))$, then μ_{X_n} is almost surely supported on some compact $K \subset \mathbb{C}$. Recall if X_n is normal that $\overline{\mu_{X_n}}$ is the spectral measure of X_n with respect to $\tau = \mathbb{E}_\mu[\frac{1}{n}\mathrm{tr}]$.

For an arbitrary $X_n \in M_n(L^{\infty-}(X,\mu))$, we still always have the formulas for analytic polynomials $P(z)$:

$$\frac{1}{n}\mathrm{tr}(P(X_n)) = \frac{1}{n}\sum_{i=1}^{n} P(\lambda_i) = \int_{\mathbb{C}} P(z)\, d\mu_{X_n}\,. \tag{1.29}$$

In this special case, the moments still determine the empirical spectral distribution, since the coefficients of the characteristic polynomial are determined by the power sums of the eigenvalues (from Newton's identities). This formula also holds when $P(z)$ is replaced by a rational function with poles off of $\sigma(x)$. Applying Runge's approximation theorem, we see that the equation also holds for any $f \in A(\sigma(x))$, where $f(x)$ is defined using the analytic functional calculus.

Now, we define what it means for a sequence of family of non-commutative random variables to converge in law:

Definition 1.21. *Let (A, ϕ) and (A_n, ϕ_n), $n \in \mathbb{N}$ be non-commutative probability spaces. Let $\{a_i^n\}_{i\in J}$ be a sequence of elements of A_n. Then, $\{a_i^n\}_{i\in J}$* ***converges in law*** *to $\{a_i\}_{i\in J}$ if and only if for all $P \in \mathbb{C}\langle X_i, X_i^* : i \in J\rangle$,*

$$\lim_{n\to\infty} \mu_{\{a_i^n\}_{i\in J}}(P) = \mu_{\{a_i\}_{i\in J}}(P)\,. \tag{1.30}$$

Consider the following examples of the convergence of non-commutative laws:

- Consider the non-commutative probability space $(L^\infty(X, \mu), d\mu)$, and $f, f_n \in L^\infty(X, d\mu)$. Then, μ_{f_n} converges to μ_f if for all polynomials $P = P(z, \overline{z})$,

$$\int_{f(X_n)} P(z, \overline{z})\, d(f_n)_*(\mu) = \mu_{f_n}(P) \to \mu_f(P) = \int_{f(X)} P(z, \overline{z})\, df_*(\mu)\,. \tag{1.31}$$

  if $f(X_n), f(X) \subset K$ for some compact $K \subset \mathbb{C}$, then this is equivalent to the convergence of $(f_n)_*(\mu)$ to $f_*(\mu)$ in the vague topology. Hence, the convergence of the non-commutative law of f_n to f is exactly the convergence of the classical law of f_n to f.

- Consider normal random matrices $X_n \in M_n(L^\infty(X, \mu))$ and a normal operator $x \in (M, \tau)$. Then, μ_{X_n} converges to μ_x if for all polynomials $P = P(z, \overline{z})$

$$\int_{\mathbb{C}} P(z, \overline{z})\, d\overline{\mu_{X_n}} = \mu_{X_n}(P) \to \mu_x(P) = \int_{\mathbb{C}} P(z, \overline{z})\, d\mu_x\,. \tag{1.32}$$

  If X_n, x are uniformly bounded, then $\overline{\mu_{X_n}}, \mu_x$ are supported on some compact set $K \subset \mathbb{C}$ and this is equivalent to the ***convergence of measures in the vague topology***, i.e. $\int_{\mathbb{C}} f\, d\overline{\mu_{X_n}} \to \int_{\mathbb{C}} f\, d\mu_x$ for all $f \in C_c(\mathbb{C})$.

1.4 Convergence in law: Hermitian random matrices

Recall that we are interested in the convergence of the laws (i.e. empirical spectral distributions μ_{X_n}) of random matrices X_n. But, we would like to not only discuss convergence in the expectation of these laws (i.e. the $\overline{\mu_{X_n}}$), but also in probability and almost surely. In this section, we define and study these concepts for Hermitian X_n. This material is summarized from [AGZ10] and [Tao12].

Definition 1.22. *Let $X_n \in M_n(L^\infty(X, \mu))$, where (X, μ) is a probability space. Let μ_{X_n} be the empirical spectral distribution of X_n. Let μ be a complex probability measure. Then,*

- *X_n converges to μ* ***in expectation*** *if $\overline{\mu_{X_n}}$ converges to μ in the vague topology.*
- *X_n converges to μ* ***in probability*** *if for every $f \in C_c(\mathbb{C})$, $\int_{\mathbb{C}} f \, d\mu_{X_n}$ converges to $\int_{\mathbb{C}} f \, d\mu$ in probability.*
- *X_n converges to μ* ***almost surely*** *if for every $f \in C_c(\mathbb{C})$, $\int_{\mathbb{C}} f \, d\mu_{X_n}$ converges to $\int_{\mathbb{C}} f \, d\mu$ almost surely.*

When the X_n are Hermitian random matrices and μ is a real probability measure, then we consider the vague convergence on $C_c(\mathbb{R})$ instead of $C_c(\mathbb{C})$.

As in the case of classical random variables,

$$\begin{aligned} \text{almost sure convergence} &\Longrightarrow \text{convergence in probability} \\ \text{convergence in probability} &\Longrightarrow \text{convergence in expectation}\,. \end{aligned} \tag{1.33}$$

In practice, for many random matrix models, these three convergences of the ESDs coincide, due to concentration of measure phenomena.

We describe 2 general methods for proving the convergence of the empirical spectral distributions of Hermitian random matrices, one of which we will adapt to prove the convergence of the law of a non-Hermitian random matrix model.

1.4.1 Moment method

Recall that for compactly supported real measures μ, the moments $m_k = \int_{\mathbb{R}} t^k \, d\mu$ determine the measure. Further, if μ_n, μ are real measures that are all supported on a compact set $K \subset \mathbb{R}$, then the convergence of the moments of μ_n to μ implies the convergence of μ_n to μ in the vague topology. Both of these facts follow from the Weierstrass approximation theorem.

This motivates the following Proposition generalizing this to the convergence of random measures. In particular, this can be applied to empirical spectral distributions of random matrices. Here we consider the simplest case, where measures are all supported on a compact set. In general, for potentially unbounded random matrix models, one needs some quantitative estimates on the tightness of the measures to turn the convergence of the moments into the vague convergence on $\mathbb{R}$.

Proposition 1.23. *Let μ_n be a sequence of random probability measures on $\mathbb{R}$ and let μ be a deterministic probability measure. Assume that μ_n, μ are almost surely supported on a compact set $K \subset \mathbb{R}$. For every $k \in \mathbb{N}$, let*

$$\begin{aligned} m_{n,k} &= \int_{\mathbb{R}} t^k \, d\mu_n(t) \\ m_k &= \int_{\mathbb{R}} t^k \, d\mu(t) \, . \end{aligned} \tag{1.34}$$

Then,

1. *If $\mathbb{E}[m_{n,k}] \to m_k$ as $n \to \infty$ for every $k \in \mathbb{N}$, then $\mu_n \to \mu$ in expectation.*
2. *If $m_{n,k} \to m_k$ in probability as $n \to \infty$ for every $k \in \mathbb{N}$, then $\mu_n \to \mu$ in probability.*
3. *If $m_{n,k} \to m_k$ almost surely as $n \to \infty$ for every $k \in \mathbb{N}$, then $\mu_n \to \mu$ almost surely.*

Proof. 1. For the first point, note that

$$\mathbb{E}[m_{n,k}] = \int_{\mathbb{R}} t^k \, d\overline{\mu_n}(t) \, . \tag{1.35}$$

Since the μ_n are almost surely supported on K, then so are $\overline{\mu_n}$. Then, the result follows from the convergence of moments of $\overline{\mu_n}$ to μ.

2. By approximating any $f \in C(K)$ uniformly by polynomials, it suffices to prove the convergence in probability for polynomials. Any polynomial $P(t)$ is a linear combination of the t^k, and since the $m_{n,k}$ converge to m_k in probability, then

$$\int_{\mathbb{R}} P(t)\, d\mu_n(t) \to \int_{\mathbb{R}} P(t)\, d\mu(t) \tag{1.36}$$

in probability also.

3. Let $f \in C(K)$ and consider polynomials $P_m(t)$ such that $\|f - P_m\|_{L^\infty(K)} < 1/m$. Then, since P_m are linear combination of the t^k, then

$$\int_{\mathbb{R}} P_m(t)\, d\mu_n(t) \to \int_{\mathbb{R}} P_m(t)\, d\mu(t) \tag{1.37}$$

almost surely. On the measure 1 set where these integrals converge for all m, then by approximating f with P_m, we see that

$$\int_{\mathbb{R}} f(t)\, d\mu_n(t) \to \int_{\mathbb{R}} f(t)\, d\mu(t)\,. \tag{1.38}$$

$\square$

We observe that the second condition in Proposition 1.23 is equivalent to the variances of $m_{n,k}$ converging to 0 as $n \to \infty$, for any $k \in \mathbb{N}$.

1.4.2 Stieltjes transform

For the second method, we consider the Stieltjes transform (also called the Cauchy transform) of a real probability measure μ:

Definition 1.24. *Let μ be a probability measure on $\mathbb{R}$. Then, the* ***Stieltjes transform***

*(also called **Cauchy transform**) of μ is the function $G_\mu : \mathbb{C} \setminus supp(\mu) \to \mathbb{C}$ given by:*

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t}\, d\mu(t)\,. \tag{1.39}$$

We list some of the well-known facts about G_μ in the following Proposition:

Proposition 1.25. *Let G_μ be the Stieltjes transform of μ. Then,*

- *G_μ is analytic on $\mathbb{C} \setminus supp(\mu)$.*
- *Let $\mathbb{H}^{\pm}(\mathbb{C})$ be the upper/lower half-planes of $\mathbb{C}$. Then, $G_\mu : \mathbb{H}^{\pm} \to \mathbb{H}^{\mp}$. In particular, $G_\mu(z) \in \mathbb{R}$ if and only if $z \in \mathbb{R} \setminus supp(\mu)$.*
- *$\overline{G_\mu(z)} = G_\mu(\overline{z})$ for $z \in \mathbb{C} \setminus supp(\mu)$.*
- *$|G_\mu(z)| \leq \frac{1}{|\mathrm{Im}(z)|}$.*
- *G_μ is analytic on $\mathbb{C} \setminus supp(\mu)$.*
- *μ is compactly supported, $G_\mu(z)$ has the following Laurent series expansion for $|z| > \sup_{\lambda \in supp(\mu)} |\lambda|$:*

$$G_\mu(z) = \sum_{n=0}^{\infty} \left(\int_{\mathbb{R}} t^n \, d\mu(t) \right) z^{-n-1}\,. \tag{1.40}$$

Thus, for compact measures, the Stieltjes transform of a measure μ contains the same information as the moments of μ.

We highlight one special fact about the Stieltjes transform. For $z = a + ib$, note that

$$\mathrm{Im}\, \frac{1}{z-t} = -\frac{b}{(t-a)^2 + b^2}\,. \tag{1.41}$$

Recall that the Poisson kernel on the upper half-plane is given by:

$$P_b(a) = \frac{1}{\pi} \frac{b}{a^2 + b^2}\,, \quad b > 0\,. \tag{1.42}$$

Combining these two facts shows that

$$-\frac{1}{\pi}\operatorname{Im} G_\mu(a+ib) = (\mu * P_b)(a)\,. \tag{1.43}$$

Recall that P_b are approximations to the identity as $b \to 0^+$, so then

$$\lim_{b\to 0^+} -\frac{1}{\pi}\operatorname{Im} G_\mu(\cdot + ib) = \mu \tag{1.44}$$

in the vague topology on $\mathbb{R}$.

By exploiting the conjugate symmetry of G_μ, we can also write this in terms of the discontinuity of G_μ across $\mathbb{R}$:

$$\lim_{b\to 0^+} -\frac{G_\mu(\cdot + ib) - G_\mu(\cdot - ib)}{2\pi i} = \mu \tag{1.45}$$

in the vague topology on $\mathbb{R}$.

Additionally, there is an explicit formula for intervals ([MS17], Theorem 6):

Proposition 1.26. *For $a, b \in \mathbb{R}$ and $a < b$,*

$$\lim_{y\to 0^+} \int_a^b -\frac{1}{\pi}\mathrm{Im}G_\mu(x+iy)\,dx = \mu((a,b)) + \frac{1}{2}\mu(\{a\}) + \frac{1}{2}\mu(\{b\})\,. \tag{1.46}$$

The analogous result to the moment method is the following result about the Stieltjes transforms ([Tao12], Exercise 2.4.10):

Proposition 1.27. *Let μ_n be a sequence of random probability measures on $\mathbb{R}$, and let μ be a deterministic probability measure. Then,*

1. *μ_n converges in expectation to μ if and only if $\mathbb{E}G_{\mu_n}(z)$ converges to $G_\mu(z)$ for every z in the upper half-plane.*

2. *μ_n converges in probability to μ if and only if $G_{\mu_n}(z)$ converges in probability to $G_\mu(z)$ for every z in the upper half-plane.*

3. *μ_n converges almost surely to μ if and only if $G_{\mu_n}(z)$ converges almost surely to $G_\mu(z)$ for every z in the upper half-plane.*

Sketch of Proof. Make the randomness of the μ_n explicit by letting $\mu_n : (\Omega, \mathbb{P}) \to \mathcal{P}(\mathbb{R})$.

For the "only if" directions, note that $(z - \cdot)^{-1}$ is not a compactly supported function on $\mathbb{R}$ but is bounded uniformly by $|\mathrm{Im} z|^{-1}$. So, we need to use the appropriate quantitative statements of tightness of the μ_n in each case. We state the appropriate tightness conditions in each case:

1. For every $\epsilon > 0$, there exists a compact set K_ϵ such that for every n, $\mathbb{E}_{\mathbb{P}}(\mu_n(\mathbb{R} \setminus K_\epsilon)) < \epsilon$.

2. For every $\epsilon, \delta > 0$ there exists a compact set K_ϵ so that for all n sufficiently large, $\mathbb{P}(\mu_n(K_\epsilon) < 1 - \epsilon) < \delta$.

3. For every m, there exists a K_m such that on a measure 1 set, $\liminf_{n\to\infty} \mu_n(K_m) > 1 - 1/m$.

For the "if" directions, recall that for $f \in C_c(\mathbb{R})$, $f * P_b \to f$ uniformly on $\mathbb{R}$ as $b \to 0^+$. Then,

$$\int_{\mathbb{R}} f \, d\mu \approx \int_{\mathbb{R}} f * P_b \, d\mu = -\frac{1}{\pi} \int_{\mathbb{R}} G_\mu(a + ib) f(a) da \,, \tag{1.47}$$

and similarly for μ_n. Then, we apply Fubini's theorem to change the conditions on the pointwise convergence in z to pointwise convergence in ω. Finally, we use the bound $|G_\mu(a + ib)| \le b^{-1}$ to argue the convergence of the integrals

$$\frac{1}{\pi} \int_{\mathbb{R}} G_{\mu_n}(a + ib) f(a) da \to \frac{1}{\pi} \int_{\mathbb{R}} G_\mu(a + ib) f(a) da \,. \tag{1.48}$$

□

We will adapt the Stieltjes transform method to complex measures in Proposition 6.3 by considering a different function.

1.5 Free probability

In this section, we discuss the definition of free independence and its applications to random matrices. This material is summarized from [AGZ10], [MS17], and [Shl].

Recall that in the classical probability space (X, μ), two random variables x_1 and x_2 are independent if for all integrable $f_1, f_2 : X \to \mathbb{C}$, $\mathbb{E}(f_1(x_1)f_2(x_2)) = \mathbb{E}(f(x_1))\mathbb{E}(f(x_2))$. Equivalently, for all $f_1, f_2 : X \to \mathbb{C}$ such that $\mathbb{E}(f_1(x_1)) = \mathbb{E}(f_2(x_2)) = 0$, $\mathbb{E}(f_1(x_1)f_2(x_2)) = 0$.

In a tracial von Neumann algebra (M, τ), the expectation becomes τ and functions $f_1(x_1)$, $f_2(x_2)$ just become elements of the algebra generated by x_1 and x_2. This motivates the following definition of free independence of subalgebras in a non-commutative probability space:

Definition 1.28. *Let (A, ϕ) be a non-commutative probability space, and let $\{A_i\}_{i\in I}$ be a family of unital subalgebras of A. $\{A_i\}_{i\in I}$ are **freely independent** if for any $a_j \in A_{k(j)}$ with $k(j) \neq k(j+1)$, $j = 1, \ldots, n = 1$ and $\phi(a_i) = 0$, then*

$$\phi(a_1 \ldots, a_n) = 0\,. \tag{1.49}$$

*Let $r, (m_k)_{1\leq k\leq r}$ be positive integers. The sets $\{X_{1,p}, \ldots, X_{m_p,p}\}_{1\leq p\leq r}$ of non-commutative random variables are **free** if the algebras they generate are free.*

If $\{A_i\}$ are freely independent and generate the algebra A, then $\phi\big|_A$ is determined by $\phi\big|_{A_i}$.

In particular, we are interested in the case that (A, ϕ) is a tracial von Neumann algebra (M, τ). In this situation, this condition can be extended to von Neumann subalgebras A_i (i.e. s.o. closures of the A_i) using the normality of τ.

In classical probability theory, one can create independent random variables by forming the product of probability spaces. There is a similar construction in non-commutative probability, where we may form the ***free product*** of (A_i, ϕ_i), $i = 1, \ldots, k$, (A, ϕ). (A, ϕ) contains embedded copies of (A_i, ϕ_i) that are free. In particular, for tracial von Neumann algebras (M_i, τ_i), we may form a tracial von Neumann algebra (M, τ) that has embedded copies of (M_i, τ_i).

1.5.1 Free probability transforms

Before considering some consequences of free independence, we define some transforms of real measures similar to the Stieltjes transform. For a Hermitian $x \in (M, \tau)$, we will abuse notation by using the subscript x to denote the integral transform with respect to the spectral measure μ_x: for example, G_x means the Stieltjes transform of the spectral measure μ_x.

Consider the Stieltjes transform $G_\mu(z)$ when μ is a compactly supported probability measure on $\mathbb{R}$. It is clear that

$$\lim_{|z|\to\infty} zG_\mu(z) = 1\,. \tag{1.50}$$

Since G_μ is analytic outside of a large disk centered at 0, then $F_\mu(z) = G_\mu(1/z)$ is analytic on a punctured disk centered at 0 and can be analytically continued to have $F_\mu(0) = 0$. Then, F_μ has a simple zero at 0, and hence F_μ is invertible in a neighborhood of 0. This implies that G_μ is invertible in a neighborhood of infinity.

Hence, we may consider the R-transform of a measure μ:

Definition 1.29. *Let μ be a compactly supported probability measure on $\mathbb{R}$. The **R-transform** of μ is the function*

$$R_\mu(w) = G_\mu^{\langle -1\rangle}(w) - \frac{1}{w}\,, \tag{1.51}$$

which is defined in a neighborhood of 0 in $\mathbb{C}$.

Computation using F_μ shows that R_μ is analytic in a neighborhood of 0, as the $1/w$ is exactly the pole of $G_\mu^{\langle -1\rangle}$ at $w = 0$.

For non-compactly supported measures μ, then $G_\mu^{\langle -1\rangle}$ can still be defined on wedge-shaped domains containing 0 ([MS17], Theorem 33), but we will not need this fact.

Similar to the Stieltjes transform, we can define ψ_μ:

Definition 1.30. *Let μ be a probability measure on $[0,\infty)$. Let $supp(\mu)^{-1} = \{1/x : x \in supp(\mu)\}$. Define $\psi_\mu : \mathbb{C} \setminus supp(\mu)^{-1} \to \mathbb{C}$ by:*

$$\psi_\mu(z) = \int_{\mathbb{R}} \frac{tz}{1-tz}\, d\mu(t)\,. \tag{1.52}$$

We list some well-known properties of ψ_μ in the following Proposition:

Proposition 1.31. *Let ψ_μ be as above. Then,*

- $\psi_\mu(0) = 0$.
- *ψ_μ is analytic on $\mathbb{C} \setminus supp(\mu)^{-1}$.*
- *$\overline{\psi_\mu(z)} = \psi_\mu(\overline{z})$ for all $z \in \mathbb{C} \setminus supp(\mu)^{-1}$.*
- *Let $\mathbb{H}^{\pm}(\mathbb{C})$ be the upper/lower half-planes of $\mathbb{C}$. If $\mu(\{0\}) < 1$, then $\psi_\mu : \mathbb{H}^{\pm}(\mathbb{C}) \to \mathbb{H}^{\pm}(\mathbb{C})$. In particular, $\psi_\mu(z) \in \mathbb{R}$ if and only if $z \in \mathbb{R} \setminus supp(\mu)^{-1}$.*
- *If μ is compactly supported, then $G_\mu(z)$ has the following Taylor series expansion for $|z| < \inf_{\lambda \in supp(\mu)^{-1}} |\lambda|$:*

$$\psi_\mu(z) = \sum_{n=1}^{\infty} \left(\int_{\mathbb{R}} t^n \, d\mu(t) \right) z^n\,. \tag{1.53}$$

We highlight the following equation, valid for $z \in \mathbb{C} \setminus \mathrm{supp}(\mu)$:

$$G_\mu(z) = \frac{1}{z} \left(\psi_\mu \left(\frac{1}{z} \right) + 1 \right)\,. \tag{1.54}$$

To consider the inverse of ψ_μ at $z = 0$, note that

$$\psi'(0) = \int_{\mathbb{R}} t\, d\mu(t)\,. \tag{1.55}$$

As long as $\mu(\{0\}) < 1$, then ψ_μ is invertible in a neighborhood of 0. In this situation, define the following functions:

Definition 1.32. *Let μ be a compactly supported probability measure on $[0,\infty)$ such that $\mu(\{0\}) < 1$. Then, ψ_μ is invertible in a neighborhood of 0 in $\mathbb{C}$. Define*

$$\begin{aligned} \chi_\mu(w) &= \psi_\mu^{\langle -1 \rangle}(w) \\ S_\mu(w) &= \chi_\mu(w)\frac{w+1}{w}\,. \end{aligned} \tag{1.56}$$

S_μ is called the ***S-transform*** *of μ.*

In particular, since $\chi_\mu(0) = 0$ then S_μ is well-defined in a neighborhood of 0.

1.5.2 Free additive and multiplicative convolution

In this section, we describe the free additive and multiplicative convolution and the free addition/multiplication laws.

First, we define the additive/multiplicative convolution for general non-commutative probability spaces:

Definition 1.33. *Let (A,ϕ) be a non-commutative probability space and $a, b \in A$ with laws μ_a, μ_b, respectively. If a and b are freely independent, then the* ***free additive convolution*** *of a and b is the law of $a+b$, denoted $\mu_a \boxplus \mu_b$. The* ***free multiplicative convolution*** *of a and b is the law of ab, denoted $\mu_a \boxtimes \mu_b$.*

We are interested in the situation where $(A,\phi) = (M,\tau)$ and when $x, y \in (M,\tau)$ are Hermitian. In this case the laws of x and y are the spectral measures $\mu_x, \mu_y \in \mathcal{P}(\mathbb{R})$. Then, $\mu_x \boxplus \mu_y$ is the spectral measure of $x+y$ when x and y are free.

Note that for x and y Hermitian, xy is not generally Hermitian. For the free multiplicative convolution consider $x, y \in (M,\tau)$ that are both positive operators (so then $\mu_x, \mu_y \in \mathcal{P}([0,\infty))$). Then, the law of xy can be identified with the law of $x^{1/2}yx^{1/2}$, which is Hermitian. Hence $\mu_x \boxtimes \mu_y$ is the spectral measure of $x^{1/2}yx^{1/2}$, which is also a measure on $[0,\infty)$.

Then, the free additive (resp. multiplicative) convolution satisfies the following addition (resp. multiplication) laws:

Theorem 1.34. *Let $x, y \in (M, \tau)$ be Hermitian and freely independent. Where the functions are defined,*

$$R_{x \boxplus y}(z) = R_{\mu_x}(z) + R_{\mu_y}(z)\,. \tag{1.57}$$

If x, y are positive, then where the functions are defined,

$$S_{x \boxtimes y}(z) = S_{\mu_x}(z) S_{\mu_y}(z)\,. \tag{1.58}$$

1.5.3 Asymptotic freeness

The connection between random matrices and free probability is that the empirical spectral distributions of several standard random matrix models become asymptotically free. First, we define what it means for non-commutative random variables to be asymptotically free:

Definition 1.35. *A sequence of non-commutative random variables $(\{a_i^n\}_{i\in J})_{n\in\mathbb{N}}$ in non-commutative probability spaces (A_n, ϕ_n) is* ***asymptotically free*** *if it converges in law to some non-commutative random variables $\{a_i\}_{i\in J}$ in a non-commutative probability space (A, ϕ), where $\{a_i\}_{i\in J}$ are free. A collection of subalgebras $\{A_i^n\}_{i\in J} \subset A_n$ is asymptotically free if for every $a_i^n \in A_i^n$, $(\{a_i^n\}_{i\in J})_{n\in\mathbb{N}}$ is asymptotically free.*

Specifically, we will consider (A_n, ϕ_n) to be $(M_n(L^\infty(X, \mu)), \mathbb{E}_\mu[\frac{1}{n}\mathrm{tr}])$. In analogy to convergence in probability and convergence almost surely, we would like to define what it means to be asymptotically free in probability and almost surely asymptotically free:

Definition 1.36. *Let $(\{X_i^n\}_{i\in J})_{n\in\mathbb{N}}$ be a sequence of random matrices in $(M_n(L^\infty(X, \mu)), \mathbb{E}_\mu[\frac{1}{n}\mathrm{tr}])$. Then, $(\{X_i^n\}_{i\in J})_{n\in\mathbb{N}}$ is* ***almost surely asymptotically free*** *(resp.* ***asymptotically free in probability****) if there exists freely independent non-commutative random variables $\{a_i\}_{i\in J}$ in some non-commutative probability space (A, ϕ), where for every non-commutative polynomial $P \in \mathbb{C}\langle X_i, X_i^* : i \in J\rangle$,*

$$\frac{1}{n}\mathrm{tr} P(\{X_i^n, (X_i^n)^*\}_{i\in J}) \to \phi(P(\{a_i, a_i^*\}_{i\in J})) \tag{1.59}$$

almost surely (resp. in probability).

A collection of subalgebras $\{A_i^n\}_{i\in J} \subset M_n(L^\infty(X,\mu))$ *is almost surely asymptotically free (resp. asymptotically free in probability) if for every* $X_i^n \in A_i^n$, $(\{a_i^n\}_{i\in J})_{n\in\mathbb{N}}$ *is almost surely asymptotically free (resp. asymptotically free in probability).*

One example of asymptotic freeness of a certain random matrix model that will be relevant later on is the following ([AGZ10], Theorem 5.4.10):

Theorem 1.37. *Let* $\{D_i^n\}_{1\le i\le p}$ *be deterministic Hermitian matrices in* $(M_n(\mathbb{C}), \frac{1}{n}\mathrm{tr})$ *with uniformly bounded eigenvalues whose law converges to the law of* $\{D_i\}_{1\le i\le p}$ *in a non-commutative probability space* (A,ϕ)*. Let* $\{U_i^n\}_{1\le i\le p}$ *be independent random unitary matrices, distributed by the Haar measure, independent from* $\{D_i^n\}_{1\le i\le p}$*. Then, the* $*$*-subalgebras* $\mathscr{U}_i^n$ *generated by the matrices* U_i^n *and the subalgebra* $\mathscr{D}^n$ *generated by* $\{D_i^n\}_{1\le i\le p}$ *are asymptotically free (resp. almost surely asymptotically free). The limit law of* $\{U_i^n, (U_i^n)^*\}$ *is* $\{U, U^*\}$*, where* $\tau((UU^*-1)^2) = 0$, $\tau(U^n) = \tau((U^*)^n) = \delta_0(n)$.

CHAPTER 2

Outline of results

In this chapter, we describe the central problem of the thesis and outline the remaining sections.

We consider operators of the form $X = p + iq$, where $p, q \in (M, \tau)$ are Hermitian, freely independent, and their spectral measures are atomic, i.e.

$$\begin{aligned}\mu_p &= a_1\delta_{\alpha_1} + \cdots + a_k\delta_{\alpha_k} \\ \mu_q &= b_1\delta_{\beta_1} + \cdots + b_l\delta_{\beta_l}\ ,\end{aligned} \tag{2.1}$$

where $\alpha_i, \beta_j \in \mathbb{R}$, $a_i, b_j \geq 0$, and $a_1 + \cdots + a_k = b_1 + \cdots b_l = 1$.

When the α_i are distinct and $a_i > 0$, we say that **p has k atoms**. Similarly, when the β_j are distinct and $b_i > 0$, we say that **q has l atoms**. When p and q are not constants, then $X = p + iq$ is not normal. We deduce this in Corollary 4.16.

There is a corresponding random matrix model X_n. To define X_n, first define the following:

- Let $\mathcal{H}_n$ denote the Haar measure on the unitary group $U(n) \subset M_n(\mathbb{C})$.
- Let $\{U_n, V_n\}$ be a sequence of independent, $\mathcal{H}_n$-distributed matrices.
- Let $P_n', Q_n' \in M_n(\mathbb{C})$ be deterministic, Hermitian, and

$$\begin{aligned}\mu_{P_n'} &= (a_1)_n\delta_{(\alpha_1)_n} + \cdots + (a_k)_n\delta_{(\alpha_k)_n} \\ \mu_{Q_n'} &= (b_1)_n\delta_{(\beta_1)_n} + \cdots + (b_l)_n\delta_{(\beta_l)_n}\ ,\end{aligned} \tag{2.2}$$

where $(\alpha_i)_n, (\beta_j)_n \in \mathbb{R}$, $(a_i)_n, (b_j)_n \geq 0$, and $(a_1)_n + \cdots + (a_k)_n = (b_1)_n + \cdots (b_l)_n = 1$.

Let

$$P_n = U_n P_n' U_n^*$$
$$Q_n = V_n Q_n' V_n^* \,. \tag{2.3}$$

Thus, $P_n, Q_n \in M_n(\mathbb{C})$ are independently Haar-rotated Hermitian matrices with the same distributions as P_n', Q_n'. Note that μ_{P_n} and μ_{Q_n} are deterministic probability measures on $\mathbb{R}$.

We define our random matrix model $X_n \in M_n(\mathbb{C})$ as:

$$X_n = P_n + iQ_n \,. \tag{2.4}$$

We summarize the definition of X_n by the following:

Definition 2.1. *The random matrix model $X_n = P_n + iQ_n$, where $P_n, Q_n \in M_n(\mathbb{C})$ are independently Haar-rotated Hermitian matrices with spectral measures*

$$\mu_{P_n} = (a_1)_n \delta_{(\alpha_1)_n} + \cdots + (a_k)_n \delta_{(\alpha_k)_n}$$
$$\mu_{Q_n} = (b_1)_n \delta_{(\beta_1)_n} + \cdots + (b_l)_n \delta_{(\beta_l)_n} \,, \tag{2.5}$$

where $(\alpha_i)_n, (\beta_j)_n \in \mathbb{R}$, $(a_i)_n, (b_j)_n \geq 0$, and $(a_1)_n + \cdots + (a_k)_n = (b_1)_n + \cdots (b_l)_n = 1$.

When P_n, Q_n are not constant, then $X_n = P_n + iQ_n$ is normal with probability 0. This is proven in Proposition 5.2

We give examples of some Mathematica plots of the empirical spectral distributions of the X_n for different μ_p, μ_q and some deterministic U_n, V_n:

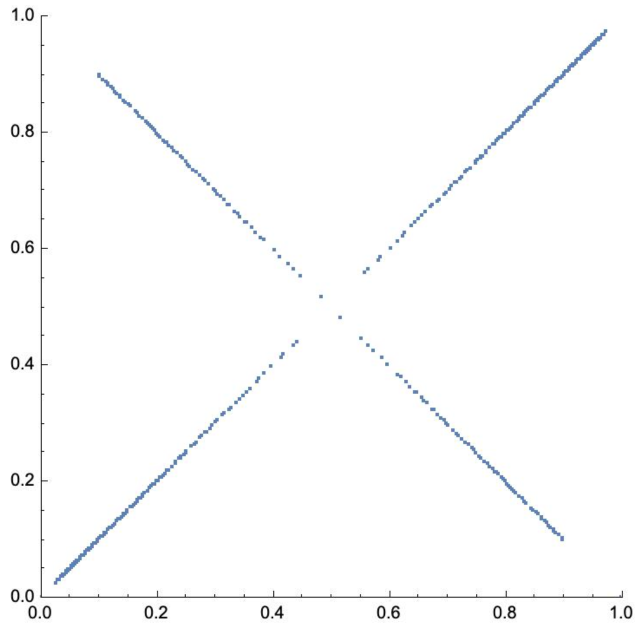


Figure 2.1: ESD of $X_n = P_n + iQ_n$
$\mu_{P_n} = (2/5)\delta_0 + (3/5)\delta_1$
$\mu_{Q_n} = (1/5)\delta_0 + (4/5)\delta_1$
$n = 1000$

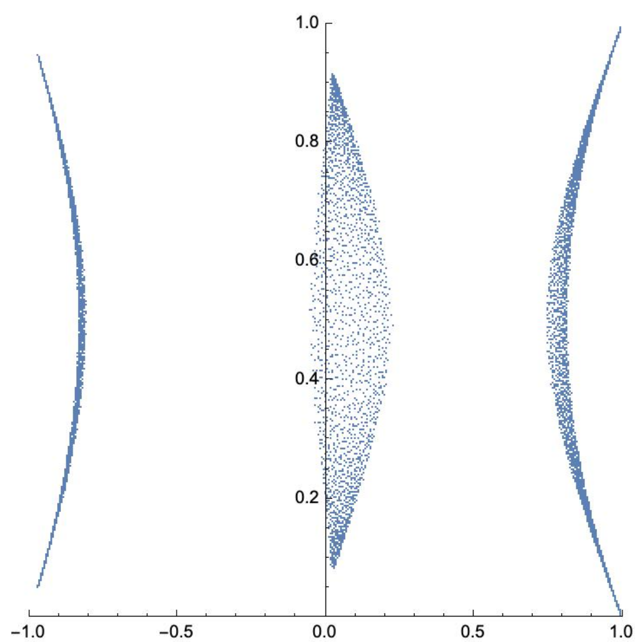


Figure 2.2: ESD of $X_n = P_n + iQ_n$
$\mu_{P_n} = (1/4)\delta_{-1} + (1/5)\delta_0 + (11/20)\delta_1$
$\mu_{Q_n} = (1/2)\delta_0 + (1/2)\delta_1$
$n = 10000$

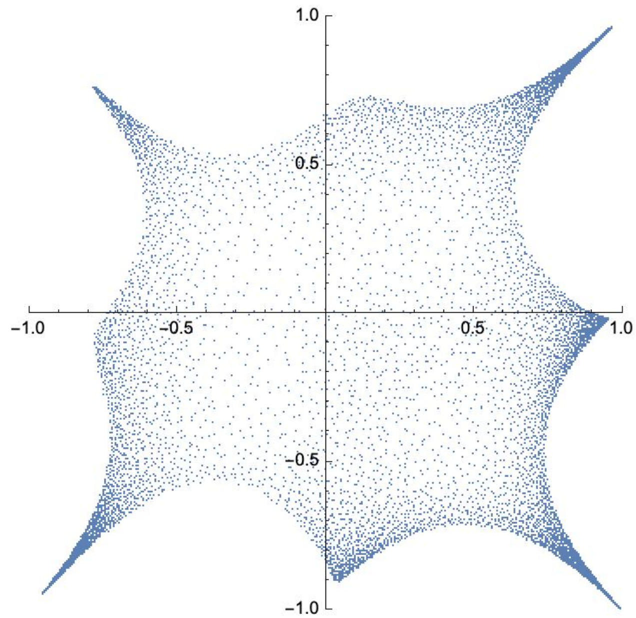


Figure 2.3: ESD of $X_n = P_n + iQ_n$
$\mu_{P_n} = (1/4)\delta_{-1} + (1/5)\delta_0 + (11/20)\delta_1$
$\mu_{Q_n} = (1/2)\delta_{-1} + (1/4)\delta_0 + (1/4)\delta_1$
$n = 10000$

We are interested in understanding the Brown measure of the $X = p + iq$ and its relationship with the empirical spectral distributions of the X_n. We will formally define the Brown measure in the next chapter. For this outline, it suffices to know that for any $X \in (M, \tau)$, the Brown measure of X is a complex probability measure that is supported on $\sigma(X)$. The Brown measure of X agrees with the spectral measure of X when X is normal and with the empirical spectral distribution of X if $X = X_n$ is a random matrix.

The Brown measure was first introduced in [Bro86]. The first interesting explicit computations for the Brown measure of non-trivial operators were provided in [HL00]. This class of operators is called "R-diagonal operators" and includes Voiculescu's circular operator. The circular law in [TV10] and the single ring theorem in [GKZ11] showed that the limit of the ESDs of these random matrix models is the Brown measure of the natural limit operator from free probability.

In general, the limit of the ESDs of random matrices is not the Brown measure of the limit operator (see (Chapter 11, Exercise 5 in [MS17]) for a simple example where this does not hold). But, in [Śn02] it was shown that any random matrix model X_n that converges in law to x can be perturbed in such a way where the new random matrix model still converges in law to x and the new ESDs converge to the Brown measure of X. Thus, we expect that in most cases, the Brown measure is the limit of the ESDs.

The explicit computation of the Brown measure of different families of operators arising from free probability is an active area of research. These computations are interesting in their own right, as they are difficult and the resulting measures are often complicated but have interesting properties (ex. the Brown measure of the R-diagonal operators from [HL00] are supported on a single ring and are rotationally symmetric). More importantly, they provide the candidate limits for the ESDs of these random matrix models, although there have been relatively few results on the convergence of the ESDs to these Brown measures besides deductions from the single ring theorem and the circular law (for a recent generalization of the single ring theorem, see [HZ23b]). Finally, the motivation for the computation of the Brown measure has led to the development of new techniques in operator-valued free probability (see [BSS18], [BMS17]) and an application of stochastic PDEs to free probability (see [Hal21]).

The operators we consider are distinctly different from other examples where the Brown measure has been explicitly computed, as they are made from purely atomic operators. Other operators considered in the past have been created from other operators (from products or non-Hermitian sums) where at least one has Brown measure that is atomless (ex. circular operator, semicircular operator, Haar unitary).

For our operators $X = p + iq$, when either p or q is constant (i.e. real) then X is normal so that the Brown measure of X is just the spectral measure of X. When p and q have 2 atoms, we can explicitly compute the Brown measure of X. In this case, the Brown measure of X is supported on hyperbolas. These seem to be one of few explicit computations of the Brown measure that have 1-dimensional support (besides Hermitian/normal operators and Haar unitaries). We also determine some interesting properties, such as the exact support of the Brown measure on the hyperbolas and the weights/positions of the atoms.

Consider our random matrix model X_n when P_n converges in law to p and Q_n converges in law to q. Then, p and q have finitely many atoms. Making p and q freely independent, then $X = p + iq$ is one of the operators we consider. From the asymptotic freeness of independent Haar unitaries from Hermitian matrices (Theorem 1.37), the law of X_n converges to the law of X. From the previous discussion, we expect that the empirical spectral distribution of X_n converges to the Brown measure of X. We can prove this when P_n and Q_n each have at most 2 atoms, using the geometry of the support of the Brown measures. This provides one of the relatively few non-trivial examples where the convergence of the ESDs to the Brown measure can be proven.

In the general case of many atoms, we use a technique from the physics literature using a quaternionic analogue of the Stieltjes transform. By a simple change of coordinates, we can relate this Quaternionic Green's function from the physics literature to the operator-valued Cauchy transform in the mathematics literature, which has been used in analyzing the Brown measure (see [BSS18], [BMS17]).

The benefit of using the Quaternionic Green's function is the algebraic structure of the domain and range being quaternions, which allows for explicit computations with the representation of quaternions as a subalgebra of complex 2×2 matrices. In particular, for $X = p + iq$, where $p, q \in (M, \tau)$ are Hermitian and freely independent, there are formulas for the Quaternionic Green's function in terms of the (complex) Stieltjes transforms of p and q.

In the physics literature, the use of the Quaternionic Green's function is oftentimes not mathematically rigorous but is useful as a guiding heuristic. In this area, formal arguments from the physics literature often are eventually made mathematically rigorous (ex. the limit of the Ginibre ensemble introduced in [Gin65] was made rigorous in [Meh67] and the single ring theorem introduced in [FZ97a] was made rigorous in [GKZ11]). In our situation, there are heuristics for how to determine the boundary of the Brown measure and the support of the Brown measure for $X = p + iq$, where $p, q \in (M, \tau)$ are Hermitian and freely independent. In [JN04], the authors showed that the heuristic produces the correct results for several examples, including the circular operator.

We will verify these heuristics in the case when p and q have 2 atoms. In the general case when p and q have finitely many atoms, we will show that the heuristic about the boundary of the Brown measure implies that the boundary of the Brown measure is an algebraic curve and provide an algorithm to compute a non-zero polynomial that defines the algebraic curve. The implications of these heuristics are not surprising: in other examples where the Brown measure has been computed, the support of the Brown measure is on a set defined by an inequality involving the trace of some relevant quantity (see [HL00], [Zho22], [BYZ24] for some examples). In our situation, the Stieltjes transforms of p and q are rational functions, so it is not surprising that the support and the boundary of the Brown measure are determined by algebraic functions. We conclude with a discussion of the atoms of the Brown measure of X and their relation to the Quaternionic Green's function using the analysis from [BSS18].

Several classes of operators that have been previously considered in the literature are of the form $X = p + iq$, where $p, q \in (M, \tau)$ are Hermitian and freely independent (for example, [BL01], [Ho22], [HZ23a]). We would be interested in seeing if the explicit computations using the Quaternionic Green's function can recover the Brown measure in these situations, or at least the boundary/support of the Brown measure. In [HH22], the authors verified that these heuristics recovered the boundary and support of the Brown measure for the operators they considered. In particular, we have discussed the heuristic about the support of the Brown measure with another author, and it is an open problem to determine if this heuristic always holds.

We briefly mention that there is another method that has been recently developed for analyzing the Brown measure of non-Hermitian operators $X \in (M, \tau)$ that come from free probability (see [Hal21] for an introduction to the technique). It involves introducing a time parameter in the operator $X = X_t$ and letting X_t be a free Brownian motion with an initial condition. Then, f_ϵ from (3.38) now also depends on t. If one can derive a stochastic partial differential equation involving $f_{\epsilon,t}(z)$ and understand the solutions of this equation as $\epsilon \to 0^+$, then by taking Laplacians, one can understand the Brown measure of X_t. We were unable to apply this method, as we could not find a stochastic PDE for a suitable X_t for our situation.

We give an outline of the remaining chapters of the thesis:

- Chapter 3: We define the Brown measure of X and prove some basic properties.
- Chapter 4: We compute the Brown measure of $X = p + iq$ when p and q have 2 atoms.
- Chapter 5: We provide some facts about the random matrices X_n and their empirical spectral distributions.
- Chapter 6: We show that when p and q have 2 atoms, the empirical spectral distribution of X_n converges to the Brown measure of X.
- Chapter 7: We describe the quaternionic method of computing the Brown measure and provide the heuristics in determining the boundary and support of the Brown measure for $X = p + iq$, where $p, q \in (M, \tau)$ are Hermitian and freely independent.
- Chapter 8: We compute the relevant functions for the quaternionic method for $X = p+iq$ when p and q have 2 atoms.
- Chapter 9: We consider the heuristic in determining the boundary of the Brown measure of $X = p + iq$, where $p, q \in (M, \tau)$ are Hermitian and freely independent. First, we verify the heuristic when p and q have 2 atoms. Then, we show that the heuristic implies the boundary of the Brown measure lies on algebraic curves and explicitly provide an algorithm to compute a non-zero polynomial defining the curve.
- Chapter 10: We consider the heuristic in determining the support of the Brown measure of $X = p + iq$, where $p, q \in (M, \tau)$ are Hermitian and freely independent. We verify the heuristic holds when p and q have 2 atoms with equal weights.
- Chapter 11: We discuss the atoms of the Brown measure in the context of the quaternionic Green's function and earlier results.

CHAPTER 3

The Brown measure

In this chapter, we define the Brown measure and review some properties about it. Much of this material is taken from [MS17], [Tao12], [HS07].

First, we need to define a generalization of the determinant for tracial von Neumann algebras.

3.1 The Fuglede-Kadison determinant

In [FK52], Fuglede and Kadison defined a positive, normalized determinant for elements in a general tracial von Neumann algebra (M, τ). It is referred to as the Fuglede-Kadison determinant. We motivate this definition by first considering (deterministic) matrices $X_n \in M_n(\mathbb{C})$.

We wish to define the determinant of a matrix X_n in terms of the trace, as the analogue of this is available in a tracial von Neumann algebra. As the trace is the sum of the eigenvalues while the determinant is the product of the eigenvalues, it is natural to consider the logarithm and exponential of a matrix.

Recall for a positive invertible matrix $X_n \in M_n(\mathbb{C})$, $\log X_n$ is well-defined. When X_n is diagonal with entries $\lambda_1, \ldots, \lambda_n$, $\log X_n$ is diagonal with entries $\log \lambda_1, \ldots, \log \lambda_n$. In general, if $X_n = V_n D_n V_n^*$ for a unitary V_n and diagonal D_n, then $\log X_n = V_n \log D_n V_n^*$. Note that the existence of $\log X_n$ also follows from the continuous functional calculus.

For positive matrices X_n,

$$\operatorname{tr}(\log X_n) = \log(\det X_n)\,. \tag{3.1}$$

To generalize this formula, we will need to use the functional calculus to make sense of $\log X_n$. The easiest situation would be if X_n is positive, as then $\log x$ is well-defined on the spectrum of X_n. Then, our determinant will only be a positive determinant (defined on positive elements and taking positive values).

In the case of matrices, for a general matrix X_n, we can consider the positive matrix $|X_n| = (X_n^* X_n)^{1/2}$. When X_n is invertible,

$$\operatorname{tr}\log(|X_n|) = \operatorname{tr}\log((X_n^* X_n)^{1/2}) = \log(\det(X_n^* X_n)^{1/2}) = \log|\det X_n|\,. \tag{3.2}$$

The formula still holds when X_n is not invertible, as both sides are $-\infty$.

Recall the analogous trace to the tracial von Neumann algebra is the normalized trace $\frac{1}{n}\operatorname{tr}$. Thus, we can recover the normalized positive determinant:

$$\exp\left(\frac{1}{n}\operatorname{tr}\log(|X_n|)\right) = |\det(X_n)|^{1/n}\,. \tag{3.3}$$

Note that this normalized determinant has the advantage that it scales linearly when X_n is multiplied by a constant. This is necessary in the general situation as the von Neumann algebra is normalized to have "dimension" (i.e. trace) 1.

On a general tracial von Neumann algebra (M, τ), we define the Fuglede-Kadison determinant in the analogous way:

Definition 3.1. *Let $x \in (M, \tau)$. Let $\mu_{|x|}$ be the spectral measure of $|x| = (x^* x)^{1/2}$. Then, the* ***Fuglede-Kadison determinant*** *of x, $\Delta(x)$, is given by:*

$$\Delta(x) = \exp\left[\int_0^\infty \log t\, d\mu_{|x|}(t)\right]. \tag{3.4}$$

We make a few notes about the definition:

- When $x = X_n \in (M_n(\mathbb{C}), \frac{1}{n}\mathrm{tr})$, then $\Delta(x) = |\det(X_n)|^{1/n}$.

- When x is invertible (with bounded inverse), then $\sigma(|x|) \subset [c, \infty)$ for some $c > 0$, so then $\Delta(x)$ is finite and non-zero.

- When x not invertible, since x is bounded, then $\log^+ t$ is always integrable. Thus, the integral of $\log t$ makes sense, even if it could be infinite. To compute the integral in practice, we can consider $f_\epsilon(t) = \log(t + \epsilon)$ for $\epsilon > 0$. The f_ϵ are continuous on $[0, \infty)$ and decrease to $\log(t)$, so

$$\lim_{\epsilon \to 0^+} \tau(\log(|x| + \epsilon)) = \lim_{\epsilon \to 0^+} \int_0^\infty \log(t + \epsilon)\, d\mu_{|x|}(t) = \int_0^\infty \log t\, d\mu_{|x|}(t)\,. \tag{3.5}$$

  We could also choose $g_\epsilon(t) = \chi_{[\epsilon,\infty)} \log(t)$, but these do not have the differentiability properties of the f_ϵ that we will need later.

- In practice, it is easier to work with $|x|^2 = x^* x$ instead of its square root. Observe that $\mu_{|x|^2}$ is just the pushforward measure of $\mu_{|x|}$ under t^2, since for continuous f on $[0, \infty)$,

$$\int_0^\infty f(t^2)\, d\mu_{|x|}(t) = \tau(f(|x|^2)) = \int_0^\infty f(t)\, d\mu_{|x|^2}(t)\,. \tag{3.6}$$

  Note that we have used that the functional calculus respects composition of functions. Then, x with bounded inverse we immediately obtain the formula:

$$\int_0^\infty \log t\, d\mu_{|x|}(t) = \frac{1}{2} \int_0^\infty \log t\, d\mu_{|x|^2}(t)\,. \tag{3.7}$$

  In the general case, we consider the f_ϵ as before and let $\epsilon \to 0^+$ to obtain the same formula.

In [FK52], several properties of this determinant are proved. We list some of them here:

Proposition 3.2. *Let $x \in (M, \tau)$ and let $\Delta(x)$ be the Fuglede-Kadison determinant of x. Call x **regular** if it has bounded inverse in M. Then, the determinant satisfies the following:*

- $\Delta(x) \leq |\sigma(x)|$ *and* $\|x^{-1}\|^{-1} \leq \Delta(x)$ *for regular* x.

- $\Delta(\lambda x) = |\lambda|\,\Delta(x)$ *for* $\lambda \neq 0$.

- $\Delta(x^*) = \Delta(x) = \Delta(x^*x)^{1/2}$.

- $\Delta(xy) = \Delta(x)\Delta(y)$ *for regular* x, y.

- $\Delta(x^{-1}) = \Delta(x)^{-1}$ *for regular* x.

- $\Delta(x) \geq \Delta(y)$ *if* $x \geq y \geq 0$.

- $\limsup_{n\to\infty} \Delta(x_n) \leq \Delta(x)$ *when* x_n *converges to* x *uniformly.*

- $\lim_{n\to\infty} \Delta(x_n) = \Delta(x)$ *if* $x_n \geq x \geq 0$ *and* x_n *converges to* x *uniformly.*

3.2 Construction of the Brown measure

In this subsection, we motivate and define the Brown measure of an operator $x \in (M, \tau)$.

For motivation, we again consider $X_n \in (M_n(\mathbb{C}), \frac{1}{n}\mathrm{tr})$. Let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ be the eigenvalues of X_n. Recall that the empirical spectral distribution of X_n is

$$\mu_{X_n} = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i}\,. \tag{3.8}$$

To recover this from $\Delta(X_n)$, consider for $z \in \mathbb{C}$, the following equalities:

$$\log \Delta(z - X_n) = \frac{1}{n}\mathrm{tr}\log(|z - X_n|) = \frac{1}{n}\log|\det(z - X_n)| = \frac{1}{n}\sum_{i=1}^{n}\log|z - \lambda_i|\,. \tag{3.9}$$

Recall that as distributions on $\mathcal{D}(\mathbb{C})$,

$$\frac{1}{2\pi}\nabla^2 \log|\cdot - \lambda| = \delta_\lambda\,. \tag{3.10}$$

Then, we recover μ_{X_n} by:

$$\mu_{X_n} = \frac{1}{2\pi}\nabla^2 \log \Delta(z - X_n)\,. \tag{3.11}$$

Note that we recovered the empirical spectral distribution of X_n with only the computation of the positive determinant of $|z - X_n|$. The key reason why this worked is that the absolute value of the product of the eigenvalues of X_n is equal to the product of the singular values of X_n.

In a general von Neumann algebra, we will construct the Brown measure of $x \in (M, \tau)$ with the same formula,

$$\mu_x = \frac{1}{2\pi}\nabla^2 \log \Delta(z - x)\,. \tag{3.12}$$

We need to check that this construction defines a probability measure. For this, consider $f_\epsilon(z) : \mathbb{C} \to \mathbb{R}$ for $\epsilon > 0$, given by:

$$f_\epsilon(z) = \frac{1}{2}\tau\left[\log((z-x)^*(z-x)+\epsilon)\right]\,. \tag{3.13}$$

Note that

$$f_\epsilon(z) = \frac{1}{2}\int_0^\infty \log(t^2+\epsilon)\, d\mu_{|z-x|}(t)\,. \tag{3.14}$$

so then for every $z \in \mathbb{C}$, as $\epsilon \to 0^+$, $f_\epsilon(z)$ decreases to $\log \Delta(z - x)$.

We claim that $f_\epsilon(z)$ is subharmonic on $\mathbb{C}$ for every $\epsilon > 0$. We first observe the following Lemma:

Lemma 3.3. *Let $x \in (M, \tau)$ and $z \in \mathbb{C}$. Define $x_z = z - x$. Then, for every $n \in \mathbb{N}$,*

$$\begin{aligned} \frac{\partial}{\partial z}\tau\left[(x_z^* x_z)^n\right] &= n\tau\left[(x_z^* x_z)^{n-1}x_z^*\right] \\ \frac{\partial}{\partial \bar{z}}\tau\left[(x_z^* x_z)^n x_z^*\right] &= \sum_{i=0}^{n}\tau\left[(x_z x_z^*)^{n-i}(x_z^* x_z)^i\right]\,. \end{aligned} \tag{3.15}$$

Proof. For $h \in \mathbb{C}$, observe the following equalities:

$$\begin{aligned} x_{z+h} - x_z &= h \\ x^*_{z+h} - x^*_z &= h^* \\ x^*_{z+h}x_{z+h} - x^*_z x_z &= h^* x_{z+h} + x^*_z h\,. \end{aligned} \tag{3.16}$$

For the first derivative, note that for $n = 0$ the result is trivial. For $n \geq 1$, rewrite the difference quotient:

$$(x^*_{z+h}x_{z+h})^n - (x^*_z x_z) = \sum_{i=0}^{n-1} (x^*_z x_z)^i \left[x^*_{z+h}x_{z+h} - x^*_z x_z\right] (x^*_{z+h}x_{z+h})^{n-i-1}. \tag{3.17}$$

By using the formula for $x^*_{z+h}x_{z+h} - x^*_z x_z$ for each derivative and using the cyclic property of τ, we see that

$$\begin{aligned} \frac{\partial}{\partial x}\tau\left[(x^*_z x_z)^n\right] &= \sum_{i=0}^{n-1} \tau\left[(x^*_z x_z)^i (x_z + x^*_z)(x^*_z x_z)^{n-i-1}\right] \\ &= n\tau\left[(x^*_z x_z)^{n-1}(x_z + x^*_z)\right] \end{aligned} \tag{3.18}$$

$$\begin{aligned} \frac{\partial}{\partial (iy)}\tau\left[(x^*_z x_z)^n\right] &= \sum_{i=0}^{n-1} \tau\left[(x^*_z x_z)^i (-x_z + x^*_z)(x^*_z x_z)^{n-i-1}\right] \\ &= n\tau\left[(x^*_z x_z)^{n-1}(-x_z + x^*_z)\right]. \end{aligned} \tag{3.19}$$

Averaging these two quantities gives the first derivative.

For the second derivative, the formula for $n = 0$ follows from $x^*_{z+h} - x^*_z = h^*$. For $n \geq 1$, rewrite the difference quotient:

$$\begin{aligned} &(x^*_{z+h}x_{z+h})^n x^*_{z+h} - (x^*_z x_z)x^*_z \\ &= \left(\sum_{i=0}^{n-1} (x^*_z x_z)^i \left[x^*_{z+h}x_{z+h} - x^*_z x_z\right] (x^*_{z+h}x_{z+h})^{n-i-1} x^*_{z+h}\right) + \\ &\qquad (x^*_z x_z)^n (x^*_{z+h} - x^*_z). \end{aligned} \tag{3.20}$$

By using the formula for $x_{z+h}^* x_{z+h} - x_z^* x_z$ for each derivative, we see that

$$\begin{aligned} &\frac{\partial}{\partial x}\tau\left[(x_z^* x_z)^n x_z^*\right] \\ &= \left(\sum_{i=0}^{n-1}\tau\left[(x_z^* x_z)^i (x_z + x_z^*)(x_z^* x_z)^{n-i-1} x_z^*\right]\right) + \tau\left[(x_z^* x_z)^n\right]. \end{aligned} \tag{3.21}$$

$$\begin{aligned} &\frac{\partial}{\partial (iy)}\tau\left[(x_z^* x_z)^n x_z^*\right] \\ &= \left(\sum_{i=0}^{n-1}\tau\left[(x_z^* x_z)^i (-x_z + x_z^*)(x_z^* x_z)^{n-i-1} x_z^*\right]\right) - \tau\left[(x_z^* x_z)^n\right]. \end{aligned} \tag{3.22}$$

Taking half of the difference between these two and using the cyclic property of τ,

$$\begin{aligned} \frac{\partial}{\partial \overline{z}}\tau\left[(x_z^* x_z)^n x_z^*\right] &= \left(\sum_{i=0}^{n-1}\tau\left[(x_z^* x_z)^i x_z (x_z^* x_z)^{n-i-1} x_z^*\right]\right) + \tau\left[(x_z^* x_z)^n\right] \\ &= \left(\sum_{i=0}^{n-1}\tau\left[(x_z x_z^*)^{n-i} (x_z^* x_z)^i\right]\right) + \tau\left[(x_z^* x_z)^n\right] \\ &= \sum_{i=0}^{n}\tau\left[(x_z x_z^*)^{n-i} (x_z^* x_z)^i\right]. \end{aligned} \tag{3.23}$$

□

Proposition 3.4. *Define $x_z = z - x$. For $\epsilon > 0$, let $f_\epsilon : \mathbb{C} \to \mathbb{R}$ where*

$$f_\epsilon(z) = \frac{1}{2}\tau\left[\log(x_z^* x_z + \epsilon)\right]. \tag{3.24}$$

Then,

$$\begin{aligned} \frac{\partial}{\partial z} f_\epsilon(z) &= \frac{1}{2}\tau\left[(x_z^* x_z + \epsilon)^{-1} x_z^*\right] \\ \frac{\partial^2}{\partial \overline{z}\partial z} f_\epsilon(z) &= \frac{\epsilon}{2}\tau\left[(x_z x_z^* + \epsilon)^{-1}(x_z^* x_z + \epsilon)^{-1}\right]. \end{aligned} \tag{3.25}$$

Thus, $f_\epsilon(z)$ is subharmonic on $\mathbb{C}$.

Proof. We first verify the equations for $|\epsilon| > \|x_z^* x_z\|$, $\mathrm{Re}(\epsilon) > 0$. In this case, letting log be the principal branch of the logarithm defined on $\mathbb{C} \setminus (-\infty, 0]$,

$$\log(x_z^* x_z + \epsilon) = \log(\epsilon) + \log\left(1 + \frac{x_z^* x_z}{\epsilon}\right). \tag{3.26}$$

Then, the following power series representation is absolutely convergent:

$$\log\left(1 + \frac{x_z^* x_z}{\epsilon}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{x_z^* x_z}{\epsilon}\right)^n. \tag{3.27}$$

Thus, we may take the trace of both sides:

$$\tau\left[\log\left(1 + \frac{x_z^* x_z}{\epsilon}\right)\right] = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{\tau\left[(x_z^* x_z)^n\right]}{\epsilon^n}. \tag{3.28}$$

A standard result in the interchange of sum and derivative ([Rud76], Theorem 7.17) shows that the term-by-term derivative on the right-hand side is valid if the resulting sum is locally uniformly convergent. Thus, from Lemma 3.3,

$$\begin{aligned} \frac{\partial}{\partial z}\tau\left[\log(x_z^* x_z + \epsilon)\right] &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\epsilon^n} \tau\left[(x_z^* x_z)^{n-1} x_z^*\right] \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{\epsilon^{n+1}} \tau\left[(x_z^* x_z)^n x_z^*\right]. \end{aligned} \tag{3.29}$$

Observe that the following power series formula is valid:

$$\epsilon(x_z^* x_z + \epsilon)^{-1} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{x_z^* x_z}{\epsilon}\right)^n. \tag{3.30}$$

Recognizing this expression in the derivative, we see that

$$\frac{\partial}{\partial z} f_\epsilon(z) = \frac{1}{2}\frac{\partial}{\partial z}\tau\left[\log(x_z^* x_z + \epsilon)\right] = \frac{1}{2}\tau\left[(x_z^* x_z + \epsilon)^{-1} x_z^*\right]. \tag{3.31}$$

To compute the second derivative, we differentiate both sides of (3.29) term-by-term using Lemma 3.3. Again, the term-by-term differentiation of the sum is valid because the resulting sum is locally uniformly convergent:

$$\frac{\partial^2}{\partial\bar{z}\partial z}\tau\left[\log(x_z^*x_z+\epsilon)\right]=\sum_{n=0}^{\infty}\frac{(-1)^n}{\epsilon^{n+1}}\left(\sum_{i=0}^{n}\tau\left[(x_zx_z^*)^{n-i}(x_z^*x_z)^i\right]\right). \tag{3.32}$$

Observe that the following power series formula is valid:

$$\epsilon^2(x_zx_z^*+\epsilon)^{-1}(x_z^*x_z+\epsilon)^{-1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{\epsilon^n}\left(\sum_{i=0}^{n}(x_zx_z^*)^{n-i}(x_z^*x_z)^i\right). \tag{3.33}$$

Recognizing this expression in the derivative, we see that

$$\frac{\partial^2}{\partial\bar{z}\partial z}f_\epsilon(z)=\frac{1}{2}\frac{\partial^2}{\partial\bar{z}\partial z}\tau\left[\log(x_z^*x_z+\epsilon)\right]=\frac{\epsilon}{2}\tau\left[(x_zx_z^*+\epsilon)^{-1}(x_z^*x_z+\epsilon)^{-1}\right]. \tag{3.34}$$

To extend these formulas for all $\mathrm{Re}(\epsilon)>0$, we use analyticity of both sides of these equations for $\mathrm{Re}(\epsilon)>0$. As an example, consider the argument for $\frac{\partial}{\partial x}f_\epsilon(z)$: Fix $z=x+iy$. For $\mathrm{Re}(\epsilon)>0$ and $|\epsilon|$ sufficiently large, the following formula holds:

$$\begin{aligned}f_\epsilon(z)&=f_\epsilon(iy)+\int_0^x\frac{\partial}{\partial x}f_\epsilon(z)\Big|_{z=t+iy}dt\\&=f_\epsilon(iy)+\int_0^x\frac{1}{2}\,\tau\left[(x_z^*x_z+\epsilon)^{-1}(x_z+x_z^*)\right]\Big|_{z=t+iy}dt\,.\end{aligned} \tag{3.35}$$

From the power series representations of log and $(A+\cdot)^{-1}$, the both sides of this equation are analytic for $\mathrm{Re}(\epsilon)>0$. Since this identity holds for $|\epsilon|>\|x_z^*x_z\|$, the the identity holds for all $\mathrm{Re}(\epsilon)>0$. To show that f_ϵ is subharmonic,

$$\nabla^2f_\epsilon(z)=4\frac{\partial^2}{\partial\bar{z}\partial z}f_\epsilon(z)=2\epsilon\,\tau\left[(x_zx_z^*+\epsilon)^{-1}(x_z^*x_z+\epsilon)^{-1}\right]\geq 0\,. \tag{3.36}$$

This last quantity is non-negative, being the trace of a product of 2 positive operators. As this expression is continuous in z, f_ϵ is subharmonic. □

We return to showing that $\mu_x=\frac{1}{2\pi}\nabla^2\log\Delta(z-x)$ defines a positive measure. As $f_\epsilon(z)$ decreases to $\log\Delta(z-x)$ and the $f_\epsilon(z)$ are subharmonic, then $\log\Delta(z-x)$ is either subharmonic or $-\infty$. But, for $|z| > \|x\|$, $\log \Delta(z-x) > -\infty$, so then $\log \Delta(z-x)$ is subharmonic.

In particular, $\log \Delta(z-x) \in L^1_{\text{loc}}(\mathbb{C})$, and since $f_\epsilon(z)$ decrease to $\log \Delta(z-x)$, then $f_\epsilon(z)$ converges to $\log \Delta(z-x)$ in $L^1_{\text{loc}}(\mathbb{C})$. Hence, as distributions, $f_\epsilon(z)$ converges to $\log \Delta(z-x)$. This implies that $\nabla^2 f_\epsilon(z)$ converges to $\nabla^2 \log \Delta(z-x)$. As $\nabla^2 f_\epsilon(z)$ are positive distributions, then so is $\nabla^2 \log \Delta(z-x)$. This implies that $\frac{1}{2\pi}\nabla^2 \log \Delta(z-x)$ is a positive measure, as desired. Thus, we may define the Brown measure as follows:

Definition 3.5. *Let $x \in (M, \tau)$. Then, the **Brown measure** of x is defined as:*

$$\mu_x = \frac{1}{2\pi}\nabla^2 \log \Delta(z-x)\,. \tag{3.37}$$

Using the f_ϵ, the Brown measure can be computed as the distributional limit:

$$\mu_x = \lim_{\epsilon \to 0^+} \frac{1}{2\pi}\nabla^2 f_\epsilon(z) = \lim_{\epsilon \to 0^+} \frac{\epsilon}{\pi}\,\tau\left[(x_z x_z^* + \epsilon)^{-1}(x_z^* x_z + \epsilon)^{-1}\right]. \tag{3.38}$$

3.3 Properties of the Brown measure

First, we consider some examples of the Brown measure:

- When $X_n \in M_n(\mathbb{C})$, then μ_{X_n} is the empirical spectral distribution of X_n. Thus, the notation for Brown measure is consistent with the previous notation for empirical spectral distributions. When X_n is random, the same is true.

- When $x \in (M, \tau)$ is Hermitian, let μ be the spectral measure of x. Then,

$$f_\epsilon(z) = \frac{1}{2}\tau\left[\log((z-x)^*(z-x)+\epsilon)\right] = \frac{1}{2}\int_{\mathbb{R}} \log(|z-t|^2 + \epsilon)\, d\mu(t)\,. \tag{3.39}$$

  Since $f_\epsilon(z)$ decreases to $\log \Delta(z-x)$, then

$$\log \Delta(z-x) = \int_{\mathbb{R}} \log|z-t|\; d\mu(t)\,. \tag{3.40}$$

Taking the Laplacian of the right-hand side with respect to z shows that the Brown measure of x is equal to the spectral measure of x. The same argument holds when x is normal. Thus, the notation for Brown measure is consistent with the previous notation for spectral measures.

- When $f \in (L^\infty(X,\mu), d\mu)$, then

$$f_\epsilon(z) = \frac{1}{2}\tau\left[\log((z-f)^*(z-f)+\epsilon)\right] = \frac{1}{2}\int_X \log(|z-f(x)|^2+\epsilon)\, d\mu(x)\,, \tag{3.41}$$

so then

$$\log\Delta(z-x) = \int_X \log|z-f(x)|\; d\mu(x) = \int_{\mathbb{C}} \log|z-w|\; df_*(\mu)(w). \tag{3.42}$$

Hence, the Brown measure of f is the pushforward measure $f_*(\mu)$.

At this point, we know that the Brown measure is a positive measure, but it still needs to be verified that the Brown measure is a probability measure. It also turns out that the Brown measure is supported on $\sigma(X)$, just like the spectral measure of a normal operator.

First, we highlight the following Lemma:

Lemma 3.6. *Let $x \in (M,\tau)$ and $z \in \mathbb{C}$. For $\epsilon > 0$, let $f_\epsilon : \mathbb{C} \to [-\infty,\infty)$ where*

$$f_\epsilon(z) = \frac{1}{2}\tau\left[\log((z-x)^*(z-x)+\epsilon)\right]\,. \tag{3.43}$$

Suppose that $z \notin \sigma(x)$. Then, as $\epsilon \to 0^+$, we have the following convergences locally uniformly:

$$\begin{aligned}\frac{\partial}{\partial z} f_\epsilon(z) &\to \frac{1}{2}\tau\left[(z-x)^{-1}\right]\\ \frac{\partial^2}{\partial\bar{z}\partial z} f_\epsilon(z) &\to 0\,.\end{aligned} \tag{3.44}$$

Thus, for $z \notin \sigma(x)$,

$$\begin{aligned} \frac{\partial}{\partial z} \log \Delta(z-x) &= \frac{1}{2}\tau\left[(z-x)^{-1}\right] \\ \frac{\partial^2}{\partial \bar{z}\partial z} \log \Delta(z-x) &= 0\,. \end{aligned} \tag{3.45}$$

Proof. Define $x_z = z - x$. For the convergence of the first derivative, using the formula in Proposition 3.4,

$$\begin{aligned} &\frac{\partial}{\partial z} f_\epsilon(z) - \frac{1}{2}\tau\left[(z-x)^{-1}\right] \\ &= \frac{1}{2}\left(\tau\left[(x_z^* x_z + \epsilon)^{-1} x_z^*\right] - \tau\left[x_z^{-1}\right]\right) \\ &= \frac{1}{2}\left(\tau\left[x_z^{-1}(x_z^* x_z + \epsilon)^{-1} x_z^* x_z\right] - \tau\left[x_z^{-1}(x_z^* x_z + \epsilon)^{-1}(x_z^* x_z + \epsilon)\right]\right) \\ &= -\frac{\epsilon}{2}\tau\left[x_z^{-1}(x_z^* x_z + \epsilon)^{-1}\right] . \end{aligned} \tag{3.46}$$

To bound the absolute value of the final expression, note that for any $\epsilon > 0$,

$$\left|\tau\left[x_z^{-1}(x_z^* x_z + \epsilon)^{-1}\right]\right| \le \left\|x_z^{-1}\right\| \left\|(x_z^* x_z + \epsilon)^{-1}\right\| \le \left\|x_z^{-1}\right\| \left\|(x_z^* x_z)^{-1}\right\| . \tag{3.47}$$

This last expression is locally bounded, so then the convergence in the first derivative is locally uniform for $z \notin \sigma(x)$. The formula for the first derivative follows from a standard result about the interchange of the limit and derivative ([Rud76], Theorem 7.17).

For the second derivative, from the formula in Proposition 3.4,

$$\frac{\partial^2}{\partial \bar{z}\partial z} f_\epsilon(z) = \frac{\epsilon}{2}\tau\left[(x_z x_z^* + \epsilon)^{-1}(x_z^* x_z + \epsilon)^{-1}\right] . \tag{3.48}$$

To bound this in absolute value, consider the following inequalities:

$$\begin{aligned} \left|\tau\left[(x_z x_z^* + \epsilon)^{-1}(x_z^* x_z + \epsilon)^{-1}\right]\right| &\le \left\|(x_z x_z^* + \epsilon)^{-1}\right\| \left\|(x_z^* x_z + \epsilon)^{-1}\right\| \\ &\le \left\|(x_z x_z^*)^{-1}\right\| \left\|(x_z^* x_z)^{-1}\right\| . \end{aligned} \tag{3.49}$$

This final quantity is locally bounded, so then the convergence in the second derivative is locally uniform for $z \notin \sigma(x)$. The formula follows again from the result about the interchange of the limit and derivative. $\square$

Now, we will show that μ_x is a compactly supported probability measure and μ_x recovers the moments of x.

Proposition 3.7. *Let $x \in (M, \tau)$. Let μ_x be the Brown measure of x. Then, supp$(\mu_x) \subset \sigma(x)$. Let U be a neighborhood of $\sigma(x)$. If $f : U \to \mathbb{C}$ is an analytic function, then*

$$\tau(f(x)) = \int_{\mathbb{C}} f(z)\, d\mu_x(z)\,. \tag{3.50}$$

In particular, for all $n \in \mathbb{N}$,

$$\begin{aligned} \tau(x^n) &= \int_{\mathbb{C}} z^n\, d\mu_x(z) \\ \tau((x^*)^n) &= \int_{\mathbb{C}} \overline{z}^n\, d\mu_x(z) \end{aligned} \tag{3.51}$$

and μ_x is a probability measure.

Proof. From Lemma 3.6, when $z \notin \sigma(x)$, then $\mu_x = \frac{1}{2\pi}\nabla^2 \log \Delta(z-x)$ is zero in a neighborhood of z. Hence, $\mathrm{supp}(\mu_x) \subset \sigma(x)$.

Choose a compact set K such that the interior of K contains $\sigma(x)$ and $K \subset U$ and K has piecewise smooth boundary. Choose $g \in C_c(\mathbb{C})$ so that $g(z) = f(z)$ on K.

Then,

$$\int_{\mathbb{C}} f(z)\, d\mu_x(z) = \int_{\mathbb{C}} g(z)\, d\mu_x(z) = \lim_{\epsilon \to 0^+} \frac{1}{2\pi} \int_{\mathbb{C}} g(z) \nabla^2 f_\epsilon(z)\, dz\,. \tag{3.52}$$

From Lemma 3.6, on $\mathrm{supp}(f) \setminus K$, $\nabla^2 f_\epsilon(z) \to 0$ uniformly as $\epsilon \to 0^+$. Hence,

$$\begin{aligned} \lim_{\epsilon \to 0^+} \frac{1}{2\pi} \int_{\mathbb{C}} g(z) \nabla^2 f_\epsilon(z)\, dz &= \lim_{\epsilon \to 0^+} \frac{1}{2\pi} \int_K g(z)\, \nabla^2 f_\epsilon(z)\, dz \\ &= \lim_{\epsilon \to 0^+} \frac{1}{2\pi} \int_K f(z)\, \nabla^2 f_\epsilon(z)\, dz\,. \end{aligned} \tag{3.53}$$

Recall that $\nabla^2 = 4\frac{\partial^2}{\partial\bar{z}\partial z}$ and $\frac{\partial}{\partial\bar{z}}f(z) = 0$. Thus,

$$\begin{aligned} \frac{1}{2\pi}\int_K f(z)\,\nabla^2 f_\epsilon(z)\,dz &= \frac{2}{\pi}\int_K f(z)\,\frac{\partial^2}{\partial\bar{z}\partial z}f_\epsilon(z)\,dz \\ &= \frac{2}{\pi}\int_K \frac{\partial}{\partial\bar{z}}\left(f(z)\frac{\partial}{\partial z}f_\epsilon(z)\right)\,dz\,. \end{aligned} \tag{3.54}$$

From Green's theorem,

$$\frac{2}{\pi}\int_K \frac{\partial}{\partial\bar{z}}\left(f(z)\,\frac{\partial}{\partial z}f_\epsilon(z)\right)\,dz = \frac{1}{\pi i}\oint_{\partial K} f(z)\,\frac{\partial}{\partial z}f_\epsilon(z)\,dz\,. \tag{3.55}$$

From Lemma 3.6, $\frac{\partial}{\partial z}f_\epsilon(z) \to \frac{1}{2}\tau[(z-x)^{-1}]$ uniformly on ∂D_R, so putting all of these steps together,

$$\int_{\mathbb{C}} f(z)\,d\mu_x(z) = \frac{1}{2\pi i}\oint_{\partial K} f(z)\,\tau[(z-x)^{-1}]\,dz\,. \tag{3.56}$$

From the analytic functional calculus,

$$\frac{1}{2\pi i}\oint_{\partial K} f(z)\,\tau[(z-x)^{-1}]\,dz = \tau\left[\frac{1}{2\pi i}\oint_{\partial K} f(z)(z-x)^{-1}\,dz\right] = \tau(f(x))\,. \tag{3.57}$$

The moment formula follows from considering $f(z) = z^n$. The $*$-moment formula follows from conjugating the moment formula. The fact that μ_x is a probability measure follows from considering the 0-th moment. $\square$

Note that the fact that the Brown measure recovers the moments of x generalizes the fact for empirical spectral distributions of (random) matrices.

Now, we consider the function: $L : \mathbb{C} \to \mathbb{C}$ given by:

$$L(z) = \int_{\mathbb{C}} \log|z-w|\;d\mu_x(w)\,. \tag{3.58}$$

Since μ_x is a compactly supported probability measure, $L \in L^1_{\text{loc}}(\mathbb{C})$. This follows from the fact that $\log|\cdot - w| \in L^1_{\text{loc}}(\mathbb{C})$ uniformly for w in a compact set. Thus, L is a well-defined distribution and since $\frac{1}{2\pi}\nabla^2 \log|\cdot - w| = \delta_w$, then

$$\frac{1}{2\pi}\nabla^2 L(z) = \mu_x \,. \tag{3.59}$$

so that on $\mathbb{C}$,

$$\nabla^2(\log \Delta(z-x) - L(z)) = 0 \,. \tag{3.60}$$

From Weyl's lemma, this implies that

$$\log \Delta(z-x) = L(z) + u(z) = \int_{\mathbb{C}} \log|z-w| \; d\mu_x(w) \; + u(z) \,. \tag{3.61}$$

for some harmonic function $u : \mathbb{C} \to \mathbb{R}$. Note that this is exactly the factorization in the Riesz representation theorem for subharmonic functions.

Now, we proceed to show that $u(z) = 0$ so that $\log \Delta(z-x) = L(z)$.

We need the following general lemma about subharmonic functions ([HS07], Lemma 2.10):

Lemma 3.8. *Let $g : \mathbb{C} \to [-\infty, \infty)$ be a subharmonic function. For $r > 0$, define:*

$$\begin{aligned} m(g,r) &= \frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})\, d\theta \\ M(g,r) &= \sup_{|z|=r} g(z) \,. \end{aligned} \tag{3.62}$$

Then,

$$g(0) = \lim_{r\to 0} m(g,r) = \lim_{r\to 0} M(g,r) \,. \tag{3.63}$$

Proof. Clearly $m(g,r) \le M(g,r)$. Since g is subharmonic, $g(0) \le m(g,r)$. Hence, for $r > 0$,

$$g(0) \le m(g,r) \le M(g,r) \,. \tag{3.64}$$

Since upper semicontinuous functions attain their maxima on compact sets, $M(g,r) = g(z_r)$

for some $z_r \in \mathbb{C}$ where $|z_r| = r$. Since g is upper semicontinuous, then

$$\limsup_{r\to 0} M(g,r) = \limsup_{r\to 0} g(z_r) \le g(0)\,. \tag{3.65}$$

The result follows by considering $\limsup_{r\to 0}$ and $\liminf_{r\to 0}$ of $m(g,r)$ and $M(g,r)$. $\square$

Next, we analyze $\log \Delta(z-x)$ and $L(z)$ in a neighborhood of ∞:

Lemma 3.9. *Let $x \in (M,\tau)$ and μ_x be the Brown measure of x. Define $g,h : \mathbb{C} \to [-\infty,\infty)$ by:*

$$\begin{aligned} g(z) &= \log \Delta(1-xz) \\ h(z) &= \int_{\mathbb{C}} \log|1-zw|\; d\mu_x(w)\,. \end{aligned} \tag{3.66}$$

Then, g and h are subharmonic on $\mathbb{C}$ and

$$\begin{aligned} \log \Delta(z-x) &= \log|z| + g\left(\frac{1}{z}\right) \\ L(z) &= \log|z| + h\left(\frac{1}{z}\right)\,. \end{aligned} \tag{3.67}$$

Proof. Note that it suffices to prove the result for $g(z)$ since $h(z)$ is just the specific case where $(M,\tau) = (L^\infty(\mathbb{C},\mu_x), d\mu_x)$ with the random variable $f(w) = w$.

Recall that for $\epsilon > 0$, $f_\epsilon : \mathbb{C} \to \mathbb{R}$ given by

$$f_\epsilon(z) = \frac{1}{2}\tau\left[\log((z-x)^*(z-x)+\epsilon)\right]\,. \tag{3.68}$$

is subharmonic. Define

$$g_\epsilon(z) = \frac{1}{2}\tau\left[\log((1-xz)^*(1-xz)+\epsilon\,|z|^2)\right]\,. \tag{3.69}$$

Then, for $z \neq 0$,

$$f_\epsilon(z) = \log|z| + g_\epsilon\left(\frac{1}{z}\right)\,. \tag{3.70}$$

Since $\log|z|$ is harmonic on $\mathbb{C}\backslash\{0\}$, then $g_\epsilon(1/z)$ is subharmonic on $\mathbb{C}\backslash\{0\}$, i.e. $\nabla^2 g_\epsilon(1/z) \ge 0$.

By changing variables in the Laplacian in polar coordinates, we conclude that $\nabla^2 g_\epsilon(z) \geq 0$, so that $g_\epsilon(z)$ is subharmonic. Finally, $g_\epsilon(z)$ decreases to $g(z)$ so that $g(z)$ is subharmonic. $\square$

By taking $|z| \to \infty$ and using Lemmas 3.8 and 3.9, we see that $L(z), \log \Delta(z-x) = O(\log |z|)$.

Hence, $u(z) = \log \Delta(z-x) - L(z) = O(\log |z|)$. Now, we observe that a harmonic function with this growth condition is constant:

Lemma 3.10. *Suppose that $u : \mathbb{C} \to \mathbb{R}$ is harmonic and $u(z) \leq a \left|\log |z|\right| + b$ for some $a, b > 0$. Then, u is constant.*

Proof. By considering $\tilde{u} = \frac{u-b}{a}$, we may assume that $u(z) \leq \left|\log |z|\right|$. Let f be the entire function where $\mathrm{Re} f(z) = u(z)$. Then, for $g(z) = \exp(f(z))$,

$$|g(z)| = \exp(\mathrm{Re} f(z)) = \exp(u(z)) \leq \exp(\left|\log |z|\right|) = \max(|z|, |z|^{-1}). \tag{3.71}$$

For $|z| \geq 1$, $|g(z)| \leq |z|$, so $g(z)$ is a linear polynomial. But, $g(z)$ does not have a zero, so $g(z)$ must be constant. This implies that $f(z)$ is constant, and hence $u(z)$ is constant. $\square$

Thus, it suffices to show that this constant is 0. We do that in the following Proposition:

Proposition 3.11. *Let $x \in (M, \tau)$ and μ_x be the Brown measure of x. Then, for $z \in \mathbb{C}$,*

$$\log \Delta(z-x) = \int_{\mathbb{C}} \log |z-w| \, d\mu_x(w). \tag{3.72}$$

The Brown measure μ_x is the unique complex measure to satisfy this equation.

Proof. Let $u : \mathbb{C} \to \mathbb{R}$ be given by $u(z) = \log \Delta(z-x) - L(z)$. Thus far we have shown that $u(z) = c$ for some $c \in \mathbb{R}$. From Lemma 3.8,

$$u(z) = \log \Delta(z-x) - L(z) = g\left(\frac{1}{z}\right) - h\left(\frac{1}{z}\right). \tag{3.73}$$

Using the notation of Lemma 3.8,

$$c = m(u, r) = m(g, 1/r) - m(h, 1/r)\,. \tag{3.74}$$

Taking $r \to \infty$ and applying Lemma 3.8,

$$\begin{aligned} \lim_{r\to\infty} m(g, 1/r) &= g(0) = 0 \\ \lim_{r\to\infty} m(h, 1/r) &= h(0) = 0\,. \end{aligned} \tag{3.75}$$

Hence, $c = 0$.

Uniqueness follows from taking Laplacians of both sides. □

As a corollary, we obtain a result about the pushforward of the Brown measure under polynomials:

Corollary 3.12. *Let $x \in (M, \tau)$ and consider a polynomial $p(z)$. Then, the Brown measure of $p(x)$ is the pushforward of the Brown measure of x under p, i.e.*

$$\mu_{p(x)} = p_*(\mu_x)\,. \tag{3.76}$$

Proof. From Proposition 3.11, it suffices to show that for all $z \in \mathbb{C}$,

$$\log \Delta(z - x) = \int_{\mathbb{C}} \log |z - p(w)|\; d\mu_x(w)\,. \tag{3.77}$$

Let $z - p(w) = c\prod_{i=1}^{n}(r_i - w)$. Then, the following equalities verify the identity:

$$\begin{aligned}
\int_{\mathbb{C}} \log|z - p(w)|\, d\mu_x(w) &= \int_{\mathbb{C}} \log\left|c\prod_{i=1}^{n}(r_i - w)\right| d\mu_x(w) \\
&= \log|c| + \sum_{i=1}^{n}\int_{\mathbb{C}} \log|r_i - w|\, d\mu_x(w) \\
&= \log\Delta(c) + \sum_{i=1}^{n}\log\Delta(r_i - x) \\
&= \log\left(\Delta(c)\prod_{i=1}^{n}\Delta(r_i - x)\right) \\
&= \log\Delta\left(c\prod_{i=1}^{n}(r_i - w)\right) \\
&= \log\Delta(z - p(w))\,.
\end{aligned} \tag{3.78}$$

□

For the final result in this chapter, we will deduce that the Brown measure of the operators we consider, $X = p + iq$ from (2.1), is contained in the convex hull of the points $\{\alpha_i + i\beta_j : 1 \le i \le k, 1 \le j \le l\}$, i.e. the rectangle:

$$R = \{z = x + iy : x \in [\min_i \alpha_i, \max_i \alpha_i], y \in [\min_i \beta_i, \max_i, \beta_i]\}\,. \tag{3.79}$$

This follows from the following general Proposition:

Proposition 3.13. *Let $x \in (M,\tau)$, where $x = p + iq$ for p, q Hermitian. Let $I, J \subset \mathbb{R}$ be closed intervals such that $\sigma(p) \subset I$ and $\sigma(q) \subset J$. Let $R = \{z = x + iy : x \in I, y \in J\}$. Then, $\sigma(x) \subset R$. Thus, the Brown measure of x is supported on R.*

Proof. Let $\lambda = \lambda_1 + i\lambda_2$ where $\lambda_1, \lambda_2 \in \mathbb{R}$. Suppose that $\lambda \notin R$. Let $x_\lambda = x - \lambda$. We will show that x_λ is invertible.

From the polar decomposition and functional calculus, it suffices to show that $x_\lambda^* x_\lambda$ is invertible, i.e. $x_\lambda^* x_\lambda \geq c$ for some $c > 0$. Let $M \subset B(H)$. Then, $x_\lambda^* x_\lambda \geq c$ is equivalent to

$$\|x_\lambda(\xi)\|^2 = \langle x_\lambda^* x_\lambda \xi, \xi\rangle \geq c\,\|\xi\|^2\ , \quad \xi \in H\,. \tag{3.80}$$

We proceed to show $\inf_{|\xi|=1} \|x_\lambda(\xi)\| \geq c$ for some $c > 0$.

For this, note that for $\xi \in H$ where $|\xi| = 1$,

$$\begin{aligned} \|x_\lambda(\xi)\| &\geq |\langle x_\lambda(\xi), \xi\rangle| \\ &= |\langle ((p-\lambda_1) + i(q-\lambda_2))\,\xi, \xi\rangle| \\ &= |\langle (p-\lambda_1)\xi, \xi\rangle + i\langle (q-\lambda_2)\xi, \xi\rangle|\ . \end{aligned} \tag{3.81}$$

As p and q are Hermitian, then $\langle (p-\lambda_1)\xi, \xi\rangle, \langle (q-\lambda_2)\xi, \xi\rangle \in \mathbb{R}$. Without loss of generality, assume that $\lambda_1 \notin I$. Then, either $p - \lambda_1 \geq c$ or $p - \lambda_1 \leq -c$ for some $c > 0$. In either case, $|\langle (p-\lambda_1)\xi, \xi\rangle| \geq c$ and hence

$$|\langle (p-\lambda_1)\xi, \xi\rangle + i\langle (q-\lambda_2)\xi, \xi\rangle| \geq |\langle (p-\lambda_1)\xi, \xi\rangle| \geq c\,. \tag{3.82}$$

Thus, $\inf_{|\xi|=1} \|x_\lambda(\xi)\| \geq c$, so that x_λ is invertible, as desired. $\square$

CHAPTER 4

Brown measure of $X = p + iq$

In this chapter, we compute the Brown measure of $X = p + iq$, where p and q are Hermitian, freely independent, and have at most 2 atoms, i.e.

$$\begin{aligned} \mu_p &= a\delta_\alpha + (1-a)\delta_{\alpha'} \\ \mu_q &= b\delta_\beta + (1-b)\delta_{\beta'}\,. \end{aligned} \tag{4.1}$$

where $a, b \in [0,1]$ and $\alpha, \alpha', \beta, \beta' \in \mathbb{R}$. When either p or q is a constant, then X is normal, so the Brown measure of X is just the spectral measure of X. **Thus, we will assume throughout this section that $a, b \in (0,1)$, $\alpha \neq \alpha'$, and $\beta \neq \beta'$.**

Recall the Brown measure of X is defined as:

$$\mu = \frac{1}{2\pi}\nabla^2 \log \Delta(z - X) = \frac{1}{2\pi}\nabla^2 \frac{1}{2}\int_0^\infty \log(x)\, d\nu_z(x)\,, \tag{4.2}$$

where ν_z is the spectral measure of $H_z(X) = (z-X)^*(z-X)$.

To compute the Brown measure, we need to complete the following steps:

1. Compute ν_z.

2. Compute $\log \Delta(z - X) = \frac{1}{2}\int_0^\infty \log(x)\, d\nu_z(x)$.

3. Compute $\mu = \frac{1}{2\pi}\nabla^2 \log \Delta(z - X)$.

First, we need to describe some results about the tracial von Neumann algebra generated by two projections. This is described in the next section.

4.1 The von Neumann algebra generated by two projections

We summarize the results from ([Voi99], Section 12) that describe the tracial von Neumann algebra generated by two projections.

Let (M, τ) be the von Neumann algebra generated by two projections p and q.

Consider the following elements of M:

$$\begin{aligned} e_{00} &= (1-p) \wedge (1-q) \\ e_{01} &= (1-p) \wedge q \\ e_{10} &= p \wedge (1-q) \\ e_{11} &= p \wedge q \\ e &= 1 - (e_{00} + e_{01} + e_{10} + e_{11}) \,. \end{aligned} \tag{4.3}$$

These elements are central, mutually orthogonal projections and

$$e_{00} + e_{01} + e_{10} + e_{11} + e = 1 \,. \tag{4.4}$$

If we consider the matrix of $m \in M$ with respect to the e_{ij}, e (i.e. the 5×5 matrix with entries of the form fmf' where $f, f' \in \{e_{i,j}, e\}$), then this matrix is diagonal. Further, the 4 diagonal terms $e_{ij}me_{ij}$ are constants. Hence, the only interesting part of M is the subalgebra eMe. In particular,

$$\begin{aligned} p &= p \wedge (1-q) + p \wedge q + epe = e_{10} + e_{11} + epe \\ q &= (1-p) \wedge q + p \wedge q + eqe = e_{01} + e_{11} + eqe \,. \end{aligned} \tag{4.5}$$

When $e \neq 0$, consider $(eMe, \tau\big|_{eMe})$ as a von Neumann subalgebra with identity e and $\tau\big|_{eMe}(eme) = \frac{\tau(eme)}{\tau(e)}$. Then,

$$(eMe, \tau\big|_{eMe}) \cong \left(M_2\left(L^{\infty}((0,1), \nu^*)\right), \mathbb{E}_{\nu^*}[\frac{1}{n}\mathrm{tr}] \right) \,. \tag{4.6}$$

where ν^* is a Borel measure on $(0, 1)$.

For any $m \in M$, let $\tilde{m} = eme \in eMe$. The isomorphism has the following correspondence between $\tilde{m} \in eMe$ and matrix-valued functions of $t \in (0, 1)$:

$$\begin{aligned} \tilde{p} &\leftrightarrow \begin{pmatrix} t & (t-t^2)^{1/2} \\ (t-t^2)^{1/2} & 1-t \end{pmatrix} \\ \tilde{q} &\leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} . \end{aligned} \tag{4.7}$$

The measure ν^* is recovered by noting that for the central element $x = pqp + (1-p)(1-q)(1-p) = 1 - p - q + pq + qp$,

$$\tilde{x} \leftrightarrow \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} . \tag{4.8}$$

Hence, ν^* is the spectral measure of the element $\tilde{x}$ in $(eMe, \tau\big|_{eMe})$.

Observe that with the change of variables $t = \cos^2\theta$, $\theta \in (0, \pi/2)$,

$$\tilde{p} = R_\theta \tilde{q} R_\theta^{-1} , \tag{4.9}$$

where $R_\theta \in M_2(\mathbb{C})$ is the rotation matrix with angle θ.

In practice, we will work with θ instead of t. Let $(\cos^2)^{-1} : [0, 1] \to [0, \pi/2]$ be the inverse of $\cos^2 : [0, \pi/2] \to [0, 1]$ and consider the measure ν on $(0, \pi/2)$ given by:

$$\nu = \left((\cos^2)^{-1}\right)_* (\nu^*) . \tag{4.10}$$

Then, we will use the isomorphism:

$$(eMe, \tau\big|_{eMe}) \cong \left(M_2\left(L^\infty((0, \pi/2), \nu)\right), \mathbb{E}_\nu[\frac{1}{n}\mathrm{tr}] \right) . \tag{4.11}$$

Let us highlight a few facts about the algebra generated by 2 projections:

First, we explain why in the matrix algebras the domains are open instead of closed. This is because of the following fact:

Lemma 4.1. *The measure ν^* does not have atoms at 0 or 1, i.e. $\nu^*(\{0,1\}) = 0$. This is equivalent to ν not having atoms at 0 or $\pi/2$.*

Proof. Let $\tilde{p} = epe$ and $\tilde{q} = eqe$. Since e commutes with p and q, $\tilde{p} = p \wedge e$ and $\tilde{q} = q \wedge e$.

Consider $\tau((\tilde{q}\tilde{p}\tilde{q})^n) \to \tau(\tilde{p} \wedge \tilde{q}) = \tau(e \wedge (p \wedge q))$ as $n \to \infty$. Recall that e and $p \wedge q$ are mutually orthogonal, so $\tau(e \wedge (p \wedge q)) = 0$. Under the isomorphism, computation shows that

$$\tau((\tilde{q}\tilde{p}\tilde{q})^n) = \int_0^1 t^n \, d\nu^*(t) \,. \tag{4.12}$$

As $n \to \infty$ the integral on the right-hand side decreases to $\nu^*(\{1\})$. Hence, $\nu^*(\{1\}) = 0$.

A similar argument considering $\tau(((1-\tilde{q})(\tilde{p})(1-\tilde{q}))^n) \to \tau((1-\tilde{q}) \wedge \tilde{p})$ using that e and $(1-q) \wedge p$ are mutually orthogonal shows that $\nu^*(\{0\}) = 0$.

Finally, we note that from the definition of pushforward measure, $\nu^*(\{0,1\}) = \nu(\{0, \pi/2\})$.

□

Next, we note that on eMe, epe and eqe have trace $1/2$. While this is also easy to see from the isomorphism, it follows from e being mutually orthogonal to all of the e_{ij}:

Lemma 4.2. *Let τ be the trace on eMe. Then, $\tau(epe) = \tau(eqe) = 1/2$.*

Proof. If this is false, then we may choose one of epe or $1 - epe$ and one of eqe or $1 - eqe$ so that the sum of the traces is greater than 1. Without loss of generality assume that $\tau(epe) + \tau(eqe) > 1$. Since e commutes with p and q, $epe = p \wedge e$ and $eqe = q \wedge e$. Then, from the parallelogram law, we have the following contradiction:

$$\begin{aligned} 0 &= \tau(e \wedge (p \wedge q)) = \tau(epe \wedge eqe) = \tau(epe) + \tau(eqe) - \tau(epe \vee eqe) \\ &\geq \tau(epe) - \tau(eqe) - 1 > 0 \,. \end{aligned} \tag{4.13}$$

□

4.2 Computation of Brown measure of X (up to ν and weights)

In this subsection, we consider an arbitrary $X = p + iq$ where p, q are Hermitian and each have 2 atoms, i.e. we do not assume that p and q are freely independent. Since p and q each have 2 atoms, then p and q are affine transformations of two projections. Then, we can use the isomorphism in Section 4.1 to compute the Brown measure of $p + iq$.

In this section, we will describe the Brown measure as a convex combination of 4 atoms and another probability measure, μ'. The measure μ' depends on the measure ν in the isomorphism $(eMe, \tau\big|_{eMe}) \cong \left(M_2\left(L^\infty((0,\pi/2),\nu)\right), \mathbb{E}_\nu[\frac{1}{n}\mathrm{tr}]\right)$. The weights in the convex combination are $\tau(e_{ij}), \tau(e)$ from the previous section. We will determine the Brown measure up to determining the weights $\tau(e_{ij}), \tau(e)$ and the measure ν. These will depend on the joint law of p and q. In the next subsection, we will compute these quantities when p and q are freely independent.

For the computation, fix the projections p' and q'. We will use the following definition frequently:

Definition 4.3. *Let $p, q \in (M, \tau)$ be Hermitian with laws:*

$$\begin{aligned} \mu_p &= a\delta_\alpha + (1-a)\delta_{\alpha'} \\ \mu_q &= b\delta_\beta + (1-b)\delta_{\beta'} \,. \end{aligned} \tag{4.14}$$

Let $p', q' \in (M, \tau)$ be the following projections:

$$\begin{aligned} p' &= \chi_{\{\alpha'\}}(p) \\ q' &= \chi_{\{\beta'\}}(q) \,. \end{aligned} \tag{4.15}$$

As a consequence,

$$\begin{aligned} 1 - p' &= \chi_{\{\alpha\}}(p) \\ 1 - q' &= \chi_{\{\beta\}}(q) \,. \end{aligned} \tag{4.16}$$

Hence,

$$p = \alpha(1-p') + \alpha' p' = (\alpha' - \alpha)p' + \alpha$$
$$q = \beta(1-q') + \beta' q' = (\beta' - \beta)q' + \beta \tag{4.17}$$

or equivalently

$$p' = \frac{p-\alpha}{\alpha'-\alpha}$$
$$q' = \frac{q-\beta}{\beta'-\beta}\,. \tag{4.18}$$

Throughout this section we will use the notation of Section 4.1 with the general pair of projections (p, q) replaced by the projections (p', q').

With this notation, observe that

$$\begin{aligned}
e_{00} &= \chi_{\{\alpha\}}(p) \wedge \chi_{\{\beta\}}(q) \\
e_{01} &= \chi_{\{\alpha\}}(p) \wedge \chi_{\{\beta'\}}(q) \\
e_{10} &= \chi_{\{\alpha'\}}(p) \wedge \chi_{\{\beta\}}(q) \\
e_{11} &= \chi_{\{\alpha'\}}(p) \wedge \chi_{\{\beta'\}}(q) \\
e &= 1 - (e_{00} + e_{01} + e_{10} + e_{11})\,.
\end{aligned} \tag{4.19}$$

To compute the Brown measure of X, recall we need to first compute ν_z, the spectral measure of $H_z(X) = (z - X)^*(z - X)$.

The result of the computation is the following:

Proposition 4.4. *If $e = 0$,*

$$\nu_z = \tau(e_{00})\delta_{|z-(\alpha+i\beta)|^2} + \tau(e_{01})\delta_{|z-(\alpha+i\beta')|^2} + \tau(e_{10})\delta_{|z-(\alpha'+i\beta)|^2} + \tau(e_{11})\delta_{|z-(\alpha'+i\beta')|^2}\,. \tag{4.20}$$

If $e \neq 0$, there exists continuous functions $\sigma_{z,1}, \sigma_{z,2} : (0, \pi/2) \to [0, \infty)$ such that $\sigma_{z,1}(\theta), \sigma_{z,2}(\theta)$ are the singular values of the element of $\left(M_2\left(L^\infty((0, \pi/2), \nu)\right), \mathbb{E}_\nu[\frac{1}{n}\mathrm{tr}]\right)$ corresponding to $e(z - (p + iq))e \in (eMe, \tau\big|_{eMe})$.

Then,

$$\nu_z = \tau(e_{00})\delta_{|z-(\alpha+i\beta)|^2} + \tau(e_{01})\delta_{|z-(\alpha+i\beta')|^2} + \tau(e_{10})\delta_{|z-(\alpha'+i\beta)|^2} + \tau(e_{11})\delta_{|z-(\alpha'+i\beta')|^2} + \tau(e)\frac{(\sigma^2_{z,1})_*(\nu) + (\sigma^2_{z,2})_*(\nu)}{2}\,. \tag{4.21}$$

Proof. Let $H_z = H_z(X) = (z-X)^*(z-X)$. Since the e_{ij}, e are central, mutually orthogonal projections that sum to 1, then

$$\begin{aligned}(H_z)^n &= (e_{00}+e_{01}+e_{10}+e_{11}+e)(H_z)^n(e_{00}+e_{01}+e_{10}+e_{11}+e)\\ &= e_{00}(H_z)^n e_{00} + e_{01}(H_z)^n e_{01} + e_{10}(H_z)^n e_{10} + e_{11}(H_z)^n e_{11} + e(H_z)^n e\\ &= (e_{00}H_z e_{00})^n + (e_{01}H_z e_{01})^n + (e_{10}H_z e_{10})^n + (e_{11}H_z e_{11})^n + (eH_z e)^n\,.\end{aligned} \tag{4.22}$$

Taking traces of both sides,

$$\begin{aligned}&\tau((H_z)^n)\\ &= \tau((e_{00}H_z e_{00})^n) + \tau((e_{01}H_z e_{01})^n) +\\ &\qquad \tau((e_{10}H_z e_{10})^n) + \tau((e_{11}H_z e_{11})^n) + \tau((eH_z e)^n)\,.\end{aligned} \tag{4.23}$$

From the definitions of the e_{ij},

$$\begin{aligned}(e_{00}H_z e_{00})^n &= (|z-(\alpha+i\beta)|^2)^n e_{00}\\ (e_{01}H_z e_{01})^n &= (|z-(\alpha+i\beta')|^2)^n e_{01}\\ (e_{10}H_z e_{10})^n &= (|z-(\alpha'+i\beta)|^2)^n e_{10}\\ (e_{11}H_z e_{11})^n &= (|z-(\alpha'+i\beta')|^2)^n e_{11}\,.\end{aligned} \tag{4.24}$$

Combining the previous two equations,

$$\begin{aligned}\tau((H_z)^n) = {} & \tau(e_{00})(\left|z-(\alpha+i\beta)\right|^2)^n \\ & + \tau(e_{01})(\left|z-(\alpha+i\beta')\right|^2)^n \\ & + \tau(e_{10})(\left|z-(\alpha'+i\beta)\right|^2)^n \\ & + \tau(e_{11})(\left|z-(\alpha'+i\beta')\right|^2)^n \\ & + \tau(e)\tau\big|_{eMe}((eH_ze)^n)\,.\end{aligned} \tag{4.25}$$

The right-hand side is the n-th moment of the following convex combination of probability measures:

$$\begin{aligned}&\tau(e_{00})\delta_{|z-(\alpha+i\beta)|^2} + \tau(e_{01})\delta_{|z-(\alpha+i\beta')|^2} + \\ &\quad + \tau(e_{10})\delta_{|z-(\alpha'+i\beta)|^2} + \tau(e_{11})\delta_{|z-(\alpha'+i\beta')|^2} + \tau(e)\mu_{eH_ze}\,,\end{aligned} \tag{4.26}$$

where μ_{eH_ze} is the spectral measure of eH_ze in $\left(eMe, \tau\big|_{eMe}\right)$.

As moments determine compact measures on $\mathbb{R}$, then this is ν_z.

If $e = 0$, then ν_z is the desired convex combination of atoms.

If $e \neq 0$, consider μ_{eH_ze} using the isomorphism described in Section 4.1, i.e. we proceed by assuming that $(eMe, \tau\big|_{eMe}) = \left(M_2\left(L^\infty((0,\pi/2),\nu)\right), \mathbb{E}_\nu[\frac{1}{n}\mathrm{tr}]\right)$.

For any $m \in M$, let $\tilde{m} = eme \in eMe$. Then, $\tilde{H}_z = (z-(\tilde{p}+i\tilde{q}))^*(z-(\tilde{p}+i\tilde{q}))$.

Now, we claim that it is possible to choose continuous functions $\sigma_{z,1}(\theta), \sigma_{z,2}(\theta)$ that are the singular values of $(z-(\tilde{p}+i\tilde{q})) = e(z-(p+iq))e$ for $\theta \in (0,\pi/2)$. First, note that from the expressions for $\tilde{p'}, \tilde{q'}$ in Section 4.1 that $\tilde{H}_z$ is continuous in θ. Hence, the characteristic polynomial of $\tilde{H}_z$ is a monic polynomial with coefficients that are continuous in θ. From the quadratic formula, the eigenvalues of $\tilde{H}_z$ have expressions in terms of the coefficients of the characteristic polynomial of $\tilde{H}_z$. As $\tilde{H}_z$ are positive operators, then the eigenvalues are real, i.e. the discriminant of the characteristic polynomial is non-negative. Hence, the two branches of the square root of the discriminant are continuous in θ. Thus, we may choose continuous expressions for the two eigenvalues of $\tilde{H}_z$. Taking square roots of these two non-negative continuous functions produces $\sigma_{z,1}(\theta), \sigma_{z,2}(\theta)$.

Thus,

$$\begin{aligned}\tau\big|_{eMe}((\tilde{H}_z)^n) &= \int_0^{\pi/2} \frac{1}{2}\mathrm{tr}((\tilde{H}_z)^n)\, d\nu(\theta) \\ &= \int_0^{\pi/2} \frac{(\sigma_{z,1}^2(\theta))^n + (\sigma_{z,2}^2(\theta))^n}{2}\, d\nu(\theta)\,.\end{aligned} \tag{4.27}$$

The right-hand side is the n-th moment of the following measure:

$$\frac{(\sigma_{z,1}^2)_*(\nu) + (\sigma_{z,2}^2)_*(\nu)}{2}\,. \tag{4.28}$$

Since moments determine compact measures on $\mathbb{R}$, then

$$\mu_{eH_ze} = \frac{(\sigma_{z,1}^2)_*(\nu) + (\sigma_{z,2}^2)_*(\nu)}{2}\,. \tag{4.29}$$

By combining this with (4.26), we get the desired result:

$$\begin{aligned}\nu_z = \tau(e_{00})\delta_{|z-(\alpha+i\beta)|^2} + \tau(e_{01})\delta_{|z-(\alpha+i\beta')|^2} + \tau(e_{10})\delta_{|z-(\alpha'+i\beta)|^2} + \\ \tau(e_{11})\delta_{|z-(\alpha'+i\beta')|^2} + \tau(e)\frac{(\sigma_{z,1}^2)_*(\nu) + (\sigma_{z,2}^2)_*(\nu)}{2}\,.\end{aligned} \tag{4.30}$$

□

As an intermediate step in computing the Brown measure of X, we compute $\log\Delta(z-X)$:

Proposition 4.5. *If $e = 0$,*

$$\begin{aligned}\log\Delta(z-X) &= \frac{1}{2}\int_0^\infty \log(x)\, d\nu_z(x) \\ &= \tau(e_{00})\log|z-(\alpha+i\beta)| \\ &\quad + \tau(e_{01})\log|z-(\alpha+i\beta')| \\ &\quad + \tau(e_{10})\log|z-(\alpha'+i\beta)| \\ &\quad + \tau(e_{11})\log|z-(\alpha'+i\beta')|\,.\end{aligned} \tag{4.31}$$

If $e \neq 0$, there exists continuous functions $\lambda_1, \lambda_2 : (0, \pi/2) \to \mathbb{C}$ such that $\lambda_1(\theta), \lambda_2(\theta)$ are the eigenvalues of the element of $\left(M_2\left(L^\infty((0,\pi/2),\nu)\right), \mathbb{E}_\nu[\frac{1}{n}\mathrm{tr}]\right)$ corresponding to $e(p+iq)e \in$

$(eMe, \tau\big|_{eMe})$

Let $\mathscr{A} = \alpha' - \alpha$ *and* $\mathscr{B} = \beta' - \beta$ *and* $\sqrt{z}$ *denote the principal branch of the square root defined on* $\mathbb{C} \setminus (-\infty, 0)$*. Then,*

$$\lambda_1(\theta) = \begin{cases} \dfrac{\alpha+\alpha'}{2} + i\dfrac{\beta+\beta'}{2} - \dfrac{1}{2}\sqrt{\mathscr{A}^2 - \mathscr{B}^2 + 2i\mathscr{A}\mathscr{B}\cos(2\theta)} & \textit{when } |\mathscr{A}| \geq |\mathscr{B}| \\ \dfrac{\alpha+\alpha'}{2} + i\dfrac{\beta+\beta'}{2} - \dfrac{i}{2}\sqrt{\mathscr{B}^2 - \mathscr{A}^2 - 2i\mathscr{A}\mathscr{B}\cos(2\theta)} & \textit{when } |\mathscr{A}| < |\mathscr{B}| \end{cases} \tag{4.32}$$

$$\lambda_2(\theta) = \begin{cases} \dfrac{\alpha+\alpha'}{2} + i\dfrac{\beta+\beta'}{2} + \dfrac{1}{2}\sqrt{\mathscr{A}^2 - \mathscr{B}^2 + 2i\mathscr{A}\mathscr{B}\cos(2\theta)} & \textit{when } |\mathscr{A}| \geq |\mathscr{B}| \\ \dfrac{\alpha+\alpha'}{2} + i\dfrac{\beta+\beta'}{2} + \dfrac{i}{2}\sqrt{\mathscr{B}^2 - \mathscr{A}^2 - 2i\mathscr{A}\mathscr{B}\cos(2\theta)} & \textit{when } |\mathscr{A}| < |\mathscr{B}|\,. \end{cases} \tag{4.33}$$

Then,

$$\begin{aligned} \log\Delta(z - X) = \frac{1}{2}\int_0^\infty \log(x)\,d\nu_z(x) \\ &= \tau(e_{00})\log|z - (\alpha + i\beta)| \\ &\quad + \tau(e_{01})\log|z - (\alpha + i\beta')| \\ &\quad + \tau(e_{10})\log|z - (\alpha' + i\beta)| \\ &\quad + \tau(e_{11})\log|z - (\alpha' + i\beta')| \\ &\quad + \tau(e)\int_0^{\pi/2} \frac{\log|z - \lambda_1(\theta)| + \log|z - \lambda_2(\theta)|}{2}\,d\nu(\theta)\,. \end{aligned} \tag{4.34}$$

Proof. When $e = 0$, the formula for $\log\Delta(z - X)$ follows easily from Proposition 4.4.

For $e \neq 0$, from Proposition 4.4, for any continuous $f : [0, \infty) \to \mathbb{C}$,

$$\begin{aligned} \int_0^\infty f(x)\,d\nu_z(x) &= \tau(e_{00})f(|z - (\alpha + i\beta)|^2) \\ &\quad + \tau(e_{01})f(|z - (\alpha + i\beta')|^2) \\ &\quad + \tau(e_{10})f(|z - (\alpha' + i\beta)|^2) \\ &\quad + \tau(e_{11})f(|z - (\alpha' + i\beta')|^2) \\ &\quad + \tau(e)\int_0^\infty f(x)\,d\left(\frac{(\sigma_{z,1}^2)_*(\nu) + (\sigma_{z,2}^2)_*(\nu)}{2}\right)\,. \end{aligned} \tag{4.35}$$

Rewriting the final integral using the change of variables formula,

$$
\begin{aligned}
&\int_0^\infty f(x)\, d\left(\frac{(\sigma_{z,1}^2)_*(\nu) + (\sigma_{z,2}^2)_*(\nu)}{2}\right) \\
&= \frac{1}{2}\left(\int_0^\infty f(x)\, d(\sigma_{z,1}^2)_*(\nu) + \int_0^\infty f(x)(\sigma_{z,2}^2)_*(\nu)\right) \\
&= \frac{1}{2}\left(\int_0^{\pi/2} f(\sigma_{z,1}^2(\theta))\, d\nu(\theta) + \int_0^{\pi/2} f(\sigma_{z,2}^2(\theta))\, d\nu(\theta)\right) \\
&= \frac{1}{2}\int_0^{\pi/2} f(\sigma_{z,1}^2(\theta)) + f(\sigma_{z,2}^2(\theta))\, d\nu(\theta)\,.
\end{aligned}
\tag{4.36}
$$

Let $f_n(x) = \log(x + 1/n)/2$ for $n = 1, 2, \ldots$. Then, $f_n : [0, \infty) \to \mathbb{R}$ are continuous and decrease to $\log(x)/2$. Applying two previous equations for $f = f_n$ and using the monotone convergence theorem to take the limit as $n \to \infty$,

$$
\begin{aligned}
\log \Delta(z - X) &= \frac{1}{2}\int_0^\infty \log(x)\, d\nu_z(x) \\
&= \tau(e_{00}) \log|z - (\alpha + i\beta)| \\
&\quad + \tau(e_{01}) \log|z - (\alpha + i\beta')| \\
&\quad + \tau(e_{10}) \log|z - (\alpha' + i\beta)| \\
&\quad + \tau(e_{11}) \log|z - (\alpha' + i\beta')| \\
&\quad + \frac{\tau(e)}{2}\int_0^{\pi/2} \frac{\log(\sigma_{z,1}^2(\theta)) + \log(\sigma_{z,2}^2(\theta))}{2}\, d\nu(\theta)\,.
\end{aligned}
\tag{4.37}
$$

For the rest of the proof, assume that $(eMe, \tau\big|_{eMe}) = \left(M_2\left(L^\infty((0,\pi/2),\nu)\right), \mathbb{E}_\nu[\frac{1}{n}\mathrm{tr}]\right)$.

Recall that $\sigma_{z,1}(\theta), \sigma_{z,2}(\theta)$ are the singular values of $e(z - X)e$. Then, the integrand of the final integral can be simplified as:

$$
\begin{aligned}
\frac{\log(\sigma_{z,1}^2(\theta)) + \log(\sigma_{z,2}^2(\theta))}{2} &= \frac{\log(\sigma_{z,1}^2(\theta)\sigma_{z,2}^2(\theta))}{2} \\
&= \frac{\log(\det((e(z-X)e)^* e(z-X)e))}{2} \\
&= \frac{\log(|\det((e(z-X)e))|^2)}{2} \\
&= \log|\det((e(z-X)e))| \\
&= \log|\det(z-eXe)|\,.
\end{aligned} \tag{4.38}
$$

Given that we can verify the formulas for $\lambda_1(\theta)$, $\lambda_2(\theta)$, then $z-\lambda_1(\theta), z-\lambda_2(\theta)$ are the eigenvalues for $z-eXe$. Thus, combining the previous two equations,

$$
\begin{aligned}
&\int_0^{\pi/2} \frac{\log(\sigma_{z,1}^2(\theta)) + \log(\sigma_{z,2}^2(\theta))}{2}\, d\nu(\theta) \\
&= \int_0^{\pi/2} \log|\det(z-eXe)|\; d\nu(\theta) \\
&= \int_0^{\pi/2} \log|z-\lambda_1(\theta)| + \log|z-\lambda_2(\theta)|\; d\nu(\theta)\,.
\end{aligned} \tag{4.39}
$$

Substituting this expression into (4.37) produces the desired formula for $\log\Delta(z-X)$.

We return to verifying the formulas for $\lambda_1(\theta)$, $\lambda_2(\theta)$. Straightforward computation shows that the characteristic polynomial of $eXe \in \left(M_2\left(L^\infty((0,\pi/2),\nu)\right), \mathbb{E}_\nu[\frac{1}{n}\mathrm{tr}]\right)$ is:

$$
p(\lambda) = \lambda^2 - ((\alpha+\alpha') + i(\beta+\beta'))\lambda + \left(\alpha\alpha' - \beta\beta' + \frac{i}{2}((\alpha+\alpha')(\beta+\beta') - \mathscr{A}\mathscr{B}\cos(2\theta))\right). \tag{4.40}
$$

The eigenvalues of eXe are:

$$
\frac{\alpha+\alpha'}{2} + i\frac{\beta+\beta'}{2} \pm \frac{1}{2}\sqrt{\mathscr{A}^2 - \mathscr{B}^2 + 2i\mathscr{A}\mathscr{B}\cos(2\theta)}\,. \tag{4.41}
$$

where the square root is any branch of the square root.

When $|\mathscr{A}| \geq |\mathscr{B}|$, then $\mathrm{Re}(\mathscr{A}^2 - \mathscr{B}^2 + 2i\mathscr{A}\mathscr{B}\cos(2\theta)) \geq 0$ and the expression for the eigenvalues of $e(p+iq)e$ is continuous and well-defined if we take the square root to be the principal branch of the square root. This gives the formulas for $\lambda_1(\theta), \lambda_2(\theta)$ for $|\mathscr{A}| \geq |\mathscr{B}|$.

When $|\mathscr{A}| < |\mathscr{B}|$, then $\pm i\sqrt{\mathscr{B}^2 - \mathscr{A}^2 - 2i\mathscr{A}\mathscr{B}\cos(2\theta)}$ are also expressions for the square roots of $\mathscr{A}^2 - \mathscr{B}^2 + 2i\mathscr{A}\mathscr{B}\cos(2\theta)$, where the square root is the principal branch of the square root. Since $|\mathscr{A}| < |\mathscr{B}|$, then $\mathrm{Re}(\mathscr{B}^2 - \mathscr{A}^2 - 2i\mathscr{A}\mathscr{B}\cos(2\theta)) > 0$ and this expression is continuous and well-defined.. This gives the formulas for $\lambda_1(\theta), \lambda_2(\theta)$ for $|\mathscr{A}| < |\mathscr{B}|$. $\square$

Finally, we compute the Brown measure of X, $\mu = \frac{1}{2\pi}\nabla^2 \log \Delta(z - X)$ in the following Proposition:

Proposition 4.6. *Let μ be the Brown measure of X. If $e = 0$,*

$$\begin{aligned} \mu &= \frac{1}{2\pi}\nabla^2 \log \Delta(z - X) \\ &= \tau(e_{00})\delta_{\alpha+i\beta} + \tau(e_{01})\delta_{\alpha+i\beta'} + \tau(e_{10})\delta_{\alpha'+i\beta} + \tau(e_{11})\delta_{\alpha'+i\beta'}. \end{aligned} \tag{4.42}$$

If $e \neq 0$, let $\lambda_1, \lambda_2 : (0, \pi/2) \to \mathbb{C}$ be as in Proposition 4.5. Then,

$$\begin{aligned} \mu &= \frac{1}{2\pi}\nabla^2 \log \Delta(z - X) \\ &= \tau(e_{00})\delta_{\alpha+i\beta} + \tau(e_{01})\delta_{\alpha+i\beta'} + \tau(e_{10})\delta_{\alpha'+i\beta} + \tau(e_{11})\delta_{\alpha'+i\beta'} \\ &\quad + \tau(e)\mu'\,, \end{aligned} \tag{4.43}$$

where

$$\mu' = \frac{(\lambda_1)_*(\nu) + (\lambda_2)_*(\nu)}{2}\,. \tag{4.44}$$

Additionally, $\mu'(\{\alpha + i\beta, \alpha' + i\beta, \alpha + i\beta', \alpha' + i\beta'\}) = 0$.

Proof. If $e = 0$, the result follows from directly applying $\frac{1}{2\pi}\nabla^2 \log|\cdot - \lambda| = \delta_\lambda$ to the expression for $\log \Delta(z - X)$ in Proposition 4.5.

If $e \neq 0$, we take the distributional Laplacian of the result from Proposition 4.5:

$$\begin{aligned}
\frac{1}{2\pi}\nabla^2 \log \Delta(z-X) &= \frac{1}{2\pi}\nabla^2 \frac{1}{2}\int_0^\infty \log(x)\, d\nu_z(x) \\
&= \frac{1}{2\pi}\nabla^2 \Bigg(\tau(e_{00}) \log|z-(\alpha+i\beta)| \\
&\quad + \tau(e_{01}) \log|z-(\alpha+i\beta')| \\
&\quad + \tau(e_{10}) \log|z-(\alpha'+i\beta)| \\
&\quad + \tau(e_{11}) \log|z-(\alpha'+i\beta')| \\
&\quad + \tau(e) \int_0^{\pi/2} \frac{\log|z-\lambda_1(\theta)| + \log|z-\lambda_2(\theta)|}{2}\, d\nu(\theta) \Bigg).
\end{aligned} \tag{4.45}$$

As in the case when $e = 0$, applying $\frac{1}{2\pi}\nabla^2 \log|\cdot - \lambda| = \delta_\lambda$ directly to the first 4 atomic terms of $\log\Delta(z-X)$ produces the weighted sum of the 4 atoms in μ.

To apply $\frac{1}{2\pi}\nabla^2 \log|\cdot - \lambda| = \delta_\lambda$ for the final integral, we apply Fubini's theorem. Consider $f \in C_c^\infty(\mathbb{C})$. Since $\log|z-w| \in L^1_{\text{loc}}(\mathbb{C})$ for all $w \in \mathbb{C}$, then we may apply Fubini's theorem for $i = 1, 2$:

$$\begin{aligned}
&\langle f, \frac{1}{2\pi}\nabla^2 \int_0^{\pi/2} \log|z-\lambda_i(\theta)|\; d\nu(\theta)\rangle \\
&= \langle \nabla^2 f, \frac{1}{2\pi}\int_0^{\pi/2} \log|z-\lambda_i(\theta)|\; d\nu(\theta)\rangle \\
&= \int_{\mathbb{C}} \nabla^2 f(z) \left(\frac{1}{2\pi}\int_0^{\pi/2} \log|z-\lambda_i(\theta)|\; d\nu(\theta)\right) d\lambda(z) \\
&= \int_0^{\pi/2} \left(\int_{\mathbb{C}} \frac{1}{2\pi}\nabla^2 f(z) \log|z-\lambda_i(\theta)|\; dd\lambda(z)\right) d\nu(\theta) \\
&= \int_0^{\pi/2} f(\lambda_i(\theta))\, d\nu(\theta)\,.
\end{aligned} \tag{4.46}$$

Hence, for $i = 1, 2$,

$$\frac{1}{2\pi}\nabla^2 \int_0^{\pi/2} \log|z-\lambda_i(\theta)|\; d\nu(\theta) = (\lambda_i)_*(\nu)\,. \tag{4.47}$$

Thus, the Laplacian of the integral is:

$$\frac{1}{2\pi}\nabla^2 \int_0^{\pi/2} \frac{\log|z-\lambda_1(\theta)| + \log|z-\lambda_2(\theta)|}{2}\, d\nu(\theta) = \frac{(\lambda_1)_*(\nu) + (\lambda_2)_*(\nu)}{2} = \mu'. \tag{4.48}$$

Combining this with the atomic terms gives the desired Brown measure for X.

For the final note, it follows from Lemma 4.1 and the fact that $\lambda_i(\{0,\pi/2\}) = \{\alpha+i\beta, \alpha'+i\beta, \alpha+i\beta', \alpha'+i\beta'\}$ that

$$\mu'(\{\alpha+i\beta, \alpha'+i\beta, \alpha+i\beta', \alpha'+i\beta'\}) = \nu(\{0,\pi/2\}) = 0\,. \tag{4.49}$$

$\square$

Even though the weights $\tau(e_{ij}), \tau(e)$ and the measure ν have not been determined yet, we can already say something about the support of the Brown measure in general.

First, we prove a lemma about a relevant hyperbola and rectangle:

Lemma 4.7. *Let $\alpha, \alpha', \beta, \beta' \in \mathbb{R}$, where $\alpha \neq \alpha'$ and $\beta \neq \beta'$. Let $\mathscr{A} = \alpha' - \alpha$ and $\mathscr{B} = \beta' - \beta$.*

Let

$$\begin{aligned} H &= \left\{ z = x+iy \in \mathbb{C} : \left(x - \frac{\alpha+\alpha'}{2}\right)^2 - \left(y - \frac{\beta+\beta'}{2}\right)^2 = \frac{\mathscr{A}^2 - \mathscr{B}^2}{4} \right\} \\ R &= \{z = x+iy \in \mathbb{C} : x \in [\alpha\wedge\alpha', \alpha\vee\alpha'], y \in [\beta\wedge\beta', \beta\vee\beta']\}\ . \end{aligned} \tag{4.50}$$

The equation of H is equivalent to:

$$(x-\alpha)(x-\alpha') = (y-\beta)(y-\beta')\,. \tag{4.51}$$

The equation of H in coordinates

$$\begin{aligned} x' &= x - \frac{\alpha+\alpha'}{2} \\ y' &= y - \frac{\beta+\beta'}{2} \end{aligned} \tag{4.52}$$

is

$$(x')^2 - \frac{\mathscr{A}^2}{4} = (y')^2 - \frac{\mathscr{B}^2}{4}. \tag{4.53}$$

It follows that for $(x,y) \in H$,

$$(x,y) \in R \iff (4.53) \leq 0 \iff x \in [\alpha\wedge\alpha', \alpha\vee\alpha'] \text{ or } y \in [\beta\wedge\beta', \beta\vee\beta']\,. \tag{4.54}$$

Similarly,

$$(x, y) \in \mathring{R} \iff (4.53) < 0 \iff x \in (\alpha \wedge \alpha', \alpha \vee \alpha') \text{ or } y \in (\beta \wedge \beta', \beta \vee \beta')\,. \tag{4.55}$$

Alternatively, the equation of the hyperbola is:

$$\operatorname{Re}\left(\left(z - \frac{\alpha + \alpha'}{2} - i\frac{\beta + \beta'}{2}\right)^2\right) = \frac{\mathscr{A}^2 - \mathscr{B}^2}{4}\,. \tag{4.56}$$

If $z \in H$, then $z \in R$ if and only if

$$\left|\operatorname{Im}\left(\left(z - \frac{\alpha + \alpha'}{2} - i\frac{\beta + \beta'}{2}\right)^2\right)\right| \leq \frac{|\mathscr{A}\mathscr{B}|}{2}\,. \tag{4.57}$$

Proof. The equivalent equations for the hyperbola are straightforward to check.

The equivalences for the closed conditions follow from the following equivalences and the equation of the hyperbola in x', y' coordinates

$$(x, y) \in R \iff (x')^2 - \frac{\mathscr{A}^2}{4} \leq 0 \text{ and } (y')^2 - \frac{\mathscr{B}^2}{4} \leq 0 \tag{4.58}$$

$$\begin{aligned} (x')^2 - \frac{\mathscr{A}^2}{4} \leq 0 &\iff |x'| \leq \frac{|\mathscr{A}|}{2} &&\iff x \in [\alpha \wedge \alpha', \alpha \vee \alpha'] \\ (y')^2 - \frac{\mathscr{B}^2}{4} \leq 0 &\iff |y'| \leq \frac{|\mathscr{B}|}{2} &&\iff y \in [\beta \wedge \beta', \beta \vee \beta']\,. \end{aligned} \tag{4.59}$$

The equivalences for the open conditions follow from similar equivalences with the closed conditions replaced by open conditions.

The last equation of the hyperbola follows from direct computation. For the inequality of the rectangle, observe that

$$\operatorname{Im}\left(\left(z - \frac{\alpha + \alpha'}{2} - i\frac{\beta + \beta'}{2}\right)^2\right) = 2x'y'\,. \tag{4.60}$$

In light of what was previously shown,

$$x'y' \leq \frac{|\mathscr{A}\mathscr{B}|}{4} \Longrightarrow x' \leq \frac{|\mathscr{A}|}{2} \text{ or } y' \leq \frac{|\mathscr{B}|}{2} \Longrightarrow z \in R\,. \tag{4.61}$$

Conversely,

$$z \in R \Longrightarrow x' \leq \frac{|\mathscr{A}|}{2} \text{ and } y' \leq \frac{|\mathscr{B}|}{2} \Longrightarrow x'y' \leq \frac{|\mathscr{A}\mathscr{B}|}{4}\,. \tag{4.62}$$

□

Now, we will show that the support of the Brown measure is contained in $H \cap R$:

Corollary 4.8. *Let* $\mathscr{A} = \alpha' - \alpha$ *and* $\mathscr{B} = \beta' - \beta$.

The continuous functions $\lambda_1, \lambda_2 : [0, \pi/2] \to \mathbb{C}$ *in Proposition 4.5 parameterize the intersection of the hyperbola*

$$H = \left\{ z = x + iy \in \mathbb{C} : \left(x - \frac{\alpha + \alpha'}{2}\right)^2 - \left(y - \frac{\beta + \beta'}{2}\right)^2 = \frac{\mathscr{A}^2 - \mathscr{B}^2}{4} \right\} \tag{4.63}$$

with the rectangle

$$R = \{z = x + iy \in \mathbb{C} : x \in [\alpha \wedge \alpha', \alpha \vee \alpha'], y \in [\beta \wedge \beta', \beta \vee \beta']\}\ . \tag{4.64}$$

When $|\mathscr{A}| \geq |\mathscr{B}|$, λ_1 *parameterizes the left component of* $H \cap R$ *and* λ_2 *parameterizes the right component of* $H \cap R$. *When* $|\mathscr{A}| < |\mathscr{B}|$, λ_1 *parameterizes the bottom component of* $H \cap R$ *and* λ_2 *parameterizes the top component of* $H \cap R$.

The support of the Brown measure is contained in $H \cap R$.

Proof. From Lemma 4.7, $z = x + iy$ is on $H \cap R$ if and only if

$$\left(z - \frac{\alpha + \alpha'}{2} - i\frac{\beta + \beta'}{2}\right)^2 = \frac{\mathscr{A}^2 - \mathscr{B}^2 + 2i\mathscr{A}\mathscr{B}\cos(2\theta)}{4} \quad \text{for } \theta \in [0, \pi/2]\,. \tag{4.65}$$

Note that $\lambda_1(\theta), \lambda_2(\theta)$ are exactly the solutions to this equation. Hence, the $\lambda_i(\theta)$ parameterize the intersection of the hyperbola and rectangle.

For the cases of λ_i parameterizing the left/right or top/bottom components, it is easy to see from the formulas that the λ_i map into the left/right or top/bottom components, and since the $\lambda_i(\theta)$ parameterize all of $H \cap R$, then the λ_i have to parameterize the entire left/right or top/bottom components.

As the λ_i parameterize $H \cap R$, then μ' is supported on $H \cap R$. The 4 atoms in the Brown measure are on $(\partial R) \cap H$ (at the 4 corners of the rectangle).

Thus, we conclude that the Brown measure is supported on $H \cap R$. $\square$

Motivated by the hyperbola and rectangle appearing in Corollary 4.8, we introduce the definition of the hyperbola and rectangle associated with X:

Definition 4.9. *Let $p, q \in (M, \tau)$ be Hermitian with laws:*

$$\begin{aligned} \mu_p &= a\delta_\alpha + (1-a)\delta_{\alpha'} \\ \mu_q &= b\delta_\beta + (1-b)\delta_{\beta'} \,. \end{aligned} \tag{4.66}$$

Let $\mathscr{A} = \alpha' - \alpha$ and $\mathscr{B} = \beta' - \beta$.

*The **hyperbola associated with** X is*

$$H = \left\{ z = x + iy \in \mathbb{C} : \left(x - \frac{\alpha + \alpha'}{2} \right)^2 - \left(y - \frac{\beta + \beta'}{2} \right)^2 = \frac{\mathscr{A}^2 - \mathscr{B}^2}{4} \right\} . \tag{4.67}$$

*The **rectangle associated with** X is*

$$R = \{ z = x + iy \in \mathbb{C} : x \in [\alpha \wedge \alpha', \alpha \vee \alpha'], y \in [\beta \wedge \beta', \beta \vee \beta'] \} . \tag{4.68}$$

While the weights $\tau(e_{ij}), \tau(e)$ are as of yet undetermined in the case p and q are free, there are some general relationships between the weights and traces of the spectral projections of p and q:

Proposition 4.10. *Let μ be the Brown measure of X and let μ' be as in Proposition 4.6.*

Then, the μ' measure of each of the two components of $H \cap R$ is equal to $1/2$. Additionally,

$$\begin{aligned}
\tau\left(\chi_{\{\alpha\}}(p)\right) &= \mu(\{\alpha + i\beta, \alpha + i\beta'\}) + \tau(e)/2 \\
\tau\left(\chi_{\{\alpha'\}}(p)\right) &= \mu(\{\alpha' + i\beta, \alpha' + i\beta'\}) + \tau(e)/2 \\
\tau\left(\chi_{\{\beta\}}(q)\right) &= \mu(\{\alpha + i\beta, \alpha' + i\beta\}) + \tau(e)/2 \\
\tau\left(\chi_{\{\beta'\}}(q)\right) &= \mu(\{\alpha + i\beta', \alpha' + i\beta'\}) + \tau(e)/2 \,.
\end{aligned} \tag{4.69}$$

If $|\mathscr{A}| \geq |\mathscr{B}|$, the Brown measure of the left component of $H \cap R$ is $\tau(\chi_{\{\alpha \wedge \alpha'\}(p)})$ and the Brown measure of the right component of $H \cap R$ is $\tau(\chi_{\{\alpha \vee \alpha'\}}(p))$.

If $|\mathscr{A}| < |\mathscr{B}|$, the Brown measure of the bottom component of $H \cap R$ is $\tau(\chi_{\{\beta \wedge \beta'\}}(q))$ and the Brown measure of the top component of $H \cap R$ is $\tau(\chi_{\{\beta \vee \beta'\}}(q))$.

Proof. Recall that

$$\mu' = \frac{(\lambda_1)_*(\nu) + (\lambda_2)_*(\nu)}{2}\,, \tag{4.70}$$

where ν is a probability measure on $(0, \pi/2)$. From Corollary 4.8, the λ_i each parameterize one of the components of $H \cap R$. Hence, the μ' measure of each component of $H \cap R$ is $1/2$.

For the equations of the traces of spectral projections of p and q, we will just prove the first equation, the others are similar. From (4.19),

$$\chi_{\{a\}}(p) = e_{00} + e_{01} + e\chi_{\{a\}}(p)e\,. \tag{4.71}$$

From Proposition 4.6, $\mu(\{\alpha + i\beta\}) = \tau(e_{00})$ and $\mu(\{\alpha + i\beta'\}) = \tau(e_{01})$. From Lemma 4.2, $\tau(e\chi_{\{a\}}(p)e) = \tau(e)/2$. The desired equation follows from taking the trace of the equation and using these facts.

For the last point, we consider the Brown measure of the left component of $H \cap R$ when

$|\mathscr{A}| \geq |\mathscr{B}|$ as the other cases are similar. Let L be this left component. Then,

$$\begin{aligned} \mu(L) &= \mu(\{\alpha \wedge \alpha' + i\beta, \alpha \wedge \alpha' + i\beta'\}) + \tau(e)\mu'(L) \\ &= \mu(\{\alpha \wedge \alpha' + i\beta, \alpha \wedge \alpha' + i\beta'\}) + \tau(e)/2 \\ &= \tau(\chi_{\{\alpha\wedge\alpha'\}}(p)) \,. \end{aligned} \tag{4.72}$$

□

4.3 Computation of ν and weights

In this section, we will determine the weights $\tau(e_{ij}), \tau(e)$ and the measure ν in Proposition 4.6 under the assumption that p and q are freely independent.

We will use the functions ψ_μ, χ_μ, and S_μ that were introduced in Section 1.5.1. Recall that for $x \in (M, \tau)$, we will use ψ_x, χ_x, and S_x to denote the respective functions with respect to μ_x, the spectral measure of x.

Recall that the measure ν and the weights $\tau(e_{ij}), \tau(e)$ are computed in terms of projections p', q' where

$$\begin{aligned} p' &= \chi_{\{\alpha'\}}(p) \\ q' &= \chi_{\{\beta'\}}(q) \,. \end{aligned} \tag{4.73}$$

Since p and q are freely independent, then p' and q' are also freely independent. This along with the traces $\tau(p') = 1 - a$ and $\tau(q') = 1 - b$ determine the joint law of p', q'.

For notational convenience, in this section we will consider two general projections p and q that are freely independent and $\tau(p) = a$, $\tau(q) = b$ for some $a, b \in (0, 1)$. This is a natural continuation of Section 4.1. It is easy to translate the results in this section to the p', q' in Section 4.2.

Recall from Section 4.1 that the weights $\tau(e_{ij}), \tau(e)$ are:

$$\begin{aligned}
\tau(e_{00}) &= \tau((1-p) \wedge (1-q)) \\
\tau(e_{01}) &= \tau((1-p) \wedge q) \\
\tau(e_{10}) &= \tau(p \wedge (1-q)) \\
\tau(e_{11}) &= \tau(p \wedge q) \\
\tau(e) &= 1 - (\tau(e_{00}) + \tau(e_{01}) + \tau(e_{10}) + \tau(e_{11}))\,.
\end{aligned} \tag{4.74}$$

Recall from Section 4.1 that when $e \neq 0$, $((0, \pi/2), \nu)$ is the pushforward measure of $((0,1), \nu^*)$ under the inverse of of $\cos^2(\theta)$ and ν^* is the spectral measure of exe, where $x = pqp + (1-p)(1-q)(1-p)$.

Since $\tau((pqp)^n) \to \tau(p \wedge q)$ as $n \to \infty$, then understanding the laws of pqp and $(1-p)(1-q)(1-p)$ are relevant for computing both ν and the weights $\tau(e_{ij}), \tau(e)$. The first step to this is computing the relevant free probability functions:

Proposition 4.11. *Let $p, q \in (M, \tau)$ be two freely independent projections with $\tau(p) = a$, $\tau(q) = b$, $a, b \in (0,1)$. Then,*

$$\begin{aligned}
\psi_p(z) &= \frac{az}{1-z} \qquad & \psi_q(z) &= \frac{bz}{1-z} \\
\chi_p(w) &= \frac{w}{w+a} \qquad & \chi_q(w) &= \frac{w}{w+b}
\end{aligned} \tag{4.75}$$

$$\chi_{pqp}(w) = \frac{w(1+w)}{(w+a)(w+b)}\,. \tag{4.76}$$

Let

$$f(z) = 1 + (4ab - 2(a+b))z + (a-b)^2 z^2\,. \tag{4.77}$$

Then, $\psi_{pqp}, \psi_{(1-p)(1-q)(1-p)}$ are analytic on $\mathbb{C} \setminus [1, \infty)$ and

$$\begin{aligned}
\psi_{pqp}(z) &= \frac{1 - (a+b)z - \sqrt{f(z)}}{2(z-1)} \\
\psi_{(1-p)(1-q)(1-p)}(z) &= \frac{1 - (2-a-b)z - \sqrt{f(z)}}{2(z-1)}\,,
\end{aligned} \tag{4.78}$$

where $\sqrt{f(z)}$ is an analytic branch of the square root of $f(z)$ on $\mathbb{C}\setminus[1,\infty)$ where $\sqrt{1}=+1$. In particular, the root(s) of $f(z)$ are in $[1,\infty)$ and are distinct when f is quadratic.

Proof. The formula for ψ_p follows from

$$\mu_p = (1-a)\delta_0 + a\delta_1 \tag{4.79}$$

and similarly for ψ_q. Since ψ_p, ψ_q are linear fractional transformations, the formulas for their inverses (χ_p, χ_q, respectively) are easily computed.

Since p and q are freely independent, then the formula for χ_{pqp} follows from the multiplicativity of the S-transform and its relationship with χ.

Since $\|pqp\| \leq 1$ then μ_{pqp} is supported on $[0,1]$. So, the formula

$$\psi_{pqp}(z) = \int_0^\infty \frac{tz}{1-tz}\, d\mu_{pqp}(t) = \int_0^1 \frac{tz}{1-tz}\, d\mu_{pqp}(t) \tag{4.80}$$

defines an analytic function for $z \in \mathbb{C}\setminus[1,\infty)$.

To compute the formula for ψ_{pqp}, recall that $\psi_{pqp} = \chi_{pqp}^{-1}$ in a neighborhood of 0. Then,

$$z = \chi_{pqp}(w) = \frac{w(1+w)}{(w+a)(w+b)} \tag{4.81}$$

if and only if

$$z(w+a)(w+b) = w(1+w)\,. \tag{4.82}$$

This is clearly true for $w \neq -a, -b$ and the latter equation is not satisfied for $w = -a, -b \in (-1,0)$ at any z.

This last equation is true if and only if

$$(z-1)w^2 + (z(a+b)-1)w + zab = 0\,. \tag{4.83}$$

Fixing a z and solving for w, then from the quadratic formula and simplifying, we obtain the desired formula for ψ_{pqp}. The sign of the square root follows from the general fact that $\psi_\mu(0) = 0$.

The formula for $\psi_{(1-p)(1-q)(1-p)}$ follows from the formula for ψ_{pqp} and noting that $1-p$ and $1-q$ are freely independent projections with traces $\tau(p) = 1-a$, $\tau(q) = 1-b$. Observe that $f(z)$ is invariant under changing the pair (a, b) to $(1-a, 1-b)$.

Finally, note that the $\sqrt{f(z)}$ in both ψ_{pqp} and $\psi_{(1-p)(1-q)(1-p)}$ are identical since they are defined on the domain $\mathbb{C} \setminus [1, \infty)$ and agree at $z = 0$.

The fact that $f(z)$ has root(s) in $[1, \infty)$ follows from the fact that $\sqrt{w}$ is not analytic in any neighborhood of 0, so the root(s) of f cannot be on $\mathbb{C} \setminus [1, \infty)$.

For distinctness of the roots when f is quadratic, the discriminant is $16ab(1-a)(1-b) > 0$ for $a, b \in (0, 1)$. $\square$

Now, we proceed to determine $\tau(e_{ij}), \tau(e)$. First, we need the following Lemma ([MS17], Proposition 8):

Lemma 4.12. *Let μ be a finite measure on the real line and $s \in \mathbb{R}$. For a sequence $z_n \to s$ non-tangentially to $\mathbb{R}$, $(z_n - s)G_\mu(z_n) \to \mu(\{s\})$.*

Proof. Let $z_n = x_n + iy_n$. The condition that $z_n \to s$ non-tangentially to $\mathbb{R}$ is equivalent to $|(x_n - s)/y_n| \leq M$ for some M. In particular, $y_n \neq 0$.

Computation shows that

$$(z_n - s)G_\mu(z_n) = \int_{-\infty}^{\infty} \frac{z_n - s}{z_n - t}\, d\mu(t)\,. \tag{4.84}$$

Let $f_n : \mathbb{R} \to \mathbb{C}$ where

$$f_n(t) = \frac{z_n - s}{z_n - t}\,. \tag{4.85}$$

Note that $f_n(s) = 1$ for all n and $\lim\limits_{n\to\infty} f_n(t) = 0$ for all $t \neq s$. It suffices to show that the $f_n(t)$ are uniformly bounded, as then the result follows from the bounded convergence theorem.

First, rewrite $f_n(t)$:

$$f_n(t) = \frac{z_n - s}{z_n - t} = \frac{(x_n - s) + iy_n}{(x_n - t) + iy_n} = \frac{(x_n - s)/y_n + i}{(x_n - t)/y_n + i}\,. \tag{4.86}$$

Since $|(x_n - s)/y_n + i| \le |(x_n - s)/y_n| + 1 = 1 + M$ and $|(x_n - t)/y_n + i| \ge 1$,

$$|f_n(t)| = \frac{|(x_n - s)/y_n + i|}{|(x_n - t)/y_n + i|} \le \frac{1 + M}{1} = 1 + M\,. \tag{4.87}$$

as desired. ☐

Next, we determine $\tau(e_{ij}), \tau(e)$ using the free independence of p and q:

Proposition 4.13. *Let $p, q \in (M, \tau)$ be two freely independent projections with $\tau(p) = a$, $\tau(q) = b$, $a, b \in (0, 1)$. Then,*

$$\begin{aligned}
\tau(e_{00}) &= \tau((1-p) \wedge (1-q)) = \max(0, (1-a) + (1-b) - 1) \\
\tau(e_{01}) &= \tau((1-p) \wedge q) = \max(0, (1-a) + b - 1) \\
\tau(e_{10}) &= \tau(p \wedge (1-q)) = \max(0, a + (1-b) - 1) \\
\tau(e_{11}) &= \tau(p \wedge q) = \max(0, a + b - 1) \\
\tau(e) &= 1 - (\tau(e_{00}) + \tau(e_{01}) + \tau(e_{10}) + \tau(e_{11}))\,.
\end{aligned} \tag{4.88}$$

Proof. By replacing p with $1 - p$ and/or q with $1 - q$, it suffices to just prove

$$\tau(p \wedge q) = \max(0, a + b - 1)\,. \tag{4.89}$$

Recall that $\tau((pqp)^n) \to \tau(p \wedge q)$ as $n \to \infty$. Since $x^n \to \chi_{\{1\}}(x)$ on $[0, 1]$ and $\sigma(pqp) \subset [0, 1]$, then $\tau((pqp)^n) \to \mu_{pqp}(\{1\})$. Hence,

$$\tau(p \wedge q) = \mu_{pqp}(\{1\})\,. \tag{4.90}$$

We proceed to use Proposition 4.11 and Lemma 4.12 to determine $\mu_{pqp}(\{1\})$.

In general, for $z \in \mathbb{C} \setminus \sigma(pqp)$,

$$G_{pqp}(z) = \frac{1}{z}\left(\psi_{pqp}\left(\frac{1}{z}\right) + 1\right). \tag{4.91}$$

From Proposition 4.11, the right-hand side of this equation is:

$$\frac{1}{z}\left(\psi_{pqp}\left(\frac{1}{z}\right) + 1\right) = \frac{z + (a+b-2) + z\sqrt{f(1/z)}}{2z}. \tag{4.92}$$

and is defined on $\mathbb{C} \setminus [0,1]$. Since $\sigma(pqp) \subset [0,1]$, then G_{pqp} is also defined on $\mathbb{C} \setminus [0,1]$, so then the following equality holds for $z \in \mathbb{C} \setminus [0,1]$:

$$G_{pqp}(z) = \frac{z + (a+b-2) + z\sqrt{f(1/z)}}{2z(z-1)}. \tag{4.93}$$

Thus, we may use this formula and Lemma 4.12 to obtain:

$$\begin{aligned} \mu_{pqp}(\{1\}) &= \lim_{z\to 1}(z-1)G_{pqp}(z) \\ &= \lim_{z\to 1}\frac{z + (a+b-2) + z\sqrt{f(1/z)}}{2z} \\ &= \frac{a+b-1+|a+b-1|}{2} \\ &= \max(0, a+b-1). \end{aligned} \tag{4.94}$$

□

Proposition 4.13 can be summarized by: "free projections intersect as little as possible." For $\tau(p \wedge q) = \max(0, a+b-1)$, the term $\max(0, a+b-1)$ is just the minimum trace of the intersection between two projections p and q where of $\tau(p) = a$ and $\tau(q) = b$:

Recall from the parallelogram law that for projections $p, q \in (M, \tau)$,

$$\tau(p \wedge q) = \tau(p) + \tau(q) - \tau(p \vee q) \geq \tau(p) - \tau(q) - 1 = a + b - 1. \tag{4.95}$$

As $\tau(p \wedge q) \geq 0$, then for any projections $p, q \in (M, \tau)$, $\tau(p \wedge q) \geq \max(0, a+b-1)$.

As a corollary, we will no longer need to consider the possibility that $e = 0$:

Corollary 4.14. *Let $p, q \in (M, \tau)$ be two freely independent projections. Then, $e = 0$ if and only if one of $p, 1 - p, q, 1 - q$ is 0.*

Proof. If one of $p, 1 - p, q, 1 - q$ is 0, then it is clear that $e = 0$.

Suppose that none of $p, 1 - p, q, 1 - q$ is 0. Then, we may apply Proposition 4.13. As τ is faithful, it suffices to conclude that $\tau(e) = 0$.

Consider the expressions of a and b in Proposition 4.13. Observe that the expressions inside $\tau(e_{00})$ and $\tau(e_{11})$ are negatives of each other and hence

$$\tau(e_{00}) + \tau(e_{11}) = \max(0, -(a + b - 1)) + \max(0, a + b - 1) = |a + b - 1| . \tag{4.96}$$

Similarly,

$$\tau(e_{01}) + \tau(e_{10}) = \max(0, b - a) + \max(0, a - b) = |a - b| . \tag{4.97}$$

Thus,

$$\tau(e) = 0 \iff |a + b - 1| + |a - b| = 1 . \tag{4.98}$$

By considering the four cases of $|a + b - 1| = \pm(a + b - 1)$ and $|a - b| = \pm(a - b)$, we see that $|a + b - 1| + |a - b| = 1$ if and only if one of $p, 1 - p, q, 1 - q$ is 0. Hence, $\tau(e) \neq 0$. □

As a further corollary, we see that $X = p + iq$ is not normal:

Corollary 4.15. *Let $p, q \in (M, \tau)$ be two freely independent projections with $\tau(p) = a$, $\tau(q) = b$, $a, b \in (0, 1)$. Then, $X = p + iq$ is not normal.*

Proof. It suffices to show that p and q do not commute. For this, recall that p and q commute if and only if $pq = p \wedge q$. Applying this to the other three pairs of commuting projections, $(1 - p, q)$, $(p, 1 - q)$, and $(1 - p, 1 - q)$, then $(1 - p)q = (1 - p) \wedge q$, $p(1 - q) = p \wedge (1 - q)$, and $(1 - p)(1 - q) = (1 - p) \wedge (1 - q)$.

Thus if p and q commute, the following equalities hold:

$$\begin{aligned} 1 &= (p + (1-p))(q + (1-q)) \\ &= pq + (1-p)q + p(1-q) + (1-p)(1-q) \\ &= p \wedge q + (1-p) \wedge q + p \wedge (1-q) + (1-p) \wedge (1-q)\,. \end{aligned} \tag{4.99}$$

In the notation of the projections e_{ij}, e, this means that $e = 0$. From Corollary 4.14, this cannot happen if $a, b \in (0,1)$. Thus, $X = p + iq$ is not normal. $\square$

We can now deduce that in general, when p and q have finitely many atoms, $X = p + iq$ is not normal unless p or q is constant:

Corollary 4.16. *Let $X = p + iq$, where $p, q \in (M, \tau)$ are Hermitian, freely independent, and their spectral measures are atomic i.e.*

$$\begin{aligned} \mu_p &= a_1\delta_{\alpha_1} + \cdots + a_k\delta_{\alpha_k} \\ \mu_q &= b_1\delta_{\beta_1} + \cdots + b_l\delta_{\beta_l}\,, \end{aligned} \tag{4.100}$$

where $\alpha_i, \beta_j \in \mathbb{R}$, $a_i, b_j \geq 0$, and $a_1 + \cdots + a_k = b_1 + \cdots b_l = 1$.

Then, X is normal if and only if one of p or q is constant.

Proof. Suppose that p has k atoms and q has l atoms, so that $a_i, b_j > 0$ for all i, j.

Recall that two Hermitian operators $p, q \in (M, \tau)$ commute if and only if all of their spectral projections commute. Thus, X is normal if and only if $\chi_{\{\alpha_i\}}(p)$ and $\chi_{\{\beta_j\}}(q)$ commute. But if one of these pairs commutes, then from Corollary 4.15, one of $a_i = 0, 1$ or $b_j = 0, 1$ must hold. Since $a_i, b_j > 0$, then either $a_i = 1$ or $b_j = 1$. This means that either $p = \alpha_i$ or $q = \beta_j$. Hence, if X is normal, then one of p or q is constant. The converse is obviously true. $\square$

Returning to the case when p and q have 2 atoms, we proceed to compute ν with the knowledge that $e \neq 0$. This requires computing ν^*, the spectral measure of exe in

$(eMe, \tau\big|_{eMe})$, where $x = pqp + (1-p)(1-q)(1-p)$, and then pushing forward the measure under the inverse of $\cos^2$ onto $(0, \pi/2)$. The result of the computation is the following:

Proposition 4.17. *Let $p, q \in (M, \tau)$ be two freely independent projections with $\tau(p) = a, \tau(q) = b$, $a, b \in (0,1)$. Let $f : \mathbb{R} \to \mathbb{R}$ be given by:*

$$f(x) = 1 + (4ab - 2(a+b))x + (a-b)^2 x^2 \,. \tag{4.101}$$

The measure ν on $(0, \pi/2)$ is a probability measure with density with respect to Lebesgue measure θ:

$$\frac{d\nu}{d\theta} = \frac{2}{\pi}\frac{1}{\tau(e)}\mathrm{Im}\left(\sqrt{f(\sec^2\theta)}\right)\cot(\theta)\,, \tag{4.102}$$

where the square root of a negative number is on the positive imaginary axis.

Proof. To compute ν^*, first note that

$$\begin{aligned} pqp &= p \wedge q + e(pqp)e \\ (1-p)(1-q)(1-p) &= (1-p) \wedge (1-q) + e((1-p)(1-q)(1-p))e \,. \end{aligned} \tag{4.103}$$

Taking n-th powers of each side,

$$\begin{aligned} (pqp)^n &= (p \wedge q)^n + (e(pqp)e)^n \\ ((1-p)(1-q)(1-p))^n &= ((1-p) \wedge (1-q))^n + (e(1-p)(1-q)(1-p)e)^n. \end{aligned} \tag{4.104}$$

Computing $\psi_{e(pqp)e}, \psi_{e(1-p)(1-q)(1-p)e}$ in $(eMe, \tau\big|_{eMe})$ and using the power series expansions of general ψ_μ, then in a neighborhood of 0,

$$\begin{aligned} \psi_{pqp}(z) &= \psi_{p\wedge q}(z) + \tau(e)\psi_{e(pqp)e}(z) \\ \psi_{(1-p)(1-q)(1-p)}(z) &= \psi_{(1-p)\wedge(1-q)}(z) + \tau(e)\psi_{e(1-p)(1-q)(1-p)e}(z) \,. \end{aligned} \tag{4.105}$$

Since the spectra of $pqp, e(pqp)e, (1-p)(1-q)(1-p), e(1-p)(1-q)(1-p)e, p\wedge q, (1-p)\wedge(1-q)$ are all contained in $[0,1]$, then this equality holds on $\mathbb{C} \setminus [1, \infty)$.

From Corollary 4.14, $e \neq 0$, so

$$\begin{aligned}
\psi_{e(pqp)e}(z) &= \frac{\psi_{pqp}(z) - \psi_{p\wedge q}(z)}{\tau(e)} \\
&= \frac{\psi_{pqp}(z)}{\tau(e)} - \frac{\tau(p \wedge q)}{\tau(e)} \frac{z}{1-z} \\
\psi_{e(1-p)(1-q)(1-p)e}(z) &= \frac{\psi_{(1-p)(1-q)(1-p)}(z) - \psi_{(1-p)\wedge(1-q)}(z)}{\tau(e)} \\
&= \frac{\psi_{(1-p)(1-q)(1-p)}(z)}{\tau(e)} - \frac{\tau((1-p) \wedge (1-q))}{\tau(e)} \frac{z}{1-z}\,.
\end{aligned} \tag{4.106}$$

Since e is central and $x = pqp + (1-p)(1-q)(1-p)$, then for $n \geq 1$,

$$(exe)^n = (e(pqp)e)^n + (e(1-p)(1-q)(1-p)e)^n. \tag{4.107}$$

From Proposition 4.11, in $(eMe, \tau\big|_{eMe})$,

$$\begin{aligned}
\psi_{exe}(z) &= \psi_{e(pqp)e}(z) + \psi_{e(1-p)(1-q)(1-p)e}(z) \\
&= \frac{1}{\tau(e)} \left(\psi_{pqp}(z) + \psi_{(1-p)(1-q)(1-p)}(z)\right) \\
&\quad - \frac{\tau(p \wedge q)}{\tau(e)} \frac{z}{1-z} - \frac{\tau((1-p) \wedge (1-q))}{\tau(e)} \frac{z}{1-z} \\
&= \frac{1}{\tau(e)} \left(-1 - \frac{\sqrt{f(z)}}{z-1}\right) - \frac{\tau(p \wedge q) + \tau((1-p) \wedge (1-q))}{\tau(e)} \frac{z}{1-z}\,,
\end{aligned} \tag{4.108}$$

where

$$f(z) = 1 + (4ab - 2(a+b))z + (a-b)^2 z^2. \tag{4.109}$$

From Proposition 4.11, $\sqrt{f(z)}$ is analytic on $\mathbb{C} \setminus [1, \infty)$, so this formula for ψ_{exe} is valid on $\mathbb{C} \setminus [1, \infty)$.

Hence, the following formula for $G_{\nu^*} = G_{exe}$ is valid on $\mathbb{C} \setminus [0,1]$:

$$\begin{aligned} G_{\nu^*}(z) &= G_{exe}(z) \\ &= \frac{1}{z}\left(\psi_{exe}\left(\frac{1}{z}\right) + 1\right) \\ &= \frac{1}{\tau(e)}\frac{\sqrt{f(1/z)}}{z-1} - \frac{1}{\tau(e)z} - \frac{\tau(p \wedge q) + \tau((1-p) \wedge (1-q))}{\tau(e)}\frac{1}{z(z-1)}\,. \end{aligned} \tag{4.110}$$

Recall that ν^* is supported on $[0,1]$. We observe that the measure ν^* has no atoms: For $t \in (0,1)$, this follows from Lemma 4.12 and computing that $\lim_{z \to t}(z-t)G_{\nu^*}(z) = 0$. Recall from Section 4.1 that $\nu^*(\{0,1\}) = 0$ is true in general. Hence, ν^* has no atoms.

Thus, we may recover the measure completely with the formula:

$$\begin{aligned} \lim_{y \to 0^+} \int_a^b -\frac{1}{\pi}\mathrm{Im}\, G_{\nu^*}(x+iy)\, d\mu(x) &= \nu^*((a,b)) + \frac{1}{2}\left(\nu^*(\{a\}) + \nu^*(\{b\})\right) \\ &= \nu^*([a,b])\,. \end{aligned} \tag{4.111}$$

where $a, b \in (0,1)$.

On compact subsets of $(0,1)$, $G_{\nu^*}(x+iy)$ is uniformly bounded as $y \to 0^+$. Given that the following pointwise limit exists for $x \in (0,1)$:

$$\lim_{y \to 0^+} -\frac{1}{\pi}\mathrm{Im}\, G_{\nu^*}(x+iy)\,, \tag{4.112}$$

then this limit will be the density of ν^* with respect to the Lebesgue measure.

Using the formula for $G_{\nu^*}(z)$, we notice that for any $x \in (0,1)$, only one term is non-zero in this limit:

$$\begin{aligned} \lim_{y \to 0^+} -\frac{1}{\pi}\mathrm{Im}\, G_{\nu^*}(x+iy) &= \lim_{y \to 0^+} \frac{1}{\pi}\mathrm{Im}\left(\frac{1}{\tau(e)}\frac{\sqrt{f(1/(x+iy))}}{1-(x+iy)}\right) \\ &= \frac{1}{\pi\tau(e)}\frac{\lim_{y \to 0^+} \mathrm{Im}\sqrt{f(1/(x+iy))}}{1-x}\,. \end{aligned} \tag{4.113}$$

We proceed by showing $\lim_{y\to 0^+} \operatorname{Im}\sqrt{f(1/(x+iy))}$ exists. It suffices to show the following limit exists:

$$\lim_{\substack{z\to t\\ \operatorname{Im}(z)<0}} g(z)\,, \tag{4.114}$$

where $t\in(0,1)$ and g is defined an analytic on the lower half-plane and $g(z)^2 = f(z)$. First, note that for any sequence $z_n \to t$, $g(z_n)$ is bounded, because $g(z_n)^2 = f(z_n)$. Considering a convergent subsequence where $g(z_{n_k})$ converges, then if $g(z_{n_k}) \to s$, then $s^2 = \lim_{n_k\to\infty} g(z_{n_k})^2 = \lim_{n_k\to\infty} f(z_{n_k}) = f(t)$. Hence, $g(z_{n_k})$ converges to a square root of $f(t)$. When $f(t)=0$ this is enough to prove the desired limit. When $f(t)\neq 0$, we consider if there are two sequences in the lower half-plane, $z_n, z_n' \to t$, where $g(z_n), g(z_n')$ converge to the different square roots of $f(t)$. Taking derivatives of $g(z)^2 = f(z)$, then $2g(z)g'(z) = f'(z)$. As $|f'(z)|$ is bounded from above and $|g(z)|$ bounded from below near t where $f(t)\neq 0$, then $|g'(z)|$ is bounded from above near t. Then, $g(z)$ is Lipschitz near t. Thus, $g(z_n), g(z_n')$ converging to different square roots is a contradiction.

Since the measure is positive, then $\lim_{y\to 0^+} \operatorname{Im}\sqrt{f(1/(x+iy))}$ has to be non-negative, at least Lebesgue almost everywhere $t\in(0,1)$. It is possible to extend this to all $t\in(0,1)$ by noting this set of t is dense and using g being Lipschitz near t where $f(t)\neq 0$.

Thus, for $x\in(0,1)$,

$$\frac{d\nu^*}{d\lambda} = \frac{1}{\pi\tau(e)}\frac{\operatorname{Im}\sqrt{f(1/x)}}{1-x}\,, \tag{4.115}$$

where the square root of a negative number is on the positive imaginary axis.

By applying the change of variables formula, the measure ν on $(0,\pi/2)$ is given by:

$$\frac{d\nu}{d\theta} = \frac{2}{\pi}\frac{1}{\tau(e)}\operatorname{Im}\left(\sqrt{f(\sec^2\theta)}\right)\cot\theta\,. \tag{4.116}$$

Note that this is a probability measure, being the spectral measure of a non-zero element of eMe. $\square$

4.4 The Brown measure of X

By combining Propositions 4.6 and 4.17, we can determine the Brown measure of X when p and q have 2 atoms:

Theorem 4.18. *Let $p, q \in (M, \tau)$ be Hermitian, freely independent and*

$$\begin{aligned} \mu_p &= a\delta_\alpha + (1-a)\delta_{\alpha'} \\ \mu_q &= b\delta_\beta + (1-b)\delta_{\beta'} \,, \end{aligned} \tag{4.117}$$

where $a, b \in (0,1)$, $\alpha \neq \alpha'$, $\beta \neq \beta'$, and $\alpha, \alpha', \beta, \beta' \in \mathbb{R}$.

Let

$$\begin{aligned} \epsilon_{00} &= \max(0, a+b-1) \\ \epsilon_{01} &= \max(0, a+(1-b)-1) \\ \epsilon_{10} &= \max(0, (1-a)+b-1) \\ \epsilon_{11} &= \max(0, (1-a)+(1-b)-1) \\ \epsilon &= 1 - (\epsilon_{00}+\epsilon_{01}+\epsilon_{10}+\epsilon_{11})\,. \end{aligned} \tag{4.118}$$

Then, $\epsilon > 0$.

Let $\mathscr{A} = \alpha' - \alpha$ and $\mathscr{B} = \beta' - \beta$ and $\sqrt{z}$ denote the principal branch of the square root defined on $\mathbb{C} \setminus (-\infty, 0)$. Then,

$$\lambda_1(\theta) = \begin{cases} \dfrac{\alpha+\alpha'}{2} + i\dfrac{\beta+\beta'}{2} - \dfrac{1}{2}\sqrt{\mathscr{A}^2 - \mathscr{B}^2 + 2i\mathscr{A}\mathscr{B}\cos(2\theta)} & \text{when } |\mathscr{A}| \geq |\mathscr{B}| \\ \dfrac{\alpha+\alpha'}{2} + i\dfrac{\beta+\beta'}{2} - \dfrac{i}{2}\sqrt{\mathscr{B}^2 - \mathscr{A}^2 - 2i\mathscr{A}\mathscr{B}\cos(2\theta)} & \text{when } |\mathscr{A}| < |\mathscr{B}| \end{cases} \tag{4.119}$$

$$\lambda_2(\theta) = \begin{cases} \dfrac{\alpha+\alpha'}{2} + i\dfrac{\beta+\beta'}{2} + \dfrac{1}{2}\sqrt{\mathscr{A}^2 - \mathscr{B}^2 + 2i\mathscr{A}\mathscr{B}\cos(2\theta)} & \text{when } |\mathscr{A}| \geq |\mathscr{B}| \\ \dfrac{\alpha+\alpha'}{2} + i\dfrac{\beta+\beta'}{2} + \dfrac{i}{2}\sqrt{\mathscr{B}^2 - \mathscr{A}^2 - 2i\mathscr{A}\mathscr{B}\cos(2\theta)} & \text{when } |\mathscr{A}| < |\mathscr{B}|\,. \end{cases} \tag{4.120}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be given by:

$$f(x) = 1 + (4ab - 2(a+b))x + (a-b)^2x^2\,. \tag{4.121}$$

Let ν be a probability measure on $(0, \pi/2)$ with density with respect to Lebesgue measure θ:

$$\frac{d\nu}{d\theta} = \frac{2}{\pi}\frac{1}{\epsilon}\mathrm{Im}\left(\sqrt{f(\sec^2\theta)}\right)\cot(\theta)\,, \tag{4.122}$$

where the square root of a negative number is on the positive imaginary axis.

Let μ' be a complex probability measure given by:

$$\mu' = \frac{(\lambda_1)_*(\nu) + (\lambda_2)_*(\nu)}{2}\,. \tag{4.123}$$

Then, the Brown measure of $X = p + iq$ is:

$$\mu = \epsilon_{00}\delta_{\alpha+i\beta} + \epsilon_{01}\delta_{\alpha+i\beta'} + \epsilon_{10}\delta_{\alpha'+i\beta} + \epsilon_{11}\delta_{\alpha'+i\beta'} + \epsilon\mu'\,. \tag{4.124}$$

Proof. The Theorem follows from combining several previous results:

The general form of μ is given in Proposition 4.6. The $\tau(e_{ij}), e$ are relabeled as ϵ_{ij}, ϵ in light of Proposition 4.13 and interchanging general p, q for $p' = \chi_{\{\alpha'\}}(p), q' = \chi_{\{\beta'\}}(q)$. Additionally, the fact that $\epsilon \neq 0$ comes from Corollary 4.14. Finally, the measure ν comes from Proposition 4.17 (after switching p, q for p', q'). □

We observe that the measure ν is only dependent on the weights of the measures of p and q (i.e. a and b), and the "shape" of the measure (positions of the atoms and λ_i) is only dependent on the positions of the atoms p and q (i.e. $\alpha, \alpha', \beta, \beta'$)

We will use the definition of μ' from Theorem 4.18 for what follows:

Definition 4.19. *Let $X = p + iq$, where $p, q \in (M, \tau)$ are Hermitian, freely independent, and have 2 atoms. Define μ' to be the measure as in 4.18.*

Now, we make some observations about the Brown measure of $p + iq$ in the following Corollaries. First, we consider the atoms of the Brown measure:

Corollary 4.20. *Let μ be the Brown measure of $p + iq$ where p and q have 2 atoms. Then,*

1. *μ' does not have atoms.*

2. *μ is never atomic.*

3. *μ can have atoms only at the points $\alpha + i\beta, \alpha' + i\beta, \alpha + i\beta', \alpha' + i\beta'$.*

4. *μ has no atoms at $\alpha + i\beta$ and $\alpha' + i\beta'$ if and only if $a + b = 1$. If $a + b \neq 1$, then μ has exactly 1 atom at either $\alpha + i\beta$ or $\alpha' + i\beta'$, with size $|a + b - 1|$.*

5. *μ has no atoms at $\alpha + i\beta'$ and $\alpha' + i\beta$ if and only if $a = b$. If $a \neq b$, then μ has exactly 1 atom at either $\alpha + i\beta'$ or $\alpha' + i\beta$, with size $|a - b|$.*

6. *μ has 0, 1, or 2 atoms. μ has 0 atoms if and only if $a = b = 1/2$. μ has 1 atom if and only if one of $a + b = 1$ or $a = b$. μ has 2 atoms if and only if $a + b \neq 1$ and $a \neq b$.*

7. *Changing $a \mapsto 1 - a$ and/or $b \mapsto 1 - b$ permutes the ϵ_{ij}.*

Proof. 1. ν is absolutely continuous and λ_i are injective, so $(\lambda_i)_*(\nu)$ does not have atoms. Hence, μ' does not have atoms.

2. Since $\epsilon \neq 0$, then $\mu' \neq 0$, so μ is never atomic

3. Since μ' does not have atoms, the only atoms of μ can occur at the points $\alpha + i\beta, \alpha' + i\beta, \alpha + i\beta', \alpha' + i\beta'$.

4. Note that

$$\epsilon_{00} + \epsilon_{11} = \max(0, -(a + b - 1)) + \max(0, a + b - 1) = |a + b - 1| . \tag{4.125}$$

Hence, when $a + b = 1$, $\epsilon_{00} + \epsilon_{11} = 0$ so μ has no atoms at $\alpha + i\beta$ and $\alpha' + i\beta'$. If $a + b \neq 1$, then only one of $a + b - 1$ and $-(a + b - 1)$ is positive and equal to $|a + b - 1|$ and hence one of $\epsilon_{00}, \epsilon_{11}$ is equal to $|a + b - 1|$.

5. Follows similarly to 4, with the equation

$$\epsilon_{01} + \epsilon_{10} = \max(0, b - a) + \max(0, a - b) = |a - b| . \tag{4.126}$$

6. Directly follows from 4. and 5.

7. Follows directly from the formulas for the ϵ_{ij}.

$\square$

Now, we consider the symmetries of μ':

Corollary 4.21. *Let μ be the Brown measure of $p+iq$ where p and q have 2 atoms*

1. *Swapping the roles of p and q fixes ν.*

2. *Changing one of $a \mapsto 1-a$ or $b \mapsto 1-b$ changes ν by changing $\theta \to \pi/2 - \theta$. These correspond changing $p \mapsto \alpha + \alpha' - p$ and $q \mapsto \beta + \beta' - q$, respectively.*

3. *Changing both $a \mapsto 1-a$ and $b \mapsto 1-b$ fixes ν. This corresponds to changing both $p \mapsto \alpha + \alpha' - p$ and $q \mapsto \beta + \beta' - q$. This amounts to changing $p + iq \mapsto (\alpha+\alpha') + i(\beta+\beta') - (p+iq)$.*

Proof. 1. Swapping the roles of p and q amounts to swapping a and b in the formula for $f(x)$. But, the formula for $f(x)$ is symmetric with respect to a, b, so ν is fixed.

2. Since f is symmetric with respect to changing a and b, it suffices to check just $a \mapsto 1-a$. Denote f by $f_{a,b}$ to refer to the coefficients. For this, we just need to check the identity:

$$\sqrt{f_{1-a,b}(\sec^2\theta)}\cot\theta = \sqrt{f_{a,b}(\csc^2\theta)}\tan\theta \tag{4.127}$$

for $\theta \in (0, \pi/2)$. As $\cot\theta, \tan\theta > 0$ for $\theta \in (0,\pi/2)$, it is equivalent to check

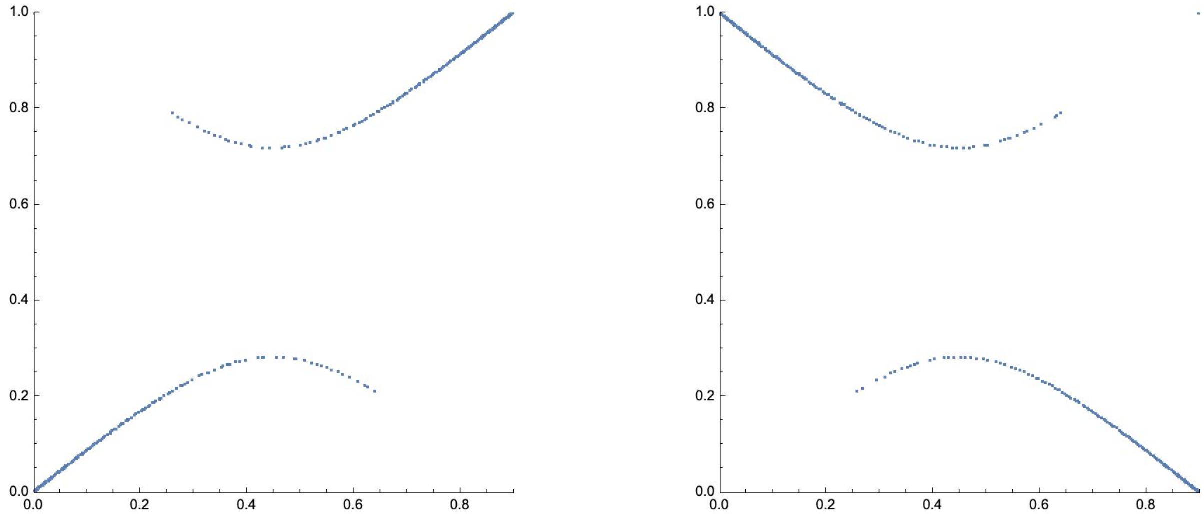
$$f_{1-a,b}(\sec^2\theta)\cot^2\theta = f_{a,b}(\csc^2\theta)\tan^2\theta\,. \tag{4.128}$$

Checking this identity is a straightforward calculation.

3. Follows from applying 2. twice.

$\square$

Figure 4.1 illustrates the behavior in Corollary 4.21 where the Brown measure of $X = p+iq$ is approximated by the ESD of $X_n = P_n + iQ_n$. Note the symmetry between the two ESDs.



(a) ESD of $X_n = P_n + iQ_n$
$\mu_{P_n} = (4/5)\delta_0 + (1/5)\delta_{9/10}$
$\mu_{Q_n} = (1/5)\delta_0 + (4/5)\delta_1$
$n = 1000$

(b) ESD of $X_n = P_n + iQ_n$
$\mu_{P_n} = (1/5)\delta_0 + (4/5)\delta_{9/10}$
$\mu_{Q_n} = (1/5)\delta_0 + (4/5)\delta_1$
$n = 1000$

Figure 4.1: ESDs for Corollary 4.21

Finally, we consider when the density of μ' extends all the way to the 4 corners of the intersection of the hyperbola with the boundary of the rectangle:

Corollary 4.22. *Let μ be the Brown measure of $p + iq$ where p and q have 2 atoms*

1. *μ' has density extending to $\alpha' + i\beta$ and $\alpha + i\beta'$ if and only if $a = b$.*

2. *μ' has density extending to $\alpha + i\beta$ and $\alpha' + i\beta'$ if and only if $a = 1 - b$.*

3. *μ' has density extending to all 4 corners of the intersection of the hyperbola with the boundary of the rectangle if and only if $a = b = 1/2$. Hence, the support of μ is equal to $H \cap R$ if and only if $a = b = 1/2$.*

Proof. Let $\mathscr{A} = \alpha' - \alpha$ and $\mathscr{B} = \beta' - \beta$.

1. For $z = \alpha' + i\beta, \alpha + i\beta'$,

$$\left(z - \frac{\alpha+\alpha'}{2} - i\frac{\beta+\beta'}{2}\right)^2 = \frac{\mathscr{A}^2 - \mathscr{B}^2 - 2i\mathscr{A}\mathscr{B}}{4}. \tag{4.129}$$

   Hence, $z = \lambda_i(\pi/2)$.

   Thus, it is equivalent to determine for which a, b the measure ν has density approaching $\pi/2$. Note that ν has density approaching $\pi/2$ if and only if $f(\sec^2\theta) < 0$ for $\theta \to \pi/2^-$. Note that $\lim\limits_{\theta\to\pi/2^-} \sec^2\theta = \infty$. Recall that $f(z)$ is a polynomial of degree at most 2. If f is quadratic, then $\lim\limits_{z\to\infty} f(z) = \infty$, so then ν does not have density approaching $\pi/2$. Hence, we must have $a = b$. Conversely, if $a = b$, then $f(z) = 1 + 4(a^2 - a)z = 1 + 4(a-1)az$, and the linear term has negative coefficient for $a \in (0,1)$. Thus, $f(z) < 0$ as $z \to \infty$ and ν has density approaching $\pi/2$.

2. For $z = \alpha + i\beta, \alpha' + i\beta'$,

$$\left(z - \frac{\alpha+\alpha'}{2} - i\frac{\beta+\beta'}{2}\right)^2 = \frac{\mathscr{A}^2 - \mathscr{B}^2 + 2i\mathscr{A}\mathscr{B}}{4}. \tag{4.130}$$

   Hence, $z = \lambda_i(0)$.

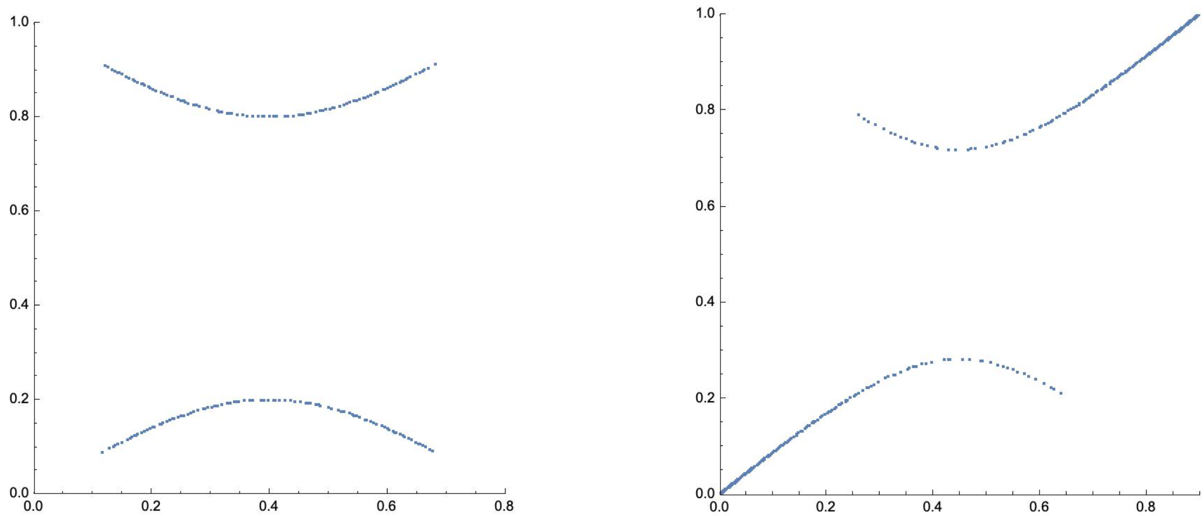
   Thus, it is equivalent to determine for which a, b the measure ν has density approaching 0. Note that ν has density approaching 0 if and only if $f(\sec^2\theta) < 0$ for $\theta \to 0^+$. Since $\lim\limits_{\theta\to 0^+} \sec^2\theta = 1$, then we need that $f(x) < 0$ for $x \to 1^+$. Since $f(1) = (a+b-1)^2$, then $f(1) = 0$, i.e. $a = 1 - b$. Conversely, if $a = 1 - b$, then $f(1) = 0$ and since f is either quadratic with positive leading coefficient or f is linear with negative slope, then f must be negative to the right of 1. Note that in the case where f is quadratic, f cannot have a double root at 1 (from Proposition 4.11).

3. Follows from 1. and 2.

$\square$

Figure 4.2 illustrates the behavior in Corollary 4.22 where the Brown measure of $X = p + iq$

is approximated by the ESD of $X_n = P_n + iQ_n$, for a deterministic $P_n, Q_n \in M_n(\mathbb{C})$. The left ESD does not have density approaching the corners of R, but the right ESD does.



(a) ESD of $X_n = P_n + iQ_n$
$\mu_{P_n} = (9/10)\delta_0 + (1/10)\delta_{4/5}$
$\mu_{Q_n} = (1/2)\delta_0 + (1/2)\delta_1$
$n = 1000$

(b) ESD of $X_n = P_n + iQ_n$
$\mu_{P_n} = (4/5)\delta_0 + (1/5)\delta_{9/10}$
$\mu_{Q_n} = (1/5)\delta_0 + (4/5)\delta_1$
$n = 1000$

Figure 4.2: ESDs for Corollary 4.22

Finally, we conclude that the Brown measure of $X = p + iq$ uniquely determines the laws of p and q. Here we allow p and q to be possibly constant.

Corollary 4.23. *Let $p, q \in (M, \tau)$ be Hermitian and freely independent and*

$$\begin{aligned} \mu_p &= a\delta_\alpha + (1-a)\delta_{\alpha'} \\ \mu_q &= b\delta_\beta + (1-b)\delta_{\beta'} \,. \end{aligned} \tag{4.131}$$

where $a, b \in [0,1]$ and $\alpha, \alpha', \beta, \beta' \in \mathbb{R}$. Let μ be the Brown measure of $X = p + iq$. Then, the assignment $(\mu_p, \mu_q) \mapsto \mu$ is 1 to 1.

Proof. From Corollary 4.20, μ is atomic if and only if one of p or q is constant. In this case, it is easy to determine the weights and atoms of both μ_p and μ_q from μ.

Thus, we may consider μ which is the Brown measure of $X = p + iq$ where p and q are not constant.

First, we show that μ determines the positions of the atoms of p and q. Since the support of μ' on $H \cap R$ contains at least 5 points, then the equation of H is uniquely determined. If $a = b$ or $a + b = 1$, then from Corollary 4.22, we can determine the positions of the atoms of p and q. Thus, assume that $a \neq b$ and $a + b \neq 1$. From Corollary 4.20, μ has 2 atoms at the points $\{\alpha + i\beta, \alpha' + i\beta, \alpha + i\beta', \alpha' + i\beta'\}$. From these two points, at least 3 of $\alpha, \alpha', \beta, \beta'$ are determined. To determine the last one, we can use the equation of the hyperbola and look at either the coefficient of x or y. Thus, μ determines the positions of the atoms of p and q.

We can determine the weights of the atoms of p and q directly from Proposition 4.10. $\square$

CHAPTER 5

The random matrix model X_n

In this chapter, we discuss some of the properties of the random matrix model $X_n = P_n + iQ_n$, especially their empirical spectral distributions. In the case when P_n, Q_n have 2 atoms, we will provide the counterparts of Corollaries 4.20, 4.21, and 4.22.

We first need the following Lemma describing the dimension of the intersection of Haar-rotated subspaces:

Lemma 5.1. *Let $P_n, Q_n \in M_n(\mathbb{C})$ be independently Haar-rotated projections. Then, $\tau(P_n \wedge Q_n) = \max(0, \tau(P_n) + \tau(Q_n) - 1)$ with probability 1.*

Proof. We may assume that $P_n = U_n P_n' U_n^*$ and $Q_n = V_n Q_n' V_n^*$, where $P_n', Q_n' \in M_n(\mathbb{C})$ are deterministic projections. In particular, we may assume that Q_n' is the diagonal matrix with all 1's coming before 0's on the diagonal. By conjugating by U_n^*, we can reduce to the case where $P_n = P_n'$ is deterministic and Q_n is as before.

Let $W = P_n\,\mathbb{C}^n$ and let $V = Q_n\,\mathbb{C}^n$. It suffices to show that $\dim(V + W)$ is as large as possible with probability 1, i.e. $\dim(V + W) = \min(n, \dim(V) + \dim(W))$ with probability 1. For this, it suffices to show that if W is a subspace of $\mathbb{C}^n$, V is a span of the first m columns of a Haar-distributed unitary, and $\dim(W) \leq n - m$, then $V \cap W = \{0\}$ with probability 1.

We recall the following process that produces a Haar-distributed unitary V_n (see [Mec19], Chapter 1): Pick the first column of V_n uniformly from the sphere $S^{n-1} \subset \mathbb{C}^n$, call it v_1. Having chosen the first k columns $v_1, \ldots, v_k$, choose the $(k + 1)$-th column v_{k+1} uniformly from $S^{n-1} \cap \operatorname{span}(v_1, \ldots, v_k)^{\perp}$, which can be identified with S^{n-k-1}.

We return to showing that $V \cap W = \{0\}$ with probability 1. Let V_n be a Haar distributed unitary and $V_n e_i = v_i$, so that $V = \text{span}\{v_1, \ldots, v_m\}$. We proceed by induction on $\dim(V) = m$, $1 \leq m \leq n$. For $m = 1$, $\dim(W) \leq n-1$. Then, $V \cap W \neq \{0\}$ would mean that $v_1 \in W \cap S^{n-1}$. Note that $W \cap S^{n-1}$ can be identified with $S^{\dim(W)-1}$, a sphere of smaller dimension. Since v_1 is chosen uniformly from S^{n-1}, this happens with probability 0. For the inductive step, suppose $V = \text{span}\{v_1, \ldots, v_{m+1}\}$. From the case $m = 1$, the first vector v_1 intersects W with probability 0. Now, consider the projection of $\mathbb{C}^n$ onto $(v_1)^\perp$. If W' is the image of W under this projection, then $\dim(W') = \dim(W)$ with probability 1 (when $v_1 \notin W$). The projection reduces the dimension of $\mathbb{C}^n$ to $n-1$. Then, the process of choosing the columns $v_2, \ldots, v_{m+1}$ reduces to the case where $\dim(V) = m$ and $\dim(W') \leq n-(m+1) = (n-1)-m$ inside $\mathbb{C}^{n-1}$. Thus, with probability 1, $\text{span}(v_2, \ldots, v_{m+1}) \cap W' = \{0\}$. Hence, $V \cap W = \{0\}$ with probability 1. $\square$

Note that the result in Lemma 5.1 is the same as the analogous quantities for freely independent projections in Proposition 4.13.

We now show that the random matrices $X_n = P_n + iQ_n$ are normal with probability 0, except in the obvious cases when P_n or Q_n are constant:

Proposition 5.2. *Let $X_n = P_n + iQ_n$, where $P_n, Q_n \in M_n(\mathbb{C})$ are independently Haar-rotated Hermitian matrices with distributions*

$$\begin{aligned} \mu_{P_n} &= (a_1)_n \delta_{(\alpha_1)_n} + \cdots + (a_k)_n \delta_{(\alpha_k)_n} \\ \mu_{Q_n} &= (b_1)_n \delta_{(\beta_1)_n} + \cdots + (b_l)_n \delta_{(\beta_l)_n}\,, \end{aligned} \tag{5.1}$$

where $(\alpha_i)_n, (\beta_j)_n \in \mathbb{R}$, $(a_i)_n, (b_j)_n \geq 0$, and $(a_1)_n + \cdots + (a_k)_n = (b_1)_n + \cdots (b_l)_n = 1$. If μ_{P_n}, μ_{Q_n} are not constant, then X_n is normal with probability 0.

Proof. First, we recall some general facts: Hermitian operators $p, q \in (M, \tau)$ commute if and only if all of their spectral projections commute. If p, q are projections, then p and q commute if and only if $pq = p \wedge q$. Applying this to the other three pairs of commuting projections, $(1-p, q)$, $(p, 1-q)$, and $(1-p, 1-q)$, then $(1-p)q = (1-p) \wedge q$, $p(1-q) = p \wedge (1-q)$, and $(1-p)(1-q) = (1-p) \wedge (1-q)$.

Thus, if p and q commute, the following equalities hold:

$$\begin{aligned}1 &= (p + (1-p))(q + (1-q)) \\ &= pq + (1-p)q + p(1-q) + (1-p)(1-q) \\ &= p \wedge q + (1-p) \wedge q + p \wedge (1-q) + (1-p) \wedge (1-q)\,.\end{aligned} \tag{5.2}$$

Hence, if $p, q \in B(H)$, then we have the orthogonal decomposition:

$$H = (p \wedge q)H \oplus ((1-p) \wedge q)H \oplus (p \wedge (1-q))H \oplus ((1-p) \wedge (1-q))H\,. \tag{5.3}$$

Returning to the Proposition, without loss of generality, assume that $(a_1)_n = \min_i (a_i)_n$ and $(b_1)_n = \min_j (b_j)_n$. If P_n and Q_n commute, then $p = \chi_{\{(\alpha_1)_n\}}(P_n)$ and $q = \chi_{\{(\beta_1)_n\}}(Q_n)$ commute. Note that $\tau(p) = (a_1)_n$ and $\tau(q) = (b_1)_n$. Since P_n and Q_n are not constant and $(a_1)_n, (b_1)_n$ are minimal, then $(a_1)_n, (b_1)_n \in (0, 1/2]$. Applying Lemma 5.1, $p \wedge q = 0$ with probability 1. Let $W = q\,\mathbb{C}^n$ and $V = p\,\mathbb{C}^n$. From the orthogonal decomposition of $\mathbb{C}^n$ by commuting projections, $W \subset V^{\perp}$. Since $\tau(q) \neq 0$, then W contains a non-zero vector v that is uniformly distributed on S^{n-1}. Applying Lemma 5.1, $v \in W$ with probability 0. Hence, p and q commute with probability 0. Thus, P_n and Q_n commute with probability 0, and we conclude X_n is normal with probability 0. $\square$

Now, we consider the case when P_n and Q_n have 2 atoms. In this case, we will use the following notation:

$$\begin{aligned}\mu_{P_n} &= a_n \delta_{\alpha_n} + (1-a_n)\delta_{\alpha'_n} \\ \mu_{Q_n} &= b_n \delta_{\beta_n} + (1-b_n)\delta_{\beta'_n}\,,\end{aligned} \tag{5.4}$$

for $a_n, b_n \in (0,1)$, $\alpha_n \neq \alpha'_n$, $\beta_n \neq \beta'_n$, and $\alpha_n, \alpha'_n, \beta_n, \beta'_n \in \mathbb{R}$.

$$
\begin{aligned}
\tilde{P}_n &= P_n - \frac{\alpha_n + \alpha'_n}{2} \\
\tilde{Q}_n &= Q_n - \frac{\beta_n + \beta'_n}{2} \\
\tilde{X}_n &= \tilde{P}_n + i\tilde{Q}_n \\
\mathscr{A}_n &= \alpha'_n - \alpha_n \\
\mathscr{B}_n &= \beta'_n - \beta_n \,.
\end{aligned} \tag{5.5}
$$

Analogous to H and R from Definition 4.9, let H_n and R_n be the hyperbola and rectangle associated with X_n:

Definition 5.3. *Let $P_n, Q_n \in M_n(\mathbb{C})$ be Hermitian with laws:*

$$
\begin{aligned}
\mu_{P_n} &= a_n \delta_{\alpha_n} + (1 - a_n)\delta_{\alpha'_n} \\
\mu_{Q_n} &= b_n \delta_{\beta_n} + (1 - b_n)\delta_{\beta'_n} \,.
\end{aligned} \tag{5.6}
$$

Let $\mathscr{A}_n = \alpha'_n - \alpha_n$ and $\mathscr{B}_n = \beta'_n - \beta_n$.

The ***hyperbola associated with*** *X_n is*

$$
H_n = \left\{ z = x + iy \in \mathbb{C} : \left(x - \frac{\alpha_n + \alpha'_n}{2} \right)^2 - \left(y - \frac{\beta_n + \beta'_n}{2} \right)^2 = \frac{\mathscr{A}_n^2 - \mathscr{B}_n^2}{4} \right\}. \tag{5.7}
$$

The ***rectangle associated with*** *X_n is*

$$
R_n = \{ z = x + iy \in \mathbb{C} : x \in [\alpha_n \wedge \alpha'_n, \alpha_n \vee \alpha'_n], y \in [\beta_n \wedge \beta'_n, \beta_n \vee \beta'_n] \} \,. \tag{5.8}
$$

We will show that the empirical spectral distribution of X_n is supported on $H_n \cap R_n$.

First, we consider $\tilde{X}_n{}^2$:

Proposition 5.4. *$\tilde{X}_n{}^2$ is normal and*

$$
\begin{aligned}
\mathrm{Re}(\tilde{X}_n{}^2) &= \frac{\mathscr{A}_n^2 - \mathscr{B}_n^2}{4} \\
\left\| \mathrm{Im}(\tilde{X}_n{}^2) \right\| &\leq \frac{|\mathscr{A}_n \mathscr{B}_n|}{2} \,.
\end{aligned} \tag{5.9}
$$

If ρ is an eigenvalue of $\tilde{X_n}^2$ then

$$\begin{aligned} \mathrm{Re}(\rho) &= \frac{\mathscr{A}_n^2 - \mathscr{B}_n^2}{4} \\ |\mathrm{Im}(\rho)| &\leq \frac{|\mathscr{A}_n \mathscr{B}_n|}{2} \,. \end{aligned} \tag{5.10}$$

Proof. Observe that $\tilde{P_n}^2 = \mathscr{A}_n^2/4$ and $\tilde{Q_n}^2 = \mathscr{B}_n^2/4$.

Hence,

$$\begin{aligned} \tilde{X_n}^2 &= (\tilde{P_n} + i\tilde{Q_n})^2 \\ &= (\tilde{P_n}^2 - \tilde{Q_n}^2) + i(\tilde{P_n}\tilde{Q_n} + \tilde{Q_n}\tilde{P_n}) \\ &= \frac{\mathscr{A}_n^2 - \mathscr{B}_n^2}{4} + i(\tilde{P_n}\tilde{Q_n} + \tilde{Q_n}\tilde{P_n}) \,. \end{aligned} \tag{5.11}$$

As $\tilde{P_n}\tilde{Q_n} + \tilde{P_n}\tilde{Q_n}$ is Hermitian then

$$\begin{aligned} \mathrm{Re}(\tilde{X_n}^2) &= \frac{\mathscr{A}_n^2 - \mathscr{B}_n^2}{4} \\ \mathrm{Im}(\tilde{X_n}^2) &= \tilde{P_n}\tilde{Q_n} + \tilde{Q_n}\tilde{P_n} \,. \end{aligned} \tag{5.12}$$

Note that from the spectra of $\tilde{P_n}, \tilde{Q_n}$ that $\left\|\tilde{P_n}\right\| = |\mathscr{A}_n|/2$ and $\left\|\tilde{Q_n}\right\| = |\mathscr{B}_n|/2$. Hence,

$$\begin{aligned} \left\|\mathrm{Im}(\tilde{X_n}^2)\right\| &= \left\|\tilde{P_n}\tilde{Q_n} + \tilde{Q_n}\tilde{P_n}\right\| \\ &\leq \left\|\tilde{P_n}\right\| \left\|\tilde{Q_n}\right\| + \left\|\tilde{Q_n}\right\| \left\|\tilde{P_n}\right\| \\ &= \frac{|\mathscr{A}_n \mathscr{B}_n|}{2} \,. \end{aligned} \tag{5.13}$$

Since $\mathrm{Re}(\tilde{X_n})$ is a constant, then clearly it commutes with $\mathrm{Im}(\tilde{X_n})$ and it follows that $\tilde{X_n}$ is normal. Hence, $\tilde{X_n}$ is diagonalizable and its eigenvalues are of the form $\rho = \rho_1 + i\rho_2$, where ρ_1 is an eigenvalue of $\mathrm{Re}(\tilde{X_n})$ and ρ_2 is an eigenvalue of $\mathrm{Im}(\tilde{X_n})$. Thus,

$$\begin{aligned} \mathrm{Re}(\rho) &= \frac{\mathscr{A}_n^2 - \mathscr{B}_n^2}{4} \\ |\mathrm{Im}(\rho)| &\leq \left\|\mathrm{Im}(\tilde{X_n})\right\| \leq \frac{|\mathscr{A}_n \mathscr{B}_n|}{2} \,. \end{aligned} \tag{5.14}$$

$\square$

We conclude that the eigenvalues of $X_n = P_n + iQ_n$ lie on $H_n \cap R_n$.

Proposition 5.5. *The eigenvalues of X_n lie on $H_n \cap R_n$.*

Proof. If λ is an eigenvalue for $X_n = P_n + iQ_n$, then

$$\left(\lambda - \frac{\alpha_n + \alpha_n'}{2} - i\frac{\beta_n + \beta_n'}{2}\right)^2. \tag{5.15}$$

is an eigenvalue for $\tilde{X_n}^2$. Hence, from Proposition 5.4,

$$\begin{aligned} \mathrm{Re}\left(\left(\lambda - \frac{\alpha_n + \alpha_n'}{2} - i\frac{\beta_n + \beta_n'}{2}\right)^2\right) &= \frac{\mathscr{A}_n^2 - \mathscr{B}_n^2}{4} \\ \left|\mathrm{Im}\left(\left(\lambda - \frac{\alpha_n + \alpha_n'}{2} - i\frac{\beta_n + \beta_n'}{2}\right)^2\right)\right| &\leq \frac{|\mathscr{A}_n\mathscr{B}_n|}{2}. \end{aligned} \tag{5.16}$$

From Lemma 4.7, $\lambda \in H_n \cap R_n$. $\square$

The symmetries described in Corollary 4.21 are trivial for the empirical spectral distributions of X_n, as they correspond to replacing P_n (resp. Q_n) by $\alpha_n + \alpha_n' - P_n$ (resp. $\beta_n + \beta_n' - Q_n$).

We introduce the following notation for the eigenspaces of a (random) matrix $Y_n \in M_n(\mathbb{C})$:

Definition 5.6. *Let $Y_n \in M_n(\mathbb{C})$. Define the following:*

- *Let $E_\lambda(Y_n)$ be the **λ-eigenspace** for Y_n, i.e. $E_\lambda(Y_n) = \ker(Y_n - \lambda I_n)$.*
- *Let $V_\lambda(Y_n)$ be the **generalized** **λ-eigenspace** for Y_n, i.e. $E_\lambda(Y_n) = \bigcup_{k\geq 0} \ker((Y_n - \lambda I_n)^k) = \ker((Y_n - \lambda I_n)^n)$.*

*In general, we allow for $E_\lambda(Y_n) = \{0\}$ or $V_\lambda(Y_n) = \{0\}$, but if we specifically say that λ is an **eigenvalue**, then it is implied that $E_\lambda(Y_n) \neq \{0\}$ (and hence $V_\lambda(Y_n) \neq \{0\}$).*

For the atoms of X_n, the following result shows the restriction of the empirical spectral distribution of X_n to ∂R_n is atomic, with the weight of the atoms corresponding to the dimension of the intersection of randomly rotated eigenspaces of P_n and Q_n. This behavior matches the result of Corollary 4.20 about the Brown measure of $X = p + iq$ when p and q have 2 atoms.

Proposition 5.7. *Let $X_n = P_n + iQ_n$, where P_n, Q_n are independently Haar-rotated Hermitian matrices with distributions:*

$$\begin{aligned} \mu_{P_n} &= (a_1)_n \delta_{(\alpha_1)_n} + \cdots + (a_k)_n \delta_{(\alpha_k)_n} \\ \mu_{Q_n} &= (b_1)_n \delta_{(\beta_1)_n} + \cdots + (b_l)_n \delta_{(\beta_l)_n} , \end{aligned} \tag{5.17}$$

where $(\alpha_i)_n, (\beta_j)_n \in \mathbb{R}$, $(a_i)_n, (b_j)_n \geq 0$, and $(a_1)_n + \cdots + (a_k)_n = (b_1)_n + \cdots (b_l)_n = 1$. Further, assume that the $(\alpha_i)_n$ (resp. $(\beta_j)_n$) are distinct and the $(a_i)_n, (b_j)_n > 0$.

Let $i_{\min}, i_{\max}$ be indices such that $(\alpha_{i_{\min}})_n = \min_i (\alpha_i)_n$ and $(\alpha_{i_{\max}})_n = \max_i (\alpha_i)_n$. Similarly, let $j_{\min}, j_{\max}$ be indices such that $(\beta_{j_{\min}})_n = \min_j (\beta_j)_n$ and $(\beta_{j_{\max}})_n = \max_j (\beta_j)_n$. Then, the empirical spectral distribution of X_n on ∂R_n is:

$$\begin{aligned} &\sum_{j=1}^{l} \max(0, (a_{i_{\min}})_n + (b_j)_n - 1) \delta_{(\alpha_{i_{\min}})_n + i(b_j)_n} + \\ &\sum_{j=1}^{l} \max(0, (a_{i_{\max}})_n + (b_j)_n - 1) \delta_{(\alpha_{i_{\max}})_n + i(b_j)_n} + \\ &\sum_{i=1}^{k} \max(0, (a_i)_n + (b_{j_{\min}})_n - 1) \delta_{(\alpha_i)_n + i(b_{j_{\min}})_n} + \\ &\sum_{i=1}^{k} \max(0, (a_i)_n + (b_{j_{\max}})_n - 1) \delta_{(\alpha_i)_n + i(b_{j_{\max}})_n} . \end{aligned} \tag{5.18}$$

Proof. First, we will show the only eigenvalues of X_n on ∂R_n are of the form $(\alpha_i)_n + i(\beta_j)_n$, and $E_{(\alpha_i)_n + i(\beta_j)_n}(X_n) = E_{(\alpha_i)_n}(P_n) \cap E_{(\beta_j)_n}(Q_n)$. Let v be a unit eigenvector for an eigenvalue $\lambda \in \partial R_n$. Let $\lambda = \lambda_1 + i\lambda_2$, where $\lambda_1, \lambda_2 \in \mathbb{R}$. Consider the following equalities:

$$\lambda_1 + i\lambda_2 = \langle \lambda v, v \rangle = \langle (P_n + iQ_n) v, v \rangle = \langle P_n v, v \rangle + i \langle Q_n v, v \rangle . \tag{5.19}$$

Since P_n, Q_n are Hermitian, then $\lambda_1 = \langle P_n v, v\rangle$ and $\lambda_2 = \langle Q_n v, v\rangle$. Since $\lambda \in \partial R_n$, then one of λ_1 or λ_2 is an extremal eigenvalue for P_n or Q_n, respectively. Without loss of generality assume that λ_1 is an extremal eigenvalue for P_n. Then, $\langle P_n v, v\rangle = \lambda_1$ implies that $P_n v = \lambda_1 v$. Hence, $Q_n v = \lambda_2 v$. Thus, we conclude that $E_\lambda(X_n) \subset E_{\lambda_1}(P_n) \cap E_{\lambda_2}(Q_n)$. The reverse inclusion is always true, so we have equality. Since $\lambda_1 = (\alpha_i)_n$ and $\lambda_2 = (\beta_j)_n$, then $\lambda = (\alpha_i)_n + i(\beta_j)_n$.

To compute the dimension of $E_{(\alpha_i)_n + i(\beta_j)_n}(X_n) = E_{(\alpha_i)_n}(P_n) \cap E_{(\beta_j)_n}(Q_n)$, we may assume that $P_n = U_n P_n' U_n^*$ and $Q_n = V_n Q_n' V_n^*$, where P_n', Q_n' are deterministic and U_n, V_n are independent Haar-distributed unitaries. Then, $E_{(\alpha_i)_n}(P_n) = U_n E_{(\alpha_i)_n}(P_n')$ and $E_{(\beta_j)_n}(Q_n) = V_n E_{(\beta_j)_n}(Q_n')$. From Lemma 5.1,

$$\dim(E_{(\alpha_i)_n + i(\beta_j)_n}(X_n)) = \max(0, \dim(E_{(\alpha_i)_n}(P_n')) + \dim(E_{(\beta_j)_n}(Q_n')) - n)\,. \tag{5.20}$$

Dividing by n and using the definition of the $(a_i)_n, (b_j)_n$ verifies the weights of the eigenvalues in the empirical spectral distribution. □

For the counterpart of Corollary 4.22, we will use the following description of two subspaces in terms of their principal angles (for more information about these angles, see [BG73], Introduction).

Proposition 5.8. *Let V, W be two subspaces of $\mathbb{C}^n$, where $\dim(V) = k$, $\dim(W) = l$, and $k \le l$. Then, there exists $\{v_1, \dots, v_k\}, \{w_1, \dots, w_l\} \subset \mathbb{C}^n$ and $0 \le \theta_1 \le \theta_2 \le \cdots \le \theta_k \le \pi/2$, such that:*

1. *$\{v_1, \dots, v_k\}$ are an orthogonal basis for V.*
2. *$\{w_1, \dots, w_l\}$ are an orthogonal basis for W.*
3. $$\langle v_i, w_j\rangle = \begin{cases} \cos\theta_i & i = j \\ 0 & i \neq j\,. \end{cases} \tag{5.21}$$

For any other such decomposition, the θ_i are the same. These angles are called the ***principal angles*** *between V and W.*

Suppose that $\theta_m = 0$ and $\theta_{m+1} > 0$ (or $m = n$). Then, $\dim(V \cap W) = m$.

Conversely, for any $1 \leq m \leq k \leq l \leq n$ such that $k + l - m \leq n$ and $\theta_1 = \cdots = \theta_m = 0 < \theta_{m+1} \leq \cdots \leq \theta_k \leq \pi/2$, there exists subspaces $V, W \subset \mathbb{C}^n$ that satisfy the previous properties.

Proof. Let $\mathcal{P}_n$ be a $n \times k$ matrix whose columns form an orthogonal basis for V and let $\mathcal{Q}_n$ be a $n \times l$ matrix whose columns form an orthogonal basis for W. Consider the singular value decomposition of $\mathcal{Q}_n^* \mathcal{P}_n$. First, $\|\mathcal{Q}_n^* \mathcal{P}_n\| \leq 1$, since for $\xi \in \mathbb{C}^k$, $\eta \in \mathbb{C}^l$ such that $\|\xi\| = 1$, $\|\eta\| = 1$, $|\langle \mathcal{Q}_n^* \mathcal{P}_n \xi, \eta \rangle| = |\langle \mathcal{P}_n \xi, \mathcal{Q}_n \eta \rangle| \leq \|\mathcal{P}_n \xi\| \, \|\mathcal{Q}_n \eta\| = |\xi| \, |\eta| = 1$. Hence, we may parameterize the singular values of $\mathcal{Q}_n^* \mathcal{P}_n$ by $\cos\theta_1, \ldots, \cos\theta_k$, where $0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_k \leq \pi/2$.

From the singular value decomposition, there exists an orthogonal basis $\{v_1', \ldots, v_k'\} \subset \mathbb{C}^k$ and an orthogonal set $\{w_1', \ldots, w_k'\} \subset \mathbb{C}^l$ such that:

$$\mathcal{Q}_n^* \mathcal{P}_n v_i' = \cos(\theta_i) w_i' \, . \tag{5.22}$$

By adding $w_{k+1}', \ldots, w_l'$ to complete the orthogonal set to a basis of $\mathbb{C}^l$, then taking $v_i = \mathcal{P}_n v_i'$ and $w_j = \mathcal{Q}_n w_j'$ satisfies the desired properties for v_i and w_j.

For any other such decomposition of V and W, $\{\tilde{v}_1, \ldots, \tilde{v}_k\}$ and $\{\tilde{w}_1, \ldots, \tilde{w}_l\}$, let $\tilde{\mathcal{P}}_n$ be the matrix whose columns are the $\tilde{v}_i$ and let $\tilde{\mathcal{Q}}_n$ be the matrix whose columns are the $\tilde{w}_j$. Then, $\tilde{\mathcal{P}}_n = \mathcal{P}_n U$ for some unitary $U \in U(k)$ and $\tilde{\mathcal{Q}}_n = \mathcal{Q}_n V$ for some unitary $V \in U(l)$. Thus, the singular values of $\mathcal{Q}_n^* \mathcal{P}_n$ and the singular values of $\tilde{\mathcal{Q}}_n^* \tilde{\mathcal{P}}_n$ are identical. As the singular values of each matrix are the cosines of the principal angles from their respective decompositions, then we conclude that the principal values are unique to the pair V, W.

Next, we show that $V \cap W = \operatorname{span}\{v_1, \ldots, v_m\} = \operatorname{span}\{w_1, \ldots, w_m\}$. First, since $\|v_i\| = \|w_i\| = 1$ and $\langle v_i, w_i \rangle = 1$ for $i \leq m$, then $v_i = w_i$. Now, consider an element in $V \cap W$,

$$a_1 v_1 + \cdots + a_k v_k = b_1 w_1 + \cdots + b_l w_l \, . \tag{5.23}$$

By dotting both sides of the equation with $\overline{v_i}$, $a_i = b_i \cos\theta_i$ for $i \leq k$. By dotting both sides of the equation with $\overline{w_j}$, $a_j \cos\theta_j = b_j$ for $j \leq k$ and $b_j = 0$ for $j > k$. For $m < i \leq k$, $a_i = b_i \cos\theta_i = a_i \cos^2\theta_i$ and $\cos\theta_i \in [0,1)$ implies that $a_i = b_i = 0$. Thus, $V \cap W = \operatorname{span}\{v_1, \ldots, v_m\} = \operatorname{span}\{w_1, \ldots, w_m\}$.

For the converse statement, let $\{e_1, \ldots, e_n\}$ be an orthogonal basis for $\mathbb{C}^n$. Let $v_i = e_i$ for $i = 1, \ldots, k$, and let

$$w_j = \begin{cases} e_j & j = 1, \ldots, m \\ \cos\theta_j e_j + \sin\theta_j e_{j+k-m} & j = m+1, \ldots, k \\ e_{j+k-m} & j = k+1, \ldots, l \end{cases} \tag{5.24}$$

Choosing $V = \operatorname{span}(v_1, \ldots, v_n)$ and $W = \operatorname{span}(w_1, \ldots, w_l)$ satisfies the desired properties. $\square$

Now, we can discuss the analogue of Corollary 4.22 when P_n and Q_n have 2 atoms. The fact that the Brown measure of X usually does not extend to the corners of the rectangle R is a phenomenon that only appears in the limit: the empirical spectral distributions of X_n always have full support on $H_n \cap R_n$:

Proposition 5.9. *Let $X_n = P_n + iQ_n$, where P_n, Q_n are independently Haar-rotated Hermitian matrices with distributions:*

$$\begin{aligned} \mu_{P_n} &= a_n \delta_{\alpha_n} + (1-a_n)\delta_{\alpha_n'} \\ \mu_{Q_n} &= b_n \delta_{\beta_n} + (1-b_n)\delta_{\beta_n'}\,, \end{aligned} \tag{5.25}$$

for $a_n, b_n \in (0,1)$, $\alpha_n \neq \alpha_n'$, $\beta_n \neq \beta_n'$, and $\alpha_n, \alpha_n', \beta_n, \beta_n' \in \mathbb{R}$.

The support of X_n is $H_n \cap R_n$.

Proof. First, we will show that for any $\theta \in (0, \pi/2)$, there exists deterministic P_n, Q_n with distributions

$$\begin{aligned} \mu_{P_n} &= a_n \delta_{\alpha_n} + (1-a_n)\delta_{\alpha_n'} \\ \mu_{Q_n} &= b_n \delta_{\beta_n} + (1-b_n)\delta_{\beta_n'}\,, \end{aligned} \tag{5.26}$$

where $X_n = P_n + iQ_n$ has eigenvalues $\lambda_1(\theta), \lambda_2(\theta)$ from Proposition 4.5.

Since $a_n, b_n > 0$, then from Proposition 5.8, there exists subspaces V, W where $\dim(V) =$

$n(1-a_n)$, $\dim(W) = n(1-b_n)$, and θ is one of the principal angles between V and W. We may choose P_n, Q_n so that $P'_n = \chi_{\{\alpha'_n\}}(P_n)$ and $Q'_n = \chi_{\{\beta'_n\}}(Q_n)$ are the projections onto V and W, respectively. Let $v \in V$, $w \in W$ be from the decomposition from Proposition 5.8 such that $\langle v, w\rangle = \cos\theta$. Then, P'_n and Q'_n fix $U = \mathrm{span}(v, w)$, and hence so do P_n, Q_n, and $X_n = P_n + iQ_n$. We claim that the restriction of X_n to U has eigenvalues $\lambda_1(\theta)$, $\lambda_2(\theta)$. The restriction of P'_n to U is the projection onto v and the restriction of Q'_n to U is the projection onto w. By conjugating by a unitary, we may assume that $v = (\cos\theta, \sin\theta)$ and $w = (1, 0)$. In this situation, the matrices of P'_n and Q'_n with respect to the standard basis are exactly those of $\tilde{p}$ and $\tilde{q}$ from Section 4.1. Then, the computations proceed identically as in the proof of Proposition 4.5 to show that the eigenvalues of X_n on U are $\lambda_1(\theta), \lambda_2(\theta)$.

We return to the situation where X_n is the random matrix model. From Corollary 4.8, $\lambda_1, \lambda_2 : [0, \pi/2] \to \mathbb{C}$ parameterize $H_n \cap R_n$. Note that λ_1, λ_2 map $\{0, \pi/2\}$ to the 4 corners of R_n. Let μ_n be the empirical spectral distribution of X_n. It suffices to show that for any neighborhood U of $\lambda_1(\theta)$ or $\lambda_2(\theta)$, $\theta \in (0, \pi/2)$, $\mu_n(U) > 0$.

For this, let $P_n = U_n P'_n U_n^*$, $Q_n = V_n Q'_n V_n^*$, where P'_n, Q'_n are deterministic and U_n, V_n are independent Haar-distributed unitaries. Then, there exists a point where U_n, V_n make $X_n = P_n + iQ_n$ have eigenvalue $\lambda_1(\theta)$. As $(U_n, V_n) \mapsto X_n$ is continuous, then for positive probability neighborhoods of U_n and V_n, X_n has an eigenvalue in U. Hence, $\mu_n(U) > 0$. $\square$

CHAPTER 6

Convergence of ESD of X_n to Brown measure of X

In this chapter, we consider the convergence of empirical spectral distributions of X_n when P_n and Q_n have at most 2 atoms.

If the law of P_n converges to the law of some Hermitian $p \in (M, \tau)$, then μ_p is supported on at most 2 points: if the support of μ_p contained more points, we get a contradiction by testing the convergence on non-negative compactly supported continuous functions that are 1 on sufficiently small neighborhoods of each of these points. The same result holds if the law of Q_n converges to the law of q. By considering the joint law of P_n and Q_n, then we will show the joint law of p is q is that they are freely independent.

Hence, we consider as in the previous section $X = p+iq \in (M, \tau)$, where p, q are Hermitian and freely independent, with

$$\begin{aligned} \mu_p &= a\delta_\alpha + (1-a)\delta_{\alpha'} \\ \mu_q &= b\delta_\beta + (1-b)\delta_{\beta'}\,, \end{aligned} \tag{6.1}$$

where $a, b \in [0,1]$ and $\alpha, \alpha', \beta, \beta' \in \mathbb{R}$.

The main result of this chapter is: if the law of P_n converges to the law of p and the law of Q_n converges to the law of q, then the empirical spectral distribution of X_n converges to the Brown measure of X.

Theorem 6.1. *Consider the random matrix model $X_n = P_n + iQ_n$, where $P_n, Q_n \in M_n(\mathbb{C})$ are independently Haar-rotated Hermitian matrices with distributions with at most 2 atoms. Suppose that the law of P_n converges to the law of p and the law of Q_n converges to the law of q. Let $p, q \in (M, \tau)$ where p and q are freely independent. Then, the empirical spectral distribution of X_n converges almost surely in the vague topology to the Brown measure of*

$X = p + iq$.

Recall that p and q have at most 2 atoms. If p and q each have 2 atoms, the condition that the law of P_n converges to p and the law of Q_n converges to q is equivalent to the positions and weights of the atoms of P_n and Q_n converging to the position and weights of the atoms of p and q. Then, we may assume that $\alpha_n \to \alpha$, $\beta_n \to \beta$, $\alpha'_n \to \alpha'$, $\beta'_n \to \beta'$, $a_n \to a$, and $b_n \to b$. Hence, for n sufficiently large, P_n and Q_n also have 2 atoms.

We will spend the vast majority of our effort dealing with the situation when p and q each have 2 atoms. The situation where either p or q is constant (and hence real numbers) are special cases that can be handled individually by analyzing limiting cases of the Brown measure. We will prove the intermediate results exclusively in the case where p and q have 2 atoms (resp. P_n and Q_n have 2 atoms), even if the results can be generalized for the case when p or q is constant (resp. P_n or Q_n is constant).

Once we prove Theorem 6.1, we can deduce the following converse result:

Theorem 6.2. *Consider the random matrix model $X_n = P_n + iQ_n$, where $P_n, Q_n \in M_n(\mathbb{C})$ are independently Haar-rotated Hermitian matrices with distributions with at most 2 atoms. Suppose that the empirical spectral distribution of X_n converges in probability in the vague topology to some deterministic probability measure μ. Then, the law of P_n converges to the law of p and the law of Q_n converges to the law of q for some $p, q \in (M, \tau)$ where p and q are freely independent. Hence, μ is the Brown measure of $X = p + iq$.*

Note that in Theorem 6.2 the hypothesis that the limit measure is a probability measure is essential to keep the atoms from running off to infinity. For instance, if $Q_n = 0$ and $\mu_{P_n} = a\delta_0 + (1-a)\delta_n$ for $a \in (0, 1)$, then $\mu_n = \mu_{P_n} \to a\delta_0$, which is not a probability measure.

6.1 Outline of proof of convergence

In this section, we describe the strategy for proving Theorem 6.1. This strategy is well-known and has been used in [TV10], [GKZ11] to prove analogous convergence results for the limit laws of various non-Hermitian random matrix models. In [TV10], [GKZ11] the limit measures were Brown measures of operators in a tracial von Neumann algebra, and our case is similar. The Brown measure of the operator which is the natural limit of the random matrix model is the typical candidate for the limit measure.

We recall from Proposition 1.27 that if μ_n is a sequence of random probability measures on $\mathbb{R}$ and μ is a deterministic measure, then the convergence of μ_n to μ is equivalent to the convergence of the Stieltjes transforms G_{μ_n} to G_μ.

We wish to use a function with similar properties to the Stieltjes transform when the measures are complex instead of real (i.e. the random matrices are not necessarily Hermitian). Call this function L_μ, where μ is a complex measure. We are looking for an L_μ that has the following properties:

- μ can be recovered from L_μ.
- Convergence of L_{μ_n} to L_μ implies convergence of μ_n to μ.
- The convergence L_{μ_n} to L_μ can be proven in practice, where μ is the Brown measure of an operator.

Since the case where μ are real measures is well-understood, then ideally L_μ could be related to a real measure.

One such L_μ is the ***logarithmic potential*** $L_\mu : \mathbb{C} \to \mathbb{C}$, given by:

$$L_\mu(z) = \int_{\mathbb{C}} \log |z - w| \, d\mu(w) \,. \tag{6.2}$$

In general, recall from Proposition 3.11 that if μ is the Brown measure of X, then

$$L_\mu(z) = \log \Delta(z - X) \,. \tag{6.3}$$

In particular, this holds for a random matrix X_n and its empirical spectral distribution μ_{X_n}.

When μ is a compactly supported probability measure, $L_\mu \in L^1_{\text{loc}}(\mathbb{C})$. When μ is the empirical spectral distribution of a random matrix, this follows from the fact that $\log|\cdot - w| \in L^1_{\text{loc}}(\mathbb{C})$ for any $w \in \mathbb{C}$. In general $L_\mu \in L^1_{\text{loc}}(\mathbb{C})$ follows from the fact that $\log|\cdot - w| \in L^1_{\text{loc}}(\mathbb{C})$ uniformly for w in a compact set.

In particular, L_μ is a well-defined distribution and since $\frac{1}{2\pi}\nabla^2 \log|\cdot - w| = \delta_w$, then

$$\frac{1}{2\pi}\nabla^2 L_\mu = \mu\,, \tag{6.4}$$

so we can recover μ from L_μ.

Thus, if we can show the limit:

$$L_{\mu_n}(z) \to L_\mu(z)\,, \tag{6.5}$$

then we should be able to take Laplacians and conclude that $\mu_n \to \mu$. The following Proposition ([Tao12], Theorem 2.8.3) makes this precise:

Proposition 6.3. *Let μ_n be a sequence of random probability measures on $\mathbb{C}$ and suppose that μ is a deterministic probability measure on $\mathbb{C}$. Assume that μ_n, μ are almost surely supported on some compact set. Suppose for Lebesgue almost every $z \in \mathbb{C}$, $L_{\mu_n}(z) \to L_\mu(z)$ almost surely (resp. in probability). Then, μ_n converges almost surely (resp. in probability) in the vague topology.*

Proof. Let λ be the Lebesgue measure on $\mathbb{C}$. Make the randomness of μ_n explicit, so that $\mu_n : (\Omega, \mathbb{P}) \to \mathcal{P}(\mathbb{C})$. Let μ_n, μ be almost surely supported on K'.

Recall that $L_{\mu_n}, L_\mu \in L^1_{\text{loc}}(\mathbb{C})$. We will need something slightly stronger, say $L_{\mu_n}, L_\mu \in L^2_{\text{loc}}(\mathbb{C})$ uniformly almost surely, i.e. for any compact set $K \subset \mathbb{C}$, $\|L_{\mu_n}\|_{L^2(K)}, \|L_\mu\|_{L^2(K)} \leq M$ for some $M > 0$, almost surely.

From Minkowski's integral inequality:

$$
\begin{aligned}
\|L_\mu(z)\|_{L^2(K)} &= \left(\int_K |L_\mu(z)|^2 \, d\lambda(z)\right)^{1/2} \\
&= \left(\int_K \left(\int_{K'} \log|z-w| \, d\mu(w)\right)^2 d\lambda(z)\right)^{1/2} \\
&\le \int_{K'} \left(\int_K \log|z-w|^2 \, d\lambda(z)\right)^{1/2} d\mu(w)\,.
\end{aligned}
\tag{6.6}
$$

We have a similar inequality where μ is replaced by μ_n. The uniform bound follows by noting that $\log|\cdot - w| \in L^2(K)$ uniformly as w ranges over K'.

First, we consider the almost sure convergence. To determine the convergence of μ_n in the vague topology, it suffices to prove convergence for $f \in C_c^\infty(\mathbb{C})$: For arbitrary $f \in C_c(\mathbb{C})$, we may take a sequence $f_k \to f$ uniformly, where $f_k \in C_c^\infty(\mathbb{C})$. Then, on the measure 1 set where $\langle f_k, \mu_n\rangle \to \langle f_k, \mu\rangle$ for all k, a standard $\epsilon/3$ argument shows that $\langle f, \mu_n\rangle \to \langle f, \mu\rangle$.

For $f \in C_c^\infty(\mathbb{C})$, since $\frac{1}{2\pi}\nabla^2 L_{\mu_n}(z) = \mu_n$, then $\langle f, \mu_n\rangle = \langle \frac{1}{2\pi}\nabla^2 f, L_{\mu_n}\rangle$ and similarly for μ. Hence, it suffices to prove that $\langle f, L_{\mu_n}\rangle \to \langle f, L_\mu\rangle$ almost surely for $f \in C_c^\infty(\mathbb{C})$. For this, it suffices to prove that for any $K \subset \mathbb{C}$, $L_{\mu_n} \to L_\mu$ in $L^1(K)$ almost surely.

Now, we claim that for almost every $\omega \in \Omega$, $L_{\mu_n}(z) \to L_\mu(z)$ for almost every $z \in K$. This amounts to switching the "almost every" quantifiers in the hypothesis. This follows from the following application of Fubini's theorem:

$$
\begin{aligned}
0 &= \int_{\mathbb{C}} \mathbb{P}\left(\{\omega \in \Omega : L_{\mu_n}(z) \not\to L_\mu(z)\}\right) d\lambda(z) \\
&= (\mathbb{P} \times \lambda)\left(\{(\omega, z) \in (\Omega, \mathbb{C}) : L_{\mu_n}(z) \not\to L_\mu(z)\}\right) \\
&= \int_\Omega \lambda\left(\{z \in \mathbb{C} : L_{\mu_n}(z) \not\to L_\mu(z)\}\right) d\omega\,.
\end{aligned}
\tag{6.7}
$$

We return to showing that $L_{\mu_n} \to L_\mu$ in $L^1(K)$ almost surely. Consider an ω such that $L_{\mu_n}(z) \to L_\mu(z)$ for almost every $z \in K$. Let $f_n = |L_{\mu_n}(z) - L_\mu(z)|$, so our goal is to show that $f_n \to 0$ in $L^1(K)$. For $M > 0$, consider:

$$f_n = \min(f_n, M) + (f_n - M)\chi_{f>M}(z)\,. \tag{6.8}$$

For any $M > 0$, the first term goes to 0 in $L^1(K)$ as $n \to \infty$ from the bounded convergence theorem.

For the second term,

$$\begin{aligned}
\int_K (f_n - M)\chi_{f_n>M}(z)\,d\lambda(z) &\le \int_{f_n>M} f_n\,d\lambda(z) \\
&\le \int_{f_n>M} f_n \frac{f_n}{M}\,d\lambda(z) \\
&\le \frac{1}{M}\,\|f_n\|_{L^2(K)} \\
&\le \frac{\|L_{\mu_n}\|_{L^2(K)} + \|L_\mu\|_{L^2(K)}}{M}\,.
\end{aligned} \tag{6.9}$$

Thus, choosing M sufficiently large and taking $n \to \infty$ shows that $L_{\mu_n} \to L_\mu$ in $L^1(K)$.

Next, we consider the convergence in probability. First, we note it suffices to prove the convergence for $f \in C_c^\infty(\mathbb{C})$. For arbitrary $g \in C_c(\mathbb{C})$, let $f \in C_c^\infty(\mathbb{C})$ so that $\|f - g\|_\infty \le \epsilon/3$. Then,

$$\begin{aligned}
\{\omega : |\langle g, \mu_n\rangle - \langle g, \mu\rangle| > \epsilon\} \subset &\{\omega : |\langle f, \mu_n\rangle - \langle g, \mu_n\rangle| > \epsilon/3\} \\
&\cup \{\omega : |\langle f, \mu_n\rangle - \langle f, \mu\rangle| > \epsilon/3\} \\
&\cup \{\omega : |\langle f, \mu\rangle - \langle g, \mu\rangle| > \epsilon/3\}\,.
\end{aligned} \tag{6.10}$$

But, two of the sets are empty, so then

$$\mathbb{P}(\{\omega : |\langle g, \mu_n\rangle - \langle g, \mu\rangle| > \epsilon\}) \le \mathbb{P}(\{\omega : |\langle f, \mu_n\rangle - \langle f, \mu\rangle| > \epsilon/3\})\,. \tag{6.11}$$

so the weak convergence of $\langle f, \mu_n\rangle \to \langle f, \mu\rangle$ implies the weak convergence of $\langle g, \mu_n\rangle \to \langle g, \mu\rangle$.

For $f \in C_c^\infty(\mathbb{C})$, since $\frac{1}{2\pi}\nabla^2 L_{\mu_n}(z) = \mu_n$, then $\langle f, \mu_n\rangle = \langle \frac{1}{2\pi}\nabla^2 f, L_{\mu_n}\rangle$ and similarly for μ. Hence, it suffices to prove that $\langle f, L_{\mu_n}\rangle \to \langle f, L_\mu\rangle$ in probability for $f \in C_c^\infty(\mathbb{C})$.

We again need a Fubini argument to switch the quantifiers of $\omega \in \Omega$ and $z \in \mathbb{C}$ in the hypothesis. Let f be supported on compact $K \subset \mathbb{C}$. Fix $\epsilon > 0$ and consider the following application of Fubini's theorem:

$$
\begin{aligned}
&\int_K \mathbb{P}(\{\omega \in \Omega : |L_{\mu_n}(z) - L_\mu(z)| > \epsilon\})\, d\lambda(z) \\
&= (\mathbb{P} \times \lambda)(\{(\omega, z) \in (\Omega, K) : |L_{\mu_n}(z) - L_\mu(z)| > \epsilon\}) \\
&= \int_\Omega \lambda(\{z \in K : |L_{\mu_n}(z) - L_\mu(z)| > \epsilon\})\, d\omega \,.
\end{aligned}
\tag{6.12}
$$

As $n \to \infty$ the integrand of the first integral goes to 0. Hence, from the bounded convergence theorem, the integral goes to 0. For $\omega \in \Omega$, let $K_{n,\epsilon} = \{z \in K : |L_{\mu_n}(z) - L_\mu(z)| > \epsilon\}$. As $\|\lambda(K_{n,\epsilon})\|_{L^1(\Omega)} \to 0$ as $n \to \infty$ then for any $\delta > 0$, $\mathbb{P}(\{\omega \in \Omega : \lambda(K_{n,\epsilon}) > \delta\}) \to 0$ as $n \to \infty$.

Consider an $\omega \in \Omega$ such that $\lambda(K_{n,\epsilon}) \leq \delta$. Then, the following inequalities hold:

$$
\begin{aligned}
&\int_K |f(z)(L_{\mu_n}(z) - L_\mu(z))|\ d\lambda(z) \\
&= \int_{K_{n,\epsilon}} |f(z)(L_{\mu_n}(z) - L_\mu(z))|\ d\lambda(z) + \int_{K \setminus K_{n,\epsilon}} |f(z)(L_{\mu_n}(z) - L_\mu(z))|\, d\lambda(z) \\
&\leq \|f\|_{L^\infty(K)} \int_K |L_{\mu_n}(z) - L_\mu(z)|\, \chi_{K_{n,\epsilon}}(z)\, d\lambda(z) + \epsilon \int_{K \setminus K_{n,\epsilon}} |f(z)|\ d\lambda(z) \\
&\leq \|f\|_{L^\infty(K)} \|L_{\mu_n}(z) - L_\mu(z)\|_{L^2(K)}\, \lambda(K_{n,\epsilon})^{1/2} + \epsilon \|f\|_{L^1(K)} \\
&\leq \|f\|_{L^\infty(K)} (\|L_{\mu_n}(z)\|_{L^2(K)} + \|L_\mu(z)\|_{L^2(K)}) \delta^{1/2} + \epsilon \|f\|_{L^1(K)} \,.
\end{aligned}
\tag{6.13}
$$

Thus, the weak convergence follows by choosing ϵ, δ sufficiently small and taking $n \to \infty$. $\square$

Thus, in order to prove the convergence of μ_n to μ, it suffices to show for almost every $z \in \mathbb{C}$

$$
L_{\mu_n}(z) \to L_\mu(z) \,. \tag{6.14}
$$

almost surely.

From $L_{\mu_n}(z) = \log(\Delta(z - X_n))$ and $L_{\mu_n}(z) = \log(\Delta(z - X))$, to prove Theorem 6.1, it suffices to show the following convergence of integrals for almost every $z \in \mathbb{C}$ and almost surely:

$$\frac{1}{2}\int_0^\infty \log x \, d\nu_{n,z}(x) \to \frac{1}{2}\int_0^\infty \log x \, d\nu_z(x) \,. \tag{6.15}$$

In general, the strength of the convergence of these integrals (i.e. whether the convergence is in probability or almost surely) determines the strength of the convergence of μ_n to μ.

Note also that we have reduced the problem of the convergence of the law of a non-Hermitian X_n to the convergence in law of the Hermitian $H_z(X_n) = (z - X_n)^*(z - X_n)$ for $z \in \mathbb{C}$. Thus, we may employ Hermitian techniques to calculate the limit and identify an X for which ν_z is the spectral measure for $H_z(X) = (z - X)^*(z - X)$.

There is usually already a candidate $X \in (M, \tau)$ where the laws $\nu_{n,z} \to \nu_z$ for $z \in \mathbb{C}$. In this case, this $X = p + iq$, where p and q are as in Chapter 4. This comes from Theorem 1.37 about the asymptotic freeness of independent Hermitian operators and Haar unitaries.

The main difficulty in justifying the convergence of the integrals is the fact that $\log x$ is not continuous at 0: $\log x \to -\infty$ as $x \to 0^-$. So, the logarithmic integrals may not converge if the measures $\nu_{n,z}$ have too much mass near 0. Note that the measure $\nu_{n,z}$ is supported on $[\sigma_{\min}(z - X_n)^2, \|z - X_n\|]$, so in order to prove the convergence of the logarithmic integrals, it suffices bound the minimum singular value from below.

In practice, this is the most difficult part of these convergence arguments. In our situation, we use the geometry of the support of the Brown measure of X (see Corollary 4.8).

Thus, given we can complete the following steps:

1. Find a suitable $X \in (M, \tau)$ so that for almost every $z \in \mathbb{C}$, $\nu_{n,z} \to \nu_z$ almost surely in the vague topology.

2. Bound the minimum singular value of $z - X_n$ from below to justify the convergence

   $$\frac{1}{2}\int_0^\infty \log x \, d\nu_{n,z}(x) \to \frac{1}{2}\int_0^\infty \log x \, d\nu_z(x) \,. \tag{6.16}$$

   for almost every $z \in \mathbb{C}$ almost surely.

then we can conclude that $\mu_n \to \mu$, where μ is the Brown measure of X.

6.2 The law ν_z

In this section, we assume that p and q have 2 atoms, i.e.

$$\begin{aligned} \mu_p &= a\delta_\alpha + (1-a)\delta_{\alpha'} \\ \mu_q &= b\delta_\beta + (1-b)\delta_{\beta'} \,. \end{aligned} \tag{6.17}$$

where $a, b \in (0,1)$, $\alpha \neq \alpha'$, $\beta \neq \beta'$, and $\alpha, \alpha', \beta, \beta' \in \mathbb{R}$. We have the convergences $\alpha_n \to \alpha$, $\beta_n \to \beta$, $\alpha'_n \to \alpha'$, $\beta'_n \to \beta'$, $a_n \to a$, and $b_n \to b$.

Let $\nu_{n,z}$ be the spectral measure of $H_z(X_n) = (z - X_n)^*(z - X_n)$ and let ν_z be the spectral measure of $H_z(X) = (z - X)^*(z - X)$. We will show that the laws $\nu_{n,z}$ converge to ν_z

This fact follows from Theorem 1.37, which we apply to observe the following Proposition:

Proposition 6.4. *The joint law of P_n, Q_n in $\left(M_n(\mathbb{C}), \mathbb{E}\left[\frac{1}{n}\mathrm{tr}\right]\right)$ (resp. $\left(M_n(\mathbb{C}), \frac{1}{n}\mathrm{tr}\right)$) is asymptotically free (resp. almost surely asymptotically free), converging to the law of two free Hermitian operators $p, q \in (M, \tau)$ where*

$$\begin{aligned} \mu_p &= a\delta_\alpha + (1-a)\delta_{\alpha'} \\ \mu_q &= b\delta_\beta + (1-b)\delta_{\beta'} \,. \end{aligned} \tag{6.18}$$

Proof. Recall that $P_n = U_n P'_n U_n^*$, $Q_n = V_n Q'_n V_n^*$, where U_n, V_n are independent Haar-distributed unitaries and P'_n, Q'_n are deterministic, Hermitian, and

$$\begin{aligned} \mu_{P'_n} &= a_n\delta_{\alpha_n} + (1-a_n)\delta_{\alpha'_n} \\ \mu_{Q'_n} &= b_n\delta_{\beta_n} + (1-b_n)\delta_{\beta'_n} \,. \end{aligned} \tag{6.19}$$

Because of the invariance property of the Haar measure, any choice of the P'_n, Q'_n gives the same law for X_n. Thus, let P'_n, Q'_n be given by:

$$\begin{aligned} P'_n &= (\alpha'_n - \alpha_n)\tilde{P}_n + \alpha_n \\ Q'_n &= (\beta'_n - \beta_n)\tilde{Q}_n + \beta_n \, , \end{aligned} \tag{6.20}$$

where $\tilde{P}_n$ is the matrix of the projection onto the first $n(1-a_n)$ standard basis vectors of $\mathbb{C}^n$ and $\tilde{Q}_n$ is the matrix of the projection onto the first $n(1-b_n)$ standard basis vectors of $\mathbb{C}^n$.

Then, $\tau(\tilde{P}_n) = 1 - a_n$, $\tau(\tilde{Q}_n) = 1 - b_n$ and

$$\begin{aligned} \tilde{P}_n &= \tilde{P}_n \wedge \tilde{Q}_n + \tilde{P}_n \wedge (1 - \tilde{Q}_n) \\ \tilde{Q}_n &= \tilde{P}_n \wedge \tilde{Q}_n + (1 - \tilde{P}_n) \wedge \tilde{Q}_n \end{aligned} \tag{6.21}$$

$$\begin{aligned} \tau(\tilde{P}_n \wedge \tilde{Q}_n) &= \min(\tau(\tilde{P}_n), \tau(\tilde{Q}_n)) \\ \tau(\tilde{P}_n \wedge (1 - \tilde{Q}_n)) &= \tau(\tilde{P}_n) - \tau(\tilde{P}_n \wedge \tilde{Q}_n) \\ \tau((1 - \tilde{P}_n) \wedge \tilde{Q}_n) &= \tau(\tilde{Q}_n) - \tau(\tilde{P}_n \wedge \tilde{Q}_n) \\ \tau((1 - \tilde{P}_n) \wedge (1 - \tilde{Q}_n)) &= 1 - \max(\tau(\tilde{P}_n), \tau(\tilde{Q}_n)) \, . \end{aligned} \tag{6.22}$$

The projections $\tilde{P}_n \wedge \tilde{Q}_n, \tilde{P}_n \wedge (1 - \tilde{Q}_n), (1 - \tilde{P}_n) \wedge \tilde{Q}_n, (1 - \tilde{P}_n) \wedge (1 - \tilde{Q}_n)$ are mutually orthogonal and sum to 1. Hence, their joint law converges to the joint law of mutually orthogonal projections $e_{11}, e_{10}, e_{01}, e_{00}$ that sum to 1, where

$$\begin{aligned} \tau(e_{11}) &= \min(1-a, 1-b) \\ \tau(e_{10}) &= 1 - a - \min(1-a, 1-b) \\ \tau(e_{01}) &= 1 - b - \min(1-a, 1-b) \\ \tau(e_{00}) &= 1 - \max(1-a, 1-b) \, . \end{aligned} \tag{6.23}$$

By forming the variables:

$$\begin{aligned} \tilde{p} &= e_{11} + e_{10} \\ \tilde{q} &= e_{11} + e_{01} \\ p' &= (\alpha' - \alpha)\tilde{p} + \alpha \\ q' &= (\beta' - \beta)\tilde{q} + \beta \,. \end{aligned} \tag{6.24}$$

then it follows that the law of P'_n, Q'_n converges to the law of p', q', where

$$\begin{aligned} \mu_{p'} &= a\delta_\alpha + (1-a)\delta_{\alpha'} \\ \mu_{q'} &= b\delta_\beta + (1-b)\delta_{\beta'} \,. \end{aligned} \tag{6.25}$$

Since P'_n, Q'_n are deterministic, then they are independent from U_n, V_n. Further, $\|P'_n\| = \max(\alpha_n, \alpha'_n)$ and $\|Q'_n\| = \max(\beta_n, \beta'_n)$ and since these sequences converge, then $\|P'_n\|$, $\|Q'_n\|$ are uniformly bounded. Hence, from Theorem 1.37, the law of P'_n, Q'_n becomes (almost surely) asymptotically free from U_n, V_n. Let u, v be the limit variables for U_n, V_n. It is easy to check that $p = up'u*$ and $vq'v^*$ are freely independent. Hence, the joint law of P_n, Q_n converges to the law of p, q, where p and q are free and

$$\begin{aligned} \mu_p &= a\delta_\alpha + (1-a)\delta_{\alpha'} \\ \mu_q &= b\delta_\beta + (1-b)\delta_{\beta'} \,. \end{aligned} \tag{6.26}$$

□

As a direct corollary, we observe the following:

Corollary 6.5. *Let $\nu_{n,z}$ be the spectral measure of $H_z(X_n) = (z - X_n)^*(z - X_n)$ and let ν_z be the spectral measure of $H_z(X) = (z - X)^*(z - X)$. For every $z \in \mathbb{C}$, $\nu_{n,z}$ converges to ν_z almost surely in the vague topology.*

Proof. This follows from Proposition 6.4 and noting that the convergence of the laws almost surely implies convergence of the μ_n almost surely. □

6.3 Bounds on the minimum singular value

Recall that for our random matrix model, $X_n = P_n + iQ_n$, where $P_n, Q_n \in M_n(\mathbb{C})$ are Hermitian and

$$\begin{aligned} \mu_{P_n} &= a_n\delta_{\alpha_n} + (1-a_n)\delta_{\alpha'_n} \\ \mu_{Q_n} &= b_n\delta_{\beta_n} + (1-b_n)\delta_{\beta'_n}\,, \end{aligned} \tag{6.27}$$

where $a_n, b_n \in [0,1]$, and $\alpha, \alpha'_n, \beta_n, \beta'_n \in \mathbb{R}$. In this section, we will assume that P_n and Q_n have 2 atoms, so that $a_n, b_n \in (0,1)$ and $\alpha_n \neq \alpha'_n$, $\beta_n \neq \beta'_n$.

Recall the following notation from Chapter 5:

$$\begin{aligned} \tilde{P_n} &= P_n - \frac{\alpha_n + \alpha'_n}{2} \\ \tilde{Q_n} &= Q_n - \frac{\beta_n + \beta'_n}{2} \\ \tilde{X_n} &= \tilde{P_n} + i\tilde{Q_n} \\ \mathscr{A}_n &= \alpha'_n - \alpha_n \\ \mathscr{B}_n &= \beta'_n - \beta_n\,. \end{aligned} \tag{6.28}$$

In this section, we will bound the minimum singular value of $z - X_n$ from below. We could use the principal angles between two subspaces from Chapter 5 to obtain the bound, but we provide a different proof that analyzes the restriction of X to the eigenspaces of $\tilde{X_n}^2$.

First, we examine the relationship between the (generalized) eigenspaces of X_n and $\tilde{X_n}^2$ to see that X_n is almost diagonalizable:

Proposition 6.6. *Let $\rho \in \mathbb{C}$. If $\rho \neq 0$, let*

$$\begin{aligned} \lambda_+(\rho) &= \frac{\alpha_n + \alpha'_n}{2} + i\frac{\beta_n + \beta'_n}{2} + \sqrt{\rho} \\ \lambda_-(\rho) &= \frac{\alpha_n + \alpha'_n}{2} + i\frac{\beta_n + \beta'_n}{2} - \sqrt{\rho}\,, \end{aligned} \tag{6.29}$$

where $\sqrt{\rho}$ is a chosen square root of ρ. Then,

$$E_\rho(\tilde{X_n}^2) = \begin{cases} E_{\lambda_+(\rho)}(X_n) + E_{\lambda_-(\rho)}(X_n) & \rho \neq 0 \\ V_{\frac{\alpha_n+\alpha'_n}{2}+i\frac{\beta_n+\beta'_n}{2}}(X_n) & \rho = 0\,. \end{cases} \tag{6.30}$$

Further,

$$V_{\frac{\alpha_n+\alpha'_n}{2}+i\frac{\beta_n+\beta'_n}{2}}(X_n) = \ker\left(\left(X_n - \frac{\alpha_n+\alpha'_n}{2} - i\frac{\beta_n+\beta'_n}{2}\right)^2\right)\,. \tag{6.31}$$

Proof. Since $\tilde{X_n}^2$ and X_n commute, then X_n fixes the $E_\rho(\tilde{X_n}^2)$.

Consider $X_n\big|_{E_\rho(\tilde{X_n}^2)} : E_\rho(\tilde{X_n}^2) \to E_\rho(\tilde{X_n}^2)$. Then, $X_n\big|_{E_\rho(\tilde{X_n}^2)}$ satisfies the polynomial

$$p(x) = \left(x - \frac{\alpha_n+\alpha'_n}{2} - i\frac{\beta_n+\beta'_n}{2}\right)^2 - \rho\,. \tag{6.32}$$

When $\rho \neq 0$, $p(x)$ is separable with roots $\lambda_+(\rho), \lambda_-(\rho)$. Hence,

$$E_\rho(\tilde{X_n}^2) = E_{\lambda_+(\rho)}\left(X_n\big|_{E_\rho(\tilde{X_n}^2)}\right) + E_{\lambda_-(\rho)}\left(X_n\big|_{E_\rho(\tilde{X_n}^2)}\right)\,. \tag{6.33}$$

For $\rho = 0$,

$$0 = p\left(X_n\big|_{E_\rho(\tilde{X_n}^2)}\right) = \left(X_n\big|_{E_\rho(\tilde{X_n}^2)} - \frac{\alpha_n+\alpha'_n}{2} - i\frac{\beta_n+\beta'_n}{2}\right)^2\,. \tag{6.34}$$

Hence,

$$\begin{aligned} E_0(\tilde{X_n}^2) &= \ker\left(\left(X_n\big|_{E_0(\tilde{X_n}^2)} - \frac{\alpha_n+\alpha'_n}{2} + i\frac{\beta_n+\beta'_n}{2}\right)^2\right) \\ &= V_{\frac{\alpha_n+\alpha'_n}{2}+i\frac{\beta_n+\beta'_n}{2}}\left(X_n\big|_{E_0(\tilde{X_n}^2)}\right)\,. \end{aligned} \tag{6.35}$$

Recall from Proposition 5.4 that $\tilde{X_n}^2$ is normal so that $\oplus_\rho E_\rho(\tilde{X_n}^2)$ is an orthogonal decomposition of the domain of X_n. For $\rho \neq 0$, the $\lambda_+(\rho), \lambda_-(\rho)$ are distinct and not equal to

$\frac{\alpha_n+\alpha_n'}{2} + i\frac{\beta_n+\beta_n'}{2}$. Hence,

$$\begin{aligned} E_{\lambda_+(\rho)}\left(X_n\big|_{E_\rho(\tilde{X_n}^2)}\right) &= E_{\lambda_+(\rho)}(X_n) \\ E_{\lambda_-(\rho)}\left(X_n\big|_{E_\rho(\tilde{X_n}^2)}\right) &= E_{\lambda_-(\rho)}(X_n) \\ V_{\frac{\alpha_n+\alpha_n'}{2}+i\frac{\beta_n+\beta_n'}{2}}\left(X_n\big|_{E_0(\tilde{X_n}^2)}\right) &= V_{\frac{\alpha_n+\alpha_n'}{2}+i\frac{\beta_n+\beta_n'}{2}}(X_n)\,. \end{aligned} \tag{6.36}$$

□

Since $\tilde{X_n}^2$ is normal (Proposition 5.4), then $\oplus_\rho E_\rho(\tilde{X_n}^2)$ is an orthogonal decomposition of the domain of $z - X_n$. Thus, in order to bound $\sigma_{\min}(z - X_n)$ from below, it suffices to bound $\sigma_{\min}\left((z - X_n)\big|_{E_\rho(\tilde{X_n}^2)}\right)$ from below. No meaningful bound can be given when z is an eigenvalue of X_n, as this is exactly when $\sigma_{\min}(z - X_n) = 0$. Hence, it suffices to consider when z is not an eigenvalue of X_n. Then, the following Proposition bounds $\sigma_{\min}\left((z - X_n)\big|_{E_\rho(\tilde{X_n}^2)}\right)$ from below when z is not an eigenvalue of X_n. The idea is to consider $z - X_n$ on invariant subspaces with small dimension.

Proposition 6.7. *Let $z \in \mathbb{C}$. Consider $\rho \in \mathbb{C}$ that is an eigenvalue of $\tilde{X_n}^2$. For $\rho \neq 0$, let*

$$\begin{aligned} \lambda_+(\rho) &= \frac{\alpha_n+\alpha_n'}{2} + i\frac{\beta_n+\beta_n'}{2} + \sqrt{\rho} \\ \lambda_-(\rho) &= \frac{\alpha_n+\alpha_n'}{2} + i\frac{\beta_n+\beta_n'}{2} - \sqrt{\rho}\,, \end{aligned} \tag{6.37}$$

where $\sqrt{\rho}$ is a chosen square root of ρ.

Suppose that z is not an eigenvalue of $X_n\big|_{E_\rho(\tilde{X_n}^2)}$. Then, the minimum singular value of $(z - X_n)\big|_{E_\rho(\tilde{X_n}^2)}$ is bounded from below by the following:

$$\sigma_{\min}\left((z - X_n)\big|_{E_\rho(\tilde{X_n}^2)}\right) \geq \begin{cases} \dfrac{\min\left(|z-\lambda_+(\rho)|\,,|z-\lambda_-(\rho)|\right)^2}{\left\|(z-X_n)\big|_{E_\rho(\tilde{X_n}^2)}\right\|} & \rho \neq 0 \\ \dfrac{\left|z - \frac{\alpha_n+\alpha_n'}{2} - i\frac{\beta_n+\beta_n'}{2}\right|^2}{\left\|(z-X_n)\big|_{E_0(\tilde{X_n}^2)}\right\|} & \rho = 0\,. \end{cases} \tag{6.38}$$

Proof. Recall that

$$\sigma_{\min}\left(\left.(z-X_n)\right|_{E_\rho(\tilde{X_n}^2)}\right) = \min_{\substack{\zeta\in E_\rho(\tilde{X_n}^2)\\ \zeta\neq 0}} \frac{\|(z-X_n)(\zeta)\|}{\|\zeta\|}. \tag{6.39}$$

First, consider if $\rho \neq 0$. From Proposition 6.6, $E_\rho(\tilde{X_n}^2) = E_{\lambda_+(\rho)}(X_n) + E_{\lambda_-(\rho)}(X_n)$. Let $V = E_{\lambda_+(\rho)}(X_n)$, $W = E_{\lambda_-(\rho)}(X_n)$. Then,

$$\min_{\substack{\zeta\in E_\rho(\tilde{X_n}^2)\\ \zeta\neq 0}} \frac{\|(z-X_n)(\zeta)\|}{\|\zeta\|} = \min_{\substack{v\in V, w\in W\\ (v,w)\neq(0,0)}} \frac{\|(z-X_n)(v+w)\|}{\|v+w\|}. \tag{6.40}$$

Fix some $v \in V, w \in W$. Let $U = \operatorname{span}(v, w)$. Then, $z - X_n$ fixes U. We proceed to show that

$$\sigma_{\min}\left(\left.(z-X_n)\right|_U\right) \geq \frac{\min\left(|z-\lambda_+(\rho)|, |z-\lambda_-(\rho)|\right)^2}{\left\|\left.(z-X_n)\right|_U\right\|}. \tag{6.41}$$

Note that $\left\|\left.(z-X_n)\right|_U\right\| \neq 0$ since z is not an eigenvalue of $\left.(z-X_n)\right|_U$. If either $v = 0$ or $w = 0$, then $\left.(z-X_n)\right|_U = z - \lambda_+(\rho)$ or $\left.(z-X_n)\right|_U = z - \lambda_-(\rho)$. Without loss of generality assume that $\left.(z-X_n)\right|_U = z - \lambda_+(\rho)$. Then, the following verifies (6.41):

$$\begin{aligned}
\sigma_{\min}\left(\left.(z-X_n)\right|_U\right) &= |z-\lambda_+(\rho)| \\
&= \frac{|z-\lambda_+(\rho)|^2}{\left\|\left.(z-X_n)\right|_U\right\|} \\
&\geq \frac{\min\left(|z-\lambda_+(\rho)|, |z-\lambda_-(\rho)|\right)^2}{\left\|\left.(z-X_n)\right|_U\right\|}.
\end{aligned} \tag{6.42}$$

If $v \neq 0$ and $w \neq 0$ then $\dim(U) = 2$ and the eigenvalues of $\left.(z-X_n)\right|_U$ are $z-\lambda_+(\rho), z-\lambda_-(\rho)$. Since the absolute value of the product of the eigenvalues of $\left.(z-X_n)\right|_U$ is equal to the product of the singular values of $\left.(z-X_n)\right|_U$, then

$$\begin{aligned}
|z-\lambda_+(\rho)|\,|z-\lambda_-(\rho)| &= \sigma_1(\left.(z-X_n)\right|_U)\sigma_2(\left.(z-X_n)\right|_U) \\
&= \left\|\left.(z-X_n)\right|_U\right\| \sigma_{\min}\left(\left.(z-X_n)\right|_U\right).
\end{aligned} \tag{6.43}$$

Hence,

$$\begin{aligned}\sigma_{\min}\left((z-X_n)\big|_U\right) &= \frac{|z-\lambda_+(\rho)|\,|z-\lambda_-(\rho)|}{\left\|(z-X_n)\big|_U\right\|} \\ &\geq \frac{\min\left(|z-\lambda_+(\rho)|\,,|z-\lambda_-(\rho)|\right)^2}{\left\|(z-X_n)\big|_U\right\|}\,.\end{aligned} \tag{6.44}$$

Thus, in all cases of $\dim(U)$, (6.41) holds. The following inequalities complete the proof in the case where $\rho \neq 0$:

$$\begin{aligned}\min_{\substack{v\in V, w\in W\\(v,w)\neq(0,0)}} \frac{\|(z-X_n)(v+w)\|}{\|v+w\|} &\geq \min_{\substack{v\in V, w\in W\\(v,w)\neq(0,0)}} \min_{\substack{\zeta\in U\\ \zeta\neq 0}} \frac{\|(z-X_n)(\zeta)\|}{\|\zeta\|} \\ &= \min_{\substack{v\in V, w\in W\\(v,w)\neq(0,0)}} \sigma_{\min}\left((z-X_n)\big|_U\right) \\ &\geq \min_{\substack{v\in V, w\in W\\(v,w)\neq(0,0)}} \frac{\min\left(|z-\lambda_+(\rho)|\,,|z-\lambda_-(\rho)|\right)^2}{\left\|(z-X_n)\big|_U\right\|} \\ &\geq \frac{\min\left(|z-\lambda_+(\rho)|\,,|z-\lambda_-(\rho)|\right)^2}{\left\|(z-X_n)\big|_{E_\rho(\tilde{X_n}^2)}\right\|}\,.\end{aligned} \tag{6.45}$$

Next, consider if $\rho = 0$. For $\zeta \in E_0(\tilde{X_n}^2)$, let $V = \operatorname{span}(\zeta, \tilde{X_n}\zeta)$. Since $E_0(\tilde{X_n}^2) = \ker(\tilde{X_n}^2)$, then $\tilde{X_n}$ fixes V, so $z - X_n$ also fixes V. We proceed to show that:

$$\sigma_{\min}\left((z-X_n)\big|_V\right) = \frac{\left|z - \frac{\alpha_n+\alpha_n'}{2} - i\frac{\beta_n+\beta_n'}{2}\right|^2}{\left\|(z-X_n)\big|_V\right\|}\,. \tag{6.46}$$

Note that $\left\|(z-X_n)\big|_V\right\| \neq 0$ since z is not an eigenvalue of $(z-X_n)\big|_V$. If $\tilde{X_n}\zeta = 0$, then $(z-X_n)\big|_V = z - \frac{\alpha_n+\alpha_n'}{2} - i\frac{\beta_n+\beta_n'}{2}$. In this case, the equality clearly holds.

If $\tilde{X_n}\zeta \neq 0$, then $\dim(V) = 2$. The only eigenvalue of $(z-X_n)\big|_V$ is $z - \frac{\alpha_n+\alpha_n'}{2} - i\frac{\beta_n+\beta_n'}{2}$. Since the absolute value of the product of the eigenvalues of $(z-X_n)\big|_V$ is equal to the product of the singular values of $(z-X_n)\big|_V$, then

$$\begin{aligned}\left|z - \frac{\alpha_n+\alpha_n'}{2} - i\frac{\beta_n+\beta_n'}{2}\right|^2 &= \sigma_1((z-X_n)\big|_V)\sigma_2((z-X_n)\big|_V) \\ &= \left\|(z-X_n)\big|_V\right\| \sigma_{\min}\left((z-X_n)\big|_V\right)\,.\end{aligned} \tag{6.47}$$

Rearranging this produces (6.46). Hence, in all cases of $\dim(V)$, (6.46) holds. The following inequalities complete the proof in the case where $\rho = 0$:

$$\begin{aligned}
\min_{\substack{\zeta \in E_\rho(\tilde{X_n}^2) \\ \zeta \neq 0}} \frac{\|(z - X_n)(\zeta)\|}{\|\zeta\|} &\geq \min_{\substack{\zeta \in E_\rho(\tilde{X_n}^2) \\ \zeta \neq 0}} \min_{\substack{v \in V \\ v \neq 0}} \frac{\|(z - X_n)(v)\|}{\|v\|} \\
&= \min_{\substack{\zeta \in E_\rho(\tilde{X_n}^2) \\ \zeta \neq 0}} \sigma_{\min}((z - X_n)\big|_V) \\
&= \min_{\substack{\zeta \in E_\rho(\tilde{X_n}^2) \\ \zeta \neq 0}} \frac{\left| z - \frac{\alpha_n + \alpha_n'}{2} - i\frac{\beta_n + \beta_n'}{2} \right|^2}{\left\| (z - X_n)\big|_V \right\|} \\
&\geq \frac{\left| z - \frac{\alpha_n + \alpha_n'}{2} - i\frac{\beta_n + \beta_n'}{2} \right|^2}{\left\| (z - X_n)\big|_{E_0(\tilde{X_n}^2)} \right\|} \,.
\end{aligned} \tag{6.48}$$

Both of the inequalities in the above chain are actually equalities, but we will not need this fact. $\square$

We conclude by unifying the cases $\rho \neq 0$ and $\rho = 0$ in Proposition 6.7 and presenting a bound on the entire domain of $z - X_n$ for all $z \in \mathbb{C}$:

Theorem 6.8. *Let $z \in \mathbb{C}$. The minimum singular value of $z - X_n$ satisfies the following inequality:*

$$\sigma_{\min}(z - X_n) \geq \frac{\operatorname{dist}(z, H_n \cap R_n)^2}{\|z - X_n\|} \,. \tag{6.49}$$

Proof. If z is an eigenvalue of X_n, then the left-hand side of the inequality is 0 and from Proposition 5.5 the right-hand side of the inequality is also 0. Thus, we may assume that z is not an eigenvalue of X_n.

Consider $\rho \in \mathbb{C}$ that is an eigenvalue for $\tilde{X_n}^2$. Since z is not an eigenvalue for X_n, then z

is not an eigenvalue of $X_n\big|_{E_\rho(\tilde{X_n}^2)}$. Hence, from Proposition 6.7,

$$\sigma_{\min}\left((z-X_n)\big|_{E_\rho(\tilde{X_n}^2)}\right) \geq \begin{cases} \dfrac{\min\left(|z-\lambda_+(\rho)|\,,|z-\lambda_-(\rho)|\right)^2}{\left\|(z-X_n)\big|_{E_\rho(\tilde{X_n}^2)}\right\|} & \rho \neq 0 \\ \dfrac{\left|z-\frac{\alpha_n+\alpha'_n}{2}-i\frac{\beta_n+\beta'_n}{2}\right|^2}{\left\|(z-X_n)\big|_{E_0(\tilde{X_n}^2)}\right\|} & \rho = 0\,. \end{cases} \tag{6.50}$$

Now, we proceed to make these estimates independent of ρ.

If $\rho \neq 0$, recall from Proposition 6.6 that $E_\rho(\tilde{X_n}^2) = E_{\lambda_+(\rho)}(X_n) + E_{\lambda_-(\rho)}(X_n)$. Since ρ is an eigenvalue of $\tilde{X_n}^2$ then at least one of $\lambda_+(\rho), \lambda_-(\rho)$ is an eigenvalue for $X_n\big|_{E_\rho(\tilde{X_n}^2)}$. From Proposition 5.5, any eigenvalue of X_n is on $H_n \cap R_n$. Hence,

$$\begin{aligned} \operatorname{dist}(z, H_n\cap R_n) &\leq \min\left(|z-\lambda_+(\rho)|\,,|z-\lambda_-(\rho)|\right) \\ &\leq \left\|(z-X_n)\big|_{E_\rho(\tilde{X_n}^2)}\right\| \\ &\leq \|z-X_n\|\;. \end{aligned} \tag{6.51}$$

Since z is not an eigenvalue of X_n, then $\|z-X_n\|\neq 0$ and thus the following inequalities hold:

$$\frac{\min\left(|z-\lambda_+(\rho)|\,,|z-\lambda_-(\rho)|\right)^2}{\left\|(z-X_n)\big|_{E_\rho(\tilde{X_n}^2)}\right\|} \geq \frac{\operatorname{dist}(z,H_n\cap R_n)^2}{\|z-X_n\|}\,. \tag{6.52}$$

If $\rho = 0$, recall from Proposition 6.6 that $E_0(\tilde{X_n}^2) = V_{\frac{\alpha_n+\alpha'_n}{2}+i\frac{\beta_n+\beta'_n}{2}}(X_n)$ so that $\frac{\alpha_n+\alpha'_n}{2}+i\frac{\beta_n+\beta'_n}{2}$ is an eigenvalue for $X_n\big|_{E_0(\tilde{X_n}^2)}$. From Proposition 5.5, any eigenvalue of X_n is on $H_n\cap R_n$. Hence,

$$\begin{aligned} \operatorname{dist}(z, H_n\cap R_n) &\leq \left|z-\frac{\alpha_n+\alpha'_n}{2}-i\frac{\beta_n+\beta'_n}{2}\right| \\ &\leq \left\|(z-X_n)\big|_{E_0(\tilde{X_n}^2)}\right\| \\ &\leq \|z-X_n\|\;. \end{aligned} \tag{6.53}$$

Since z is not an eigenvalue of X_n, then $\|z-X_n\|\neq 0$ and thus the following inequalities hold:

$$\frac{\left|z - \frac{\alpha_n+\alpha_n'}{2} - i\frac{\beta_n+\beta_n'}{2}\right|^2}{\left\|(z-X_n)\big|_{E_\rho(\tilde{X_n}^2)}\right\|} \geq \frac{\operatorname{dist}(z, H_n \cap R_n)^2}{\|z-X_n\|}\,. \tag{6.54}$$

We conclude that for any ρ that is an eigenvalue for $\tilde{X_n}^2$,

$$\sigma_{\min}\left((z-X_n)\big|_{E_\rho(\tilde{X_n}^2)}\right) \geq \frac{\operatorname{dist}(z, H_n \cap R_n)^2}{\|z-X_n\|}\,. \tag{6.55}$$

Since $\tilde{X_n}^2$ is normal (Proposition 5.4), then $\oplus_\rho E_\rho(\tilde{X_n}^2)$ is an orthogonal decomposition of the domain of $z - X_n$. Hence,

$$\begin{aligned}
\sigma_{\min}(z-X_n) &= \min_{\{\rho : E_\rho(\tilde{X_n}^2)\neq\{0\}\}} \sigma_{\min}\left((z-X_n)\big|_{E_\rho(\tilde{X_n}^2)}\right) \\
&\geq \min_{\{\rho : E_\rho(\tilde{X_n}^2)\neq\{0\}\}} \frac{\operatorname{dist}(z, H_n \cap R_n)^2}{\|z-X_n\|} \\
&= \frac{\operatorname{dist}(z, H_n \cap R_n)^2}{\|z-X_n\|}\,.
\end{aligned} \tag{6.56}$$

□

6.4 Proofs of convergence and converse

In this section, we complete the proof of Theorem 6.1 and then deduce Theorem 6.2.

Proof of Theorem 6.1. Let

$$\begin{aligned}
\mu_{P_n} &= a_n\delta_{\alpha_n} + (1-a_n)\delta_{\alpha_n'} \\
\mu_{Q_n} &= b_n\delta_{\beta_n} + (1-b_n)\delta_{\beta_n'}
\end{aligned} \tag{6.57}$$

for $a_n, b_n \in [0,1]$, $\alpha_n, \alpha_n', \beta_n, \beta_n' \in \mathbb{R}$.

Let μ_n be the empirical spectral distribution of X_n.

Recall that μ_p and μ_q have at most 2 points in their supports, so then

$$\begin{aligned}\mu_p &= a\delta_\alpha + (1-a)\delta_{\alpha'} \\ \mu_q &= b\delta_\beta + (1-b)\delta_{\beta'}\,.\end{aligned} \tag{6.58}$$

for $a, b \in [0,1]$, $\alpha, \alpha', \beta, \beta' \in \mathbb{R}$.

First, we consider the case where p and q have 2 atoms, i.e. $(a,b) \in (0,1)$ and $\alpha \neq \alpha'$, $\beta \neq \beta'$.

In this situation, using the convergence $\mu_{P_n} \to \mu_p$ on non-negative $f \in C_c(\mathbb{R})$ that is supported on a neighborhood of α and $f(\alpha) = 1$, then an atom of μ_{P_n} converges to α. A similar argument shows that an atom of μ_{P_n} converges to α'. By choosing f to be 1 on neighborhoods of α and α', we see that the weights of the atoms of P_n converge to the weights of the corresponding atoms of p. Thus, we may assume that $\alpha_n \to \alpha$, $\beta_n \to \beta$, $\alpha'_n \to \alpha'$, $\beta'_n \to \beta'$, $a_n \to a$, and $b_n \to b$.

Recall that we need to complete the following steps to complete the proof of Theorem 6.1 when p and q have 2 atoms:

1. Find a suitable $X \in (M, \tau)$ so that for almost every $z \in \mathbb{C}$, $\nu_{n,z} \to \nu_z$ almost surely in the vague topology.

2. Bound the minimum singular value of $z - X_n$ from below to justify the convergence

$$\frac{1}{2}\int_0^\infty \log x \, d\nu_{n,z}(x) \to \frac{1}{2}\int_0^\infty \log x \, d\nu_z(x)\,. \tag{6.59}$$

   for almost every $z \in \mathbb{C}$ almost surely.

The first step was completed in Corollary 6.5, where we identified $X = p + iq$, where $p, q \in (M, \tau)$ were Hermitian, freely independent, and

$$\begin{aligned}\mu_p &= a\delta_\alpha + (1-a)\delta_{\alpha'} \\ \mu_q &= b\delta_\beta + (1-b)\delta_{\beta'}\,.\end{aligned} \tag{6.60}$$

For the second step, it suffices to prove the convergence of the logarithmic integrals for $z \notin H \cap R$, where H and R are the hyperbola and rectangle associated with $X = p + iq$. Fix a $z \notin H \cap R$. Let H_n and R_n be the hyperbola and rectangle associated with X_n.

From the triangle inequality, for arbitrary $z \in \mathbb{C}$, the following inequality holds:

$$\operatorname{dist}(z, H \cap R) \leq \operatorname{dist}(z, H_n \cap R_n) + \sup_{w \in H_n \cap R_n} \operatorname{dist}(w, H \cap R) . \tag{6.61}$$

Hence,

$$\operatorname{dist}(z, H_n \cap R_n) \geq \operatorname{dist}(z, H \cap R) - \sup_{w \in H_n \cap R_n} \operatorname{dist}(w, H \cap R) . \tag{6.62}$$

As $n \to \infty$, $\sup_{w \in H_n \cap R_n} \operatorname{dist}(w, H \cap R) \to 0$. This can be seen by using the parameterization of $H_n \cap R_n$, $H \cap R$ in Corollary 4.8 and comparing the points on $H_n \cap R_n$, $H \cap R$ with the same θ. Since $z \notin H \cap R$, then $\operatorname{dist}(z, H \cap R) > 0$. So, for n sufficiently large, $\operatorname{dist}(z, H_n \cap R_n) > \delta$ for some $\delta > 0$.

Since $\|X_n\| \leq \|P_n\| + \|Q_n\| = \max(\alpha_n, \alpha_n') + \max(\beta_n, \beta_n')$ and the sequences all converge, then the X_n are uniformly bounded. Hence, the $z - X_n$ are uniformly bounded.

Combining these two facts and Theorem 6.8, then for n sufficiently large, $\sigma_{\min}(z - X_n) > \delta$ almost surely, for some $\delta > 0$. As $H_z(X_n) = (z - X_n)^*(z - X_n)$ are uniformly bounded, then for n sufficiently large, $\nu_{n,z}$ is almost surely supported on $[\delta, M]$ for some $M > 0$.

By applying the convergence $\nu_{n,z} \to \nu_z$ in the vague topology to $f \in C_c([0, \infty))$ where $f \equiv 0$ on $[\delta, M]$, we see that ν_z is also supported on $[\delta, M]$.

Hence, we may choose $f \in C_c([0, \infty))$ such that $f(x) = \log(x)$ on $[\delta, M]$, so that the logarithmic integrals converge almost surely:

$$\begin{gathered} \frac{1}{2} \int_0^\infty \log x \, d\nu_{n,z}(x) = \frac{1}{2} \int_0^\infty f(x) \, d\nu_{n,z}(x) \longrightarrow \\ \frac{1}{2} \int_0^\infty f(x) \, d\nu_z(x) = \frac{1}{2} \int_0^\infty \log x \, d\nu_z(x) . \end{gathered} \tag{6.63}$$

This completes the proof in the case where p and q both have 2 atoms.

Next, we consider the case where $p, q \in \mathbb{R}$. By applying the convergence $\mu_{P_n} \to \mu_p = \delta_p$

on f that is 1 on small a neighborhood of p and supported on a slightly larger neighborhood, we see that the sum of the weights of the atoms of P_n in any neighborhood U converges to 1 as $n \to \infty$. The same result applies for Q_n and q with neighborhood V. Let n be sufficiently large so that the sum of the weights of the atoms of P_n in U is larger than $1-\epsilon$ and the sum of the weights of the atoms of Q_n in V is larger than $1-\epsilon$. If $\alpha_n, \alpha'_n \in U$ and $\beta_n, \beta'_n \in V$, then μ_n is supported on $U \times V$, so $\mu_n(U \times V) = 1$ and it is clear that μ_n tends to δ_{p+iq}. In the other case, suppose that $\alpha_n \in U$ but $\alpha'_n \notin U$. Then, $a_n = \tau(\chi_{\{\alpha_n\}}(P_n)) > 1-\epsilon$, so then from the parallelogram law,

$$\begin{aligned}
\mu_n(\{\alpha_n + i\beta_n\}) &= \tau(\chi_{\{\alpha_n\}}(P_n) \wedge \chi_{\{\beta_n\}}(Q_n)) \\
&= \tau(\chi_{\{\alpha_n\}}(P_n)) + \tau(\chi_{\{\beta_n\}}(Q_n)) - \tau(\chi_{\{\alpha_n\}}(P_n) \vee \chi_{\{\beta_n\}}(Q_n)) \\
&> (1-\epsilon) + \tau(\chi_{\{\beta_n\}}(Q_n)) - 1 \\
&= \tau(\chi_{\{\beta_n\}}(Q_n)) - \epsilon \\
&= \mu_{Q_n}(\{\beta_n\}) - \epsilon\,.
\end{aligned} \tag{6.64}$$

Similarly, $\mu_n(\{\alpha_n + i\beta'_n\}) > \mu_{Q_n}(\{\beta'_n\}) - \epsilon$. Then, the μ_n-measure of the atom(s) of $\{\alpha_n + i\beta_n, \alpha_n + i\beta'_n\}$ in $U \times V$ has measure greater than $1 - 3\epsilon$, so again μ_n tends to δ_{p+iq}.

Finally, the case where exactly one of p and q is constant follows in a similar manner. Suppose that q has 2 atoms and $p \in \mathbb{R}$. Then $b \in (0,1)$, $\beta \neq \beta'$, $b_n \to b$, $\beta_n \to \beta$, and $\beta'_n \to \beta'$. For the case that both atoms of P_n are in a small neighborhood of p, the branches of $H_n \cap R_n$ are in small neighborhoods of $\{p+i\beta, p+i\beta'\}$, and from Proposition 4.10 these branches have the appropriate measures $\mu_{Q_n}(\{\beta_n\}) = b_n$ and $\mu_{Q_n}(\{\beta'_n\}) = 1 - b_n$ so that μ_n tends towards $\mu = b\delta_\beta + (1-b)\delta_{\beta'}$. For the case that α_n is in a small neighborhood of p and $\mu_{P_n}(\{\alpha_n\}) \approx 1$, then we use the previous argument to show that the $\mu_n(\{\alpha_n + i\beta_n\}) \approx \mu_{Q_n}(\{\beta_n\})$ and $\mu_n(\{\alpha_n + i\beta'_n\}) \approx \mu_{Q_n}(\{\beta'_n\})$. □

Now, we prove the converse result, Theorem 6.2:

Proof of Theorem 6.2. Let μ_n be the empirical spectral distribution of X_n.

The majority of the proof is spent showing that μ_n converges vaguely to a probability measure μ, then μ_{P_n} and μ_{Q_n} are tight, i.e. for any subsequence of these sequences, there exists a further subsequence that converges in the vague topology.

Having shown this, the conclusion follows by passing to subsequences, using Theorem 6.1, and using that the Brown measure of $X = p + iq$ determines μ_p and μ_q (Corollary 4.23).

Now, it suffices to show a subsequence of μ_n has $\mu_{P_{n_k}}$ and $\mu_{Q_{n_k}}$ converge.

First, if the atoms of P_n and Q_n, $\alpha_n, \alpha_n', \beta_n, \beta_n'$ are uniformly bounded then we may extract a convergent subsequence where all atoms and weights $a_n, b_n \in [0,1]$ converge. Then, it is clear that P_{n_k} and Q_{n_k} converge in the vague topology.

Thus, we may assume that $(\alpha_n, \alpha_n', \beta_n, \beta_n') \in \mathbb{R}^4$ is not uniformly bounded. By passing to a subsequence, we may assume that $\|(\alpha_n, \alpha_n', \beta_n, \beta_n')\| \to \infty$. Also assume without loss of generality that $|\alpha_n| = \max(|\alpha_n|, |\alpha_n'|, |\beta_n|, |\beta_n'|)$. We proceed to show that a_n, the weight of α_n in μ_{P_n}, converges to 0.

Let $X_n' = X_n/\alpha_n$, so that $X_n' = P_n' + iQ_n'$, where $P_n' = P_n/\alpha_n$, $Q_n' = Q_n/\alpha_n$. Then, the atoms of P_n' are at $1, \alpha_n'/\alpha_n$ with respective weights $a_n, 1 - a_n$ and the atoms of Q_n' are at $\beta_n/\alpha_n, \beta_n'/\alpha_n$ with respective weights $b_n, 1 - b_n$. By construction, the atoms of P_n', Q_n' are in $[-1, 1]$. Let μ_n' be the empirical spectral distribution of X_n'.

Since μ is a probability measure and $\mu_n \to \mu$ in probability in the vague topology, then we deduce that for every $\epsilon > 0$ and $\delta > 0$ there exists a compact set $K_\epsilon \subset \mathbb{C}$ such that $\mu(K_\epsilon) > 1 - \epsilon$ and for n sufficiently large, $\mathbb{P}(\mu_n(K_\epsilon) > 1 - \epsilon) > 1 - \delta$.

Now, note that $\mu_n'(\frac{1}{\alpha_n} K_\epsilon) = \mu_n(K_\epsilon)$. For any neighborhood U of 0 in $\mathbb{C}$, we may choose n sufficiently large to that $\frac{1}{\alpha_n} K_\epsilon \subset U$. Hence, for n sufficiently large, $\mathbb{P}(\mu_n'(U) > 1 - \epsilon) > 1 - \delta$. This implies that μ_n' converges to δ_0 in probability.

Any subsequence of X_n' has a convergent subsequence where the atoms and weights of P_{n_k}', Q_{n_k}' converge. From Theorem 6.1, this implies that μ_{n_k}' converges almost surely to μ', the Brown measure of $X' = p' + iq'$, where $\mu_{P_{n_k}'}$ converges to $\mu_{p'}$ and $\mu_{Q_{n_k}'}$ converges to $\mu_{q'}$. Hence, $\mu = \delta_0$. Since the assignment $(\mu_{p'}, \mu_{q'}) \mapsto \mu'$ is 1-1, then $p = q = 0$. Hence, $\mu_{P_{n_k}} \to \delta_0$.

This implies that $\mu_{P_n} \to \delta_0$. For this to happen, the weight of the atom at 1, a_n, has to tend to 0. Thus we conclude that if $(\alpha_n, \alpha'_n, \beta_n, \beta'_n) \in \mathbb{R}^4$ is not uniformly bounded then the weight of one of these atoms goes to 0.

Without loss of generality, let us keep assuming that $a_n \to 0$. If α'_n is not bounded, then by passing a subsequence where $|\alpha'_n| \to \infty$, then the atoms $\alpha'_n + i\beta_n$ and $\alpha'_n + i\beta'_n$ go to ∞. As the weight of α_n in P_n tends to 1, then from the previous argument, the measure of the atoms $\alpha'_n + i\beta_n$ and $\alpha'_n + i\beta'_n$ tends to 1. Then the limit measure has to be $\mu = 0$, a contradiction. Hence, α'_n is bounded, and by passing to a subsequence we may assume that $\alpha'_n \to \alpha'$. Thus, $\mu_{P_n} \to \delta_{\alpha'}$ for some $\alpha' \in \mathbb{R}$.

Now, we proceed to show that μ_{Q_n} is a tight sequence of measures. Since $\mu_n \to \mu$ in probability and μ is a probability measure, then for any $\epsilon > 0$, we may choose K_ϵ so that for all n sufficiently large, $\mathbb{P}(\mu_n(K_\epsilon) > 1 - \epsilon) > 0$. Also assume n is sufficiently large so that $a_n < \epsilon$. Hence,

$$
\begin{aligned}
\mu_n(\{\alpha'_n + i\beta_n\}) &= \tau(\chi_{\{\alpha'_n\}}(P_n) \wedge \chi_{\{\beta_n\}}(Q_n)) \\
&= \tau(\chi_{\{\alpha'_n\}}(P_n)) + \tau(\chi_{\{\beta_n\}}(Q_n)) - \tau(\chi_{\{\alpha'_n\}}(P_n) \vee \chi_{\{\beta_n\}}(Q_n)) \\
&= (1 - a_n) + \tau(\chi_{\{\beta_n\}}(Q_n)) - \tau(\chi_{\{\alpha'_n\}}(P_n) \vee \chi_{\{\beta_n\}}(Q_n)) \\
&> (1 - \epsilon) + \tau(\chi_{\{\beta_n\}}(Q_n)) - 1 \\
&= \tau(\chi_{\{\beta_n\}}(Q_n)) - \epsilon \\
&= \mu_{Q_n}(\{\beta_n\}) - \epsilon \\
&= b_n - \epsilon\,.
\end{aligned}
\tag{6.65}
$$

Similarly, $\mu_n(\{\alpha'_n + i\beta_n\}) > (1 - b_n) - \epsilon$. Thus, for $\epsilon < 1/3$, at least one of $\alpha'_n + i\beta_n, \alpha'_n + i\beta_n$ is in K_ϵ. Let C_ϵ be the projection of K_ϵ onto the y coordinate. If both $\alpha'_n + i\beta_n, \alpha_n + i\beta_n \in K_\epsilon$, then $\mu_{Q_n}(C_\epsilon) > 1 - 2\epsilon$. Assume that $\alpha'_n + i\beta_n \notin K_\epsilon$. Then, with positive probability, $\mu_n(K_\epsilon) > 1 - \epsilon$, so $\mu_n(\{\alpha'_n + i\beta_n\}) < \epsilon$. This implies that $b_n < 2\epsilon$, so that $\mu_{Q_n}(C_\epsilon) > 1 - 2\epsilon$. Hence, Q_n is tight.

Then, by passing to subsequences, using Theorem 6.1, and using that the Brown measure of $X = p + iq$ determines μ_p and μ_q, then we conclude that the law of Q_n converges to the law of q. Using that the law of P_n converged to $\delta_{\alpha'}$, then this completes the proof. $\square$

CHAPTER 7

Quaternions and Quaternionic Green's function

In this chapter, we introduce the Quaternions and the Quaternionic Green's function for an arbitrary $X \in (M, \tau)$. Then, we will discuss how this relates to the Brown measure of X. We will give an outline of how to compute this function when $X = p + iq$, where $p, q \in (M, \tau)$ are Hermitian and freely independent. Finally, we provide the heuristics describing the boundary and support of the Brown measure of $X = p + iq$ in terms of the Quaternionic Green's function.

7.1 Quaternions and notation

In this section, we will set the notation for the quaternions and list some basic facts.

The ***Quaternions*** is a real 4-dimensional algebra generated by the elements 1, $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, with the relations

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1\,. \tag{7.1}$$

A general quaternion Q can be written as:

$$Q = x_0 + x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k}, \quad x_0, x_1, x_2, x_3 \in \mathbb{R}\,. \tag{7.2}$$

From physics, the Pauli matrices in $M_2(\mathbb{C})$ are defined by:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.3}$$

It is easy to see that $i\sigma_1$, $i\sigma_2$, and $i\sigma_3$ have the desired relations of the quaternionic generators $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$. The subalgebra of $M_2(\mathbb{C})$ these elements and the identity matrix generate is 4-dimensional over $\mathbb{R}$ and hence is a representation of the quaternions. Explicitly, we have the following definitions for the quaternions that will be used for the rest of the thesis:

Definition 7.1. *The **quaternions** are the real subalgebra of $M_2(\mathbb{C})$ consisting of the following matrices:*

$$\mathbb{H} = \left\{ Q = \begin{pmatrix} A & i\overline{B} \\ iB & \overline{A} \end{pmatrix} : A, B \in \mathbb{C} \right\}. \tag{7.4}$$

In terms of the coefficients x_i in (7.2),

$$A = x_0 + ix_3 \qquad B = x_1 + ix_2, \quad x_0, x_1, x_2, x_3 \in \mathbb{R}\,. \tag{7.5}$$

*The coefficients x_i are called the **real coefficients** of Q and A, B are the **complex coefficients** of Q.*

*We will refer to the **real numbers** in the quaternions as the subset where $x_1 = x_2 = x_3 = 0$ (i.e. $A \in \mathbb{R}$ and $B = 0$). Similarly, we will refer to the **complex numbers** in the quaternions as the subset where $x_1 = x_2 = 0$ (i.e. $B = 0$).*

We highlight the fact that the quaternions are not a complex algebra, since the center of $\mathbb{H}$ is $\mathbb{R}$ and not $\mathbb{C}$. In particular, this means $\mathbb{H}$ is not a von Neumann algebra.

There are some relevant operations on quaternions:

- the ***inverse*** of a quaternion is just the matrix inverse.
- The ***conjugate*** of a quaternion $Q = x_0 + x_1 i + x_2 j + x_3 k$ is $\overline{Q} = x_0 - x_1 i - x_2 j - x_3 k$. In the matrix representation of Q, this corresponds to the matrix Q^*.
- The ***norm*** of a quaternion, $|Q|$, is defined by:

$$|Q| = (Q\overline{Q})^{1/2} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2} \tag{7.6}$$

and is equal to the square root of the determinant of Q. From the multiplicative property of the determinant, it easily follows that $|Q_1Q_2| = |Q_1|\,|Q_2|$ for quaternions Q_1, Q_2.

- The norm on quaternions induces a metric, and when we speak of the ***convergence*** of a sequence of quaternions $Q_k \to Q$, it is with respect to this metric.

Every quaternion Q is diagonalizable with eigenvalues

$$\begin{aligned} g &= x_0 + i\sqrt{x_1^2 + x_2^2 + x_3^2} \\ \overline{g} &= x_0 - i\sqrt{x_1^2 + x_2^2 + x_3^2}\,. \end{aligned} \tag{7.7}$$

From the (7.7), we observe the following facts:

- The eigenvalues of Q come in conjugate pairs.
- The eigenvalues of Q are distinct if and only if $Q \notin \mathbb{R}$.
- The eigenvalues of Q are real if and only if $Q \in \mathbb{R}$.
- $|g|^2 = \det Q = |Q|^2$ (so $|g| = |Q|$) and $\det(Q) = |A|^2 + |B|^2$.

In what follows, it is convenient to define g^I to be the eigenvalue of Qi with non-negative imaginary part. Using the notation in (7.4) and (7.5),

$$Qi = \begin{pmatrix} iA & \overline{B} \\ -B & -i\overline{A} \end{pmatrix} = \begin{pmatrix} iA & i\overline{(iB)} \\ i(iB) & \overline{iA} \end{pmatrix}. \tag{7.8}$$

Hence,

$$g^I = -x_3 + i\sqrt{x_2^2 + x_1^2 + x_0^2}\,. \tag{7.9}$$

Remark 7.2. *Given a quaternion Q, our convention (unless stated otherwise) will be to use g (resp g^I) to denote the eigenvalue of Q (resp. Qi) with non-negative imaginary part, i.e. as in (7.7) and (7.9).*

It is easy to see directly from formulas that $|g| = \left|g^I\right|$. This also follows from the fact that g^I is an eigenvalue of Qi and $|i| = 1$. There is another condition between g and g^I that is evident from the formulas: $\left|\mathrm{Re}(g^I)\right| \leq \mathrm{Im}(g)$. These conditions characterize g and g^I, as discussed in the following Proposition:

Proposition 7.3. *Let $z, w \in \mathbb{C}$ such that*

- $\mathrm{Im}(z) \geq 0$.
- $\mathrm{Im}(w) \geq 0$.
- $|z| = |w|$.
- $|\mathrm{Re}(w)| \leq \mathrm{Im}(z)$.

Then, $(z, w) = (g, g^I)$ for some $Q \in \mathbb{H}$. The converse is also true. In polar coordinates where $z = re^{i\theta}$ and $w = se^{i\phi}$, $r, s \geq 0$, $\theta, \phi \in [0, 2\pi)$, the conditions become:

- $\theta \in [0, \pi]$.
- $\phi \in [0, \pi]$.
- $r = s$.
- $\phi \in [|\pi/2 - \theta|, \pi - |\pi/2 - \theta|]$.

Proof. Let $\mathrm{Re}(z) = x_0$, $\mathrm{Re}(w) = -x_3$, for some $x_0, x_3 \in \mathbb{R}$. Since $|\mathrm{Re}(w)| \leq \mathrm{Im}(z)$ and $\mathrm{Im}(z) \geq 0$, then we may choose $x_1, x_2 \in \mathbb{R}$ such that $\mathrm{Im}(z) = \sqrt{x_1^2 + x_2^2 + x_3^2}$. From $|z| = |w|$ and $\mathrm{Im}(w) \geq 0$, then $\mathrm{Im}(w) = \sqrt{x_2^2 + x_1^2 + x_0^2}$. Hence, $(z, w) = (g, g^I)$ for $Q = x_0 + x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k}$. The converse and the polar coordinates condition are straightforward to check. $\square$

We also record the following observations about when $g, g^I \in \mathbb{R}$ implies that $Q \in \mathbb{R}$ or $Q \in i\mathbb{R}$:

Proposition 7.4. *For $Q \in \mathbb{H}$, $g \in \mathbb{R}$ implies that $Q \in \mathbb{R}$ and $g^I \in \mathbb{R}$ implies that $Q \in i\mathbb{R}$. But, $g^I \in i\mathbb{R}$ does not imply that $Q \in \mathbb{R}$. But, $Q \in \mathbb{C}$ and $g^I \in i\mathbb{R}$ does imply that $Q \in \mathbb{R}$.*

Proof. The first statements about $g \in \mathbb{R}$ and $g^I \in \mathbb{R}$ follow from the fact that Q has real eigenvalues if and only if $Q \in \mathbb{R}$ (and similarly for Qi). If $Q = \boldsymbol{i}$, then $g^I \in i\mathbb{R}$ but $Q \notin \mathbb{R}$. If $Q \in \mathbb{C}$, then g^I is either ig or $\overline{ig}$ depending on the sign of $\mathrm{Im}(ig)$, and so $g^I \in i\mathbb{R}$ implies $g \in \mathbb{R}$, and hence $Q \in \mathbb{R}$. $\square$

The conventions $\mathrm{Im}(g) \geq 0$ and $\mathrm{Im}(g^I) \geq 0$ are useful for some general computations, but it has a drawback: if $Q \in \mathbb{C}$, then the eigenvalue g may not be equal to Q: $g = Q$ when $\mathrm{Im}(Q) \geq 0$ and $g = \overline{Q}$ otherwise. Similarly, $Qi \in \mathbb{C}$, but we can only conclude that either $g^I = Qi$ or $g^I = \overline{Qi}$.

But, many of the complex-valued functions of g we will consider arise from context as functions of a quaternion Q. This means that these functions only depend on the *set* of eigenvalues of Q, i.e. these functions are invariant under changing g and $\overline{g}$. We record the following Lemma for these situations:

Lemma 7.5. *Let $Q \in \mathbb{C}$, g (resp. $\overline{g}$) as in (7.7) (resp. (7.9), and $f_1, f_2 : \mathbb{C} \to \mathbb{C}$ where*

$$f_1(g) = f_1(\overline{g}) \qquad f_2(g) = f_2(\overline{g})\,. \tag{7.10}$$

Then,

$$f_1(g) = f_1(Q) \qquad f_2(g^I) = f_2(Qi)\,. \tag{7.11}$$

In particular, when we evaluate f_1 and f_2 at g and g^I respectively, we may assume that $g = Q$ and $g^I = ig$.

Proof. Recall that $g = Q$ when $\mathrm{Im}(Q) \geq 0$ and $g = \overline{Q}$ otherwise. Since $f_1(g) = f_1(\overline{g})$, in either case $f_1(g) = f_1(Q)$. Similar logic shows that $f_2(g^I) = f_2(Qi)$. Hence, for purposes of evaluating $f_1(g)$ and $f_2(g^I)$, we may assume that $g = Q$ and $g^I = Qi = ig$. $\square$

We also record the following fact about the convergence of quaternions to a real number:

Lemma 7.6. *Let Q_k be a sequence of quaternions where the eigenvalues of Q_k converge to some real number $g \in \mathbb{C}$. Then, Q_k converges to $g \in \mathbb{C}$.*

Proof. Let $g_k, \overline{g_k}$ be the eigenvalues of Q_k, where

$$g_k = (x_0)_k + i\sqrt{(x_1)_k^2 + (x_2)_k^2 + (x_3)_k^2}\,. \tag{7.12}$$

g_k converges to a real $t \in \mathbb{R}$ if and only if $(x_1)_k \to 0$, $(x_2)_k \to 0$, $(x_3)_k \to 0$, and $(x_0)_k \to t$. This implies that Q_k converges to t. □

7.2 Quaternionic Green's function

In this section, we will define the Quaternionic Green's function of an arbitrary $X \in (M, \tau)$ in terms of the operator-valued Cauchy transform. It generalizes the Stieltjes transform from Section 1.4.2 by considering a non-Hermitian operator and being a function defined on quaternions. The term "Quaternionic Green's function" comes from the physics literature (see [BS15], [JN04], [JN06], [FZ97b], and [JNP97]), and we will use it instead of "Quaternionic Stieltjes transform/Cauchy transform." In the mathematics literature, this method was first suggested in [Gir84] and is now commonly known as the ***Hermitization method*** ([BSS18], [GKZ11]). Applied to a specific problem, it is known as the ***linearization trick*** ([BMS17]).

We will briefly define the relevant concepts in operator-valued free probability and then apply them in our situation (summarized from ([MS17], Chapters 9, 10) and ([AP], Chapter 9)). We will only consider the case of a von Neumann algebra, but there is a more general notion of an operator-valued probability space that generalizes the non-commutative probability spaces from Section 1.3. Then, we will show how to pass from the conventions in the mathematics literature to the conventions in the physics literature in our setup.

First, we define the conditional expectation on von Neumann algebras:

Definition 7.7. *Let M be a von Neumann algebra and B a von Neumann subalgebra. A* ***conditional expectation*** *from E to B is a linear map $E : M \to B$ that satisfies the* *following properties:*

1. *If $x \geq 0$, then $E(x) \geq 0$.*

2. *$E(b) = b$ for $b \in B$.*

3. *$E(b_1 x b_2) = b_1 E(x) b_2$ for $b_1, b_2 \in B$, $x \in M$.*

We will refer to (M, E, B) as an ***operator-valued probability space****.*

When (M, τ) is a tracial von Neumann algebra and $B = \mathbb{C}$, we can take $E = \tau$.

In general, when (M, τ) is a tracial von Neumann algebra and B is a von Neumann subalgebra, there exists a unique conditional expectation $E : M \to B$ such that $\tau \circ E = \tau$. Recall that $L^2 M$ is the Hilbert space obtained by completing M with respect to the inner product $\langle a, b \rangle = \tau(b^* a)$. Then, this conditional expectation E is the restriction to M of the orthogonal projection from $L^2 M$ onto $L^2 B$.

Note that this generalizes the conditional expectation in the classical probability space $M = L^\infty(X, \mathcal{A}, \mu)$, where $\mathcal{A}$ is the σ-algebra of μ-measurable sets. In this case, $B = L^\infty(X, \mathcal{B}, \mu)$, where $\mathcal{B}$ is a sub σ-algebra of $\mathcal{A}$, and the conditional expectation is the restriction to M of the orthogonal projection of $L^2 M = L^2(X, \mathcal{A}, \mu)$ onto $L^2 B = L^2(X, \mathcal{B}, \mu)$.

As in the case when $B = \mathbb{C}$, there is a notion of freeness with respect to E:

Definition 7.8. *Let (M, E, B) be an operator-valued probability space. Let $\{A_i\}_{i \in I}$ be a family of von Neumann subalgebras of M that each contain B. Then, $\{A_i\}_{i \in I}$ are* ***freely independent with amalgamation over B*** *if for any $a_j \in A_{k(j)}$ with $k(j) \neq k(j+1)$, $j = 1, \ldots, n = 1$ and $E(a_i) = 0$, then*

$$E(a_1 \ldots, a_n) = 0 \,. \tag{7.13}$$

Let $r, (m_k)_{1 \leq k \leq r}$ be positive integers. The sets $\{X_{1,p}, \ldots, X_{m_p,p}\}_{1 \leq p \leq r}$ of non-commutative random variables are ***free with amalgamation over B*** *if the algebras they generate with B are free.*

Now, we define the operator-valued Cauchy transform and consider its domain of definition.

Recall that the usual complex-valued Cauchy transform for a Hermitian $x \in (M, \tau)$ is a complex analytic function $G_x : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C} \setminus \mathbb{R}$ given by:

$$G_x(z) = \tau \left[(z - x)^{-1} \right]. \tag{7.14}$$

Letting $\mathbb{H}^+(\mathbb{C})$ be the upper half-plane and $\mathbb{H}^-(\mathbb{C})$ be the lower half-plane, then in particular, $G_x : \mathbb{H}^+(\mathbb{C}) \to \mathbb{H}^-(\mathbb{C})$.

We can generalize the complex-valued Cauchy transform by considering a Cauchy transform that takes values in a subalgebra $B \subset M$, where τ is replaced by the conditional expectation E. We state the definition of the operator-valued Cauchy transform and some facts about it:

Definition 7.9. *Let (M, E, B) be an operator-valued probability space. For $x \in (M, \tau)$, let $\mathrm{Im}(x) = \frac{x+x^*}{2i}$. Let the **operator upper/lower half-planes** be:*

$$\begin{aligned} \mathbb{H}^+(B) &= \{x \in B : \mathrm{Im}(x) > 0\} \\ \mathbb{H}^-(B) &= \{x \in B : \mathrm{Im}(x) < 0\}. \end{aligned} \tag{7.15}$$

*Let $x \in (M, \tau)$ where x is Hermitian. Then, the **operator-valued Cauchy transform** $\mathcal{G}_{\mathbf{X}} : \mathbb{H}^+(B) \to \mathbb{H}^-(B)$ is given by:*

$$\mathcal{G}_{\mathbf{X}}(b) = E \left[(b - X)^{-1} \right]. \tag{7.16}$$

Further, the function $\mathcal{G}_{\mathbf{X}}$ is Fréchet differentiable on $\mathbb{H}^+(B)$.

Analogous to the complex-valued case, there is also an operator-valued R-transform:

Definition 7.10. *Let (M, E, B) be an operator-valued probability space. Let $x \in (M, \tau)$ where x is Hermitian.*

Then for $b \in \mathbb{H}^+(B)$ in a neighborhood of infinity, $\mathcal{G}_x(b)$ is invertible. Thus, for $c \in \mathbb{H}^-(B)$

in a neighborhood of 0*, we may define the* ***operator-valued R-transform***

$$\mathcal{R}_x(c) = \mathcal{G}_x^{\langle -1 \rangle}(c) - c^{-1}\,. \tag{7.17}$$

Finally, the connection between the freeness and the R-transform in the operator-valued case is the operator-valued addition law:

Theorem 7.11. *Let* (M, E, B) *be an operator-valued probability space, and let* $x, y \in (M, E, B)$ *be Hermitian and freely independent with amalgamation over* B*. Where the functions are defined,*

$$\mathcal{R}_{x+y}(c) = \mathcal{R}_x(c) + \mathcal{R}_y(c)\,. \tag{7.18}$$

Now, we introduce the setup in the mathematical literature to our problem of determining the Brown measure of $X = p + iq$: Let (M, τ) be a tracial von-Neumann algebra. Then, $\left(M_2(M), \tau\left[\frac{1}{2}\mathrm{tr}\right]\right)$ is another tracial von-Neumann algebra. The unique trace-preserving conditional expectation from $M_2(M)$ onto $M_2(\mathbb{C})$ is the ***block trace***: $\mathrm{bTr} : M_2(M) \to M_2(\mathbb{C})$, given by:

$$\mathrm{bTr}\left[\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\right] = \begin{pmatrix} \tau(m_{11}) & \tau(m_{12}) \\ \tau(m_{21}) & \tau(m_{22}) \end{pmatrix}\,. \tag{7.19}$$

Thus, we will work in the operator-valued probability space $(M_2(M), \mathrm{bTr}, M_2(\mathbb{C}))$. We will use $\sim$ over the notation coming from the mathematics literature, then drop it for the corresponding objects from the physics literature.

Let $X \in M$. Then, consider

$$\tilde{\mathbf{X}} = \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} \in M_2(M)\,. \tag{7.20}$$

Note that $\tilde{\mathbf{X}}$ is Hermitian, so we may consider the operator-valued Cauchy transform $\mathcal{G}_{\tilde{\mathbf{X}}} : \mathbb{H}^+(M_2(\mathbb{C})) \to \mathbb{H}^-(M_2(\mathbb{C}))$ given by:

$$\mathcal{G}_{\tilde{\mathbf{X}}}(\tilde{Q}) = \mathrm{bTr}[(\tilde{Q} - \tilde{\mathbf{X}})^{-1}]\,, \tag{7.21}$$

where $\tilde{Q} \in \mathbb{H}^+(M_2(\mathbb{C}))$.

We consider $\tilde{Q} \in M_2(\mathbb{C})$ of the form:

$$\tilde{Q} = \begin{pmatrix} i\overline{B} & A \\ \overline{A} & iB \end{pmatrix}, \qquad A, B \in \mathbb{C}. \tag{7.22}$$

Then,

$$\mathrm{Im}(\tilde{Q}) = \begin{pmatrix} \mathrm{Re}(B) & 0 \\ 0 & \mathrm{Re}(B) \end{pmatrix} \tag{7.23}$$

and hence $\tilde{Q} \in \mathbb{H}^+(M_2(\mathbb{C}))$ when $\mathrm{Re}(B) > 0$.

To pass from the mathematics notation to the physics notation, let $J \in M_2(\mathbb{C})$ be the following matrix:

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{7.24}$$

Note that $J^2 = I$, so that $J = J^{-1}$. Then, let the corresponding physics quantities be:

$$\begin{aligned} \mathbf{X} = \tilde{\mathbf{X}} J &= \begin{pmatrix} X & 0 \\ 0 & X^* \end{pmatrix} \in M_2(M) \\ Q = \tilde{Q} J &= \begin{pmatrix} A & i\overline{B} \\ iB & \overline{A} \end{pmatrix} \in M_2(\mathbb{C}). \end{aligned} \tag{7.25}$$

Note that Q is exactly the general form of a quaternion from (7.4). Then, we can define the ***Quaternionic Green's function*** by:

$$\mathcal{G}_{\mathbf{X}}(Q) = \mathrm{bTr}[(Q - \mathbf{X})^{-1}]. \tag{7.26}$$

The Quaternionic Green's function $\mathcal{G}_{\mathbf{X}}(Q)$ and operator-valued Cauchy transform $\mathcal{G}_{\tilde{\mathbf{X}}}(\tilde{Q})$ are related by the following formula:

$$\mathcal{G}_{\mathbf{X}}(Q) = J\, \mathcal{G}_{\tilde{\mathbf{X}}}(\tilde{Q}). \tag{7.27}$$

From general facts about the operator-valued Cauchy transform, we know that $\mathcal{G}_{\mathbf{X}}(Q)$ is well-defined for $B > 0$ and maps into $J\,\mathbb{H}^{-}(M_2(\mathbb{C}))$. But, we can say more: $\mathcal{G}_{\mathbf{X}}(Q)$ is defined for quaternions Q where $B \neq 0$, and in this case $\mathcal{G}_{\mathbf{X}}(Q)$ is a quaternion:

To check that $Q - \mathbf{X}$ is invertible when $B \neq 0$, first let $X_A = A - X$. Note that for any $A, B \in \mathbb{C}$ where $B \neq 0$, $(X_A)^* X_A + |B|^2 \geq |B|^2 > 0$, and hence is invertible. The same applies for $X_A(X_A)^* + |B|^2$.

Direct computation shows that a left inverse of $Q - \mathbf{X}$ is:

$$\begin{pmatrix} ((X_A)^* X_A + |B|^2)^{-1}(X_A)^* & -i\overline{B}((X_A)^* X_A + |B|^2)^{-1} \\ -iB(X_A(X_A)^* + |B|^2)^{-1} & (X_A(X_A)^* + |B|^2)^{-1} X_A \end{pmatrix}. \tag{7.28}$$

Similarly, a right inverse of $Q - \mathbf{X}$ is:

$$\begin{pmatrix} (X_A)^*(X_A(X_A)^* + |B|^2)^{-1} & -i\overline{B}((X_A)^* X_A + |B|^2)^{-1} \\ -iB(X_A(X_A)^* + |B|^2)^{-1} & X_A((X_A)^* X_A + |B|^2)^{-1} \end{pmatrix}. \tag{7.29}$$

These two are the same and define a two-sided inverse given that we can verify the following equations:

$$\begin{gathered} ((X_A)^* X_A + |B|^2)^{-1}(X_A)^* = (X_A)^*(X_A(X_A)^* + |B|^2)^{-1} \\ (X_A(X_A)^* + |B|^2)^{-1} X_A = X_A((X_A)^* X_A + |B|^2)^{-1}\,. \end{gathered} \tag{7.30}$$

These two equations can be verified by multiplying on the left and right by $(X_A)^* X_A + |B|^2$ and $X_A(X_A)^* + |B|^2$ to clear the inverses.

Next, we observe that $\mathcal{G}_{\mathbf{X}}(Q)$ is a quaternion. This is equivalent to the following equations:

$$\begin{gathered} \tau[(X_A)^* X_A + |B|^2)^{-1}(X_A)^*] = \overline{\tau[X_A((X_A)^* X_A + |B|^2)^{-1}]} \\ \tau[((X_A)^* X_A + |B|^2)^{-1}] = \tau[(X_A(X_A)^* + |B|^2)^{-1}]\,. \end{gathered} \tag{7.31}$$

The first equation follows from using the tracial property of τ and noting that the traces of an adjoint pair of operators are conjugate. The second equation follows from the fact that the spectral measures of $(X_A)^* X_A$ and $X_A(X_A)^*$ are the same and applying the continuous functional calculus.

When $B = 0$ in the quaternion Q, then $Q = A \in \mathbb{C}$ and

$$\mathcal{G}_{\mathbf{X}}(Q) = \mathcal{G}_{\mathbf{X}}(A) = \tau[(A - X)^{-1}]\,, \tag{7.32}$$

and this formula is well-defined even for non-Hermitian X when $A \notin \sigma(X)$. We will discuss the restriction of $\mathcal{G}_{\mathbf{X}}$ to $\mathbb{C}$ in a later section.

Thus, we have the following definition of the Quaternionic Green's function:

Definition 7.12. *Let $X \in (M, \tau)$. For $A \in \mathbb{C}$, let $X_A = A - X$. Then, the* ***Quaternionic Green's function*** $\mathcal{G}_{\mathbf{X}} : \mathbb{H} \setminus \sigma(X) \to \mathbb{H}$ *is given by:*

$$\begin{aligned}
\mathcal{G}_{\mathbf{X}}(Q) &= \mathrm{bTr}[(Q - \mathbf{X})^{-1}] \\
&= \mathrm{bTr}\left[\left(\begin{pmatrix} A & i\overline{B} \\ iB & \overline{A} \end{pmatrix} - \begin{pmatrix} X & 0 \\ 0 & X^* \end{pmatrix}\right)^{-1}\right] \\
&= \begin{pmatrix} \tau[((X_A)^* X_A + |B|^2)^{-1}(X_A)^*] & -i\overline{B}\,\tau[((X_A)^* X_A + |B|^2)^{-1}] \\ -iB\,\tau[((X_A)^* X_A + |B|^2)^{-1}] & \tau[X_A((X_A)^* X_A + |B|^2)^{-1}] \end{pmatrix}.
\end{aligned} \tag{7.33}$$

Let $\mathbb{H}^+$ be the ***upper Quaternionic half-plane*** (and $\mathbb{H}^-$ be the ***lower Quaternionic half-plane***):

$$\begin{aligned}
\mathbb{H}^+ &= \{Q \in \mathbb{H} : B > 0\} \\
\mathbb{H}^- &= \{Q \in \mathbb{H} : B < 0\}\,.
\end{aligned} \tag{7.34}$$

Note that the Quaternionic upper/lower half-planes are not the operator upper/lower half-planes of $M_2(\mathbb{C})$ restricted to $\mathbb{H}$.

As a generalization of the fact that the complex Cauchy transform swaps the upper/lower half-planes, it is easy to see from the formula for $\mathcal{G}_{\mathbf{X}}$ that:

$$\mathcal{G}_{\mathbf{X}} : \mathbb{H}^{\pm} \to \mathbb{H}^{\mp}\,. \tag{7.35}$$

Similarly, as a consequence of the formula for $\mathcal{G}_{\mathbf{X}}$ and (7.30) there is a generalization of the conjugate symmetry of the complex Cauchy transform:

$$\mathcal{G}_{\mathbf{X}^*}(Q^*) = \mathcal{G}_{\mathbf{X}}(Q)^* \,. \tag{7.36}$$

Now, we provide the relevant formulas relating the Inverse Quaternionic Green's function with the inverse operator-valued Cauchy transform and the Quaternionic R-transform with the operator-valued R-transform. These formulas follow directly from (7.25) and (7.27):

Definition 7.13. *Let $X \in (M,\tau)$. The **Inverse Quaternionic Green's function**, $\mathcal{B}_{\mathbf{X}} = \mathcal{G}_{\mathbf{X}}^{\langle -1\rangle}$, is defined for quaternions $W \in J\,\mathbb{H}^-(M_2(\mathbb{C}))$ in a neighborhood of 0. Its relationship with the inverse operator-valued Cauchy transform $\mathcal{G}_{\tilde{\mathbf{X}}}^{\langle -1\rangle}$ is:*

$$\mathcal{B}_{\mathbf{X}}(W) = \mathcal{G}_{\mathbf{X}}^{\langle -1\rangle}(W) = \mathcal{G}_{\tilde{\mathbf{X}}}^{\langle -1\rangle}(J\,W)\,J. \tag{7.37}$$

*The **Inverse Quaternionic R-transform**, $\mathcal{R}_{\mathbf{X}}$, is defined for quaternions $W \in J\,\mathbb{H}^-(M_2(\mathbb{C}))$ in a neighborhood of 0. Its relationship with the inverse operator-valued Cauchy transform $\mathcal{R}_{\tilde{\mathbf{X}}}$ is:*

$$\mathcal{R}_{\mathbf{X}}(W) = \mathcal{G}_{\mathbf{X}}^{\langle -1\rangle}(W) - W^{-1} = \mathcal{R}_{\tilde{\mathbf{X}}}(J\,W)\,W. \tag{7.38}$$

We conclude that there is an addition law for the Quaternionic R-transform:

Proposition 7.14. *Let $x, y \in (M,\tau)$ be freely independent. For quaternions $W \in J\,\mathbb{H}^-(M_2(\mathbb{C}))$ in a neighborhood of 0,*

$$\mathcal{R}_{\mathbf{x}+\mathbf{y}}(W) = \mathcal{R}_{\mathbf{x}}(W) + \mathcal{R}_{\mathbf{y}}(W)\,. \tag{7.39}$$

Proof. Consider the operator-valued probability space $(M_2(M), \mathrm{bTr}, M_2(\mathbb{C}))$. If x and y are free in (M,τ), then it is easy to see that $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ are free with amalgamation over $M_2(\mathbb{C})$. Hence, from the operator-valued addition law,

$$\mathcal{R}_{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}(c) = \mathcal{R}_{\tilde{\mathbf{x}}}(c) + \mathcal{R}_{\tilde{\mathbf{y}}}(c)\,, \tag{7.40}$$

and this identity holds for $c \in \mathbb{H}^-(M_2(\mathbb{C}))$ in a neighborhood of 0. Then, for quaternions $W \in J\,\mathbb{H}^-(M_2(\mathbb{C}))$ in a neighborhood of 0,

$$\begin{aligned} \mathcal{R}_{\mathbf{x}+\mathbf{y}}(W) &= \mathcal{R}_{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}(J\,W)\,W \\ &= \mathcal{R}_{\tilde{\mathbf{x}}}(J\,W)\,W + \mathcal{R}_{\tilde{\mathbf{y}}}(J\,W)\,W \\ &= \mathcal{R}_{\mathbf{x}}(W) + \mathcal{R}_{\mathbf{y}}(W)\,. \end{aligned} \tag{7.41}$$

□

7.3 Quaternionic Green's function and Brown measure

There is a strong analogy between the Quaternionic Green's function for an arbitrary $X \in (M, \tau)$ and the usual complex Cauchy transform for a Hermitian $x \in (M, \tau)$. Recall that the key utility of G_x is that the spectral measure of x is a distributional limit of the Cauchy transform approaching the real axis:

$$\mu_x = \lim_{b \to 0^+} -\frac{1}{\pi} \mathrm{Im}\, G_\mu(\cdot + ib)\,. \tag{7.42}$$

To complete the analogy, we describe how to recover the Brown measure of X as a limit of the Quaternionic Green's function approaching the complex plane.

First, let $F : \mathbb{H} \to \mathbb{H}$ capture the first part of the quaternion, i.e. for a quaternion Q as in (7.4), $F(Q) = A$.

For $z \in \mathbb{C}$ and $\epsilon > 0$, consider the quaternion

$$z_\epsilon = \begin{pmatrix} z & i\epsilon \\ i\epsilon & \overline{z} \end{pmatrix} \in \mathbb{H}. \tag{7.43}$$

Then,

$$F(\mathcal{G}_{\mathbf{X}}(z_\epsilon)) = \tau[((X_z)^* X_z + \epsilon^2)^{-1} (X_z)^*] = 2\,\frac{\partial}{\partial z} f_{\epsilon^2}(z)\,, \tag{7.44}$$

where $f_{\epsilon^2} : \mathbb{C} \to \mathbb{R}$ is from Proposition 3.4, given by:

$$f_{\epsilon^2}(z) = \frac{1}{2}\tau\left[\log(X_z^* X_z + \epsilon^2)\right] . \tag{7.45}$$

Then, from (3.38), the Brown measure of X, μ_X, is given by the following distributional limit:

$$\mu_X = \lim_{\epsilon \to 0^+} \frac{1}{\pi}\frac{\partial}{\partial \bar{z}} F(\mathcal{G}_{\mathbf{X}}(z_\epsilon)) . \tag{7.46}$$

In the mathematics notation, $\tilde{z}_\epsilon = z_\epsilon\, J \in \mathbb{H}^+(M_2(\mathbb{C}))$ for $\epsilon > 0$ and we can recover the Brown measure from a similar formula in terms of the operator-valued Cauchy transform as $\epsilon \to 0^+$.

7.4 Outline for computing Inverse Quaternionic Green's function for $X = p + iq$

We return to the situation where $X = p + iq$, where $p, q \in (M, \tau)$ are Hermitian and freely independent. Consider the operator-valued probability space $(M_2(M), \mathrm{bTr}, M_2(\mathbb{C}))$ and the following elements of $M_2(M)$:

$$\mathbf{p} = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \qquad i\mathbf{q} = \begin{pmatrix} iq & 0 \\ 0 & -iq \end{pmatrix} . \tag{7.47}$$

From the addition law for the Quaternionic R-transform,

$$\mathcal{R}_{\mathbf{X}}(Q) = \mathcal{R}_{\mathbf{p}}(Q) + \mathcal{R}_{i\mathbf{q}}(Q) . \tag{7.48}$$

In terms of the Inverse Quaternionic Green's function,

$$\mathcal{B}_{\mathbf{X}}(Q) = \mathcal{B}_{\mathbf{p}}(Q) + \mathcal{B}_{i\mathbf{q}}(Q) - Q^{-1} . \tag{7.49}$$

In this section, we will outline how to obtain expressions for $\mathcal{B}_{\mathbf{p}}(Q)$ and $\mathcal{B}_{i\mathbf{q}}(Q)$, and hence how to obtain an expression for $\mathcal{B}_{\mathbf{X}}(Q)$.

The computations in this section are taken from ([JN04], Section 4), so we will justify the domains where the computations make sense and state their results.

7.4.1 (Inverse) Quaternionic Green's function at a complex number

In this section, we make some observations about the restriction of the (Inverse) Quaternionic Green's function of an arbitrary $X \in (M, \tau)$ to the complex numbers.

Recall that the Cauchy transform of a Hermitian $x \in (M, \tau)$ has $G_x(z) \in \mathbb{R}$ if and only if $z \in \mathbb{R} \setminus \sigma(x)$. We generalize this fact for $\mathcal{G}_{\mathbf{X}}$:

Let $X \in (M, \tau)$ and consider a complex number quaternion, $z \in \mathbb{C} \setminus \sigma(X)$. From the definition of the Quaternionic Green's function,

$$\mathcal{G}_{\mathbf{X}}(z) = \tau[(z - X)^{-1}]\,. \tag{7.50}$$

Note that this is the same formula for the Cauchy transform $G_X(z)$, except X is potentially non-Hermitian. Even though X is not necessarily Hermitian, this function is defined and analytic for $z \notin \sigma(X)$. In this context, we will refer to this function as the ***complex Green's function*** (or ***Holomorphic Green's function***, as used in [JN04]) of X and still use the notation G_X. The significance of the term "Holomorphic Green's function" will be discussed in a later section.

Conversely, consider $\mathcal{G}_{\mathbf{X}}(Q)$ for $Q \in \mathbb{H} \setminus \mathbb{C}$, i.e. $B \neq 0$. The off-diagonal term of $\mathcal{G}_{\mathbf{X}}(Q)$ is:

$$-i\overline{B}\,\tau[((X_A)^* X_A + |B|^2)^{-1}]\,, \tag{7.51}$$

where $X_A = A - X$. If $B \neq 0$, then this term is non-zero. Hence,

$$\mathcal{G}_{\mathbf{X}}(Q) \in \mathbb{C} \iff Q \in \mathbb{C}\,, \tag{7.52}$$

and in this case $\mathcal{G}_{\mathbf{X}}(z) = G_X(z)$.

When X is non-Hermitian, $G_X(z)$ still satisfies $zG_X(z) \to 1$ as $|z| \to \infty$. In particular, G_X is still invertible in a neighborhood of infinity, and its inverse is defined in a neighborhood of 0. Let $B_X = G_X^{\langle -1 \rangle}$. Let $\mathcal{B}_{\mathbf{X}} : V \to \mathbb{H}$. For $Q \in V$,

$$\mathcal{B}_{\mathbf{X}}(Q) \in \mathbb{C} \iff Q = \mathcal{G}_{\mathbf{X}}(\mathcal{B}_{\mathbf{X}}(Q)) \in \mathbb{C}\,. \tag{7.53}$$

Let $U, V \subset \mathbb{H}$ be domains where $\mathcal{G}_{\mathbf{X}} : U \to V$, $\mathcal{B}_{\mathbf{X}} : V \to U$, $\mathcal{B}_{\mathbf{X}} \circ \mathcal{G}_{\mathbf{X}} = \mathrm{id}_U$, and $\mathcal{G}_{\mathbf{X}} \circ \mathcal{B}_{\mathbf{X}} = \mathrm{id}_V$. Then, for $z \in U \cap \mathbb{C}$ and $w \in V \cap \mathbb{C}$,

$$\begin{aligned} G_X(\mathcal{B}_{\mathbf{X}}(w)) = \mathcal{G}_{\mathbf{X}}(\mathcal{B}_{\mathbf{X}}(w)) = w \\ z = \mathcal{B}_{\mathbf{X}}(\mathcal{G}_{\mathbf{X}}(z)) = \mathcal{B}_{\mathbf{X}}(G_X(z))\,. \end{aligned} \tag{7.54}$$

Hence, $\mathcal{B}_{\mathbf{X}}$ is an inverse for G_X on $U \cap \mathbb{C}$, so $\mathcal{B}_{\mathbf{X}} = B_X$ on $U \cap \mathbb{C}$.

We summarize all of this in the following Proposition:

Proposition 7.15. *Let $X \in (M, \tau)$. Then, $\mathcal{G}_{\mathbf{X}}(Q) \in \mathbb{C}$ if and only if $Q \in \mathbb{C} \setminus \sigma(X)$. For $z \in \mathbb{C} \setminus \sigma(X)$,*

$$\mathcal{G}_{\mathbf{X}}(z) = G_X(z)\,. \tag{7.55}$$

Let $B_X = G_X^{\langle -1 \rangle}$ and let $\mathcal{B}_{\mathbf{X}} = \mathcal{G}_{\mathbf{X}}^{\langle -1 \rangle}$. Let $\mathcal{B}_{\mathbf{X}} : V \to \mathbb{H}$. Then, $\mathcal{B}_{\mathbf{X}}(Q) \in \mathbb{C}$ if and only if $Q \in V \cap \mathbb{C}$. For $w \in V \cap \mathbb{C}$,

$$\mathcal{B}_{\mathbf{X}}(w) = B_X(w)\,. \tag{7.56}$$

7.4.2 (Inverse) Quaternionic Green's function for a Hermitian operator

In this section, we describe how to compute the Quaternionic Green's Function for a Hermitian $H \in (M, \tau)$, $\mathcal{G}_{\mathbf{H}}$, in terms of the usual Cauchy transform G_H. Using analogous computations, we will determine where the Inverse Green's Function for H, $\mathcal{B}_{\mathbf{H}}$, is well-defined and compute $\mathcal{B}_{\mathbf{H}}$ in terms of the inverse of the usual Cauchy transform, $B_H = G_H^{\langle -1 \rangle}$.

To compute $\mathcal{G}_{\mathbf{H}}(Q)$, recall when $Q = g \in \mathbb{C} \setminus \sigma(H)$,

$$\mathcal{G}_{\mathbf{H}}(g) = G_H(g)\,. \tag{7.57}$$

For an arbitrary quaternion $Q \notin \mathbb{R}$, we may diagonalize Q by:

$$Q = S^{-1} g S\,, \tag{7.58}$$

where

$$S = \begin{pmatrix} iB & g - A \\ \overline{g} - \overline{A} & i\overline{B} \end{pmatrix}. \tag{7.59}$$

Since H is Hermitian, then $\mathbf{H}$ commutes $M_2(\mathbb{C})$. This combined with the bimodularity of bTr shows that

$$\mathcal{G}_{\mathbf{H}}(Q) = \mathcal{G}_{\mathbf{H}}(S^{-1} g S) = S^{-1} \mathcal{G}_{\mathbf{H}}(g) S\,. \tag{7.60}$$

Note that since $Q \notin \mathbb{R}$, then $g \notin \mathbb{R}$, so $\mathcal{G}_{\mathbf{H}}(g)$ is well-defined.

Thus, we can express $\mathcal{G}_{\mathbf{H}}(Q)$ in terms of Q and $G_H(g)$. The result of the computation is:

$$\mathcal{G}_{\mathbf{H}}(Q) = \begin{cases} \gamma_H(g) 1 - \gamma'_H(g) Q^* & Q \notin \mathbb{R} \\ G_H(g) & Q = g \in \mathbb{C} \setminus \sigma(H)\,, \end{cases} \tag{7.61}$$

where

$$\begin{aligned} \gamma_H(g) &= \frac{g G_H(g) - \overline{g} G_H(\overline{g})}{g - \overline{g}} \\ \gamma'_H(g) &= \frac{G_H(g) - G_H(\overline{g})}{g - \overline{g}}\,. \end{aligned} \tag{7.62}$$

It is straightforward to check that both formulas for $\mathcal{G}_{\mathbf{H}}(Q)$ agree for $Q \in \mathbb{C} \setminus \mathbb{R}$.

While the computation is carried out with the convention that $\mathrm{Im}(g) \geq 0$, γ_H and γ'_H only depends on the *set* of eigenvalues of Q since $\gamma_H(g) = \gamma_H(\overline{g})$ and $\gamma'_H(g) = \gamma'_H(\overline{g})$. Finally, note that γ_H, γ'_H are real-valued.

We observe the properties $\mathcal{G}_{\mathbf{H}}(g) = G_H(g)$ and $\mathcal{G}_{\mathbf{H}}(Q) = S^{-1}\mathcal{G}_{\mathbf{H}}(g) S$ lead to the following coordinate-free characterization of $\mathcal{G}_{\mathbf{H}}$ that will be useful in defining $\mathcal{B}_{\mathbf{H}}$. We will use the notation for eigenspaces of matrices from Definition 5.6.

Proposition 7.16. *Let $H \in (M, \tau)$ be Hermitian. Let $Q \in \mathbb{H}$ be a quaternion such that $Q \notin \sigma(X)$. Then, $\mathcal{G}_{\mathbf{H}} : \mathbb{H} \setminus \sigma(X) \to \mathbb{H}$ is defined by the property that*

$$E_{G_H(g)}(\mathcal{G}_{\mathbf{H}}(Q)) = E_g(Q)\,. \tag{7.63}$$

Proof. First, note that the definition is well-defined, i.e. that G_H is 1-1 on the set of eigenvalues of Q. When $Q \in \mathbb{R}$, then there is only one eigenvalue. When $Q \notin \mathbb{R}$, then Q has two eigenvalues $g, \overline{g} \in \mathbb{C} \setminus \mathbb{R}$. Then, since $G_H(g) \in \mathbb{R}$ if and only if $g \in \mathbb{R} \setminus \sigma(H)$, then $G_H(\overline{g}) = \overline{G_H(g)} \neq G_H(g)$. As Q is a quaternion, it is diagonalizable, and hence this property defines $\mathcal{G}_{\mathbf{H}}(Q)$. □

This leads to a coordinate-free definition of $\mathcal{B}_{\mathbf{H}} = \mathcal{G}_{\mathbf{H}}^{\langle -1 \rangle}$ and a domain where it is defined. Then, the analogous computations as for $\mathcal{G}_{\mathbf{H}}(Q)$ gives an explicit formula for $\mathcal{B}_{\mathbf{H}}$ in terms of Q and $B_H(g)$. The results of this are summarized in the following Proposition:

Proposition 7.17. *Let $H \in (M, \tau)$ be Hermitian. Let $B_H : U \to \mathbb{C}$ be the inverse of G_H, where U is an open neighborhood of 0 that is fixed under complex conjugation. Then, $\mathcal{B}_{\mathbf{H}} = \mathcal{G}_{\mathbf{H}}^{\langle -1 \rangle}$ is defined on $\mathbb{H}_U = \{Q \in \mathbb{H} : g \in U\}$ and $\mathcal{B}_{\mathbf{H}}$ is defined by the property that:*

$$E_{B_H(g)}(\mathcal{B}_{\mathbf{H}}(Q)) = E_g(Q)\,. \tag{7.64}$$

In particular, $\mathcal{B}_{\mathbf{H}}(g) = B_H(g)$ for $g \in U$ and $\mathcal{B}_{\mathbf{H}}(Q) = S^{-1}\mathcal{B}_{\mathbf{H}}(Q)S$ for $Q \in \mathbb{H}_U$, $S \in GL_2(\mathbb{C})$.

$\mathcal{B}_{\mathbf{H}}$ is continuous on $\mathbb{H}_U$ and an explicit formula for $\mathcal{B}_{\mathbf{H}}$ is given by:

$$\mathcal{B}_{\mathbf{H}}(Q) = \begin{cases} \beta_H(g)1 - \beta'_H(g)Q^* & Q \notin \mathbb{R} \\ B_H(x_0)1 & Q = x_0 1 \in \mathbb{R}\,, \end{cases} \tag{7.65}$$

where

$$\beta_H(g) = \frac{gB_H(g) - \overline{g}B_H(\overline{g})}{g - \overline{g}}$$
$$\beta'_H(g) = \frac{B_H(g) - B_H(\overline{g})}{g - \overline{g}}\,. \tag{7.66}$$

Finally, $\beta_H(g) = \beta_H(\overline{g})$, $\beta'_H(g) = \beta'_H(\overline{g})$, *and* β_H, β'_H *are real-valued.*

Proof. First, note that we may extend the domain of B_H to one that is fixed by complex conjugation, because of the conjugate symmetry of G_H.

The definition is well-defined because B_H is 1-1 on the set of eigenvalues of Q. When $Q \in \mathbb{R}$, there is only one eigenvalue. When $Q \notin \mathbb{R}$, then Q has two eigenvalues $g, \overline{g} \in \mathbb{C} \setminus \mathbb{R}$. Then, since $B_H(g) \in \mathbb{R}$ implies that $g \in \mathbb{R}$. As $\mathcal{B}_{\mathbf{H}}(Q)$ is diagonalizable, then this defines $\mathcal{B}_{\mathbf{H}}(Q)$.

This definition gives an inverse for $\mathcal{G}_{\mathbf{H}}$, since

$$\begin{aligned} E_g(\mathcal{G}_{\mathbf{H}}(\mathcal{B}_{\mathbf{H}}(Q))) &= E_{G_H(B_H(g))}(\mathcal{G}_{\mathbf{H}}(\mathcal{B}_{\mathbf{H}}(Q))) \\ &= E_{B_H(g)}(\mathcal{B}_{\mathbf{H}}(Q)) \\ &= E_g(Q)\,. \end{aligned} \tag{7.67}$$

implies that $\mathcal{G}_{\mathbf{H}}(\mathcal{B}_{\mathbf{H}}(Q)) = Q$ and similar equalities show that $\mathcal{B}_{\mathbf{H}}(\mathcal{G}_{\mathbf{H}}(Q)) = Q$.

For $g \in \mathbb{H}^+(\mathbb{C}) \cap U$,

$$\begin{aligned} E_{B_H(g)}(\mathcal{B}_{\mathbf{H}}(g)) &= E_g(g) = e_1 \\ E_{B_H(\overline{g})}(\mathcal{B}_{\mathbf{H}}(g)) &= E_{\overline{g}}(g) = e_2\,. \end{aligned} \tag{7.68}$$

so then $\mathcal{B}_{\mathbf{H}}(g) = B_H(g)$. When $g \in \mathbb{H}^-(\mathbb{C}) \cap U$, e_1 and e_2 are swapped in the above equalities. For $S \in GL_2(\mathbb{C})$ and $Q \in \mathbb{H}_U$,

$$\begin{aligned} E_{B_H(g)}(\mathcal{B}_{\mathbf{H}}(S^{-1}QS)) &= E_g(S^{-1}QS) \\ &= S^{-1}E_g(Q) \\ &= S^{-1}E_{B_H(g)}(\mathcal{B}_{\mathbf{H}}(Q)) \\ &= E_{B_H(g)}(S^{-1}\mathcal{B}_{\mathbf{H}}(Q)S)\,. \end{aligned} \tag{7.69}$$

As $\mathcal{B}_{\mathbf{H}}(Q)$ is diagonalizable, then $\mathcal{B}_{\mathbf{H}}(S^{-1}QS) = S^{-1}\mathcal{B}_{\mathbf{H}}(Q)S$.

For continuity of $\mathcal{B}_{\mathbf{H}}$, let $Q_n \to Q$, where $Q_n, Q \in \mathbb{H}_U$.

If $Q \in \mathbb{R}$, then $g_n, \overline{g_n} \to g = Q \in \mathbb{R}$ and $B_H(g_n), B_H(\overline{g_n}) \to B_H(g) = B_H(Q) \in \mathbb{R}$. Hence, the eigenvalues of $\mathcal{B}_{\mathbf{H}}(Q_n)$ converge to $B_H(Q)$, and from Lemma 7.6, this implies that $\mathcal{B}_{\mathbf{H}}(Q_n)$ converges to $\mathcal{B}_{\mathbf{H}}(Q) = B_H(Q)$.

If $Q \notin \mathbb{R}$, then $g_n \to g$, $\overline{g_n} \to \overline{g}$, and the diagonalizing transforms for Q_n,

$$S_n = \begin{pmatrix} iB_n & g_n - A_n \\ \overline{g_n} - \overline{A_n} & i\overline{B_n} \end{pmatrix} \tag{7.70}$$

converge to S, the diagonalizing transform for Q. Then, using the conjugation property of $\mathcal{B}_{\mathbf{H}}$ shows that $\mathcal{B}_{\mathbf{H}}(Q_n)$ converges to $\mathcal{B}_{\mathbf{H}}(Q)$.

The explicit formula for $\mathcal{B}_{\mathbf{H}}$ follows from the analogous computation as for $\mathcal{G}_{\mathbf{H}}$. The properties for β_H, β'_H are self-evident. □

7.4.3 (Inverse) Quaternionic Green's function and multiplication by a complex number

In this section, we consider an arbitrary $X \in (M, \tau)$. For $c \in \mathbb{C}$, we compare $\mathcal{G}_{c\mathbf{X}}$ with $\mathcal{G}_{\mathbf{X}}$, and similarly $\mathcal{B}_{c\mathbf{X}}$ with $\mathcal{B}_{\mathbf{X}}$.

For $c = 0$ and $Q \in \mathbb{H} \setminus \{0\}$,

$$\mathcal{G}_{c\mathbf{X}}(Q) = \mathcal{G}_0(Q) = Q^{-1}\,. \tag{7.71}$$

For $c \neq 0$ and $Q \in \mathbb{H} \setminus \sigma(cX) = \mathbb{H} \setminus (c\sigma(X))$,

$$\mathcal{G}_{c\mathbf{X}}(Q) = \mathcal{G}_{\mathbf{X}}\left(\frac{1}{c}Q\right)\frac{1}{c}\,. \tag{7.72}$$

The order of multiplication is important, as the quaternion Q may not commute with $c \in \mathbb{C}$.

This generalizes the formula for the complex Green's function,

$$G_{cX}(z) = \frac{1}{c} G_X\left(\frac{1}{c} z\right). \tag{7.73}$$

The formula for the inverse Green's function $\mathcal{B}_{cX}$ can be obtained by inverting this formula. This is summarized in the following Proposition:

Proposition 7.18. *Let $X \in (M, \tau)$ and let $\mathcal{B}_{\mathbf{X}} : V \to \mathbb{H}$, where $V \subset \mathbb{H}$.*

For $c = 0$, $\mathcal{B}_{c\mathbf{X}} : \mathbb{H} \setminus \{0\} \to \mathbb{H}$ and

$$\mathcal{B}_{c\mathbf{X}}(Q) = Q^{-1}. \tag{7.74}$$

For $c \neq 0$, $\mathcal{B}_{c\mathbf{X}} : V\left(\frac{1}{c}\right) \to \mathbb{H}$, and

$$\mathcal{B}_{c\mathbf{X}}(Q) = c\mathcal{B}_{\mathbf{X}}(Qc). \tag{7.75}$$

Proof. The result for $c = 0$ follows from $\mathcal{G}_{0\mathbf{X}} = Q^{-1}$, so we consider $c \neq 0$.

Let $U, V \subset \mathbb{H}$ be domains where $\mathcal{G}_{\mathbf{X}} : U \to V$, $\mathcal{B}_{\mathbf{X}} : V \to U$, $\mathcal{B}_{\mathbf{X}} \circ \mathcal{G}_{\mathbf{X}} = \mathrm{id}_U$, and $\mathcal{G}_{\mathbf{X}} \circ \mathcal{B}_{\mathbf{X}} = \mathrm{id}_V$. From (7.72), $\mathcal{G}_{c\mathbf{X}} : cU \to V\left(\frac{1}{c}\right)$. Similarly, defining $\mathcal{B}_{c\mathbf{X}}(Q) = c\mathcal{B}_{\mathbf{X}}(Qc)$ shows that $\mathcal{B}_{c\mathbf{X}} : V\left(\frac{1}{c}\right) \to cU$ and it is straightforward to check that $\mathcal{B}_{c\mathbf{X}} \circ \mathcal{G}_{c\mathbf{X}} = \mathrm{id}_{cU}$, and $\mathcal{G}_{c\mathbf{X}} \circ \mathcal{B}_{c\mathbf{X}} = \mathrm{id}_{V\left(\frac{1}{c}\right)}$. □

For the complex B_{cX}, there is a similar formula:

$$B_{cX}(z) = cB_X(cz). \tag{7.76}$$

This can be proved analogously from the formula for the complex Green's function.

7.4.4 Inverse Quaternionic Green's function for $X = p + iq$

We summarize how to compute $\mathcal{B}_{\mathbf{X}}$ when $X = p + iq$, where $p, q \in (M, \tau)$ are Hermitian and freely independent. Recall the definition of g^I from (7.9).

From the addition law for the Quaternionic R-transform, and Proposition 7.18,

$$\begin{aligned} \mathcal{B}_{\mathbf{X}}(Q) &= \mathcal{B}_{\mathbf{p}}(Q) + \mathcal{B}_{i\mathbf{q}}(Q) - Q^{-1} \\ &= \mathcal{B}_{\mathbf{p}}(Q) + i\mathcal{B}_{\mathbf{q}}(Qi) - Q^{-1}\,. \end{aligned} \tag{7.77}$$

Let $B_p : U \to \mathbb{C}$ and $B_q : V \to \mathbb{C}$. Then, the right-hand side of (7.77) is well-defined for $g \in U$, $g^I \in V$.

Using (7.65) to rewrite $\mathcal{B}_{\mathbf{p}}$ and $\mathcal{B}_{\mathbf{q}}$ when $g, g^I \notin \mathbb{R}$,

$$\mathcal{B}_{\mathbf{X}}(Q) = \beta_p(g) + \beta_q(g^I)i - \left(\beta_p'(g) + \beta_q'(g^I) + \frac{1}{\det Q}\right) Q^*\,. \tag{7.78}$$

Expanding this completely,

$$\mathcal{B}_{\mathbf{X}}(Q) = \mathcal{B}_{\mathbf{X}}\left(\begin{pmatrix} A & i\overline{B} \\ iB & \overline{A} \end{pmatrix}\right) = \begin{pmatrix} k + ik' - l\overline{A} & li\overline{B} \\ liB & k - ik' - lA \end{pmatrix}, \tag{7.79}$$

where

$$\begin{aligned} k &= \beta_p(g) \\ k' &= \beta_q(g^I) \\ l &= \beta_p'(g) + \beta_q'(g^I) + \frac{1}{\det Q}\,. \end{aligned} \tag{7.80}$$

We conclude this section by making some observations:

- k, k', and l are real-valued.
- The addition law (7.49) defines $\mathcal{B}_{\mathbf{X}}$ and works for all Q in the domain of $\mathcal{B}_{\mathbf{X}}$, but the expanded formula (7.79) only works when $g, g^I \notin \mathbb{R}$.
- We can apply Lemma 7.5 to $\beta_p, \beta_q, \beta_p', \beta_q'$, which are all real-valued and respect conju-

gation. Then, when evaluating these functions at some $Q \in \mathbb{C}$, we can assume that $g = Q$ and $g^I = ig$ in (7.78) and (7.79).

- Due to the presence of $\mathcal{B}_{\mathbf{q}}(Qi)$ in (7.75), it does not follow that $\mathcal{B}_{\mathbf{X}}(S^{-1}QS) = S^{-1}\mathcal{B}_{\mathbf{X}}(Q)S$ as in the case when X is Hermitian. In particular, $\mathcal{B}_{\mathbf{X}}$ is not determined by how it acts on diagonal (i.e. complex) quaternions. Further, it is not necessarily true like in the Hermitian case that the eigenvalues of $\mathcal{B}_{\mathbf{X}}$ are just $B_X(g)$, $B_X(\overline{g})$.

In summary, we will complete the following steps to compute $\mathcal{B}_{\mathbf{X}}$:

1. Compute B_p (resp. B_q).
2. Use (7.66) to compute β_p, β_p' (resp. β_q, β_q').
3. Use (7.65) to compute $\mathcal{B}_{\mathbf{p}}$ (resp. $\mathcal{B}_{\mathbf{q}}$).
4. Use (7.75) to compute $\mathcal{B}_{i\mathbf{q}}$.
5. Use (7.79) to compute $\mathcal{B}_{\mathbf{X}}$.

7.5 Heuristics for the support and boundary of the Brown measure

In this section, we provide the heuristics for the boundary and support of the Brown measure of $X = p + iq$ when $p, q \in (M, \tau)$ are Hermitian and freely independent.

Recall that the Brown measure of $X \in (M, \tau)$ can be defined by:

$$\mu_X = \lim_{\epsilon \to 0^+} \frac{1}{\pi} \frac{\partial}{\partial \overline{z}} F(\mathcal{G}_{\mathbf{X}}(z_\epsilon)) \,. \tag{7.81}$$

Let $X = p + iq$ where p, q are Hermitian and freely independent. Consider $z \in \mathbb{C}$ such that $\mathcal{G}_{\mathbf{X}}(z_\epsilon)$ is in the domain of $\mathcal{B}_{\mathbf{X}}$ for all sufficiently small $\epsilon > 0$.

Let

$$\mathcal{G}_{\mathbf{X}}(z_\epsilon) = \begin{pmatrix} A_\epsilon & i\overline{B_\epsilon} \\ iB_\epsilon & \overline{A_\epsilon} \end{pmatrix} , \tag{7.82}$$

where A_ϵ and B_ϵ are implicitly understood to depend on z. We wish to analyze $\mathcal{G}_{\mathbf{X}}(z_\epsilon)$ as $\epsilon \to 0^+$.

From the free independence of p and q and the Quaternionic addition law, it is more natural to consider $\mathcal{B}_{\mathbf{X}}$. If $\mathcal{G}_{\mathbf{X}}(z_\epsilon)$ is in the domain of $\mathcal{B}_{\mathbf{X}}$, then

$$\mathcal{B}_{\mathbf{X}}\left(Q_\epsilon\right) = z_\epsilon \tag{7.83}$$

has the unique solution of

$$Q_\epsilon = \mathcal{G}_{\mathbf{X}}(z_\epsilon)\,. \tag{7.84}$$

Since we can explicitly compute $\mathcal{B}_{\mathbf{X}}$, then we can use (7.83) to understand $\mathcal{G}_{\mathbf{X}}(z_\epsilon)$. Note that we can always use (7.79) to expand $\mathcal{B}_{\mathbf{X}}(Q_\epsilon)$: if Q_ϵ has $g_\epsilon \in \mathbb{R}$ or $g_\epsilon^I \in \mathbb{R}$, then $Q_\epsilon \in \mathbb{C}$ and from Proposition 7.15, $\mathcal{B}_{\mathbf{X}}(Q_\epsilon) \in \mathbb{C}$, a contradiction to $\epsilon > 0$.

By expanding the left-hand side of (7.83) with (7.79), the equation for the off-diagonal terms is:

$$l(Q_\epsilon) i B_\epsilon = i\epsilon\,. \tag{7.85}$$

As $\epsilon \to 0^+$, then either $l_\epsilon = l(Q_\epsilon)$ is small or B_ϵ is small. If we consider a sequence of ϵ_k to 0, we may extract a subsequence where either $l_{\epsilon_k} = l(Q_{\epsilon_k})$ converges to 0 or B_{ϵ_k} converges to 0.

To understand the situation heuristically, assume that

$$\lim_{\epsilon \to 0^+} Q_\epsilon = Q = \begin{pmatrix} A & i\overline{B} \\ iB & \overline{A} \end{pmatrix}\,. \tag{7.86}$$

Then, either

$$\lim_{\epsilon \to 0^+} B_\epsilon = B = 0 \qquad \text{or} \qquad \lim_{\epsilon \to 0^+} l_\epsilon = 0\,. \tag{7.87}$$

Assume it is possible to interchange the limit and derivative in (7.81). Then,

$$\mu_X = \frac{1}{\pi}\frac{\partial}{\partial \overline{z}} F(Q)\,. \tag{7.88}$$

If $\mathcal{B}_{\mathbf{X}}$ is defined and continuous at Q, then

$$z = \lim_{\epsilon \to 0^+} z_\epsilon = \lim_{\epsilon \to 0^+} \mathcal{B}_{\mathbf{X}}(Q_\epsilon) = \mathcal{B}_{\mathbf{X}}(Q)\,. \tag{7.89}$$

Thus,

$$\mathcal{G}_{\mathbf{X}}(z) = Q\,, \tag{7.90}$$

so that $B = 0$. As G_X is holomorphic at z, then

$$\frac{\partial}{\partial \overline{z}} F(Q) = \frac{\partial}{\partial \overline{z}} G_X(z) = 0\,. \tag{7.91}$$

Our first heuristic is that whenever $B = 0$ (i.e. not just in the continuous case), the Brown measure has zero density at z. Then, the Brown measure is supported on the set where $B \neq 0$. We also must consider the case where Q_ϵ has no limit as $\epsilon \to 0^+$. We state the heuristic formally:

Heuristic 7.19. *Let $p, q \in (M, \tau)$ where p and q are Hermitian and freely independent. Then, the support of the Brown measure of $X = p + iq$ is the closure of the set of $z \in \mathbb{C}$ such that:*

$$\lim_{\epsilon \to 0^+} \mathcal{G}_{\mathbf{X}}(z_\epsilon) = \begin{pmatrix} A & i\overline{B} \\ iB & \overline{A} \end{pmatrix} \tag{7.92}$$

for some $B \neq 0$ or where the limit does not exist.

We can verify this heuristic when p and q have 2 atoms that have equal weights (Theorem 10.1).

For the second heuristic, first note that:

$$\lim_{\epsilon \to 0^+} B_\epsilon = B \neq 0 \Longrightarrow \lim_{\epsilon \to 0^+} l_\epsilon = 0\,. \tag{7.93}$$

Hence, given the first heuristic, the support of the Brown measure should also be contained in the closure of the set of $z \in \mathbb{C}$ where $\lim_{\epsilon \to 0^+} l_\epsilon = 0$. But, it could be possible that this set has non-empty intersection with the set of $z \in \mathbb{C}$ where $B = 0$, where the measure has zero density. In particular, when $\mathcal{B}_{\mathbf{X}}$ is continuous at Q, then the intersection of these sets corresponds to a second order zero in the off-diagonal terms of (7.79). Our second heuristic is that the intersection of these two sets is the boundary of the Brown measure:

Heuristic 7.20. *Let $p, q \in (M, \tau)$ where p and q are Hermitian and freely independent. Then, the boundary of the Brown measure of $X = p + iq$ is the closure of the intersection of the set of $z \in \mathbb{C}$ such that:*

$$\lim_{\epsilon \to 0^+} \mathcal{G}_{\mathbf{X}}(z_\epsilon) = Q \in \mathbb{C} \tag{7.94}$$

with the set of $z \in \mathbb{C}$ such that $\lim_{\epsilon \to 0^+} l_\epsilon = 0$.

In ([JN04], Section 6), the authors verify the boundary heuristic for some random matrix models. With some restrictions on z, we will see that the intersection of the sets corresponds to the solutions of a system of equations. When p and q have 2 atoms, Proposition 9.2 states that its closure is the boundary (i.e. support) of μ'.

When p and q have an arbitrary number of atoms, Mathematica simulations suggest that the solution set to the system of equations contains the boundary of the Brown measure. When p and q have generic positions of atoms and weights, this solution set is an algebraic curve. We will provide an algorithm to produce a non-zero polynomial whose zero set contains this solution set. This is the content of Theorem 9.3.

We can now explain the terminology of the "Holomorphic Green's function" for $G_X(z)$. Consider $z \in \mathbb{C}$ where the following limit exists:

$$\lim_{\epsilon \to 0^+} \mathcal{G}_{\mathbf{X}}(z_\epsilon) = \begin{pmatrix} A(z) & i\overline{B(z)} \\ iB(z) & \overline{A(z)} \end{pmatrix}. \tag{7.95}$$

When $B(z) = 0$ and $\mathcal{B}_{\mathbf{X}}$ is defined and continuous at the limit point, recall that $A(z) = G_X(z)$. Hence, the ***Holomorphic Green's function*** is the limit of the Quaternionic Green's function towards the complex plane where the resulting complex function is holomorphic.

When $B(z) \neq 0$, then from Heuristic 7.19, the Brown measure of X is not zero in a neighborhood of z, so then $A(z)$ is not holomorphic in a neighborhood of z. This limit is the ***Non-Holomorphic Green's function***.

Remark 7.21. *We will use the following conventions throughout Chapters 9 and 10:*

- *Quantities that depend on a continuous limit as* $\epsilon \to 0^+$ *and are in the context of (7.83) have a subscript* ϵ.
- *Quantities that depend on a sequence* $\epsilon_k \to 0^+$ *and are in the context of (7.83) have a subscript* ϵ_k.
- *Quantities that are just general sequences with no implicit context have a subscript* k.
- *Unless specified otherwise,* p, q, *and* X *are implied to mean the specific case of* p, q *Hermitian and freely independent with 2 atoms, and* $X = p + iq$.
- *General sequences of quaternions* Q_k *are implied to be in the domain of* $\mathcal{B}_{\mathbf{X}}$.

CHAPTER 8

Computing $\mathcal{B}_{\mathbf{X}}$ for $X = p + iq$

In this chapter, we follow the outline for computing $\mathcal{B}_{\mathbf{X}}$ when p and q have 2 atoms.

Recall that we may assume that:

$$\begin{aligned} \mu_p &= a\delta_\alpha + (1-a)\delta_{\alpha'} \\ \mu_q &= b\delta_\beta + (1-b)\delta_{\beta'}\,, \end{aligned} \tag{8.1}$$

for $a, b \in (0,1)$, $\alpha_n, \alpha'_n, \beta_n, \beta'_n \in \mathbb{R}$, $\alpha \neq \alpha'$, and $\beta \neq \beta'$.

8.1 Notation and conventions

Let us highlight some notation and conventions we will use for the rest of the thesis:

The notation $\sqrt{z}$ will always denote the principal square root, defined on $\mathbb{C} \setminus (-\infty, 0)$ and taking $\sqrt{1} = +1$.

We will also use the following definitions:

Definition 8.1. *Let D_p, D_q be the following polynomials:*

$$\begin{aligned} D_p(w) &= ((\alpha' - \alpha)w + (1-2a))^2 + 4a(1-a) \\ D_q(w) &= ((\beta' - \beta)w + (1-2b))^2 + 4b(1-b)\,. \end{aligned} \tag{8.2}$$

Definition 8.2. *Let I_p, I_q be the following subsets of $\mathbb{C}$:*

$$I_p = \left\{ -\frac{1-2a}{\alpha' - \alpha} + iy : |y| > \frac{2\sqrt{a(1-a)}}{|\alpha' - \alpha|} \right\}$$
$$I_q = \left\{ -\frac{1-2b}{\beta' - \beta} + iy : |y| > \frac{2\sqrt{b(1-b)}}{|\beta' - \beta|} \right\}. \tag{8.3}$$

8.2 Auxiliary functions

Before computing the relevant functions, we will prove some lemmas about some related functions:

Lemma 8.3. *For $w \in \mathbb{C} \setminus I_p$, let*

$$f(w) = \sqrt{D_p(w)}\,. \tag{8.4}$$

Then, f is continuous on $\mathbb{C} \setminus I_p$ and analytic on $\mathbb{C} \setminus \overline{I_p}$.

Let $\operatorname{sgn}(x) = x/|x|$ *for $x \in \mathbb{R} \setminus \{0\}$.*

Let $w_0 \in I_p$.

If w approaches w_0 from the right, then

$$\lim_{w \to w_0^+} f(w) = \operatorname{sgn}(\operatorname{Im}(w_0)) i \sqrt{-D_p(w_0)}\,. \tag{8.5}$$

If w approaches w_0 from the left, then

$$\lim_{w \to w_0^-} f(w) = -\operatorname{sgn}(\operatorname{Im}(w_0)) i \sqrt{-D_p(w_0)}\,. \tag{8.6}$$

Proof. Note that f is a composition of the principal square root with $D_p(w)$. $D_p(w)$ is well-defined and continuous everywhere, the principal square root is well-defined and continuous except at $\mathbb{C} \setminus (-\infty, 0)$ and analytic on $\mathbb{C} \setminus (-\infty, 0]$. The limit to some $w \in (-\infty, 0)$ from above is $+\sqrt{-w}i$ and the limit from below is $-\sqrt{-w}i$.

Hence, f is defined and continuous except when:

$$\begin{aligned}
D_p(w) &= ((\alpha' - \alpha)w + (1-2a))^2 + 4a(1-a) \in (-\infty, 0) \\
\iff &((\alpha' - \alpha)w + (1-2a))^2 \in (-\infty, -4a(1-a)) \\
\iff &(\alpha' - \alpha)w + (1-2a) \in \left\{iy : |y| > 2\sqrt{a(1-a)}\right\} \\
\iff &(\alpha' - \alpha)w \in \left\{-(1-2a) + iy : |y| > 2\sqrt{a(1-a)}\right\} \\
\iff &w \in \left\{-\frac{1-2a}{\alpha' - \alpha} + iy : |y| > \frac{2\sqrt{a(1-a)}}{|\alpha' - \alpha|}\right\} \\
\iff &w \in I_p\,.
\end{aligned} \tag{8.7}$$

For analyticity,

$$D_p^{-1}(\{0\}) = \left\{-\frac{1-2a}{\alpha' - \alpha} \pm i\frac{2\sqrt{a(1-a)}}{|\alpha' - \alpha|}\right\}, \tag{8.8}$$

so we just need to remove those two points from the domain to make f analytic.

For the last points on limits approaching w_0, it follows from casework considering cases of $\operatorname{sgn}(\alpha' - \alpha)$, $\operatorname{sgn}(\operatorname{Im}(w_0))$, and w approaching w_0 from the left or right. Let us just verify the case when $\operatorname{sgn}(\alpha' - \alpha) = \operatorname{sgn}(\operatorname{Im}(w_0)) = +1$ and w approaches w_0 from the right.

Then, $(\alpha' - \alpha)w + (1-2a)$ approaches $(\alpha' - \alpha)w_0 + (1-2a) \in i\mathbb{R}$ from the right. Since $\operatorname{Im}((\alpha' - \alpha)w_0 + (1-2a)) > 0$, then $D_p(w)$ approaches $D_p(w_0) \in (-\infty, 0)$ from above. Finally, taking square roots, $f(w)$ approaches $i\sqrt{-D_p(w_0)}$ (from the right). $\square$

Lemma 8.4. *For $w \in \mathbb{C} \setminus (I_p \cup \mathbb{R})$, let*

$$f(w) = \frac{\sqrt{D_p(w)} - \sqrt{D_p(\overline{w})}}{w - \overline{w}} = \frac{\operatorname{Im}\left(\sqrt{D_p(w)}\right)}{\operatorname{Im}(w)}\,. \tag{8.9}$$

The formula for f is continuous on $\mathbb{C} \setminus (I_p \cup \mathbb{R})$ and can be extended continuously to $\mathbb{R}$ by:

$$f(t) = \frac{(\alpha' - \alpha)((\alpha' - \alpha)t + (1-2a))}{\sqrt{D_p(t)}}\,. \tag{8.10}$$

Let $w_0 \in I_p$.

If w approaches w_0 from the right, then

$$\lim_{w \to w_0^+} f(w) = \frac{\sqrt{-D_p(w_0)}}{|\mathrm{Im}(w_0)|}\,. \tag{8.11}$$

If w approaches w_0 from the left, then

$$\lim_{w \to w_0^-} f(w) = -\frac{\sqrt{-D_p(w_0)}}{|\mathrm{Im}(w_0)|}\,. \tag{8.12}$$

Proof. From Lemma 8.3 and the fact that I_p is fixed by complex conjugation, $\sqrt{D_p(w)}$, $\sqrt{D_p(\overline{w})}$ are defined and continuous except on I_p. Since the denominator vanishes on $\mathbb{R}$, the expression for f is valid and continuous on $\mathbb{C} \setminus (I_p \cup \mathbb{R})$.

Consider $w \in \mathbb{C} \setminus (I_p \cup \mathbb{R})$ approaching some $t \in \mathbb{R}$.

As D_p is a polynomial with real coefficients, from the property of the principal square root,

$$\sqrt{D_p(\overline{w})} = \sqrt{\overline{D_p(w)}} = \overline{\sqrt{D_p(w)}}\,. \tag{8.13}$$

Hence,

$$f(w) = \frac{\mathrm{Im}\left(\sqrt{D_p(w)}\right)}{\mathrm{Im}(w)}\,. \tag{8.14}$$

By taking real parts of $\left(\sqrt{D_p(w)}\right)^2 = D_p(w)$,

$$\begin{aligned}
&2\,\mathrm{Re}\left(\sqrt{D_p(w)}\right)\mathrm{Im}\left(\sqrt{D_p(w)}\right)\\
&= \mathrm{Im}(D_p(w))\\
&= \mathrm{Im}(((\alpha'-\alpha)w + (1-2a))^2 + 4a(1-a))\\
&= \mathrm{Im}(((\alpha'-\alpha)w + (1-2a))^2)\\
&= 2\,\mathrm{Re}((\alpha'-\alpha)w + (1-2a))\,\mathrm{Im}((\alpha'-\alpha)w + (1-2a))\\
&= 2((\alpha'-\alpha)\mathrm{Re}(w) + (1-2a))(\alpha'-\alpha)\mathrm{Im}(w)\,.
\end{aligned} \tag{8.15}$$

Dividing both sides by $2\,\mathrm{Im}(w)$,

$$\mathrm{Re}\left(\sqrt{D_p(w)}\right)\frac{\mathrm{Im}\left(\sqrt{D_p(w)}\right)}{\mathrm{Im}(w)} = ((\alpha'-\alpha)\mathrm{Re}(w)+(1-2a))(\alpha'-\alpha)\,. \tag{8.16}$$

As $w \to t \in \mathbb{R}$, $D_p(w)$ converges to $D_p(t) \geq 4a(1-a) > 0$ since $a \in (0,1)$. Thus, it is possible to divide both sides by $\mathrm{Re}\left(\sqrt{D_p(w)}\right)$ for w sufficiently close to t. Then,

$$f(w) = \frac{(\alpha'-\alpha)((\alpha'-\alpha)\mathrm{Re}(w)+(1-2a))}{\mathrm{Re}\left(\sqrt{D_p(w)}\right)}\,. \tag{8.17}$$

Finally, taking $w \to t \in \mathbb{R}$,

$$\lim_{w\to t} f(w) = \frac{(\alpha'-\alpha)((\alpha'-\alpha)t+(1-2a))}{\sqrt{D_p(t)}}\,. \tag{8.18}$$

This final expression is continuous on $\mathbb{R}$, so f may be extended continuously to $\mathbb{R}$ by this expression.

For the final limits towards I_p, using (8.14) and Lemma 8.3,

$$\begin{aligned}\lim_{w\to w_0^\pm} f(w) &= \lim_{w\to w_0^\pm}\frac{\mathrm{Im}\left(\sqrt{D_p(w)}\right)}{\mathrm{Im}(w)}\\ &= \pm\frac{\mathrm{sgn}(\mathrm{Im}(w_0))\sqrt{-D_p(w_0)}}{\mathrm{Im}(w_0)}\\ &= \pm\frac{\sqrt{-D_p(w_0)}}{|\mathrm{Im}(w_0)|}\,.\end{aligned} \tag{8.19}$$

□

Lemma 8.5. *For $w \in \mathbb{C}\setminus(I_p\cup\mathbb{R})$, let*

$$f(w) = \frac{\overline{w}\sqrt{D_p(w)} - w\sqrt{D_p(\overline{w})}}{w-\overline{w}} = \frac{\mathrm{Im}(\overline{w}\sqrt{D_p(w)})}{\mathrm{Im}(w)}\,, \tag{8.20}$$

The formula for f is continuous on $\mathbb{C} \setminus (I_p \cup \mathbb{R})$ and can be extended continuously to $\mathbb{R}$ by:

$$f(t) = \frac{-(1-2a)(\alpha' - \alpha)t - 1}{\sqrt{D_p(t)}}\,. \tag{8.21}$$

Let $w_0 \in I_p$.

If w approaches w_0 from the right, then

$$\lim_{w \to w_0^+} f(w) = -\frac{(1-2a)\sqrt{-D_p(w_0)}}{(\alpha' - \alpha)\,|\mathrm{Im}(w_0)|}\,. \tag{8.22}$$

If w approaches w_0 from the left, then

$$\lim_{w \to w_0^-} f(w) = \frac{(1-2a)\sqrt{-D_p(w_0)}}{(\alpha' - \alpha)\,|\mathrm{Im}(w_0)|}\,. \tag{8.23}$$

Hence, f extends continuously to I_p if and only if $a = 1/2$, in which case letting $f \equiv 0$ on I_p makes f continuous on $\mathbb{C}$.

Proof. From Lemma 8.3 and the fact that I_p is fixed by complex conjugation, $\sqrt{D_p(w)}$, $\sqrt{D_p(\overline{w})}$ are defined and continuous except on I_p. Since the denominator vanishes on $\mathbb{R}$, the expression for f is valid and continuous on $\mathbb{C} \setminus (I_p \cup \mathbb{R})$.

Consider $w \in \mathbb{C} \setminus (I_p \cup \mathbb{R})$ approaching some $t \in \mathbb{R}$.

As D_p is a polynomial with real coefficients, from the property of the principal square root,

$$\sqrt{D_p(\overline{w})} = \sqrt{\overline{D_p(w)}} = \overline{\sqrt{D_p(w)}}\,. \tag{8.24}$$

Hence,

$$f(w) = \frac{\mathrm{Im}(\overline{w}\sqrt{D_p(w)})}{\mathrm{Im}(w)}\,. \tag{8.25}$$

Expanding the numerator,

$$\begin{aligned}\operatorname{Im}\left(\overline{w}\sqrt{D_p(w)}\right) &= \operatorname{Im}(\overline{w})\operatorname{Re}\left(\sqrt{D_p(w)}\right) + \operatorname{Re}(\overline{w})\operatorname{Im}\left(\sqrt{D_p(w)}\right) \\ &= -\operatorname{Im}(w)\operatorname{Re}\left(\sqrt{D_p(w)}\right) + \operatorname{Re}(w)\operatorname{Im}\left(\sqrt{D_p(w)}\right) .\end{aligned} \tag{8.26}$$

Dividing both sides by $\operatorname{Im}(w)$,

$$f(w) = -\operatorname{Re}\left(\sqrt{D_p(w)}\right) + \frac{\operatorname{Re}(w)\operatorname{Im}\left(\sqrt{D_p(w)}\right)}{\operatorname{Im}(w)} . \tag{8.27}$$

The right-hand side contains the function in Lemma 8.4. As this function has a continuous extension to $\mathbb{R}$, taking the limit as $w \to t \in \mathbb{R}$ and simplifying yields:

$$\begin{aligned}\lim_{w\to t} f(w) &= -\sqrt{D_p(t)} + \frac{t(\alpha'-\alpha)((\alpha'-\alpha)t + (1-2a))}{\sqrt{D_p(t)}} \\ &= \frac{-(1-2a)(\alpha'-\alpha)t - 1}{\sqrt{D_p(t)}} .\end{aligned} \tag{8.28}$$

This expression is continuous on $\mathbb{R}$, so f may be extended continuously to $\mathbb{R}$ by this expression.

For the limits as w approaches $w_0 \in I_p$, note that

$$\lim_{w\to w_0} \operatorname{Re}\left(\sqrt{D_p(w)}\right) = 0 \qquad \lim_{w\to w_0} \operatorname{Re}(w) = -\frac{1-2a}{\alpha'-\alpha} . \tag{8.29}$$

Applying Lemma 8.4 to (8.27) produces:

$$\begin{aligned}\lim_{w\to w_0^\pm} f(w) &= \lim_{w\to w_0^\pm} -\operatorname{Re}\left(\sqrt{D_p(w)}\right) + \lim_{w\to w_0^\pm} \operatorname{Re}(w) \lim_{w\to w_0^\pm} \frac{\operatorname{Im}\left(\sqrt{D_p(w)}\right)}{\operatorname{Im}(w)} \\ &= 0 - \frac{1-2a}{\alpha'-\alpha} \lim_{w\to w_0^\pm} \frac{\operatorname{Im}\left(\sqrt{D_p(w)}\right)}{\operatorname{Im}(w)} \\ &= \mp\frac{(1-2a)\sqrt{-D_p(w_0)}}{(\alpha'-\alpha)\left|\operatorname{Im}(w_0)\right|} .\end{aligned} \tag{8.30}$$

The left/right limits are negatives of each other, so they are equal if and only if both are zero. As $D_p(w_0) < 0$ for $w_0 \in I_p$, the limits are zero if and only if $a = 1/2$. In this situation, defining $f \equiv 0$ on I_p makes f continuous on all of $\mathbb{C}$. $\square$

8.3 Computing $\mathcal{B}_{\mathbf{X}}$

In this section, we follow the steps described in Section 7.4 to compute $\mathcal{B}_{\mathbf{X}}$. Since the computations for $\{B_p, \beta_p, \beta_p', \mathcal{B}_{\mathbf{p}}\}$ and $\{B_q, \beta_q, \beta_q', \mathcal{B}_{\mathbf{q}}\}$ are analogous, for these functions, we will just prove and state the results for p.

The first step is to compute B_p:

Proposition 8.6. *For $w \in \mathbb{C} \setminus (I_p \cup \{0\})$, define*

$$B_p(w) = \frac{\alpha + \alpha'}{2} + \frac{1 + \sqrt{D_p(w)}}{2w}\,. \tag{8.31}$$

For z in a neighborhood of infinity, $B_p(w) = G_p^{\langle -1 \rangle}(w) = z$. Further, B_p is continuous on $\mathbb{C} \setminus (I_p \cup \{0\})$ and analytic on $\mathbb{C} \setminus (\overline{I_p} \cup \{0\})$.

Let $w_0 \in I_p$.

If w approaches w_0 from the right, then

$$\lim_{w \to w_0^+} B_p(w) = \frac{\alpha + \alpha'}{2} + \frac{1 + \operatorname{sgn}(\operatorname{Im}(w_0)) i \sqrt{-D_p(w_0)}}{2w_0}\,. \tag{8.32}$$

If w approaches w_0 from the left, then

$$\lim_{w \to w_0^-} B_p(w) = \frac{\alpha + \alpha'}{2} + \frac{1 - \operatorname{sgn}(\operatorname{Im}(w_0)) i \sqrt{-D_p(w_0)}}{2w_0}\,. \tag{8.33}$$

Additionally,

$$\lim_{w \to 0} |B_p(w)| = \infty\,. \tag{8.34}$$

Proof. As

$$\mu_p = a\delta_\alpha + (1-a)\delta_{\alpha'}\,, \tag{8.35}$$

then the complex Green's function for p is given by:

$$w = G_p(z) = \tau[(z-p)^{-1}] = \frac{a}{z-\alpha} + \frac{1-a}{z-\alpha'}\,. \tag{8.36}$$

Recall that in general, $G_p(z)$ is invertible in a neighborhood of ∞. To invert $G_p(z)$, note that (8.36) holds if and only if

$$w(z-\alpha)(z-\alpha') = a(z-\alpha') + (1-a)(z-\alpha)\,. \tag{8.37}$$

This is true for $z \neq \alpha, \alpha'$ and the polynomial equation is not satisfied at $z = \alpha, \alpha'$ at any w. Hence, the solutions to the polynomial equation are solutions to $w = G_p(z)$ for all $z \in \mathbb{C}$.

The polynomial equation can be rewritten:

$$wz^2 - (w(\alpha+\alpha') + 1)z + (w\alpha\alpha' + \alpha\alpha' + (1-a)\alpha) = 0\,. \tag{8.38}$$

Fixing a w and solving for z, then from the quadratic formula and simplifying,

$$\begin{aligned} z &= \frac{\alpha+\alpha'}{2} + \frac{1 \pm \sqrt{((\alpha-\alpha')w + (1-2a))^2 + 4a(1-a)}}{2w} \\ &= \frac{\alpha+\alpha'}{2} + \frac{1 \pm \sqrt{D_p(w)}}{2w}\,. \end{aligned} \tag{8.39}$$

To determine the sign of the square root of the inverse defined for z in a neighborhood of infinity, note that for $\lim_{|z|\to\infty} G_p(z) = 0$. When the sign is $-$, the quotient involving the square root is $1/2$ times the derivative of $\sqrt{D_p(w)}$ at $w = 0$, which is finite as $w \to 0$. Hence, the inverse of $G_p(z)$ for z in a neighborhood of ∞ must take the $+$ sign. Hence,

$$B_p(w) = \frac{\alpha+\alpha'}{2} + \frac{1 + \sqrt{D_p(w)}}{2w} \tag{8.40}$$

and

$$\lim_{w\to 0} |B_p(w)| = \infty\,. \tag{8.41}$$

From Lemma 8.3, the formula in (8.39) is defined and continuous on $\mathbb{C} \setminus (I_p \cup \{0\})$ and analytic on $\mathbb{C} \setminus (\overline{I_p} \cup \{0\})$. As this set is connected and the formula forms an inverse of $G_p(z)$ for z in a neighborhood of infinity, then this formula is an inverse for $G_p(z)$ on its domain. Hence,

$$B_p(w) = \frac{\alpha + \alpha'}{2} + \frac{1 + \sqrt{D_p(w)}}{2w}. \tag{8.42}$$

From Lemma 8.3, B_p and the fact that B_p is an inverse for G_p in a neighborhood of $w = 0$, then B_p is defined and continuous on $\mathbb{C} \setminus (I_p \cup \{0\})$ and analytic on $\mathbb{C} \setminus (\overline{I_p} \cup \{0\})$. The left/right limits of B_p can also be computed using Lemma 8.3. $\square$

Next, we compute β_p, β_p' and determined the domains where they are well-defined and continuous. The results for β_q, β_q' are completely analogous.

Proposition 8.7. *For $w \in \mathbb{C} \setminus (I_p \cup \mathbb{R})$,*

$$\beta_p(g) = \frac{\alpha + \alpha'}{2} + \frac{1}{2}\frac{\sqrt{D_p(g)} - \sqrt{D_p(\overline{g})}}{g - \overline{g}}. \tag{8.43}$$

β_p can be extended to be continuously to $\mathbb{R}$ by:

$$\beta_p(t) = \frac{\alpha + \alpha'}{2} + \frac{1}{2}\frac{(\alpha' - \alpha)((\alpha' - \alpha)t + (1 - 2a))}{\sqrt{D_p(t)}}. \tag{8.44}$$

Let $g_0 \in I_p$.

If g approaches g_0 from the right, then

$$\lim_{g \to g_0^+} \beta_p(g) = \frac{\alpha + \alpha'}{2} + \frac{\sqrt{-D_p(g_0)}}{2\,|\mathrm{Im}(g_0)|}. \tag{8.45}$$

If g approaches g_0 from the left, then

$$\lim_{g \to g_0^-} \beta_p(g) = \frac{\alpha + \alpha'}{2} - \frac{\sqrt{-D_p(g_0)}}{2\,|\mathrm{Im}(g_0)|}. \tag{8.46}$$

Proof. Computation using (7.66) and Proposition (8.6) yields

$$\beta_p(g) = \frac{\alpha + \alpha'}{2} + \frac{1}{2} \frac{\sqrt{D_p(g)} - \sqrt{D_p(\overline{g})}}{g - \overline{g}}\,. \tag{8.47}$$

Everything else follows from the fact that

$$\beta_p(g) = \frac{\alpha + \alpha'}{2} + \frac{1}{2} f(g)\,, \tag{8.48}$$

where f is from Lemma 8.4 □

Proposition 8.8. *For $w \in \mathbb{C} \setminus (I_p \cup \mathbb{R})$,*

$$\beta'_p(g) = \frac{1}{2\,|g|^2} \left(-1 + \frac{\overline{g}\sqrt{D_p(g)} - g\sqrt{D_p(\overline{g})}}{g - \overline{g}} \right). \tag{8.49}$$

β'_p can be extended continuously to $\mathbb{R} \setminus \{0\}$ by:

$$\beta'_p(t) = \frac{1}{2t^2} \left(-1 + \frac{-(1-2a)(\alpha' - \alpha)t - 1}{\sqrt{D_p(t)}} \right). \tag{8.50}$$

At $g = 0$,

$$\lim_{g \to 0} \beta'_p(g) = -\infty\,. \tag{8.51}$$

Let $g_0 \in I_p$.

If g approaches g_0 from the right, then

$$\lim_{g \to g_0^+} \beta'_p(g) = \frac{1}{2\,|g_0|^2} \left(-1 - \frac{(1-2a)\sqrt{-D_p(g_0)}}{(\alpha' - \alpha)\,|\mathrm{Im}(g_0)|} \right). \tag{8.52}$$

If g approaches g_0 from the left, then

$$\lim_{g \to g_0^-} \beta'_p(g) = \frac{1}{2\,|g_0|^2} \left(-1 + \frac{(1-2a)\sqrt{-D_p(g_0)}}{(\alpha' - \alpha)\,|\mathrm{Im}(g_0)|} \right). \tag{8.53}$$

β'_p can be extended continuously to I_p if and only if $a = 1/2$, in which case on I_p,

$$\beta'_p(t) = -\frac{1}{2\,|g|^2}\,. \tag{8.54}$$

Proof. Computation using (7.66) and Proposition (8.6) yields

$$\beta'_p(g) = \frac{1}{2\,|g|^2}\left(-1 + \frac{\overline{g}\sqrt{D_p(g)} - g\sqrt{D_p(\overline{g})}}{g - \overline{g}}\right). \tag{8.55}$$

Most of the claims for the domains where β' is well-defined and continuous follows from the fact that

$$\beta'_p(g) = \frac{1}{2\,|g|^2}\left(-1 + f(g)\right)\,, \tag{8.56}$$

where f is from Lemma 8.5.

The only extra complication is $g = 0$. For this, we use Lemma 8.5 to see that the limit of the term of $\beta'_p(g)$ inside the parentheses is:

$$\begin{aligned}\lim_{g\to 0} -1 + \frac{\overline{g}\sqrt{D_p(g)} - g\sqrt{D_p(\overline{g})}}{g - \overline{g}} &= -1 + \left.\frac{-(1-2a)(\alpha' - \alpha)t - 1}{\sqrt{D_p(t)}}\right|_{t=0} \\ &= -1 + \frac{-1}{\sqrt{D_p(0)}}\,.\end{aligned} \tag{8.57}$$

This final quantity is negative, so $\beta'_p(g) \to -\infty$ as $g \to 0$. Hence, $g = 0$ is not in the domains of definition and continuity for β_p.

The left and right limits towards some $g_0 \in I_p$ follow directly from Lemma 8.5. $\square$

Next, we compute $\mathcal{B}_{\mathbf{p}}$ and $\mathcal{B}_{i\mathbf{q}}$ and determine the domains where they are well-defined and continuous. In particular, these domains agree with the ones from Propositions 7.17 and 7.18.

When we refer to $\mathcal{B}_{\mathbf{p}}$ (resp $\mathcal{B}_{i\mathbf{q}}$) being continuous for $g \in S$ for some $S \subset \mathbb{C}$, it meant to be understood as $\mathcal{B}_{\mathbf{p}}$ (resp $\mathcal{B}_{i\mathbf{q}}$) being continuous for quaternions Q where its eigenvalue $g \in S$. The well-definedness and continuity of $\mathcal{B}_{\mathbf{p}}(Q)$ depends mainly on the well-defined and continuous of β_p, β'_p at g. This is summarized in the following Proposition:

Proposition 8.9. *For $Q \in \mathbb{H}$ such that $g \in \mathbb{C} \setminus (I_p \cup \{0\})$,*

$$\mathcal{B}_{\mathbf{p}}(Q) = \begin{cases} B_p(g_0) & Q = g_0 \in \mathbb{C} \\ \beta_p(g) - \beta'_p(g) Q^* & Q \notin \mathbb{R}\,, \end{cases} \tag{8.58}$$

where

$$\begin{aligned} B_p(g_0) &= \frac{\alpha+\alpha'}{2} + \frac{1+\sqrt{D_p(g_0)}}{2g_0} \\ \beta_p(g) &= \begin{cases} \dfrac{\alpha+\alpha'}{2} + \dfrac{1}{2}\dfrac{\sqrt{D_p(g)}-\sqrt{D_p(\overline{g})}}{g-\overline{g}} & g \in \mathbb{C} \setminus (I_p \cup \mathbb{R}) \\ \dfrac{\alpha+\alpha'}{2} + \dfrac{1}{2}\dfrac{(\alpha'-\alpha)((\alpha'-\alpha)t + (1-2a))}{\sqrt{D_p(t)}} & g = t \in \mathbb{R} \end{cases} \\ \beta'_p(g) &= \begin{cases} \dfrac{1}{2\,|g|^2}\left(-1 + \dfrac{\overline{g}\sqrt{D_p(g)} - g\sqrt{D_p(\overline{g})}}{g-\overline{g}}\right) & g \in \mathbb{C} \setminus (I_p \cup \mathbb{R}) \\ \dfrac{1}{2t^2}\left(-1 + \dfrac{-(1-2a)(\alpha'-\alpha)t - 1}{\sqrt{D_p(t)}}\right) & g = t \in \mathbb{R} \setminus \{0\}\,. \end{cases} \end{aligned} \tag{8.59}$$

Then, $\mathcal{B}_{\mathbf{p}}$ is continuous on this subset of $\mathbb{H}$ and in no larger domain, and

$$\lim_{Q \to 0} |\mathcal{B}_{\mathbf{p}}(Q)| = \infty\,. \tag{8.60}$$

Proof. The formulas follow from (7.65), (7.66) and Propositions 8.7 and 8.8.

For the continuity points of $\mathcal{B}_{\mathbf{p}}$, it follows from the formulas for β_p and β'_p that $\mathcal{B}_{\mathbf{p}}$ is well-defined and continuous for $g \in \mathbb{C} \setminus (I_p \cup \mathbb{R})$. From Propositions 8.7 and 8.8, $\mathcal{B}_{\mathbf{p}}$ does have a limit as g approaches a non-zero real number. But, $\mathcal{B}_{\mathbf{p}}$ is already defined on $\mathbb{R} \setminus \{0\}$. It is a straightforward computation to verify that:

$$\lim_{Q \to g_0 \in \mathbb{R}\setminus\{0\}} \mathcal{B}_{\mathbf{p}}(Q) = \lim_{g \to g_0} \beta_p(g) - \beta'_p(g) Q^* = B_p(g_0)\,, \tag{8.61}$$

so that $\mathcal{B}_{\mathbf{p}}$ is continuous on $\mathbb{C} \setminus (I_p \cup \{0\})$.

To analyze $\mathcal{B}_{\mathbf{p}}(Q)$ as $Q \to 0$, recall from Proposition 7.17 that the eigenvalues of $\mathcal{B}_{\mathbf{p}}(Q)$ are $B_p(g), B_p(\overline{g})$. As $Q \to 0$, $g \to 0$. From Proposition 8.6, as $g \to 0$, $|B_p(g)| \to \infty$. Hence, the eigenvalues of $\mathcal{B}_{\mathbf{p}}(Q)$ diverge to ∞. Hence, as $Q \to 0$, $|\mathcal{B}_{\mathbf{p}}(Q)| \to \infty$.

To analyze the limit as Q approaches Q_0 where g tends to $g_0 \in I_p$, it suffices to consider the difference between the left and right limits as g approaches $g_0 \in I_p$ for $\mathcal{B}_{\mathbf{p}}$:

$$\lim_{g \to g_0^+} \mathcal{B}_{\mathbf{p}}(Q) - \lim_{g \to g_0^-} \mathcal{B}_{\mathbf{p}}(Q)\,. \tag{8.62}$$

From Propositions 7.17, 8.7, and 8.8, the diagonal term of this difference is:

$$\begin{aligned}\frac{\sqrt{-D_p(g_0)}}{|\mathrm{Im}(g_0)|} + \frac{(1-2a)\sqrt{-D_p(g_0)}}{(\alpha'-\alpha)\,|g_0|^2\,|\mathrm{Im}(g_0)|}\overline{A} &= \frac{\sqrt{-D_p(g_0)}}{|\mathrm{Im}(g_0)|}\left(1 + \frac{1-2a}{\alpha'-\alpha}\frac{A}{|g_0|^2}\right) \\ &= \frac{\sqrt{-D_p(g_0)}}{|\mathrm{Im}(g_0)|}\left(1 - \frac{\mathrm{Re}(g_0)}{|g_0|}\frac{A}{|g_0|}\right)\,.\end{aligned} \tag{8.63}$$

The factor in front of the last term is non-zero, so it suffices to analyze when the term inside the parentheses is zero. It is clear that $\mathrm{Re}(g_0) \leq |g_0|$, with equality occurring only when g_0 is real. Recall that $g_0 = \sqrt{|A|^2 + |B|^2}$, so $|A| \leq |g_0|$. Hence, the term inside the parentheses can be zero only when g_0 is real. As $I_p \cap \mathbb{R} = \emptyset$, this quantity is non-zero, so $\mathcal{B}_{\mathbf{p}}$ is discontinuous at I_p. $\square$

The results for $\mathcal{B}_{i\mathbf{q}}$ follow immediately from Propositions 7.18 and 8.9:

Proposition 8.10. *For $Q \in \mathbb{H}$ such that $g^I \in \mathbb{C} \setminus (I_p \cup \{0\})$,*

$$\mathcal{B}_{i\mathbf{q}}(Q) = \begin{cases} iB_q(ig_0) & Qi = ig_0 \in \mathbb{C} \\ i\beta_q(g^I) - \beta_q'(g^I)Q^* & Qi \notin \mathbb{R}\,, \end{cases} \tag{8.64}$$

where

$$
\begin{aligned}
B_q(ig_0) &= \frac{\beta+\beta'}{2} + \frac{1+\sqrt{D_q(ig_0)}}{2ig_0}\,,\\
\beta_q(g^I) &= \begin{cases} \dfrac{\beta+\beta'}{2} + \dfrac{1}{2}\dfrac{\sqrt{D_q(g^I)}-\sqrt{D_q(\overline{g^I})}}{g^I-\overline{g^I}} & g^I \in \mathbb{C}\setminus(I_p\cup\mathbb{R})\\ \dfrac{\beta+\beta'}{2} + \dfrac{1}{2}\dfrac{(\beta'-\beta)((\beta'-\beta)t+(1-2b))}{\sqrt{D_q(t)}} & g^I = t\in\mathbb{R} \end{cases}\\
\beta'_q(g^I) &= \begin{cases} \dfrac{1}{2\left|g^I\right|^2}\left(-1+\dfrac{\overline{g^I}\sqrt{D_q(g^I)}-g^I\sqrt{D_q(\overline{g^I})}}{g^I-\overline{g^I}}\right) & g^I\in\mathbb{C}\setminus(I_q\cup\mathbb{R})\\ \dfrac{1}{2t^2}\left(-1+\dfrac{-(1-2b)(\beta'-\beta)t-1}{\sqrt{D_q(t)}}\right) & g^I = t\in\mathbb{R}\setminus\{0\}\,. \end{cases}
\end{aligned}
\tag{8.65}
$$

Then, $\mathcal{B}_{i\mathbf{q}}$ is continuous on this subset of $\mathbb{H}$ and in no larger domain, and

$$\lim_{Q\to 0}|\mathcal{B}_{i\mathbf{q}}(Q)| = \infty\,. \tag{8.66}$$

All that is left is to determine a formula for $\mathcal{B}_{\mathbf{X}}$ and determine the domains where it is defined and continuous.

First, let us state the formula and domains of well-definedness and continuity of the complex Green's function B_X:

Proposition 8.11. *For $w\in\mathbb{C}$ such that $w\notin I_p$ and $iw\notin I_q$ and $w\neq 0$,*

$$B_X(w) = \frac{\alpha+\alpha'}{2} + i\frac{\beta+\beta'}{2} + \frac{\sqrt{D_p(w)}+\sqrt{D_q(iw)}}{2w} \tag{8.67}$$

is analytic. Further, B_X cannot be continuously extended to any larger domain. Additionally,

$$\lim_{w\to 0} wB_X(w) = 1\,. \tag{8.68}$$

Proof. From the addition law for the Quaternionic R-transform,

$$\mathcal{B}_{\mathbf{X}}(Q) = \mathcal{B}_{\mathbf{p}}(Q) + \mathcal{B}_{i\mathbf{q}}(Q) - Q^{-1}\,. \tag{8.69}$$

Using Proposition 7.15 to restrict this formula $w \in \mathbb{C}$ (and taking note of the domains of $\mathcal{B}_{\mathbf{p}}$ and $\mathcal{B}_{i\mathbf{q}}$ from Propositions 8.9 and 8.10), then

$$B_X(w) = B_p(w) + iB_q(iw) - w^{-1}\,, \tag{8.70}$$

Then, the formula follows from Proposition 8.6.

For continuity and analyticity, we need to check what happens if $w \in I_p$ and $iw \in I_q$, in case the discontinuities of $\sqrt{D_p(w)}$ and $\sqrt{D_q(iw)}$ cancel out.

We may choose different sequences approaching w such that all four combinations of limits from applying Lemma 8.3 to p and q are possible:

$$\frac{\alpha+\alpha'}{2} + i\frac{\beta+\beta'}{2} \pm i\frac{\sqrt{-D_p(w)}}{2w} \pm i\frac{\sqrt{-D_q(iw)}}{2w}\,. \tag{8.71}$$

Setting pairs of these expressions equal to each other, then for B_X to be continuous at w,

$$-D_p(w) = -D_q(iw) = 0\,. \tag{8.72}$$

But this is impossible for $w \in I_p$ and $iw \in I_q$.

Hence, when $w \in I_p$ and $iw \in I_q$, B_X is discontinuous at w.

The limit follows from the fact that B_X is defined in a punctured neighborhood centered at 0 and

$$\begin{aligned} \lim_{|z|\to\infty} G_X(z) &= 0 \\ \lim_{|z|\to\infty} zG_X(z) &= 1\,. \end{aligned} \tag{8.73}$$

□

The addition law (7.49) and Propositions 8.9 and 8.10 imply that $\mathcal{B}_{\mathbf{X}}$ is defined and continuous where $g \notin I_p \cup \{0\}$ and $g^I \notin I_q \cup \{0\}$. In order to show that this domain is the maximal domain of definition for $\mathcal{B}_{\mathbf{X}}$, we need to analyze the following limits more carefully:

1. $g \to 0$ (and hence $g^I \to 0$).

2. $g \to g_0 \in I_p$ and $g^I \to g_0^I \in I_q$ (in case the discontinuities of $\mathcal{B}_{\mathbf{p}}$ and $\mathcal{B}_{i\mathbf{q}}$ to cancel each other out).

We will use (7.79) when $g^I, g \notin \mathbb{R}$. In particular, we need to first analyze $l = l(g, g^I)$:

Proposition 8.12. *Let S_p, S_q be the following subsets of $\mathbb{C}$:*

$$\begin{aligned} S_p &= \begin{cases} I_p \cup \{0\} & a \neq \dfrac{1}{2} \\ \{0\} & a = \dfrac{1}{2} \end{cases} \\ S_q &= \begin{cases} I_q \cup \{0\} & b \neq \dfrac{1}{2} \\ \{0\} & b = \dfrac{1}{2} \end{cases}. \end{aligned} \tag{8.74}$$

For $Q \in \mathbb{H}$ such that $g \in \mathbb{C} \setminus (S_p \cup \mathbb{R})$ or $g^I \in \mathbb{C} \setminus (S_q \cup \mathbb{R})$,

$$l(Q) = l(g, g^I) = \frac{1}{2\,|g|^2} \left(\frac{\overline{g}\sqrt{D_p(g)} - g\sqrt{D_p(\overline{g})}}{g - \overline{g}} + \frac{\overline{g^I}\sqrt{D_q(g^I)} - g^I\sqrt{D_q(\overline{g^I})}}{g^I - \overline{g^I}} \right). \tag{8.75}$$

Then, l can be extended to be continuous at $Q \in \mathbb{H}$ such that $g \in \mathbb{C} \setminus S_p$ or $g^I \in \mathbb{C} \setminus S_q$ and in no larger domain and

$$\lim_{g, g^I \to 0} |g|^2\, l(g, g^I) = -2\,. \tag{8.76}$$

Proof. For $g \in \mathbb{C} \setminus (I_p \cup \mathbb{R})$ and $g^I \in \mathbb{C} \setminus (I_p \cup \mathbb{R})$, using the definition of $l(Q)$ and Proposition 8.8 yields (8.75).

From Lemma 8.5, the formula for l can be continuously extended to at least when $g \in \mathbb{C} \setminus S_p$ and $g^I \in \mathbb{C} \setminus S_q$. If exactly one of $g \in S_p$ or $g^I \in S_q$ then the limit also will not exist, as l will be a sum of two terms, one where the limit exists and one where the limit doesn't.

Thus, we just need to consider what happens when:

1. $g \to 0$ (and hence $g^I \to 0$).
2. $g \to g_0 \in S_p \setminus \{0\}$ and $g^I \to g_0^I \in S_q \setminus \{0\}$. Note that this only makes sense when $a \neq 1/2$ and $b \neq 1/2$, in which case we may assume $g_0 \in I_p$ and $g_0^I \in I_q$.

For the first case when $g \to 0$ (and $g^I \to 0$), using Lemma 8.5, the term inside the parentheses in (8.75) tends towards

$$-\frac{1}{\sqrt{D_p(0)}} - \frac{1}{\sqrt{D_q(0)}} = -1 - 1 = -2\,. \tag{8.77}$$

Hence, as $g \to 0$, $|g|^2\, l(g, g^I) \to -2$, so $l(g, g^I) \to -\infty$.

For the second case when $g_0 \in I_p$ and $g_0^I \in I_q$, recall that

$$\begin{aligned} g_0 &= x_0 + i\sqrt{x_1^2 + x_2^2 + x_3^2} \\ g_0^I &= -x_3 + i\sqrt{x_2^2 + x_1^2 + x_0^2}\,, \end{aligned} \tag{8.78}$$

By choosing if x_0 is approached from left or right and if x_3 is approached from left or right, all four combinations of limits from applying Lemma 8.5 to p and q are possible:

$$\pm\frac{(1-2a)\sqrt{-D_p(g)}}{(\alpha' - \alpha)\,|\mathrm{Im}(g)|} \pm \frac{(1-2b)\sqrt{-D_q(g^I)}}{(\beta' - \beta)\,|\mathrm{Im}(g^I)|}\,. \tag{8.79}$$

By setting pairs of these expressions equal to each other, then for the limit to exist,

$$\frac{(1-2a)\sqrt{-D_p(g)}}{(\alpha' - \alpha)\,|\mathrm{Im}(g)|} = \frac{(1-2b)\sqrt{-D_q(g^I)}}{(\beta' - \beta)\,|\mathrm{Im}(g^I)|} = 0\,. \tag{8.80}$$

However, this can only happen if $a = b = 1/2$, a contradiction. □

We conclude this section by proving the result for $\mathcal{B}_{\mathbf{X}}(Q)$:

Theorem 8.13. *For $Q \in \mathbb{H}$ such that $g \notin I_p \cup \{0\}$ and $g^I \notin I_q \cup \{0\}$,*

$$\mathcal{B}_{\mathbf{X}}(Q) = \mathcal{B}_{\mathbf{X}}\left(\begin{pmatrix} A & i\overline{B} \\ iB & \overline{A} \end{pmatrix}\right) = \begin{pmatrix} k + ik' - l\overline{A} & li\overline{B} \\ liB & k - ik' - lA \end{pmatrix}, \tag{8.81}$$

where

$$\begin{aligned} k &= \beta_p(g) \\ k' &= \beta_q(g^I) \\ l &= \beta_p'(g) + \beta_q'(g^I) + \frac{1}{\det Q}\,. \end{aligned} \tag{8.82}$$

and $\beta_p(g), \beta_q(g)$ are defined in Proposition 8.9, and l is defined in Proposition 8.12.

$\mathcal{B}_{\mathbf{X}}$ is continuous on this subset of $\mathbb{H}$ and in no larger domain, and

$$\lim_{Q \to 0} |\mathcal{B}_{\mathbf{X}}(Q)| = \infty\,. \tag{8.83}$$

Proof. Recall that all that remains is to examine the limit of $\mathcal{B}_{\mathbf{X}}(Q)$ when g, g^I approaches the following limits:

1. $g \to 0$ (and hence $g^I \to 0$).
2. $g \to g_0 \in I_p$ and $g^I \to g_0^I \in I_q$ (in case the discontinuities of $\mathcal{B}_{\mathbf{p}}$ and $\mathcal{B}_{i\mathbf{q}}$ to cancel each other out).

For the first limit, when $g \in \mathbb{R}$ or $g^I \in \mathbb{R}$, then either $Q \in \mathbb{R}$ or $Q \in i\mathbb{R}$, and we can apply Proposition 8.11 to see that $|\mathcal{B}_{\mathbf{X}}(Q)| \to \infty$. When $g, g^I \notin \mathbb{R}$, then we can use (7.79). From Proposition 8.7, k and k' have limits as $g, g^I \to 0$. Hence, it suffices to consider the limit:

$$\lim_{Q \to 0} \begin{pmatrix} l\overline{A} & -li\overline{B} \\ -liB & lA \end{pmatrix} = \lim_{Q \to 0} l \begin{pmatrix} \overline{A} & -i\overline{B} \\ -iB & A \end{pmatrix} = \lim_{Q \to 0} lQ^*. \tag{8.84}$$

Taking quaternionic norms and using Proposition 8.12,

$$\lim_{Q\to 0}|lQ^*| = \lim_{Q\to 0}\left|l\,|Q|^2\,\frac{Q^*}{|Q|^2}\right| = \lim_{Q\to 0}\left|l\,|Q|^2\right|\left|\frac{Q^*}{|Q|^2}\right| = \lim_{Q\to 0}\frac{2}{|Q|} = \infty\,. \tag{8.85}$$

Hence, $\lim_{Q\to 0}|\mathcal{B}_{\mathbf{X}}(Q)| = \infty$.

Finally, consider if Q approaches Q_0 where $g_0 \in I_p$ and $g_0^I \in I_q$.

From the addition law and Proposition 7.18,

$$\mathcal{B}_{\mathbf{X}}(Q) = \mathcal{B}_{\mathbf{p}}(Q) + i\mathcal{B}_{\mathbf{q}}(Qi) - Q^{-1}\,. \tag{8.86}$$

Recall that:

$$\begin{aligned} g_0 &= x_0 + i\sqrt{x_1^2 + x_2^2 + x_3^2} \\ g_0^I &= -x_3 + i\sqrt{x_2^2 + x_1^2 + x_0^2}\,, \end{aligned} \tag{8.87}$$

By choosing if x_0 is approached from left or right and if x_3 is approached from left or right, all four combinations of left/right limits are possible for choices as Q approaches Q_0:

$$\lim_{Q\to Q_0}\mathcal{B}_{\mathbf{X}}(Q) = \lim_{g\to g_0^{\pm}}\mathcal{B}_{\mathbf{p}}(Q) + i\lim_{g^I\to g_0^{I\pm}}\mathcal{B}_{\mathbf{q}}(Qi) - Q_0^{-1}. \tag{8.88}$$

By setting pairs of these expressions equal to each other, then for the limit to exist:

$$\begin{aligned} \lim_{g\to g_0^+}\mathcal{B}_{\mathbf{p}}(Q) - \lim_{g\to g_0^-}\mathcal{B}_{\mathbf{p}}(Q) &= 0 \\ \lim_{g^I\to g_0^{I+}}\mathcal{B}_{\mathbf{q}}(Qi) - \lim_{g\to g_0^{I-}}\mathcal{B}_{\mathbf{q}}(Qi) &= 0\,. \end{aligned} \tag{8.89}$$

From (8.63) in the proof of Proposition 8.9, this cannot happen when $g_0 \in I_p$ or $g_0^I \in I_q$. Hence, $\mathcal{B}_{\mathbf{X}}$ is discontinuous for $g \in I_p \cup \{0\}$ or $g^I \in I_q \cup \{0\}$. $\square$

CHAPTER 9

Boundary of the Brown measure

In this Chapter, we consider Heuristic 7.20 about the boundary of the Brown measure of $X = p + iq$ where $p, q \in (M, \tau)$ are Hermitian and freely independent.

First, we restrict to only considering certain z that satisfy some continuity conditions. Then, we will verify that the heuristic is almost true in the case when p and q have 2 atoms: the closure of the intersection of the sets in the heuristic is the boundary (i.e. support) of μ'.

The main result of this section is that for $X = p + iq$, where p, q have arbitrarily many atoms, the intersection of the two sets in the heuristic is an algebraic curve. Mathematica computations of the empirical spectral distributions of X_n suggest that this algebraic curve contains the boundary of the Brown measure of X.

9.1 General p and q

We first consider a general $X = p + iq$, where p, q are Hermitian and freely independent.

With some restrictions on z, we first present a computation that characterizes the z where both $l_\epsilon \to 0$ and $B_\epsilon \to 0$:

We only consider $z \in \mathbb{C}$ where:

1. $\lim\limits_{\epsilon \to 0} \mathcal{G}_{\mathbf{X}}(z_\epsilon) = \lim\limits_{\epsilon \to 0} Q_\epsilon = Q$ for some $Q \in \mathbb{H}$.

2. $\mathcal{B}_{\mathbf{X}}$ and all related functions are defined and continuous in a neighborhood of Q.

3. $g, g^I \notin \mathbb{R}$ (i.e. $Q \notin \mathbb{R} \cup i\mathbb{R}$).

The first condition is the continuity assumption made in stating the heuristic. The second condition is needed to use $\mathcal{B}_{\mathbf{X}}$. The third condition is relevant in order to use (7.79). In general, since we are taking closures of the intersection of $\{z : l_\epsilon \to 0\}$ and $\{z : B_\epsilon \to B = 0\}$, the third condition should not be significant in recovering almost all of the boundary of the Brown measure. In the case when p and q have two atoms, this condition causes our computation to not recover the atoms of the measure.

The first and second conditions allow us to pass to the limit to see that:

$$\mathcal{B}_{\mathbf{X}}(Q) = \lim_{\epsilon \to 0^+} \mathcal{B}_{\mathbf{X}}(Q_\epsilon) = \lim_{\epsilon \to 0^+} z_\epsilon = z\,. \tag{9.1}$$

By passing to the limit, the condition $l_\epsilon \to 0$ implies that $l(Q) = 0$.

The conditions $B_\epsilon \to 0$ and $g, g^I \notin \mathbb{R}$ imply that $Q \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$.

First, we assume $Q \in \mathbb{H}$ where $l(Q) = 0$ and $\mathcal{B}_{\mathbf{X}}(Q) = z$. Let $z = x + iy$. The third assumption allows us to apply (7.79) to $\mathcal{B}_{\mathbf{X}}(Q) = z$, which yields:

$$\begin{aligned} x = k = \beta_p(g) &= \frac{gB_p(g) - \overline{g}B_p(\overline{g})}{g - \overline{g}} \\ y = k' = \beta_q(g^I) &= \frac{g^I B_q(g^I) - \overline{g^I} B_q(\overline{g^I})}{g - \overline{g^I}}\,. \end{aligned} \tag{9.2}$$

Let m describe the intercept of the (complex) line passing through $(g, gB_p(g))$ and $(\overline{g}, \overline{g}B_p(\overline{g}))$, i.e.

$$\begin{aligned} gB_p(g) &= xg + m \\ \overline{g}B_p(\overline{g}) &= x\overline{g} + m\,. \end{aligned} \tag{9.3}$$

Conjugating the first equation and comparing with the second equation shows that $m \in \mathbb{R}$. Noting that $B_p(\overline{g}) = \overline{B_p(g)}$ and $g \neq 0$, then:

$$x = \beta_p(g) \iff B_p(g) = x + \frac{m}{g} \text{ for some } m \in \mathbb{R}\,. \tag{9.4}$$

Similarly, let m' describe the intercept of the line passing through $(g^I, gB_q(g^I))$ and $(\overline{g^I}, \overline{g^I}B_q(\overline{g^I}))$:

$$\begin{aligned} g^I B_q(g^I) &= y g^I + m' \\ \overline{g^I} B_q(\overline{g^I}) &= y \overline{g^I} + m' \,. \end{aligned} \tag{9.5}$$

Similar arguments show:

$$y = \beta_q(g^I) \iff B_q(g^I) = y + \frac{m'}{g^I} \text{ for some } m' \in \mathbb{R}\,. \tag{9.6}$$

Referring to our original definitions of β_p and l in (7.66) and (7.80),

$$l = \frac{B_p(g) - B_p(\overline{g})}{g - \overline{g}} + \frac{B_q(g^I) - B_q(\overline{g^I})}{g - \overline{g^I}} + \frac{1}{|g|^2}\,. \tag{9.7}$$

Inserting the two equations (9.4) and (9.6) into this expression and simplifying shows that

$$l = 0 \iff m + m' = 1\,. \tag{9.8}$$

Recall that we assumed initially that $g, g^I \notin \mathbb{R}$, i.e. $Q \notin \mathbb{R} \cup i\mathbb{R}$. Hence,

$$\begin{aligned} &\mathcal{B}_{\mathbf{X}}(Q) = z \text{ for } Q \in \mathbb{H} \setminus (\mathbb{R} \cup i\mathbb{R}) \text{ where } l(Q) = 0 \\ &\qquad\iff \\ &B_p(g) = x + \frac{m}{g} \\ &B_q(g^I) = y + \frac{1-m}{g^I} \text{ for some } m \in \mathbb{R}\,. \end{aligned} \tag{9.9}$$

Heuristic 7.19 suggests that the closure of the set of z that satisfy this system of equations should contain the support of the Brown measure of X.

If we impose the additional condition that $B = 0$, then $Q \in \mathbb{C}$. Applying Lemma 7.5 to $B_p(g) - (x + m/g)$ and $B_q(g^I) - (y + (1-m)/g^I)$, we may assume that $Q = g$ and $g^I = ig$.

Then,

$$
\begin{aligned}
&\mathcal{B}_{\mathbf{X}}(g) = z \text{ for } g \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}) \text{ where } l(g) = 0 \\
&\qquad \Longleftrightarrow \\
&B_p(g) = x + \frac{m}{g} \\
&B_q(ig) = y + \frac{1-m}{ig} \text{ for some } m \in \mathbb{R}\,.
\end{aligned}
\tag{9.10}
$$

Thus, Heuristic 7.20 with the continuity assumptions suggests that the closure of the set of z that satisfy this system of equations should be the boundary of the Brown measure of X.

The support of a general measure on $\mathbb{C}$ is 2-dimensional and the boundary of the support of a general measure is 1-dimensional. This agrees with the dimensions of generic solutions to systems of equations with the same number of equations and variables as in (9.9) and (9.10):

Applying G_p to both sides in (9.9), and using $\left|g^I\right|^2 = |g|^2$, the following system of equations contains the solutions to (9.9):

$$
\begin{aligned}
g &= G_p\left(x + \frac{m}{g}\right) \\
g^I &= G_q\left(y + \frac{1-m}{g^I}\right) \\
|g|^2 &= \left|g^I\right|^2 .
\end{aligned}
\tag{9.11}
$$

This is a system of 5 real equations (taking the real/imaginary parts of the first two equations) with 7 real variables ($g, g^I \in \mathbb{C}$, $x, y, m \in \mathbb{R}$). Thus, we expect in general the solution set to be 2-dimensional over $\mathbb{R}$, like the support of a generic measure on $\mathbb{C}$.

Adding the condition $B = 0$, then the system of equations (9.10) is equivalent to:

$$
\begin{aligned}
g &= G_p\left(x + \frac{m}{g}\right) \\
g^I &= G_q\left(y + \frac{1-m}{g^I}\right) . \\
g^I &= ig\,.
\end{aligned}
\tag{9.12}
$$

This system of equations has one more real equation than the previous system, with 6 real equations with 7 real variables. So, the solution set is a subset of the previous solution set and we expect it to be 1-dimensional over $\mathbb{R}$, like the boundary of the support of a generic measure on $\mathbb{C}$.

9.2 When p and q have 2 atoms

We apply the computation of the previous section to the case when p and q have 2 atoms. Our results are that the z that solve the system of equations in (9.9) is a set that is contained in the intersection of the hyperbola and open rectangle associated with $X = p + iq$ (from Definition 4.9). It is unclear whether this set is actually the support of the Brown measure or not, we will discuss the difficulties in determining this. But, adding the extra condition $B = 0$ to produce the system of equations in (9.10) does recover the support of μ'. We will also discuss the atoms of the Brown measure of X.

First, we consider the z that solve the system of equations in (9.9):

Proposition 9.1. *Let $X = p + iq$, where $p, q \in (M, \tau)$ are Hermitian, freely independent, and have 2 atoms, i.e.*

$$\begin{aligned} \mu_p &= a\delta_\alpha + (1-a)\delta_{\alpha'} \\ \mu_q &= b\delta_\beta + (1-b)\delta_{\beta'}\,, \end{aligned} \tag{9.13}$$

where $a, b \in (0,1)$, $\alpha_n, \alpha'_n, \beta_n, \beta'_n \in \mathbb{R}$, $\alpha \neq \alpha'$, and $\beta \neq \beta'$.

The set of $z \in \mathbb{C}$ such that

$$\mathcal{B}_{\mathbf{X}}(Q) = z \tag{9.14}$$

for some $Q \in \mathbb{H} \setminus (\mathbb{R} \cup i\mathbb{R})$ where $l(Q) = 0$ is contained in the intersection of the hyperbola and open rectangle associated with X.

Proof. Applying G_p to both sides in (9.9) produces the equivalent system of equations:

$$\begin{aligned} g &= G_p\left(x+\frac{m}{g}\right) \\ g^I &= G_q\left(y+\frac{1-m}{g^I}\right). \end{aligned} \tag{9.15}$$

From simplifying these rational expressions and using the formulas for G_p and G_q (see (8.36)), these equations are equivalent to:

$$\begin{aligned} (x-\alpha)(x-\alpha')g^2+[(x-\alpha)(m-(1-a))+(x-\alpha')(m-a)]g+m^2-m=0 \\ (y-\beta)(y-\beta')(g^I)^2-[(y-\beta)(m-b)+(y-\beta')(m-(1-b))]g^I+m^2-m=0\,. \end{aligned} \tag{9.16}$$

We proceed to show that these two equations are quadratic equations with real coefficients and a non-zero constant term: Note that $Q \notin (\mathbb{R} \cup i\mathbb{R})$ if and only if $g, g^I \notin \mathbb{R}$. Then, g and g^I and their conjugates also satisfy their respective equations, so the polynomials have two distinct roots. The only other possibility besides the polynomials being quadratic is that they are the zero polynomial. Without loss of generality, consider the first polynomial. The degree 2 term being zero implies that $x=\alpha$ or $x=\alpha'$. The constant term being zero implies that $m=0$ or $m=1$. But then, this implies that the linear term is non-zero, a contradiction.

Then, from $|g|^2=\left|g^I\right|^2$,

$$\frac{m^2-m}{(x-\alpha)(x-\alpha')}=\frac{m^2-m}{(y-\beta)(y-\beta')}\,. \tag{9.17}$$

As $g \neq 0$, then $m^2-m \neq 0$ (i.e. $m \neq 0,1$), so:

$$(x-\alpha)(x-\alpha')=(y-\beta)(y-\beta')\,. \tag{9.18}$$

From Lemma 4.7, this is equivalent to the equation of the hyperbola.

Finally, we will verify that $z=x+iy$ lies in the open rectangle associated with X. The first expression in (9.16) viewed as a polynomial in g has two roots at $g \neq \overline{g}$. Hence, the discriminant is negative:

$$
\begin{aligned}
0 &> ((x-\alpha)(m-(1-a)) + (x-\alpha')(m-a))^2 - 4(x-\alpha)(x-\alpha')(m^2-m) \\
&= (\alpha'-\alpha)^2 m^2 + 2(\alpha'-\alpha)[2(a-1/2)x + (\alpha(1-a) - \alpha'\alpha)]m + \\
&\qquad + (x - (\alpha(1-a) + \alpha' a))^2 \,.
\end{aligned}
\tag{9.19}
$$

Viewed as a polynomial in m over $\mathbb{R}$, this final expression attains a negative value at some real m. Hence, there must be two distinct real roots, so the discriminant is positive:

$$
0 < -16(1-a)a(x-\alpha)(x-\alpha')(\alpha'-\alpha)^2 \,. \tag{9.20}
$$

As $a \in (0,1)$, then

$$
(x-\alpha)(x-\alpha') < 0 \,, \tag{9.21}
$$

i.e. $x \in (\alpha \wedge \alpha', \alpha \vee \alpha')$. From Lemma 4.7, $z = x + iy$ lies on the open rectangle associated with X. Hence, we conclude that when $z = \mathcal{B}_{\mathbf{X}}(Q)$ for a quaternion $Q \notin \mathbb{R} \cup i\mathbb{R}$ such that $l(Q) = 0$, z lies on the intersection of the hyperbola and open rectangle. $\square$

From Corollary 4.22, the intersection of the hyperbola with the open rectangle is contained in the support of the Brown measure only if $a = b = 1/2$, so in general, the set in Proposition 9.1 only contains the support of μ'.

The potential atoms of the Brown measure, $\{\alpha + i\beta, \alpha' + i\beta, \alpha + i\beta', \alpha' + i\beta'\}$ correspond to the two equations in (9.16) becoming linear equations. This can be viewed as a degenerate situation. Even if we allow solutions where $g, g^I \in \mathbb{R}$, these two linear equations are not satisfied, since it would require both $g, g^I \in \mathbb{R}$. This degeneracy of the 4 corners of the Brown measure also appears when we examine the heuristic for the support of the Brown measure (Proposition 10.12).

In order to determine the precise set of z for which the condition in Proposition 9.1 holds, we need to not only analyze individually which x (resp. y) have the discriminants of the quadratic equations in g (resp. g^I) attain a negative value at a real m, but we need to find an m that *simultaneously* works for both equations. We also need to analyze which g, g^I that solve the equations actually come from a Q (recall from Proposition 7.3 that it not sufficient that $|g|^2 = |g^I|$). Finally, we need to eliminate those x (resp y) where $g \in I_p$ (resp $g^I \in I_q$), as the original equations are not defined for those g, g^I. In general, this turns out to be difficult, but if we intersect with the condition that $B = 0$, then we recover almost all of the boundary of the Brown measure:

Proposition 9.2. *Let $X = p + iq$, where $p, q \in (M, \tau)$ are Hermitian, freely independent, and have 2 atoms, i.e.*

$$\begin{aligned} \mu_p &= a\delta_\alpha + (1-a)\delta_{\alpha'} \\ \mu_q &= b\delta_\beta + (1-b)\delta_{\beta'}\,, \end{aligned} \tag{9.22}$$

where $a, b \in (0,1)$, $\alpha_n, \alpha_n', \beta_n, \beta_n' \in \mathbb{R}$, $\alpha \neq \alpha'$, and $\beta \neq \beta'$.

The set of $z \in \mathbb{C}$ such that

$$\mathcal{B}_{\mathbf{X}}(g) = z \tag{9.23}$$

for some $g \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$ where $l(g) = 0$ is equal to the support of the Brown measure of X with at most finitely many points removed. The closure of this set is the support of μ' in Theorem 4.18.

Proof. Let

$$\begin{aligned} \mathscr{A} &= \alpha' - \alpha \\ \mathscr{B} &= \beta' - \beta \\ x' &= x - \frac{\alpha + \alpha'}{2} \\ y' &= y - \frac{\beta + \beta'}{2} \\ \tilde{m} &= m - 1/2 \\ \tilde{a} &= a - 1/2 \\ \tilde{b} &= b - 1/2\,. \end{aligned} \tag{9.24}$$

Since the set in the Proposition is a subset of the set in Proposition 9.1, from Lemma 4.7,

$$\mathscr{H} = (x')^2 - \frac{\mathscr{A}^2}{4} = (y')^2 - \frac{\mathscr{B}^2}{4} < 0\,. \tag{9.25}$$

Apply G_p and G_q to the equations in (9.10) to obtain the equivalent equations:

$$\begin{aligned} g &= G_p\left(x + \frac{m}{g}\right) \\ ig &= G_q\left(y + \frac{1-m}{ig}\right)\,. \end{aligned} \tag{9.26}$$

Recall that the set in the Proposition is equal to the set of $z = x + iy$ such that there exists $g \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$, $m \in \mathbb{R}$ that satisfy these equations.

From simplifying these rational expressions and using the formulas for G_p and G_q (see (8.36)), these equations are equivalent to:

$$\begin{aligned} (x-\alpha)(x-\alpha')g^2 + [(x-\alpha)(m-(1-a)) + (x-\alpha')(m-a)]g + m^2 - m = 0 \\ -(y-\beta)(y-\beta')g^2 - [(y-\beta)(m-b) + (y-\beta')(m-(1-b))]ig + m^2 - m = 0\,. \end{aligned} \tag{9.27}$$

In the new variables, the previous two quadratic equations are:

$$\begin{aligned} \mathscr{H} g^2 + (2x'\tilde{m} + \mathscr{A}\tilde{a})g + \tilde{m}^2 - 1/4 = 0 \\ -\mathscr{H} g^2 - (2y'\tilde{m} - \mathscr{B}\tilde{b})ig + \tilde{m}^2 - 1/4 = 0\,. \end{aligned} \tag{9.28}$$

As $g \neq \overline{g}$ are solutions to the first equation, then

$$\tilde{m}^2 - 1/4 = \mathscr{H}\,|g|^2 < 0\,. \tag{9.29}$$

Taking the sum and difference of the equations in (9.28) and letting $z' = x' + iy'$ produces the equivalent system of equations:

$$\begin{aligned}(2\overline{z'}\tilde{m} + \mathscr{A}\tilde{a} + i\mathscr{B}\tilde{b})g + 2(\tilde{m}^2 - 1/4) &= 0 \\ 2\mathscr{H}g^2 + (2z'\tilde{m} + \mathscr{A}\tilde{a} - i\mathscr{B}\tilde{b})g &= 0\,.\end{aligned} \tag{9.30}$$

The second equation can be factored:

$$g(2\mathscr{H}g + (2z'\tilde{m} + \mathscr{A}\tilde{a} - i\mathscr{B}\tilde{b})) = 0\,. \tag{9.31}$$

Then, (9.30) has a solution for some $g \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$, $\tilde{m} \in \mathbb{R}$ if and only if the following system of linear equations has a solution for some $g \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$, $\tilde{m} \in \mathbb{R}$:

$$\begin{aligned}(2\overline{z'}\tilde{m} + \mathscr{A}\tilde{a} + i\mathscr{B}\tilde{b})g + 2(\tilde{m}^2 - 1/4) &= 0 \\ 2\mathscr{H}g + (2z'\tilde{m} + \mathscr{A}\tilde{a} - i\mathscr{B}\tilde{b}) &= 0\,.\end{aligned} \tag{9.32}$$

By taking the determinant of the associated 2×2 matrix, this system has a solution for some $g \in \mathbb{C}$, $\tilde{m} \in \mathbb{R}$ if and only if

$$\left| z'\tilde{m} + \frac{\mathscr{A}\tilde{a} - i\mathscr{B}\tilde{b}}{2} \right|^2 = \mathscr{H}(\tilde{m}^2 - 1/4) \tag{9.33}$$

for some $\tilde{m} \in \mathbb{R}$.

We proceed to show that the $z \in C$ where there exists a $\tilde{m} \in \mathbb{R}$ that solves this equation is a set whose closure is the support of μ'. Afterwards, we will consider the possibilities where $g \in \mathbb{R} \cup i\mathbb{R}$, $g \in I_p$, or $ig \in I_q$.

Expanding out the absolute value in the previous equation,

$$\left(x'\tilde{m} + \frac{\mathscr{A}\tilde{a}}{2} \right)^2 + \left(y'\tilde{m} - \frac{\mathscr{B}\tilde{b}}{2} \right)^2 = \mathscr{H}(\tilde{m}^2 - 1/4)\,. \tag{9.34}$$

Rewriting as a polynomial in $\tilde{m}$,

$$((x')^2 + (y')^2 - \mathscr{H})\tilde{m}^2 + (\mathscr{A}\tilde{a}x' - \mathscr{B}\tilde{b}y')\tilde{m} + \frac{\mathscr{A}^2\tilde{a}^2 + \mathscr{B}^2\tilde{b}^2 + \mathscr{H}}{4} = 0\,. \tag{9.35}$$

There is an $\tilde{m} \in \mathbb{R}$ that solves this if and only if the discriminant is non-negative. Simplifying the discriminant using (9.25),

$$\begin{aligned} 0 &\leq (\mathscr{A}\tilde{a}x' - \mathscr{B}\tilde{b}y')^2 - ((x')^2 + (y')^2 - \mathscr{H})(\mathscr{A}^2\tilde{a}^2 + \mathscr{B}^2\tilde{b}^2 + \mathscr{H}) \\ &= -(x')^2(y')^2 - 2ab\mathscr{A}\mathscr{B}x'y' - \frac{\mathscr{A}^2\mathscr{B}^2(4a^2 + 4b^2 - 1)}{16}\,. \end{aligned} \tag{9.36}$$

Define the new variables:

$$\begin{aligned} \tilde{x} &= \frac{x'}{\mathscr{A}} \\ \tilde{y} &= \frac{y'}{\mathscr{B}}\,. \end{aligned} \tag{9.37}$$

The previous inequality is equivalent to:

$$(\tilde{x}\tilde{y})^2 + 2\tilde{a}\tilde{b}(\tilde{x}\tilde{y}) + \frac{4\tilde{a}^2 + 4\tilde{b}^2 - 1}{16} \leq 0\,. \tag{9.38}$$

We now show that this condition along with $z = x + iy$ being on the intersection of the hyperbola with the open rectangle is equivalent to the following condition:

From Theorem 4.18, the support μ' is the closure of the set of $z \in \mathbb{C}$ on the hyperbola such that:

$$\operatorname{Im}\left(\left(z - \frac{\alpha + \alpha'}{2} - i\frac{\beta + \beta'}{2}\right)^2\right) = \frac{\mathscr{A}\mathscr{B}\cos(2\theta)}{2}\,, \tag{9.39}$$

where $\theta \in (0, \pi/2)$ satisfies

$$f(\sec^2(\theta)) \leq 0\,, \tag{9.40}$$

where

$$f(t) = (a - b)^2 t^2 + (4ab - 2(a + b))t + 1\,. \tag{9.41}$$

From Lemma 4.7, for $z = x + iy$ on the hyperbola, (9.39) is equivalent to being on the open rectangle. The condition can be rewritten in the new variables as:

$$4\tilde{x}\tilde{y} = \cos(2\theta)\,. \tag{9.42}$$

Next, by using the new variables,

$$f(t) = (\tilde{a} - \tilde{b})^2 t^2 + (4\tilde{a}\tilde{b} - 1)t + 1\,. \tag{9.43}$$

From the double-angle formula,

$$\sec^2(\theta) = \frac{1}{\cos^2(\theta)} = \frac{2}{1+\cos(2\theta)} = \frac{2}{1+4\tilde{x}\tilde{y}}\,. \tag{9.44}$$

Combining these final two expressions, a straightforward computation shows that $f(\sec^2(\theta)) \leq 0$ is equivalent to (9.38). Hence, the closure of the $z = x + iy$ on the intersection of the hyperbola and open rectangle and satisfying (9.38) is the support of μ'.

Finally, we need to remove the $z \in \mathbb{C}$ where there exists a $g \in \mathbb{R} \cup i\mathbb{R}$, $g \in I_p$, or $ig \in I_q$ that solves (9.32). It suffices to remove solutions where $\operatorname{Im}(g) = c$ or $\operatorname{Re}(g) = c$ for some $c \in \mathbb{R}$. We claim that for any $c \in \mathbb{R}$ this removes only finitely many points. Since the support of μ' has no isolated points, then the closure of the set is still the support of μ'. We will prove the case where $\operatorname{Im}(g) = c$, the case where $\operatorname{Re}(g) = c$ is similar.

If there exists a g solving (9.32), then since $\mathscr{H} < 0$ and $\tilde{m}^2 - 1/4 < 0$ (from (9.29)) so there are 2 equations for g:

$$-\frac{z'\tilde{m} + \frac{\mathscr{A}\tilde{a} - i\mathscr{B}\tilde{b}}{2}}{\mathscr{H}} = g = -\frac{\tilde{m}^2 - 1/4}{\overline{z'}\tilde{m} + \frac{\mathscr{A}\tilde{a} + i\mathscr{B}\tilde{b}}{2}}\,. \tag{9.45}$$

Using the first equation,

$$\operatorname{Im}(g) = c \iff \mathscr{H}c = -y'\tilde{m} + \frac{\mathscr{B}\tilde{b}}{2}\,. \tag{9.46}$$

We may assume without loss of generality that $y' \neq 0$, as we can remove the finitely many points where this occurs on the hyperbola. Then, rewriting the final expression as $\tilde{m}$ in terms of y',

$$\tilde{m} = \frac{1}{y'}\left(\frac{\mathscr{B}\tilde{b}}{2} - \mathscr{H}c\right) = -y'c + \frac{1}{y'}\left(\frac{\mathscr{B}\tilde{b}}{2} + \frac{\mathscr{B}^2 c}{4}\right). \tag{9.47}$$

Using that

$$\begin{aligned} x' &= \pm\sqrt{\frac{\mathscr{A}^2 - \mathscr{B}^2}{4} + (y')^2} \\ H &= (y')^2 - \frac{\mathscr{B}^2}{4}, \end{aligned} \tag{9.48}$$

we can rewrite (9.34) in terms of y' and then manipulate the expression to see that y' is a root of a rational equation. We claim that this rational equation is non-zero, so then there are only finitely many y' that solve this equation. Hence, when we remove $\mathrm{Im}(g) = c$ for some c, we only remove finitely many points.

If (9.34) and (9.46) hold, then the following equation also holds:

$$\left(\mathscr{H}(\tilde{m}^2 - 1/4) - (\mathscr{H}c)^2 - (x'm)^2 - \frac{\mathscr{A}^2\tilde{a}^2}{4}\right)^2 - (x'\tilde{m}\mathscr{A}\tilde{a})^2 = 0\,. \tag{9.49}$$

For $c \neq 0$, by rewriting $x', H, \tilde{m}$ in terms of y' using the previous equations, then the left-hand side is a rational expression in y'. The highest degree term is $(y')^8$ with coefficient c^4, so the rational expression is non-zero.

If $c = 0$, the highest degree term on the left-hand side is $(y')^4$ with coefficient $1/16$, so the rational expression is non-zero.

We conclude that when we remove the possible solutions of (9.45) where $g \in \mathbb{R} \cup i\mathbb{R}$, $g \in I_p$, or $ig \in I_q$ only removes finitely many points. Hence, it does not affect the closure of the set. $\square$

When $a = b = 1/2$ (equivalently, $\tilde{a} = \tilde{b} = 0$), we conclude from Proposition 9.2 and Corollary 4.22 that the closure of the set in Proposition 9.1 is also the support of μ' (which is also the support of the Brown measure in this case). It is also easy to check directly (9.16) can be solved with $m = 1/2$ for all z on the support of the Brown measure. Note that in these solutions $g, g^I \in i\mathbb{R}$, so these are not the same solutions for Proposition 9.2.

9.3 When p and q have finitely many atoms

Recall that our heuristic from (9.10) is that the boundary of the Brown measure of $X = p + iq$ is the closure of the set of $z = x + iy$ for which there exists $g \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$ and $m \in \mathbb{R}$ such that the following system of equations is satisfied:

$$\begin{aligned} B_p(g) &= x + \frac{m}{g} \\ B_q(ig) &= y + \frac{1-m}{ig}\,. \end{aligned} \tag{9.50}$$

In Proposition 9.2, we verified this claim was almost true when p and q have two atoms (the set was missing the atoms of the Brown measure).

Next, consider $X = p + iq$ where p and q have arbitrarily many atoms. Let

$$\begin{aligned} \mu_p &= a_1\delta_{\alpha_1} + \cdots + a_n\delta_{\alpha_n} \\ \mu_q &= b_1\delta_{\beta_1} + \cdots + b_k\delta_{\beta_k}\,, \end{aligned} \tag{9.51}$$

where $\alpha_i, \beta_j \in \mathbb{R}$, $a_i, b_j \geq 0$ and $a_1 + \cdots + a_n = b_1 + \cdots + b_k = 1$. It is significant to what follows that we do not assume the α_i (resp. β_j) are distinct.

Let

$$\begin{aligned} \boldsymbol{\alpha} &= (\alpha_1, \ldots, \alpha_n) \\ \boldsymbol{\beta} &= (\beta_1, \ldots, \beta_k) \\ \boldsymbol{a} &= (a_1, \ldots, a_n) \\ \boldsymbol{b} &= (b_1, \ldots, b_k) \end{aligned} \tag{9.52}$$

be shorthand for the positions and weights of the atoms in the measures.

The (complex) Green's functions for p and q are:

$$\begin{aligned} G_p(z) &= \frac{a_1}{z-\alpha_1} + \cdots + \frac{a_n}{z-\alpha_n} \\ G_q(z) &= \frac{b_1}{z-\beta_1} + \cdots + \frac{b_k}{z-\beta_k}\,. \end{aligned} \tag{9.53}$$

Applying G_p and G_q, an equivalent system of equations is:

$$\begin{aligned} G_p\left(x+\frac{m}{g}\right) &= g \\ G_q\left(y+\frac{1-m}{ig}\right) &= ig\,. \end{aligned} \tag{9.54}$$

Let

$$\Omega_{p,q} = \{z = x+iy \in \mathbb{C} : \text{ there exists } g \in \mathbb{C}\setminus(\mathbb{R}\cup i\mathbb{R}), m \in \mathbb{R} \text{ satisfying (9.54)}\}\ . \tag{9.55}$$

From (9.10), $\overline{\Omega_{p,q}}$ is heuristically understood to be the boundary of the Brown measure of $X = p + iq$.

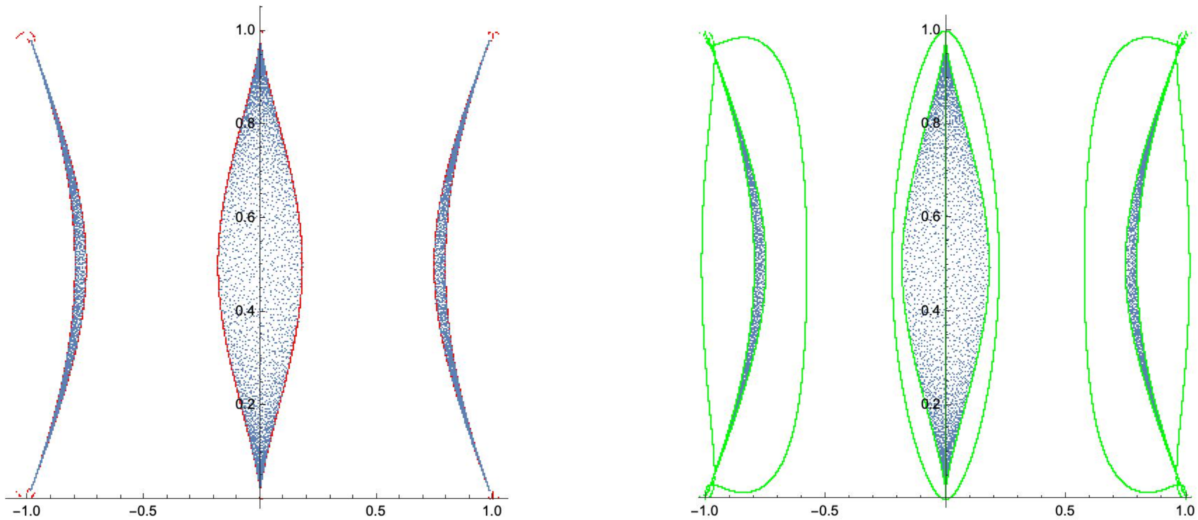
The main result of the section is the following:

Theorem 9.3. *Fix $\boldsymbol{a}, \boldsymbol{b}$. Then, for Lebesgue almost every $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbb{R}^n \times \mathbb{R}^k = \mathbb{R}^{n+k}$, $\Omega_{p,q}$ lies on a real algebraic curve, i.e. $\Omega_{p,q}$ lies in the zero set of some non-zero two-variable polynomial with real coefficients. In particular, we provide an explicit algorithm to produce such a polynomial.*

The proof is split up into the following parts:

1. State the algorithm to find the two-variable polynomial whose zero set contains $\Omega_{p,q}$.
2. Check the algorithm does indeed produce a two-variable polynomial whose zero set contains $\Omega_{p,q}$.
3. Prove that in the generic situation, this algorithm produces a non-zero polynomial.

First, we provide some figures generated using Mathematica comparing the empirical spectral distribution of a deterministic $X_n = P_n + iQ_n$ with the sets $\overline{\Omega_{p,q}}$ and the algebraic curve the algorithm produces.



(a) ESD of X_n vs. $\overline{\Omega_{p,q}}$ from (9.55)

(b) ESD of X_n vs. algebraic curve

Figure 9.1: $X_n = P_n + iQ_n$
$\mu_{P_n} \approx (1/3)\delta_{-1} + (1/3)\delta_0 + (1/3)\delta_1$
$\mu_{Q_n} = (1/2)\delta_0 + (1/2)\delta_1$
$n = 10000$

9.3.1 Algorithm

In this section, we will state the algorithm that produces a two-variable polynomial whose zero set contains $\Omega_{p,q}$.

First, the system of equations defining $\Omega_{p,q}$, (9.54), can be written as a system of polynomial equations:

$$\begin{gathered} \prod_{i=1}^{n}(g(x-\alpha_i)+m) - \sum_{i=1}^{n}\left(a_i \prod_{s\neq i}(g(x-\alpha_s)+m)\right) = 0 \\ \prod_{j=1}^{l}(ig(y-\beta_j)+1-m) - \sum_{j=1}^{n}\left(b_j \prod_{s\neq j}(ig(y-\beta_s)+1-m)\right) = 0\,. \end{gathered} \tag{9.56}$$

Next, we recall the resultant of two polynomials and some basic properties:

Definition 9.4. *Let $A(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ and $B(x) = b_k x^k + b_{k-1}x^{k-1} + \cdots + b_0$ be one-variable polynomials with coefficients in a commutative ring R. The **resultant** of A and B, $\mathrm{Res}(A, B)$, is the determinant of the $(n + k) \times (n + k)$ **Sylvester matrix**:*

$$
\operatorname{Res}(A,B) = \begin{vmatrix}
a_n & & & & b_k & & & \\
a_{n-1} & a_n & & & b_{k-1} & b_k & & \\
a_{n-2} & a_{n-1} & \ddots & & b_{k-2} & b_{k-1} & \ddots & \\
\vdots & \vdots & & a_n & \vdots & \vdots & & b_k \\
a_0 & a_1 & & \vdots & b_0 & b_1 & & \vdots \\
 & a_0 & \ddots & \vdots & & b_0 & \ddots & \vdots \\
 & & \ddots & a_1 & & & \ddots & b_1 \\
 & & & a_0 & & & & b_0
\end{vmatrix} \tag{9.57}
$$

Suppose that R is an integral domain. Then, it makes sense to talk about the roots of $A(x)$ and $B(x)$ in some algebraically closed field containing R.

We state the following well-known result:

Proposition 9.5. *Let $a_i, b_j \in R$, where R is an integral domain. Let λ_i, μ_j be the roots of $A(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ and $B(x) = b_k x^k + b_{k-1}x^{k-1} + \cdots + b_0$ in some algebraically closed field containing R, respectively.*

$$
\begin{aligned}
A(x) &= a_n(x-\lambda_1)\cdots(x-\lambda_n) \\
B(x) &= b_k(x-\mu_1)\cdots(x-\mu_k)\,.
\end{aligned} \tag{9.58}
$$

Then,

$$
\operatorname{Res}(A,B) = a_n^k b_k^n \prod_{\substack{1\le i\le n\\ 1\le j\le k}} (\lambda_i - \mu_j)\,. \tag{9.59}
$$

A corollary of this result is:

Corollary 9.6. *Let $a_i, b_j \in R$, where R is an integral domain. Let $A(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ and $B(x) = b_k x^k + b_{k-1}x^{k-1} + \cdots + b_0$. Then, A and B have a common root in some algebraically closed field containing R if and only if* $\operatorname{Res}(A,B) = 0$.

We can now state the algorithm to produce the two-variable polynomial:

Algorithm 9.7. *The following algorithm takes p, q as in (9.51) and produces a two-variable real polynomial $f(x, y)$:*

1. *Take the resultants of the polynomials in (9.56) with respect to g. Let $f_1(m, x, y)$ be this resultant.*

2. *Divide $f_1(m, x, y)$ by $m^{n-1}(m-1)^{k-1}$. Let $f_2(m, x, y)$ be the result of this, so that $f_2(m, x, y)m^{n-1}(m-1)^{k-1} = f_1(m, x, y)$.*

3. *Take the real and imaginary parts of $f_2(m, x, y)$ assuming that $m, x, y \in \mathbb{R}$. This produces real polynomials $\mathrm{Re} f_2(m, x, y)$ and $\mathrm{Im} f_2(m, x, y)$.*

4. *Take the resultant of $\mathrm{Re} f_2(m, x, y)$ and $\mathrm{Im} f_2(m, x, y)$ with respect to m. Return this polynomial as $f(x, y)$, a real two-variable polynomial.*

This idea of reducing the number of variables in a system of polynomial equations by taking resultants is not new (see [Sti] for a description of the technique), but the main issue is that by computing resultants one may introduce too many new solutions in the system. In particular, we wish to avoid the situation that the resultant is the zero polynomial, which gives no information about the solutions of the original system.

The second step of dividing by $m^{n-1}(m-1)^{k-1}$ avoids the resultant being the zero polynomial (in general). There is also the detail that x, y, m are real variables, but g is complex and we start with two complex equations in (9.56). This is handled by taking $\mathrm{Re} f_2$ and $\mathrm{Im} f_2$ in Step 3.

9.3.2 Proof of correctness for algorithm

Now, we will prove that Algorithm 9.7 produces a polynomial whose zero set contains $\Omega_{p,q}$.

Consider $z_0 = x_0 + iy_0 \in \Omega_{p,q}$ where (x_0, y_0, g_0, m_0) solves (9.56). Recall that $x_0, y_0, m_0 \in \mathbb{R}$ and $g_0 \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$.

In Step 1, substitute $x = x_0$, $y = y_0$ and $m = m_0$ into the polynomials in (9.56) and treat them as polynomials in g. Then, these polynomials both have a root at $g = g_0$, so their resultant with respect to g must be zero. Hence, $f_1(m_0, x_0, y_0) = 0$.

In Step 2, we must show that $m^{n-1}(m-1)^{k-1}$ divides $f_1(m, x, y)$ and also that $f_2(m_0, x_0, y_0) = 0$. Supposing we have proven the first statement, the second statement follows immediately from the fact that $m_0 \neq 0, 1$:

If $m_0 = 0$, then the first equation in (9.54) is $G_p(x_0) = g_0$ for $x_0 \in \mathbb{R}$, so then $g_0 \in \mathbb{R}$, a contradiction. Similarly, if $m_0 = 1$, the second equation is $G_q(y_0) = ig_0$ for $y_0 \in \mathbb{R}$, so $g_0 \in i\mathbb{R}$, a contradiction.

The fact that $m^{n-1}(m-1)^{k-1}$ divides $f_1(m, x, y)$ depends on the following Lemma:

Lemma 9.8. *As elements of* $\mathbb{Z}[a_0, \ldots, a_n, b_0, \ldots, b_k, m]$*,* $m^{n-1}(m-1)^{k-1}$ *divides the following determinant:*

$$\begin{vmatrix}
a_n & & & b_k & & \\
a_{n-1} & \ddots & & b_{k-1} & \ddots & \\
a_{n-2}m & & a_n & b_{k-2}(m-1) & & b_k \\
\vdots & & \vdots & \vdots & & \vdots \\
a_0 m^{n-1}(m-1) & & \vdots & b_0(m-1)^{k-1}m & & \vdots \\
 & \ddots & a_1 m^{n-2} & & \ddots & b_1(m-1)^{k-2} \\
 & & a_0 m^{n-1}(m-1) & & & b_0(m-1)^{k-1}m
\end{vmatrix} \tag{9.60}$$

More precisely, this is the determinant of the matrix obtained from the Sylvester matrix for $A(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ *and* $B(x) = b_k x^k + b_{k-1}x^{k-1} + \cdots + b_0$ *with the following changes:*

1. *Replace all* a_i *with* $a_i m^{n-1-i}$ *for* $i = 1, \ldots, n-1$.
2. *Replace all* b_i *with* $b_i(m-1)^{k-1-i}$ *for* $i = 1, \ldots, k-1$.
3. *Replace all* a_0 *with* $a_0 m^{n-1}(m-1)$.
4. *Replace all* b_0 *with* $b_0(m-1)^{k-1}m$.

Proof. As the polynomial ring is a unique factorization domain and m, $m-1$ are primes, it suffices to check that m^{n-1} and $(m-1)^{k-1}$ each individually divide the determinant.

We will just prove that m^{n-1} divides the determinant, as the other case can be obtained from switching the roles of A and B and using $m' = 1 - m$ instead of m

Consider the terms that come from evaluating the determinant using the Leibniz formula. To show that m^{n-1} divides the determinant, it suffices to check that m^{n-1} divides any term that is a product of some non-zero entries of the matrix.

To simplify matters, consider only the power of m in each coordinate, i.e. it suffices to check that m^{n-1} divides the non-zero terms of the Leibniz formula in the following determinant:

$$\begin{vmatrix}
1 & & & 1 & & & \\
1 & \ddots & & 1 & \ddots & & \\
m & & 1 & 1 & & 1 & \\
\vdots & & \vdots & \vdots & & \vdots & 1 \\
\vdots & & \vdots & 1 & & \vdots & \vdots \\
m^{n-2} & & \vdots & m & & 1 & \vdots \\
m^{n-1} & & m^{n-3} & & \ddots & 1 & 1 \\
 & \ddots & m^{n-2} & & & m & 1 \\
 & & m^{n-1} & & & & m
\end{vmatrix} \tag{9.61}$$

The proof follows by induction on n (with k fixed). The base case $n = 1$ is trivial as $m^{n-1} = 1$. For the inductive step, suppose that the claim has been verified for $n - 1$ and consider the statement for n. Any term in the Leibniz formula where one of the m^{n-1} terms is part of the product is clearly divisible by m^{n-1}.

Hence, it suffices to consider only those terms in the Leibniz formula that are a product of matrix entries that are not m^{n-1}. This corresponds to changing all instances of m^{n-1} to 0 and looking for terms in the Leibniz formula that are products of non-zero entries of this new matrix:

$$\begin{vmatrix}
1 & & & 1 & & & \\
1 & \ddots & & 1 & \ddots & & \\
m & & 1 & 1 & & 1 & \\
\vdots & & \vdots & \vdots & & \vdots & 1 \\
\vdots & & \vdots & 1 & & \vdots & \vdots \\
m^{-2} & & \vdots & m & & 1 & \vdots \\
0 & & m^{n-3} & & \ddots & 1 & 1 \\
 & \ddots & m^{n-2} & & & m & 1 \\
 & & 0 & & & & m
\end{vmatrix} \tag{9.62}$$

In this new matrix, there is only one non-zero entry in the last row, the lower right m. Hence, all non-zero terms in the Leibniz formula for the new matrix are equal to m multiplied with a non-zero term in the Leibniz formula for the $(n+k-1) \times (n+k-1)$ minor of the first $n+k-1$ rows and columns:

$$\begin{vmatrix}
1 & & & 1 & & \\
1 & \ddots & & 1 & \ddots & \\
m & & 1 & 1 & & 1 \\
\vdots & & \vdots & \vdots & & \vdots \\
\vdots & & \vdots & 1 & & \vdots \\
m^{n-2} & & \vdots & m & & 1 \\
 & \ddots & m^{n-3} & & \ddots & 1 \\
 & & m^{n-2} & & & m
\end{vmatrix} \tag{9.63}$$

This $(n+k-1) \times (n+k-1)$ minor is just the original matrix in (9.61) but with $n-1$ instead of n. Hence, from induction, all non-zero terms in the Leibniz formula for this matrix are divisible by m^{n-2}. Once this is multiplied by the lower right m, then all of these terms are divisible by m^{n-1}, as desired. $\square$

Now, we can prove that $m^{n-1}(m-1)^{k-1}$ divides $f_1(m, x, y)$, which completes the verification of Step 2 of the algorithm:

Proposition 9.9. *Let $f_1(m,x,y)$ be the resultant of the polynomials in (9.56) with respect to g. Then, $m^{n-1}(m-1)^{k-1}$ divides $f_1(m,x,y)$ in $\mathbb{C}[m,x,y]$.*

Proof. Consider the two polynomials in (9.56): For the first polynomial, the first term is homogeneous in g and m with degree n and the second term is homogeneous in g and m with degree $n-1$. Similarly, for the second polynomial, the first term is homogeneous in g and $1-m$ with degree k and the second term is homogeneous in g and $1-m$ with degree $k-1$. Thus, by expanding the products in g and factoring out m and $m-1$ respectively, these polynomials can be written respectively as:

$$\begin{aligned} &p_n(x,m)g^n + p_{n-1}(x,m)g^{n-1} + p_{n-2}(x,m)mg^{n-2}+ \\ &\qquad + p_{n-3}(x,m)m^2g^{n-3}\cdots + p_0(x,m)m^{n-1} \\ &q_k(y,m)g^k + q_{k-1}(y,m)g^{k-1} \qquad + q_{k-2}(y,m)(1-m)g^{k-2}+ \\ &\qquad + q_{k-3}(x,m)(1-m)^2g^{k-3}\cdots + q_0(y,m)(1-m)^{k-1}. \end{aligned} \tag{9.64}$$

for some complex polynomials $p_i(x,m), q_i(y,m)$.

Using that $a_1+\cdots+a_n = b_1+\cdots+b_k = 1$, then $p_0(x,m) = m-1$ and $q_0(y,m) = -m$.

The resultant of these polynomials with respect to g is exactly the determinant in Lemma 9.8 with $a_0 = 1$, $b_0 = (-1)^k$, $a_i = p_i(x,m)$ for $i = 1,\ldots,n$ and $b_i = q_i(y,m)$ for $i = 1,\ldots,k$. Hence, $m^{n-1}(m-1)^{k-1}$ divides $f_1(m,x,y)$ in $\mathbb{C}[m,x,y]$. □

Returning to the verification of the algorithm, consider Step 3. $f_2(m_0,x_0,y_0) = 0$ is equivalent to $\mathrm{Re}f_2(m_0,x_0,y_0) = \mathrm{Im}f_2(m_0,x_0,y_0) = 0$.

Treating $\mathrm{Re}f_2(m,x_0,y_0)$ and $\mathrm{Im}f_2(m,x_0,y_0)$ as polynomials in m, then these polynomials have a common root at $m = m_0$. Hence, the resultant of these two polynomials vanishes in m vanishes at (x_0,y_0). This completes the proof of the correctness of the algorithm.

9.3.3 A specific case

In order to prove Theorem 9.3 for the generic case, we first apply the algorithm in the specific case where $\boldsymbol{\alpha} = (0, \ldots, 0) \in \mathbb{R}^n$ and $\boldsymbol{\beta} = (0, \ldots, 0) \in \mathbb{R}^k$. We will see that in this case, the algorithm produces a non-zero polynomial:

Proposition 9.10. *The algorithm for any $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{\alpha} = \mathbf{0}$, $\boldsymbol{\beta} = \mathbf{0}$ produces a non-zero polynomial.*

Proof. For any $\boldsymbol{a}$ and $\boldsymbol{b}$, (9.56) is:

$$\begin{aligned} 0 &= (gx+m)^n - (gx+m)^{n-1} = (gx+m)^{n-1}(gx+m-1) \\ 0 &= (igy+1-m)^k - (igy+1-m)^{k-1} = (igy+1-m)^{k-1}(igy-m)\,. \end{aligned} \tag{9.65}$$

Step 1 of the algorithm is taking the resultant of the polynomials in (9.65) with respect to g. From continuity, it suffices to take the resultant when $x \neq 0$ and $y \neq 0$. In this situation, we can factor the polynomials:

$$\begin{aligned} (gx+m)^{n-1}(gx+m-1) &= x^n \left(g + \frac{m}{x}\right)^{n-1} \left(g + \frac{m-1}{x}\right) \\ (igy+1-m)^{k-1}(igy-m) &= (iy)^k \left(g - \frac{i(1-m)}{y}\right)^{k-1} \left(g + \frac{im}{y}\right)\,. \end{aligned} \tag{9.66}$$

From Proposition 9.5,

$$
\begin{aligned}
f_1(m,x,y) &= x^{nk}(iy)^{nk}\left(\frac{m}{x}+\frac{i(1-m)}{y}\right)^{(n-1)(k-1)}\left(\frac{m}{x}-\frac{im}{y}\right)^{n-1}\times \\
&\qquad \left(\frac{m-1}{x}+\frac{i(1-m)}{y}\right)^{k-1}\left(\frac{m-1}{x}-\frac{im}{y}\right) \\
&= (imy+(m-1)x)^{(n-1)(k-1)}(imy+mx)^{n-1}\times \\
&\qquad (i(m-1)y+(m-1)x)^{k-1}(i(m-1)y+mx) \qquad (9.67) \\
&= m^{n-1}(m-1)^{k-1}(imy+(m-1)x)^{(n-1)(k-1)}(x+iy)^{n-1}\times \\
&\qquad (x+iy)^{k-1}(i(m-1)y+mx) \\
&= m^{n-1}(m-1)^{k-1}(x+iy)^{n+k-2}\times \\
&\qquad ((m-1)x+imy)^{(n-1)(k-1)}(mx+i(m-1)y)\,.
\end{aligned}
$$

In Step 2 of the algorithm, we divide $f_1(m,x,y)$ by $m^{n-1}(m-1)^{k-1}$. This is easy to do from the final expression for $f_1(m,x,y)$:

$$
f_2(m,x,y) = (x+iy)^{n+k-2}((m-1)x+imy)^{(n-1)(k-1)}(mx+i(m-1)y)\,. \qquad (9.68)
$$

In Step 3 of the algorithm, we compute $\mathrm{Re}f_2(m,x,y)$ and $\mathrm{Im}f_2(m,x,y)$ assuming $m,x,y\in\mathbb{R}$. In Step 4, we compute the resultant of these two real polynomials in m, resulting in a polynomial in x and y. We will not compute these explicitly, but just argue that the result of the algorithm is a non-zero polynomial.

First, consider a general $f_2(m,x,y)\in\mathbb{C}[m,x,y]$:

$$
f_2(m,x,y) = \sum_{j,k,l} c_{j,k,l}m^jx^ky^l\,, \qquad c_{j,kl}\in\mathbb{C}\,. \qquad (9.69)
$$

The result of applying Step 3 of the algorithm to $f_2(m, x, y)$ is:

$$\begin{aligned} \mathrm{Re} f_2(m, x, y) &= \sum_{j,k,l} \mathrm{Re}(c_{j,k,l}) m^j x^k y^l \\ \mathrm{Im} f_2(m, x, y) &= \sum_{j,k,l} \mathrm{Im}(c_{j,k,l}) m^j x^k y^l \,. \end{aligned} \tag{9.70}$$

Thus, the following equalities hold as polynomials in $\mathbb{C}[m, x, y]$:

$$\begin{aligned} f_2(m, x, y) &= \mathrm{Re} f_2(m, x, y) + i\, \mathrm{Im} f_2(m, x, y) \\ \overline{f_2}(m, x, y) &= \mathrm{Re} f_2(m, x, y) - i\, \mathrm{Im} f_2(m, x, y) \,, \end{aligned} \tag{9.71}$$

where $\overline{f_2}(m, x, y) \in \mathbb{C}[m, x, y]$ is computed assuming that $m, x, y \in \mathbb{R}$.

In Step 4, we compute the resultant of $\mathrm{Re} f_2(m, x, y)$ and $\mathrm{Im} f_2(m, x, y)$ with respect to m and this is the result of the algorithm. This resultant vanishes at (x_0, y_0) if and only if there is some $m_0 \in \mathbb{C}$ such that $\mathrm{Re} f_2(m_0, x_0, y_0) = \mathrm{Im} f_2(m_0, x_0, y_0) = 0$. This happens if and only if there is some $m_0 \in \mathbb{C}$ such that $f_2(m_0, x_0, y_0) = \overline{f_2}(m, x_0, y_0) = 0$. Hence, it suffices to show there exists $(x_0, y_0) \in \mathbb{R}$ where there does not exist $m \in \mathbb{C}$ such that:

$$\begin{aligned} 0 &= f_2(m, x_0, y_0) \\ &= (x_0 + iy_0)^{n+k-2}((m-1)x_0 + imy_0)^{(n-1)(k-1)}(x_0 m + i(m-1)y_0) \\ 0 &= \overline{f_2}(m, x_0, y_0) \\ &= (x_0 - iy_0)^{n+k-2}((m-1)x_0 - imy_0)^{(n-1)(k-1)}(x_0 m - i(m-1)y_0) \,. \end{aligned} \tag{9.72}$$

As f_2 and $\overline{f_2}$ are factored, it is easy to see that

$$\begin{aligned} f_2(m, x_0, y_0) = 0 &\iff x_0 = y_0 = 0 \quad \text{or} \quad m = \frac{x_0}{x_0 + iy_0} \quad \text{or} \quad m = \frac{iy_0}{x_0 + iy_0} \\ \overline{f_2}(m, x_0, y_0) = 0 &\iff x_0 = y_0 = 0 \quad \text{or} \quad m = \frac{x_0}{x_0 - iy_0} \quad \text{or} \quad m = \frac{-iy_0}{x_0 - iy_0} \,. \end{aligned} \tag{9.73}$$

Consider $x_0 + iy_0 = e^{i\theta}$. Then,

$$\begin{aligned} \frac{x_0}{x_0+iy_0} &= \cos\theta e^{-i\theta}\,, \qquad \frac{iy_0}{x_0+iy_0} = i\sin\theta e^{-i\theta} = \sin\theta e^{i(\pi/2-\theta)} \\ \frac{x_0}{x_0-iy_0} &= \cos\theta e^{i\theta}\,, \qquad \frac{-iy_0}{x_0-iy_0} = -i\sin\theta e^{i\theta} = \sin\theta e^{i(\theta-\pi/2)}\,. \end{aligned} \tag{9.74}$$

The roots of $f_2(m, x_0, y_0)$ and $\overline{f_2}(m, x_0, y_0)$ occur at angles $-\theta, \pi/2 - \theta, \theta, \theta - \pi/2$. When $\theta \in (0, \pi/4)$, these four angles are all distinct because $\theta - \pi/2 < -\theta < \theta < \pi/2 - \theta$. Thus, $f_2(m, x_0, y_0)$ and $\overline{f_2}(m, x_0, y_0)$ cannot have a common root, and we conclude that the algorithm produces a non-zero polynomial in this instance. □

9.3.4 Extending to generic case

We are now ready to prove Theorem 9.3:

Proof of Theorem 9.3. We have already presented the algorithm and proved that it does produce a real two-variable that vanishes on $\Omega_{p,q}$.

All that is left to prove is that for any fixed $\boldsymbol{a}, \boldsymbol{b}$, for Lebesgue almost every $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbb{R}^n \times \mathbb{R}^k$, the polynomial from the algorithm is non-zero.

First, change coordinates as follows:

Let S^{n-1} and S^{k-1} be the unit spheres in $\mathbb{R}^n$ and $\mathbb{R}^k$:

$$\begin{aligned} S^{n-1} &= \{\boldsymbol{u} \in \mathbb{R}^n : \|\boldsymbol{u}\| = 1\} \\ S^{k-1} &= \{\boldsymbol{v} \in \mathbb{R}^k : \|\boldsymbol{v}\| = 1\}\,. \end{aligned} \tag{9.75}$$

Consider the map $\phi : [0, \infty) \times S^{n-1} \times [0, \infty) \times S^{k-1} \to \mathbb{R}^n \times \mathbb{R}^k$ given by:

$$\phi\left((r, \boldsymbol{u}, s, \boldsymbol{v})\right) = (r\boldsymbol{u}, s\boldsymbol{v})\,. \tag{9.76}$$

Endow $[0, \infty) \times S^{n-1} \times [0, \infty) \times S^{k-1}$ with the product of the Lebesgue measures on the intervals and the normalized spherical measures and endow $\mathbb{R}^n \times \mathbb{R}^k$ with the usual Lebesgue measure. From the Change of Variables formula, ϕ maps sets of measure 0 to sets of measure 0. Hence, it suffices to prove the generic statement of the Theorem in the $(r, \boldsymbol{u}, s, \boldsymbol{v})$ coordinates.

It suffices to show that for any $\boldsymbol{a}, \boldsymbol{b}$ and $(\boldsymbol{u}, \boldsymbol{v}) \in S^{n-1} \times S^{k-1}$ and Lebesgue almost every $(r, s) \in [0, \infty) \times [0, \infty)$, the algorithm applied to $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\alpha} = r\boldsymbol{u}, \boldsymbol{\beta} = s\boldsymbol{v}$ produces a non-zero polynomial.

Fix $\boldsymbol{a}, \boldsymbol{b}$ and $(\boldsymbol{u}, \boldsymbol{v}) \in S^{n-1} \times S^{k-1}$. It is straightforward to check that if we consider $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ as functions of r and s, respectively: $\boldsymbol{\alpha}(r) = r\boldsymbol{u}$, $\boldsymbol{\beta}(s) = s\boldsymbol{v}$, then the algorithm produces a real polynomial in x, y, r, s. Additionally, doing the algorithm and evaluation at a specific $r = r_0$, $s = s_0$ commute.

Hence, the algorithm produces a polynomial $p(x, y, r, s)$, where from Proposition 9.10, $p(x, y, \mathbf{0}, \mathbf{0})$ is a non-zero polynomial.

Viewing $p(x, y, r, s)$ as a polynomial in x, y with coefficients in $\mathbb{R}[r, s]$, we see that there is at least one coefficient that is not the zero polynomial. As the zero set of any non-zero polynomial in $\mathbb{R}[r, s]$ is Lebesgue measure 0, then for Lebesgue almost every $(r, s) \in [0, \infty) \times [0\infty)$, $p(x, y, r, s)$ has a non-zero coefficient. Hence, for almost every r, s, the result of the algorithm is a non-zero polynomial. $\square$

CHAPTER 10

Support of the Brown measure

In this chapter, we consider Heuristic 7.19 about the support of the Brown measure of $X = p + iq$, where $p, q \in (M, \tau)$ are Hermitian and freely independent.

The main result of this chapter is that in the case when p, q have 2 atoms that have equal weights, then $X = p + iq$ satisfies this property when we restrict to points where we can use $\mathcal{B}_{\mathbf{X}}$:

Theorem 10.1. *Suppose that $p, q \in (M, \tau)$ are Hermitian, freely independent operators such that their spectral measures are:*

$$\begin{aligned} \mu_p &= (1/2)\delta_\alpha + (1/2)\delta_{\alpha'} \\ \mu_q &= (1/2)\delta_\beta + (1/2)\delta_{\beta'} \end{aligned} \tag{10.1}$$

for some $\alpha \neq \alpha', \beta \neq \beta' \in \mathbb{R}$.

Considering only points $z \in \mathbb{C}$ where $\mathcal{G}_{\mathbf{X}}(z_\epsilon)$ is in the domain of $\mathcal{B}_{\mathbf{X}}$ for sufficiently small $\epsilon > 0$, the support of the Brown measure of $X = p + iq$ is the closure of the set of z such that

$$\lim_{\epsilon \to 0^+} \mathcal{G}_{\mathbf{X}}(z_\epsilon) = \begin{pmatrix} A & i\overline{B} \\ iB & \overline{A} \end{pmatrix} \tag{10.2}$$

for some $B \neq 0$ or where the limit does not exist.

Recall that from Corollary 4.22 we know what the support of the Brown measure of X is in this situation, it is the intersection of the hyperbola

$$\left\{z = x + iy : \left(x - \frac{\alpha+\alpha'}{2}\right)^2 - \left(y - \frac{\beta+\beta'}{2}\right)^2 = \frac{(\alpha'-\alpha)^2 - (\beta'-\beta)^2}{4}\right\} \tag{10.3}$$

with the rectangle

$$\{z = x + iy : x \in [\alpha \wedge \alpha', \alpha \vee \alpha'], y \in [\beta \wedge \beta', \beta \vee \beta']\} \; . \tag{10.4}$$

Remark 10.2. *In the following sections comprising the proof of Theorem 10.1, we will assume $z \in \mathbb{C}$ is as described in the Theorem: $\mathcal{G}_{\mathbf{X}}(z_\epsilon)$ is in the domain of $\mathcal{B}_{\mathbf{X}}$ for sufficiently small $\epsilon > 0$.*

We give a brief outline of the proof:

Fix some z and consider a sequence $\epsilon_k \to 0^+$. Recall that we use the notation $Q_\epsilon = \mathcal{G}_{\mathbf{X}}(z_\epsilon)$. There are some preliminary steps to reduce to the case where $Q_{\epsilon_k} \to Q \in \mathbb{H}$, $Q \neq 0$. This is discussed in the next section.

After passing to a subsequence, the following two cases follow from (7.85):

1. There exists a sequence $\epsilon_k \to 0^+$ where $l_{\epsilon_k} \to 0$.

2. There exists a sequence $\epsilon_k \to 0^+$ where $B_{\epsilon_k} \to 0$.

We will classify which z is in each of these two cases: these cases impose conditions on the limit Q, which in turn impose conditions on z. Note that $\mathcal{B}_{\mathbf{X}}$ may not be well-defined and/or discontinuous at Q. The proof of Theorem 10.1 follows once all of these cases are understood.

Before we continue, let us highlight that letting $a = b = 1/2$ in Definitions 8.1 and 8.2 makes D_p, D_q, I_p, and I_q particularly simple:

$$\begin{aligned} D_p(w) &= ((\alpha'-\alpha)w)^2 + 1 \\ D_q(w) &= ((\beta'-\beta)w)^2 + 1 \, . \end{aligned} \tag{10.5}$$

$$
\begin{aligned}
I_p &= \left\{ iy : |y| > \frac{1}{|\alpha' - \alpha|} \right\} \\
I_q &= \left\{ iy : |y| > \frac{1}{|\beta' - \beta|} \right\} .
\end{aligned}
\tag{10.6}
$$

In particular, observe that $I_p, I_q \subset i\mathbb{R}$.

Additionally, the ***hyperbola*** will always mean (10.3), and the ***rectangle*** will always mean (10.4).

10.1 Preliminary reductions

In this section, we will reduce to the case where Q_{ϵ_k} converges to some $Q \in \mathbb{H}$, $Q \neq 0$.

For a general sequence $\{Q_k\}$, there are three cases:

1. The sequence $\{Q_k\}$ is not bounded.

2. The sequence $\{Q_k\}$ converges to 0.

3. The sequence $\{Q_k\}$ is bounded but does not converge to 0.

By bounded/unbounded, we refer to the boundedness/unboundedness of the quaternionic norm.

In the third case, we may just pass to a subsequence where $Q_k \to Q$, where $Q \neq 0$, which is what we desired.

The second case is not possible, for $Q_k = Q_{\epsilon_k}$ as from Theorem 8.13,

$$
|z| = \lim_{k\to\infty} |z_{\epsilon_k}| = \lim_{k\to\infty} |\mathcal{B}_{\mathbf{X}}(Q_{\epsilon_k})| = \infty\,. \tag{10.7}
$$

Thus, all that remains is the first case, where we may pass to a subsequence and assume that $|Q_k| \to \infty$. This is the subject of the following Proposition:

Proposition 10.3. *Consider a sequence $\{Q_k\} \subset \mathbb{H}$ where $|Q_k| \to \infty$ and $\mathcal{B}_{\mathbf{X}}(Q_k)$ converges. Then, $\mathcal{B}_{\mathbf{X}}(Q_k)$ converges to one of: $\{\alpha + i\beta, \alpha + i\beta', \alpha' + i\beta, \alpha' + i\beta'\}$.*

Proof. From (7.49) and $|Q_k| \to \infty$ implying $\left|Q_k^{-1}\right| \to 0$, it suffices to analyze

$$\lim_{k\to\infty} \mathcal{B}_{\mathbf{p}}(Q_k) \qquad \lim_{k\to\infty} \mathcal{B}_{i\mathbf{q}}(Q_k)\,. \tag{10.8}$$

Using Proposition 7.18 and noting that $|Q_k i| = |Q_k|$, then it suffices just to analyze the first limit and apply the result to the second limit.

We proceed to show that there exists a subsequence Q_{k_j} where $\mathcal{B}_{\mathbf{p}}(Q_{k_j})$ converges to one of α, α'. From Lemma 7.6, it suffices to show that the eigenvalues of $\mathcal{B}_{\mathbf{p}}(Q_{k_j})$, $B_p(g_{k_j}), B_p(\overline{g_{k_j}}) = \overline{B_p(g_{k_j})}$ converge to one of α, α'.

Since $|Q_k| \to \infty$, then $|g_k| \to \infty$ also. From the expression for B_p, (Proposition 8.6),

$$\begin{aligned}\lim_{k\to\infty} B_p(g_k) &= \lim_{k\to\infty} \frac{\alpha+\alpha'}{2} + \frac{1+\sqrt{D_p(g_k)}}{2g_k} \\ &= \frac{\alpha+\alpha'}{2} + \lim_{k\to\infty} \frac{\sqrt{(\alpha'-\alpha)^2 g_k^2 + 1}}{2g_k}\,.\end{aligned} \tag{10.9}$$

The square of the quantity inside the final limit is:

$$\frac{(\alpha'-\alpha)^2 g_k^2 + 1}{4g_k^2}\,, \tag{10.10}$$

which converges to $(\alpha'-\alpha)^2/4$ as $k \to \infty$. Hence, we may choose a subsequence g_{k_j} such that

$$\lim_{j\to\infty} \frac{\sqrt{(\alpha'-\alpha)^2 g_{k_j}^2 + 1}}{2g_{k_j}} = \pm\frac{\alpha'-\alpha}{2}\,. \tag{10.11}$$

Then,

$$\lim_{j\to\infty} B_p(g_{k_j}) = \frac{\alpha+\alpha'}{2} \pm \frac{\alpha'-\alpha}{2}\,, \tag{10.12}$$

i.e. $B_p(g_{k_j})$ converges to one of α, α'. Hence, $\mathcal{B}_{\mathbf{p}}(Q_{k_j})$ converges to one of α, α'.

By applying the same argument to $\mathcal{B}_{i\mathbf{q}}(Q_k)$, there exists a subsequence k_{j_l} where $\mathcal{B}_{\mathbf{X}}(Q_{k_{j_l}})$ converges to one of $\{\alpha+i\beta, \alpha+i\beta', \alpha'+i\beta, \alpha'+i\beta'\}$. Hence, $\mathcal{B}_{\mathbf{X}}(Q_k)$ also converges to one of $\{\alpha+i\beta, \alpha+i\beta', \alpha'+i\beta, \alpha'+i\beta'\}$. $\square$

Thus, for what follows, we may assume that we are considering a sequence $Q_k \to Q$ and $Q \neq 0$.

We return to the original dichotomy, which follows from (7.85):

1. There exists a sequence $\epsilon_k \to 0^+$ where $l_{\epsilon_k} \to 0$.
2. There exists a sequence $\epsilon_k \to 0^+$ where $B_{\epsilon_k} \to 0$.

10.2 $l_k \to 0$

In this section, we determine the z such that there exists a sequence $\epsilon_k \to 0^+$ where $Q_{\epsilon_k} \to Q \neq 0$ and $l_{\epsilon_k} \to 0$.

We summarize the results of the three cases when $l_k \to 0$:

1. If $\mathcal{B}_{\mathbf{X}}$ is discontinuous at Q, then $\mathcal{B}_{\mathbf{X}}(Q_k)$ converges to z on the intersection of the hyperbola with the open rectangle. (Proposition 10.4)
2. If $\mathcal{B}_{\mathbf{X}}$ is continuous at Q:
   (a) If $g \in \mathbb{R}$ or $g^I \in \mathbb{R}$, then $l(Q_k) \not\to 0$. (Proposition 10.5)
   (b) If $g \notin \mathbb{R}$ and $g^I \notin \mathbb{R}$, then $\mathcal{B}_{\mathbf{X}}(Q_k) \to \mathcal{B}_{\mathbf{X}}(Q) = z$, which is on the intersection of the hyperbola with the open rectangle. (Proposition 10.6)

For the first case, we have the following result:

Proposition 10.4. *If Q_k converges to $Q \neq 0$, $l_k \to 0$, and $\mathcal{B}_{\mathbf{X}}$ is discontinuous at Q, then one of the following is true:*

1. $g_k \to I_p$ *and* $g_k^I \to \overline{I_q}$.
2. $g_k \to \overline{I_p}$ *and* $g_k^I \to I_q$.

In either case, $\mathcal{B}_{\mathbf{X}}(Q_k)$ converges to $z = x + iy$ on the intersection of the hyperbola

$$H = \left\{ z = x + iy : \left(x - \frac{\alpha + \alpha'}{2} \right)^2 - \left(y - \frac{\beta + \beta'}{2} \right)^2 = \frac{(\alpha' - \alpha)^2 - (\beta' - \beta)^2}{4} \right\} \tag{10.13}$$

with the open rectangle

$$\mathring{R} = \{ z = x + iy : x \in (\alpha \wedge \alpha', \alpha \vee \alpha'), y \in (\beta \wedge \beta', \beta \vee \beta') \} . \tag{10.14}$$

Proof. From Theorem 8.13, if $\mathcal{B}_{\mathbf{X}}$ is discontinuous at Q, then either g_k converges to I_p or g_k^I converges to I_q. We will prove the second case, the first case is similar.

Recall that for $g_k, g_k^I \notin \mathbb{R}$,

$$l_k = \frac{1}{2\left|g_k\right|^2} \left(\frac{\overline{g_k}\sqrt{D_p(g_k)} - g_k\sqrt{D_p(\overline{g_k})}}{g_k - \overline{g_k}} + \frac{\overline{g_k^I}\sqrt{D_q(g_k^I)} - g_k^I\sqrt{D_q(\overline{g_k^I})}}{g_k^I - \overline{g_k^I}} \right) , \tag{10.15}$$

and there exists continuous extensions in the case where $g_k \in \mathbb{R} \setminus \{0\}$ or $g_k^I \in \mathbb{R} \setminus \{0\}$ (see Proposition 8.12).

If g_k^I converges to I_q, then from Lemma 8.5, the second term inside the parenthesis converges to 0. As $l_k \to 0$, then the first term inside the parentheses (or its continuous extension to $\mathbb{R}$) also must converge to 0:

$$\tilde{l}(g_k) = \begin{cases} \dfrac{\overline{g_k}\sqrt{D_p(g_k)} - g_k\sqrt{D_p(\overline{g_k})}}{g_k - \overline{g_k}} & g_k \notin \mathbb{R} \\ \dfrac{-1}{\sqrt{D_p(g_k)}} & g_k \in \mathbb{R} \end{cases} \longrightarrow 0 . \tag{10.16}$$

Let g_k converge to $g \neq 0$. From Lemma 8.5, $\tilde{l}$ is continuous on $\mathbb{C} \setminus \{0\}$ when $a = 1/2$. Hence, $\tilde{l}(g) = 0$. We proceed to show that $g \in \overline{I_p}$ by considering what happens if $g \in \mathbb{C} \setminus \overline{I_p}$:

If $g \in \mathbb{R}$, then

$$\tilde{l}(g) = \frac{-1}{D_p(g)} < 0 . \tag{10.17}$$

If $g \in \mathbb{C} \setminus (\mathbb{R} \cup \overline{I_p})$, then consider the following equivalences:

$$\begin{aligned}
\tilde{l}(g) = 0 \iff & \operatorname{Im}\left(\overline{g}\sqrt{D_p(g)}\right) = 0 \\
\iff & \overline{g}\sqrt{D_p(g)} = t, \qquad t \in \mathbb{R} \\
\iff & |g|^2 \sqrt{D_p(g)} = tg \\
\iff & \sqrt{D_p(g)} = \frac{t}{|g|^2} g \\
\implies & (\alpha' - \alpha)^2 g^2 + 1 = \frac{t^2}{|g|^4} g^2 \\
\iff & g^2 = \frac{1}{\frac{t^2}{|g|^4} - (\alpha' - \alpha)^2}\,.
\end{aligned} \tag{10.18}$$

The denominator in the last term is non-zero, or else the equality in the previous line is incorrect. Thus, for $\tilde{l}(g) = 0$, $g^2 \in \mathbb{R}$. As we assumed $g \notin \mathbb{R}$, then $g \in i\mathbb{R}$. Since we assumed $g \notin \overline{I_p}$, then $g = iy$ for some $y \in \mathbb{R} \setminus \{0\}$, $y < 1/|\alpha' - \alpha|$. But, for such a g, $D_p(g) > 0$, so

$$\operatorname{Im}\left(\overline{g}\sqrt{D_p(g)}\right) = -\sqrt{D_p(iy)}y \neq 0\,. \tag{10.19}$$

Hence, $\tilde{l}(g) \neq 0$.

Similar analysis switching p, g for q, g^I shows that there are two possibilities:

1. $g_k \to \overline{I_p}$ and $g_k^I \to I_q$.
2. $g_k \to I_p$ and $g_k^I \to \overline{I_q}$.

In either situation, g_k, g_k^I converge to $i\mathbb{R}$, so that in the real coefficients of Q_k, $(x_0)_k \to 0$ and $(x_3)_k \to 0$. In particular, from (7.7) and (7.9) this implies that g_k and g_k^I both converge to some it, $t > 0$. Since $l_k \to 0$, then using (7.79),

$$\lim_{k\to\infty} \mathcal{B}_{\mathbf{X}}(Q_k) = \lim_{k\to\infty} \beta_p(g_k) + i\beta_q(g_k^I)\,. \tag{10.20}$$

From Lemma 8.7, depending on if g_k and g_k^I approach it from the left or right, all 4 limits are possible:

$$
\begin{aligned}
\lim_{k\to\infty} \mathcal{B}_{\mathbf{X}}(Q_k) &= \lim_{k\to\infty} \beta_p(g_k) + i\beta_q(g_k^I) \\
&= \left(\frac{\alpha+\alpha'}{2} \pm \frac{\sqrt{-D_p(it)}}{2t}\right) + i\left(\frac{\beta+\beta'}{2} \pm \frac{\sqrt{-D_p(it)}}{2t}\right).
\end{aligned}
\tag{10.21}
$$

Hence, if the limit is $z = x + iy$, then

$$
\begin{aligned}
x &= \frac{\alpha+\alpha'}{2} \pm \frac{\sqrt{-D_p(it)}}{2t} \\
y &= \frac{\beta+\beta'}{2} \pm \frac{\sqrt{-D_q(it)}}{2t}.
\end{aligned}
\tag{10.22}
$$

Finally, we check that this z is on the intersection of the hyperbola and the open rectangle. For the hyperbola equation:

$$
\begin{aligned}
&\left(x - \frac{\alpha+\alpha'}{2}\right)^2 - \left(y - \frac{\beta+\beta'}{2}\right)^2 \\
&= \left(\frac{\sqrt{-D_p(it)}}{2t}\right)^2 - \left(\frac{\sqrt{-D_q(it)}}{2t}\right)^2 \\
&= \left(\frac{\sqrt{(\alpha'-\alpha)^2t^2-1}}{2t}\right)^2 - \left(\frac{\sqrt{(\beta'-\beta)^2t^2-1}}{2t}\right)^2 \\
&= \frac{(\alpha'-\alpha)^2t^2-1}{4t^2} - \frac{(\beta'-\beta)^2t^2-1}{4t^2} \\
&= \frac{(\alpha'-\alpha)^2-(\beta'-\beta)^2}{4}.
\end{aligned}
\tag{10.23}
$$

For the rectangle condition, from Lemma 4.7 it suffices to check $x \in (\alpha \wedge \alpha', \alpha \vee \alpha')$:

$$\begin{aligned}
\left| x - \frac{\alpha + \alpha'}{2} \right| &= \frac{\sqrt{-D_p(it)}}{2t} \\
&= \frac{\sqrt{(\alpha' - \alpha)^2 t^2 - 1}}{2t} \\
&= \sqrt{\frac{(\alpha' - \alpha)^2 t^2 - 1}{4t^2}} \\
&= \sqrt{\frac{(\alpha' - \alpha)^2}{4} - \frac{1}{4t^2}} \\
&< \sqrt{\frac{(\alpha' - \alpha)^2}{4}} \\
&= \frac{|\alpha' - \alpha'|}{2} \, .
\end{aligned} \tag{10.24}$$

□

Now, consider the case where $\mathcal{B}_{\mathbf{X}}$ is continuous at Q and $g \in \mathbb{R}$ or $g^I \in \mathbb{R}$. We can rule out this case from happening, so we can use (7.79) to analyze the general case:

Proposition 10.5. *If Q_k converges to $Q \neq 0$, $\mathcal{B}_{\mathbf{X}}$ is continuous at Q, and $g \in \mathbb{R}$ or $g^I \in \mathbb{R}$, then $l_k \not\to 0$.*

Proof. The cases $g \in \mathbb{R}$ and $g^I \in \mathbb{R}$ are similar, so we will just prove the case when $g \in \mathbb{R}$. From Proposition 8.12 and Lemma 8.5, l can be extend continuously to (g, g^I) and l_k converges to $l(g, g^I)$:

$$l(g, g^I) = \frac{1}{2\,|g|^2} \left(\frac{-1}{\sqrt{D_p(g)}} + \frac{\mathrm{Im}(\overline{g^I}\sqrt{D_q(g^I)})}{\mathrm{Im}(g^I)} \right) . \tag{10.25}$$

From Lemma 7.5, we can just assume that $g^I = ig$ for the purpose of evaluating l:

$$l(g, g^I) = l(g, ig) = \frac{1}{2\,|g|^2} \left(\frac{-1}{\sqrt{D_p(g)}} + \frac{\mathrm{Im}(\overline{ig}\sqrt{D_q(ig)})}{\mathrm{Im}(ig)} \right) . \tag{10.26}$$

Since $\mathcal{B}_{\mathbf{X}}$ is continuous at Q, $g^I \notin I_q$, so $ig \notin I_q$. Combining this with $g \in \mathbb{R}$, then $D_q(ig) \geq 0$, and

$$\frac{\mathrm{Im}(\overline{ig}\sqrt{D_q(ig)})}{\mathrm{Im}(ig)} = -\sqrt{D_q(ig)} \leq 0 \, . \tag{10.27}$$

Therefore,

$$l(g, ig) \leq \frac{1}{2\,|g|^2}\left(\frac{-1}{\sqrt{D_p(g)}}\right) < 0\,. \tag{10.28}$$

$\square$

For the final case, consider the following Proposition:

Proposition 10.6. *If Q_k converges to $Q \neq 0$, $l_k \to 0$, and $\mathcal{B}_{\mathbf{X}}$ is continuous at Q, then $\mathcal{B}_{\mathbf{X}}(Q_k)$ converges to $\mathcal{B}_{\mathbf{X}}(Q) = z = x + iy$ on the intersection of the hyperbola*

$$H = \left\{ z = x + iy : \left(x - \frac{\alpha + \alpha'}{2}\right)^2 - \left(y - \frac{\beta + \beta'}{2}\right)^2 = \frac{(\alpha' - \alpha)^2 - (\beta' - \beta)^2}{4} \right\} \tag{10.29}$$

with the open rectangle

$$\mathring{R} = \{z = x + iy : x \in (\alpha \wedge \alpha', \alpha \vee \alpha'), y \in (\beta \wedge \beta', \beta \vee \beta')\}\ . \tag{10.30}$$

Proof. Since $\mathcal{B}_{\mathbf{X}}$ is continuous at Q, then

$$\lim_{k \to \infty} \mathcal{B}_{\mathbf{X}}(Q_k) = \mathcal{B}_{\mathbf{X}}(Q)\,. \tag{10.31}$$

Since $Q \neq 0$, then from Proposition 8.12, l is continuous at Q, so

$$l(Q) = \lim_{k \to \infty} l_k = 0\,. \tag{10.32}$$

From Proposition 10.5, since $l_k \to 0$, then $g, g^I \notin \mathbb{R}$, i.e. $Q \notin \mathbb{R} \cup i\mathbb{R}$.

Hence, (7.79) applies and shows that $\mathcal{B}_{\mathbf{X}} = z$ for some $z \in \mathbb{C}$.

Applying Proposition 9.1 completes the proof. $\square$

10.3 $B_k \to 0$

In this section, we determine the z such that there exists a sequence $\epsilon_k \to 0^+$ where $Q_{\epsilon_k} \to Q \neq 0$ and $B_{\epsilon_k} \to 0$, i.e. $Q \in \mathbb{C}$.

We summarize the results of the two cases when $B_k \to 0$:

1. If $\mathcal{B}_{\mathbf{X}}$ is continuous at Q, then $\mathcal{B}_{\mathbf{X}}(Q_k) \to \mathcal{B}_{\mathbf{X}}(Q) = z$, and whenever z is on the hyperbola, z is not in the rectangle. (Proposition 10.7)

2. If $\mathcal{B}_{\mathbf{X}}$ is discontinuous at Q, then $\mathcal{B}_{\mathbf{X}}(Q_k)$ does not converge to a z on the hyperbola. (Proposition 10.10)

Recall that $g \in \mathbb{C} \subset \mathbb{H}$ is a continuity point of $\mathcal{B}_{\mathbf{X}}$ if and only if $g \notin I_p \cup \{0\}$ and $g^I \notin I_q \cup \{0\}$.

For the first case, we have the following result:

Proposition 10.7. *If Q_k converges to $Q \in \mathbb{C}$ where $Q \neq 0$ and $\mathcal{B}_{\mathbf{X}}$ is continuous at Q, then $\mathcal{B}_{\mathbf{X}}(Q_k)$ converges to $\mathcal{B}_{\mathbf{X}}(Q) = z = x + iy$ such that:*

$$\left(x - \frac{\alpha + \alpha'}{2}\right)^2 - \left(y - \frac{\beta + \beta'}{2}\right)^2 = \frac{(\alpha' - \alpha)^2 - (\beta' - \beta)^2}{4}, \tag{10.33}$$

then

$$\left|\left(x - \frac{\alpha + \alpha'}{2}\right)\left(y - \frac{\beta + \beta'}{2}\right)\right| > \frac{(\alpha' - \alpha)(\beta' - \beta)}{4}. \tag{10.34}$$

In particular, this implies that z is not on the intersection of the hyperbola and rectangle.

Proof. Since $\mathcal{B}_{\mathbf{X}}$ is continuous at Q, then $\mathcal{B}_{\mathbf{X}}(Q_k) \to \mathcal{B}_{\mathbf{X}}(Q)$. Since $Q \in \mathbb{C}$, then $\mathcal{B}_{\mathbf{X}}(Q) = z \in \mathbb{C}$ also. Let $Q = g \in \mathbb{C}$.

Applying the addition law (7.49),

$$z = \mathcal{B}_{\mathbf{X}}(g) = \mathcal{B}_{\mathbf{p}}(g) + i\mathcal{B}_{\mathbf{q}}(ig) - \frac{1}{g}. \tag{10.35}$$

Hence,

$$\begin{aligned}
z &= B_X(g) \\
&= B_p(g) + iB_q(ig) - \frac{1}{g} \\
&= \left(\frac{\alpha+\alpha'}{2} + \frac{1+\sqrt{(\alpha'-\alpha)^2g^2+1}}{2g}\right) + \\
&\qquad i\left(\frac{\beta+\beta'}{2} + \frac{1+\sqrt{(\beta'-\beta)^2(ig)^2+1}}{2ig}\right) - \frac{1}{g} \\
&= \frac{\alpha+\alpha'}{2} + i\frac{\beta+\beta'}{2} + \frac{\sqrt{(\alpha'-\alpha)^2g^2+1}}{2g} + \frac{\sqrt{1-(\beta'-\beta)^2g^2}}{2g}\,.
\end{aligned} \tag{10.36}$$

We proceed to show that if z lies on the hyperbola in (10.3), then z lies outside of the rectangle in (10.4).

Let

$$\begin{aligned}
\mathscr{A} &= \alpha' - \alpha \\
\mathscr{B} &= \beta' - \beta\,.
\end{aligned} \tag{10.37}$$

From Lemma 4.7, the equation of the hyperbola for $z = x + iy$ can be written as

$$\mathrm{Re}\left(\left(z - \frac{\alpha+\alpha'}{2} - i\frac{\beta+\beta'}{2}\right)^2 - \frac{\mathscr{A}^2-\mathscr{B}^2}{4}\right) = 0\,. \tag{10.38}$$

For $z = x + iy$ on the hyperbola, z is on the rectangle if and only if

$$\left|\mathrm{Im}\left(\left(z - \frac{\alpha+\alpha'}{2} - i\frac{\beta+\beta'}{2}\right)^2 - \frac{\mathscr{A}^2-\mathscr{B}^2}{4}\right)\right| \le \frac{\mathscr{A}\mathscr{B}}{2}\,. \tag{10.39}$$

The relevant quantity can be simplified to:

$$\left(z - \frac{\alpha+\alpha'}{2} - i\frac{\beta+\beta'}{2}\right)^2 - \frac{\mathscr{A}^2-\mathscr{B}^2}{4} = \frac{1+\sqrt{\mathscr{A}^2g^2+1}\sqrt{1-\mathscr{B}^2g^2}}{2g^2} \tag{10.40}$$

Hence, the hyperbola equation is:

$$\mathrm{Re}\left(\frac{1+\sqrt{\mathscr{A}^2g^2+1}\sqrt{1-\mathscr{B}^2g^2}}{2g^2}\right) = 0\,, \tag{10.41}$$

i.e.

$$\frac{1+\sqrt{\mathscr{A}^2 g^2+1}\sqrt{1-\mathscr{B}^2 g^2}}{2g^2}=\eta\,, \tag{10.42}$$

for some $\eta \in i\mathbb{R}$.

For z on the hyperbola, z is on the rectangle when

$$\left|\frac{1+\sqrt{\mathscr{A}^2 g^2+1}\sqrt{1-\mathscr{B}^2 g^2}}{2g^2}\right| \leq \frac{\mathscr{A}\mathscr{B}}{2}\,. \tag{10.43}$$

Simplifying (10.42) further, z is on the hyperbola if and only if

$$\sqrt{\mathscr{A}^2 g^2+1}\sqrt{1-\mathscr{B}^2 g^2}=2g^2\eta-1\,, \tag{10.44}$$

Squaring both sides and rearranging, g must satisfy:

$$0=g^2((4\eta^2-\mathscr{A}^2\mathscr{B}^2)g^2-(4\eta+(\mathscr{A}^2-\mathscr{B}^2)))\,. \tag{10.45}$$

As $g \neq 0$, then:

$$g^2=\frac{4\eta+(\mathscr{A}^2-\mathscr{B}^2)}{4\eta^2+\mathscr{A}^2\mathscr{B}^2}\,. \tag{10.46}$$

All steps were reversible except for the step where both sides were squared. We proceed to determine which values of η make (10.41) true, and show for these η, (10.43) is not satisfied.

Computation shows that the relevant quantities in (10.42) are:

$$\begin{aligned}\mathscr{A}^2 g^2+1&=\frac{(\mathscr{A}^2+2\eta)^2}{4\eta^2+\mathscr{A}^2\mathscr{B}^2}\\ 1-\mathscr{B}^2 g^2&=\frac{(\mathscr{B}^2-2\eta)^2}{4\eta^2+\mathscr{A}^2\mathscr{B}^2}\,.\end{aligned} \tag{10.47}$$

To take the square root of these quantities, consider the following cases for $\eta \in i\mathbb{R}$:

1. $|\eta| < \mathscr{A}\mathscr{B}/2$, i.e. $|\mathrm{Im}(\eta)| \leq \mathscr{A}\mathscr{B}/2$.

2. $|\eta| > \mathscr{A}\mathscr{B}/2$, i.e. $\mathrm{Im}(\eta) > \mathscr{A}\mathscr{B}/2$ or $\mathrm{Im}(\eta) < -\mathscr{A}\mathscr{B}/2$.

For the first case, note that the denominator in (10.47) is positive and $\mathscr{A}^2 + 2\eta, \mathscr{B}^2 - 2\eta$ are in the right half-plane, so

$$
\begin{aligned}
\sqrt{\mathscr{A}^2 g^2 + 1} &= \frac{\mathscr{A}^2 + 2\eta}{\sqrt{4\eta^2 + \mathscr{A}^2\mathscr{B}^2}} \\
\sqrt{1 - \mathscr{B}^2 g^2} &= \frac{\mathscr{B}^2 - 2\eta}{\sqrt{4\eta^2 + \mathscr{A}^2\mathscr{B}^2}} .
\end{aligned} \tag{10.48}
$$

Hence,

$$
\frac{1 + \sqrt{\mathscr{A}^2 g^2 + 1}\sqrt{1 - \mathscr{B}^2 g^2}}{2g^2} = \frac{\mathscr{A}^2\mathscr{B}^2 + (\mathscr{B}^2 - \mathscr{A}^2)\eta}{4\eta + (\mathscr{A}^2 - \mathscr{B}^2)} . \tag{10.49}
$$

If $\mathscr{A}^2 - \mathscr{B}^2 = 0$, then

$$
\frac{\mathscr{A}^2\mathscr{B}^2 + (\mathscr{B}^2 - \mathscr{A}^2)\eta}{4\eta + (\mathscr{A}^2 - \mathscr{B}^2)} = \frac{\mathscr{A}^2\mathscr{B}^2}{4\eta} . \tag{10.50}
$$

For any $\eta \in i\mathbb{R}$, this quantity is purely imaginary, and so z lies on the hyperbola. But, for $|\eta| < \mathscr{A}\mathscr{B}/2$, $|\mathscr{A}^2\mathscr{B}^2/(4\eta)| > \mathscr{A}\mathscr{B}/2$, so z is not on the rectangle.

If $\mathscr{A}^2 - \mathscr{B}^2 \neq 0$, the right-hand side in (10.49) is purely imaginary when:

$$
\frac{-4\,\mathrm{Im}(\eta)}{\mathscr{A}^2\mathscr{B}^2} = \frac{\mathscr{A}^2 - \mathscr{B}^2}{(\mathscr{B}^2 - \mathscr{A}^2)\,\mathrm{Im}(\eta)} , \tag{10.51}
$$

i.e. when $\mathrm{Im}(\eta) = \pm\mathscr{A}\mathscr{B}/2$. This is impossible for $|\eta| < \mathscr{A}\mathscr{B}/2$, so z is not on the hyperbola.

Thus, when $|\eta| < \mathscr{A}\mathscr{B}/2$, it is impossible for z to be on both the hyperbola and rectangle.

For $|\eta| > \mathscr{A}\mathscr{B}/2$, first consider when $\mathrm{Im}(\eta) > \mathscr{A}\mathscr{B}/2$. Then, the denominator in (10.47) is negative, $\mathscr{A}^2 + 2\eta$ is in the first quadrant, and $\mathscr{B}^2 - 2\eta$ is in the fourth quadrant, so

$$
\begin{aligned}
\sqrt{\mathscr{A}^2 g^2 + 1} &= \frac{-i(\mathscr{A}^2 + 2\eta)}{\sqrt{-4\eta^2 - \mathscr{A}^2\mathscr{B}^2}} \\
\sqrt{1 - \mathscr{B}^2 g^2} &= \frac{i(\mathscr{B}^2 - 2\eta)}{\sqrt{-4\eta^2 - \mathscr{A}^2\mathscr{B}^2}} .
\end{aligned} \tag{10.52}
$$

When $\mathrm{Im}(\eta) < -\mathscr{A}\mathscr{B}/2$, the denominator in (10.47) is negative, $\mathscr{A}^2 + 2\eta$ is in the fourth quadrant, and $\mathscr{B}^2 - 2\eta$ is in the first quadrant, so

$$\begin{aligned}
\sqrt{\mathscr{A}^2 g^2 + 1} &= \frac{i(\mathscr{A}^2 + 2\eta)}{\sqrt{-4\eta^2 - \mathscr{A}^2\mathscr{B}^2}} \\
\sqrt{1 - \mathscr{B}^2 g^2} &= \frac{-i(\mathscr{B}^2 - 2\eta)}{\sqrt{-4\eta^2 - \mathscr{A}^2\mathscr{B}^2}} \,.
\end{aligned} \tag{10.53}$$

In either case, when $|\eta| > \mathscr{A}\mathscr{B}/2$,

$$\frac{1 + \sqrt{\mathscr{A}^2 g^2 + 1}\sqrt{1 - \mathscr{B}^2 g^2}}{2g^2} = \frac{4\eta^2 + \eta(\mathscr{A}^2 - \mathscr{B}^2)}{4\eta + (\mathscr{A}^2 - \mathscr{B}^2)} = \eta \,. \tag{10.54}$$

From (10.46), the denominator $4\eta + (\mathscr{A}^2 - \mathscr{B}^2)$ is zero exactly when $g = 0$, which is impossible, so this expression makes sense. Thus, for all η where $|\eta| > \mathscr{A}\mathscr{B}/2$, z is on the hyperbola. But, since $|\eta| > \mathscr{A}\mathscr{B}/2$, then (10.43) does not hold and z is not on the rectangle. $\square$

Finally, consider when Q_k converges to $Q \in \mathbb{C}$, $Q \neq 0$, such that Q is not a continuity point of $\mathcal{B}_{\mathbf{X}}$. Assuming that $Q = g$ and $g^I = ig$, this happens when either $g \in I_p$ or $ig \in I_q$, i.e. $g \in i\mathbb{R} \cup \mathbb{R}$. Then, in the addition law (7.49), one of $\mathcal{B}_{\mathbf{p}}$ or $\mathcal{B}_{i\mathbf{q}}$ is continuous. We can handle the other term with the following general Lemma:

Lemma 10.8. *Let $p \in (M, \tau)$ be Hermitian and consider a sequence $\{Q_k\} \subset \mathbb{H}$ such that $Q_k \to Q \in \mathbb{C}$, where $Q \neq 0$ either satisfies $Q \in \mathbb{C} \setminus \mathbb{R}$ or B_p is continuous at Q. Then,*

$$\lim_{k\to\infty} \mathcal{B}_{\mathbf{p}}(Q_k) = \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix} , \tag{10.55}$$

where $\lim_{k\to\infty} B_p(\zeta_k) = z$ for some sequence $\{\zeta_k\} \subset \mathbb{C}$ where $\zeta_k \to Q$.

It follows that if G_p is defined and continuous at z, then $Q = G_p(z)$.

Proof. We will show the sequence

$$\zeta_k = \begin{cases} g_k & \mathrm{Im}(Q) \geq 0 \\ \overline{g_k} & \mathrm{Im}(Q) < 0 \end{cases} \tag{10.56}$$

satisfies the conclusion of the Lemma.

When $Q \in \mathbb{R}$, then from hypothesis, B_p is continuous at Q. Then, $g_k, \overline{g_k}$ converge to Q, so the eigenvalues of $\mathcal{B}_{\mathbf{p}}(Q_k)$, $B_p(g_k), B_p(\overline{g_k})$ converge to $B_p(Q)$. Since $Q \in \mathbb{R}$, then $B_p(Q) \in \mathbb{R}$. From Lemma 7.6, $\mathcal{B}_{\mathbf{p}}(Q_k)$ converges to $B_p(Q)$. Thus, our choice of ζ_k satisfies the conclusion of the Lemma.

For $Q \notin \mathbb{R}$, Consider a diagonalization of Q_k:

$$Q_k = S_k^{-1} g_k S_k \,. \tag{10.57}$$

Then,

$$\begin{aligned} \mathcal{B}_{\mathbf{p}}(Q_k) &= \mathcal{B}_{\mathbf{p}}(S_k^{-1} g_k S_k) \\ &= S_k^{-1} \mathcal{B}_{\mathbf{p}}(g_k) S_k \\ &= S_k^{-1} B_p(g_k) S_k \,. \end{aligned} \tag{10.58}$$

Recall that g_k is the eigenvalue of Q_k with $\mathrm{Im}(g_k) \geq 0$. From the definition of ζ_k, it suffices to show that we can choose suitable S_k for each of the following cases for $x_3 = \mathrm{Im}(Q)$:

1. If $x_3 < 0$, S_k converges to an invertible matrix S that switches the diagonal entries of $B_p(g_k)$, i.e. $SB_p(g_k)S^{-1} = B_p(\overline{g_k})$.

2. If $x_3 > 0$, S_k converges to an invertible matrix that fixes the diagonal entries of $B_p(g_k)$, i.e. $SB_p(g_k)S^{-1} = B_p(g_k)$.

For the first case, where $x_3 < 0$, computation shows that we may choose

$$S_k = \begin{pmatrix} iB_k & g_k - A_k \\ \overline{g} - \overline{A_k} & iB_k \end{pmatrix} \tag{10.59}$$

to diagonalize the matrix. Since $Q_k \to Q \in \mathbb{C}$, the diagonal terms of S_k converge to 0. When

$(x_3)_k < 0$, the off-diagonal terms are:

$$\begin{aligned} g_k - A_k &= i\left(\sqrt{(x_1)_k^2 + (x_2)_k^2 + (x_3)_k^2} - (x_3)_k\right) \\ &= i\left(\sqrt{(x_1)_k^2 + (x_2)_k^2 + (x_3)_k^2} + |(x_3)_k|\right) \end{aligned} \tag{10.60}$$

As $B_k \to 0$, then $(x_1)_k, (x_2)_k \to 0$ and $(x_3)_k \to x_3$, so this term converges to $2i\,|x_3|$. Hence,

$$S_k \to \begin{pmatrix} 0 & 2i\,|x_3| \\ -2i\,|x_3| & 0 \end{pmatrix}. \tag{10.61}$$

This matrix is invertible and switches the diagonal entries of the matrix.

If $x_3 > 0$, we alter the previous S_k by dividing by B_k, choosing

$$S_k = \begin{cases} \begin{pmatrix} i & \frac{g_k - A_k}{B_k} \\ \frac{\overline{g} - \overline{A_k}}{\overline{B_k}} & i \end{pmatrix} & B_k \neq 0 \\ \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} & B_k = 0\,. \end{cases} \tag{10.62}$$

It suffices to check that S_k tends to its value at $B_k = 0$. For this, as the off-diagonal terms are conjugates, we examine just one of them:

$$\frac{g_k - A_k}{B_k} = \frac{i\left(\sqrt{(x_1)_k^2 + (x_2)_k^2 + (x_3)_k^2} - (x_3)_k\right)}{(x_1)_k + i(x_2)_k}. \tag{10.63}$$

Taking the absolute value of the right-hand side,

$$\left|\frac{i\left(\sqrt{(x_1)_k^2 + (x_2)_k^2 + (x_3)_k^2} - (x_3)_k\right)}{(x_1)_k + i(x_2)_k}\right| = \frac{\sqrt{(x_1)_k^2 + (x_2)_k^2 + (x_3)_k^2} - (x_3)_k}{\sqrt{(x_1)_k^2 + (x_2)_k^2}}. \tag{10.64}$$

When $(x_3)_k > 0$, applying the Mean Value Theorem to the function $f(t) = \sqrt{(x_3)_k^2 + t^2}$ for

$t \in [0, \sqrt{(x_1)_k^2 + (x_2)_k^2}]$ yields

$$\frac{\sqrt{(x_1)_k^2 + (x_2)_k^2 + (x_3)_k^2} - (x_3)_k}{(x_1)_k^2 + (x_2)_k^2} = \frac{t'}{\sqrt{(x_3)_k^2 + (t')^2}}\,, \tag{10.65}$$

for some $t' \in (0, \sqrt{(x_1)_k^2 + (x_2)_k^2})$. Since $(x_3)_k \to x_3 > 0$ and $B_k = (x_1)_k + (x_2)_k \to 0$, then the following inequalities show that the off-diagonal terms converge to 0:

$$\frac{t'}{\sqrt{(x_3)_k^2 + (t')^2}} \leq \frac{\sqrt{(x_1)_k^2 + (x_2)_k^2}}{|(x_3)_k|} \to 0\,. \tag{10.66}$$

For the final point, if G_p is continuous at z, then

$$Q = \lim_{k\to\infty} \zeta_k = \lim_{k\to\infty} G_p(B_p(\zeta_k)) = G_p(z)\,. \tag{10.67}$$

$\square$

Consider the previous result, but with p replaced with $X = p + iq$. From the addition law (7.49) and the previous result, we could prove a result with two sequences ζ_k and ζ_k' converging to Q and Qi for $\mathcal{B}_{\mathbf{p}}$ and $\mathcal{B}_{i\mathbf{q}}$. For general p, q we may not be able to use only one sequence by replacing ζ_k' with $i\zeta_k$, as the limits of $\mathcal{B}_{i\mathbf{q}}$ along these sequences may be different. But, in our situation, we can:

Corollary 10.9. *Consider a sequence $\{Q_k\} \subset \mathbb{H}$ such that $Q_k \to Q \in \mathbb{C}$, where $Q \neq 0$. Then,*

$$\lim_{k\to\infty} \mathcal{B}_{\mathbf{X}}(Q_k) = \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix}\,, \tag{10.68}$$

where $\lim_{k\to\infty} B_X(\zeta_k) = z$ for some sequence $\{\zeta_k\} \subset \mathbb{C}$ where $\zeta_k \to Q$.

It follows that if G_X is defined and continuous at z, then $Q = G_X(z)$.

Proof. When $\mathcal{B}_{\mathbf{X}}$ is continuous at Q, we can just take ζ_k to be any sequence converging to Q.

When $\mathcal{B}_{\mathbf{X}}$ is discontinuous at Q, either $g \in I_p$ or $g^I \in I_q$. In either case, in the addition law (7.49), only one of $\mathcal{B}_{\mathbf{p}}$ or $\mathcal{B}_{i\mathbf{q}}$ is discontinuous at Q.

Lemma 10.8 produces the appropriate ζ_k that works for the one of $\mathcal{B}_{\mathbf{p}}$ or $\mathcal{B}_{i\mathbf{q}}$ that is discontinuous at Q, and that ζ_k will also work for the other function that is continuous at Q.

For the final point, if G_X is continuous at z, then

$$Q = \lim_{k\to\infty} \zeta_k = \lim_{k\to\infty} G_X(B_X(\zeta_k)) = G_X(z)\,. \tag{10.69}$$

□

Finally, we are ready to prove the following Proposition:

Proposition 10.10. *If Q_k converges to $Q \in \mathbb{C}$ where $Q \neq 0$ and $\mathcal{B}_{\mathbf{X}}$ is discontinuous at Q, then $\mathcal{B}_{\mathbf{X}}(Q_k) \to z \in \mathbb{C}$ where $z = x + iy$ has*

$$\left(x - \frac{\alpha+\alpha'}{2}\right)^2 - \left(y - \frac{\beta+\beta'}{2}\right)^2 \neq \frac{(\alpha'-\alpha)^2 - (\beta'-\beta)^2}{4}\,. \tag{10.70}$$

Proof. From Theorem 8.13, $g \in I_p$ or $g^I \in I_q$. We will prove the first case, the second case is similar.

From Corollary 10.9,

$$\mathcal{B}_{\mathbf{X}}(Q_k) \to \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix}\,, \tag{10.71}$$

where $\lim_{k\to\infty} B_X(\zeta_k) = z$ for some $\zeta_k \to g$. From the addition law (7.49),

$$B_X(\zeta_k) = B_p(\zeta_k) + iB_q(\zeta_k i) - \frac{1}{\zeta_k}\,. \tag{10.72}$$

If $g \in I_p$, then B_p is discontinuous at g. From Proposition 8.6,

$$\lim_{k\to\infty} B_p(\zeta_k) = \frac{\alpha+\alpha'}{2} + \frac{1 \pm i\sqrt{-D_p(g)}}{2g}\,. \tag{10.73}$$

Since $\zeta_k i \to ig \in \mathbb{R}$, then B_q is continuous at ig. From Proposition 8.6,

$$\lim_{k\to\infty} iB_q(\zeta_k i) = iB_q(gi) = i\frac{\beta+\beta'}{2} + \frac{1+\sqrt{D_q(ig)}}{2g}\,. \tag{10.74}$$

Hence,

$$z = \frac{\alpha + \alpha'}{2} + i\frac{\beta + \beta'}{2} \pm i\frac{\sqrt{-D_p(g)}}{2g} + \frac{\sqrt{D_q(ig)}}{2g}\,. \tag{10.75}$$

From Lemma 4.7, z is on the hyperbola if and only if:

$$\mathrm{Re}\left(\left(z - \frac{\alpha+\alpha'}{2} - i\frac{\beta+\beta'}{2}\right)^2 - \frac{(\alpha'-\alpha)^2 - (\beta'-\beta)^2}{4}\right) = 0\,. \tag{10.76}$$

Substituting the expression for z into this and simplifying, z is on the hyperbola if and only if:

$$\mathrm{Re}\left(\frac{1 \pm i\sqrt{-D_p(g)}\sqrt{D_q(ig)}}{2g^2}\right) = 0\,. \tag{10.77}$$

Since $g \in I_p$, then $\sqrt{-D_p(g)}, \sqrt{D_q(ig)}, g^2 \in \mathbb{R}$, so the real part of the previous equation is $1/(2g^2)$, which is non-zero. Hence, $\mathcal{B}_{\mathbf{X}}(Q_k)$ does not converge to a point on the hyperbola. $\square$

10.4 Proof when p and q have 2 atoms

In this section, we prove Theorem 10.1 and also make some observations about the 4 corners of the rectangle R.

First, a summary of the results in the previous sections:

1. If $Q_k \to 0$, then $|\mathcal{B}_{\mathbf{X}}(Q_k)| \to \infty$. (Theorem 8.13)
2. If $|Q_k| \to \infty$ and $\mathcal{B}_{\mathbf{X}}(Q_k)$ converges, then $\mathcal{B}_{\mathbf{X}}(Q_k)$ converges to one of $\{\alpha + i\beta, \alpha + i\beta', \alpha' + i\beta, \alpha' + i\beta'\}$. (Proposition 10.3)
3. If $Q_k \to Q \neq 0$:
   (a) If $l_k \to 0$:
      i. If $\mathcal{B}_{\mathbf{X}}$ is discontinuous at Q, then $\mathcal{B}_{\mathbf{X}}(Q_k)$ converges to z on the intersection of the hyperbola with the open rectangle. (Proposition 10.4)
      ii. If $\mathcal{B}_{\mathbf{X}}$ is continuous at Q:

A. If $g \in \mathbb{R}$ or $g^I \in \mathbb{R}$, then $l(Q_k) \not\to 0$. (Proposition 10.5)

B. If $g \notin \mathbb{R}$ and $g^I \notin \mathbb{R}$, then $\mathcal{B}_{\mathbf{X}}(Q_k) \to \mathcal{B}_{\mathbf{X}}(Q) = z$, which is on the intersection of the hyperbola with the open rectangle. (Proposition 10.6)

(b) If $B_k \to 0$:

i. If $\mathcal{B}_{\mathbf{X}}$ is continuous at Q, then $\mathcal{B}_{\mathbf{X}}(Q_k) \to \mathcal{B}_{\mathbf{X}}(Q) = z$, and whenever z is on the hyperbola, z is not in the rectangle. (Proposition 10.7)

ii. If $\mathcal{B}_{\mathbf{X}}$ is discontinuous at Q, then $\mathcal{B}_{\mathbf{X}}(Q_k)$ does not converge to a z on the hyperbola. (Proposition 10.10)

As a corollary to these facts, we make an observation about the 4 corners of the intersection of the hyperbola and rectangle:

Corollary 10.11. *Let $z \in \{\alpha + i\beta, \alpha' + i\beta, \alpha + i\beta', \alpha' + i\beta'\}$ and suppose that $\mathcal{G}_{\mathbf{X}}(z_\epsilon)$ is in the domain of $\mathcal{B}_{\mathbf{X}}$ for sufficiently small $\epsilon > 0$. Then,*

$$\lim_{\epsilon \to 0^+} |\mathcal{G}_{\mathbf{X}}(z_\epsilon)| = \infty \,. \tag{10.78}$$

Proof. The proof is based on the results in the previous sections and noticing that only in the situation where $z \in \{\alpha + i\beta, \alpha' + i\beta, \alpha + i\beta', \alpha' + i\beta'\}$ is possible is when $|Q_k| \to \infty$ and $\mathcal{B}_{\mathbf{X}}(Q_k)$ converges.

Recall that $Q_\epsilon = \mathcal{G}_{\mathbf{X}}(z_\epsilon)$. It suffices to show that for every sequence $\epsilon_k \to 0^+$, $|Q_{\epsilon_k}| \to \infty$.

Suppose for the sake of contradiction that Q_{ϵ_k} is bounded for some $\epsilon_k \to 0^+$.

If $Q_{\epsilon_k} \to 0$, then

$$z = \lim_{k\to\infty} z_{\epsilon_k} = \lim_{k\to\infty} \mathcal{B}_{\mathbf{X}}(Q_{\epsilon_k}) \,, \tag{10.79}$$

but from Theorem 8.13, the final limit diverges.

If $Q_{\epsilon_k} \not\to 0$, then we may pass to a subsequence and assume that $Q_{\epsilon_k} \to Q \neq 0$. Then, from the results in the previous sections, $z \notin \{\alpha + i\beta, \alpha' + i\beta, \alpha + i\beta', \alpha' + i\beta'\}$, a contradiction.

Thus, it must be the case that

$$\lim_{\epsilon \to 0^+} |\mathcal{G}_{\mathbf{X}}(z_\epsilon)| = \infty\,. \tag{10.80}$$

$\square$

We can verify the domain condition in Corollary 10.11 to get the following concrete result:

Proposition 10.12. *Let $z \in \{\alpha + i\beta, \alpha' + i\beta, \alpha + i\beta', \alpha' + i\beta'\}$. Then,*

$$\lim_{\epsilon \to 0^+} |\mathcal{G}_{\mathbf{X}}(z_\epsilon)| = \infty\,. \tag{10.81}$$

Proof. Let $X_z = z - X$. Computation using Definition 7.12 shows that:

$$\mathcal{G}_{\mathbf{X}}(z_\epsilon) = \begin{pmatrix} \tau[((X_z)^* X_z + \epsilon^2)^{-1}(X_z)^*] & -i\epsilon\tau[((X_z)^* X_z + \epsilon^2)^{-1}] \\ i\epsilon\tau[((X_z)^* X_z + \epsilon^2)^{-1}] & \tau[X_z((X_z)^* X_z + \epsilon^2)^{-1}] \end{pmatrix}. \tag{10.82}$$

In light of Corollary 10.11 and the domain of $\mathcal{B}_{\mathbf{X}}$ described in Theorem 8.13, it suffices to show that for $z \in \{\alpha + i\beta, \alpha' + i\beta, \alpha + i\beta', \alpha' + i\beta'\}$ and $\epsilon > 0$, $\tau[((X_z)^* X_z + \epsilon^2)^{-1}(X_z)^*] \notin \mathbb{R} \cup i\mathbb{R}$.

We will just show that $\tau[((X_z)^* X_z + \epsilon^2)^{-1}(X_z)^*] \notin i\mathbb{R}$, the other case is similar. For this, let $X_z = p + iq$, where p, q are Hermitian, freely independent, and have 2 atoms. For $z \in \{\alpha + i\beta, \alpha' + i\beta, \alpha + i\beta', \alpha' + i\beta'\}$, p and q both have 0 as an atom. Hence, p and q are either positive or negative operators. At this point, we also drop the subscript z so that $X = p + iq$.

Assume for the sake of contradiction that $\tau[(X^*X + \epsilon^2)^{-1}X^*] \in i\mathbb{R}$. This is equivalent to

$$0 = \operatorname{Re}\tau[(X^*X + \epsilon^2)^{-1}X^*] = \tau[(X^*X + \epsilon^2)^{-1}p]\,. \tag{10.83}$$

Since $p \geq 0$ or $p \leq 0$, without loss of generality assume that $p \geq 0$. Then,

$$\tau[((X^*X + \epsilon^2)^{-1/2}p^{1/2})^*(X^*X + \epsilon^2)^{-1/2}p^{1/2}] = \tau[(X^*X + \epsilon^2)^{-1}p] = 0\,. \tag{10.84}$$

Since τ is faithful, then $(X^*X + \epsilon^2)^{-1/2}p^{1/2} = 0$. Hence,

$$p = (X^*X + \epsilon^2)^{1/2}[(X^*X + \epsilon^2)^{-1/2}p^{1/2}]p^{1/2} = 0\,. \tag{10.85}$$

But, this is impossible, as p has 2 atoms. Hence, $\tau[(X^*X + \epsilon^2)^{-1}X^*] \notin i\mathbb{R}$. □

We prove one final Proposition before the proof of Theorem 10.1:

Proposition 10.13. *Let $z \in \mathbb{C}$ and suppose there exists a sequence $\epsilon_k \to 0^+$, such that $Q_{\epsilon_k} \to Q \neq 0$, where $Q \in \mathbb{C}$. Then,*

$$\lim_{\epsilon \to 0^+} \mathcal{G}_{\mathbf{X}}(z_\epsilon) = \lim_{\epsilon \to 0^+} Q_\epsilon = Q\,. \tag{10.86}$$

In particular, when G_X is continuous at z, $Q = G_X(z)$.

Proof. If there is a sequence $Q_{\epsilon_k} \to Q \neq 0$ where $Q \in \mathbb{C}$, then $B_{\epsilon_k} \to 0$. We proceed to upgrade the convergence $B_{\epsilon_k} \to 0$ along a specific sequence to $B_\epsilon \to 0$ as $\epsilon \to 0^+$: If $B_\epsilon \not\to 0$, then from (7.85), on some sequence $\epsilon'_k \to 0^+$, $l_{\epsilon'_k} \to 0$. From $B_{\epsilon_k} \to 0$ and Propositions 10.7 and 10.10, z does not lie on the intersection of the hyperbola and rectangle. But, from $l_{\epsilon'_k} \to 0$ and Propositions 10.4 and 10.6, z does lie on the intersection of the hyperbola and rectangle. This is a contradiction, so we conclude that $B_\epsilon \to 0$.

Next, consider an arbitrary sequence $\epsilon'_k \to 0$ such that $Q_{\epsilon'_k} \to Q' \neq 0$, where $Q' \in \mathbb{C}$. To prove the limit in the statement of the Proposition, it suffices to show that $Q' = Q$.

Corollary 10.9 gives two sequences $\zeta_{\epsilon_k}, \zeta_{\epsilon'_k} \subset \mathbb{C}$ such that $\zeta_{\epsilon_k} \to Q$, $\zeta_{\epsilon'_k} \to Q'$, and:

$$\lim_{k\to\infty} B_X(\zeta_{\epsilon_k}) = \lim_{k\to\infty} B_X(Q_{\epsilon_k}) = z = \lim_{k\to\infty} B_X(Q_{\epsilon'_k}) = \lim_{k\to\infty} B_X(\zeta_{\epsilon'_k})\,. \tag{10.87}$$

We consider the cases when B_X is continuous or discontinuous at Q and Q':

If B_X is continuous at both Q and Q', then since B_X is invertible on its domain, it is injective. Hence,

$$B_X(Q) = \lim_{k\to\infty} B_X(\zeta_{\epsilon_k}) = z = \lim_{k\to\infty} B_X(\zeta_{\epsilon'_k}) = B_X(Q') \tag{10.88}$$

implies that $Q = Q'$.

If B_X is continuous at exactly one of Q and Q', assume B_X is continuous at Q but not Q'. From Proposition 8.11, B_X is analytic in a neighborhood of Q, so from the Open Mapping Theorem, for $U \subset \mathbb{C}$ open where $Q \in U$, $Q' \notin U$, $B_X(U)$ is an open set containing $B_X(Q) = z$. Since $B_X(\zeta_{\epsilon'_k})$ converges to z also, then for sufficiently large k, $B_X(\zeta_{\epsilon'_k}) \in B_X(U)$, but $\zeta_{\epsilon'_k} \notin U$. This contradicts the injectivity of B_X.

Finally, consider if B_X is discontinuous at both Q and Q', i.e. $Q \in I_p$ or $Qi \in I_q$, and $Q' \in I_p$ or $Q'i \in I_q$.

From the proof of Proposition 10.10, the possible limits for $B_X(\zeta_k)$ are:

$$\begin{aligned} z &= \lim_{k\to\infty} \mathcal{B}_{\mathbf{X}}(Q_{\epsilon_k}) \\ &= \begin{cases} \dfrac{\alpha+\alpha'}{2} + i\dfrac{\beta+\beta'}{2} \pm i\dfrac{\sqrt{-D_p(g)}}{2g} + \dfrac{\sqrt{D_q(ig)}}{2g} & g \in I_p \\ \dfrac{\alpha+\alpha'}{2} + i\dfrac{\beta+\beta'}{2} + \dfrac{\sqrt{D_p(g)}}{2g} \pm i\dfrac{\sqrt{-D_q(ig)}}{2g} & ig \in I_q\,. \end{cases} \end{aligned} \tag{10.89}$$

Computation shows that:

$$\begin{aligned} \tilde{z} &= \left(z - \frac{\alpha+\alpha'}{2} - i\frac{\beta+\beta'}{2}\right)^2 - \frac{(\alpha'-\alpha)^2 - (\beta'-\beta)^2}{4} \\ &= \begin{cases} \dfrac{1 \pm i\sqrt{-D_p(g)}\sqrt{D_q(ig)}}{2g^2} & g \in I_p \\ \dfrac{1 \pm i\sqrt{D_p(g)}\sqrt{-D_q(ig)}}{2g^2} & ig \in I_q\,. \end{cases} \end{aligned} \tag{10.90}$$

There are analogous equations for Q', where g is replaced with g'.

In these equations, we can determine if $g \in I_p$ or $ig \in I_q$ by observing that $\mathrm{Re}(\tilde{z}) < 0$ for $g \in I_p$ and $\mathrm{Re}(\tilde{z}) > 0$ for $ig \in I_q$. Then, by observing $\mathrm{Re}(\tilde{z}) = 1/(2g^2)$, we can recover g

up to a sign. Finally, by examining $\mathrm{Re}(z)$, we can determine what g is. Hence, $Q = Q'$, as desired.

The last point follows from Corollary 10.9. $\square$

Finally, we can prove Theorem 10.1:

Proof of Theorem 10.1. First, consider z on the support of the Brown measure of X. From Theorem 4.18, $z = x + iy$ lies on the intersection of the hyperbola

$$H = \left\{ z = x + iy : \left(x - \frac{\alpha + \alpha'}{2} \right)^2 - \left(y - \frac{\beta + \beta'}{2} \right)^2 = \frac{(\alpha' - \alpha)^2 - (\beta' - \beta)^2}{4} \right\} \quad (10.91)$$

with the rectangle

$$R = \{ z = x + iy : x \in [\alpha \wedge \alpha', \alpha \vee \alpha'], y \in [\beta \wedge \beta', \beta \vee \beta'] \} . \quad (10.92)$$

Recall that

$$Q_\epsilon = \mathcal{G}_{\mathbf{X}}(z_\epsilon) = \mathcal{G}_{\mathbf{X}}\left(\begin{pmatrix} z & i\epsilon \\ i\epsilon & \overline{z} \end{pmatrix} \right) = \begin{pmatrix} A_\epsilon & i\overline{B_\epsilon} \\ iB_\epsilon & \overline{A_\epsilon} \end{pmatrix} . \quad (10.93)$$

If Q_ϵ converges to some Q, then from Theorem 8.13, $Q \neq 0$. Further, $B \neq 0$: if $B = 0$, then from Propositions 10.7 and 10.10, $z_\epsilon = \mathcal{B}_{\mathbf{X}}(Q_\epsilon)$ converges to z not on the intersection of the hyperbola and rectangle, a contradiction.

Conversely, assume z has that $Q_\epsilon \to Q$ where $B \neq 0$. Then, from (7.79), $l_\epsilon \to 0$. From Propositions 10.4 and 10.6, $z_\epsilon = \mathcal{B}_{\mathbf{X}}(Q_\epsilon)$ converges to z on the intersection of the hyperbola and rectangle, and hence the support of the Brown measure of X.

All that remains to show is that when Q_ϵ does not have a limit as $\epsilon \to 0^+$, then z is on the intersection of the hyperbola and rectangle.

First, we consider when Q_ϵ does not stay bounded as $\epsilon \to 0^+$. Choose a sequence $\epsilon_k \to 0^+$ such that $|Q_{\epsilon_k}| \to \infty$. From Proposition 10.3, z is one of $\{\alpha + i\beta, \alpha + i\beta', \alpha' + i\beta, \alpha' + i\beta'\}$, which are the boundary points of the intersection of the hyperbola with the rectangle.

Now, suppose that Q_ϵ remains bounded as $\epsilon \to 0^+$ but has no limit. From Propositions 10.4 and 10.6, it suffices to show that $l_\epsilon \to 0$. For this, it suffices to show that for any sequence $\epsilon_k \to 0^+$, $B_{\epsilon_k} \not\to 0$.

For the sake of contradiction, assume that there is some $\epsilon_k \to 0^+$ where $B_{\epsilon_k} \to 0$. Passing to a subsequence, we may assume that Q_{ϵ_k} converges to $Q \neq 0$. From Proposition (10.13), Q_ϵ converges to Q, a contradiction to the assumption that Q_ϵ had no limit. $\square$

CHAPTER 11

Atoms of the Brown measure

In this final chapter, we discuss the atoms of the Brown measure of $X = p + iq$ in the framework of the Quaternionic Green's function $\mathcal{G}_{\mathbf{X}}$ and relate them to some of our previous results.

In ([BV98], Theorem 7.4), the authors proved that an atom of $\mu \boxplus \nu$ can occur at γ if and only if there exists $\alpha, \beta \in \mathbb{R}$ such that $\gamma = \alpha + \beta$ and $\mu(\{\alpha\}) + \nu(\{\beta\}) > 1$, in which case $(\mu \boxplus \nu)(\{\gamma\}) = \mu(\{\alpha\}) + \nu(\{\beta\}) - 1$.

In ([BSS18], Proposition 1), an analogue of this result was proven for the operator-valued Cauchy transform and the sum of free operators that are not necessarily Hermitian:

Proposition 11.1. *Let $x, y \in (M, \tau)$, where $x, y \notin \mathbb{C}$ and are freely independent. If there exists a projection $p \neq 0$ and $\lambda \in \mathbb{C}$ such that $(x + y)p = \lambda p$, then:*

1. *p is the projection onto $\ker((x + y - \lambda)^*(x + y - \lambda))$.*
2. *There exists projections $p_1, p_2 \in (M, \tau)$ and $u_1, u_2 \in \mathbb{C}$ so that:*
   - *$xp_1 = u_1 p_1$, $yp_2 = u_2 p_2$.*
   - *$u_1 + u_2 = \lambda$.*
   - *$\tau(p_1) + \tau(p_2) = \tau(p) + 1$.*

We wish to conclude that the $\lambda \in \mathbb{C}$ where there exists such a $p \neq 0$ correspond to the atoms of the Brown measure of $x+y$, and that $\mu(\{\lambda\}) = \tau(p)$. If this were true, then we could conclude that the atoms of the Brown measure of $X = p+iq$ are exactly of the form $\alpha_i + i\beta_j$, where α_i is an atom of p and β_j is an atom of q. Further, we could conclude exactly what the weights of these atoms were. Note that these weights correspond to the intersection of free projections of certain traces (Proposition 4.13) and the intersection of randomly rotated subspaces of certain dimension (Lemma 5.1). These are the weights of the atoms when p and q have 2 atoms (Theorem 4.18). When the atom is on ∂R, this also matches the weights given in Proposition 5.7.

Recall from Lemma 4.12 that for a real probability measure μ and a sequence $z_n \to s$ non-tangentially to $\mathbb{R}$, $(z_n - s)G_\mu(z_n) \to \mu(\{s\})$. We proceed to analyze the analogous limit for the Quaternionic Green's function for $X \in (M, \tau)$. This analysis comes from [BSS18], translated to our notation for $\mathcal{G}_{\mathbf{X}}$.

We wish to analyze the following limit for fixed $z \in \mathbb{C}$:

$$\lim_{\epsilon \to 0^+} (z_\epsilon - z)\mathcal{G}_{\mathbf{X}}(z_\epsilon)\,, \tag{11.1}$$

where

$$z_\epsilon = \begin{pmatrix} z & i\epsilon \\ i\epsilon & \overline{z} \end{pmatrix}. \tag{11.2}$$

By analogy from Lemma 4.12, we expect this limit to be $\mu(\{z\})$.

Let $a = X_z = z - X$. Computation with Definition 7.12 shows that

$$(z_\epsilon - z)\mathcal{G}_{\mathbf{X}}(z_\epsilon) = \begin{pmatrix} \epsilon^2\tau[(a^*a + \epsilon^2)^{-1}] & i\epsilon\tau[a(a^*a + \epsilon^2)^{-1}] \\ i\epsilon\tau[a^*(a^*a + \epsilon^2)^{-1}] & \epsilon^2\tau[(a^*a + \epsilon^2)^{-1}] \end{pmatrix}. \tag{11.3}$$

For the diagonal terms, we rewrite the expression as an integral over the spectral measure of a^*a:

$$\epsilon^2\tau[(a^*a + \epsilon^2)^{-1}] = \int_0^\infty \frac{\epsilon^2}{t + \epsilon^2}\, d\mu_{a^*a}(t)\,. \tag{11.4}$$

Since $\epsilon^2/(t+\epsilon^2) \leq 1$, then from the dominated convergence theorem,

$$\begin{aligned}\lim_{\epsilon\to 0^+} \epsilon^2\tau[(a^*a+\epsilon^2)^{-1}] &= \int_0^\infty \lim_{\epsilon\to 0^+} \frac{\epsilon^2}{t+\epsilon^2}\, d\mu_{a^*a}(t) \\ &= \int_0^\infty \chi_{\{0\}}(t)\, d\mu_{a^*a}(t) \\ &= \tau(p)\,,\end{aligned} \tag{11.5}$$

where p is the projection onto $\ker(a^*a) = \ker(a) = \ker(z - X)$.

For the off-diagonal terms, we use the polar decomposition $a = v\,|a|$ and the Cauchy-Schwarz inequality with v and $i\epsilon\,|a|\,(a^*a+\epsilon^2)^{-1}$ to see that:

$$\left|i\epsilon\tau[a(a^*a+\epsilon^2)^{-1}]\right| \leq \tau[v^*v]^{1/2}\tau[\epsilon^2 a^*a(a^*a+\epsilon^2)^{-2}]^{1/2}\,. \tag{11.6}$$

Again, rewriting in terms of the spectral measure of a^*a,

$$\tau[\epsilon^2 a^*a(a^*a+\epsilon^2)^{-1}] = \int_0^\infty \frac{t\epsilon^2}{(t+\epsilon^2)^2}\, d\mu_{a^*a}(t)\,. \tag{11.7}$$

Since $2t\epsilon^2 \leq (t+\epsilon^2)^2$, then from the dominated convergence theorem,

$$\begin{aligned}\lim_{\epsilon\to 0^+} \tau[\epsilon^2 a^*a(a^*a+\epsilon^2)^{-1}] &= \int_0^\infty \lim_{\epsilon\to 0^+} \frac{t\epsilon^2}{(t+\epsilon^2)^2}\, d\mu_{a^*a}(t) \\ &= \int_0^\infty 0\, d\mu_{a^*a}(t) \\ &= 0\,.\end{aligned} \tag{11.8}$$

Hence,

$$\lim_{\epsilon\to 0^+} (z_\epsilon - z)\mathcal{G}_{\mathbf{X}}(z_\epsilon) = \tau(p)\,, \tag{11.9}$$

where p is the projection onto $\ker(z - X)$.

In particular, when $p \neq 0$, then

$$\lim_{\epsilon\to 0^+} |\mathcal{G}_{\mathbf{X}}(z_\epsilon)| = \infty\,. \tag{11.10}$$

This can applied to the Brown measure of $X = p + iq$ when the $\chi_{\{\alpha_i\}}(p)$ and $\chi_{\{\beta_j\}}(q)$ are guaranteed to intersect because their traces are large. Note that this result does not imply Proposition 10.12, since in that case the traces of all atoms of p and q are $1/2$ and $X = p + iq$ does not have any atoms. In particular, the situation of Proposition 10.12 gives an example where:

$$\lim_{\epsilon \to 0^+} |\mathcal{G}_{\mathbf{X}}(z_\epsilon)| = \infty \,, \tag{11.11}$$

but

$$\lim_{\epsilon \to 0^+} (z_\epsilon - z)\mathcal{G}_{\mathbf{X}}(z_\epsilon) = 0 \,. \tag{11.12}$$

To possibly relate this limit to the Brown measure, we observe from the computation that:

$$\lim_{\epsilon \to 0^+} (z_\epsilon - z)\mathcal{G}_{\mathbf{X}}(z_\epsilon) = \lim_{\epsilon \to 0^+} \epsilon^2 \tau[(a^*a + \epsilon^2)^{-1}] \,. \tag{11.13}$$

This final limit can be related to the Brown measure when X is normal or $X = X_n$ is a random matrix. For instance, when X is normal,

$$\epsilon^2 \tau[(a^*a + \epsilon^2)^{-1}] = \int_{\mathbb{C}} \frac{\epsilon^2}{|z - w|^2 + \epsilon^2} \, d\mu_X(w) \,, \tag{11.14}$$

and from the dominated convergence theorem the right-hand side converges to $\mu_X(\{z\})$ as $\epsilon \to 0^+$. When $X = X_n$ is a random matrix where a^*a has empirical spectral distribution $\frac{1}{n}\sum_{i=1}^n \delta_{\lambda_i}$,

$$\epsilon^2 \tau[(a^*a + \epsilon^2)^{-1}] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n \frac{\epsilon^2}{\lambda_i + \epsilon^2}\right] \,. \tag{11.15}$$

Applying the dominated convergence theorem as $\epsilon \to 0^+$,

$$\lim_{\epsilon \to 0^+} \epsilon^2 \tau[(a^*a + \epsilon^2)^{-1}] = \mathbb{E}\left[\lim_{\epsilon \to 0+} \frac{1}{n}\sum_{i=1}^n \frac{\epsilon^2}{\lambda_i + \epsilon^2}\right] = \mathbb{E}[\tau(p)] \,, \tag{11.16}$$

where p is the projection onto $\ker(a^*a) = \ker(a) = \ker(z - X)$. Hence,

$$\mathbb{E}[\tau(p)] = \mathbb{E}\left[\int_{\mathbb{C}} \chi_{\{0\}}(z - X) \, d\mu_{X_n}(z)\right] = \overline{\mu_{X_n}}(\{z\}) \,. \tag{11.17}$$

In general, it is unclear if

$$\lim_{\epsilon \to 0^+} \epsilon^2 \tau[(a^* a + \epsilon^2)^{-1}] = \mu_X(\{z\})\,. \tag{11.18}$$

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