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Input-Output-to-State Stability Tools for Hybrid Systems and their Interconnections

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Abstract—**We present results for the analysis of input/output properties of a general class of hybrid systems given by a flow set, a flow map, a jump set, a jump map, and an output map. For this class of systems, the notion of input-output-to-state stability is introduced in the first part of the paper. Under mild assumptions on the functions and sets defining a hybrid system, sufficient conditions for this notion in terms of Lyapunov functions are derived. Equivalences between Lyapunov functions for input-output-to-state stability for asymptotic and exponential decay rates are established. The sufficient conditions and equivalences are linked to the existence of norm observers for hybrid systems. These results are used in the second part of the paper to study interconnections of hybrid systems. An interconnection result in terms of a Lyapunov-based small gain theorem is also presented. Examples illustrate the results.**

I. Introduction

Notions relating input, outputs, and the state of dynamical systems have been widely applied to the analysis and design of nonlinear control systems and their interconnections. The ISS concept has been found to be of particular importance for the study of such interconnections. Pioneering work on interconnections of ISS systems appeared in [1], [2] for continuous-time systems in the form of small gain theorems. In the results therein, the ISS properties of the individual systems are expressed in terms of $\mathcal{KL}$ estimates and involve a condition on the gains of the individual systems, called the small gain condition. This line of work was followed by its output to state counterpart, output-to-state stability (OSS) [3], which provides a tool to establish bounds on the state in terms of the system outputs as well as a link to detectability. The combination of ISS and OSS notions led to the concept of input-output-to-state stability (IOSS) in [4], where bounds on solutions are given in terms of bounds on the inputs and outputs; see [5] for a survey.

In recent years, research on sufficient conditions for ISS in terms of Lyapunov functions and extensions of ISS-like notions to other classes of systems received great attention. In particular, sufficient conditions for ISS in terms of Lyapunov functions have been shown to be a powerful tool not only to guarantee the ISS bound but also to systematically study interconnections. Interconnection results that exploit ISS properties of the individual systems were presented for continuous-time systems in [6], for discrete-time systems in [7], for switched systems in [8], and for hybrid systems in [9]. A Lyapunov-based small gain theorem for interconnections of ISS systems appeared in [10] for continuous-time systems, and later extended to discrete-time and hybrid systems in [11] and [12], respectively; see also [13], [14], [15]. The former notion of OSS in [3] was extended to hybrid systems in [16]. Among other issues about interconnections of hybrid systems, the notion of input-to-output stability (IOS) and its characterization in terms of Lyapunov functions were discussed in [17]. IOSS was recently extended to discrete-time systems in [18] and to switched systems in [19]. Lyapunov characterizations of IOSS were reported for continuous-time systems in [20], for discrete-time systems in [21], and for switched systems in [22]. Small gain results in terms of $\mathcal{KL}$ estimates were presented for interconnections of IOSS nonlinear continuous-time systems in [1] and for a class of systems with jumps in [23].

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In this paper, we consider hybrid systems denoted as $\mathcal{H}$ and written as

$$\mathcal{H}\left\{\begin{array}{lcll} \dot{x} & = & f(x,u) & (x,u)\in C \\ x^+ & = & g(x,u) & (x,u)\in D \\ y & = & h(x). & \end{array}\right. \tag{1}$$

The state space for the state x is $\mathbb{R}^n$ and the space for inputs u is the set $\mathcal{U}\subset\mathbb{R}^m$. The set $C\subset\mathbb{R}^n\times\mathcal{U}$ is the *flow set*, and defines the set of points in $\mathbb{R}^n\times\mathcal{U}$ on which continuous evolution or flow is possible. The function $f: C\to\mathbb{R}^n$ is the *flow map*, and defines the motion during flow. The set $D\subset\mathbb{R}^n\times\mathcal{U}$ is the *jump set*, and defines the set of points in $\mathbb{R}^n\times\mathcal{U}$ from where discrete evolution or jumps are possible. The function $g: D\to\mathbb{R}^n$ is the *jump map*, and defines the value of the state after the jump. Finally, $h:\mathbb{R}^n\to\mathbb{R}^p$ is the *output map* and defines the output. In this framework, the data of the hybrid system $\mathcal{H}$ is given by (C,f,D,g,h).

For this class of systems, the first part of this paper (Section III) introduces an IOSS notion as well as the following key results: 1) The existence of an IOSS Lyapunov function is equivalent to the existence of an exponential-decay IOSS Lyapunov function and implies the existence of a state-norm estimator; 2) The existence of an IOSS Lyapunov function implies IOSS. In the second part of this paper (Section IV), we establish the following: 3) A Lyapunov-based small gain theorem for interconnections of two hybrid systems with Lyapunov functions satisfying IOSS-like bounds.

II. Preliminaries

Notation: $\mathbb{R}^n$ denotes n-dimensional Euclidean space; $\mathbb{R}$ denotes real numbers. $\mathbb{R}_{\geq 0}$ denotes nonnegative real numbers. $\mathbb{N}$ denotes natural numbers including 0. Given a set S, $\overline{S}$ denotes its closure, ess sup S denotes its essential supremum, and $\mathrm{int}(S)$ its interior. Given a vector $x\in\mathbb{R}^n$, $|x|$ denotes the Euclidean vector norm. Given vectors x and y, we write $[x^\top y^\top]^\top$ with the shorthand notation (x,y). Given a set $S\subset\mathbb{R}^n$ and a point $x\in\mathbb{R}^n$, $|x|_S := \inf_{y\in S}|x-y|$. Id is the identity function. A function $\alpha:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{K}$ ($\alpha\in\mathcal{K}$) if it is continuous, zero at zero, and strictly increasing. It is said to belong to class-$\mathcal{K}_\infty$ ($\alpha\in\mathcal{K}_\infty$) if it belongs to class-$\mathcal{K}$ and is unbounded. A function $\beta:\mathbb{R}_{\geq 0}\times\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{KL}$ ($\beta\in\mathcal{KL}$) if it is continuous, nondecreasing in its first argument, nonincreasing in its second argument, and $\lim_{s\searrow 0}\beta(s,r) = \lim_{r\to\infty}\beta(s,r) = 0$. For a locally Lipschitz function V, $V^\circ(x,w)$ denotes the Clarke generalized derivative of V at x in the direction w [24], i.e., $V^\circ(x,w) = \max_{\zeta\in\partial V(x)}\langle\zeta,w\rangle$, where $\partial V(x)$ is the generalized gradient of V in the sense of Clarke, which is a closed, convex, and nonempty set equal to the convex hull of all limit sequences of $\nabla V(x_i)$ with $x_i\to x$ taking value away from every set of measure zero in which V is nondifferentiable. △

In this paper, we consider hybrid systems as in [25] with solutions that can evolve continuously (flow) and/or discretely (jump) depending on the continuous and discrete dynamics as well as on the sets where those dynamics apply. In general, a hybrid system $\mathcal{H}$ is given by data (C,f,D,g,h) and can be written in the compact form (1). Solutions (ϕ,u) to $\mathcal{H}$ will be given on *hybrid time domains*, which are denoted $\mathrm{dom}(\phi,u)$ and are subsets of $\mathbb{R}_{\geq 0}\times\mathbb{N}$ with the following structure: for each $(T,J)\in\mathrm{dom}(\phi,u)$, the truncation $\mathrm{dom}(\phi,u)\cap([0,T]\times\{0,1,...J\})$ can be written as $\bigcup_{j=0}^{J-1}([t_j,t_{j+1}],j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 ... \leq t_J$. A solution is a function defined on $\mathrm{dom}(\phi, u)$ that satisfies the dynamics of $\mathcal{H}$ with the property that, for each $j \in \mathbb{N}$, $t \mapsto \phi(t,j)$ is absolutely continuous and $t \mapsto u(t,j)$ is Lebesgue measurable and locally essentially bounded on $\{t \ : \ (t,j) \in \mathrm{dom}(\phi,u)\ \}$; see [25], [17] for more details. The $\mathcal{L}_\infty$ norm of ϕ and u – in general, of a hybrid signal r – is given by[1]

$$\|r\|_{(t,j)} \quad := \quad \max\left\{ \max_{j' \leq j} \ \operatorname*{ess\,sup}_{t' \ \text{s.t.}\ (t',j') \in \mathrm{dom}\, r} |r(t',j')|, \right.$$
$$\left. \sup_{(t',j') \in \Gamma(r),\ t'+j' \leq t+j} |r(t',j')| \right\},$$

where $\Gamma(r) := \{(t,j) \in \mathrm{dom}\, r \ : \ (t,j+1) \in \mathrm{dom}\, r\ \}$. For notational convenience, $\|r\|$ denotes $\lim_{t+j \to N} \|r\|_{(t,j)}$, where $N = \sup_{(t,j)\in \mathrm{dom}\, r} t+j \in [0,+\infty]$.

A solution pair (ϕ,u) to $\mathcal{H}$ is said to be *complete* if $\mathrm{dom}(\phi,u)$ is unbounded and *maximal* if there does not exist another pair $(\phi,u)'$ such that (ϕ,u) is a truncation of $(\phi,u)'$ to some proper subset of $\mathrm{dom}(\phi,u)'$. Given $\xi \in \mathbb{R}^n$, $\mathcal{S}_\mathcal{H}(\xi)$ denotes the set of maximal solution pairs (ϕ,u) to $\mathcal{H}$ with $\phi(0,0) = \xi$ and u with finite $\|u\|$. For a solution pair $(\phi,u) \in \mathcal{S}_\mathcal{H}(\xi)$, when convenient, we denote by $\phi(t,j,\xi,u)$ its value at $(t,j) \in \mathrm{dom}(\phi,u)$, i.e., the third entry corresponds to the initial condition for the state and the fourth entry denotes the input.

The following mild conditions on $\mathcal{H}$ will be imposed in some of the results of this paper.

Assumption 2.1: The data (C,f,D,g,h) of the hybrid system $\mathcal{H}$ is such that

(A1) C and D are closed sets;

(A2) $f : C \to \mathbb{R}^n$, $g : D \to \mathbb{R}^n$, and $h : \mathbb{R}^n \to \mathbb{R}^p$ are continuous.

In addition to enabling the developments in this paper, the conditions in Assumption 2.1 assure several good structural properties of the solution set of $\mathcal{H}$; see [25] for more details.

III. IOSS for Hybrid Systems

Below, X denotes the projection of the closure of $C \cup D \cup (g(D) \times \mathcal{U})$ onto $\mathbb{R}^n$ and it is assumed that, given a nonempty compact set $\mathcal{A} \subset \mathbb{R}^n$, the output $h : \mathbb{R}^n \to \mathbb{R}^p$ is such that $h(\mathcal{A}) = \{0\}$.

A. IOSS definitions and results

Input-output-to-state stability is a property that guarantees that the internal state components of the system are bounded when it is known that its output and input are bounded [4], [18]. The following definition introduces the property of IOSS for hybrid systems $\mathcal{H}$. It does not insist on all maximal solutions to be complete.

Definition 3.1 (input-output-to-state stability): Given a compact set $\mathcal{A} \subset \mathbb{R}^n$, the hybrid system $\mathcal{H}$ is *input-output-to-state stable (IOSS)* if there exist $\beta \in \mathcal{KL}$ and $\gamma_1, \gamma_2 \in \mathcal{K}$ such that, for each $\xi \in \mathbb{R}^n$, each $(\phi,u) \in \mathcal{S}_\mathcal{H}(\xi)$ satisfies, for each $(t,j) \in \mathrm{dom}(\phi,u)$,

$$|\phi(t,j,\xi,u)|_\mathcal{A} \leq \max\left\{\beta(|\xi|_\mathcal{A}, t+j), \gamma_1(\|u\|_{(t,j)}), \gamma_2(\|y\|_{(t,j)})\right\}. \quad (2)$$

IOSS Lyapunov functions for hybrid systems $\mathcal{H}$ are given by locally Lipschitz functions; see [26] for their use in invariance principles and stability, and [27], [28] for their use in ISS/IOS.

[1]The use of ess sup allows for hybrid signals r such that $t \mapsto r(t,j)$ is only Lebesgue measurable for each j. Note that ess sup is first evaluated over $[t_{j'}, t_{j'+1}]$ for each fixed $j' \leq j$, and then the max over each such j' is computed. Since, when j' is fixed, hybrid signals $t' \mapsto r(t',j')$ are defined on subsets of the real line, using the Lebesgue measure space, the standard definition of ess sup over a set $E \subset \mathbb{R}$ can be used, namely, $\operatorname{ess\,sup}_{t' \in E} |f(t')| = \inf\{M \in \mathbb{R}_{\geq 0} \ : \ \mu\{t' \in E \ : \ |f(t')| > M\} = 0\}$, where μ is the Lebesgue measure.

Definition 3.2 (IOSS Lyapunov function): Given a compact set $\mathcal{A} \subset \mathbb{R}^n$, a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is an *IOSS Lyapunov function* for the hybrid system $\mathcal{H}$ if there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\sigma_1, \sigma_2 \in \mathcal{K}$, such that

$$\alpha_1(|x|_\mathcal{A}) \leq V(x) \leq \alpha_2(|x|_\mathcal{A}) \qquad \forall x \in X, \quad (3)$$
$$V^\circ(x, f(x,u)) \leq -\alpha_3(|x|_\mathcal{A}) + \sigma_1(|u|) + \sigma_2(|h(x)|) \quad \forall (x,u) \in C, \quad (4)$$
$$V(g(x,u)) - V(x) \leq -\alpha_3(|x|_\mathcal{A}) + \sigma_1(|u|) + \sigma_2(|h(x)|) \quad \forall (x,u) \in D. \quad (5)$$

Remark 3.3: IOSS Lyapunov functions are defined in a dissipative form (cf. the strict form in Proposition 3.4). In this way, this definition reduces to the one for continuous-time systems in [4, Definition 2.2] when $C := \mathbb{R}^n \times \mathbb{R}^m$ and $D := \emptyset$, that is, only flow of the hybrid system $\mathcal{H}$ is possible. It also reduces to the one for systems in [18, Definition 3.7] when $C := \emptyset$ and $D := \mathbb{R}^n \times \mathbb{R}^m$.

Proposition 3.4: Given a hybrid system $\mathcal{H}$ satisfying Assumption 2.1, a C^1 function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfies (3)-(5) *with $\widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\alpha}_3 \in \mathcal{K}_\infty$ and $\widetilde{\sigma}_1, \widetilde{\sigma}_2 \in \mathcal{K}$ if and only if there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\chi_1, \chi_2 \in \mathcal{K}$ such that* (3) *holds and*

$$V^\circ(x, f(x,u)) \leq -\alpha_3(|x|_\mathcal{A}) \quad \forall (x,u) \in C, V(x) \geq \max\{\chi_1(|u|), \chi_2(|h(x)|)\}; \quad (6)$$
$$V(g(x,u)) - V(x) \leq -\alpha_3(|x|_\mathcal{A}) \quad \forall (x,u) \in D, V(x) \geq \max\{\chi_1(|u|), \chi_2(|h(x)|)\}. \quad (7)$$

Proof: Let $\widetilde{\alpha}_1, \widetilde{\alpha}_2 \in \mathcal{K}_\infty$ satisfy (3), and $\widetilde{\alpha}_3 \in \mathcal{K}_\infty$ and $\widetilde{\sigma}_1, \widetilde{\sigma}_2 \in \mathcal{K}$ satisfy (4) and (5) for the given function V. It is left to show that (6) and (7) hold for some $\chi_1, \chi_2 \in \mathcal{K}$ and $\alpha_3 \in \mathcal{K}_\infty$. Let $\hat{\chi}_1 := \widetilde{\alpha}_3^{-1} \circ (4\widetilde{\sigma}_1)$, $\hat{\chi}_2 := \widetilde{\alpha}_3^{-1} \circ (4\widetilde{\sigma}_2)$. Note that with the definitions of $\hat{\chi}_1$ and $\hat{\chi}_2$, we have that $|x|_\mathcal{A} \geq \max\{\hat{\chi}_1(|u|), \hat{\chi}_2(|h(x)|)\}$ implies that $\widetilde{\sigma}_1(|u|) \leq \frac{1}{4}\widetilde{\alpha}_3(|x|_\mathcal{A})$ and $\widetilde{\sigma}_2(|h(x)|) \leq \frac{1}{4}\widetilde{\alpha}_3(|x|_\mathcal{A})$. Then, from inequalities (3)-(5), we have that (6) and (7) hold with $\alpha_3 := \frac{1}{2}\widetilde{\alpha}_3$, $\chi_1 := \widetilde{\alpha}_2 \circ \hat{\chi}_1$, and $\chi_2 := \widetilde{\alpha}_2 \circ \hat{\chi}_2$.

We now prove the other direction. Let V, $\mathcal{K}_\infty$-functions $\alpha_1, \alpha_2, \alpha_3$, and $\mathcal{K}$-functions χ_1, χ_2 satisfying (3), (6), and (7) be given. If $|x|_\mathcal{A} > \max\{\alpha_1^{-1} \circ \chi_1(|u|), \alpha_1^{-1} \circ \chi_2(|h(x)|)\}$ then, using (3), we have that (6) and (7) hold. Then, (4) and (5) hold with α_3 and any $\sigma_1, \sigma_2 \in \mathcal{K}$. Now, consider the case $|x|_\mathcal{A} \leq \max\{\alpha_1^{-1} \circ \chi_1(|u|), \alpha_1^{-1} \circ \chi_2(|h(x)|)\}$.

a) If $\alpha_1^{-1} \circ \chi_1(|u|) > \alpha_1^{-1} \circ \chi_2(|h(x)|)$, then $|x|_\mathcal{A} \leq \alpha_1^{-1} \circ \chi_1(|u|)$. For each $r \geq 0$, if the value $\theta_c = \max\{V^\circ(x, f(x,u)) + \alpha_3(|x|_\mathcal{A}) \ : \ (x,u) \in C, |u| \leq r, |x|_\mathcal{A} \leq \alpha_1^{-1} \circ \chi_1(|u|)\}$ exists, then define $\sigma_{c,1}(r) := \max\{\theta_c, 0\}$; otherwise $\sigma_{c,1}(r) = 0$. For each $r \geq 0$, if the value $\theta_d = \max\{V(g(x,u)) - V(x) + \alpha_3(|x|_\mathcal{A}) \ : \ (x,u) \in D, |u| \leq r, |x|_\mathcal{A} \leq \alpha_1^{-1} \circ \chi_1(|u|)\}$ exists, then define $\sigma_{d,1}(r) = \max\{\theta_d, 0\}$; otherwise $\sigma_{d,1}(r) = 0$.

b) If $\alpha_1^{-1} \circ \chi_1(|u|) \leq \alpha_1^{-1} \circ \chi_2(|h(x)|)$, then $\chi_2^{-1} \circ \chi_1(|u|) \leq |h(x)|$ and $|x|_\mathcal{A} \leq \alpha_1^{-1} \circ \chi_2(|h(x)|)$. For each $r \geq 0$, if the value $\theta_c = \max\{V^\circ(x, f(x,u)) + \alpha_3(|x|_\mathcal{A}) \ : \ (x,u) \in C, \chi_2^{-1} \circ \chi_1(|u|) \leq r, |x|_\mathcal{A} \leq \alpha_1^{-1} \circ \chi_2(r)\}$ exists, then define $\sigma_{c,2}(r) = \max\{\theta_c, 0\}$; otherwise $\sigma_{c,2}(r) = 0$. For each $r \geq 0$, if the value $\theta_d = \max\{V(g(x,u)) - V(x) + \alpha_3(|x|_\mathcal{A}) \ : \ (x,u) \in D, \chi_2^{-1} \circ \chi_1(|u|) \leq r, |x|_\mathcal{A} \leq \alpha_1^{-1} \circ \chi_2(r)\}$ exists, define $\sigma_{d,2}(r) = \max\{\theta_d, 0\}$; otherwise $\sigma_{d,2}(r) = 0$.

Since χ_1, χ_2, and α_1 are strictly increasing, we have that $\chi_2^{-1} \circ \chi_1$ and $\alpha_1^{-1} \circ \chi_2$ are strictly increasing, which in turn implies that, for each $i = 1,2$, $\sigma_{c,i}$ and $\sigma_{d,i}$ are nondecreasing. From the definitions of $\sigma_{d,1}$ and $\sigma_{d,2}$ above, $\sigma_{d,1}(0) = 0$ and $\sigma_{d,2}(0) = 0$ if $(x,0) \notin D$. If $(x,0) \in D$ then, from the definitions of $\sigma_{d,1}$ and $\sigma_{d,2}$, we have that $r = 0$ implies $|x|_\mathcal{A} = 0$ and, since $h(\mathcal{A}) = 0$, using (7), we have $V(g(x,0)) \leq V(x) - \alpha_3(|x|_\mathcal{A}) = V(x)$. Then, using the upper bound in (3), we have that $V(g(x,0)) = 0$. Then, $\sigma_{d,1}(0) = 0$ and

$\sigma_{d,2}(0)=0$. The continuity of g and V implies that $\sigma_{d,1}$ and $\sigma_{d,2}$ are continuous. Then, since $\sigma_{d,1}$ and $\sigma_{d,2}$ are continuous, zero at zero, and nondecreasing, we can majorize them by class-$\mathcal{K}_\infty$ functions $\widetilde{\sigma}_{d,1}$ and $\widetilde{\sigma}_{d,2}$, respectively. Then, for each $(x,u)\in D$ such that $|x|_{\mathcal{A}}\le \alpha_1^{-1}\circ\chi_1(|u|)$:

$$\begin{aligned}&\sigma_{d,1}(|u|)\ge V(g(x,u))-V(x)+\alpha_3(|x|_{\mathcal{A}})\\ &\qquad\Rightarrow\quad V(g(x,u))-V(x)\le -\alpha_3(|x|_{\mathcal{A}})+\sigma_{d,1}(|u|);\end{aligned}$$

while for each $(x,u)\in D$ such that $|x|_{\mathcal{A}}\le \alpha_1^{-1}\circ\chi_2(|h(x)|)$:

$$\begin{aligned}&\sigma_{d,2}(|h(x)|)\ge V(g(x,u))-V(x)+\alpha_3(|x|_{\mathcal{A}})\\ &\qquad\Rightarrow V(g(x,u))-V(x)\le -\alpha_3(|x|_{\mathcal{A}})+\sigma_{d,2}(|h(x)|).\end{aligned}$$

Proceeding similarly for flows, the proof follows by combining these bounds. ∎

An IOSS Lyapunov function that guarantees an exponential decay in the IOSS bound (2) is called *exponential-decay IOSS Lyapunov function*.

Definition 3.5: Given a compact set $\mathcal{A}\subset\mathbb{R}^n$, a locally Lipschitz function $V:\mathbb{R}^n\to\mathbb{R}_{\ge 0}$ satisfying (3) for some $\alpha_1,\alpha_2\in\mathcal{K}_\infty$, and for some $\sigma_1,\sigma_2\in\mathcal{K}$ and $\varepsilon\in(0,1]$ the following:

$$\begin{aligned}V^\circ(x,f(x,u)) &\le -\varepsilon V(x)+\sigma_1(|u|)+\sigma_2(|h(x)|)\\ &\qquad\qquad \forall (x,u)\in C, \quad (8)\\ V(g(x,u))-V(x) &\le -\varepsilon V(x)+\sigma_1(|u|)+\sigma_2(|h(x)|)\\ &\qquad\qquad \forall (x,u)\in D \quad (9)\end{aligned}$$

is said to be an *exponential-decay IOSS Lyapunov function* for the hybrid system $\mathcal{H}$.

The following result shows an equivalence between the existence of IOSS and exponential-decay IOSS Lyapunov functions for hybrid systems.

Proposition 3.6: Let $\mathcal{A}\subset\mathbb{R}^n$ be a compact set and $\mathcal{H}$ be a hybrid system satisfying Assumption 2.1. The hybrid system $\mathcal{H}$ admits an IOSS Lyapunov function if and only if it admits an exponential-decay IOSS Lyapunov function.

Proof: The $\Leftarrow$ direction is trivial and omitted for brevity. The other direction follows from (4) and (5) by appropriately combining the ideas in [4], [18]. Suppose that the IOSS Lyapunov function V satisfies (3) with $\overline{\alpha}_1,\overline{\alpha}_2$ and (4)-(5) with $\overline{\alpha}_3$. Following [29], there exists a $\mathcal{K}_\infty$ function κ such that $\frac{d\kappa}{ds}(r)\overline{\alpha}_3(r)\ge 2\kappa(r)$ for all $r\ge 0$, with $s\mapsto\kappa(s)$ such that $r\mapsto\frac{d\kappa}{ds}(r)$ is nonnegative and nondecreasing. Let $W:=\kappa\circ V$. Using (4), [4, Lemma 5.2] implies the existence of $\mathcal{K}_\infty$ functions $\hat{\sigma}_1$ and $\hat{\sigma}_2$ for which W satisfies, for all $(x,u)\in C$, $W^\circ(x,f(x,u))\le -W(x)+\hat{\sigma}_1(|u|)+\hat{\sigma}_2(|h(x)|)$. Using (5), [18, Section C.2] implies the existence of $\mathcal{K}_\infty$ functions $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ for which W satisfies, for all $(x,u)\in D$, $W(g(x,u))-W(x)\le -\frac{1}{2}W(x)+\widetilde{\sigma}_1(|u|)+\widetilde{\sigma}_2(|h(x)|)$. It follows that W is an exponential-decay IOSS Lyapunov function for $\mathcal{H}$ with $\varepsilon=\frac{1}{2}$, $\alpha_1=\kappa\circ\overline{\alpha}_1$, $\alpha_2=\kappa\circ\overline{\alpha}_2$, $\sigma_1=\max\{\hat{\sigma}_1,\widetilde{\sigma}_1\}$, and $\sigma_2=\max\{\hat{\sigma}_2,\widetilde{\sigma}_2\}$. ∎

The next result establishes that, under mild assumptions on $\mathcal{H}$ and for a compact set $\mathcal{A}$, the existence of an IOSS Lyapunov function implies IOSS; see Section III-B for a proof.

Theorem 3.7: Let $\mathcal{H}$ satisfy Assumption 2.1 and $\mathcal{A}\subset\mathbb{R}^n$ be compact. If there exists an IOSS Lyapunov function for $\mathcal{H}$ then $\mathcal{H}$ is IOSS.

Example 3.8: Consider a hybrid system $\mathcal{H}$ with state $x=(x_1,x_2)\in\mathbb{R}^2$, output $y\in\mathbb{R}$, input $u=(u_1,u_2)\in\mathbb{R}^2=:\mathcal{U}$, and data $f(x,u):=(x_2,-\gamma-bx_2+u_1)$, $g(x,u):=(x_1+ax_2^2, e|x_2|+u_2)$, $h(x):=x_1$, $C:=\{(x,u)\,:\,x_1\ge u_1, u_1\ge 0\}$, $D:=\{(x,u)\,:\,x_1=u_1,u_1\ge 0\}$, $X:=\mathbb{R}_{\ge 0}\times\mathbb{R}$, $\gamma,b,a>0$, $e\in[0,1)$. It is straightforward to check that $\mathcal{H}$ satisfies Assumption 2.1.

Consider the C^1 function $V(x)=\gamma x_1+\frac{1}{2}x_2^2$ and $\mathcal{A}=\{0\}\in\mathbb{R}^2$. Then, (3) holds with $\alpha_1(s)=\min\{s^2/4,\gamma s/\sqrt{2}\}$, $\alpha_2(s)=\frac{1}{2}s^2+\gamma s$, for all $s\ge 0$. For each $(x,u)\in C$ we have

$$\begin{aligned}\langle\nabla V(x),f(x,u)\rangle &= -bx_2^2+x_2u_1\le -bx_2^2+\frac{b}{2}x_2^2+\frac{1}{2b}u_1^2\\ &\le -\frac{b}{2}V(x)+\frac{1}{2b}u_1^2+\frac{b\gamma}{2}h(x).\end{aligned}$$

For each $(x,u)\in D$ we have

$$\begin{aligned}V(g(x,u))-V(x) &\le \gamma(1-e^2-2\gamma a-\varepsilon)h(x)\\ &\quad -(1-e^2-2\gamma a-\varepsilon)V(x)+\tfrac{1}{2}\left(1+\tfrac{e^2}{\varepsilon}\right)u_2^2\end{aligned}$$

where $\varepsilon>0$. Then, (4) and (5) hold with $\alpha_3(s):=\min\left\{\frac{b}{2},1-e^2-2\gamma a-\varepsilon\right\}s$, $\sigma_1(s):=\max\left\{\frac{1}{2b},\frac{1}{2}\left(1+\frac{e^2}{\varepsilon}\right)\right\}s^2$, $\sigma_2(s):=\gamma\max\left\{\frac{b}{2},1-e^2-2\gamma a-\varepsilon\right\}s$ for all $s\ge 0$ and parameters such that $1-e^2-2\gamma a-\varepsilon>0$. By Theorem 3.7, $\mathcal{H}$ is IOSS. △

B. State-norm estimators

State-norm estimators are useful for the purposes of control when the full state is not available for measurement, but rather, a function of the state defining an output. As shown in the literature (see, e.g., [3], [4] and the references therein), their existence is linked to the OSS and IOSS properties of the system. Here, a state-norm estimator for a hybrid system $\mathcal{H}$ is given by a hybrid system that has flow map $f_\circ$ and jump map $g_\circ$, and has a state $\zeta\in\mathbb{R}^{n_\circ}$ that flows and jumps when the state of $\mathcal{H}$ flows and jumps, respectively. The interconnection between the hybrid system and the norm observer results in the hybrid system $\mathcal{H},\mathcal{H}_\circ$

$$\left.\begin{aligned}\dot{x}&=f(x,u)\\ \dot{\zeta}&=f_\circ(\zeta,u,y)\end{aligned}\right\}\quad (x,u)\in C,\qquad \left.\begin{aligned}x^+&=g(x,u)\\ \zeta^+&=g_\circ(\zeta,u,y)\end{aligned}\right\}\quad (x,u)\in D, \tag{10}$$

where C and D are the flow set and jump set associated to $\mathcal{H}$, respectively.[2] The input to this interconnection is given by u. Solutions to $\mathcal{H},\mathcal{H}_\circ$ are given by pairs $((\phi,\zeta),u)\in\mathcal{S}_{\mathcal{H},\mathcal{H}_\circ}(\xi,z)$, where (ϕ,ζ) is the state part of the solution starting from $(\xi,z)=(\phi(0,0),\zeta(0,0))$, while u is the input part.

Definition 3.9: Given a compact set $\mathcal{A}\subset\mathbb{R}^n$, a *state-norm estimator for a hybrid system $\mathcal{H}$* consists of a function $\psi:\mathbb{R}^{n_\circ}\times\mathbb{R}^p\to\mathbb{R}$ and a hybrid system with state $\zeta\in\mathbb{R}^{n_\circ}$ and input (u,y) leading to $\mathcal{H},\mathcal{H}_\circ$ as in (10), for which

- There exist $\hat{\rho}_1,\hat{\rho}_2\in\mathcal{K}$ and $\hat{\beta}\in\mathcal{KL}$ such that, for every $(\xi,z)\in\mathbb{R}^n\times\mathbb{R}^{n_\circ}$, every solution pair $((\phi,\zeta),u)\in\mathcal{S}_{\mathcal{H},\mathcal{H}_\circ}(\xi,z)$ satisfies, for all $(t,j)\in\operatorname{dom}((\phi,\zeta),u)$,[3]

$$|\psi(\zeta(t,j),y(t,j))|\le\hat{\beta}(|z|,t+j)+\hat{\rho}_1(\|u\|_{(t,j)})+\hat{\rho}_2(\|y\|_{(t,j)}); \tag{11}$$

- There exist $\widetilde{\rho}\in\mathcal{K}$ and $\widetilde{\beta}\in\mathcal{KL}$ such that, for every $(\xi,z)\in\mathbb{R}^n\times\mathbb{R}^{n_\circ}$, every solution pair $((\phi,\zeta),u)\in\mathcal{S}_{\mathcal{H},\mathcal{H}_\circ}(\xi,z)$ satisfies, for all $(t,j)\in\operatorname{dom}((\phi,\zeta),u)$,

$$|\phi(t,j)|_{\mathcal{A}}\le\widetilde{\beta}(|\xi|_{\mathcal{A}}+|z|,t+j)+\widetilde{\rho}(|\psi(\zeta(t,j),y(t,j))|). \tag{12}$$

Input-output-to-state stability and the existence of a state-norm estimator are related as follows.

Proposition 3.10: Given a compact set $\mathcal{A}\subset\mathbb{R}^n$ and a hybrid system $\mathcal{H}$, the following hold:

1) *If $\mathcal{H}$ satisfies Assumption 2.1 and admits an exponential-decay IOSS Lyapunov function then $\mathcal{H}$ admits a state-norm estimator.*
2) *If $\mathcal{H}$ admits a state-norm estimator then $\mathcal{H}$ is IOSS.*

Proof: To prove item 1, let $\varepsilon\in(0,1]$, $\sigma_1,\sigma_2\in\mathcal{K}$ satisfy (8) and (9) for a given exponential-decay IOSS Lyapunov function

[2] In terms of (x,ζ,u), these sets are given by $\{(x,\zeta,u)\,:\,(x,u)\in C\}$ and $\{(x,\zeta,u)\,:\,(x,u)\in D\}$, respectively.

[3] For simplicity, we write $\phi(t,j)$ instead of $\phi(t,j,(\xi,z),u)$ (same for ζ) and $y(t,j)$ instead of $h\circ\phi(t,j,(\xi,z),u)$.

V. Consider the candidate state-norm estimator $\mathcal{H}_\circ$ with $n_\circ = 1$, $\psi(\zeta, y) := \zeta$, $f_\circ(\zeta, u, y) := -\varepsilon\zeta + \sigma_1(|u|) + \sigma_2(|y|)$, and $g_\circ(\zeta, u, y) = (1-\varepsilon)\zeta + \sigma_1(|u|) + \sigma_2(|y|)$. Given a solution pair (ϕ, u) to $\mathcal{H}$, the state ζ of the norm estimator satisfies

$$\begin{aligned}\dot{\zeta}(t,j) &= -\varepsilon\zeta(t,j) + \sigma_1(|u(t,j)|) + \sigma_2(|y(t,j)|) \\ &\qquad \forall t \in (t_j, t_{j+1}) \times \{j\}, \\ \zeta(t,j+1) &= (1-\varepsilon)\zeta(t,j) + \sigma_1(|u(t,j)|) + \sigma_2(|y(t,j)|) \\ &\qquad \forall (t,j) \in \Gamma(\phi).\end{aligned} \tag{13}$$

Condition (11) holds since the jumps of ζ are triggered externally (when $(\phi, u) \in D$), and the continuous and discrete dynamics of ζ are linear and, when the inputs (ϕ, u) are zero, $\zeta(t,j)$ exponentially converges to zero.

Now, we show that (12) holds. By construction, jumps of ζ and x occur simultaneously. For any given $(\xi, z) \in \mathbb{R}^n \times \mathbb{R}^{n_\circ}$, evaluating (8) and (9) along an arbitrary solution pair $((\phi,\zeta),u) \in \mathcal{S}_{\mathcal{H},\mathcal{H}_\circ}(\xi, z)$ we have that, for each $j \in \mathbb{N}$ and for almost all t such that $(t,j) \in \operatorname{dom}((\phi,\zeta),u)$, $\frac{d}{dt}V(\phi(t,j)) \le V^\circ(\phi(t,j), \dot{\phi}(t,j))$ (see [24] and [26, Section IV.B]), and, consequently,

$$\frac{d}{dt}\left(V(\phi(t,j)) - \zeta(t,j)\right) \;\le\; -\varepsilon\left(V(\phi(t,j)) - \zeta(t,j)\right).$$

For each $(t,j) \in \operatorname{dom}((\phi,\zeta),u)$ such that $(t,j+1) \in \operatorname{dom}((\phi,\zeta),u)$, we have that

$$V(\phi(t,j+1)) - \zeta(t,j+1) \le (1-\varepsilon)(V(\phi(t,j)) - \zeta(t,j)).$$

Using the upper bound in (3), it follows that, for all $(t,j) \in \operatorname{dom}((\phi,\zeta),u)$, $V(\phi(t,j)) \le \zeta(t,j) + \exp(-\varepsilon t)(1-\varepsilon)^j\left(\alpha_2(|\xi|_\mathcal{A}) - z\right)$. Assuming, without loss of generality, that $\alpha_2(s) \ge s$ for all $s \ge 0$ gives (12) since we have

$$V(\phi(t,j)) \le |\zeta(t,j)| + 2(\max\{\exp(-\varepsilon), (1-\varepsilon)\})^{t+j}\alpha_2(|\xi|_\mathcal{A} + |z|).$$

We now show item 2. Let ψ and $\mathcal{H}_\circ$ define the generic state-norm estimator for $\mathcal{H}$. Let $(\phi, u) \in \mathcal{S}_\mathcal{H}(\xi)$ and denote $y(t,j) = h(\phi(t,j,\xi,u))$. Then, for any $\xi \in \mathbb{R}^n$ and every solution pair $(\zeta,(u,y)) \in \mathcal{S}_{\mathcal{H}_\circ}(z)$, (11) for $z = 0$ implies that, for all $(t,j) \in \operatorname{dom}(\zeta,(u,y))$,

$$|\psi(\zeta(t,j,0,(u,y)), y(t,j))| \;\le\; \hat{\rho}_1(\|u\|_{(t,j)}) + \hat{\rho}_2(\|y\|_{(t,j)}) \tag{14}$$

Since $((\phi,\zeta),u)$ is also a solution to $\mathcal{S}_{\mathcal{H},\mathcal{H}_\circ}(\xi, z)$, using (14) in (12), we get the IOSS bound $|\phi(t,j,\xi,u)|_\mathcal{A} \le \max\left\{2\widetilde{\beta}(|\xi|_\mathcal{A}, t+j), 2\widetilde{\rho}\circ(2\hat{\rho}_1)(\|u\|_{(t,j)}), 2\widetilde{\rho}\circ(2\hat{\rho}_2)(\|y\|_{(t,j)})\right\}$. ■

With Proposition 3.10 in place, we are ready to prove Theorem 3.7.

Proof of Theorem 3.7: Since $\mathcal{H}$ admits an IOSS Lyapunov function, by Proposition 3.6, it admits an exponential-decay IOSS Lyapunov function. Then, IOSS follows from Proposition 3.10: under the assumptions, item 1 implies that $\mathcal{H}$ admits a state-norm estimator, from where item 2 implies that $\mathcal{H}$ is IOSS. ■

IV. Small Gain Theorem for the Construction of IOSS Lyapunov Functions

We consider hybrid systems $\mathcal{H}$ with state x, input v, and output y that can be written as

$$\left.\begin{aligned}\dot{x}_1 &= f_1(x_1, h_2(x_2), v_1) \\ \dot{x}_2 &= f_2(x_2, h_1(x_1), v_2)\end{aligned}\right\}\ (x,v) \in C, \qquad \left.\begin{aligned} x_1^+ &= g_1(x_1, h_2(x_2), v_1) \\ x_2^+ &= g_2(x_1, h_2(x_2), v_1)\end{aligned}\right\}\ (x,v) \in D, \tag{15}$$

with output $y = (h_1(x_1), h_2(x_2))$, where $x := (x_1, x_2)$, $x_i \in \mathbb{R}^{n_i}$, $i = 1, 2$, $n := n_1 + n_2$, $v := (v_1, v_2) \in \mathcal{V}_1 \times \mathcal{V}_2 =: \mathcal{V}$, and $y_i \in \mathbb{R}^{p_i}$, $p := p_1 + p_2$. The data of $\mathcal{H}$ is (C, f, D, g, h), where $f = (f_1, f_2)$, $g = (g_1, g_2)$, and $h = (h_1, h_2)$. In particular, appropriate decompositions of hybrid systems $\mathcal{H}$ as in (1) can be written as (15). Interconnections of two hybrid systems $\mathcal{H}_1 = (C_1, g_1, D_1, g_1, h_1)$ and $\mathcal{H}_2 = (C_2, g_2, D_2, g_2, h_2)$ with the property that jumps of the individual systems and of the interconnection coincide can be written as (15).[4] Example 4.4 illustrates such a situation. The interconnections of two hybrid systems considered in [12] for the study of ISS properties as well as the systems with jumps considered in [23] can also be written as in (15).

Before we introduce a Lyapunov-based small gain result for hybrid systems in the form (15), we present a lemma that follows from [27, Section 3.3] establishing the existence of two $\mathcal{K}_\infty$ functions and their general properties. As a difference to [10, Lemma A.1], it assures the existence of class-$\mathcal{K}_\infty$ functions that are continuously differentiable on $[0, +\infty)$, rather than on $(0, \infty)$, which is key in defining a locally Lipschitz Lyapunov function for (15).

Lemma 4.1: Let $\chi_1, \chi_2 \in \mathcal{K}_\infty$ satisfy $\chi_1 \circ \chi_2(s) < s\ \forall s > 0$. Then, $\exists\ \rho_1, \rho_2 \in \mathcal{K}_\infty$ such that

1) *$\chi_1(s) < \rho_1^{-1} \circ \rho_2(s)\ \forall s > 0$;* 2) *$\chi_2(s) < \rho_2^{-1} \circ \rho_1(s)\ \forall s > 0$;*
3) *ρ_1, ρ_2 are continuously differentiable on $[0, \infty)$, and $\frac{d\rho_1}{ds}(s) > 0$ and $\frac{d\rho_2}{ds}(s) > 0\ \forall s > 0$.*

Proof: The following result is a consequence of [27, Lemma A.6] that follows from the arguments in the proof of [31, Lemma 4.3].

Claim 1: Suppose $\rho \in \mathcal{K}_\infty$ is C^1 on $(0, \infty)$ and has strictly positive derivative on $(0, \infty)$. Then, there exists a C^1 function $\sigma \in \mathcal{K}_\infty$ such that: 1) σ has strictly positive derivative on $(0, \infty)$; 2) $\sigma \circ \rho$ is C^1 and of class $\mathcal{K}_\infty$; and 3) $\sigma \circ \rho$ has strictly positive derivative on $(0, \infty)$. △

Now, from $\chi_1 \circ \chi_2 < \mathrm{Id}$, we have $\chi_2(s) < \chi_1^{-1}(s)\ \forall s \in (0, \infty)$. Using [10, Lemma B.1], we have the existence of $\widetilde{\rho} \in \mathcal{K}_\infty$ that, on $(0, \infty)$, satisfies the following: is C^1, has strictly positive derivative, and $\chi_2(s) < \widetilde{\rho}(s) < \chi_1^{-1}(s)$. Now, use *Claim 1* with $\widetilde{\rho}$ to get σ. Then, we have $\sigma \circ \chi_2(s) < \sigma \circ \widetilde{\rho}(s) < \sigma \circ \chi_1^{-1}(s)\ \forall s \in (0, \infty)$. Using the properties of σ, $\widetilde{\rho}$, and the identity function, the claim follows with $\rho_1 = \sigma \circ \widetilde{\rho}$ and $\rho_2 = \sigma$. ■

Let X_1, X_2, and $X_\circ$ be the projection of $\overline{C \cup D \cup (g(D) \times \mathcal{V})}$ onto $\mathbb{R}^{n_1}$, $\mathbb{R}^{n_2}$, and $\mathbb{R}^n$, resp.

Theorem 4.2: Consider the hybrid system $\mathcal{H}$ in (15). Suppose there exist locally Lipschitz functions $V_i : \mathbb{R}^{n_i} \to \mathbb{R}_{\ge 0}$, $i = 1, 2$, such that:

A) *There exist functions $\alpha_{i1}, \alpha_{i2} \in \mathcal{K}_\infty$ and $\phi_{i1}, \phi_{i2} : \mathbb{R}^{n_i} \to \mathbb{R}^{p_i}$ such that for all $x_i \in X_i$*
$$\alpha_{i1}(|\phi_{i1}(x_i)|) \;\le\; V_i(x_i) \;\le\; \alpha_{i2}(|\phi_{i2}(x_i)|). \tag{16}$$
B) *There exist functions $\chi_i \in \mathcal{K}_\infty$, $\gamma_i, \varphi_i, \lambda_i, \nu_i \in \mathcal{K}$, and positive definite functions α_i s.t.*
   - *For all $(x,v) \in C$, if $V_1(x_1) \ge \max\{\chi_1(V_2(x_2)), \gamma_1(|v_1|), \varphi_1(|h_1(x_1)|)\}$ then*
   $$V_1^\circ(x_1, f_1(x_1, h_2(x_2), v_1)) \le -\alpha_1(V_1(x_1)); \tag{17}$$
   - *For all $(x,v) \in C$, if $V_2(x_2) \ge \max\{\chi_2(V_1(x_1)), \gamma_2(|v_2|), \varphi_2(|h_2(x_2)|)\}$ then*
   $$V_2^\circ(x_2, f_2(x_2, h_1(x_1), v_2)) \le -\alpha_2(V_2(x_2)); \tag{18}$$
   - *For all $(x,v) \in D$, if $V_1(x_1) \ge \max\{\chi_1(V_2(x_2)), \gamma_1(|v_1|), \varphi_1(|h_1(x_1)|)\}$ then*
   $$V_1(g_1(x_1, h_2(x_2), v_1)) - V_1(x_1) \le -\alpha_1(V_1(x_1)), \tag{19}$$
   otherwise
   $$V_1(g_1(x_1, h_2(x_2), v_1)) \le \max\{\lambda_1(V_1(x_1)), \nu_1(|v|), \nu_1(|h(x)|)\}; \tag{20}$$

[4] System (15) can be interpreted as the interconnection of $\mathcal{H}_1$ with input $u_1 = (w_1, v_1)$ and $\mathcal{H}_2$ with input $u_2 = (w_2, v_2)$ via the assignment $w_1 = y_2$ and $w_2 = y_1$. Then, the flow set for the interconnection between $\mathcal{H}_1$ and $\mathcal{H}_2$ is given by $C := \{(x,v) : (x_1, h_2(x_2), v_1) \in C_1, (x_2, h_1(x_1), v_2) \in C_2\}$, while the flow map is given by the stack of the flow maps f_1 and f_2, i.e., (f_1, f_2). When the jumps of the individual systems and of the interconnection coincide, i.e., $(x_1, h_2(x_2), v_1) \in D_1$ if and only if $(x_2, h_1(x_1), v_2) \in D_2$, then the jump set for the interconnection between $\mathcal{H}_1$ and $\mathcal{H}_2$ is given by $D := \{(x,v) : (x_1, h_2(x_2), v_1) \in D_1, (x_2, h_1(x_1), v_2) \in D_2\}$ and the jump map is given by (g_1, g_2); see [30] and [17].

– *For all* $(x,v) \in D$, *if* $V_2(x_2) \geq \max\{\chi_2(V_1(x_1)), \gamma_2(|v_2|), \varphi_2(|h_2(x_2)|)\}$ *then*

$$V_2(g_2(x_2,h_1(x_1),v_2)) - V_2(x_2) \leq -\alpha_2(V_2(x_2)), \quad (21)$$

otherwise

$$V_2(g_2(x_2,h_1(x_1),v_2)) \leq \max\{\lambda_2(V_2(x_2)), \nu_2(|v|), \nu_2(|h(x)|)\}. \quad (22)$$

– *For all* $s > 0$

$$\chi_1 \circ \chi_2(s) < s \quad (23)$$

and, with $\rho_1, \rho_2 \in \mathcal{K}_\infty$ *generated by Lemma 4.1 using* χ_1, χ_2, *for all* $s > 0$ [5]

$$\lambda_1 \circ \chi_1 \circ \rho_2^{-1}(s) < \rho_1^{-1}(s), \;\; \lambda_2 \circ \chi_2 \circ \rho_1^{-1}(s) < \rho_2^{-1}(s). \quad (24)$$

Then, the locally Lipschitz function

$$V(x) := \max\{\rho_1(V_1(x_1)), \rho_2(V_2(x_2))\} \qquad \forall x \in X_\circ \quad (25)$$

is such that the following hold:

1) *There exist functions* $\widetilde{\alpha}_1, \widetilde{\alpha}_2 \in \mathcal{K}_\infty$ *such that, for all* $x \in X_\circ$,

$$\widetilde{\alpha}_1(|(\phi_{11}(x_1), \phi_{21}(x_2))|) \;\leq\; V(x) \;\leq\; \widetilde{\alpha}_2(|(\phi_{12}(x_1), \phi_{22}(x_2))|); \quad (26)$$

2) *There exist a positive definite function* α, *and functions* $\widetilde{\gamma}, \widetilde{\varphi} \in \mathcal{K}$ *such that*

$$V^\circ(x, f(x,v)) \;\leq\; -\alpha(V(x)) \qquad (x,v) \in C, V(x) \geq \max\{\widetilde{\gamma}(|v|), \widetilde{\varphi}(|h(x)|)\}; \quad (27)$$

$$V(g(x,v)) - V(x) \;\leq\; -\alpha(V(x)) \qquad (x,v) \in D, V(x) \geq \max\{\widetilde{\gamma}(|v|), \widetilde{\varphi}(|h(x)|)\}. \quad (28)$$

Proof: Item 1) follows from the fact that (16) implies that, for all $x \in X_\circ$, $\rho_1 \circ \alpha_{11}(|\phi_{11}(x_1)|)/2 + \rho_2 \circ \alpha_{21}(|\phi_{21}(x_2)|)/2 \leq V(x) \leq \max\{\rho_1 \circ \alpha_{12}(|\phi_{12}(x_1)|), \rho_2 \circ \alpha_{22}(|\phi_{22}(x_2)|)\}$. Let $S_1 = \{(x,v) \;:\; \rho_1(V_1(x_1)) < \rho_2(V_2(x_2))\}$, $S_2 = \{(x,v) \;:\; \rho_1(V_1(x_1)) > \rho_2(V_2(x_2))\}$, $S_3 = \{(x,v) \;:\; \rho_1(V_1(x_1)) = \rho_2(V_2(x_2))\}$. Consider the case $(x,v) \in S_1$. Then, $V(x) = \rho_2(V_2(x_2))$ and, using (23), we have that item 2 in Lemma 4.1 implies $V_2(x_2) > \chi_2(V_1(x_1))$. From (18) and (25), $(x,v) \in C$ and $V(x) \geq \max\{\rho_2 \circ \gamma_2(|v_2|), \rho_2 \circ \varphi_2(|h_2(x_2)|)\}$ imply

$$V^\circ(x, f(x,v)) \;\leq\; -\frac{d\rho_2}{dr} \circ \rho_2^{-1}(V(x)) \; \alpha_2 \circ \rho_2^{-1}(V(x)) =: -\hat{\alpha}_2(V(x)).$$

Proceeding similarly for the case $(x,v) \in S_2$, we obtain $(x,v) \in C$ and $V(x) \geq \max\{\rho_1 \circ \gamma_1(|v_1|), \rho_1 \circ \varphi_1(|h_1(x_1)|)\}$ imply $V^\circ(x, f(x,v)) \leq -\frac{d\rho_1}{dr} \circ \rho_1^{-1}(V(x)) \; \alpha_1 \circ \rho_1^{-1}(V(x)) =: -\hat{\alpha}_1(V(x))$. Finally, for the case $(x,v) \in S_3$, using [12, Proposition 1.1] and the computations above, $(x,v) \in C$ and $V(x) \geq \max\{\rho_1 \circ \gamma_1(|v_1|), \rho_1 \circ \varphi_1(|h_1(x_1)|), \rho_2 \circ \gamma_2(|v_2|), \rho_2 \circ \varphi_2(|h_2(x_2)|)\} =: \max\{\widetilde{\gamma}_1(|v|), \widetilde{\varphi}_1(|h(x)|)\}$ imply $V^\circ(x, f(x,v)) \leq -\min\{\hat{\alpha}_1(V(x)), \hat{\alpha}_2(V(x))\}$. Note that for every $(x,v) \in D$ we have, using (19)-(20) and (21)-(22), respectively,

$$V_1(g_1(x_1,h_2(x_2),v_1)) \;\leq\; \max\{(\mathrm{Id} - \alpha_1)(V_1(x_1)), \lambda_1 \circ \chi_1(V_2(x_2)), \lambda_1 \circ \gamma_1(|v_1|), \lambda_1 \circ \varphi_1(|h_1(x_1)|), \nu_1(|v|), \nu_1(|h(x)|)\},$$

$$V_2(g_2(x_2,h_1(x_1),v_2)) \;\leq\; \max\{(\mathrm{Id} - \alpha_2)(V_2(x_2)), \lambda_2 \circ \chi_2(V_1(x_1)), \lambda_2 \circ \gamma_2(|v|), \lambda_2 \circ \varphi_2(|h(x)|), \nu_2(|v|), \nu_2(|h(x)|)\}.$$

Then, from the definition of V in (25) and using the inequalities right above, for all $(x,v) \in D$ we have

$$V(g(x,v)) \;=\; \max\{\rho_1(V_1(g_1(x_1,h_2(x_2),v_1))), \rho_2(V_2(g_2(x_2,h_1(x_1),v_2)))\} \;\leq\; \max\{\widetilde{\varrho}(V(x)), \widetilde{\gamma}_2(|v|), \widetilde{\varphi}_2(|h(x)|)\},$$

where $\widetilde{\varrho} = \max\{\rho_1 \circ (\mathrm{Id} - \alpha_1) \circ \rho_1^{-1}, \rho_1 \circ \lambda_1 \circ \chi_1 \circ \rho_2^{-1}, \rho_2 \circ (\mathrm{Id} - \alpha_2) \circ \rho_2^{-1}, \rho_2 \circ \lambda_2 \circ \chi_2 \circ \rho_1^{-1}\}$, $\widetilde{\gamma}_2 = \max\{\rho_1 \circ \lambda_1 \circ \gamma_1, \rho_2 \circ \lambda_2 \circ \gamma_2, \rho_1 \circ \nu_1, \rho_2 \circ \nu_2\}$, $\widetilde{\varphi}_2 = \max\{\rho_1 \circ \lambda_1 \circ \varphi_1, \rho_2 \circ \lambda_2 \circ \varphi_2, \rho_1 \circ \nu_1, \rho_2 \circ \nu_2\}$. Since $\rho_i \in \mathcal{K}_\infty$ and $\lambda_i, \gamma_i, \varphi_i, \nu_i \in \mathcal{K}$, for each $i = 1,2$, we have that $\widetilde{\gamma}_2, \widetilde{\varphi}_2 \in \mathcal{K}$. With [7, Lemma B.1] we have that $(\mathrm{Id} - \alpha_i) \in \mathcal{K}$ for each $i = 1,2$. Using (24), $\widetilde{\varrho}(s) < s$ for all $s > 0$ and $\widetilde{\varrho} \in \mathcal{K}$. Then, we have that $(x,v) \in D$ and $V(x) \geq \max\{\widetilde{\varrho}^{-1} \circ \widetilde{\gamma}_2(|v|), \widetilde{\varrho}^{-1} \circ \widetilde{\varphi}_2(|h(x)|)\}$ imply $V(g(x,v)) - V(x) \leq -(\mathrm{Id} - \widetilde{\varrho})(V(x))$. It follows that (27) and (28) hold with $\alpha = \min\{\hat{\alpha}_1, \hat{\alpha}_2, \mathrm{Id} - \widetilde{\varrho}\}$, $\widetilde{\gamma} = \max\{\widetilde{\gamma}_1, \widetilde{\varrho}^{-1} \circ \widetilde{\gamma}_2\}$, and $\widetilde{\varphi} = \max\{\widetilde{\varphi}_1, \widetilde{\varrho}^{-1} \circ \widetilde{\varphi}_2\}$. ■

Remark 4.3: The proof of Theorem 4.2 combines ideas from the proof of [10, Theorem 3.1] for ISS continuous-time systems, from results in [11] for ISS discrete-time systems, and from [12, Theorem 2.1] for ISS hybrid systems. By following the construction and results from [27, Section 3.3], the function V in (25) is locally Lipschitz, while the constructions in [10], [12] do not have such a property at points where V vanishes. Then, when $|(\phi_{11}(x_1), \phi_{21}(x_2))|$ and $|(\phi_{12}(x_1), \phi_{22}(x_2))|$ vanish on a compact subset $\mathcal{A}$ of $\mathbb{R}^n$, [6] and $\alpha \in \mathcal{K}_\infty$, Theorem 4.2 implies that V satisfies (6)-(7). The additional property on α is guaranteed when the assumptions are strengthened so that $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and the function $\mathrm{Id} - \widetilde{\varrho}$ constructed in the proof is of class $\mathcal{K}_\infty$ – Example 4.4 illustrates this point.[7] Moreover, the following special cases are of interest: i) For each $i = 1,2$, $\phi_{i,1} = \phi_{i,2}$ and $\varphi_i \equiv 0$; ii) For each $i = 1,2$, $\phi_{i,1} = \phi_{i,2} = \mathrm{Id}$; iii) For each $i = 1,2$, $\phi_{i,1} = h_i$ and $\phi_{i,2} = \mathrm{Id}$. The special case i) coincides with [12, Theorem 2.1]. The conditions in Theorem 4.2 for the ii) case are in terms of IOSS Lyapunov functions in (strict) inequality form (see (6)-(7)), with the additional boundedness conditions (20),(22) at jumps, which are required in Lyapunov-based small gain results for discrete-time systems [11]. The conditions for case iii) are related to IOS Lyapunov functions for hybrid systems; see [17]. △

Example 4.4: Consider the hybrid system $\mathcal{H}_1$, as given in Example 3.8, with state $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, output $y_1 \in \mathbb{R}$, input for interconnection $w_1 \in \mathbb{R}$ and additional exogenous input $v_1 = (v_{11}, v_{12}) \in \mathbb{R}^2$, and data $f_1(\xi, w_1, v_1) := (\xi_2, -\gamma - b\xi_2 + v_{11})$, $g_1(\xi, w_1, v_1) := (\xi_1 + a_1\xi_2^2, e_1|\xi_2| + v_{12})$, $h_1(\xi) := \xi_1$, $C_1 := \{(\xi, w_1) \;:\; \xi_1 \geq w_1, w_1 \geq 0\}$, $D_1 := \{(\xi, w_1) \;:\; \xi_1 = w_1, w_1 \geq 0\}$, where $\gamma, b, a_1 > 0$, $e_1 \in [0,1)$. Consider also the hybrid system $\mathcal{H}_2$ with state $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$, output $y_2 \in \mathbb{R}$, input $w_2 \in \mathbb{R}$ and $v_2 = (v_{21}, v_{22}) \in \mathbb{R}^2$, and data $f_2(\eta, w_2, v_2) := (\eta_2, -\eta_2 + v_{21})$, $g_2(\eta, w_2, v_2) := (\eta_1 - a_2|\eta_2|, -e_2|\eta_2| + v_{22})$, $h_2(\eta) := \eta_1$, $C_2 := \{(\eta, w_2) \;:\; \eta_1 \leq w_2, \eta_1 \geq 0\}$, $D_2 := \{(\eta, w_2) \;:\; \eta_1 = w_2, \eta_1 \geq 0\}$, where $a_2 > 0$, $e_2 \in [0,1)$. We apply Theorem 4.2 to the hybrid system $\mathcal{H}$ as in (15) resulting from the assignment[8] $w_1 = y_2$, $w_2 = y_1$. Let $V_1(\xi) = \gamma\xi_1 + \frac{1}{2}\xi_2^2$ and $V_2(\eta) = \frac{1}{2}\eta^\top\eta$. Then, for each $i \in \{1,2\}$, condition (16) holds with $\phi_{i1}(s) = \phi_{i2}(s) := s$, $\alpha_{11}(s) = \min\{s^2/4, \gamma s/\sqrt{2}\}$, $\alpha_{12}(s) = \frac{1}{2}s^2 + \gamma s$, $\alpha_{21}(s) = \alpha_{22}(s) := \frac{1}{2}s^2$ for all $s \geq 0$. Now consider V_1 along the flows of $\mathcal{H}_1$ and V_2 along the flows of $\mathcal{H}_2$. Following the computations in Example 3.8, for each $(\xi, \eta) \in C$ we have

$$\langle \nabla V_1(\xi), f_1(\xi, h_2(\eta), v_1) \rangle \;\leq\; -\frac{b}{2}V_1(\xi) + \max\left\{b\gamma h_1(\xi), \frac{1}{b}v_{11}^2\right\}$$

$$\langle \nabla V_2(\eta), f_2(\eta, h_1(\xi), v_2) \rangle \;\leq\; -\frac{1}{2}V_2(\eta) + \max\left\{\frac{5}{2}h_2(\eta)^2, 2v_{21}^2\right\}.$$

[5] In particular, condition (24) holds when functions λ_1, λ_2 are given by the identity function (Id).

[6] Such is the case, e.g., when $\phi_{11} = \phi_{12} = |\cdot|_{\mathcal{A}_1}$ and $\phi_{21} = \phi_{22} = |\cdot|_{\mathcal{A}_2}$ with $\mathcal{A}_1$ and $\mathcal{A}_2$ compact and $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$.

[7] With the properties of V guaranteed by Theorem 3.7 and associated functions, it might be possible to pass to a new locally Lipschitz function V for which (4)-(5) hold for some functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\sigma_1, \sigma_2 \in \mathcal{K}$.

[8] This interconnection resembles the vertical interaction between a ball, which is modeled by $\mathcal{H}_1$ (with ξ_1 being the position and ξ_2 the velocity), and a moving arm, which is modeled as $\mathcal{H}_2$ (with η_1 being the position and η_2 the velocity), evolving for positive height η_1. In this setting, γ denotes the gravitational constant, b captures the viscous friction due to air, a_1, a_2 the compression effect and e_1, e_2 the restitution at impacts. The exogenous signals v_{11}, v_{21} and v_{12}, v_{22} represent disturbances.

With the chosen input assignment, it follows that $(\xi, \eta) \in D$ if and only if $(\xi, \eta_1) \in D_1$ and $(\eta, \xi_1) \in D_2$. Then, following the computations in Example 3.8, for each $(\xi, \eta) \in D$ we have

$$\begin{aligned} V_1(g_1(\xi, h_2(\eta), v_1)) &\leq (e_1^2 + 2\gamma a_1 + \varepsilon_1) V_1(\xi) \\ &\quad + \gamma(1 - e_1^2 - 2\gamma a_1 - \varepsilon_1) h_1(\xi) + \frac{1}{2}\left(1 + \frac{e_1^2}{\varepsilon_1}\right) v_{12}^2 \end{aligned}$$

where $\varepsilon_1 > 0$. For each $(\xi, \eta) \in D$ we also have

$$\begin{aligned} V_2(g_2(\eta, h_1(\xi), v_2)) &\leq (e_2^2 + a_2^2 + \varepsilon_2) V_2(\eta) + \frac{1}{2}\left(1 + \frac{e_2^2}{\varepsilon_2}\right) v_{22}^2 \\ &\quad + \frac{h_2(\eta)^2}{2}, \end{aligned}$$

where $\varepsilon_2 > 0$. Combining the computations above, conditions 1)-3) of Theorem 4.2 hold with $\alpha_1(s) = \min\left\{\frac{b}{4}, \frac{1}{2}(1 - e_1^2 - 2\gamma a_1 - \varepsilon_1)\right\} s$, $\varphi_1(s) = 4\gamma s$, $\gamma_1(s) = \max\left\{\frac{4}{b^2}, \frac{2\left(1+e_1^2/\varepsilon_1\right)}{(1-e_1^2-2\gamma a_1-\varepsilon_1)}\right\} s^2$, $\lambda_1(s) = 2(e_1^2 + 2\gamma a_1 + \varepsilon_1)s$, $\nu_1(s) = \max\left\{2\gamma(1 - e_1^2 - 2\gamma a_1 - \varepsilon_1)s, \left(1 + \frac{e_1^2}{\varepsilon_1}\right) s^2\right\}$, $\alpha_2(s) = \min\left\{\frac{1}{4}, \frac{1}{2}(1 - e_2^2 - a_2^2 - \varepsilon_2)\right\} s$, $\varphi_2(s) = 10s^2$, $\gamma_2(s) = \max\left\{8, \frac{2\left(1+e_2^2/\varepsilon_2\right)}{(1-e_2^2-a_2^2-\varepsilon_2)}\right\} s^2$, $\lambda_2(s) = 2(e_2^2 + a_2^2 + \varepsilon_2)s$, $\nu_2(s) = \left(1 + \frac{e_2^2}{\varepsilon_2}\right) s^2$, parameters satisfying $e_1^2 + 2\gamma a_1 + \varepsilon_1 < 1$, $\varepsilon_1 > 0$, $e_2^2 + a_2^2 + \varepsilon_2 < 1$, $\varepsilon_2 > 0$, functions χ_1, χ_2 satisfying (23) and such that, for ρ_1, ρ_2 generated via Lemma 4.1, (24) holds. In particular, the conditions hold for $\chi_1(s) = k_1 s$, $\chi_2(s) = k_2 s$, $\rho_1(s) = k_3 s$, $\rho_2(s) = k_4 s$ with $k_i > 0$, $i = 1, \ldots, 4$, $k_1 < \min\left\{\frac{k_4}{k_3}, \frac{k_4}{2(e_1^2+2\gamma a_1+\varepsilon_1)k_3}\right\}$, $k_2 < \min\left\{\frac{k_3}{k_4}, \frac{k_3}{2(e_2^2+a_2^2+\varepsilon_2)k_4}\right\}$, $k_1 k_2 < 1$. From the proof of Theorem 3.7, we have that $\widetilde{\varrho}(s) = k_5 s$ with $k_5 = \max\{1 - \min\{b/4, (1 - e_1^2 - 2\gamma a_1 - \varepsilon_1)/2\}, \frac{2k_1k_3}{k_4}(e_1^2 + 2\gamma a_1 + \varepsilon_1), 1 - \min\{1/4, (1 - e_2^2 - a_2^2 - \varepsilon_2)/2\}, \frac{2k_2k_4}{k_3}(e_2^2 + a_2^2 + \varepsilon_2)\}$, which, with the above bounds on the constants, is less than one, in turn, implying that $\alpha \in \mathcal{K}_\infty$. Using the definition of $\phi_{i1}, \phi_{i,2}$, $\mathcal{A} = \{0\} \in \mathbb{R}^4$, the fact that $\mathrm{Id} - \alpha_1$ and $\mathrm{Id} - \alpha_2$ are of class $\mathcal{K}_\infty$, the conditions of Theorem 4.2 are satisfied. $\triangle$

V. Conclusions

For a general class of hybrid systems, we presented an input-output-to-state stability notion, sufficient conditions, and a small gain result for the study of an interconnection of two hybrid systems systems. The nature of the results and the general hybrid systems framework under study, which cover classical continuous and discrete-time systems, and the mild assumptions imposed in the data suggest wide applicability of the results. Moreover, the conditions for IOSS of the individual systems and their interconnection are given in terms of Lyapunov functions and involve conditions that can be checked without knowledge of solutions of hybrid systems.

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References

[1] Z.-P. Jiang, A. R. Teel, and L. Praly. Small-gain theorem for ISS systems and applications. *Math. Control Signals Syst.*, 7:95–120, 1994.

[2] A. R. Teel. A nonlinear small gain theorem for the analysis of control systems with saturation. *IEEE Transactions on Automatic Control*, 41(9):1256–1270, September 1996.

[3] E. D. Sontag and Y. Wang. Output-to-state stability and detectability of nonlinear systems. *Systems & Control Letters*, 29:279–290, 1997.

[4] M. Krichman, E. D. Sontag, and Y. Wang. Input-output-to-state stability. *SIAM J. Control Optim.*, 39:1874–1928, 2000.

[5] S. N. Dashkovskiy, D. V. Efimov, and E. D. Sontag. Input to state stability and allied system properties. *Automation and Remote Control*, 72(8):1579–1614, 2011.

[6] E. D. Sontag and Y. Wang. New characterizations of input-to-state stability. *IEEE Transactions on Automatic Control*, 41(9):1283–1294, 1996.

[7] Z-P. Jiang and Y. Wang. Input-to-state stability for discrete-time nonlinear systems. *Automatica*, 37:857–869, 2001.

[8] J. L. Mancilla-Aguilar and R. A. Garcia. On converse Lyapunov theorems for ISS and iISS switched nonlinear systems. *Systems & Control Letters*, 42(1):47–53, 2001.

[9] C. Cai and A. R. Teel. Characterizations of input-to-state stability for hybrid systems. *Syst. & Cont. Letters*, 58:47–53, 2009.

[10] Z. P. Jiang, I. M. Y. Mareels, and Y. Wang. A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems. *Automatica-London*, 32(8):1211–1216, 1996.

[11] D. S. Laila and D. Nesic. Lyapunov-based small-gain theorem for parameterized discrete-time interconnected ISS systems. *IEEE Transactions of Automatic Control*, 48:1783–1788, 2003.

[12] D. Nesic and A. R. Teel. A Lyapunov-based small-gain theorem for hybrid ISS systems. In *Proc. 47th IEEE Conference on Decision and Control*, pages 3380–3385, 2008.

[13] S. Dashkovskiy and M. Kosmykov. Stability of networks of hybrid ISS systems. In *Proc. Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, pages 3870–3875, 2009. See also http://arxiv.org/abs/1012.5528.

[14] S. Dashkovskiy, M. Kosmykov, and F. Wirth. A small gain condition for interconnections of ISS systems with mixed ISS characterizations. *IEEE Transactions on Automatic Control*, 56(6):1247–1258, 2011.

[15] I. Karafyllis and Z.P. Jiang. A vector small-gain theorem for general non-linear control systems. *IMA Journal of Mathematical Control and Information*, 28(3):309–344, 2011.

[16] C. Cai and A. R. Teel. Output-to-state stability for hybrid systems. *Systems & Control Letters*, 60:62–68, 2011.

[17] R. G. Sanfelice. Interconnections of hybrid systems: Some challenges and recent results. *Journal of Nonlinear Systems and Applications*, pages 111–121, 2011.

[18] C. Cai and A. R. Teel. Input-output-to-state stability for discrete-time systems. *Automatica*, 44:326–336, 2008.

[19] J. L. Mancilla-Aguilar, R. Garcia, E. D. Sontag, and Y. Wang. On the representation of switched systems with inputs by perturbed control systems. *Nonlinear Analysis: Theory, Methods & Applications*, 60(6):1111–1150, 2005.

[20] D. Angeli, B. Ingalls, E. D. Sontag, and Y. Wang. Separation principles for input-output and integral-input-to-state stability. *SIAM J. Control Optim.*, 43:256–276, 2004.

[21] C. Cai and A. R. Teel. Asymptotic characterization of input-output-to-state stability for discrete-time systems. *Systems & Control Letters*, 56:408–415, 2007.

[22] M. A. Muller and D. Liberzon. Input/output-to-state stability of switched nonlinear systems. In *Proc. 2010 American Control Conference*, pages 1708–1712, 2010.

[23] D. Nesic and A.R. Teel. Input-output stability properties of networked control systems. *IEEE Transactions on Automatic Control*, 49:1650–1667, 2004.

[24] F.H. Clarke. *Optimization and Nonsmooth Analysis*. SIAM's Classic in Applied Mathematics, Philadelphia, 1990.

[25] R. Goebel, R.G. Sanfelice, and A.R. Teel. Hybrid dynamical systems. *IEEE Control Systems Magazine*, 29(2):28–93, April 2009.

[26] R. G. Sanfelice, R. Goebel, and A. R. Teel. Invariance principles for hybrid systems with connections to detectability and asymptotic stability. *IEEE Transactions on Automatic Control*, 52(12):2282–2297, 2007.

[27] L. Praly. *Fonctions de Lyapunov, Stabilite, Stabilisation et Attenuation de Perturbations - Part I.* Lecture Notes (in French), 2007. Available via private communication.

[28] B. Rueffer. *Monotone dynamical systems, graphs, and stability of large-scale interconnected systems*. PhD thesis, University of Bremen, 2007.

[29] L. Praly and Y. Wang. Stabilization in spite of matched unmodeled dynamics and an equivalent definition of input-to-state stability. *Mathematics of Control, Signals, and Systems*, 9:1–33, 1996.

[30] S. Dashkovskiy and M. Kosmykov. Input-to-state stability of interconnected hybrid systems. *Automatica (to appear)*.

[31] Y. Lin, E. D. Sontag, and Y. Wang. A smooth converse Lyapunov theorem for robust stability. *SIAM J. Control and Optimization*, 34(1):124–160, 1996.