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On Cobordism Maps in Embedded Contact Homology

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by

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ABSTRACT OF THE DISSERTATION

On Cobordism Maps in

Embedded Contact Homology

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Professor Ko Honda, Chair

In this thesis, we give a proposed construction of cobordism maps in embedded contact homology via a count of J-holomorphic curves. We prove a new index formula in the L-supersimple setting of Bao-Honda and use it to classify degenerations of 1-dimensional moduli spaces of curves in exact symplectic cobordisms. We correct these degenerations by using obstruction bundle techniques of Hutchings-Taubes to continue the moduli spaces by gluing in branched covers of trivial cylinders. We then use a new evaluation map to cut out a 1-dimensional family in the resulting moduli space, modulo two conjectures on the behavior of this map. Finally, we analyze the new endpoints of the continuations and use them to complete our proposed definition of the cobordism maps.

The dissertation of Jacob Hunter Rooney is approved.

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To my parents, Patrick and Patricia,

and to Michael

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CHAPTER 1

Introduction

Embedded contact homology (ECH) is a homology theory for contact 3-manifolds defined by Michael Hutchings as a symplectic analogue of Seiberg-Witten Floer homology. It is a topological invariant that is intimately connected to the dynamics of Reeb vector fields, in particular the existence of closed, periodic orbits. Recently, Cristofaro-Gardiner-Hutchings-Pomerleano [8] used ECH to show that every contact structure on a smooth 3-manifold with torsion first Chern class has either two or infinitely many closed, embedded orbits. The theory has also yielded obstructions for the existence of symplectic embeddings in dimension 4; see [13] and [7].

ECH is isomorphic to both Heegaard Floer homology and Seiberg-Witten Floer homology (see [19], [20], [21], [22], [23], [4]), and the latter isomorphism was used by Hutchings-Taubes in [17] to define maps induced by exact symplectic cobordisms between contact 3-manifolds. Unfortunately, a self-contained definition of such maps, i.e., in terms of counting J-holomorphic curves, has proved elusive. In [9], Chris Gerig has given a construction in a special case, and Hutchings has given an example where a curve count must take into account multi-level SFT buildings as defined in [3].

In this thesis, we propose a way to construct chain maps on ECH via a count of J-holomorphic curves. We confine ourselves to the L-supersimple setting used by Bao-Honda in [2] to define contact homology. In Chapter 3, we define a new rescaling of the evaluation map defined by Bao-Honda [1] that detects long necks in J-holomorphic curves. In Chapter 4, we prove a combinatorial formula for the ECH index that is useful for analyzing degenerations of curves with ECH index 1 in exact symplectic cobordisms. Finally, in Chapters 5 and 6, we use the obstruction bundle gluing techniques developed in [15] and [12] to correct for these degenerations, module two conjectures on the behavior of an evaluation map at the ends of punctured J-holomorphic curves, yielding a chain map.

Remark 1.0.1. We do not discuss invariance of the proposed chain map under 1-dimensional families of exact symplectic cobordisms. The analysis required to define chain homotopies is likely significant even in the L-supersimple setting, and we content ourselves with focusing on chain maps in this thesis.

CHAPTER 2

Background

In this chapter, we review the definition and properties of embedded contact homology as defined by Michael Hutchings. Our main references for this material are [11] and [12].

2.1 Basic definitions

Let Y be a smooth 3-manifold and $\xi = \ker \lambda$ a contact structure on Y. We assume that λ induces a positive volume form $\lambda \wedge d\lambda$ on Y. Let R_λ denote the Reeb vector field of λ, defined as the unique vector field on Y satisfying $\lambda(R_\lambda) = 1$ and $d\lambda(R_\lambda, \cdot\,) = 0$.

Definition 2.1.1. We say that a contact form λ is **non-degenerate** if the following holds for all closed, periodic orbits of the Reeb vector field R_λ. If α is such an orbit of period T and ϕ is the time-T flow of R_λ around α, then 1 is not an eigenvalue of the **linearized first return map** $d\phi_p \colon \xi_p \to \xi_p$ for any $p \in \alpha$.

We will usually refer to such closed, periodic orbits of R_λ as **Reeb orbits**. Any contact form on Y can be perturbed to a non-degenerate form, and we will assume throughout that all contact forms are as such. There are two classes of Reeb orbits for non-degenerate contact forms, determined by the eigenvalues of $d\phi_p$.

Definition 2.1.2. A Reeb orbit α is said to be **hyperbolic** if the eigenvalues of the linearized first return map are real, and **elliptic** if the eigenvalues are complex and of unit norm.

Definition 2.1.3. The **action** of a Reeb orbit α is defined as

$$\mathcal{A}(\alpha) = \int_\alpha \lambda.$$

The differential in embedded contact homology of Y counts J-holomorphic curves in the symplectization $\mathbb{R} \times Y$, and we will need to restrict our choice of almost complex structure on $\mathbb{R} \times Y$ to achieve the necessary transversality.

Definition 2.1.4. An almost complex structure J on the symplectization $\mathbb{R} \times Y$ is **admissible** if it satisfies the following three properties:

1. J is invariant under translation in the $\mathbb{R}$-direction;

2. $J(\partial_s) = \mathbb{R}_\lambda$, where s is the $\mathbb{R}$-coordinate of $\mathbb{R} \times Y$; and

3. J restricts to an orientation-preserving isomorphism on ξ.

All almost complex structures considered in this thesis are implicitly assumed to be admissible.

Definition 2.1.5. Let γ be a Reeb orbit and τ a trivialization of $\gamma^*\xi$. The **Conley-Zehnder index** of γ with respect to the trivialization τ is defined as follows. If γ is hyperbolic, then in the trivialization τ, the linearized Reeb flow rotates ξ by an angle $n\pi$ for some integer n, and we define

$$CZ_\tau(\gamma^k) = kn.$$

If γ is elliptic, then in the trivialization τ, the linearized Reeb flow rotates ξ by an angle $2\pi\theta$ for some irrational number θ, and we define

$$CZ_\tau(\gamma^k) = 2\lfloor k\theta \rfloor + 1.$$

2.2 Punctured holomorphic curves

Let (Σ, j) be a closed Riemann surface, i.e., a complex manifold of dimension 1, with complex structure j. Let $\boldsymbol{p} \subset \Sigma$ be a finite set of points, called **punctures**, partitioned into sets $\boldsymbol{p}_+$ of **positive** and $\boldsymbol{p}_-$ of **negative** punctures, and define $\dot{\Sigma} = \Sigma \setminus \boldsymbol{p}$. Let J be an admissible almost complex structure on $\mathbb{R} \times Y$. A **punctured holomorphic curve** is a smooth map

$$u \colon \dot{\Sigma} \to \mathbb{R} \times Y$$

such that

$$du + J \circ du \circ j = 0,$$

i.e., so that the derivative du is (j, J)-linear.

A J-holomorphic curve $u\colon \dot{\Sigma} \to \mathbb{R} \times Y$ is said to be **multiply covered** if there is a punctured Riemann surface $\dot{\Sigma}'$, a J-holomorphic map $v\colon \dot{\Sigma}' \to \mathbb{R} \times Y$, and a (possibly branched) cover $\phi\colon \dot{\Sigma}' \to \dot{\Sigma}$ such that $u = v \circ \phi$. A curve u with connected domain is said to be **somewhere injective** if it is not multiply covered.

Definition 2.2.1. Let $u\colon \dot{\Sigma} \to \mathbb{R} \times Y$ be a J-holomorphic map and let τ be a trivialization of $u^*\xi$ over each end of u. The **relative first Chern number** $c_1(u^*\xi, \tau)$ of u with respect to τ is defined as follows. Choose a section of $u^*\xi$ that is constant in the trivialization τ near each puncture and take $c_1(u^*\xi, \tau)$ to be the number of zeros of said section, counted with sign.

Definition 2.2.2. Let $u\colon \dot{\Sigma} \to \mathbb{R} \times Y$ be a J-holomorphic map and let τ be a trivialization of $u^*\xi$ over each end of u. The **relative self-intersection number** $Q_\tau(u)$ with respect to τ is defined as follows. Let S and S' be two embedded surfaces in $\mathbb{R} \times Y$ that (1) both represent the same relative homology class as u when projected to Y, (2) intersect transversely away from small neighborhoods of the ends, and (3) such that the ends of S and S' at each Reeb orbit α approach α along distinct rays in a small tubular neighborhood of α when projected to Y. Then we define $Q_\tau(u)$ as the oriented intersection number of S with S'. See [12] for a more detailed explanation.

2.3 Moduli spaces

Let $u\colon \dot{\Sigma} \to \mathbb{R} \times Y$ be a punctured holomorphic curve asymptotic to Reeb orbits $\boldsymbol{\alpha}_+ = (\alpha_1, \ldots, \alpha_n)$ at the positive end and to $\boldsymbol{\alpha}_- = (\alpha_{-1}, \ldots, \alpha_{-m})$ at the negative end. For each Reeb orbit, choose a point $z_i \in \alpha_i$. At each puncture $z \in \boldsymbol{p}$, choose an element of $(T_z\Sigma \setminus \{0\})/\mathbb{R}_+$, called an **asymptotic marker**, that corresponds to the preimage of z_i under the map u, and let $\boldsymbol{r}$ denote the set of such markers. We will refer to markers at positive punctures as **positive markers** and markers at negative punctures as **negative markers**.

Given orbit sets $\boldsymbol{\alpha}_+$ and $\boldsymbol{\alpha}_-$, the **moduli space of punctured holomorphic curves** from $\boldsymbol{\alpha}_+$ to $\boldsymbol{\alpha}_-$ in $\mathbb{R} \times Y$, denoted $\mathcal{M}_J(\boldsymbol{\alpha}_+, \boldsymbol{\alpha}_-)$, is the space of pairs $(u, \boldsymbol{r})$ such that u is asymptotic to $\boldsymbol{\alpha}_+$ at the positive punctures and to $\boldsymbol{\alpha}_-$ at the negative punctures, modulo biholomorphisms of domains that map positive punctures and markers to positive punctures and markers and negative punctures and markers to negative punctures and markers. Maps in such moduli spaces have uniform energy bounds, and the spaces themselves can be compactified by adding in SFT buildings. See [3] for details.

A punctured holomorphic curve $u \in \mathcal{M}_J(\boldsymbol{\alpha}_+, \boldsymbol{\alpha}_-)$ has a **Fredholm index** give by

$$\operatorname{ind}(u) = -\chi(\dot{\Sigma}) + CZ_\tau(\boldsymbol{\alpha}_+) - CZ_\tau(\boldsymbol{\alpha}_-) + 2c_1(u^*\xi, \tau),$$

where τ is a trivialization of ξ over $\boldsymbol{\alpha}_+$ and $\boldsymbol{\alpha}_-$, $CZ_\tau(\boldsymbol{\alpha})$ is the Conley-Zehnder index of the orbit set $\boldsymbol{\alpha}$ with respect to τ, and $c_1(u^*\xi, \tau)$ is the first Chern class of the pullback of ξ with respect to τ. If $\mathcal{M}_J(\boldsymbol{\alpha}_+, \boldsymbol{\alpha}_-)$ is regular, i.e., transversely cut out, then $\operatorname{ind}(u)$ is the dimension of a neighborhood of u in $\mathcal{M}_J(\boldsymbol{\alpha}_+, \boldsymbol{\alpha}_-)$.

2.4 The ECH chain complex

We will now define the ECH chain complex. We use $\mathbb{Z}/2\mathbb{Z}$ coefficient groups throughout without comment. Let $\Gamma \in H_1(Y)$ and let J be a generic admissible almost complex structure on the symplectization $\mathbb{R} \times Y$. The chain groups $ECC(Y, \lambda, \Gamma, J)$ are generated by **orbit sets** $\boldsymbol{\alpha} = \alpha_1^{m_1} \cdots \alpha_k^{m_k}$, where the α_i are distinct, embedded Reeb orbits, $m_i \in \mathbb{Z}_{\geq 0}$ for all i, $m_i = 1$ whenever α_i is hyperbolic, and such that

$$[\boldsymbol{\alpha}] = \sum_{i=1}^{k} m_i[\alpha_i] = \Gamma.$$

For future reference, we define the **action** of an orbit set $\boldsymbol{\alpha} = \alpha_1^{m_1} \cdots \alpha_k^{m_k}$ as

$$\mathcal{A}(\boldsymbol{\alpha}) = \sum_{i=1}^{k} m_i \mathcal{A}(\alpha_i).$$

The last ingredient we will need to define the ECH complex is the **ECH index**. Let $u \in \mathcal{M}_J(\boldsymbol{\alpha}, \boldsymbol{\beta})$ and define

$$I(u) = c_1(u^*\xi, \tau) + Q_\tau(u) + CZ_\tau^I(\boldsymbol{\alpha}, \boldsymbol{\beta}),$$

where CZ_τ^I is defined by

$$CZ_\tau^I(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_i \sum_{k=1}^{m_i} CZ_\tau(\alpha_i^k) - \sum_j \sum_{k=1}^{n_j} CZ_\tau(\beta_j^k).$$

The differential ∂ counts punctured J-holomorphic curves with ECH index 1 in the symplectization $\mathbb{R} \times Y$. More precisely, let α be an orbit set and consider, for each orbit set $\boldsymbol{\beta}$, the moduli space $\mathcal{M} = \mathcal{M}_J^1(\boldsymbol{\alpha}, \boldsymbol{\beta})$ of punctured J-holomorphic curves u with $I(u) = 1$ asymptotic to $\boldsymbol{\alpha}$ at the positive punctures and to $\boldsymbol{\beta}$ at the negative punctures.

Lemma 2.4.1. *If the moduli space $\mathcal{M}_J^1(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is non-empty, then $\mathcal{A}(\boldsymbol{\beta}) < \mathcal{A}(\boldsymbol{\alpha})$.*

Proof. The result follows easily from Stokes' theorem, admissibility of J, and the fact that the symplectic form on $\mathbb{R} \times Y$ is exact. □

Lemma 2.4.2. *If J is generic, then $\mathcal{M}_J^1(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is a 1-dimensional manifold, i.e., the moduli space is regular and every curve in it has Fredholm index 1.*

Curves counted by the differential satisfy a rigid requirement on the multiplicities of their various ends. This requirement is crucial in the proof in [15] and [16] that $\partial^2 = 0$ and will be leveraged extensively later in this thesis.

Definition 2.4.3. Let γ be an embedded hyperbolic Reeb orbit in a contact 3-manifold (Y, λ). Let $u \colon \dot{\Sigma} \to \mathbb{R} \times Y$ be a J-holomorphic curve with positive ends of multiplicity $p_1, \ldots, p_k$ and negative ends $q_1, \ldots, q_l$ at covers of γ. We say that u satisfies the **ECH partition conditions** at ends asymptotic to covers of γ if the multiplicities p_i and q_j are as in Table 2.1.

The partition conditions for elliptic orbits are more complicated and will not be needed in this thesis. Interested readers can consult [11] for details.

	n even	n odd
γ positive hyperbolic	$(1,\ldots,1)$	$(1,\ldots,1)$
γ negative hyperbolic	$(2,\ldots,2)$	$(2,\ldots,2,1)$

Table 2.1: The partition conditions for hyperbolic Reeb orbits.

There is an $\mathbb{R}$-action on $\mathcal{M}$ induced by translation in the $\mathbb{R}$-direction of $\mathbb{R}\times Y$. The differential ∂ on the chain complex described above is defined by

$$\partial(\boldsymbol{\alpha}) = \sum_{\mathcal{A}(\boldsymbol{\beta})<\mathcal{A}(\boldsymbol{\alpha})} \#\left(\mathcal{M}^1_J(\boldsymbol{\alpha},\boldsymbol{\beta})/\mathbb{R}\right)\boldsymbol{\beta}.$$

Lemma 2.4.4. *If J is generic, the quotient $\mathcal{M}^1_J(\boldsymbol{\alpha},\boldsymbol{\beta})/\mathbb{R}$ is a compact 0-dimensional manifold. Thus, the count $\#\left(\mathcal{M}^1_J(\boldsymbol{\alpha},\boldsymbol{\beta})/\mathbb{R}\right)$ is finite if λ is non-degenerate.*

The proof that $\partial^2 = 0$ is quite difficult and uses obstruction bundle gluing to describe the boundary of 1-dimensional moduli spaces of punctured J-holomorphic curves in symplectizations. Interested readers can consult [15] and [16] for details.

2.5 The L-supersimple setting

In [5], Colin-Ghiggini-Honda develop a technique for perturbing contact forms that, for any fixed $L > 0$, eliminates elliptic orbits with action less than L. We will use a version of this result given in [1].

Theorem 2.5.1. *[1, Theorem 2.0.2] Let λ be a non-degenerate contact form for (Y,ξ). Then, for any $L > 0$ and $\epsilon > 0$, there exists a smooth function $\phi\colon Y \to \mathbb{R}_+$ such that*

1. *ϕ is ϵ-close to 1 with respect to a fixed C^1-norm;*
2. *all the orbits of $R_{\phi\lambda}$ of $\phi\lambda$-action less than L are hyperbolic.*

Moreover, we may assume that

3. *each positive hyperbolic orbit α has a neighborhood $(\mathbb{R}/\mathbb{Z})\times D^2_{\delta_0}$ with coordinates (t,x,y) such that*

(a) $D^2_{\delta_0} = \{x^2 + y^2 \leq \delta_0\}$, *where* $\delta_0 > 0$ *is small;*

(b) $\phi\lambda = H\ dt + \eta$;

(c) $H = c(\alpha) - \epsilon xy$, *with* $c(\alpha), \epsilon > 0$ *and* $c(\alpha) \gg \epsilon$;

(d) $\eta = 2x\, dy + y\, dx$;

(e) $\alpha = \{x = y = 0\}$.

4. *each negative hyperbolic orbit* α *has a neighborhood* $([0, 1] \times D^2_{\delta_0})/\sim$ *with coordinates* (t, x, y), *where* $\sim$ *identifies* $(1, x, y) \sim (0, -x, -y)$ *and the conditions (a) through (e) above hold.*

Following Bao-Honda, we make the following definition.

Definition 2.5.2. Let $L > 0$. We say that a contact form λ on a smooth 3-manifold Y is L-**supersimple** if all Reeb orbits of action less than L are non-degenerate, are hyperbolic, and satisfy the conclusions of Theorem 2.5.1.

For the remainder of this thesis, we will work exclusively in the L-supersimple setting, as it has two crucial advantages. One of the main advantages is that the Fredholm index is well-behaved under taking multiple covers.

Lemma 2.5.3. *[1, Lemma 3.3.2] Let* Y *be a smooth* 3*-manifold, let* λ *be an* L*-supersimple contact form on* Y*, and let* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ *be orbit sets of total action less than* L*. If* $u \in \mathcal{M}_J(\boldsymbol{\alpha}', \boldsymbol{\beta}')$ *is a degree* k *branched cover of a somewhere injective curve* $v \in \mathcal{M}_J(\boldsymbol{\alpha}, \boldsymbol{\beta})$*, and if* b *is the total branching order of* u*, then*

$$\operatorname{ind}(u) = k \operatorname{ind}(v) + b.$$

In particular, $\operatorname{ind}(u) \geq 0$ *for all* $u \in \mathcal{M}_J(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

Proof. Let $\dot{\Sigma}$ be the domain of v and $\dot{\Sigma}'$ the domain of u. By the Riemann-Hurwitz formula, $\chi(\dot{\Sigma}') = k \cdot \chi(\dot{\Sigma}) - b$. The Conley-Zehnder index is multiplicative for hyperbolic orbits, so

$$CZ_\tau(\boldsymbol{\alpha}') = k \cdot CZ_\tau(\boldsymbol{\alpha}) \quad \text{and} \quad CZ_\tau(\boldsymbol{\beta}') = k \cdot CZ_\tau(\boldsymbol{\beta}).$$

It follows that

$$\begin{aligned}\operatorname{ind}(u) &= -\chi(\dot{\Sigma}') + CZ_\tau(\boldsymbol{\alpha}') - CZ_\tau(\boldsymbol{\beta}') + 2c_1(u^*\xi, \tau) \\ &= -k\left(\chi(\dot{\Sigma}) + CZ_\tau(\boldsymbol{\alpha}) - CZ_\tau(\boldsymbol{\beta}) + 2c_1(v^*\xi, \tau)\right) + b \\ &= k \operatorname{ind}(v) + b,\end{aligned}$$

as desired. □

To utilize non-negativity of the Fredholm index, we work with **filtered ECH**, defined as follows. Let $L > 0$ and consider the homology of the sub-complex $ECC^L(Y, \lambda, \Gamma, J)$ generated by orbit sets with total action less than L. The differential preserves this sub-complex by Lemma 2.4.1. As noted by Hutchings-Taubes in [17], ECH is recovered from filtered ECH by passing to a direct limit.

Theorem 2.5.4. *[6, Theorem 3.2.1] Let Y be a closed oriented 3-manifold with a non-degenerate contact form λ, and let $\{f_i\}_{i=1}^{\infty}$ be a sequence of smooth, positive functions such that $1 \geq f_1 \geq f_2 \geq \cdots$ and $f_i\lambda$ is L_i-non-degenerate for an increasing sequence of positive real numbers L_i such that $\lim_{i\to\infty} L_i = +\infty$. Then there is a canonical isomorphism*

$$ECH(Y, \lambda) = \lim_{i\to\infty} ECC^{L_i}(Y, \lambda).$$

Thus, it suffices to define cobordism maps on filtered complexes $ECC^{L_i}(Y, \lambda_i, \Gamma, J)$ where $\{L_i\}$ is increasing, $L_i \to +\infty$, and the λ_i are appropriately chosen L_i-supersimple contact forms on Y.

The second advantage of the L-supersimple setting is that, if we choose the almost complex structure J appropriately, the $\overline{\partial}$-equation is linear for curves that are close to and graphical over trivial cylinders. We begin by specifying the appropriate J for which this assertion is true, following [2].

Definition 2.5.5. Let λ be a contact form on Y. An almost complex structure J on $\mathbb{R} \times Y$ is **λ-tame** if the following three conditions hold:

1. J is $\mathbb{R}$-independent;

2. $J(\partial_s) = gR_\lambda$ for some positive function g on Y; and

3. there exists a 2-plane field ξ' on Y such that J preserves ξ', $d\lambda$ is a symplectic form on ξ', and J restricts to an orientation-preserving isomorphism on ξ'.

Definition 2.5.6. Let $L > 0$, let λ be an L-supersimple contact form, and let γ be an embedded Reeb orbit of λ. A λ-tame almost complex structure J is **L-simple** for λ if, inside the neighborhood of γ given by Theorem 2.5.1, the following conditions hold:

1. $\xi' = \mathrm{Span}\,(\partial_x, \partial_y)$;

2. $J(\partial_x) = \partial_y$; and

3. the function g in Definition 2.5.5 satisfies $gR_\lambda = \partial_t + X_H$, where X_H is the Hamiltonian vector field of the function H from Theorem 2.5.1 with respect to the symplectic form $dx \wedge dy$.

Now we can precisely state the second advantage.

Proposition 2.5.7. *Let λ be an L-supersimple contact form on Y and let J be an L-simple almost complex structure for λ. If $u\colon [R, \infty) \times S^1 \to \mathbb{R} \times Y$ is a J-holomorphic half-cylinder asymptotic to a Reeb orbit γ, and if we write $u(s,t) = (s, t, \eta(s,t))$, then the function η satisfies*

$$\partial_s \eta + j_0 \partial_t \eta + S\eta = 0,$$

where j_0 is the standard complex structure on $\mathbb{R}^2$ and

$$S = \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix}.$$

For proofs, see [2]. Since the punctured holomorphic curves considered in this thesis are asymptotic to Reeb orbits at the ends, they satisfy the conditions in Proposition 2.5.7 near each puncture, and one can define evaluation maps on the moduli spaces defined above. We will examine this advantage more closely in Chapter 3.

CHAPTER 3

The Evaluation Map

In this chapter, we recall the definition of the evaluation map in [1] and define a rescaled evaluation map that detects long necks. We also discuss results on achieving transversality for such maps.

3.1 The asymptotic operator and evaluation maps

As before, let Y be a smooth 3-manifold and $\xi = \ker\lambda$ a contact structure on Y. Let R_λ denote the Reeb vector field of λ. Let γ be a Reeb orbit and assume for simplicity that γ is embedded with period 2π. There is an **asymptotic operator**

$$A_\gamma \colon W^{1,2}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^2) \to L^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^2)$$

defined by

$$A_\gamma = -j_0 \frac{\partial}{\partial t} - S(t),$$

where j_0 is the standard complex structure on $\mathbb{R}^2 = \mathbb{C}$ and $S(t)$ is a loop of 2×2 symmetric matrices. The operator A_γ is intimately connected to the geometry of J-holomorphic curves in $\mathbb{R} \times Y$ that are asymptotic to γ at some end. In fact, if $u\colon (-\infty, -R] \times S^1 \to \mathbb{R} \times Y$ is a proper embedding that is close to and graphical over the trivial cylinder $\mathbb{R} \times \gamma$, then results of Siefring (see [18]) show that there is an integer N such that, in suitable coordinates,

$$u(s,t) = \left(s, t, \sum_{k=1}^{N} e^{\lambda_i s}(e_i(t) + r_i(s,t)) \right),$$

where the λ_i are positive eigenvalues of the asymptotic operator A, the e_i are corresponding eigenfunctions, and the r_i are remainder terms that satisfy exponential decay estimates along with all derivatives.

In the L-supersimple setting, a stronger result is true. Let $\{f_i\}_{i\neq 0}$ be an orthonormal basis for $L^2(S^1,\mathbb{R}^2)$ consisting of eigenfunctions of A, with corresponding eigenvalues λ_i, and ordered so that

$$\cdots, \lambda_{-2} \leq \lambda_{-1} < 0 < \lambda_1 \leq \lambda_2 \leq \cdots.$$

As noted at the end of Chapter 2 in Proposition 2.5.7, the $\overline{\partial}$-equation is linear for half-cylinders that are close to and graphical over $\mathbb{R}\times\gamma$. Consequently, for such curves, we have an infinite series expansion of the form

$$u(s,t) = \left(s, t, \sum_{k=1}^{\infty} c_i e^{\lambda_i s} f_i(t)\right),$$

where now $\{f_i\}_{i\neq 0}$ is an orthonormal basis for $L^2(S^1,\mathbb{R}^2)$ consisting of eigenfunctions of A_γ and the c_i are real constants.

We now recall the evaluation map defined by Bao-Honda for punctured J-holomorphic curves (see [1] for details).

Definition 3.1.1. Let u be a J-holomorphic half-cylinder,

$$u\colon (-\infty,-R]\times S^1 \to \mathbb{R}\times Y$$
$$u(s,t) = (s,t,\widetilde{u}(s,t)),$$

asymptotic to γ at the negative end and assume that

$$\widetilde{u}(s,t) = \sum_{i=1}^{\infty} c_i e^{\lambda_i s} f_i(t),$$

is the infinite series expansion discussed above, where the c_i are real constants. Define the **order k evaluation map** on such half-cylinders by

$$\mathrm{ev}_-^k(u) = (c_1,\ldots,c_k).$$

A similar map can be defined for half-cylinders with a positive end.

Definition 3.1.2. If $\boldsymbol{\gamma}_+$ and $\boldsymbol{\gamma}_-$ are two hyperbolic orbit sets and $u\in\mathcal{M}_J(\boldsymbol{\gamma}_+,\boldsymbol{\gamma}_-)$, define the **order k evaluation map** at the i^{th} negative end

$$\mathrm{ev}_{-i}^k(\boldsymbol{\gamma}_+,\boldsymbol{\gamma}_-,J)\colon \mathcal{M}_J(\boldsymbol{\gamma}_+,\boldsymbol{\gamma}_-)\to\mathbb{R}^k$$
$$u\mapsto(c_1,\ldots,c_k),$$

by identifying u with a J-holomorphic half-cylinder near the i^{th} negative end and taking the resulting series coefficients. (Recall that we label the positive and negative ends of curves in $\mathcal{M}_J(\boldsymbol{\gamma}_+, \boldsymbol{\gamma}_-)$.) Similar evaluation maps can be defined at positive ends of punctured J-holomorphic curves.

Definition 3.1.3. Let $u \in \mathcal{M}_J(\boldsymbol{\gamma}_+, \boldsymbol{\gamma}_-)$ be a curve with positive ends labeled by $1, \ldots, n$ and negative ends labeled by $-1, \ldots, -m$. Let $I_+ = \{i_1, \ldots, i_p\}$ and $I_- = \{i_{-1}, \ldots, i_{-q}\}$ be subsets of positive and negative ends, respectively, and denote the l^{th} series coefficient at end ν by $c_{\nu,l}$. We define the **order k evaluation map** at the given ends by

$$\begin{aligned} \mathbf{ev}^k_{I_+,I_-}(\boldsymbol{\gamma}_+, \boldsymbol{\gamma}_-) \colon \mathcal{M}_J(\boldsymbol{\gamma}_+, \boldsymbol{\gamma}_-) &\to \left(\mathbb{R}^k\right)^p \times \left(\mathbb{R}^k\right)^q \\ u &\mapsto \Big((c_{\nu,1}, \ldots, c_{\nu,k})\Big)_{\nu \in I_+ \cup I_-}. \end{aligned}$$

Fact 3.1.4. *The evaluation maps above are all smooth.*

Of course, we can also vary the order of the evaluation map at each end with only an increase in notational complexity. We will refer to the sum of the orders of the map at each end as the **total order** of the map. We will often abuse terminology and refer to any and all of the above maps as "the evaluation map" and denote a generic evaluation map by ev. We will also often abbreviate $\mathbf{ev}^k_{I_+,I_-}(\boldsymbol{\gamma}_+, \boldsymbol{\gamma}_-)$ as $\mathbf{ev}^k$ when the relevant ends and orbit sets are clear from context.

If the evaluation map does not vanish at any point of $\mathcal{M}_J(\boldsymbol{\gamma}_+, \boldsymbol{\gamma}_-)$, then it descends to the quotient space $\mathcal{M}_J(\boldsymbol{\gamma}_+, \boldsymbol{\gamma}_-)/\mathbb{R}$. We treat the case of one negative end; the other cases are similar. For any curve $u \in \mathcal{M}_J(\boldsymbol{\gamma}_+, \boldsymbol{\gamma}_-)$, translating u in the $\mathbb{R}$-direction a distance s_0 changes the evaluation map by

$$(c_1, \ldots, c_k) \mapsto (e^{-\lambda_1 s_0} c_1, \ldots, e^{-\lambda_k s_0} c_k).$$

For any $(c_1, \ldots, c_k) \in \mathbb{R}^k \setminus \{0\}$, the curve $c(s) = (e^{-\lambda_1 s} c_1, \ldots, e^{-\lambda_k s} c_k)$ is transverse to the unit sphere S^{k-1} in $\mathbb{R}^k$ and intersects it exactly once.

Definition 3.1.5. The **quotient evaluation map** of order k is the map

$$\widetilde{\text{ev}}^k_- \colon \mathcal{M}_J(\boldsymbol{\gamma}_+, \boldsymbol{\gamma}_-)/\mathbb{R} \to S^{k-1}$$

defined by

$$[u] \mapsto (\widetilde{c}_1, \dots, \widetilde{c}_k).$$

Here, $\mathrm{ev}_-^k(u) = (c_1, \dots, c_k)$, the curve c is defined as above, and $(\widetilde{c}_1, \dots, \widetilde{c}_k)$ is the unique point where c intersects S^{k-1}.

3.2 The rescaled evaluation map

The gluing analysis in Chapter 5 will require a modified version of the evaluation map that distinguishes curves with long necks. We begin by defining a notion of width for punctured holomorphic curves.

Definition 3.2.1. Let $u \in \mathcal{M}_J(\boldsymbol{\gamma}_+, \boldsymbol{\gamma}_-)$ and let $\gamma_i \in \boldsymbol{\gamma}_+$. We say that $R \in \mathbb{R}$ is a **positive cutoff height** for u at a positive puncture p_i asymptotic to γ_i if, in a neighborhood of p_i, the map u can be identified with a J-holomorphic half-cylinder of the form

$$\begin{aligned} \widetilde{u}\colon [R, \infty) \times S^1 &\to \mathbb{R} \times Y \\ \widetilde{u}(s,t) &= (s, t, \eta(s,t)). \end{aligned}$$

We denote the set of cutoff heights for u at p_i by $\mathcal{C}_+(u, p_i)$. **Negative cutoff heights** at negative punctures are defined similarly.

Definition 3.2.2. Let $u \in \mathcal{M}_J(\boldsymbol{\gamma}_+, \boldsymbol{\gamma}_-)$. We define the **positive cutoff height** for u at a positive puncture p_i asymptotic to $\gamma_i \in \boldsymbol{\gamma}_+$ by

$$s_+(u, p_i) = 1 + \inf \mathcal{C}_+(u, p_i).$$

Define the **total positive cutoff height** for u by

$$s_+(u) = \max\{s_+(u, p_i) \,|\, p_i \in \boldsymbol{p}_+\}.$$

The **negative cutoff height** $s_-(u, p_-)$ for u at a negative puncture p_- is defined similarly, and we set the **total negative cutoff height** for u to be

$$s_-(u) = \min\{s_-(u, p_i) \,|\, p_i \in \boldsymbol{p}_-\}.$$

Definition 3.2.3. Define the **width** of a curve $u \in \mathcal{M}_J(\boldsymbol{\gamma}_+, \boldsymbol{\gamma}_-)$ to be

$$w(u) = s_+(u) - s_-(u).$$

We are now ready to define the modified evaluation map.

Definition 3.2.4. The **rescaled evaluation map of order** k

$$\mathrm{Ev}_-^k \colon \mathcal{M}_J(\boldsymbol{\gamma}_+, \boldsymbol{\gamma}_-)/\mathbb{R} \to \mathbb{R}^k$$

at a negative end $\gamma \in \boldsymbol{\gamma}_-$ is defined as follows. Let $\widetilde{w}$ be a C^0-close smoothing of the width function from Definition 3.2.3. If $[u] \in \mathcal{M}_J(\boldsymbol{\gamma}_+, \boldsymbol{\gamma}_-)/\mathbb{R}$ is an equivalence class of curves, let $u \in \mathcal{M}_J(\boldsymbol{\gamma}_+, \boldsymbol{\gamma}_-)$ be the representative such that $s_+(u) = 0$. If $\mathrm{ev}_-^k(u) = (c_1, \ldots, c_k)$, we set

$$\mathrm{Ev}_-^k(u) = \left(e^{\widetilde{w}(u)\lambda_1} c_1, \ldots, e^{\widetilde{w}(u)\lambda_k} c_k\right).$$

Fact 3.2.5. *The rescaled evaluation map is smooth.*

3.3 Transversality for the evaluation map

We now recall some results for achieving transversality for evaluation maps on ends of curves. We begin with a mild generalization of [1, Theorem 6.0.4].

Theorem 3.3.1. *Let J be generic, let $K \subset \mathcal{M}_J^{\mathrm{ind}=k,s}(\boldsymbol{\gamma}_+, \boldsymbol{\gamma}_-)$ be compact, and let $Z \subset S^{k-1}$ be a submanifold. Then there exists a generic J', arbitrarily close to J, and a compact subset $K' \subset \mathcal{M}_{J'}^{\mathrm{ind}=k,s}(\boldsymbol{\gamma}_+, \boldsymbol{\gamma}_-)$, arbitrarily close to K, such that the evaluation map $\mathbf{ev}^k_{I_+,I_-}(\boldsymbol{\gamma}_+, \boldsymbol{\gamma}_-)$ descends to a map $\widetilde{\mathbf{ev}}^k_{I_+,I_-}(\boldsymbol{\gamma}_+, \boldsymbol{\gamma}_-)$ on K' that is transverse to Z.*

Proof. Let $u \in \mathcal{M}_J^{\mathrm{ind}=k,s}(\boldsymbol{\gamma}_+, \boldsymbol{\gamma}_-)$. All moduli spaces in our setting consist of simple curves and are therefore regular. The perturbation constructed in the proof of [1, Theorem 6.0.4] is supported over a single end, so we can repeat the construction over the relevant ends of u separately. □

Proposition 3.3.2. *Let J be generic, let $\mathcal{M}$ be a moduli space of regular punctured holomorphic curves, and let* ev *be an evaluation map on $\mathcal{M}$ of total order N. The set of $u \in \mathcal{M}$*

such that $\mathrm{ev}(u)$ *intersects a coordinate hyperplane* $\{x_i = 0\}$ *has codimension* 1 *in* $\mathcal{M}$. *In particular, if* N *is greater than* $\dim \mathcal{M}$, *then the origin is not in the image of* ev.

Proof. By Theorem 3.3.1, the quotient evaluation map $\widetilde{\mathrm{ev}}$ is transverse to the great circle $\{x_i = 0\} \cap S^{N-1}$ if J is generic. This proves the first assertion. The second follows from a dimension count. □

Finally, note that all of the perturbations in the proofs of Theorem 3.3.1 and Proposition 3.3.2 are approximately normal to the J-holomorphic curves on the ends. Thus, similar results are also true for the rescaled evaluation map.

CHAPTER 4

The ECH Index Formula

A crucial piece in our analysis of SFT breaking is a new formula for the ECH index in terms of purely combinatorial data. We take advantage of the transversality of higher-order evaluation maps and turn the fundamental ECH index inequality from [12] into an equality by adding a correction term that depends on the way a J-holomorphic curve partitions its positive and negative orbit sets. While this new index formula is, as far as we know, only valid in the L-supersimple setting, a similar inequality is implicit in Hutchings' proof of the fundamental result.

4.1 The index inequality

Let u be a somewhere injective J-holomorphic curve in $\mathbb{R}\times Y$. The **ECH index inequality** states that

$$\operatorname{ind}(u) \leq I(u) - 2\delta(u), \qquad (*)$$

where $\delta(u)$ is a non-negative count of singularities of u. There are several ingredients in the proof of $(*)$, which we review and generalize below. The proofs and notation closely follow those in [11]. We analyze only negative ends in this chapter; the analysis for positive ends is similar.

Let ξ be the braid corresponding to a negative end of u with multiplicity q at an orbit γ. Let $\eta_\tau(\xi)$ denote the winding number of ξ around γ in the trivialization τ and $w_\tau(\xi)$ the asymptotic writhe of ξ with respect to τ. The first two essential inequalities are the following:

1. $\eta_\tau(\xi) \geq \left\lceil \frac{CZ_\tau(\gamma^q)}{2} \right\rceil$;

2. $w_\tau(\xi) \geq (q-1)\eta_\tau(\xi)$.

Now suppose that u has negative ends at γ with multiplicities $q_1, \dots, q_n$ and total multiplicity $m = \sum_{i=1}^n q_i$. Let $\boldsymbol{\xi}$ denote the braid determined by these ends and $\xi_1, \dots, \xi_n$ the components of $\boldsymbol{\xi}$ determined by the individual ends. Let $\ell_\tau(\xi_i, \xi_j)$ denote the linking number of ξ_i and ξ_j with respect to τ, and set $\eta_i = \eta_\tau(\xi_i)$. Then the following three inequalities are the remaining ingredients in the proof of $(*)$:

3. $\ell_\tau(\xi_i, \xi_j) \geq \min(q_i\eta_j, q_j\eta_i)$;

4. $w_\tau(\boldsymbol{\xi}) \geq \sum_{i=1}^n \eta_i(q_i - 1) + \sum_{i\neq j} \min(q_i\eta_j, q_j\eta_i)$;

5. $\sum_{i=1}^n \eta_i(q_i - 1) + \sum_{i\neq j} \min(q_i\eta_j, q_j\eta_i) \geq \sum_{k=1}^m CZ_\tau(\gamma^k) - \sum_{i=1}^n CZ_\tau(\gamma^{q_i})$.

4.2 The writhe bound for one end

Consider inequality number 1. By Proposition 3.3.2, the inequality can be strengthened into the following equality.

Lemma 4.2.1. *Let u be a somewhere injective curve in $\mathbb{R} \times Y$ with $\operatorname{ind}(u) = 1$. Assume that the contact structure on Y is L-supersimple and that u has a negative end at an orbit γ with $\mathcal{A}(\gamma) < L$. Let η_τ and ξ be defined as above. If J is generic, then*

$$\eta_\tau(\xi) = \left\lceil \frac{CZ_\tau(\gamma^q)}{2} \right\rceil. \tag{$*$}$$

Proof. Let (s,t) be cylindrical coordinates over the relevant negative end of u and take an asymptotic expansion

$$u(s,t) = \left(s, t, \sum_{i=1}^{\infty} c_i e^{-\lambda_i s} f_i(t) \right)$$

of u for $s \ll 0$, as in Chapter 3. Since u has Fredholm index 1 and J is generic, Proposition 3.3.2 implies that $c_1 \neq 0$. Thus, $\eta_\tau(\xi)$ equals the winding number of f_1 around γ in the trivialization τ. By computations in Section 3 of [10], said winding number is precisely $\lceil \frac{CZ_\tau(\gamma^q)}{2} \rceil$. □

We can also improve inequality number 2 into an equality. The proof follows the derivation of the inequality of [11, Lemma 6.7].

Lemma 4.2.2. *Let u be as in Lemma 4.2.1 and set $\eta = \eta_\tau(\xi)$. If J is generic, then*

$$w_\tau(\xi) = \eta(q-1) + (d-1), \tag{$**$}$$

where $d = \gcd(q, \eta)$.

Proof. If $d = 1$, then the proof of [11, Lemma 6.7] shows that $w_\tau = \eta(q-1)$. So assume $d > 1$. The same proof shows that ξ is the cabling of a braid ξ_1 with q/d strands and winding number η/d by a braid ξ_2 with d strands and winding number $\eta' = \eta + 1$ for a generic L-supersimple J. First assume that γ is positive hyperbolic. Then $\eta' = \eta + 1$, and it follows that $\gcd(d, \eta') = 1$ since η and $\eta + 1$ have no common divisors other that 1. Hence,

$$w_\tau(\xi_1) = \frac{\eta}{d}\left(\frac{q}{d} - 1\right) \quad \text{and} \quad w_\tau(\xi_2) = (\eta+1)(d-1).$$

Now we compute that

$$\begin{aligned} w_\tau(\xi) &= d^2 w_\tau(\xi_1) + w_\tau(\xi_2) \\ &= d^2\frac{\eta}{d}\left(\frac{q}{d} - 1\right) + (\eta+1)(d-1) \\ &= \eta(q-1) + (d-1), \end{aligned}$$

as desired.

Now assume that γ is negative hyperbolic. If $\eta' = \eta + 1$, we are done by the computation above. So assume that $\eta' = \eta$. Then $\gcd(d, \eta') = d$ and

$$w_\tau(\xi_1) = \frac{\eta}{d}\left(\frac{q}{d} - 1\right) \quad \text{and} \quad w_\tau(\xi_2) = (\eta)(d-1) + (d-1) = (\eta+1)(d-1),$$

as in the previous case. The same computation as before then shows that

$$w_\tau(\xi) = \eta(q-1) + (d-1),$$

as desired. □

Remark 4.2.3. The equality in Lemma 4.2.2 simplifies nicely with an appropriate choice of trivialization τ. If γ is negative hyperbolic and $q = 2c+1$, choose τ so that $\mu_\tau(\gamma) = 1$. Then $\eta = c+1$ and $\gcd(q, \eta) = 1$. For if $q = am$ and $\eta = bm$, then $1 = bm - c$. Hence

$$am = 2c + 1 = 2c + (bm - c) = bm + c,$$

which implies $(a-b)m = c$. Thus m divides both c and q, say $c = lm$, so

$$am = q = 2c + 1 = 2lm + 1,$$

which implies $(a - 2l)m = 1$. This is a contradiction unless $m = 1$. Hence

$$w_\tau(\xi) = \eta(q-1).$$

If γ is negative hyperbolic and $q = 2c$, then with the same choice of τ as above, $\eta = c$ and $\gcd(q, \eta) = c$, and hence

$$w_\tau(\xi) = \frac{q}{2}(q-1) + \left(\frac{q}{2} - 1\right) = \frac{q^2}{2} - 1 = 2\eta^2 - 1 = \eta q - 1.$$

If γ is positive hyperbolic, choose τ so that $\mu_\tau(\gamma) = 0$. Then $\eta = 0$, $\gcd(q, \eta) = q$, and

$$w_\tau(\xi) = q - 1.$$

4.3 Linking numbers

Now we turn our attention to inequality number 3. By [12, Proposition 3.9], for a generic almost complex structure J on $\mathbb{R} \times Y$, any Fredholm index 1, connected, simply-covered curve u in $\mathbb{R} \times Y$ has no overlapping ends. In particular, the proof of [11, Lemma 6.9] implies the following strengthened equality.

Lemma 4.3.1. *If J is generic, then*

$$\ell_\tau(\xi_i, \xi_j) = \min(q_i \eta_j, q_j \eta_i). \qquad (\dagger)$$

Now inequality number 4 can be improved to an equality as follows.

Lemma 4.3.2. *Assume that γ is negative hyperbolic. Suppose u has negative ends of multiplicity $q_1, \dots, q_n$ at γ, and order the ends of u at γ so that $q_1, \dots, q_k$ are the ends of odd multiplicity, ordered such that $q_1 \geq q_2 \geq \dots \geq q_k$, and $q_{k+1}, \dots, q_n$ are the ends with even multiplicity. Then*

$$w_\tau(\xi) = \sum_{i=1}^{m} CZ_\tau(\gamma^i) - \sum_{i=1}^{n} CZ_\tau(\gamma^{q_i}) + \sum_{i=1}^{k} \left(\frac{q_i - 1}{2} + i - 1 \right) + \sum_{i=k+1}^{n} (\eta_i - 1).$$

Proof. By equations (∗∗) and (†), along with Remark 4.2.3, we see that

$$\begin{aligned} w_\tau(\xi) &= \sum_{i=1}^{n} w_\tau(\xi_i) + \sum_{i\neq j} \ell_\tau(q_i\eta_j, q_j\eta_i) \\ &= \sum_{i=1}^{n} \left[\eta_i(q_i - 1) + (d_i - 1) \right] + \sum_{i\neq j} \min(q_i\eta_j, q_j\eta_i) \\ &= \sum_{i=k+1}^{n} (\eta_i - 1) + \sum_{i=1}^{n} \eta_i(q_i - 1) + \sum_{i\neq j} \min(q_i\eta_j, q_j, \eta_i). \end{aligned}$$

By a computation in the proof of [12, Lemma 4.19], we have

$$\sum_{i=1}^{n} \eta_i(q_i - 1) + \sum_{i\neq j} \min(q_i\eta_j, q_j, \eta_i) = \sum_{i=1}^{m} CZ_\tau(\gamma^i) - \sum_{i=1}^{n} CZ_\tau(\gamma^{q_i}) + \sum_{i=1}^{k} \left(\frac{q_i - 1}{2} + i - 1 \right),$$

and thus

$$w_\tau(\xi) = \sum_{i=1}^{m} CZ_\tau(\gamma^i) - \sum_{i=1}^{n} CZ_\tau(\gamma^{q_i}) + \sum_{i=1}^{k} \left(\frac{q_i - 1}{2} + i - 1 \right) + \sum_{i=k+1}^{n} (\eta_i - 1),$$

as desired. □

Lemma 4.3.3. *Assume that γ is positive hyperbolic and suppose that u has negative ends of multiplicity $q_1, \dots, q_n$ at γ. Then*

$$w_\tau(\xi) = \sum_{i=1}^{m} CZ_\tau(\gamma^i) - \sum_{i=1}^{n} CZ_\tau(\gamma^{q_i}) + \sum_{i=1}^{n} (q_i - 1).$$

Proof. Using Equations (∗∗) and (†), along with Remark 4.2.3, we have

$$\begin{aligned} w_\tau(\xi) &= \sum_{i=1}^{n} w_\tau(\xi_i) + \sum_{i\neq j} \ell_\tau(q_i\eta_j, q_j\eta_i) \\ &= \sum_{i=1}^{n} \left[\eta_i(q_i - 1) + (d_i - 1) \right] + \sum_{i\neq j} \min(q_i\eta_j, q_j\eta_i) \\ &= \sum_{i=1}^{n} (q_i - 1) + \sum_{i=1}^{n} \eta_i(q_i - 1) + \sum_{i\neq j} \min(q_i\eta_j, q_j, \eta_i). \end{aligned}$$

It is easy to see that

$$\sum_{i=1}^{n} \eta_i(q_i - 1) + \sum_{i \neq j} \min(q_i\eta_j, q_j, \eta_i) = \sum_{i=1}^{m} CZ_\tau(\gamma^i) - \sum_{i=1}^{n} CZ_\tau(\gamma^{q_i})$$

(choose τ so that $CZ_\tau(\gamma) = 0$), so

$$w_\tau(\xi) = \sum_{i=1}^{m} CZ_\tau(\gamma^i) - \sum_{i=1}^{n} CZ_\tau(\gamma^{q_i}) + \sum_{i=1}^{n}(q_i - 1),$$

as desired. □

4.4 The index formula

Now we come to the improved ECH index equality. Let u be a Fredholm index 1, somewhere injective, connected J-holomorphic curve in $\mathbb{R} \times Y$. Let $\Gamma^+(u)$ be the set of embedded Reeb orbits γ such that some positive end of u has an end asymptotic to a cover of γ, and let $\Gamma^-(u)$ denote the corresponding set for the negative ends of u.

Definition 4.4.1. The **ECH deficit** $\Delta(u, \gamma)$ of u at an orbit $\gamma \in \Gamma^+(u)$ is defined as follows. If γ is negative hyperbolic, suppose u has ends at (covers of) γ of multiplicities $q_1, \ldots, q_n$, ordered so that the first k ends hove odd multiplicity and the last $n - k$ ends have even multiplicity. Then

$$\Delta(u, \gamma) = \sum_{i=1}^{k} \left(\frac{q_i - 1}{2} + i - 1\right) + \sum_{i=k+1}^{n} \left(\frac{q_i}{2} - 1\right)$$

If γ is positive hyperbolic and u has ends at (covers of) γ of multiplicities $q_1, \ldots, q_n$, then

$$\Delta(u, \gamma) = \sum_{i=1}^{n}(q_i - 1).$$

The ECH deficit of u for negative ends is defined similarly.

Definition 4.4.2. The *ECH deficit* of u is

$$\Delta(u) = \sum_{\gamma \in \Gamma^+(u)} \Delta(u, \gamma) + \sum_{\gamma \in \Gamma^-(u)} \Delta(u, \gamma).$$

Thus, $\Delta(u)$ measures how much the curve u violates the ECH partition conditions at its ends. Using this new notation, the conclusions of Lemmas 4.3.2 and 4.3.3 can be rephrased as

$$w_\tau(\xi) = \sum_{i=1}^{m} CZ_\tau(\gamma^i) - \sum_{i=1}^{n} CZ_\tau(\gamma^{q_i}) + \Delta(u,\gamma).$$

If u has a positive end at γ, computations similar to those in the above lemmas show that

$$w_\tau(\xi) = \sum_{i=1}^{m} CZ_\tau(\gamma^i) - \sum_{i=1}^{n} CZ_\tau(\gamma^{q_i}) - \Delta(u,\gamma).$$

Thus, if we set

$$w_\tau(u) = \sum_{\text{positive ends}} w_\tau(\xi) - \sum_{\text{negative ends}} w_\tau(\xi),$$

we have

$$w_\tau(u) = CZ^I_\tau(\boldsymbol{\gamma}_+,\boldsymbol{\gamma}_-) - CZ_\tau(\boldsymbol{\gamma}_+,\boldsymbol{\gamma}_-) - \Delta(u).$$

We can now state our explicit combinatorial formula for the ECH index in the L-supersimple setting. The proof is similar to that for the original index inequality, using our new formulas for the asymptotic writhe instead of Hutchings' writhe inequalities.

Theorem 4.4.3. *If J is generic and $u \in \mathcal{M}_J(\boldsymbol{\gamma}_+,\boldsymbol{\gamma}_-)$ is somewhere injective and connected and has Fredholm index 1, then*

$$I(u) = \operatorname{ind}(u) + 2\delta(u) + \Delta(u),$$

where $\delta(u)$ is a count, with positive coefficients, of singularities of u. If follows that $I(u) = \operatorname{ind}(u)$ if and only if u is embedded and satisfies the ECH partition conditions.

Proof. Using the relative adjunction formula

$$c_1(u^*\xi,\tau) = \chi(\dot\Sigma) + Q_\tau(u) + w_\tau(u) - 2\delta(u)$$

for somewhere injective curves and the above formula for the asymptotic writhe, we have

$$\begin{aligned} I(u) &= c_1(u^*\xi,\tau) + Q_\tau(u) + CZ^I_\tau(\boldsymbol{\gamma}_+,\boldsymbol{\gamma}_-) \\ &= -\chi(\dot\Sigma) + 2c_1(u^*\xi,\tau) - w_\tau(u) + 2\delta(u) + CZ^I_\tau(\boldsymbol{\gamma}_+,\boldsymbol{\gamma}_-) \\ &= -\chi(\dot\Sigma) + 2c_1(u^*\xi,\tau) + CZ_\tau(\boldsymbol{\gamma}_+) - CZ_\tau(\boldsymbol{\gamma}_-) + 2\delta(u) + \Delta(u) \\ &= \operatorname{ind}(u) + 2\delta(u) + \Delta(u), \end{aligned}$$

as desired. It is clear from the proof of Lemmas 4.3.2 and 4.3.3 that $\Delta(u) = 0$ if and only if the positive and negative orbit sets of u satisfy the partition conditions. $\square$

Remark 4.4.4. Theorem 4.4.3 proves an analogue of [14, Conjecture 3.7] in our setting, without any additional assumptions on Conley-Zehnder indices.

CHAPTER 5

Gluing Computations

In this chapter, we show that a specific class of branched covers of trivial cylinders can be glued to Fredholm index 1 curves in symplectizations. These results will be used in Chapter 6 to construct ECH cobordism maps.

5.1 The prototypical gluing problem

Let Y be a smooth 3-manifold, let $L > 0$, and let λ be an L-supersimple contact form. Let $v\colon \dot{\Sigma} \to \mathbb{R} \times Y$ be a J-holomorphic curve with Fredholm index 1 in the symplectization of Y such that

1. the positive ends of v are asymptotic to an ECH generator $\boldsymbol{\alpha}$ with total action less than L;

2. the negative ends of v are asymptotic to an orbit set $\boldsymbol{\beta}$ and satisfy the ECH partition conditions except for a collection of n negative ends with multiplicity 1 at a negative hyperbolic orbit β_0;

3. $I(v) = 1 + \frac{n(n-1)}{2}$.

Note that by Theorem 4.4.3, v must be embedded. There are two cases, based on the parity of n. We use the notation of Hutchings-Taubes in the following definition (c.f. [15]).

Definition 5.1.1. Given a Reeb orbit β, let $\mathcal{M}(a_1, \ldots, a_n \,|\, a_{-1}, \ldots, a_{-m})$ denote the moduli space of branched covers $\dot{\Sigma} \to \mathbb{R} \times S^1$ with ends labeled and asymptotically marked and such that the i^{th} end is asymptotic to an a_i-fold cover of β. Note that this is a covering space of the space of abstract branched covers of $\mathbb{R} \times S^1$.

We will be interested in gluing curves from the moduli spaces $\mathcal{M}(1,\ldots,1\,|\,2,\ldots,2,1)$, when n is odd, and $\mathcal{M}(1,\ldots,1\,|\,2,\ldots,2)$, when n is even, to curves v with the properties described above. In particular, we consider the subset of branched covers whose domains have genus $(\frac{n}{2}-1)^2$, if n is even, and $(n-1)(n-3)/4$, when n is odd. For brevity, we will denote these moduli spaces by $\mathcal{M}_n$. Note that $\dim_{\mathbb{R}} \mathcal{M}_n = n(n-1)$. The following theorem is the main result of this chapter, which we prove modulo two conjectures stated at the end of Section 5.4.

Theorem 5.1.2. *Regardless of parity, the set $Z \subset \mathcal{M}_n$ of branched covers that glue to the negative ends of v is non-empty. Moreover, if we let $\widetilde{\mathcal{M}_n}$ denote the moduli space of curves obtained by gluing curves in Z to v, and if, for $i = 1,\ldots,\lfloor \frac{n}{2} \rfloor$, we set $d_i = 1+4(i-1)$ when n is even and $d_i = 3+4(i-1)$ when n is odd, there exists an evaluation map*

$$\widetilde{\mathcal{M}_n}/\mathbb{R} \to \mathbb{R}^{n(n-1)/2}$$

with order d_i at the i^{th} multiplicity 2 negative end coming from the glued branched cover, $i = 1,\ldots,\lfloor \frac{n}{2} \rfloor$, such that the preimage of

$$\begin{aligned} P_n = {} & (\{x_{-2,1}\} \times \{(x_{-2,2}, x_{-2,3})\}) \\ & \times \cdots \times (\{x_{-(\lfloor \frac{n}{2} \rfloor -1),}\} \times \{(x_{-(\lfloor \frac{n}{2} \rfloor -1),-2}, \ldots, x_{-(\lfloor \frac{n}{2} \rfloor -1), d_{\lfloor \frac{n}{2} \rfloor -1}})\}) \\ & \times (\mathbb{R} \times \{(x_{-\lfloor \frac{n}{2} \rfloor,2}, \ldots, x_{\lfloor \frac{n}{2} \rfloor, d_{\lfloor \frac{n}{2} \rfloor}})\}) \end{aligned}$$

if n is odd and

$$\begin{aligned} P_n = {} & (\{x_{-2,1}\} \times \{(x_{-2,2}, \ldots, x_{-2,5})\}) \\ & \times \cdots \times (\{x_{-(\lfloor \frac{n}{2} \rfloor -1),}\} \times \{(x_{-(\lfloor \frac{n}{2} \rfloor -1),-2}, \ldots, x_{-(\lfloor \frac{n}{2} \rfloor -1), d_{\lfloor \frac{n}{2} \rfloor -1}})\}) \\ & \times (\mathbb{R} \times \{(x_{-\lfloor \frac{n}{2} \rfloor,2}, \ldots, x_{\lfloor \frac{n}{2} \rfloor, d_{\lfloor \frac{n}{2} \rfloor}})\}) \end{aligned}$$

if n is even, where each $x_{-j,1}$ is sufficiently small and each point (x_j) is generic and in the unit sphere of the appropriate Euclidean space, is a 1-dimensional family of curves in $\widetilde{\mathcal{M}_n}$ with a signed count of gluings equal to 1.

We will need the following definition, adapted from [15], for computations in Section 5.4.

Definition 5.1.3. If $R > 0$, let $\mathcal{M}_R \subset \mathcal{M}$ denote the subset of branched covers defined by the following condition. If $b_+(u)$ is the greatest $\mathbb{R}$-coordinate among the images of all branch points of $u \in \mathcal{M}$ and $b_-(u)$ is the least such, then $u \in \mathcal{M}_R$ if and only if $b_+(u) - b_-(u) \leq R$. The boundary of $\mathcal{M}_R$ is the set $\partial\mathcal{M}_R = \{u \,|\, b_+(u) - b_-(u) = R\}$.

5.2 Setup for obstruction bundle gluing

We recall the setup for obstruction bundle gluing from [15] and [16]. Let $\mathcal{M}$ be defined as above. Let D_u^N denote the projection of the linearized $\overline{\partial}$-operator D_u of u to the normal bundle.

Definition 5.2.1. If $R \gg 0$ is sufficiently large, the *obstruction bundle*

$$\mathcal{O} \to [R, \infty) \times (\mathcal{M}/\mathbb{R})$$

is the vector bundle with fiber

$$\mathcal{O}_{(T,u)} = \hom\left(\operatorname{coker} D_u^N, \mathbb{R}\right).$$

Proposition 5.2.2. *Let u be a branched cover of a trivial cylinder over a hyperbolic orbit β, and assume that u has k branch points, counted with multiplicity. Then* $\operatorname{ind}(u) = k$ *and* $\dim \operatorname{coker} D_u^N = k$. *In particular, the obstruction bundle $\mathcal{O} \to [R, \infty) \times (\mathcal{M}_n/\mathbb{R})$ has rank $\frac{n(n-1)}{2}$.*

Proof. The computation of $\operatorname{ind}(u)$ follows immediately from Lemma 2.5.3. From [24, Theorem 3], we know that $\dim \ker D_u^N = \dim \ker D\overline{\partial}_J - 2k = 0$. From the computation immediately preceding that theorem, we also know that $\operatorname{ind}(D_u^N) = \operatorname{ind}(u) - 2k = -k$, so $\dim \operatorname{coker} D_u^N = k$, as desired. □

Let $\mathfrak{s}$ and $\mathfrak{s}_0$ be the **obstruction section** and **linearized obstruction section** for $\mathcal{O}$ as defined in [16, Definition 5.9] and [15, Definition 3.2], respectively. Recall from [15] that the $\mathfrak{s}^{-1}(0)$ consists of exactly those branched covers in $\mathcal{M}$ that glue to v, and that $\mathfrak{s}_0^{-1}(0)$ is homologous to $\mathfrak{s}^{-1}(0)$ if the latter is compact.

Suppose that the asymptotic series for the i^{th} negative end of v at β_0 has leading asymptotic eigenfunction γ_i. Suppose also that $\sigma \in \operatorname{coker} D_u^N$ has leading asymptotic eigenfunction σ_i at the i^{th} positive end of u. Since we only glue at positive ends of a branched cover, the linearized section in this setting is

$$\mathfrak{s}_0(u)(\sigma) = \sum_{i=1}^{n} \langle \sigma_i, \gamma_i \rangle. \tag{$*$}$$

To make the gluing computations a bit easier, we will replace the elements of $\operatorname{coker} D_u^N = \ker(D_u^N)^*$ with meromorphic $(0,1)$-forms by perturbing the asymptotic operator $A_{\beta_0^k}$ for covers of β_0 in the following way. Let

$$A_{\beta_0^k,\nu} = -j_0 \frac{\partial}{\partial t} - \begin{pmatrix} \pi & (1-\nu)\epsilon \\ (1-\nu)\epsilon & \pi \end{pmatrix}$$

be a homotopy of $A_{\beta_0^k}$, where $\nu \in [0,1]$. When k is odd, the operators $A_{\beta_0^k,\nu}$ are nondegenerate throughout the homotopy. However, when k is even, the operator $A_{\beta_0^k,\nu}$ is singular at a single point $\nu_0 \in (0,1)$, for the following reason. The endpoints $A_{\beta_0^k,0}$ and $A_{\beta_0^k,1}$ are, respectively, the asymptotic operators for a negative hyperbolic orbit and an elliptic orbit with monodromy angle π. The eigenspaces for the latter operator are complex vector spaces, while the former has (real) 1-dimensional eigenspaces for the smallest positive eigenvalue λ_+ and the largest negative eigenvalue λ_-. As ν varies from 0 to 1, the largest negative eigenvalue $\lambda_{-,\nu}$ of $A_{\beta_0^k,\nu}$ changes sign from negative to positive. Thus, it vanishes for some $\nu_0 \in (0,1)$.

We correct for this degeneration in the following way. The curves considered in this thesis have ends with even multiplicity k only when $k = 2$, so we confine ourselves to that case. Starting at ν_0, we put asymptotic weights $\boldsymbol{\delta}_\nu = (\delta_\nu, \ldots, \delta_\nu)$ on $(D_u^N)^*$, where δ_ν is a smoothly varying positive real number such that $\lambda_{+,\nu} > \delta_\nu > \lambda_{-,\nu}$ for all $\nu \geq \nu_0$. When ν is very close to, but not equal to, 1, the operator $(\ker D_u^N)^*$ is approximately complex-linear and the elements of $\ker(D_u^N)^*$ are, in cylindrical coordinates (s,t) near the positive ends, approximately equal to $\sigma(s,t) \otimes (ds - idt)$, where $\sigma(s,t)$ satisfies the equation

$$(\sigma_i)_s - i(\sigma_i)_t + \pi\sigma_i = 0.$$

If we set $\eta(s,t) = e^{-\pi s}\sigma(s,t)$ over such an end, we see that η is anti-meromorphic in the usual sense. To complete the argument, we must choose a real 1-dimensional subspace of the complex 1-dimensional $\lambda_{+,1}$-eigenspace that corresponds to the original $\lambda_{+,0}$-eigenspace. We do this by examining the asymptotic behavior of a meromorphic $(0,1)$-form near the negative ends of a curve: the form should follow the stable direction of β_0, in the sense that the leading asymptotic eigenfunction in the expansion of η should always be a real scalar multiple of the vector in $\mathbb{R}^2$ representing the stable direction of β_0^2 in the coordinates given by Theorem 2.5.1.

To make the computations easier still, we can deform the linearized obstruction section through a homotopy $\mathfrak{s}_0^t$ by modifying the data $\{\gamma_1, \ldots, \gamma_n\}$ in $(*)$ so that

$$|\gamma_1| \gg |\gamma_2| \gg \cdots \gg |\gamma_n|.$$

The set of $\{\gamma_1, \ldots, \gamma_n\}$ in $\mathbb{R}^{2n}$ that are not admissible in the sense of [15, Definition 3.3] has codimension at least 2, so $\{\gamma_1, \ldots, \gamma_n\}$ will remain admissible along a generic path in $\mathbb{R}^{2n}$. In the cases we consider, one can check by hand that there are no zeros of $\mathfrak{s}_0^t$ near the boundary of the moduli space for each t, so the homology class of $(\mathfrak{s}_0^t)^{-1}(0)$ is unchanged throughout the deformation.

After making this modification, most branched covers $u \in \mathcal{M}$ will have the following property: For every element $\sigma \in \ker(D_u^N)^*$, there is one positive end at β_0 that determines the behavior of the linearized obstruction section, in the following sense.

Definition 5.2.3. Let u be a branched cover of a trivial cylinder. The i^{th} positive end is said to **dominate** the linearized obstruction section $\mathfrak{s}_0$ if

$$|\langle \sigma_i, \gamma_i \rangle| \gg |\langle \sigma_j, \gamma_j \rangle|$$

for all $j \neq i$.

5.3 The distinguished boundary

We now describe, for each n, a distinguished portion of the boundary of $\mathcal{M}_n$ that will be useful in our work below. The computations involved in cutting out a 1-dimensional family of curves will be greatly simplified by forcing the family to be close to the distinguished part of the boundary. We compactify $\mathcal{M}_n$ as in [15].

We will describe the boundary of $Z = \mathfrak{s}_0^{-1}(0) \subset \mathcal{M}_n$ in the following way. Each partition $(a_1, \ldots, a_k)$ of n has an associated ECH deficit $\Delta(a_1, \ldots, a_k)$, defined in Chapter 4. This induces an ordering on the set of partitions such that $(1, \ldots, 1)$ is the unique maximal element and $(2, \ldots, 2)$ (if n is even) or $(2, \ldots, 2, 1)$ (if n is odd) is the unique minimal element. For each i, let $\mathcal{C}_n^i$ be the set of partitions $(a_1, \ldots, a_k)$ of n such that $\Delta(a_1, \ldots, a_k) = i$. A portion of the boundary of Z can then be described by choosing at most one partition from each non-empty $\mathcal{C}_n^i$. The branch points of a curve in each such portion are arranged in clusters separated by cylindrical portions respecting the chosen partitions. Note that a given partition of n may not appear in the description of any portion of the boundary of Z.

We can visualize the boundary of Z as a collection of towers of partitions. Some examples are shown below. Each tower encodes a collection of boundary facets determined by collections of partitions, as above, by choosing at most one partition from each level of the tower. For $n = 2$, $n = 3$, and $n = 4$, the tower is a tree. For $n = 5$, we have a pair of trees, and for $n \geq 6$, there are multiple towers, not all of which are trees. The diagram for $n = 6$ below serves as an example of this last case. The numbers next to each edge in the towers below are the number of branch points in the cluster between the two cylindrical portions.

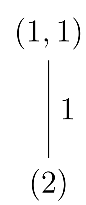

Figure 5.1: The tower for $n = 2$.

Definition 5.3.1. The **distinguished boundary** $\partial_* \mathcal{M}_n$ of $\mathcal{M}_n$ is the subset of $\partial_* \mathcal{M}_n$ determined by a certain linear tree T_n. For $n = 2$ and $n = 3$, we use the tower shown in Figure 5.1 and Figure 5.2, respectively. We define T_n for $n \geq 4$ recursively by

$$T_n = \{(a_1, \ldots, a_k, 2) \,|\, (a_1, \ldots, a_k) \in T_{n-2}\} \cup \{(1, \ldots, 1, 3), (1, \ldots, 1)\}.$$

In Figures 5.1, 5.2, 5.3, 5.4, and 5.5, the tree T_n is the leftmost tower. We note here that the rightmost 2 in each partition below $(1, \ldots, 1, 3)$ corresponds to a cylindrical portion

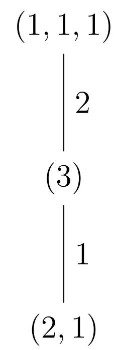

Figure 5.2: The tower for $n = 3$.

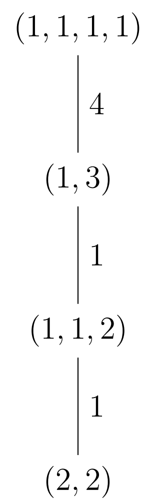

Figure 5.3: The tower for $n = 4$.

that does not interact with the rest of the branched cover below the cluster of branch points between the cylindrical portions corresponding to $(1, \ldots, 1, 3)$ and $(1, \ldots, 1, 2)$. A similar statement holds for each subsequent 2 that appears as we move down the linear tree T_n. This observation will be crucial in putting restrictions on the rescaled evaluation map, as we will see in Section 5.4.

In Section 5.4, we will see that the rescaled evaluation map from Chapter 3 can distinguish curves near the distinguished boundary $\partial_* \mathcal{M}_n$. We will also see evidence for a conjecture that the map, after postcomposing with a projection, has degree 1 in a neighborhood of a specific portion of $\partial_* \mathcal{M}_n$. For the sake of clarity, we note here that in Section 5.4, we will

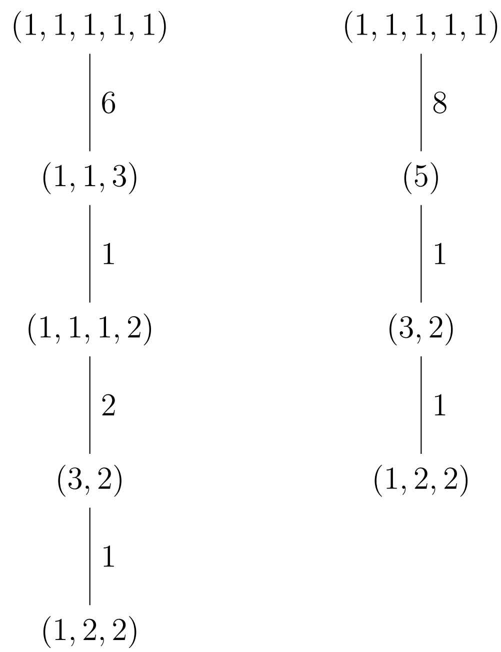

Figure 5.4: The towers for $n = 5$.

show that the zero set of $\mathfrak{s}_0$ is non-empty for each branched cover depicted in the figures above. Thus, these towers are a complete description of the boundary of the subset of curves in $\mathcal{M}_n$ mentioned in Theorem 5.1.2.

5.4 Computations and the proposed proof of Theorem 5.1.2

The goal of this section is to prove Theorem 5.1.2 modulo two conjectures stated at the end of the argument. We proceed in six steps.

Step 1. Consider the prototypical gluing problem in the case $n = 2$. In this case, our moduli space is $\mathcal{M}_2 = \mathcal{M}(1, 1 \,|\, 2)$, and it is easy to see that $\mathcal{M}/\mathbb{R} = S^1$. Thus, $\mathcal{O}$ is a real line bundle over the circle, identified with $\mathbb{R}/2\pi\mathbb{Z}$, and we need only determine if it is orientable to know the number of zeros (mod 2) of a transverse section.

We recall from [1, Section 4] some properties of elements in $\ker(D_u^N)^*$. Let τ be a trivialization of ξ along β_0. If σ is a non-zero anti-meromorphic $(0, 1)$-form representing an element of $(\ker D_u^N)^*$, then it has, over each end of u, an **asymptotic winding number**

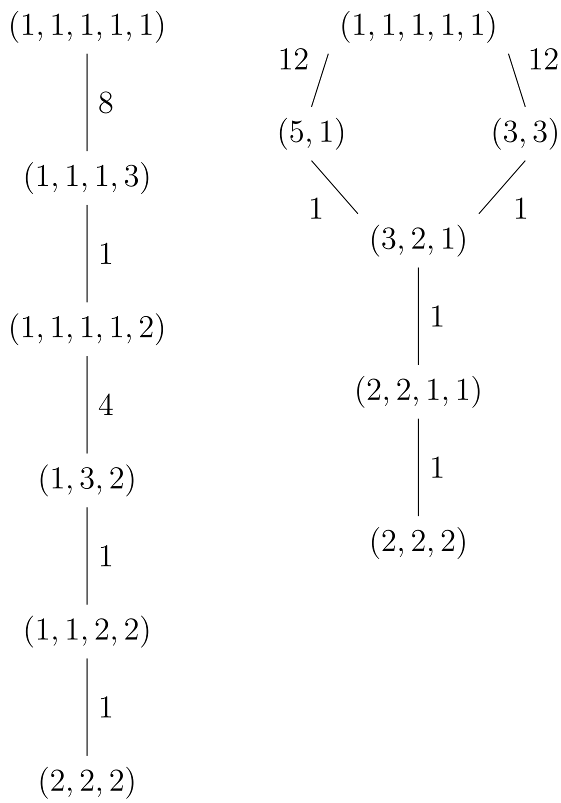

Figure 5.5: The towers for $n = 6$.

$\text{wind}_\tau(\sigma_i)$ defined as follows. As usual, label the positive ends of u by $1, \dots, n$ and the negative ends by $-1, \dots, -m$, and over each end write $\sigma = \sigma_i \otimes (ds - idt)$ in cylindrical coordinates. Then $\text{wind}_\tau(\sigma_i)$ is defined as the winding number of the leading asymptotic eigenfunction in the series expansion of σ_i. For each positive end of u, we have $2\,\text{wind}_\tau(\sigma) \geq CZ_\tau(\beta_0)$; for each negative end, we have $2\,\text{wind}_\tau(\sigma) \leq CZ_\tau(\beta_0)$.

Choose the trivialization τ of ξ over β_0 so that $CZ_\tau(\beta_0) = 1$. Since we are using anti-meromorphic forms, the zeros of σ have non-positive multiplicity. Thus, if σ is a non-zero anti-meromorphic form as above, by the proof of [15, Lemma 2.15], we have

$$\text{wind}_\tau(\sigma_1), \text{wind}_\tau(\sigma_2) \geq 1 \quad \text{and} \quad \text{wind}_\tau(\sigma_{-1}) \leq 1.$$

Thus,

$$0 \geq \sigma^{-1}(0) = \chi(\dot{\Sigma}) + \text{wind}_\tau(\sigma_1) + \text{wind}_\tau(\sigma_2) - \text{wind}_\tau(\sigma_{-1}) \geq 0$$

and σ is non-vanishing. Here, $\dot{\Sigma}$ is the domain of u, σ_i is the restriction of σ to the i^{th} end of

u, and $\operatorname{wind}_\tau(\sigma_i)$ is the winding number of σ_i around β_0 with respect to the trivialization τ.

We now write down a model for branched covers in $\mathcal{M}$ and generators for $\mathcal{O}_u$. The model is

$$u_\theta \colon \mathbb{C} \setminus \{\pm e^{i\theta}\} \to \mathbb{C} \setminus \{0\}$$
$$z \mapsto z^2 - e^{2i\theta}.$$

Here we identify $\mathbb{C} \setminus \{0\}$ with $\mathbb{R} \times S^1$ via the map $z \mapsto \log \frac{1}{z}$. Choose an asymptotic marker on β_0 and identify it with the direction determined by -1 in $S^1 = (T_0\mathbb{C} \setminus \{0\})/\mathbb{R}_+$, which is identified with the limiting orbit β_0. Now consider the union of curves $u_\theta^{-1}((-\epsilon, 0))$, where $0 < \epsilon \ll 1$. There are two components of the pre-image. Each limits to one of the two positive punctures at $z = \pm e^{i\theta}$, and each determines a direction τ at one of the punctures, which we use as the asymptotic marker of the domain at those ends. The two components of the preimage $u_\theta^{-1}((-\infty, -\frac{1}{\epsilon}))$ both determine a direction τ at the negative puncture. A continuous choice of component thus determines a choice of asymptotic marker of the domain at the negative end. Note that the marker at the negative end is independent of θ.

Since we use anti-meromorphic forms as replacements for elements of $(\ker D_u^N)^*$ and since such a form must follow the stable direction at the multiplicity 2 negative end of u_θ, a spanning vector for $\mathcal{O}_{u_\theta}$ is given by $\sigma = c d\bar{z}$, where $c \in \mathbb{C} \setminus \{0\}$ is independent of θ, interpreted as a real section of a real vector bundle. The form is independent of θ because the marker at the negative end in the domain is independent of θ. We can set $c = 1$ by fixing the stable direction at the marker in the coordinate system from Theorem 2.5.1. Thus, we may assume that

$$\sigma = d\bar{z},$$

interpreted a a real section of a real vector bundle.

Rotating the branch point through an angle 2π in the positive Reeb direction—clockwise in $\mathbb{C}\setminus\{0\}$—rotates the markers at the positive ends through an angle π in $(T_{\pm e^{i\theta}}\mathbb{C}\setminus\{0\})/\mathbb{R}_+$, which is in the negative Reeb direction. If we write the σ_θ in cylindrical coordinates $z = e^{i\theta} + e^{-s-it}$ near $z = e^{i\theta}$, we see that

$$\sigma_\theta = -e^{-s+i(t-\theta/2)} \otimes (ds - i\,dt).$$

As θ varies from 0 to 2π, σ_θ rotates through an angle π in the positive Reeb direction. Thus, the bundle $\mathcal{O}$ is non-orientable and $\mathfrak{s}_0$ has a single zero (mod 2). From our above computation, it is clear that $\mathfrak{s}_0$ has exactly one zero, but we will not need this more precise result.

Step 2. We now do a preliminary calculation for the case $n = 3$. Consider the gluing problem for branched covers in $\mathcal{M} = \mathcal{M}(3\,|\,2,1)$. The moduli space $\mathcal{M}/\mathbb{R}$ is again a circle. Due to the multiplicities of the positive end, as explained in Definition 5.1.1, $\mathcal{M}$ is identified with $\mathbb{R}/6\pi\mathbb{Z}$, and the bundle $\mathcal{O}$ is again a real line bundle. Label the positive end by 1, the multiplicity 1 negative end by -1, and the multiplicity 2 negative end by -2. As before, choose a trivialization τ of ξ over β_0 so that $CZ_\tau(\beta_0) = 1$. If σ is a non-zero anti-meromorphic form representing an element of $\ker(D_u^N)^*$, we see that

$$0 \geq \#\sigma^{-1}(0) = -\chi(\dot{\Sigma}) + \mathrm{wind}_\tau(\sigma_1) - \mathrm{wind}_\tau(\sigma_{-1}) - \mathrm{wind}_\tau(\sigma_{-2}) \geq 0,$$

so non-zero elements of $\ker(D_u^N)^*$ are non-vanishing.

The model we use for branched covers in $\mathcal{M}$ is

$$u_\theta \colon \mathbb{C} \setminus \{0, e^{i\theta}\} \to \mathbb{C} \setminus \{0\}$$
$$z \mapsto \frac{z^3}{z - e^{i\theta}}.$$

A spanning vector for $\mathcal{O}_{u_\theta}$ is given by

$$\sigma_\theta = \frac{\bar{z}}{\bar{z} - e^{-i\theta}} d\bar{z},$$

again interpreted as a section of a real vector bundle.

Each of the components of $u_\theta^{-1}((-\epsilon, 0))$ (where $0 < \epsilon \ll 1$) determines a marker at the positive end in the domain. Making a choice of component that is continuous in θ determines a continuous choice of direction τ at that end. Rotating the branch point through an angle θ in the positive Reeb direction causes the marker τ to rotate through an angle $-\frac{\theta}{6}$ in the model, which is equivalent to an angle of $\frac{\theta}{2}$ in the positive Reeb direction on the three-fold cover β_0^3.

If we write the cokernel element σ_0 in cylindrical coordinates $z = e^{-\frac{s}{3} - i\frac{t}{3}}$ near $z = 0$, we see that

$$\begin{aligned}\sigma_0 &= -\frac{e^{-\frac{2s}{3}+i\frac{2t}{3}}}{e^{-\frac{s}{3}+i\frac{t}{3}} - 1} \otimes (ds - i\,dt) \\ &\approx e^{-\frac{2s}{3}+i\frac{2t}{3}} \otimes (ds - i\,dt).\end{aligned}$$

As we rotate the branch point through an angle θ, σ_0 changes to

$$\begin{aligned}\sigma_\theta &= -\frac{e^{-\frac{2s}{3}+i\frac{2}{3}\left(t-\frac{\theta}{2}\right)}}{e^{-\frac{s}{3}+i\frac{1}{3}\left(t-\frac{\theta}{2}\right)} - e^{-i\frac{\theta}{2}}} \otimes (ds - i\,dt) \\ &\approx e^{-\frac{2s}{3}+i\left(\frac{2t}{3}+\frac{\theta}{6}\right)} \otimes (ds - i\,dt).\end{aligned}$$

Thus, when we rotate the branch point through an angle of 6π in the positive Reeb direction, the cokernel element σ again rotates through an angle of $-\pi$. It follows that $\mathcal{O}$ is non-orientable in this case as well and that $\mathfrak{s}_0$ has exactly one zero.

Step 3. We now consider the gluing problem for branched covers in $\mathcal{M} = \mathcal{M}(1,1,1\,|\,3)$. In this case, the moduli space $\mathcal{M}/\mathbb{R}$ is $\mathcal{Q} \times S^1$, where $\mathcal{Q}$ is a pair of pants homeomorphic to $\mathbb{C} \setminus \{\pm 1\}$, as illustrated below. Moving along the S^1 factor rotates both branch points in the positive Reeb direction, while moving around one of the "legs" of the pants rotates the top branch point in the positive Reeb direction and the bottom branch point in the negative Reeb direction.

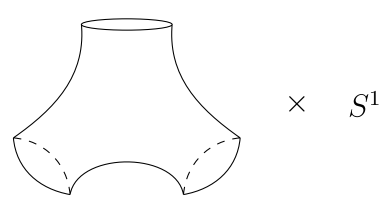

Figure 5.6: The moduli space $\mathcal{M}/\mathbb{R}$ when $n = 3$.

We claim that the obstruction bundle $\mathcal{O}$ is trivial in this example. To see this, first note that

$$0 \geq \#\sigma^{-1}(0) = \chi(\dot{\Sigma}) + \sum_{i=1}^{3} \mathrm{wind}_\tau(\sigma_i) - \mathrm{wind}_\tau(\sigma_{-1}) \geq 0,$$

so any $\sigma \in \ker(D_u^N)^*$ is non-vanishing and determined by its leading asymptotic eigenfunctions at any end. Now let W_i denote the eigenspace of the smallest positive eigenvalue of the asymptotic operator A_{β_0} at the i^{th} positive end. The above discussion shows that the bundle $\mathcal{O}$ is isomorphic to the bundle $W_i \to \mathcal{M}$, which is trivial for any $i \in \{1, 2, 3\}$.

The models we use for branched covers in $\mathcal{M}$ and for anti-meromorphic forms representing elements of $\ker(D_u^N)^*$ are both similar to those in Step 1. For the branched covers, the model is

$$\begin{aligned} u_{\rho,\theta,\phi}\colon \mathbb{C} \setminus \{0, \rho e^{i\theta} \pm e^{i\phi}\} &\to \mathbb{C} \setminus \{0\} \\ z &\mapsto z(z^2 - (\rho^2 e^{2i\theta} - e^{2i\phi})). \end{aligned}$$

We identify $\mathbb{C} \setminus \{0\}$ with $\mathbb{R} \times S^1$ as in Step 1. Since elements of $\ker(D_u^N)^*$ are non-vanishing and have winding number 1 at each end, a global frame for $\mathcal{O}$, as a real vector bundle, is given by $\{d\bar{z}, id\bar{z}\}$. Thus, the bundle $\mathcal{O}$ is complex and spanned over $\mathbb{C}$ by $d\bar{z}$.

Trivialize $\mathcal{O}$ via the bundle W_1 and let $T_{2,1}$ and $T_{3,1}$ denote the isomorphisms $W_1 \to W_2$ and $W_1 \to W_3$, respectively. Then, for any $\sigma \in \ker(D_u^N)^*$, we have

$$\begin{aligned} \mathfrak{s}_0(u)(\sigma) &= \langle \sigma_1, \gamma_1 \rangle + \langle \sigma_2, \gamma_2 \rangle + \langle \sigma_3, \gamma_3 \rangle \\ &= \langle \sigma_1, \gamma_1 \rangle + \langle T_{2,1}(\sigma_1), \gamma_2 \rangle + \langle T_{3,1}(\sigma_1), \gamma_3 \rangle \\ &= \langle \sigma_1, \gamma_1 + T_{2,1}^{\mathrm{t}}(\gamma_2) + T_{3,1}^{\mathrm{t}}(\gamma_3) \rangle, \end{aligned}$$

where t denotes the transpose of a linear map. Thus, we have a representation for $\mathfrak{s}_0$ in our trivialization of the form

$$\begin{aligned} \mathfrak{s}_0\colon \mathcal{M} &\to \mathbb{R}^2 \\ \mathfrak{s}_0(u) &= \gamma_1 + T_{2,1}^{\mathrm{t}}(\gamma_2) + T_{3,1}^{\mathrm{t}}(\gamma_3). \end{aligned}$$

Our strategy to compute $\mathfrak{s}_0^{-1}(0)$ is the following. Let $R \gg 0$, choose a generic unit vector $e \in \mathbb{R}^2$, and consider

$$\widetilde{Z} = \mathfrak{s}_0^{-1}\left(\{te \,|\, t \in [0, R]\}\right).$$

The set $\widetilde{Z}$ is a cobordism from $\mathfrak{s}_0^{-1}(0)$ to a set $Z_R \subset \partial\mathcal{M}_R$ that we will compute.

The boundary $\partial\mathcal{M}_R$ has three components, C_1, C_2 and C_3, which are described as follows. If we view branched covers of trivial cylinders as trees, a curve is in C_i if and only if the i^{th} positive leaf of the tree is adjacent to the lower of the two internal nodes of the tree.

On C_2, the map $T^{\mathrm{t}}_{2,1}$ has very small norm compared to $|\gamma_1|$ and $T^{\mathrm{t}}_{3,1}$ is approximately $-I$, so $\mathfrak{s}_0$ is approximately constant. On C_3, the map $T^{\mathrm{t}}_{3,1}$ has small norm compared to $|\gamma_1|$ and $T^{\mathrm{t}}_{2,1}$ is approximately $-I$, so $\mathfrak{s}_0$ is approximately constant there as well. Finally, on C_1, the maps $T^{\mathrm{t}}_{2,1}$ and $T^{\mathrm{t}}_{3,1}$ both have very large norm compared to $|\gamma_1|$ and $(T^{\mathrm{t}}_{2,1})^{-1} \circ T^{\mathrm{t}}_{2,1}$ is approximately $-I$. Thus, for any $\sigma \in \ker(D^N_u)^*$, the term $\langle \sigma_1, T^{\mathrm{t}}_{2,1}(\gamma_2)\rangle$ will dominate in $\mathfrak{s}_0(u)(\sigma)$.

It remains to calculate the behavior of $T^{\mathrm{t}}_{2,1}$ on C_1 as we rotate the branch points of our branched cover. The component C_1 is a smooth 2-torus and can be described as the quotient of $\mathbb{R}^2$ by the lattice

$$\Lambda = \mathrm{Span}_{\mathbb{Z}}\{(3,3),(1,3)\}.$$

Define a group homomorphism

$$\phi\colon H_1(C_i) \to \mathbb{Z}^3$$

by setting the j^{th} component of $\phi(u)$ to be the rotation number of the marker τ at the j^{th} positive end over the curve u in C_i. If we take $u_1 = (3,3)$ and $u_2 = (1,3)$ as a basis for $H_1(C_i)$, then ϕ can be represented by the matrix

$$\begin{pmatrix} -2 & -2 \\ -2 & -1 \\ -2 & -1 \end{pmatrix}.$$

Now consider the induced morphism

$$\widetilde{\phi}\colon H_1(C_i) \to \mathbb{Z}^2$$

sending a homology class to the relative rotations of the markers at ends 2 and 3 with respect to end 1. It is easy to see that, in the basis $\{u_1, u_2\}$, the map $\widetilde{\phi}$ is represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Thus, $T_{2,1}$ is constant along the curve u_1 and traces out a generator of $\pi_1(SO(2))$ along the curve u_2. Thus, the fiber of $\mathfrak{s}_0$ on C_1 is the S^1-factor u_1, while u_2 maps with degree 1 to a circle in $\mathbb{R}^2$. It follows that $[Z] = [Z_R] = [S^1]$.

Step 4. If we glue the branched covers from Step 3 to an appropriate Fredholm index 1 curve, the resulting moduli space $\widetilde{\mathcal{M}}(1,1,1\,|\,3)$ is regular and has dimension 3. We will now analyze the rescaled evaluation map on $\widetilde{\mathcal{M}}(1,1,1\,|\,3)/\mathbb{R}$ and show how to cut out a 1-dimensional family of curves. This analysis will be used in the proposed proof of Theorem 5.1.2 in the case $n = 3$.

The end of $\widetilde{\mathcal{M}}(1,1,1\,|\,3)/\mathbb{R}$ where curves are close to breaking into a Fredholm index 1 curve and a branched cover in $\mathcal{M}(1,1,1\,|\,3)$ is homeomorphic to $[R,\infty) \times Z$, where Z is the zero set computed in Step 3. Let $\mathrm{Ev}^2_- \colon \widetilde{\mathcal{M}}(1,1,1\,|\,3) \to \mathbb{R}^2$ be the order 2 rescaled evaluation map at the multiplicity 3 negative end resulting from gluing in the branched cover, and consider the restriction of Ev^2_- to the 1-dimensional slice $\{R+1\} \times Z$. We show that the restriction has winding number 1 around the origin in $\mathbb{R}^2$. A 1-dimensional family of curves is then determined by intersecting Ev^2_- with a generic ray emanating from the origin in $\mathbb{R}^2$.

First, recall that Z is homologous to S^1 and that ev^2_- is non-vanishing on $\{R+1\} \times Z$ by Proposition 3.3.2. It follows that $\mathrm{Ev}^2_- |_{\{R+1\}\times Z}$ is also non-vanishing. To compute the winding number, we consider a model for the part of the glued curve close to the branched cover. Since the curve is approximately cylindrical there, it can be represented by a section of the pullback of the normal bundle of the trivial cylinder over β_0. Recall that we perturbed the asymptotic operator $A_{\beta_0^3}$ to make it complex-linear and that we can, modulo multiplying by $e^{-\pi s}$ near the ends, assume that the section is holomorphic. Recall also that we are using asymptotic weights on the negative end, so that such a section will be unbounded near the negative end. Thus, by Proposition 3.3.2, the lower portion of a generic curve near the end of $\widetilde{\mathcal{M}}(1,1,1\,|\,3)/\mathbb{R}$ can be represented by

$$g_\theta \colon \mathbb{C} \setminus \{0, (\rho \pm e^{i\phi_0})e^{i\theta}\} \to \mathbb{C} \setminus \{0\}$$
$$z \mapsto e^{i\theta}(z - \rho e^{-i\theta})(z - \rho^2 e^{-i\theta}),$$

where ϕ_0 and $\rho \gg 1$ are fixed. The error in representing the lower portion of the glued curve by g_θ decays exponentially, so the behavior at the negative end is approximately the same. If we choose an asymptotic marker at the negative end as before, it is easy to see that, in the model, it is stationary as θ varies from 0 to 2π. Thus, the evaluation map is approximated by

$$g_\theta(-R) \approx R^2 e^{i\theta},$$

which has winding number 1. [Recall that $\mathbb{C} \setminus \{0\}$ is identified with $\mathbb{R} \times S^1$ via $z \mapsto \log \frac{1}{z}$.]

Step 5. Now we treat the prototypical gluing problem in the case $n = 3$. The moduli space is $\mathcal{M}_3 = \mathcal{M}(1,1,1\,|\,1,2)/\mathbb{R}$, which has dimension 5, and the bundle $\mathcal{O}$ has real rank 3. The zero set Z of $\mathfrak{s}_0$ is non-compact in this case, and the end of each component is diffeomorphic to $\mathbb{R} \times S^1$. [We conjecture that Z is a disk, but we will not need this precise result.] We will shortly restrict ourselves to curves that have two clusters of branch points: the top cluster is close to a curve in $\mathcal{M}(1,1,1\,|\,3)$, while the bottom is close to a curve in $\mathcal{M}(3\,|\,1,2)$. Moving in the positive $\mathbb{R}$-direction at an end of Z corresponds to separating the images of the two branch point clusters in the $\mathbb{R}$-direction in $\mathbb{R} \times Y$, while moving in the S^1-direction corresponds to rotating all three branch points simultaneously in the positive Reeb direction.

The glued curves live in a 4-dimensional moduli space $\widetilde{\mathcal{M}_3}$. The end of $\widetilde{\mathcal{M}_3}$ where curves are close to breaking into v on the top level and a branched cover in $\mathcal{M}_3$ on the bottom level is diffeomorphic to $[R, \infty) \times Z$. We will use the rescaled evaluation map to cut out a point on Z. Consider the order 3 map $\mathrm{Ev}^3_- : \widetilde{\mathcal{M}} \to \mathbb{R}^3$ at the multiplicity 2 end of the curve that results from gluing the branched cover. We claim that the conclusions of Conjectures 5.4.2 and 5.4.3 hold. To see this, write $\mathrm{Ev}^3_-(u) = (c_1(u), c_2(u), c_3(u))$. Fix $R_0 > R$ and consider the restriction of Ev^3_- to the slice $\{R_0\} \times Z$ of $\widetilde{\mathcal{M}_3}$. Let $\lambda_1 < \lambda_2$ be the smallest positive eigenfunctions of $A_{\beta_0^2}$, and let $\widetilde{\lambda}_1$ be the smallest positive eigenfunction of $A_{\beta_0^3}$. (Note that these are the unperturbed asymptotic operators.) There exists $N \gg 0$, depending on R_0, such that the restrictions

$$|c_1| < e^{-(\lambda_2 - \lambda_1)N}, \quad |(c_2, c_3)| > e^{\lambda_2 N},$$

and

$$\frac{\log |c_1|}{\log |(c_2, c_3)|} \approx \frac{\lambda_1 - \widetilde{\lambda}_1}{\lambda_2 - \widetilde{\lambda}_1},$$

force u to lie in $\{R_0\} \times (R_0, \infty) \times S^1$, where we write the end of the zero set Z as $\mathbb{R} \times S^1$ as described above. To see this, first note that if Ev^3_- is restricted to a compact subset of $\{R_0\} \times Z$, then it is bounded. Thus, since we require $|(c_2, c_3)|$ to be very large, the curve u must have a long neck different from the one involved in the gluing. Next, with any configuration of branch points other than the one described at the beginning of this step, one can show that

$$\frac{\log |c_1|}{\log |(c_2, c_3)|} \approx 0,$$

so the long neck must be at the partition (3). Finally, it is easy to see that, as in Step 4, the composition of $\mathrm{Ev}^3_- |_{\{R_0\}\times\{R_0\}\times S^1}$ with the projection $(c_1, c_2, c_3) \mapsto (c_2, c_3)$ has winding number 1 around the origin in $\mathbb{R}^2$. By Theorem 3.3.1, we can perturb J so that Ev^3_- is transverse to a line $\mathbb{R} \times \{(x_2, x_3)\}$, where $(x_2, x_3) \in \mathbb{R}^2$ is fixed and $|(x_2, x_3)|$ is sufficiently large. The preimage of this line is our desired 1-dimensional family of curves in $\widetilde{\mathcal{M}_3}$.

Step 6. We will now sketch a proposed inductive proof of Theorem 5.1.2. The cases $n = 2$ and $n = 3$ are addressed above. For convenience, let

$$\cdots < \lambda_{-3} = \lambda_{-2} < \lambda_{-1} < 0 < \lambda_1 < \lambda_2 = \lambda_3 < \cdots$$

be the eigenvalues of $A_{\beta_0^2}$ (note that this is the original asymptotic operator, not the perturbed one).

Proposition 5.4.1. *For each n, the zero set $Z_n = \mathfrak{s}_0^{-1}(0) \subset \mathcal{M}(1, \ldots, 1 \,|\, 1, \ldots, 1, 3)$ is non-empty. Here, n is the number of positive ends in the branched covers.*

Proof. We proceed by induction. Our computations above show that Z_2 and Z_3 are non-empty. If $n \geq 4$, note that, by Theorem 4.4.3, adjoining a trivial cylinder to an element of Z_{n-1} and gluing the resulting curve to v results in a curve with a single node. Therefore, there is a copy of Z_{n-1} inside Z_n in $\mathcal{N}_n$, and resolving the node gives a neighborhood of Z_{n-1} in Z_n. □

By Proposition 5.4.1, the moduli space $\mathcal{M}(1, \ldots, 1 \,|\, 1, \ldots, 1, 3)$ has non-empty zero set. By Step 2, the moduli space $\mathcal{M}(1, \ldots, 1, 3 \,|\, 1, \ldots, 1, 2)$ has non-empty zero set. By the inductive hypothesis, the moduli space $\mathcal{M}(1, \ldots, 1, 2 \,|\, 2, \ldots, 2)$ has non-empty zero set. Since we can glue curves in these three moduli spaces in succession, starting with $\mathcal{M}(1,\dots,1\,|\,1,\dots,1,3)$ and ending with $\mathcal{M}(1,\dots,1,2\,|\,2,\dots,2)$, it follows that the zero set in $\mathcal{M}_n$ is non-empty. From now on, we will denote by $\widetilde{\mathcal{M}_n}$ the moduli space obtained by gluing curves in $\mathcal{M}_n$ to v.

Base case for even n. Consider a rescaled evaluation map of total order $\frac{n(n-1)}{2}$ on $\widetilde{\mathcal{M}_n}$, with orders at the negative ends as in the statement of Theorem 5.1.2. We do not need the evaluation map in the base case $n = 2$, but we say the map has order 1 at the multiplicity 2 negative end coming from the gluing for the sake of consistency.

Consider the rescaled evaluation map from Theorem 5.1.2 in the case $n = 4$. Label the negative ends described in the statement of Theorem 5.1.2 by -1 and -2. The evaluation map has order 1 at one of the multiplicity 2 negative ends and order 5 at the other. We give restrictions on the evaluation map, after doing the same further rescaling and using the same notation as above, that will help cut out the 1-dimensional subset of $\widetilde{\mathcal{M}_4}$ that we need.

Let $(c_{i,1},\dots,c_{i,d_i})$ be the evaluation map on the i^{th} end. Write the end of the moduli space $\widetilde{\mathcal{M}_4}$ as $[R,\infty)\times Z$, where Z is the set of branched covers in $\mathcal{M}_4$ that glue to v. Fix $R_0 > R$ and let U be a small neighborhood of the largest codimension corner stratum in the distinguished boundary of $\mathcal{M}_4$. By this we mean that branched covers in U are close to breaking into a 3-level branched cover in the compactification of $\mathcal{M}_4$; we refer the reader to Definition 5.3.1 and the diagram for T_4. We will analyze the rescaled evaluation map restricted to the slice $\{R_0\}\times Z$. Since the neck length of the gluing is now fixed, we postcompose the map with a particular linear map L to compensate for said neck length. To define L, first define $d_i \times d_i$ matrices

$$L_i = \begin{pmatrix} e^{(\lambda_2-\lambda_1)R_0} & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & e^{(\lambda_2-\lambda_3)R_0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{(\lambda_2-\lambda_{d_i})R_0} \end{pmatrix}$$

for each $i = 1, \ldots, \lfloor \frac{n}{2} \rfloor$, then set

$$L = \begin{pmatrix} L_1 & 0 & 0 & \cdots & 0 \\ 0 & L_2 & 0 & \cdots & 0 \\ 0 & 0 & L_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & L_{\lfloor \frac{n}{2} \rfloor} \end{pmatrix}.$$

We will abuse notation and refer to the coefficients of the composition with L using the same notation $c_{i,1}, \ldots, c_{i,d_i}$. For convenience, let

$$\cdots < \widetilde{\lambda}_{-2} = \widetilde{\lambda}_{-1} < 0 < \widetilde{\lambda}_1 = \widetilde{\lambda}_2 < \cdots$$

be the eigenvalues of $A_{\beta_0^3}$ (note that this is the unperturbed asymptotic operator). The restrictions are given by the equations

$$\frac{|c_{-2,1}|}{|c_{-1,1}|} > e^{(\lambda_2 - \lambda_1)N}, \quad \text{and} \quad |(c_{-2,2}, c_{-2,3})| > e^{(\lambda_2 - \widetilde{\lambda})N} \tag{5.1}$$

where $\widetilde{\lambda}$ is the smallest positive eigenvalue of $A_{\beta_0^3}$ and $N \gg 0$ is a constant. For consistency with the inductive step, we also require that

$$\frac{\log |c_{-2,1}|}{\log |(c_{-2,2}, c_{-2,3})|} \approx \frac{\lambda_1 - \widetilde{\lambda}_1}{\lambda_2 - \widetilde{\lambda}_1}. \tag{5.2}$$

Base case for odd n. Consider the rescaled evaluation map from Theorem 5.1.2 in the case $n = 3$. Thus, the evaluation map has order 3 at the multiplicity 2 negative end. For completeness, we restate the restrictions given in Step 5. Write the end of the moduli space $\widetilde{\mathcal{M}_3}$ as $[R, \infty) \times Z$, where Z is the set of branched covers in $\mathcal{M}_3$ that glue to v. Fix $R_0 > R$ and let U be a small neighborhood of the largest codimension corner stratum in the distinguished boundary of $\mathcal{M}_3$. We give restrictions on the evaluation map, after doing the same further rescaling from the case $n = 4$ and using the same notation as above, that will help cut out the 1-dimensional subset of $\widetilde{\mathcal{M}_3}$ that we need. These restrictions are given by the equations

$$|c_{-2,1}| < e^{-(\lambda_2 - \lambda_1)N} \quad \text{and} \quad |(c_{-2,2}, c_{-2,3})| > e^{\lambda_2 N}, \tag{5.3}$$

where $N \gg 0$ is again a constant. For consistency with the inductive step, we also require that

$$\frac{\log|c_{-2,1}|}{\log|(c_{-2,2}, c_{-2,3})|} \approx \frac{\lambda_1 - \widetilde{\lambda}_1}{\lambda_2 - \widetilde{\lambda}_1}. \tag{5.4}$$

Inductive step. Label the negative ends of the curve by $-1, \ldots, -\lfloor \frac{n}{2} \rfloor$, where end -1 has multiplicity 1 if n is odd, and, as before, assume that the ends $-1, \ldots, -\lfloor \frac{n}{2} \rfloor + 1$ are part of the portion of the curve in $\mathcal{M}_{n-2}$. Take the orders of the evaluation map at the multiplicity 2 negative ends to be as in the statement of Theorem 5.1.2.

If $n \geq 5$, we impose the restrictions on the evaluation map from the case $n-2$ at the ends $-1, \ldots, -\lfloor \frac{n}{2} \rfloor$, except we replace the restriction

$$\frac{\log|c_{-\lfloor \frac{n}{2} \rfloor+1,1}|}{\log|(c_{-\lfloor \frac{n}{2} \rfloor+1,2}, c_{-\lfloor \frac{n}{2} \rfloor+1,3})|} \approx \frac{\lambda_1 - \widetilde{\lambda}_1}{\lambda_2 - \widetilde{\lambda}_1} \tag{5.5}$$

with

$$\log|c_{-\lfloor \frac{n}{2} \rfloor+1,1}| - \frac{\lambda_1 - \widetilde{\lambda}}{\lambda_2 - \widetilde{\lambda}} \log|(c_{-\lfloor \frac{n}{2} \rfloor+1,2}, c_{-\lfloor \frac{n}{2} \rfloor+1,3})| - \frac{\lambda_1 - \lambda_2}{\lambda_1 - \widetilde{\lambda}} \log|c_{-\lfloor \frac{n}{2} \rfloor,1}| > (\lambda_1 - \lambda_2)N. \tag{5.6}$$

We also require that

$$\frac{|c_{-\lfloor \frac{n}{2} \rfloor,1}|}{|(c_{-1,1}, \ldots, c_{-\lfloor \frac{n}{2} \rfloor+1, d_{-\lfloor \frac{n}{2} \rfloor+1}})|} > e^{(\lambda_2 - \lambda_1)N}, \quad \text{and} \quad |(c_{-\lfloor \frac{n}{2} \rfloor,2}, c_{-\lfloor \frac{n}{2} \rfloor,3})| > e^{(\lambda_2 - \widetilde{\lambda})N}, \tag{5.7}$$

as well as

$$\frac{\log|c_{-\lfloor \frac{n}{2} \rfloor,1}|}{\log|(c_{-\lfloor \frac{n}{2} \rfloor,j}, c_{-\lfloor \frac{n}{2} \rfloor,j+1})|} \approx \frac{\lambda_1 - \widetilde{\lambda}_1}{\lambda_{1+j/2} - \widetilde{\lambda}_{j/2}}, \tag{5.8}$$

where $j \in \{2, \ldots, d_{\lfloor \frac{n}{2} \rfloor - 1}\}$. Finally, we state our conjectures on the behavior of the rescaled evaluation map.

Conjecture 5.4.2. *Let U be a small neighborhood of the largest codimension corner stratum in the distinguished boundary of $\mathcal{M}_n$ and fix $R_0 > R$. There exists a constant $N \gg 0$, depending only on U, such that a curve $u \in \{R_0\} \times Z$ with all $c_{i,j}$ at least a small positive distance away from 0 lies in U if and only if the further rescaling of the evaluation map satisfies the restrictions in Equations 5.1 and 5.2, if n is even; 5.3 and 5.4, if n is odd; as well as 5.5, 5.6, 5.7, and 5.8.*

Conjecture 5.4.3. *For $n \geq 3$, let $\mathcal{E}_n$ denote the moduli space obtained by gluing curves in $\mathcal{M}(1,\ldots,1\,|\,1,\ldots,1,2)$ to v. Here n is the number of positive ends in the branched covers. Consider the rescaled evaluation map $\mathrm{Ev}_-^{d_{\lfloor \frac{n}{2} \rfloor}}$ on the multiplicity 2 negative end of curves in $\mathcal{E}_n$ that results from gluing branched covers in $\mathcal{M}(1,\ldots,1\,|\,1,\ldots,1,2)$. Writing the end of $\mathcal{E}_n$ as $[R,\infty)\times Z$, where $Z = \mathfrak{s}_0^{-1}(0) \subset \mathcal{M}(1,\ldots,1\,|\,1,\ldots,1,2)$, consider the restriction of $\mathrm{Ev}_-^{d_{\lfloor \frac{n}{2} \rfloor}}$ to the slice $\{R_0\}\times Z$ for a fixed $R_0 > R$. If we write $\mathrm{Ev}_-^{d_{\lfloor \frac{n}{2} \rfloor}} = (c_1,\ldots,c_{d_{\lfloor \frac{n}{2} \rfloor}})$ on that end, project away the first coordinate, and then project the result onto the unit sphere in $\mathbb{R}^{d_{\lfloor \frac{n}{2} \rfloor}}$, the resulting map has degree 1.*

Now, $\mathrm{Ev}^{-1}(P_n)$ is a 1-dimensional submanifold in $\widetilde{\mathcal{M}}$, and Conjectures 5.4.2 and 5.4.3 imply that only one component of the preimage intersects $\{R_0\}\times Z$. It follows that the total number of gluing is 1.

CHAPTER 6

Cobordism Maps

In this chapter, we discuss the main obstacle to defining cobordism maps on ECH in the L-supersimple setting: multiple covers of J-holomorphic planes. We first use the index formula derived in Chapter 4 to show that, in the absence of such planes, cobordism maps are well-defined. We then use the results from Chapter 5, assuming the conjectures therein, to construct cobordism maps in the L-supersimple setting.

6.1 The ideal case

Let (X^4, λ) be a compact, connected, exact symplectic cobordism, so that $d\lambda$ is a symplectic form, $\partial X = Y_+ - Y_-$, and $\lambda|_{Y_\pm} = \lambda_\pm$ is an L-supersimple contact form on $Y_\pm$. Choose a generic L-simple, admissible almost complex structure J on the completion $(\widehat{X}, \widehat{\lambda})$ that restricts to simple, admissible almost complex structures J_+ and J_- on the ends $[0, \infty) \times Y_+$ and $(-\infty, 0] \times Y_-$, respectively, of $\widehat{X}$. In this section, we further assume that all contractible Reeb orbits γ in Y have Conley-Zehnder index at least 2 with respect to any disk $u\colon \mathbb{D} \to Y$ such that $u|_{\partial\mathbb{D}}$ parametrizes γ and with respect to any disk in $\widehat{X}$ asymptotic to γ. We let $\mathcal{M}^i_J(\boldsymbol{\alpha}, \boldsymbol{\beta})$ denote the moduli space of ECH index i holomorphic curves in $\widehat{X}$ asymptotic to $\boldsymbol{\alpha}$ at the positive ends and to $\boldsymbol{\beta}$ at the negative ends. In analogy with [1], we can then define a chain map

$$\Phi_{(X,\lambda,J)}\colon ECC^L(Y_+, \lambda_+, J_+) \to ECC^L(Y_-, \lambda_-, J_-)$$

by

$$\Phi_{(X,\lambda,J)}(\boldsymbol{\alpha}) = \sum_{\mathcal{A}(\boldsymbol{\beta})<L} \#\left(\mathcal{M}^0_J(\boldsymbol{\alpha}, \boldsymbol{\beta})\right)\boldsymbol{\beta}.$$

Lemma 6.1.1. *A J-holomorphic curve u in $\widehat{X}$ with* $\operatorname{ind}(u) = 0$ *has negative ECH index if and only if it is an unbranched, disconnected cover of a J-holomorphic plane, in which case*

$$I(u) = -\frac{d(d-1)}{2},$$

where d is the degree of the covering.

Proof. Suppose that $I(u) < 0$. By the ECH index inequality from [12, Definition 4.3], somewhere injective curves in cobordisms have non-negative ECH index, so u must be a d-fold multiple cover of a somewhere injective curve $v\colon \dot{\Sigma}' \to \widehat{X}$ with $\operatorname{ind}(v) \geq 0$ and $d \geq 2$. Recall the index inequality

$$I(u) \geq d \cdot I(v) + \left(\frac{d^2-d}{2}\right)\left(2g(\dot{\Sigma}') - 2 + \operatorname{ind}(v) + h(v)\right) \qquad (*)$$

from [12], where $h(v)$ is the number of ends of v at hyperbolic orbits. Since $h(v) \geq 1$ and $\operatorname{ind}(v) \geq 0$, the only way for $I(u)$ to be negative is if $g(\dot{\Sigma}') = 0$ and $\operatorname{ind}(v) + h(v) = 1$. Since $\operatorname{ind}(u) = 0$, Lemma 2.5.3 implies that u is an unbranched cover of v and $\operatorname{ind}(v) = 0$. Hence $h(v) = 1$. It follows that u is an unbranched, disconnected cover of a plane v.

Suppose there is a component $\dot{\Sigma}$ of the domain of u such that $\dot{\Sigma} \to \dot{\Sigma}'$ is an m-fold (unbranched) covering with $m \geq 2$. Then $m = \chi(\dot{\Sigma}) = 2 - 2g(\dot{\Sigma}) - m$, so $g(\dot{\Sigma}) = 1 - m < 0$ which is impossible. It follows that every component of the domain of u maps diffeomorphically onto $\dot{\Sigma}$.

Conversely, suppose that $u\colon \dot{\Sigma} \to \widehat{X}$ is such a cover of a plane $v\colon \dot{\Sigma}' \to \widehat{X}$ with a positive end at a hyperbolic orbit γ. If we choose the trivialization τ of $\gamma^*\xi$ such that $c_1(v^*\xi, \tau) = 0$ we see that

$$\begin{aligned} 0 &= \operatorname{ind}(v) \\ &= -\chi(\dot{\Sigma}') + 2c_1(v^*\xi, \tau) + CZ_\tau(\gamma) \\ &= CZ_\tau(\gamma) - 1, \end{aligned}$$

so $CZ_\tau(\gamma) = 1$. Thus,

$$\begin{aligned} 0 &= I(v) \\ &= c_1(v^*\xi, \tau) + Q_\tau(v) + CZ_\tau(\gamma) \\ &= Q_\tau(v) + 1, \end{aligned}$$

so $Q_\tau(v) = -1$. The relative self-intersection number Q_τ is quadratic under taking multiple covers (see the discussion in Section 3.5 of [12]), so $Q_\tau(u) = -d^2$ and

$$\begin{aligned} I(u) &= c_1(u^*\xi, \tau) + Q_\tau(u) + CZ^I_\tau(\gamma) \\ &= -d^2 + \sum_{i=1}^{d} i = -\frac{d(d-1)}{2}, \end{aligned}$$

as desired. □

Lemma 6.1.2. *Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be orbit sets such that $\boldsymbol{\alpha}$ satisfies the ECH partition conditions and $\boldsymbol{\beta}$ is a generator of the ECH chain complex. Then a curve $u \in \mathcal{M}^0_J(\boldsymbol{\alpha}, \boldsymbol{\beta})$ with* $\operatorname{ind}(u) = 0$ *is a multiple cover if and only if the underlying somewhere injective curve is a J-holomorphic cylinder with ECH index* 0 *and no negative ends. In this case, the map u is an immersion.*

Proof. Assume first that u is a d-fold cover, $d \geq 2$, of a somewhere injective curve $v\colon \dot{\Sigma}' \to \widehat{X}$, which necessarily satisfies $I(v) = 0$. Since $\operatorname{ind}(u) = 0$, Lemma 2.5.3 implies that u is necessarily an unbranched cover of v. Hence u is an immersion. Since $\boldsymbol{\beta}$ is an ECH generator, it follows immediately that u has no negative ends. Since $I(u) = 0$, the inequality $(*)$ implies that $2g(\dot{\Sigma}') - 2 + h(v) \leq 0$. Thus, $h(v) = 1$ or 2. If $h(v) = 1$, then v is a plane and, by the arguments in the proof of Lemma 6.1.1, $I(u) < 0$. It follows that $h(v) = 2$ and v is a cylinder. Clearly v has no negative ends.

Now assume that u is a d-fold multiple cover, $d \geq 2$, of a J-holomorphic cylinder $v\colon \dot{\Sigma}' \to \widehat{X}$ with $I(v) = 0$ and no negative ends. Let $\boldsymbol{\gamma}_+$ be the positive orbit set of v. Note that the orbits in $\boldsymbol{\gamma}_+$ must be positive hyperbolic u will not satisfy the partition conditions at the positive ends. Choose τ such that $CZ_\tau(\gamma) = 0$ for all $\gamma \in \boldsymbol{\gamma}_+$. Then we have

$$0 = \operatorname{ind}(v) = 2c_1(v^*\xi, \tau),$$

and hence

$$0 = I(v) = Q_\tau(v).$$

It follows that $c_1(u^*\xi, \tau) = Q_\tau(u) = 0$, so $I(u) = 0$. $\square$

Theorem 6.1.3. *If all contractible Reeb orbits in Y have Conley-Zehnder index at least 2 with respect to any bounding disk in Y and with respect to any disk in $\widehat{X}$ asymptotic to γ, then $\Phi_{(X,\lambda,J)}$ is a well-defined chain map.*

Proof. We proceed in three steps. We will abbreviate $\mathcal{M}^i_J(\boldsymbol{\alpha}, \boldsymbol{\beta})$ as $\mathcal{M}^i$ and $\Phi_{(X,\lambda,J)}$ as Φ. Recall that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are assumed to be generators of the ECH chain complex.

Step 1: $\mathcal{M}^0$ is regular. Since there are no elliptic orbits of action less than L and the multiplicity of all orbits in $\boldsymbol{\alpha}$ is 1, all curves in the moduli space are somewhere injective and, therefore, regular.

Step 2: Φ is well-defined. Since $\mathcal{M}^0$ is regular by Step 1, it suffices to show that $\mathcal{M}^0$ is compact. Let $u_1, u_2, \ldots$ be a sequence representing elements of $\mathcal{M}^0$ that converges to an SFT building u_∞. By Lemma 2.5.3, each level of u_∞ has non-negative Fredholm index and each symplectization level has positive Frehdolm index. Since u_∞ has total Fredholm index 0, it follows that u_∞ is a 1-level building. This shows that $\mathcal{M}^0$ is compact.

Step 3: $\partial \circ \Phi = \Phi \circ \partial$. The moduli space $\mathcal{M}^1$ is regular. Every element of $\partial\mathcal{M}^1$ has total Fredholm index 1, hence is of the form $u_+ \cup u_0$ or $u_0 \cup u_-$, where u_0 is in the cobordism level, $\mathrm{ind}(u_0) = 0$, $u_\pm$ maps into $\mathbb{R} \times Y_\pm$, $\mathrm{ind}(u_\pm) = 1$, and the levels are arranged from top to bottom. We wish to count the number of such buildings in $\partial\mathcal{M}^1$ such that u_+ (resp. u_-) is negatively (resp. positively) asymptotic to a generator of the filtered ECH complex for Y_+ (resp. Y_-). It is easier, however, to instead count buildings that do not break at a generator. So let $\boldsymbol{\gamma}$ be the negative orbit set of u_+ and assume that some orbit in $\boldsymbol{\gamma}$ has multiplicity greater than 1. By Lemma 6.1.1, we know that $I(u_0) \geq 0$ unless it contains multiple covers of J-holomorphic planes. An easy calculation shows that such a plane must limit to an orbit with Conley-Zehnder index ± 1, contradicting our assumptions. Thus, all holomorphic curves in $\widehat{X}$ have non-negative ECH index.

Now consider the case of a building $u_+ \cup u_0$; the other case is similar. Since $I(u_+) = 1$ and u_+ maps into $\mathbb{R} \times Y_+$, it is an embedding and satisfies the ECH partition conditions. It follows that the number of ways to glue u_+ and u_0 is even: this is clear when u_0 contains no multiply covered component and follows from Lemma 6.1.2 when multiple covers occur. Thus, the number of curves in $\partial\mathcal{M}^1$ that we do not count in $\partial \circ \Phi$ or $\Phi \circ \partial$ is even, so the number of curves that are counted is even as well. □

6.2 The general case

Now we remove the restriction on J-holomorphic planes in $\widehat{X}$ with Fredholm index 0. The main difficulty now is that J-holomorphic curves in completed exact symplectic cobordisms can limit to SFT buildings that are nice from the perspective of Fredholm theory and transversality but degenerate from the perspective of ECH. More specifically, we consider moduli spaces of curves with ECH and Fredholm indices both equal to 1. The boundary points of such moduli spaces are 2-level buildings where the symplectization and cobordism levels have Fredholm indices 1 and 0, respectively. However, by Lemma 6.1.1, in the presence of multiply covered planes in the cobordism level, the symplectization level has ECH index greater than 1 and the cobordism level has negative ECH index.

The existence of degenerate boundary points causes Step 3 in the above argument to fail. An example of such a breaking with an $I = -1$ double cover of a plane is shown below. The top curve is in the symplectization level and the bottom is in the cobordism level.

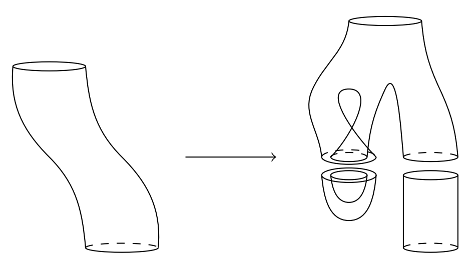

Figure 6.1: An example of degenerate breaking.

Our strategy to define a chain map in this case is the following. Suppose $u_+ \cup u_0$ is a building in $\partial\mathcal{M}^1$ such that u_0 contains a negative-index multiple cover of a plane. We **continue the moduli space** by gluing in a branched cover of a trivial cylinder, using the results of Chapter 5, and cutting out a 1-dimensional family of curves, using the results of Chapter 3. We then re-define the map $\Phi_{(X,\lambda,J)}$ to incorporate the new ends, making it a chain map. A schematic picture of this process is given below for the case illustrated in Figure 6.1. The black dot denotes the branch point of a two-fold branched cover of a trivial cylinder. Throughout, the bottom level is the cobordism level and the rest are symplectization levels.

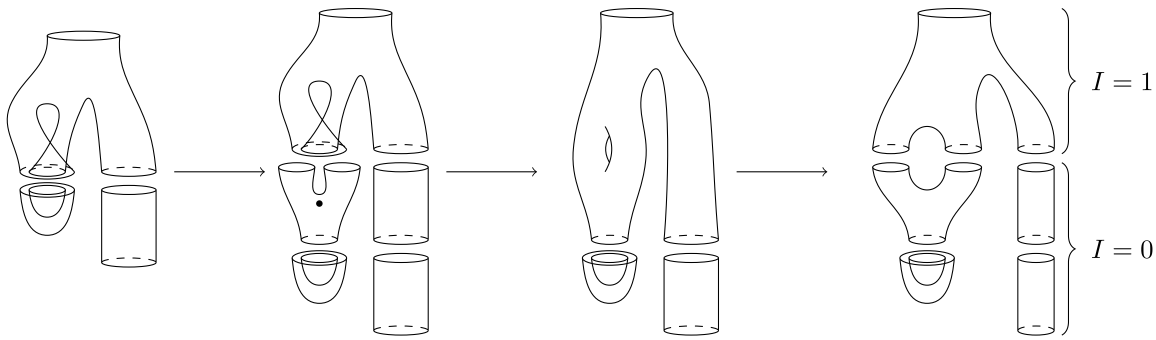

Figure 6.2: An example continuation of the moduli space.

Given such a two-level building $u_+ \cup u_0$ with $I(u_0) < 0$, we begin by examining the negative ends of u_+. The curve u_0 must contain at least one negative index multiple cover of a plane. Say the underlying embedded planes are asymptotic to Reeb orbits $\beta_1, \ldots, \beta_k$ and the covers have multiplicities $d_1, \ldots, d_k$.

Lemma 6.2.1. *For each i, the curve u_+ has exactly d_i negative ends with multiplicity* 1 *at β_i, and all other negative ends of u_+ at β_i have multiplicity* 2.

Proof. By Lemma 6.1.1,

$$I(u_0) = -\sum_{i=1}^{k} \frac{d_i(d_i-1)}{2}.$$

As in Definition 4.4.1, suppose that u_+ has negative ends of multiplicities $q_{i,1}, \ldots, q_{i,n_i}$ at covers of β_i, ordered so that, for each i, the first d_i have multiplicity 1, the next m_i are the remaining ends with odd multiplicity, and the last $n - d_i - k_i$ have even multiplicity. Then

$$I(u_+) = 1 + \sum_{i=1}^{k} \frac{d_i(d_i-1)}{2},$$

so by Theorem 4.4.3,

$$\sum_{i=1}^{k}\sum_{j=d_i+1}^{d_i+m_i}\left(\frac{q_{i,j}-1}{2}+j-1\right)+\sum_{i=1}^{k}\sum_{j=d_i+m_i+1}^{n_i}\left(\frac{q_{i,j}}{2}-1\right)=0.$$

Each term on the left-hand side is non-negative, so each must vanish. Thus, for all i, we have $m_i = 0$ and $q_{i,d_i+1} = \cdots = q_{i,n_i} = 2$. □

Suppose, for the sake of simplicity, that u_0 has exactly one negative index multiple cover of a plane, with covering multiplicity n, and let β be the Reeb orbit to which the underlying embedded plane is asymptotic. Assuming the conjectures in Chapter 5, we can glue branched covers in $\mathcal{M}_n$ to u_+ and use the evaluation map to cut out a 1-dimensional family of curves in the resulting moduli space $\widetilde{\mathcal{M}}$. It also follows that the other endpoint is in some other boundary component of $\widetilde{\mathcal{M}}$. We call the other endpoint the **re-breaking** of the glued curve. We may assume that the re-breaking is a 2-level building $v_+ \cup v_0$ with $\operatorname{ind}(v_+) = 1$.

Using the above degree calculation and Proposition 3.3.2, we may assume that the highest-order coefficients at the multiplicity 2 ends are all non-zero and distinct. It follows that v_0 is somewhere injective except possibly for components that are branched covers of trivial cylinders. By the above discussion, such branched covers must have total Fredholm index at most $\frac{n(n-1)}{2}$. Moreover, the restrictions on the rescaled evaluation map imply that they must be close to the distinguished boundary $\partial_*\mathcal{M}_n$. Conjecture 5.4.3 rules out this possibility, so by Lemma 6.2.1 and Theorem 4.4.3, v_0 is embedded and has $I(v_0) = \operatorname{ind}(v_0) = \frac{n(n-1)}{2}$.

Definition 6.2.2. An **index 0 ECH building** for the orbit sets $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is a 2-level tower of curves $v_0 \cup u_0$ satisfying the following conditions:

1. There is a J-holomorphic curve v_1 in $\mathbb{R} \times Y$ with positive orbit set $\boldsymbol{\alpha}$, with $\operatorname{ind}(v_1) = I(v_1)$, and such that v_1 glues to v_0;
2. u_0 has negative orbit set $\boldsymbol{\beta}$;
3. the partition of the negative ends of v_0 coincides with the partition of the positive ends of u_0, except for branched, disconnected multiple covers of planes in u_0;

4. $\mathrm{ind}(u_0) = 0$ and $I(u_0) < 0$;

5. $\mathrm{ind}(v_0) = I(v_0) = -I(u_0)$; and

6. the evaluation map from 5.1.2 maps v_0 into P_n and satisfies the restrictions given in Conjecture 5.4.2.

We can now re-define Φ to include a correction term counting ECH buildings $v_0 \cup u_0$ obtained by the gluing and re-breaking procedure described above. By construction, the re-defined Φ is a chain map.

Definition 6.2.3. Let $(Y_\pm, \lambda_\pm)$ be L-supersimple contact 3-manifolds and let (X, λ) be an exact symplectic cobordism from (Y_+, λ_+) to (Y_-, λ_-). Let J be a generic L-simple, admissible almost complex structure on the completion $(\widehat{X}, \widehat{\lambda})$ that restricts to simple, admissible almost complex structures J_+ and J_- on the ends $[0, \infty) \times Y_+$ and $(-\infty, 0] \times Y_-$, respectively, of $\widehat{X}$. The chain map

$$\widetilde{\Phi}_{X,\lambda,J} \colon ECC^L(Y_+, \lambda_+, J_+) \to ECC^L(Y_-, \lambda_-, J_-)$$

induced by (X, λ) is defined by

$$\widetilde{\Phi}_{X,\lambda,J}(\boldsymbol{\alpha}) = \sum_{\mathcal{A}(\boldsymbol{\beta})<L} \left[\# \left(\mathcal{M}^0_J(\boldsymbol{\alpha}, \boldsymbol{\beta})\right) + \# \left(\mathcal{B}^0_J(\boldsymbol{\alpha}, \boldsymbol{\beta})\right)\right] \boldsymbol{\beta},$$

where the correction term $\#(\mathcal{B}^0_J(\boldsymbol{\alpha}, \boldsymbol{\beta}))$ is the count of index 0 ECH buildings for the orbit sets $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

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