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by

Dylan Patrick Noack

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ABSTRACT OF THE DISSERTATION

A Characterization of Certain Bounded, Convex Domains

by

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University of California, Riverside, June 2019
Professor Bun Wong, Chairperson

Because the Riemann Mapping Theorem does not hold in several complex variables, it is of interest to fully classify the simply connected domains. By considering convex, bounded domains with noncompact automorphism groups, we can define a rescaling sequence based on the boundary-accumulating automorphism orbit. If this orbit converges nontangentially we prove the accumulation point is of finite type in the sense of D'Angelo. This both provides a partial proof to the Greene-Krantz conjecture and also classifies such domains as polynomial ellipsoids.

Contents

Chapter 1

Introduction

The Riemann Mapping Theorem states that up to biholomorphism there are only two simply connected domains in $\mathbb{C}$. There is the complex plane itself and there is the unit disk. Specifically, any open, simply-connected set that is not all of $\mathbb{C}$ is biholomorphic to the set $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. One would hope that this amazing characterization would also hold true in several variables, but unfortunately this is not the case. Even the relatively simple examples of the unit ball in $\mathbb{C}^2$ and the set $\Delta \times \Delta$ are not biholomorphic. A long-term goal for the field of several complex variables is to generalize the Riemann Mapping Theorem and ultimately come up with a classification of simply connected domains in $\mathbb{C}^n$.

In the pursuit of this classification we will assume additional hypotheses on our domains that allow us to use more specialized tools to classify them. For example, consider the group of self-biholomorphic maps on a domain. We refer to this as the automorphism group. We can split the domains into those with a *compact* automorphism group and those with a *non-compact* automorphism group. While we will state some results for more general domains, by far our interest lies in convex, bounded domains with non-compact automorphism group. We will often assume a smooth boundary as well, though there are reasons to believe a C^2 boundary is sufficient.

In the second chapter we will outline some preliminary results. In the third chapter we will discuss the Kobayashi metric, Gromov hyperbolicity and some important results about smoothly bounded convex sets. One of these results will be the main contradiction in the proof of our main result. In chapter four we discuss automorphism orbits and the Greene-Krantz conjecture, a conjecture we provide a partial solution to. If proven in full generality would be a useful tool in further classification of simply connected domains. Finally in chapter five we prove our main result:

Theorem 1.0.1. *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded, convex domain with smooth boundary. Suppose there exists $p \in \Omega$ and $\{\phi_k\}_{k\in\mathbb{N}} \subseteq \operatorname{Aut}\Omega$ such that $\lim_{k\to\infty} \phi_k(p) = q$ for $q \in \partial\Omega$ and $\phi_k(p)$ approaches q nontangentially. Then q is of finite type in the sense of D'Angelo.*

We provide a brief sketch here.

1. First we will consider a 2-dimensional slice of Ω and define a rescaling sequence for those two dimensions.

2. Next we will show there exists a holomorphic disk contained in the boundary of our rescaled 2-dimensional slice.

3. Then we apply a theorem of Frankel to show that there exists a rescaling sequence on Ω such that its blow-up also has a disk in its boundary.

4. Our rescaled domain $\hat{\Omega}$ is then shown to be biholomorphic to the original domain Ω.

5. Lastly we show there exists a holomorphic map $f : \Delta \times \Delta \to \hat{\Omega}$ that is isometric in one coordinate and isometric along a radius in another (with some error). This will result in a contradiction.

It follows as a corollary of our main result that:

Corollary 1.0.2. *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded, convex domain with smooth boundary. If there exists $p \in \Omega$ and a sequence of automorphisms $\phi_k \in \operatorname{Aut}\Omega$ such that $\phi_k(p) \to q \in \partial\Omega$ non-tangentially, then Ω is biholomorphic to a polynomial ellipsoid.*

This gives us a nice characterization of certain bounded, convex domains.

Chapter 2

Preliminaries

We begin with some basic preliminaries on several complex variables. Holomorphic functions in several variables and domains in $\mathbb{C}^n$ will be our first topics before moving onto pseudoconvexity. We also discuss the notion of finite type in the sense of D'Angelo before finishing with the methods of rescaling. Readers familiar with all of these topics can safely skip this chapter, though we suggest reviewing Theorem 2.5.3 as it will be critically important in the proof of our main result.

2.1 Holomorphic Functions of Several Complex Variables

There are several equivalent definitions for a function to be holomorphic in several complex variables. The one we take is the following:

Definition 2.1.1. Let $\Omega \subseteq \mathbb{C}^n$ be an open, connected set. A function $f : \Omega \to \mathbb{C}$ is *holomorphic* if, for each $j = 1, \dots, n$ and each fixed $z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n$, the function

$$\zeta \mapsto f(z_1, \dots, z_{j-1}, \zeta, z_{j+1}, \dots, z_n)$$

is holomorphic in the classic one-variable sense on the set

$$\Omega(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n) \equiv \{\zeta \in \mathbb{C} : (z_1, \dots, z_{j-1}, \zeta, z_{j+1}, \dots, z_n) \in \Omega\}.$$

A function $g : \Omega \to \mathbb{C}^m$ is holomorphic if $\pi_i \circ g$ is holomorphic for each $0 \leq i \leq m$ where π_i is the projection map onto the ith coordinate.

Another way to interpret this definition would be that for a function to be holomorphic in several variables, it must be holomorphic in each varaible separately. As mentioned earlier, there are a few equivalent definitions. These should come as no surprise to someone familiar with single-variable complex analysis.

Theorem 2.1.2. *Let $D^n(z_0, r) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j - z_0| < r, 0 \leq j \leq n\}$, $\Omega \subseteq \mathbb{C}^n$ be an open, connected set and $f : \Omega \to \mathbb{C}$ be continuous in each variable separately. Then the following are equivalent.*

(1) f is holomorphic

(2) f satisfies the Cauchy-Riemann equations in each variable separately.

(3) For each $z_0 \in \Omega$ there exists an $r = r(z_0) > 0$ such that $\overline{D^n(z_0, r)} \subseteq \Omega$ and f can be written as an absolutely and uniformly convergent power series

$$f(z) = \sum_{\alpha} a_\alpha (z - z_0)^\alpha$$

for all $z \in D^n(z_0, r)$.

(4) For each $w \in \Omega$ there exists $r = r(w) > 0$ such that $\overline{D^n(w, r)} \subset \Omega$ and

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\zeta_n - w_n| = r} \cdots \int_{|\zeta_1 - w_1| = r} \frac{f(\zeta_1, \cdots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} d\zeta_1 \cdots \zeta_n$$

for all $z \in D^n(w, r)$.

We point the reader to Krantz [13] for a discussion and proof of the above theorem. While such a definition may not appear to be more than a use of multi-indicies, the introduction of several variables has drastically changed the field. Many results such as the Riemann Mapping Theorem no longer hold true.

We will discuss such properties of domains in the next section, but first we mention our notion of equivalence in several complex variables.

Definition 2.1.3. Let $\Omega, \Omega' \subseteq C^n$ be open, connected sets and $f : \Omega \to \Omega'$ a holomorphic function. If f is a bijection then we refer to f as a *biholomorphism.* We refer to Ω and Ω' being *biholomorphic.* If $\Omega = \Omega'$, we refer to f as an *automorphism.*

Remark 2.1.4. It is a standard exercise to prove the existence of a holomorphic inverse given a bijective holomorphic function f.

In later chapters we will further investigate properties of automorphisms. For now denote the set of automorphisms of $\Omega \subseteq \mathbb{C}^n$ by $\operatorname{Aut}\Omega$.

2.2 Domains in $\mathbb{C}^n$

Like in single-variable complex analysis, we define a domain to be an open, connected set. Consider a domain Ω and a holomorphic function f defined on this domain. From an analytical perspective, if our function f extends to a larger domain Ω', there is little reason to consider Ω. In the same way that we would not study the real function $\frac{1}{x}$ by looking at its behavior on $[7, 12]$, we are only interested in studying functions on their maximal domain of definition.

What this means is that if we have a domain Ω such that *every* holomorphic function $f : \Omega \to \mathbb{C}$ can extend to some larger domain, there would be little purpose in studying Ω. Domains that *are* maximal domains of definition for some holomorphic function, however, would be worth studying. Thus they are referred to as *domains of holomorphy.* We provide a formal definition below.

Definition 2.2.1. $\Omega \subseteq \mathbb{C}^n$ is a *domain of holomorphy* if there do not exist nonempty open sets U_1, U_2 with U_2 connected, $U_2 \not\subset \Omega$, $U_1 \subseteq U_2 \cap \Omega$ such that for every holomorphic function g on Ω, there is a holomorphic function h on U_2 such that $g = h$ on U_1.

Remark 2.2.2. This definition is not usually brought up in a single variable complex text because in one variable, *every* domain is a domain of holomorphy. However, in several complex variables this is not true. The following example of Hartogs [8] should make this apparent.

Example 2.2.3. Consider the domain

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 3, |z_2| < 3\} \setminus \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq 1, |z_2| \leq 1\}.$$

We will show that every holomorphic function $f : \Omega \to \mathbb{C}$ extends to the domain

$$\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 3, |z_2| < 3\}.$$

For z_1 fixed, $|z_1| < 3$ we write

$$f_{z_1}(z_2) = f(z_1, z_2) = \sum_{j=-\infty}^{\infty} a_j(z_1) z_2^j$$

where the coefficients of the Laurent expansion are given by

$$a_j(z_1) = \frac{1}{2\pi i} \int_{|\zeta|=2} \frac{f(z_1, \zeta)}{\zeta^{j+1}} d\zeta.$$

Specifically, $a_j(z_1)$ depends holomorphically on z_1 by Morera's theorem. But $a_j(z_1) = 0$ for $j < 0$ and $1 < |z_1| < 3$. Thus by analytic continuation, a_j is identically zero for $j < 0$. But then the series expansion becomes

$$\sum_{j=0}^{\infty} a_j(z_1) z_2^j$$

and this series defines a holomorphic function $\hat{f}$ on $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 3, |z_2| < 3\}$ such that $\hat{f}|_\Omega = f$. Because f was arbitrary, all holomorphic functions on Ω can be continued to a larger domain, and thus Ω is not a domain of holomorphy.

Remark 2.2.4. The above example can be used to prove that every isolated singularity in several complex variables is removable. This is a major difference between single-variable complex analysis and several complex variables.

One of the first problems in several complex variables was finding a characterization of domains of holomorphy. It can be shown that any geometrically convex domain, the definition of which we will provide below, is a domain of holomorphy.

Definition 2.2.5. Let $X \subseteq \mathbb{R}^n$. X is (*geometrically*) *convex* if for all $x, y \in X$,

$$\{xt + (1-t)y : t \in [0,1]\} \subseteq X$$

It would be nice if geometric convexity was a characterization of domains of holomorphy, but unfortunately the reverse implication does not hold. Any non-convex domain in $\mathbb{C}$ serves as a counter-example. However, convexity is not too far off from a characterization. In several complex variables, a notion of "complex" convexity was developed and proved to characterize domains of holomorphy. This was known as the Levi problem, but before discussing this problem we must build up some further background.

Since such a problem is a question on whether or not holomorphic functions can extend to larger domains, it can be useful to describe our domains themselves in terms of functions.

Definition 2.2.6. Let $\Omega \subseteq \mathbb{R}^n$ be an open set with C^k boundary. A function $\rho : \mathbb{R}^n \to \mathbb{R}$ is said to be a *defining function for* Ω if ρ is C^k and

1. $\rho(x) < 0$ for all $x \in \Omega$
2. $\rho(x) > 0$ for all $x \notin \Omega$ and
3. $\nabla\rho(x) \neq 0$ for all $x \in \partial\Omega$.

A domain is said to have *smooth boundary* if its defining function is smooth in the real sense. We will refer to a domain as being *smoothly bounded* if the domain has smooth boundary and is also bounded.

With this notion of defining function, we can come up with an alternate definition of convexity that depends on the differential properties of the boundary.

Definition 2.2.7. Let $\Omega \subset \mathbb{R}^n$ have C^1 defining function ρ. Let $p \in \partial\Omega$. We consider the vector $w = (w_1, \ldots, w_n)$ to be tangent to $\partial\Omega$ at p if

$$\sum_{k=1}^{n} \left.\frac{\partial \rho}{\partial x_k}\right|_p w_k = 0.$$

In this case we write $w \in T_p\partial\Omega$.

Definition 2.2.8. Let $\Omega \subseteq \mathbb{R}^n$ have C^2 defining funtion ρ. Let $p \in \partial\Omega$. We say that $\partial\Omega$ is convex at p if

$$\sum_{j,k=1}^{n} \left.\frac{\partial^2 \rho}{\partial x_j \partial x_k}\right|_p w_j w_k \geq 0$$

for all $w = (w_1, \ldots, w_n) \in T_p\partial\Omega$. If the inequality is strict we refer to p as a point of strong convexity.

Example 2.2.9. We mention a few common domains and their defining functions. All of them are convex.June

1. The unit disk in $\mathbb{C}$ is given by $\Delta = \{z \in \mathbb{C} : |z| - 1 < 0\}$.

2. The halfplane in $\mathbb{C}$ is given by $\mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$.

3. The unit polydisk in $\mathbb{C}^n$ is given by

$$\Delta^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1| < 1, |z_2| < 1, \ldots, |z_n| < 1\}.$$

   We note for this domain the defining function is piecewise smooth.

4. The ball in $\mathbb{C}^n$ centered at a of radius r is given by

$$B_r(a) = \{(z_1, \ldots, z_n \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_n|^2 - 1 < 0\}.$$

2.2.1 Polynomial Ellipsoids

We will finish off our discussion of domains with a specific class called polynomial ellipsoids. It will follow as a corollary of our main result that domains satisfying certain properties belong to this class. Given $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{\geq 0}$ we can define a weight function $wt_m : \mathbb{Z}^n_{\geq 0} \to \mathbb{Q}$ by

$$wt_m(\alpha) = \sum_{i=1}^{n} \frac{\alpha_i}{2m}.$$

Definition 2.2.10. A domain Ω is called a *polynomial ellipsoid* if

$$\Omega = \{(w, z) \in \mathbb{C} \times \mathbb{C}^n : |w|^2 + p(z) < 1\}$$

where $p(z) : \mathbb{C}^n \to \mathbb{R}$ is a polynomial such that there exists an $m \in \mathbb{Z}^n_{\geq 0}$ so that

$$p(z_1, \ldots, z_n) = \sum_{wt_m(\alpha)=wt_m(\beta)=0.5} C_{\alpha,\beta} z^\alpha \overline{z}^\beta.$$

Example 2.2.11. The domain

$$\{(w, z) \in \mathbb{C}^2 : |w|^2 + |z|^{2m} < 1\}$$

is a polynomial ellipsoid for all integers $m \geq 1$.

2.3 Pseudoconvexity

In the early 20th ceuntury a large amount of research activity was invested in classifying domains of holomorphy. In this section we will describe the notion of pseudoconvexity, which can be thought of as a compexified definition of convexity. Pseudoconvexity is the typical classification taken today. A longer disussion of psuedoconveity and the Levi problem can be found in [13]. We start with the following definition.

Definition 2.3.1. Let $\Omega \subseteq \mathbb{C}^n$. We refer to a function $\phi : \Omega \to \mathbb{R} \cup \{-\infty\}$ as being *plurisubharmonic* if it is upper semi-continuous and given any complex line $\{a + bz\}$ the function that sends $z \mapsto \phi(a + bz)$ is subharmoinc on the set $\{z \in \mathbb{C} : a + bz \in \Omega\}$.

Another way to think about the above definition is that a function is plurisubharmonic if it is subharmonic on any complex line cutting through the domain. The connection to convexity here should be apparent. With this we can define (Hartogs) pseudoconvexity.

Definition 2.3.2. A domain Ω is *(Hartogs) pseudoconvex* if there exists a continuous plurisubharmonic function ϕ defined on Ω such that, given any $r \in \mathbb{R}$ the set $\{z \in \Omega : \phi(z) < r\}$ is relatively compact.

It turns out that with this definition, we have the following theorem of Oka [17].

Theorem 2.3.3 (Levi Problem). *Let $\Omega \subset \mathbb{C}^n$. Ω is a domain of holomorphy if and only if Ω is pseudoconvex.*

While we have stated this theorem quite briefly, it is a culmination of decades of research. The development of a characterization of domains of holomorphy was a massive undertaking that still has mathematicians interested to this day. Even with more sophisticated tools such as the $\overline{\partial}-$method, the proof of the Levi problem is highly non-trivial.

In the case that we have a domain with a twice-differentiable boundary, there is an equivalent notion of pseudoconvexity that better illustrates how it naturally arises from the definition of convexity.

Definition 2.3.4. Let $\Omega \subseteq \mathbb{C}^n$ have C^2 defining function ρ. Let $p \in \partial\Omega$. We say that p is a *point of Levi pseudoconvexity* if

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z_k}}\bigg|_p w_j \overline{w_k} \geq 0$$

for all $w \in \mathbb{C}^n$ such that

$$\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j \partial \overline{z_k}}\bigg|_p w_j = 0.$$

If the inequality is strict, we refer to p as being a point of *strong pseudoconvexity.* If every point in the boundary is strongly pseudoconvex, then the domain itself is *strongly pseudoconvex.* Otherwise, it is *(weakly) pseudoconvex.*

The notion of pseudoconvexity is fundamental to the study of several complex variables, but for our purposes we take the stronger assumption of convexity. The following proposition follows from the definitions:

Proposition 2.3.5. *Let $\Omega \subset \mathbb{C}^n$ be a domain with C^2 boundary and $p \in \partial\Omega$. If $\partial\Omega$ is convex at p, then it is pseudoconvex at p.*

2.4 Finite Type

We now discuss an algebro-geometric method of characterizing boundary points. Consider the following scenario.

Example 2.4.1. Let $f(x) = x^2$ and $g(x) = 2 - 2\sqrt{1-x^2}$ be two real-valued functions. It follows $f'(0) = 0$ and $g'(0) = 0$. Thus they are both tangent to the line $y = 0$. Furthermore, by direct calculation it follows $f''(0) = 2$ and $g''(0) = 2$. Thus they also have the same *second* derivatives.

The significance of this is that f and g have similar geometric behavior around the point $x = 0$, up to second derivatives. If we imagine that g describes the boundary of some domain, this means that we could *classify* the boundary behavior of this domain by the "base function" f.

The above is known as *order of contact.* While order of contact gives us a way to characterize points on the boundary, we will be using the more general notion of D'Angelo finite type [4].

Definition 2.4.2. Let $\Omega \subseteq \mathbb{C}^n$ and $f : \Omega \to \mathbb{C}$ be a holomorphic function. If $p \in \Omega$, then the *multiplicity of f at p* is the least positive integer k such that the k^{th} derivative of f does not vanish at p. We use the notation $\nu_p(f)$ to denote this least positive k. If $g : \Omega \to \mathbb{C}^n$, then the multiplicity of g at $p \in \Omega$ is the minimum of the multiplicities of its component functions. We may omit the subscript from ν if context makes it apparent.

Remark 2.4.3. In the case where $f : \Omega \to \mathbb{C}$ is not holomorphic, we have an alternative definition of multiplicity. A function f has multiplicity k if

$$\lim_{z \to 0} \frac{f(z)}{|z|^n} = 0$$

for all $n < k$.

Definition 2.4.4. Let $\Omega \subseteq \mathbb{C}^n$ be a smooth domain and $q \in \partial\Omega$. Let ρ be a defining function for Ω in a neighborhood of q. We say that q is of *finite type* C in the sense of D'Angelo if

$$\sup_f \left\{ \frac{\nu(\rho \circ f)}{\nu(f)} \right\} = C < \infty$$

where f ranges through nonconstant holomorphic curves with $f(0) = q$. If every point $q \in \partial\Omega$ is of finite type, we say Ω is a *finite type domain.* If $C = \infty$ we refer to the domain as being an *infinite type domain* and the point q as a *point of infinite type.*

It can be difficult to explicitly calculate the type of a given domain, even simple domains like the unit ball. Needing to consider arbitrary holomorphic curves is the source of such difficulty, but there are results that ease the scope of what we need to consider. This is one reason why we limit ourselves to convex domains in our main result. Consider the following definition.

Definition 2.4.5. Suppose that $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ where ρ is a defining function. We say a point $q \in \partial\Omega$ has *finite line type* L if

$$\sup\{\nu_0(r \circ \ell) \mid \ell : \mathbb{C} \to \mathbb{C}^n \text{ is a non-trivial affine map and } \ell(0) = q\} = L$$

where $L < \infty$. If $L = \infty$ we say x has *infinite line type.*

Remark 2.4.6. Notice that $\nu_0(r \circ \ell) \geq 2$ if and only if $\ell(\mathbb{C})$ is tangent to Ω.

Another way to think about the above definition is that instead of arbitrary holomorphic curves, we limit ourselves to complex lines in the definition of finite type. Due to a result of Jeffery McNeal [15], this is sufficient for convex domains:

Theorem 2.4.7. *Let $\Omega \subseteq \mathbb{C}^n$ be a convex domain with $q \in \partial\Omega$. Then q is a point of finite type if and only if it is of finite line type.*

Now that we can limit our scope to complex lines, we can calculate a few examples explicitly.

Example 2.4.8. Consider $B_1(0) \subseteq \mathbb{C}^2$. This has defining function $\rho(z_1, z_2) = |z_1|^2+|z_2|^2-1$. Consider the line tangent to $\partial B_1(0)$ at $(1, 0)$ given by $\ell : \mathbb{C} \to \mathbb{C}^n$, $\ell(z) = (1, z)$. It follows

$$\begin{aligned} \rho \circ \ell(z) &= 1 + |z|^2 - 1 \\ &= z\overline{z} \end{aligned}$$

and thus $\nu_0(\rho \circ \ell) = 2$. Any other line is not going to be tangent to $(1, 0)$ and thus will not have matching first partial derivatives. Thus the line type at (1,0) is equal to 2, and therefore (1,0) is of finite type. By applying a rotation to the unit ball it follows that every boundary point is of finite type, so the entire domain is of finite type.

Example 2.4.9. Consider Δ^2, the polydisk in dimension 2. This domain has local defining function $\rho(z_1, z_2) = |z_1|^2 - 1$ around the point $(1, 0)$. Consider the line $\ell : \mathbb{C} \to \mathbb{C}^n$ given by $\ell(z) = (1, z)$. It follows

$$\rho \circ \ell(z) = 1^2 - 1 = 0$$

and all derivatives will be 0. Thus $\nu(\rho \circ \ell) = \infty$, and so $(1, 0)$ is a point of infinite type.

In this example, the complex line actually overlaps the boundary of our domain. This implies the existence of a holomorphic disk contained in the boundary of Ω, and in this instance suggested the existence of a point of infinite type. This idea will be crucial in later chapters, but this is not the *only* way that points of infinite type come about.

Example 2.4.10. The *exponentially-flat domain* is given by

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + 2\exp(-|z_2|^2) - 1 < 0\}.$$

Consider the point $(1, 0)$ and the complex line $l(z) = (1, z)$. Then

$$\rho \circ l(z) = 2\exp(-|z|^2).$$

It follows

$$\lim_{z \to 0} \frac{2\exp(-|z|^2)}{|z|^n} = 0$$

for all $n \geq 0$, and thus the point 0 is a point of infinite type. However,

$$l(\mathbb{C}) \cap \partial\Omega = \{(1, 0)\},$$

and thus there is no open set where they overlap.

2.5 The Method of Rescaling

Our main result will involve rescaling a domain Ω about a point of infinite type. This means constructing a sequence of affine transformations A_j and looking at the limit under the local Hausdorff notion of set convergence.

Definition 2.5.1. For a set $A \subset \mathbb{C}^n$, let $\mathcal{N}_\epsilon(A)$ be the $\epsilon-$neighborhood of A under the standard Euclidean distance. The *Hausdorff distance* between two compact sets is given by

$$d^H(A, B) = \inf\{\epsilon > 0 \mid A \subset \mathcal{N}_\epsilon(B) \text{ and } B \subset \mathcal{N}_\epsilon(A)\}.$$

We say that a sequence of open sets $\{A_k\}$ converges to A in the local Hausdorff topology if, for all $R > 0$,

$$\lim_{k \to \infty} d^H(\overline{A_k} \cap B_R(0), \overline{A} \cap B_R(0)) = 0.$$

For notation, if we write $A_k \to A$ we mean convergence in the local Hausdorf topology. If we write $(A_k, a_k) \to (A, a)$, we mean $A_k \to A$ in the local Hausdorff topology and $a_k \to a \in A$ in the Euclidean metric.

Example 2.5.2. Consider $\Omega = \{z \in \mathbb{C} : |z-i|^2 - 1 < 0\}$. This is Δ shifted up so 0 sits on the boundary. Let us rescale this domain by the sequence of affine transformations $A_n(z) = nz$. Take $\zeta \in \mathbb{C}$ such that $\text{Im}\zeta > 0$. It follows that for large enough n, $\zeta \in A_n\Omega$. Thus $\mathcal{H} \subseteq \hat{\Omega}$. Furthermore, consider $\zeta' \in \mathbb{C}$ such that $\text{Im}\zeta' < 0$. Because every point in Ω had positive imaginary part and we are rescaling by positive integers, $\zeta' \notin \hat{\Omega}$. Thus $\hat{\Omega} = \mathcal{H}$ in the local Haudsorff topology. While it turns out that Δ is biholomorphic to $\mathcal{H}$, this example does not show it.

The most useful results can be gleaned when $\hat{\Omega}$ is biholomorphic to our original domain Ω. The following theorem of Frankel [5] gives us a condition for when this is true.

Theorem 2.5.3. *Suppose that $\Omega \subseteq \mathbb{C}^n$ is a convex set that does not contain a complex line in its boundary, $K \subseteq \Omega$ is compact and $\{\phi_k\} \subseteq \operatorname{Aut}\Omega$. If there exists a sequence $\{p_k\} \subseteq K$ and complex affine maps A_k such that*

$$A_k(\Omega, \phi_k p_k) \to (\hat{\Omega}, p)$$

where $\hat{\Omega}$ does not contain a complex line in its boundary, then Ω is biholomorphic to $\hat{\Omega}$.

Example 2.5.4. Consider the unit disk with the following sequence of automorphisms:

$$\phi_k(z) = \frac{z + \frac{k-1}{k}}{1 + \frac{k-1}{k}z}$$

We define the *Frankel rescaling sequence* as $w_k(z) = [d\phi_k|_0]^{-1}(\phi_k(z) - \phi_k(0))$. It follows $\lim_{k\to\infty} \phi_k(0) = 1$, so this sequence pushes 0 towards 1. Then the Frankel rescaling map is given explicitly by the following equation:

$$
\begin{aligned}
w_k(z) &= [d\phi_k|_0]^{-1}(\phi_k(z) - \phi_k(0)) \\
&= \left(\frac{1}{1 - |\frac{k-1}{k}|^2}\right)\left(\frac{z + \frac{k-1}{k}}{1 + \frac{k-1}{k}z} - \frac{k-1}{k}\right) \\
&= \left(\frac{1}{1 - |\frac{k-1}{k}|^2}\right)\left(\frac{z + \frac{k-1}{k} - \frac{k-1}{k} - (\frac{k-1}{k})^2 z}{1 + \frac{k-1}{k}z}\right) \\
&= \frac{z(1 - (\frac{k-1}{k})^2)}{(1 - |\frac{k-1}{k}|^2)(1 + \frac{k-1}{k}z)} \\
&= \frac{z}{1 + \frac{k-1}{k}z}.
\end{aligned}
$$

It follows by our above calculations that $w(z) = \lim_{k\to\infty} w_k(z) = \frac{z}{1+z}$. Furthermore, $w_k(0) = 0$. This gives us an explicit biholomorphism between the unit disk and the half plane. The Frankel rescaling sequence can be defined on more arbitrary domains. We point the reader to [5] for further investigation into this rescaling sequence.

Chapter 3

Geometry of Bounded Convex Sets

In this chapter we will discuss a few tools that allow us to investigate bounded, convex sets. We will assume a smooth boundary for the duration of this chapter, but having a C^2 boundary is sufficient in several of the results. We start our discussion with the Schwarz lemma from single-variable complex analysis:

Lemma 3.0.1. *Let $f : \Delta \to \mathbb{C}$ be a holomorphic map such that $f(0) = 0$ and $|f(z)| \leq 1$ on Δ. Then $|f(z)| \leq |z|$ for all $z \in \Delta$ and $|f'(0)| \leq 1$. Moreover, if $|f(z)| = |z|$ for some non-zero z or $|f'(0)| = 1$, then $f(z) = cz$ for some $c \in \mathbb{C}$ with $|c| = 1$.*

It follows from this lemma that

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

Before explaining the significance of the above inequality we introduce the definition of Poincaré metric below.

Definition 3.0.2. The Poincaré metric on the unit disk is defined by

$$K_\Delta(z; v) = \frac{|v|}{1 - |z|^2}.$$

It yields a pseudodistance function given by

$$d_\Delta(z,w) = \tanh^{-1}\left|\frac{z-w}{1-z\overline{w}}\right| = \frac{1}{2}\log\left(\frac{|1-z\overline{w}|+|z-w|}{|1-z\overline{w}|-|z-w|}\right).$$

Remark 3.0.3. Recall that a *pseudodistance* or *pseudometric* is a distance/metric that can not distinguish points. In other words, it is possible for $d_\Delta(x,y)=0$ but $x\neq y$.

One might notice a similarity between the definition of Poincaré metric and the previous inequality. Indeed it becomes the case that on the unit disk, holomorphic functions are distance-decreasing with respect to the Poincaré metric. It follows that automorphisms are thus isometries. This property would be excellent to have on domains other than the unit disk, and this is what motivated the invention of the Kobayashi metric.

3.1 Kobayashi Pseudometric

The Kobayashi pseudometric is a generalization of the Poincaré metric to more arbitrary domains $\Omega\subseteq\mathbb{C}^d$. Besides the Euclidean metric, this will be our standard way of measuring distances. The original construction of this metric can be found in [12], but we will be using the following definition instead. Let $\mathrm{Hol}(U,V)$ denote the set of holomorphic functions from U to V.

Definition 3.1.1. Given $\Omega\subseteq\mathbb{C}^n$, $p\in\Omega$ and $v\in\mathbb{C}^n$, the *Kobayashi pseudometric* is given by

$$K_\Omega(p;v) = \inf\{|\zeta| \mid f\in\mathrm{Hol}(\Delta,\Omega), f(0)=p, f'(\zeta)=v\}$$

Given any two points $z,w\in\Omega$, the *Kobayashi pseudodistance* is given by

$$d_\Omega(z,w) = \inf_\gamma \int_0^1 K_\Omega(\gamma(t),\gamma'(t))dt$$

where $z,w\in\Omega$ and $\gamma:[0,1]\to\Omega$ is a curve such that $\gamma(0)=z$ and $\gamma(1)=w$.

It is not always the case that the Kobayashi pseudodistance is an actual distance (in the sense that $d_\Omega(x, y) = 0$ if and only if $x = y$). A domain is referred to as being *Kobayashi hyperbolic* if d_Ω is an actual distance and can distinguish points. Not every domain is Kobayashi hyperbolic as the following example will demonstrate.

Example 3.1.2. Consider $(\mathbb{C}, d_{\mathbb{C}})$. For any z, w, there exists a holomorphic map $f : \Delta \to \mathbb{C}$ such that $f(0) = z$ and $f(r) = w$ which is nothing but a rescaling and a rotation. However, we could adjust this rescaling to make r arbitrarily small, and thus $d_{\mathbb{C}}(z, w) = 0$. Therefore $\mathbb{C}$ is not Kobayashi hyperbolic.

Remark 3.1.3. A domain that does not contain any complex lines is Kobayashi hyperbolic. Thus bounded domains are Kobayashi hyperbolic. Because of this we limit our scope to such domains, though when dealing with rescalings we will need to verify our new blowup domain is still Kobayashi hyperbolic.

The value in equipping our domains with the Kobayashi pseudometric is found in its distance-decreasing property.

Proposition 3.1.4 (Distance-Decreasing Property)**.** *Let U, V be domains in $\mathbb{C}^n$ and $f : U \to V$ be a holomorphic map. Then*

$$K_V(f(p); f'(v)) \leq K_U(p; v)$$

and

$$d_V(f(z), f(w)) \leq d_U(z, w).$$

Proof. Suppose that $g : \Delta \to U$ is such that $g(0) = p$ and $g'(\zeta) = v$. Then $g \circ f(0) = f(p)$ and $g' \circ f'(\zeta) = f(v)$. This implies the following set inclusion:

$$\{|\zeta| \mid h \in \mathrm{Hol}(\Delta, U), h(0) = p, h'(\zeta) = v\} \subset \{|\zeta| \mid h \in \mathrm{Hol}(\Delta, V), h(0) = p, h'(\zeta) = v\}.$$

Thus $K_V(f(p); f'(v)) \leq K_U(p; v)$.

As for the second inequality, it follows that

$$
\begin{aligned}
d_V(f(z), f(w)) &= \inf_\gamma \int_0^1 K_V(f \circ \gamma(t), f' \circ \gamma'(t))dt \\
&\leq \inf_\gamma \int_0^1 K_U(\gamma(t), \gamma'(t))dt \\
&= d_U(z, w)
\end{aligned}
$$

and the proposition is proved. □

This implies in the following corollary that the Kobayashi pseudometric is a biholomorphic invariant.

Corollary 3.1.5. *If U, V are domains in $\mathbb{C}^n$ and $f : U \to V$ is a biholomorphism, then*

$$K_V(f(p), f'(v)) = K_U(p, v)$$

and

$$d_V(f(z), f(w)) = d_U(z, w).$$

Proof. Apply f^{-1} to the previous proposition. □

Lastly we mention one final result on the Kobayashi metric regarding product domains.

Proposition 3.1.6. *Let $U, V \subset \mathbb{C}^n$ be domains. Then for any $(u, v), (u', v') \in U \times V$,*

$$d_{U\times V}((u, v), (u', v')) = \max\{d_U(u, u'), d_V(v', v')\}.$$

While the Kobayashi metric is a useful tool to use, it is often difficult to calculate explicitly. Instead we describe it in terms of estimates in the Euclidean metric. Before introducting these estimates we describe some common notations.

3.2 Notations

We describe briefly some notations that we will be using regarding different metrics.

- $d_\Omega(p, q)$ denotes the Kobayashi pseudodistance between p and q in a domain Ω.
- $d_{EUC}(p, q)$ denotes the Euclidean distance between p and q in $\mathbb{C}^n$.
- $\delta_\Omega(p)$ denotes the Euclidean distance from p to $\partial\Omega$.
- $\delta_\Omega(p; v)$ denotes the Euclidean distance from p to $\partial\Omega$ in the direction v.
- Our notations for balls will be $B_r(a) = \{z \in \mathbb{C}^n : d_{EUC}(a, z) < r\}$ for Euclidean balls of radius r centered at a, and $K_r(a) = \{z \in \Omega : d_\Omega(a, z) < r\}$ for Kobayashi balls of radius r centered at a.

Remark 3.2.1. Given a real number $r > 0$ and any domain Ω, $K_r(a) \subseteq \Omega$ for $a \in \Omega$. This follows from the definition of the Kobayashi metric. A way to visualize the Kobayashi metric on a domain is to imagine the boundary of the domain being "infinitely far away." Points close to the boundary in a Euclidean sense are, in fact, far away in a hyperbolic sense.

3.3 Kobayashi Estimates

Given an arbitrary domain, it is often difficult to calculate the Kobayashi metric directly. Luckily there are a number of estimates relating the Kobayashi metric to the Euclidean metric we can use instead. The first is the following upper bound.

Proposition 3.3.1. *Let $\Omega \subseteq \mathbb{C}^n$ be a domain, $z \in \Omega$ and $v \in \mathbb{C}^n$. Then*

$$K_\Omega(z; v) \leq \frac{\|v\|}{\delta_\Omega(z; v)}$$

Proof. Let D be the largest open disk contained in $\{z + \mathbb{C}v\} \cap \Omega$. Then $\delta_\Omega(z; v) = \delta_D(z)$. Since translations, dilations and rotations are biholomorphisms we may assume the following:

$z = 0$, $v = (v_1, 0, \ldots, 0)$ and $D = \Delta$. Thus by the distance-decreasing property of the Kobayashi metric (with the inclusion map) it follows:

$$d_\Omega(z, v) \leq d_D(0, v) = |v_1| = \frac{|v_1|}{\delta_D(0)} = \frac{\|v\|}{\delta_\Omega(z, v)}$$

□

There also exists a similar lower bound in the case of convex domains, so we would be remiss not to include it. Before that we mention the definition of the Poincaré metric on the half-plane.

Definition 3.3.2. For $z \in \mathcal{H}$ and $v \in \mathbb{C}$, the Poincaré metric for the upper half plane $\mathcal{H}$ is given by

$$K_{\mathcal{H}}(z; v) = \frac{|v|}{2\mathrm{Im}z}$$

Since there exists a biholomorphism between the unit disk and the upper half plane, it can be shown that the two metrics coincide. With this definition, we can prove the following estimate:

Proposition 3.3.3. *Let $\Omega \subseteq \mathbb{C}^n$ be a convex domain, $z \in \Omega$ and $v \in \mathbb{C}^n$. Then*

$$K_\Omega(z; v) \geq \frac{1}{2} \frac{\|v\|}{\delta_\Omega(z; v)}$$

Proof. Take $x \in \partial\Omega$ so that $\delta_\Omega(z; v) = d_{EUC}(z, x)$. Via rotation and translation, we can assume $x = 0$, $z = (z_1, 0, \cdots, 0)$ and $v = (v_1, 0, \cdots, 0)$ and $\Omega \subset \{z \in \mathbb{C}^n : \mathrm{Im}z_1 > 0\}$. Let $\pi : \mathbb{C}^n \to \mathbb{C}$ be the projection onto the first coordinate. Then

$$K_\Omega(z; v) \geq K_{\pi\Omega}(z_1, v_1) \geq K_{\mathcal{H}}(z_1, v_1) = \frac{|v_1|}{2\mathrm{Im}z_1} \geq \frac{|v_1|}{2|z_1|} = \frac{|v|}{\delta_\Omega(z; v)}$$

□

This next estimate gives a lower bound when we have two colinear points [20]. This estimate is crucial in the proof of our main result.

Theorem 3.3.4. *Suppose $\Omega \subset \mathbb{C}^n$ is an open convex set and $p, q \in \Omega$. If L is the complex line containing p, q and $\xi \in L \setminus L \cap \Omega$ then*

$$\frac{1}{2} \log \left(\frac{\|p - \xi\|}{\|q - \xi\|} \right) \leq d_\Omega(p, q)$$

Proof. Since p, q and ξ are all colinear we can apply an affine transformation to assume $\xi = 0$, $p = (p_1, 0, \cdots, 0)$, $q = (q_1, 0, \cdots, 0)$, and $\Omega \subset \{(z_1, \cdots, z_d) \in \mathbb{C}^n : \mathrm{Im} z_1 > 0\}$. If $\pi_1 : \mathbb{C}^n \to \mathbb{C}$ is the projection onto the first coordinate then we have

$$\begin{aligned}
d_\Omega(p, q) &\geq d_{\pi_1(\Omega)}(p_1, q_1) \\
&\geq d_{\mathcal{H}}(p_1, q_1) \\
&= \frac{1}{2} \operatorname{arccosh}\left(1 + \frac{|p_1 - q_1|^2}{2 \mathrm{Im} p_1 \mathrm{Im} q_1}\right) \\
&\geq \frac{1}{2} \operatorname{arccosh}\left(1 + \frac{(|p_1| - |q_1|)^2}{2|p_1||q_1|}\right) \\
&= \frac{1}{2} \operatorname{arccosh}\left(\frac{|p_1|}{2|q_1|} + \frac{|q_1|}{2|p_1|}\right) \\
&= \frac{1}{2} \left| \log \left(\frac{|p_1|}{|q_1|} \right) \right| \\
&= \frac{1}{2} \log \left(\frac{\|p - \xi\|}{\|q - \xi\|} \right)
\end{aligned}$$

∎

We provide one last lower bound used in numerous results in [21] and is particularly useful for the field. Let $T_x^{\mathbb{C}} \partial\Omega$ denote the complex tangent plane of $\partial\Omega$ at $x \in \partial\Omega$.

Lemma 3.3.5. *Suppose $\Omega \subseteq \mathbb{C}^n$ is a bounded, convex domain with C^1 boundary, $x, y \in \partial\Omega$ and $T_x^{\mathbb{C}} \partial\Omega \neq T_y^{\mathbb{C}} \partial\Omega$. Then there exists $\epsilon > 0$ and $C \geq 0$ such that*

$$K_\Omega(p, q) \geq \frac{1}{2} \log \frac{1}{\delta_\Omega(p)} + \frac{1}{2} \log \frac{1}{\delta_\Omega(q)} - C$$

when $p, q \in \Omega$, $d_{EUC}(p, T_x^{\mathbb{C}} \partial\Omega) \leq \epsilon$ and $d_{EUC}(q, T_y^{\mathbb{C}} \partial\Omega) \leq \epsilon$.

3.4 Gromov Hyperbolicity

While there exists a notion of holomorphic curvature, we will instead be using a different type of curvature known as Gromov hyperbolicity. Rather than using methods of differential geometry, Gromov hyperbolicity uses a more rudimentary method to measure hyperbolic curvature by measuring the "thinness" of triangles in a given domain.

Definition 3.4.1. Let (X, d) be a metric space. A curve $\sigma : [a, b] \to X$ is called a *geodesic* if $d(\sigma(t_1), \sigma(t_2)) = |t_1 - t_2|$ for all $t_1, t_2 \in [a, b]$. We may sometimes denote $\sigma([a, b])$ as $[\sigma(a), \sigma(b)]$. If $\alpha, \beta \in X$ we may denote a geodesic between them as $[\alpha, \beta]$ if we are not concerned with the actual map.

Definition 3.4.2. Let (X, d) be a metric spae. A *geodesic triangle* is a choice of three points $p_i \in X$ and three geodesic segments $\sigma_i : [a_i, b_i] \to X$ for $i = 0, 1, 2$ such that $\sigma_i(a_i) = p_i$ and $\sigma_i(b_i) = p_{(i+1 \mod 3)}$.

Definition 3.4.3. Let (X, d) be a metric space. We refer to X as a *geodesic metric space* if given any two points $p, q \in X$ there exists a geodesic $\sigma : [a, b] \to X$ such that $\sigma(a) = p$ and $\sigma(b) = q$.

Definition 3.4.4. Let (X, d) be a metric space and consider a geodesic triangle formed by geodesics $[a_i, b_i]$ for $a_i, b_i \in X$ and $i = 0, 2, 3$. Pick $w_i \in [a_i, b_i]$. If there exists a $w_i' \in \bigcup_j [a_j, b_j] \setminus [a_i, b_i]$ such that $d(w_i, w_i') < \delta$ then we refer to this geodesic triangle as being *δ-thin.*

With these four definitions we can introduce the idea of Gromov hyperbolicity. In short, given a metric space X we can define a notion of global hyperbolic curvature without appealing to tools of differential geometry.

Definition 3.4.5. Let (X, d) be a proper geodesic metric space. X is *δ-hyperbolic* if every geodesic triangle is *δ-thin.* If (X, d) is δ-hyperbolic for some $\delta > 0$ we refer to the space as being *Gromov hyperbolic.*

Example 3.4.6. In the upper half plane $\mathcal{H}$ equipped with the Poincaré metric, geodesics are either half-circles with their centers on the $x-$axis or vertical rays. Consider the ideal triangle formed by the geodesics $x^2 + y^2 = 1$, $x = -1$, and $x = 1$. Then the inscribed circle is $x^2 + (y-2)^2 = 1$. Consider the diameter of this circle between the points $(0, 1)$ and $(0, 3)$. Since the geodesic between these two points has parameterization $\gamma(t) = (0, e^t)$ the distance between $(0, 1)$ and $(0, 3)$ is given by $|\log 3 - \log 1| = \log 3$. Because the diameter of the inscribed circle is $\log 3$, the largest distance between a point on any side and the other two sides is less than or equal to $\log 3$. As this is an ideal triangle let $\delta = \log 3$ and it follows $\mathcal{H}$ is $\log 3-$hyperbolic.

In practice we have an equivalent definition of Gromov hyperbolicity known as the *four-point condition.* It starts with the Gromov product detailed below.

Definition 3.4.7. Given a metric space (X, d), let $x, y, z \in X$. Then the Gromov Product $(x|y)_z$ is defined as:

$$(x|y)_z = \frac{1}{2}\Big(d(z, x) + d(z, y) - d(x, y)\Big)$$

Note that in the Euclidean, hyperbolic or spherical plane, $(x|y)_z$ is the distance between the point z and where the inscribed circle intersects $\overline{zy}$ or $\overline{zx}$ (they are the same distance). Thus the Gromov product gives us a notion of "how wide" a given triangle is in our metric space. If $(x|y)_z$ is large, it implies that z is a long distance from x and y, but x and y are very close. On the other hand, a small Gromov product implies that z is close to x and y, x and y of which are far apart.

Theorem 3.4.8. *Let (X, d) be a geodesic metric space. (X, d) is Gromov hyperbolic if and only if given any four points $x, y, z, w \in X$ there exists a $\delta > 0$ such that*

$$(x|y)_w \geq \min\{(x|z)_w, (z|y)_w\} - \delta.$$

This is referred to as the four-point condition.

Proof. Assume all geodesic triangles are δ-thin and take $x, y, z, w \in X$. Consider Δwxy. Pick points $a_w \in [x, y], a_x \in [w, y]$ and $a_y \in [w, x]$ with the following properties:

$$d(y, a_w) = (w|x)_y = d(y, a_x)$$
$$d(x, a_w) = (w|y)_x = d(x, a_y)$$
$$d(w, a_x) = (x|y)_w = d(w, a_y)$$

Since our triangle Δwxy is $\delta-$thin there exists a $t \in [w, x] \cup [w, y]$ such that $d(a_w, t) \leq \delta$. Without loss of generality let $t \in [w, y]$.

Case 1: $d(a_x, y) < d(t, y)$. By the triangle inequality and our choice of points it follows that

$$d(t, y) \leq d(t, a_w) + d(a_w, y) \leq \delta + d(a_w, y) = \delta + d(a_x, y).$$

This implies $d(t, y) - d(a_x, y) = d(a_x, t) \leq \delta$ since every point lies on a geodesic.

Case 2: $d(a_x, y) \geq d(t, y)$. By the triangle inequality and our choice of points it follows that

$$d(a_x, y) = d(a_w, y) \leq d(t, a_w) + d(t, y) \leq \delta + d(t, y).$$

This implies $d(a_x, y) = d(a_x, t) \leq \delta$ since all these points lie on a geodesic. This means that

$$d(a_w, a_x) \leq d(a_w, t) + d(t, a_x) \leq 2\delta.$$

Now consider the point z and pick $t_1 \in [y, z]$ and $t_2 \in [x, z]$ closest to a_w. This means by the $\delta-$thin property, the minimum of these distances is less than δ. Putting everything together leads us to the following inequality:

$$\begin{aligned} \min\{d(w, t_1), d(w, t_2)\} &\leq \min\{d(a_w, t_1), d(a_w, t_2)\} + d(w, a_w) \\ &\leq \delta + d(w, a_w) \\ &\leq \delta + d(w, a_x) + d(a_w, a_x) \leq 3\delta + (x|y)_w. \end{aligned}$$

Now it suffices to show that $\min\{(x|z)_w, (y|z)_w\} \leq \min\{d(w,t_1), d(w,t_2)\}$. Observe that

$$\begin{aligned}
(x|z)_w &= \frac{1}{2}(d(x,w) + d(z,w) - d(x,z)) \\
&= \frac{1}{2}(d(x,w) - d(x,t_1) + d(z,w) - d(z,t_1)) \\
&\leq \frac{1}{2}(2d(w,t_1)) \\
&= d(w,t_1),
\end{aligned}$$

and

$$\begin{aligned}
(y|z)_w &= \frac{1}{2}(d(y,w) + d(z,w) - d(y,z)) \\
&= \frac{1}{2}(d(y,w) - d(y,t_2) + d(z,w) - d(z,t_2)) \\
&\leq \frac{1}{2}(2d(w,t_2)) \\
&= d(w,t_2).
\end{aligned}$$

Thus it follows

$$(x|y)_w \geq \min\{(x|z)_w, (y|z)_w\} - 3\delta.$$

Now we prove the reverse implication. Let the four-point condition hold for any four points in X. Let Δxyz be an arbitrary geodesic triangle. Without loss of geneality let $w \in [x,y]$ and suppose that $d(w,[x,z]) \leq d(w,[y,z])$. In the same manner as before we choose a_w, a_x, a_z. Again, without loss of generality we can assume $(z|a_w)_w \leq (x,a_w)_w$. We now choose three more points b_z, b_w and b_{aw} on the triangle Δwza_w in the same manner we chose a_w, a_x and a_z.

By the four-point condition we have

$$(x|y)_w \geq \min\{(x|z)_w, (y|z)_w\} - \delta = (x|z)_w - \delta,$$

but $w \in [x,y]$, so we must have $(x|y)_w = 0$. Thus $(x|z)_w \leq \delta$.

Now observe

$$\begin{aligned}
2d(a_x, b_{aw}) &= 2(a_w|z)_w - 2(x|z)_w \\
&= d(a_w, w) + d(x, z) - d(a_w, z) - d(x, w) \\
&= d(a_w, x) + d(w, b_z) + d(a_w, b_z) - d(w, a_z) - d(a_w, x) \\
&= d(a_x, b_{aw}) + d(a_w, b_w).
\end{aligned}$$

This implies $d(a_x, b_{aw}) = d(a_w, b_w)$. It follows

$$\begin{aligned}
d(w, [x, z]) &\leq d(w, a_w) \\
&= d(w, b_z) + d(b_z, a_w) \\
&= d(w, b_{aw}) + d(a_w, b_w) \\
&= d(w, a_x) + d(b_{aw}, a_x) + d(a_w, b_w) \\
&= d(w, a_x) + 2d(b_{aw}, a_x) \\
&\leq (x|z)_w + 2\delta \\
&\leq 3\delta.
\end{aligned}$$

Thus every geodesic triangle is 3δ-thin. □

The applications of Gromov hyperbolicity are best showcased by the following theorem of Andrew Zimmer [20].

Theorem 3.4.9. *Suppose $\Omega \subseteq \mathbb{C}^n$ is a bounded domain with smooth boundary. Then (Ω, d_Ω) is Gromov hyperbolic if and only if Ω has finite type in the sense of D'Angelo.*

We finish our section on Gromov hyperbolicity by providing an example of a space that is *not* Gromov hyperbolic.

Example 3.4.10. Consider $\Delta \times \Delta$ equipped with the Kobayashi metric. We note that in each coordinate we are working with the Poincaré metric ρ.

Let $p_n = (1-\frac{1}{n}, -1+\frac{1}{n})$ and $q_n = (-1+\frac{1}{n}, -1+\frac{1}{n})$. Then there is a sequence of geodesic triangles $\Delta 0 p_n q_n$. By definition of p_n, q_n the geodesic midpoint of $[p_n, q_n]$ is $z_n = (0, -1+\frac{1}{n})$. We claim that $d(z_n, [0,p_n] \cup [0,q_n]) \to \infty$.

Let $s_n = (s_n^1, s_n^2) \in [0,p_n] \cup [0,q_n]$. Without loss of generality assume $s_n \in [0,p_n]$. We observe the points $\pm(1+\frac{1}{n})$ lie along the geodesic $\gamma : \mathbb{R} \to \Delta$ parameterized as $\gamma(t) = \frac{e^t-1}{e^t+1}$ which means s_n has the form

$$s_n = \left(\frac{e^t-1}{e^t+1}, \frac{e^{-t}-1}{e^{-t}+1}\right)$$

for some $0 \leq t \leq \log(2n-1)$. It follows

$$\begin{aligned}
d(z_n, [0,p_n] \cup [0,q_n]) &= \inf_{s_n}\{d(z_n, s_n)\} \\
&= \inf_{s_n}\{\max\{\rho(0, s_n^1), \rho(-1+\frac{1}{n}, s_n^2)\}\} \\
&= \inf_{0 \leq t \leq \log(2n-1)} \left\{ \max\{\rho(0, \frac{e^t-1}{e^t+1}), \rho(-1+\frac{1}{n}, \frac{e^{-t}-1}{e^{-t}+1})\} \right\} \\
&= \inf_{0 \leq t \leq \log(2n-1)} \{\max\{t, \log(2n-1)-t\}\} \\
&= \min\left\{ \inf_{0.5\log(2n-1) \leq t \leq \log(2n-1)} \{t\}, \inf_{0 \leq t \leq 0.5\log(2n-1)} \{\log(2n-1)-t\} \right\} \\
&= 0.5\log(2n-1)
\end{aligned}$$

which goes to ∞ as $n \to \infty$. Thus this geodesic triangle is not $\delta-$thin for any δ. Our space is not Gromov hyperbolic.

3.5 Boundary-Approaching Sequences

We now discuss properties of sequences of points that approach the boundary of a bounded, convex domain. Because under the Kobayashi metric such a sequence is divergent (even if it is convergent in the Euclidean metric) it is imperative we develop ways to control that divergence.

Definition 3.5.1. Suppose (X, d) is a metric space and $I \subset \mathbb{R}$ an interval. A curve $\sigma : I \to X$ is an $(A, B)-$quasi-geodesic if

$$\frac{1}{A}|t-s| - B \leq d(\sigma(s), \sigma(t)) \leq A|t-s| + B$$

for all $s, t \in I$.

The following two results are taken directly from [21]. We reproduce the proofs here because of their relevance to our main result.

Proposition 3.5.2. *Suppose $\Omega \subseteq \mathbb{C}^n$ is a bounded convex domain with smooth boundary. Then there exists $\epsilon > 0$ and $K \geq 1$ so that if $x \in \partial\Omega$ then the curve $\sigma_x : \mathbb{R}_{\geq 0} \to \Omega$ given by*

$$\sigma_x(t) = x + \epsilon e^{-2t} n_x$$

is a $K-$quasi geodesic.

Proof. Let $x \in \partial\Omega$. Because the domain is convex and the boundary is smooth, there exists some disk $D_x \subseteq \Omega$, $D_x \cong \Delta$, such that $\partial D_x \cap \partial\Omega = \{x\}$ and the normal real line attached to x is the diameter of D_x. Let the radius of D_x be ϵ_x and consider $\inf_{x\in\partial\Omega}\{\epsilon_x\}$. Because $\partial\Omega$ is compact, if the infimum were equal to zero then there would be some point $x \in \partial\Omega$ where this is attained, which would contradict our first statement. Thus there exists some $\epsilon > 0$ such that for any $x \in \partial\Omega$, we can take D_x to have radius ϵ. If we let $\sigma_x(t) = x + \epsilon e^{-2t} n_x$ then this describes the path along the radius connecting x to the center of the disk.

Because D_x is biholomorphic to the unit disk Δ, it follows because of the distance-decreasing property of the Kobayashi metric that

$$d_\Omega(\sigma_x(t), \sigma_x(s)) = d_\Omega(x + \epsilon e^{-2t} n_x, x + \epsilon e^{-2s} n_x) \leq d_\Delta(1 - e^{-2t}, 1 - e^{-2s}).$$

We can assume without loss of generality that $0 \leq s < t$.

By the definition of the Kobayashi metric on Δ, we know

$$\begin{aligned}
d_\Delta(1-e^{-2t}, 1-e^{-2s}) &= \frac{1}{2}\log\left(\frac{(1+(1-e^{-2t}))(1-(1-e^{-2s}))}{(1-(1-e^{-2t}))(1+(1-e^{-2s}))}\right)\\
&= \frac{1}{2}\log\left(\frac{2-e^{-2t}}{2-e^{-2s}}\right) + \frac{1}{2}\log\left(\frac{e^{-2s}}{e^{-2t}}\right)\\
&\leq \frac{1}{2}\log 2 + \log(e^{t-s})\\
&= \frac{1}{2}\log 2 + |t-s|.
\end{aligned}$$

On the other hand, we note that by Theorem 3.3.4,

$$\begin{aligned}
d_\Omega(\sigma_x(t), \sigma_x(s)) &\geq \frac{1}{2}\left|\log\left(\frac{d_{Euc}(T_x^{\mathbb{C}}\partial\Omega, \sigma_x(t))}{d_{Euc}(T_x^{\mathbb{C}}\partial\Omega, \sigma_x(s))}\right)\right|\\
&= \frac{1}{2}\left|\log\left(\frac{x-(x+\epsilon e^{-2t}n_x)}{x-(x+\epsilon e^{-2t}n_x)}\right)\right|\\
&= \frac{1}{2}\left|\log\left(\frac{e^{-2t}}{e^{-2s}}\right)\right|\\
&= |s-t|\\
&\geq |s-t| - \frac{1}{2}\log 2.
\end{aligned}$$

So let $K = \frac{1}{2}\log 2$. Then σ_x is a $K-$quasi-geodesic for all $x \in \partial\Omega$. □

Theorem 3.5.3. *Suppose $\Omega \subseteq \mathbb{C}^d$ is a bounded, convex domain with a smooth boundary, $o \in \Omega$ and $p_n, q_m \in \Omega$ are sequences such that $p_n \to x \in \partial\Omega$ and $q_m \to y \in \partial\Omega$. If $x = y$ then*

$$\lim_{n,m\to\infty}(p_n|q_m)_o = \infty.$$

Furthermore, if

$$\limsup_{n,m\to\infty}(p_n|q_m) = \infty$$

then $T_x^{\mathbb{C}}\partial\Omega = T_y^{\mathbb{C}}\partial\Omega$.

Proof. Suppose $x = y$. By Proposition 3.5.2 there exists a fixed global K, ϵ, and R such that given any $z \in \partial\Omega$, there exists a K-quasi-geodesic

$$\sigma_z(t) = z + \epsilon e^{-2t} n_z.$$

Furthermore, given $z, z' \in \partial\Omega$, $d(\sigma_z(0), \sigma_{z'}(0)) \leq R$. As in the hypothesis of the theorem, let $p_n \to x$ and $q_m \to y$.

Let $\hat{p}_n$ be the closest point on the boundary to p_n. Then there exists s_n, t_m such that $p_n = \sigma_{\hat{p}_n}(s_n)$ and $q_m = \sigma_{\hat{q}_m}(t_m)$. Since $p_n, q_m \to x$ then $s_n, t_m \to \infty$. Let us fix $T > 0$. Then it follows for $s_n, t_m > T$,

$$\begin{aligned}
2(p_n|q_m)_o &= d_\Omega(p_n, o) + d_\Omega(q_m, o) - d_\Omega(p_n, q_m) \\
&\geq d_\Omega(p_n, \sigma_{\hat{p}_n}(0)) - d_\Omega(o, \sigma_{\hat{p}_n}(0)) + d_\Omega(q_m, o) - d_\Omega(p_n, q_m) \\
&\geq s_n - K - R + t_m - K - R - d_\Omega(p_n, q_m) \\
&= s_n + t_m - 2K - 2R - d_\Omega(p_n, q_m).
\end{aligned}$$

However,

$$\begin{aligned}
d_\Omega(p_n, q_m) &\leq d_\Omega(p_n, \sigma_{\hat{p}_n}(T)) + d_\Omega(\sigma_{\hat{p}_n}(T), \sigma_{\hat{q}_m}(T)) + d_\Omega(\sigma_{\hat{q}_m}(T)), q_m) \\
&\leq |s_n - T| + K + d_\Omega(\sigma_{\hat{p}_n}(T), \sigma_{\hat{q}_m}(T)) + |t_m - T| + K,
\end{aligned}$$

which implies $(p_n|q_n)_o \geq T - R - 2K - \frac{1}{2} d_\Omega(\sigma_{\hat{p}_n}(T), \sigma_{\hat{q}_m}(T))$. But the last term tends to zero since $\hat{p}_n$ and $\hat{q}_m$ are both approaching x. Since T completely arbitrary, this means $(p_n|q_m)_o \to \infty$.

Now suppose $\limsup_{n,m\to\infty}(p_n|q_m)_o = \infty$. For sake of contradiction, suppose $T_x^{\mathbb{C}}\partial\Omega \neq T_y^{\mathbb{C}}\partial\Omega$. Then by the earlier construction,

$$\begin{aligned}
d_\Omega(o, p_n) &\le d(o, \sigma_{\hat{p}_n}(0)) + d_\Omega(\sigma_{\hat{p}_n}(0), p_n) \\
&\le R + \frac{1}{2}\log\frac{\epsilon}{\delta_\Omega(p_n)} \\
&= R + \epsilon + \frac{1}{2}\log\frac{1}{\delta_\Omega(p_n)}.
\end{aligned}$$

Similarly,

$$d_\Omega(o, q_n) \le R + \epsilon + \frac{1}{2}\log\frac{1}{\delta_\Omega(q_n)},$$

but there exists a C by Lemma 3.3.5 such that

$$d_\Omega(p_n, q_m) \ge \frac{1}{2}\log\frac{1}{\delta_\Omega(p_n)} + \frac{1}{2}\log\frac{1}{\delta_\Omega(q_n)} - C$$

which together imply $2(p_n|q_m)_o \le 2R + 2\epsilon + C$. This is a contradiction since $(p_n|q_m)_o$ tends to infinity. $\square$

3.6 Two Useful Results on Bounded Convex Domains

To finish this chapter we provide two results given a bounded, convex domain. We highlight these results because they are crucial in the proof of our main result. The first is a result by Lee, Thomas and Wong [14].

Theorem 3.6.1. *Let $\Omega \subseteq \mathbb{C}^n$ be a C^2 bounded convex domain. Suppose there exists a sequence $\{\phi_j\} \subseteq \operatorname{Aut}\Omega$ such that $\{\phi_j(z)\}$ accumulates non-tangentially at some boundary point for all $z \in \Omega$. Then there does not exist a non-trivial analytic disk on $\partial\Omega$ passing through any orbit accumulation point on the boundary.*

Proof. We point the reader to [14] for a proof of this result. $\square$

We also have the following fact about bounded, convex domains proven by Zimmer [21]. This fact will be what we ultimately contradict in the main result.

Theorem 3.6.2. *Suppose $\Omega \subseteq \mathbb{C}^n$ is a bounded, convex domain with smooth boundary. Then there does not exist a holomorphic map $f : \Delta \times \Delta \to \Omega$ and an $E \geq 0$ such that for all $z, w \in \Delta$,*

$$d_\Omega(f(z,0), f(w,0)) = d_\Delta(z,w)$$

and for all $r, s \in [0,1)$,

$$d_\Delta(r,s) - E \leq d_\Omega(f(0,r), f(0,s)) \leq d_\Delta(r,s) + E$$

The idea of this theorem is that if we have a holomorphic embedding of $\Delta \times \Delta$ into a domain Ω, such an embedding cannot be isometric in one variable and also isometric up to some error term E along a radius in the other variable.

Proof. Let $r, r' \in [0,1)$. Fix $\theta \in [0, 2\pi)$. It follows

$$\begin{aligned}(f(re^{i\theta},0)|f(r'e^{i\theta},0)_{f(0,0)} &\geq d_\Delta(re^{i\theta},0) + d_\Delta(r'e^{i\theta},0) - d_{\Delta\times\Delta}((re^{i\theta},0),(r'e^{i\theta},0)) \\ &= d_\Delta(re^{i\theta},0) + d_\Delta(r'e^{i\theta},0) - d(re^{i\theta}, r'e^{i\theta}) \\ &\geq \min\{d_\Delta(re^{i\theta},0), d_\Delta(r'e^{i\theta})\}.\end{aligned}$$

Thus it follows $\lim_{r,r'\to 1}(f(re^{i\theta},0)|f(r'e^{i\theta},0)_{f(0,0)} = \infty$. Thus by the lemma above, there exists some $x_\theta \in \partial\Omega$ such that $\lim_{r\to 1} f(re^{i\theta},0) \in T^{\mathbb{C}}_{x_\theta}\partial\Omega \cap \partial\Omega$ and $\lim_{r'\to 1} f(r'e^{i\theta},0) \in T^{\mathbb{C}}_{x_\theta}\partial\Omega \cap \partial\Omega$. No matter how we approach along the radius, in the image of f that sequence will always approach the same tangent plane. The same proof works symmetrically along the real line in the second variable, except with an error term E.

Now take $x_\theta, y \in \partial\Omega$ such that $\lim_{r\to 1} f(re^{i\theta},0) \in T^{\mathbb{C}}_{x_\theta}\partial\Omega \cap \partial\Omega$ and

$$\lim_{s\to 1} f(0,s) \in T^{\mathbb{C}}_y\partial\Omega \cap \partial\Omega.$$

It follows

$$
\begin{aligned}
(f(re^{i\theta},0)|f(0,s))_{f(0,0)} &\geq d_\Delta(re^{i\theta},0)+d_\Delta(0,s)-E-d_{\Delta\times\Delta}((re^{i\theta},0),(0,s)) \\
&= d_\Delta(re^{i\theta},0)+d_\Delta(0,s)-\max\{d_\Delta(re^{i\theta},0),d_\Delta(0,s)\}-E \\
&\geq \min\{d_\Delta(re^{i\theta},0),d_\Delta(0,s)\}-E
\end{aligned}
$$

and thus

$$\lim_{r,s\to 1}(f(re^{i\theta},0)|f(0,s))_{f(0,0)}=\infty.$$

Therefore by the lemma above, $T^{\mathbb{C}}_{x_\theta}\partial\Omega = T^{\mathbb{C}}_{y}\partial\Omega$ for all θ. This means that there exists a single tangent plane $T^{\mathbb{C}}_{y}\partial\Omega$ such that given any straight-line approach from 0 to $\partial\Delta$ in the first coordinate, its image approaches $T^{\mathbb{C}}_{y}\partial\Omega\cap\partial\Omega$. We can assume without loss of generality that $T^{\mathbb{C}}_{y}\partial\Omega=\{(z_1,\cdots,z_d)\in\mathbb{C}^d : z_1=0\}$ and the Imz_1 axis points normally inward at y. Thus, every point $\boldsymbol{z}\in\Omega$ has positive imaginary z_1 component by convexity.

Let $f=(f_1,\cdots,f_d)$. It follows that, if we fix $a\in\Delta$, $f_1(z,a)$ is a function of one complex variable z. Furthermore, because Ω is bounded, $f_1(z,a)$ is bounded. Thus, by the Cauchy Integral Formula and Dominated Convergence Theorem,

$$f_1(z,a)=\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}\frac{f_1(re^{it},a)}{re^{it}-z}re^{it}dt=\frac{1}{2\pi}\int_0^{2\pi}\lim_{r\to 1}\frac{f_1(re^{it},a)}{re^{it}-z}re^{it}dt=0.$$

However, f_1 cannot be the zero function as every point in Ω must have positive imaginary z_1 component. This is a contradiction. Thus, no such map can exist. □

Chapter 4

Automorphism Orbits and the Greene-Krantz Conjecture

In the previous chapters we laid out several avenues for studying the structure of domains. We have methods that are based on the boundary, such as pseudoconvexity and finite type, and we have methods that depend more on the interior such as Gromov hyperbolicity. In this chapter we study another such interior avenue: the automorphism group. By looking at what sorts of self-biholomorphisms exist for a certain domain, we can make conclusions about its geometry. Furthermore, automorphisms prove to be crucial in the construction of biholomorphisms between domains and their rescalings.

4.1 Automorphism Groups

In this section we describe a useful connection between the orbits of automorphism groups and their compactness. We first provide the following definitions.

Definition 4.1.1. Let $\operatorname{Aut}\Omega$ denote the set of automorphisms of a domain $\Omega \subseteq \mathbb{C}^n$. It is a straightforward exercise to prove that $\operatorname{Aut}\Omega$ is a group under composition, and thus we will refer to $\operatorname{Aut}\Omega$ as *the automorphism group* of Ω.

Definition 4.1.2. Let G be a group and X a topological space. Let $x \in X$. The orbit of x under the action σ is the set

$$\{y \in X : \sigma(g, x) = y \text{ for some } g \in G\}.$$

Given a domain Ω, the automorphism group $\operatorname{Aut}\Omega$ acts on Ω by the mapping $(\phi, z) \mapsto \phi(z)$.

Example 4.1.3. Consider Δ. By the Schwarz lemma, automorphisms of Δ are generated by maps of the form

$$\phi(z) = e^{i\theta}\frac{z-a}{1-\overline{a}z}$$

where $|a| < 1$, $\theta \in [0, 2\pi)$. The automorphisms of the polydisk Δ^n are the maps whose component functions are automorphisms of Δ.

Example 4.1.4. The automorphisms of the unit ball $B_1(0) \subseteq \mathbb{C}^n$ are given by the following two classes of generators. The first generator are mappings of the form

$$\phi(z) = \left(\frac{z_1 - a}{1-\overline{a}z_1}, \frac{\sqrt{1-|a|^2}z_2}{1-\overline{a}z_1}, \cdots, \frac{\sqrt{1-|a|^2}z_n}{1-\overline{a}z_1}\right).$$

We draw the reader's attention to the denominator of each term, which is a repetition of z_1. This is the crucial difference between the unit ball and the polydisk. The polydisk has no inter-dependence between its variables, but the unit ball does (though the z_1 direction is arbitrary).

Our second generators are are complex rotations in a fixed variable. The automorphisms of the unit ball are thus generated by these two types of maps.

It is not always possible to calculate automorphism groups so explicitly, but on occasion we can get close. We can use our knowledge of the unit ball to describe the automorphisms of the exponentially flat domain in the following example.

Example 4.1.5. Let $\Omega = \{(z,w) \in \mathbb{C}^2 : |z|^2 + 2\exp(-1/|w|^2) < 1\}$. This is the exponentially flat domain.

First, we quickly demonstrate this domain is circular. Take $(z,w) \in \Omega$. If $|z'| < |z|$, then

$$|z'|^2 + 2\exp(-|w|^{-2}) < |z|^2 + 2\exp(-|w|^{-2}) < 1.$$

Similarly, because $2\exp(-1/|w|^2)$ is an increasing function of $|w|$, it follows $|w'| < |w|$ implies

$$|z|^2 + 2\exp(-|w'|^{-2}) < |z|^2 + 2\exp(-|w|^{-2}) < 1.$$

So for any $(z,w) \in \Omega$, it follows $(z',w') \in \Omega$ for all z' and w' such that $|z'| < |z|$ and$|w'| < |w|$. Thus this domain is circular.

Because our domain is circular, by Bell and Boas [2] we know automorphisms extend smoothly to the boundary. Let ϕ_k refer to the extended automorphism. Furthermore, automorphisms cannot send weakly pseudoconvex boundary points to strongly pseudoconvex ones. Let S denote the set of weakly pseudoconvex boundary points. Because S is precisely the set of $(z,w) \in \partial\Omega$ such that $w = 0$, we can conclude $\phi_k(S) = S$.

Because $\phi_k(S) = S$, it must also be true that $\phi(\{(z,0) : |z|^2 < 1\}) = \{(z,w) : |z|^2 < 1\}$ as S is the boundary. Thus, if we restrict ϕ_k to the z variable it must be an automorphism of the unit disk. So we can conclude, after perhaps composing ϕ_k with a rotation in the z variable, it has the following form:

$$\phi_k(z,0) = \left(\frac{z - a_k}{1 - \overline{a_k}z}, 0\right)$$

for some $a_k \in \mathbb{C}$, $|a_k| < 1$.

By [13] each ϕ_k commutes with rotations in the w variable. Thus, vertical disks $f_\alpha = \{(\alpha, \zeta) : |\alpha|^2 + \psi(|\zeta|) < 1\}$ are sent to other vertical disks under ϕ_k. Furthermore, the centers are sent to each other.

Because biholomorphisms of different-sized one-variable disks can only rotate and scale, we can conclude ϕ_k must have the form:

$$\phi(z, w) = \left(\frac{z - a_k}{1 - \overline{a_k} z}, w\lambda_k(z)\right)$$

Where λ_k is holomorphic and by the defining function of Ω,

$$|\lambda_k(z)| = \frac{\psi^{-1}\left(1 - \left|\frac{z - a_k}{1 - \overline{a_k} z}\right|^2\right)}{\psi^{-1}(1 - |z|^2)}$$

as λ_k must rescale each disk to the correct radius. By the definition of ψ, it follows

$$\psi^{-1}(t) = \frac{1}{\sqrt{-\log(t/2)}}.$$

Thus we have a solid grasp of the behavior of the automorphisms of the exponentially flat domain.

Definition 4.1.6. Let $\Omega \subseteq \mathbb{C}^n$ be a domain and $p \in \Omega$. We say that $q \in \mathbb{C}^n$ is an *orbit accumulation point* of $\operatorname{Aut}\Omega$ if there is a sequence $\{\phi_k\} \subset \operatorname{Aut}\Omega$ such that $\phi_k(p) \to q$. If $q \in \partial\Omega$ then we say q is a *boundary orbit accumulation point.* We may also refer to the sequence $\{\phi_k\}$ as a *boundary-accumulating automorphism orbit.*

The following theorem follows from classical results of Cartan.

Theorem 4.1.7. *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain.* $\operatorname{Aut}\Omega$ *admits a boundary orbit accumulation point if and only if* $\operatorname{Aut}\Omega$ *is non-compact.*

Proof. Suppose there exists $p \in \Omega$ and a sequence $\{\phi_k\} \subset \operatorname{Aut}\Omega$ such that $\phi(p) \to q \in \partial\Omega$. Suppose for the sake of contradiction that $\operatorname{Aut}\Omega$ is compact. Then there should exist a subsequence $\{\phi_{k_j}\}$ such that $\phi_{k_j} \to \phi \in \operatorname{Aut}\Omega$. It follows

$$\phi(p) = \lim_{j\to\infty} \phi_{k_j}(p) = q \in \partial\Omega,$$

but $\phi \in \operatorname{Aut}\Omega$. This is a contradiction, and thus the automorphism group must be non-compact.

Now for the reverse direction. Suppose $\operatorname{Aut}\Omega$ is non-compact. Then there exists a sequence $\{\phi_k\}$ such that $\phi_k \to \phi \notin \operatorname{Aut}\Omega$. It is a theorem of Cartan that either $\phi \in \operatorname{Aut}\Omega$ or $\phi(\Omega) \subseteq \partial\Omega$. See Narasimhan [16] for more details. Therefore, for any $z \in \Omega$,

$$\lim_{k\to\infty} \phi_k(z) = \phi(z) \in \partial\Omega$$

and so $\operatorname{Aut}\Omega$ admits a boundary orbit accumulation point. □

Given a domain Ω, we are inclined to characterize them by properties of their automorphism group. Two major classes are those domains who have *compact* automorphism group and those who have *non-compact* automorphism group. Thanks to the previous result the latter case can be further broken down by properties of the accumulating sequence. This leads to the Greene-Krantz conjecture and the notion of non-tangential convergence.

4.2 The Greene-Krantz Conjecture

An important conjecture in the pursuit of a generalized Riemann Mapping Theorem is the following conjecture by Greene and Krantz.

Conjecture 4.2.1 (Greene-Krantz)**.** *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain with smooth boundary. If $q \in \partial\Omega$ is a boundary orbit accumulation point for* $\operatorname{Aut}\Omega$ *then $\partial\Omega$ is of finite type at the point q.*

We can classify the bounded domains with smooth boundary into two classes, those with compact automorphism group and those with non-compact automorphism group. By the result of Cartan, if the Greene-Krantz conjecture proved to be true then it would imply every bounded domain with smooth boundary and noncompact automorphism group is of finite type. In two dimensions this would classify every such domain, and in higher dimensions it would provide an important tool for further classification.

There are numerous results that suggest this conjecture to be true. The first is of Bun Wong [19].

Theorem 4.2.2. *Let $\Omega \subseteq \mathbb{C}^2$ be a bounded domain and $\{\phi_k\} \subset \operatorname{Aut}\Omega$ be such that* (1) *$W = \{\lim_{k\to\infty} \phi_k(\Omega)\}$ is a complex variety of positive dimension contained in $\partial\Omega$,* (2) *W is contained in an open subset $U \subseteq \partial\Omega$ such that ∂U is C^1 and there is an open set $N \subset \mathbb{C}^2$ such that $N \cap \partial\Omega = U$ and $N \cap \Omega$ is convex.* (3) *There exists a point $x \in \Omega$ such that $\{\phi_k(x)\}$ converges to $p \in W \subseteq \partial\Omega$ non-tangentially. Then Ω is biholomorphic to Δ^2.*

And the next by Kim [11].

Theorem 4.2.3. *Suppose that $\Omega \subseteq \mathbb{C}^2$ is a bounded convex domain with piecewise-smooth Levi flat boundary. If $\operatorname{Aut}\Omega$ is non-compact then Ω is biholomorphic to Δ^2.*

We also mention this classical result which provides a characterization of strongly pseudoconvex domains in $\mathbb{C}^n$.

Theorem 4.2.4. *If $\Omega \subseteq \mathbb{C}^n$ is a strongly pseudoconvex bounded domain with non-compact automorphism group, then Ω is biholomorphic to the n-dimensional unit ball.*

Because of the above theorem, when studying domains with non-compact automorphism groups we can restrict ourselves to weakly pseudoconvex domains. In addition to also providing a strong characterization of domains in $\mathbb{C}^n$, because the unit ball is of finite type it provides further evidence that the Greene-Krantz conjecture is true.

The previous discussion suggests the conjecture to be true, but none prove it. A proof of this conjecture would provide a valuable tool for further characterizing domains in $\mathbb{C}^n$ and ultimately generalizing the Riemann Mapping Theorem. Our main result provides a partial proof of Greene-Krantz.

4.3 Nontangential Convergence

Since every domain with non-compact automorphism group has a boundary-accumulating automorphism orbit, we can study such domains by considering the properties of that accumulating sequence.

Given a domain Ω equipped with the Kobayashi metric, consider a sequence of points $\{p_k\}$ approaching the boundary. When rescaling a domain it is often difficult to deal with the tangential direction. For this reason the assumption of normal convergence allows us to disregard the tangential direction entirely. However, we can generalize normal convergence to *nontangential* convergence so we do not have quite as restrictive a class of domains.

Definition 4.3.1. For a domain $\Omega \subseteq \mathbb{C}^n$ with C^1 boundary, a sequence $\{q_k\} \subset \Omega$ and a point $q \in \partial\Omega$, we say that $q_k \to q$ *nontangentially* if for sufficiently large $k > 0$,

$$q_k \in \Gamma_\alpha(q) = \{z \in \Omega : \|z - q\| \leq \alpha\delta_\Omega(z)\}$$

for some $\alpha > 1$. We say that $q_k \to q$ *normally* if q_k approaches q along the real normal line to $\partial\Omega$ at q.

Remark 4.3.2. We encourage the reader to investigate the connection between this definition and proposition 3.3.1 in the previous chapter.

We can visualize this in terms of angles and a cone-like object pointing inwards. The following lemma makes this more clear.

Lemma 4.3.3. *Let $\Omega \subset \mathbb{C}^n$ be a convex domain with C^1 boundary. Let $z \in \Omega$ and $q' = q + tn_q$ for some $t > 0$. Then*

$$\Gamma_\alpha(q) \subset \left\{z \in \Omega : 0 \leq \angle zqq' \leq \arccos\left(\frac{1}{\alpha}\right)\right\}$$

Proof. We can assume without loss of generality that $q = 0$, $n_q = (i, 0, \cdots, 0)$ and $\Omega \subset \mathcal{H} \times \mathbb{C}^{n-1}$. Then $\delta_\Omega(z) \leq \delta_{\mathcal{H}\times\mathbb{C}^{d-1}}(z) = \mathrm{Im} z_1$, which implies

$$||z - q|| \leq \alpha \mathrm{Im} z_1 = \alpha \|(\mathrm{Im} z_1, 0, \dots, 0)\|.$$

Since

$$\cos(\angle zqq') = \frac{||(\mathrm{Im} z_1, 0, \cdots, 0)||}{||z - q||}$$

it follows $\angle zqq' \leq \arccos(1/\alpha)$. □

The following result of Lee, Thomas and Wong [14] shows us that in the case of non-tangential convergence, we retain control over the speed of divergence of the Kobayashi metric. Specifically, we can bind the approach of $\phi_k(p) \to \partial\Omega$ to an approach along the normal direction. This lets us define rescalings in terms of sequences approaching normally without having to worry about also controlling the tangential rescaling of our sequence.

Lemma 4.3.4. *Let $\Omega \subseteq \mathbb{C}^n$ be a convex domain with C^1 boundary. Suppose $\{\phi_k\} \subseteq \mathrm{Aut}\,\Omega$ and $\phi_k(p) \to q \in \partial\Omega$ non-tangentially for some $p \in \Omega$. Then there exists $\{p_k\} \subseteq \Omega$ such that $\phi_k(p_k) \to q$ normally and that $d_\Omega(p, p_k) \leq r$ for some $r > 0$.*

Proof. Let $\ell_q = \{q + tn_q : t \in \mathbb{R}\}$ and define $\pi : \mathbb{C}^n \to \ell_q$ as the projection mapping onto ℓ_q. Set $q_k = \phi_k(p), \tilde{q}_k = \pi(q_k)$ and $p_k = \phi^{-1}(\tilde{q}_k)$. Then $\tilde{q}_k \to q$ along the normal direction and $||\tilde{q}_k - q_k|| \leq ||q_k - q||$. Now by the previous lemma

$$\frac{1}{\alpha} \leq \cos(\angle zqq') = \frac{||\tilde{q}_k - q||}{||q_k - q||}.$$

Let $\gamma(t) = (1 - t)q_k + t\tilde{q}_k$. Then

$$\begin{aligned} d_\Omega(p, p_k) &= d_\Omega(q_k, \tilde{q}_k) \\ &\leq \int_0^1 \frac{||\gamma'(t)||}{\delta_\Omega(\gamma(t); \gamma'(t))} dt \\ &\leq \int_0^1 \frac{||\gamma'(t)||}{\delta_\Omega(\gamma(t))} dt \end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^1 \frac{||\gamma'(t)||}{||\gamma(t)-q||}dt \\
&\leq \frac{||\tilde{q}_k - q_k||\alpha}{||\tilde{q}_k - q||} \\
&\leq \frac{||\tilde{q}_k - q||\alpha}{||\tilde{q}_k - q||} \\
&\leq \alpha^2
\end{aligned}
$$

Thus $d_\Omega(p, p_k) \leq r$ for $r = \alpha^2$. $\square$

Chapter 5

Main Result

Our main result of this section will be the following theorem.

Theorem 5.0.1. *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded, convex domain with smooth boundary. Suppose there exists $p \in \Omega$ and $\{\phi_k\}_{k\in\mathbb{N}} \subseteq \mathrm{Aut}\Omega$ such that $\lim_{k\to\infty} \phi_k(p) = q$ for $q \in \partial\Omega$ and $\phi_k(p)$ approaches q nontangentially. Then q is of finite type in the sense of D'Angelo.*

We recall the brief sketch of our proof from the introduction:

1. First we will consider a 2-dimensional slice of Ω and define a rescaling sequence for those two dimensions.

2. Next we will show there exists a holomorphic disk contained in the boundary of our rescaled 2-dimensional slice.

3. Then we apply a theorem of Frankel to show that there exists a rescaling sequence on Ω such that its blow-up also has a disk in its boundary.

4. Our rescaled domain $\hat{\Omega}$ is then shown to be biholomorphic to the original domain Ω.

5. Lastly we show there exists a holomorphic map $f : \Delta \times \Delta \to \hat{\Omega}$ that is isometric in one coordinate and isometric along a radius in another (with some error). This contradicts lemma 3.6.2.

Before proving our main theorem, we prove two lemmas.

Lemma 5.0.2. *Let $\Omega \subseteq \mathbb{C}^n$ be bounded, convex with smooth boundary such that $0 \in \partial\Omega$, the positive imaginary z_n-axis points normally inward with all other directions tangent, and Ω is described locally around 0 by:*

$$\Omega \cap U = \{(z_1, \cdots, z_n) \in U : f(z_1, \cdots, z_{n-1}, \mathrm{Re} z_n) < \mathrm{Im} z_n)$$

where $f : \mathbb{C}^{n-1} \times \mathbb{R} \to \mathbb{R}$ is smooth, non-negative and convex and U is a neighborhood of 0. If $0 \in \partial\Omega$ is a point of infinite line type then there exists a change of coordinates so for all k,

$$\lim_{z \to 0} \frac{f(z, 0 \cdots, 0, 0)}{|z|^k} = 0$$

Proof. By definition of infinite line type, there exists a sequence of linear maps ℓ_k such that $\nu(r \circ \ell_k) \geq k$. By compactness of the sphere, $\ell_k \to \ell$ in subsequence after choosing the correct parameterizations. By continuity of the defining function f, $\nu(f \circ \ell) = \infty$. If we choose our coordinates so ℓ is the z_1-axis, then the conclusion follows. □

Lemma 5.0.3. *Let $\Omega \subseteq \mathbb{C}^n$ be a convex domain that contains no complex lines in its boundary, $\Delta \times \{0, \cdots, 0\} \subset \partial\Omega$, $\Delta \times \mathcal{H} \times \{0, \cdots, 0\} \subseteq \Omega$ and $(1, i, 0, \cdots, 0) \notin \Omega$. Then there exists a map $h : \Delta \times \Delta \to \Omega$ such that for all $z, w \in \Delta$,*

$$d_\Delta(z, w) = d_\Omega(h(0, z), h(0, w))$$

and there exists an $E > 0$ such that for all $0 \leq r, s < 1$,

$$d_\Delta(r, s) - E \leq d_\Omega(h(r, 0), h(s, 0)) \leq d_\Delta(r, s) + E.$$

Proof. First we show that the injection map $\iota : \mathcal{H} \to \hat{\Omega}$ defined as $\iota(z) = (0, z, 0, \cdots, 0)$ is an isometry. By the distance-decreasing property $d_\Omega(\iota(z), \iota(w)) \leq d_\mathcal{H}(z, w)$, so it suffices to show the opposite direction. Let $\pi_2 : \mathbb{C}^n \to \mathbb{C}$ be the projection map onto the second coordinate. Then $\pi_2(\Omega) = \mathcal{H}$. So $\pi_2 \circ \iota : \mathcal{H} \to \mathcal{H}$ is the identity map and thus an isometry. Since π_2 is holomorphic it is distance-decreasing under the Kobayashi metric, and so it must be that $d_{\hat{\Omega}}(\iota(z), \iota(w)) \geq d_{\mathcal{H}}(z, w)$. Therefore ι is an isometry.

Define $h : \Delta \times \Delta \to \Omega$ by $h(z, w) = (z, i\frac{1+w}{1-w}, 0, \cdots, 0)$. Then

$$\begin{aligned}
d_\Delta(w_1, w_2) &= d_{\mathcal{H}}(i\frac{1+w_1}{1-w_1}, i\frac{1+w_2}{1-w_2}) \\
&= d_\Omega(\iota(i\frac{1+w_1}{1-w_1}), \iota(i\frac{1+w_2}{1-w_2})) \\
&= d_\Omega\Big((0, i\frac{1+w_1}{1-w_1}, 0, \cdots, 0), (0, i\frac{1+w_2}{1-w_2}, 0, \cdots, 0)\Big) \\
&= d_\Omega(h(0, w_1), h(0, w_2)).
\end{aligned}$$

It follows h is isometric in the second variable. Now consider $r, s \in [0, 1)$. Then by the distance-decreasing property of the Kobayashi metric,

$$d_\Omega(h(r, 0), h(s, 0)) \leq d_{\Delta\times\Delta}((r, 0), (s, 0)) = d_\Delta(r, s).$$

It now suffices to find a lower bound. By Theorem 3.3.4, because $(1, i, 0, \cdots, 0) \notin \Omega$,

$$\begin{aligned}
d_\Omega(h(r, 0), h(s, 0)) &\geq \frac{1}{2}\left|\log\left|\frac{h(s, 0) - (1, i, 0, \cdots, 0)}{h(r, 0) - (1, i, 0, \cdots, 0)}\right|\right| \\
&= \frac{1}{2}\left|\log\left|\frac{s-1}{r-1}\right|\right| \\
&= \frac{1}{2}\left|\log\left|\frac{(1-s)(1+r)(1+s)}{(1-r)(1+r)(1+s)}\right|\right| \\
&= \frac{1}{2}\left|\log\left|\frac{(1-s)(1+r)}{(1-r)(1+s)}\right| + \frac{1}{2}\log\left|\frac{1+s}{1+r}\right|\right| \\
&= \frac{1}{2}\left|\log\left(\frac{(1-s)(1+r)(1-rs)}{(1-r)(1+s)(1-rs)}\right) + \frac{1}{2}\log\left|\frac{1+s}{1+r}\right|\right| \\
&= \frac{1}{2}\left|\log\left(\frac{(1-s)(1+r)}{1-rs}\right) + \log\left(\frac{1-rs}{(1-r)(1+s)}\right) + \frac{1}{2}\log\left|\frac{1+s}{1+r}\right|\right| \\
&= \frac{1}{2}\left|\log\left(\frac{(1-s)(1+r)}{1-rs}\right) - \log\left(\frac{(1-r)(1+s)}{1-rs}\right) + \frac{1}{2}\log\left|\frac{1+s}{1+r}\right|\right| \\
&= \frac{1}{2}\left|\log(1 + \frac{r-s}{1-rs}) - \log(1 - \frac{r-s}{1-rs}) + \frac{1}{2}\log\left|\frac{1+s}{1+r}\right|\right|
\end{aligned}$$

$$
\begin{aligned}
&= \left| d_\Delta(r,s) + \frac{1}{2}\log\left|\frac{1+s}{1+r}\right|\right| \\
&\geq d_\Delta(r,s) - \frac{1}{2}\left|\log\left|\frac{1+s}{1+r}\right|\right| \\
&\geq d_\Delta(r,s) - \frac{1}{2}\log 2.
\end{aligned}
$$

Thus $d_\Delta(r,s) - \frac{1}{2}\log 2 \leq d_\Omega(h(r,0), h(s,0)) \leq d_\Delta(r,s)$. This proves the lemma. $\square$

We will also need the following lemma from Frankel [6]. Let $\mathbb{X}_{n,0}$ denote the space of convex domains of dimension n that do not contain any complex lines.

Lemma 5.0.4. *Suppose $\Omega \subseteq \mathbb{C}^n$ is a convex domain that does not contain any complex lines. If $V \subset \mathbb{C}^n$ is a complex affine $m-$dimensional subspace intersecting Ω, $p_k \in V \cap \Omega$ and $\{A_k\}$ is a sequence of complex affine maps such that*

$$A_k(\Omega \cap V, p_k) \to (\hat{\Omega}_V, u)$$

in $\mathbb{X}_{m,0}$ then there exists complex affine maps B_k such that

$$B_k(\Omega, p_k) \to (\hat{\Omega}, u)$$

in $\mathbb{X}_{n,0}$. Furthermore, $\hat{\Omega} \cap V = \hat{\Omega}_V$.

5.1 Proof of the Main Result

Theorem 5.1.1. *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded, convex domain with smooth boundary. Suppose there exists $p \in \Omega$ and $\{\phi_k\}_{k\in\mathbb{N}} \subseteq \operatorname{Aut}\Omega$ such that $\lim_{k\to\infty}\phi_k(p) = q$ for $q \in \partial\Omega$ and $\phi_k(p)$ approaches q nontangentially. Then q is of finite type in the sense of D'Angelo.*

Proof. For the sake of contradiction suppose that q is of infinite type. By applying an affine transformation we can assume $q = 0$ and in some neighborhood U of 0,

$$\Omega \cap U = \{(z_1, \cdots, z_n) \in U : f(z_1, \mathrm{Re} z_2, z_3, \cdots, z_n) < \mathrm{Im} z_2\}$$

where $f : \mathbb{C} \times \mathbb{R} \times \mathbb{C}^{n-2} \to \mathbb{R}$ is a smooth, convex, non-negative function, the imaginary z_2-axis points normally inward in the positive direction and the remaining directions are tangent. Let $V = \{(z_1, z_2, 0, \cdots, 0) : z_1, z_2 \in \mathbb{C}\}$ and consider $\Omega_V = \Omega \cap V$. For ease of notation, let $z = z_1$ and $w = z_2$.

1. A scaling sequence for the 2-dimensional slice:

Let $\epsilon_k i$ be the projection of $\phi_k(p)$ onto the normal axis. Then $\epsilon_k > 0$ and $\epsilon_k \to 0$. Define the function $g_j : \mathbb{C} \setminus \{0\} \to \mathbb{R}$ by

$$g_j(z) = \frac{f(z,0)}{|z|^j}.$$

Because of Lemma 5.0.2, for all $j > 0$,

$$\lim_{z \to 0} g_j(z) = 0.$$

By Theorem 3.6.1, if $\partial\Omega$ contains a holomorphic disk then it is of finite type and we have a contradiction. So assume not. In particular, for all j the set $Z_j = \{z \in \mathbb{C} : f(z,0) = \epsilon_j\}$ is nonempty. Choose $z_j \in Z_j$ such that $g(z_j)$ is maximal. By this construction and the infinite type condition, $z_j \to 0$, $f_j(z_j, 0) \to 0$ and given any $|w| < |z_j|$,

$$g_j(w) \leq g_j(z_j).$$

Let us re-index by setting $k = j$ and $\epsilon_k = \epsilon_{k_j}$.

We define the linear transformation $A_k : \mathbb{C}^2 \to \mathbb{C}^2$ by

$$A_k(z,w) = \left(\frac{z}{z_k}, \frac{w}{f(z_k,0)}\right).$$

Let $C_V = \lim_{k\to\infty} A_k(\overline{\Omega_V})$ in the local Hausdorff topology, possibly in subsequence, and $\hat{\Omega}_V$ be the interior of C_V. By construction of A_k we note that $\{0\} \times \mathcal{H} \subseteq C$. We will now show that $\hat{\Omega}_V$ contains a holomorphic disk in its boundary.

2. $\hat{\Omega}_V$ contains a holomorphic disk in its boundary:

Let $\Omega_k = A_k(\Omega_V)$. Fix $a \in \Delta$ and consider the smallest real number $b_k > 0$ such that $(a, b_k i) \in \partial\Omega_k$. It follows $A_k^{-1}(a, b_k i) \in \partial\Omega_V$. Since

$$A_k^{-1}(a, b_k i) = (az_k, b_k f(z_k, 0)i)$$

it follows by definition of f that $f(az_k, 0) = b_k f(z_k, 0)$. Because $|a| < 1$, $|az_k| < |z_k|$. It follows

$$b_k = \frac{f(az_k, 0)}{f(z_k, 0)} = \frac{g_k(az_k)|az_k|^k}{g_k(z_k)|z_k|^k} = \frac{g_k(az_k)|a|^k}{g_k(z_k)} \leq \frac{g_k(z_k)|a|^k}{g_k(z_k)} = |a|^k$$

and thus $\lim_{k\to\infty} b_k = 0$. So

$$\lim_{k\to\infty} A_k(a, b_k i) = (a, 0).$$

Since a was arbitrary, it follows $\Delta \times \{0\} \subseteq \partial\hat{\Omega}_V$. By convexity $\Delta \times \mathcal{H} \subseteq \hat{\Omega}_V$.

3. Ω can be rescaled to a domain satisfying the hypothesis of lemma 5.0.3:

By construction $(1, 0) \in \partial\hat{\Omega}_V$. Furthermore, because $A_k(z_k, if(z_k, 0)) = (1, i)$ for all k it follows $(1, i) \in \partial\hat{\Omega}_V$. So by convexity, $\{1\} \times \mathcal{H} \subseteq \partial\hat{\Omega}_V$. Since $\hat{\Omega}_V$ has positive imaginary part by construction of A_n and our original choice of coordinates on Ω_V, it follows that $(\{1\} \times \mathbb{C}) \cap \hat{\Omega}_V = \emptyset$ and $\hat{\Omega}_V$ does not contain a complex line in its boundary. Thus by lemma 5.0.4 there exists a sequence of affine maps B_k such that $B_k(\Omega) \to \hat{\Omega}$ and $\hat{\Omega} \cap V = \hat{\Omega}_V$. That is, $\hat{\Omega}$ has the same properties in its first two dimensions that $\hat{\Omega}_V$ does.

In particular, $\hat{\Omega}$ is a convex domain that does not contain any complex lines in its boundary, $(1, i, 0, \cdots, 0) \notin \hat{\Omega}$ and $\Delta \times \mathcal{H} \times \{0, \cdots, 0\} \subseteq \Omega$. Thus it satisfies the hypothesis of Lemma 5.0.3 and so we have a map $h : \Delta \times \Delta \to \hat{\Omega}$ isometric in the second coordinate and isometric along a radius in the first (with some error). It now suffices to show $\hat{\Omega}$ is biholomorphic to Ω.

4. $\hat{\Omega}$ is biholomorphic to Ω:

Define the rescaling sequence:

$$\omega_k(z_1, w, z_2, \ldots, z_{n-1}) \equiv B_k \circ \phi_k(z_1, w, z_2, \ldots, z_{n-1})$$

By lemma 4.3.4, as $\phi_k(p) \to q$ nontangentially there exists an r such that $d_\Omega(e_k i, \phi_k(p)) < r$. Let $K_r(p)$ be the Kobayashi ball around p of radius r. Then there exists a sequence $p_k \in \overline{K_k(p)}$ such that $\phi_k(p_k) = e_k i$. Note that these p_k are contained within a compact set. Furthermore, it follows by lemma 5.0.4 since $A_k(0, e_k i) \to (0, i)$ that $\omega_k(0, e_k i) \to (0, i)$. Thus by theorem 2.5.3, $\omega_k \to \hat{\omega}$ where $\hat{\omega} : \Omega \to \hat{\Omega}$ is a biholomorphism.

Since $\hat{\omega}$ is a biholomorphism it follows that $\hat{\omega} \circ h$ contradicts theorem 3.6.2. Thus, the point q is of finite type in the sense of D'Angelo. $\square$

Chapter 6

Conclusion

6.1 Applications

The result of the previous chapter directly provides a partial proof to the Greene-Krantz conjecture. To recall, the Greene-Krantz conjecture states:

Conjecture 6.1.1 (Greene-Krantz). *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain with smooth boundary. If $q \in \partial\Omega$ is a boundary orbit accumulation point for $\operatorname{Aut}\Omega$ then $\partial\Omega$ is of finite type at the point q.*

We have proved this conjecture under the additional hypotheses that the domain is convex and there exists a sequence of automorphisms such that $\phi_n(p) \to q$ non-tangentially. The hypothesis of non-tangential convergence is likely unneeded, and we suspect the techniques of the proof in the previous chapter are on the right track for a complete proof in the convex case. Unfortunately, in the non-convex case the results of Frankel do not necessarily prove our rescaled domain is biholomorphic to our original. Thus further investigation into the convergence of rescaling sequences is necessary before being able to discuss non-convex domains.

In addition to the partial solution to the Greene-Krantz conjecture, we can apply a result of Andrew Zimmer along with results of Bedford and Pinchuk to provide a characterization of certain domains in $\mathbb{C}^n$. The following theorem is taken from [21].

Proposition 6.1.2. *Suppose $\Omega \subseteq \mathbb{C}^n$ is a bounded, convex domain with smooth boundary. If there exists $x \in \partial\Omega$ with finite line type, $o \in \Omega$ and $\phi_k \in \operatorname{Aut}\Omega$ such that $\phi_k o \to x$ non-tangentially, then $\partial\Omega$ has finite line type.*

We also have this useful result from Bedford and Pinchuk [1].

Theorem 6.1.3. *Suppose Ω is a bounded convex domain with smooth boundary and finite type in the sense of D'Angelo. Then $\operatorname{Aut}\Omega$ is non-compact if and only if Ω is biholomorphic to a polynomial ellipsoid.*

Combining these two results with ours, we obtain the following corollary:

Corollary 6.1.4. *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded, convex domain with smooth boundary. If there exists $p \in \Omega$ and a sequence of automorphisms $\phi_k \in \operatorname{Aut}\Omega$ such that $\phi_k(p) \to q \in \partial\Omega$ non-tangentially, then Ω is biholomorphic to a polynomial ellipsoid.*

6.2 Further Results

We would like to remove the hypothesis of non-tangential convergence from our main result. Currently it is needed to provide an upper bound on the distance between our boundary-accumulating automorphism orbit and our normally-approaching sequence. Because we defined our rescaling sequence based on the normal approach, were this distance to be unbounded it would sabotage our result. We suspect a different type of rescaling sequence will be needed to tackle the case where the sequence approaches tangentially.

Beyond that, working to replace the convexity condition with pseudoconvexity is the next goal. This requires further investigation into the Frankel rescaling sequence, as it is only known to converge for convex domains. Work by Seungro Joo [10] suggests that the Frankel rescaling sequence can be tinkered with to converge on more domains than just convex ones. A new way to rescale pseudoconvex domains would open more possibilities for classifying simply connected domains.

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