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Authors
Grove, Karsten
Petersen, Peter

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ALEXANDROV SPACES WITH MAXIMAL RADIUS

KARSTEN GROVE AND PETER PETERSEN

Abstract. In this paper we prove several rigidity theorems related to and including Lytchak's problem. The focus is on Alexandrov spaces with curv ≥ 1, nonempty boundary, and maximal radius $\frac{\pi}{2}$. We exhibit many such spaces that indicate that this class is remarkably flexible. Nevertheless, we also show that when the boundary is either geometrically or topologically spherical, then it is possible to obtain strong rigidity results. In contrast to this one can show that with general lower curvature bounds and strictly convex boundary only cones can have maximal radius. We also mention some connections between our problems and the positive mass conjectures. This paper is an expanded version and replacement of [14].

Introduction

It is a basic fact that any Alexandrov space X with lower curvature bound 1, $\operatorname{curv} X \geq 1$, has diameter, $\operatorname{diam} X \leq \pi$ and hence radius, $\operatorname{rad} X \leq \pi$. Moreover, in case of equality X is a spherical suspension in the first case and the unit sphere in the second. With a non-positive lower curvature bound $k \leq 0$ no such upper bounds for diameter or radius exist in general. When X has non-empty boundary ∂X and $\operatorname{curv} X \geq 1$, then its radius is further restricted to $\operatorname{rad} X \leq \frac{\pi}{2}$. In fact, if $\operatorname{rad} X > \frac{\pi}{2}$, then X is homeomorphic to a sphere [13], [26, Corollary 5.2.2]. Again, in the case of lower curvature bound $k \leq 0$, there are no such bounds. It is, however, possible to control the radius when the boundary is λ-convex. In this case we will see that indeed there is an $r = r(k, \lambda)$ such that $\operatorname{rad} X \leq r(k, \lambda)$, where $r(1, \lambda) \nearrow \frac{\pi}{2}$ as $\lambda \searrow 0$.

Our aim is to establish rigidity theorems for Alexandrov spaces with boundary and maximal radius in the situations described above.

The most diverse case is when $\operatorname{curv} X \geq 1$ and $\operatorname{rad} X = \frac{\pi}{2}$. The simplest examples are spherical joins $E * S$, where E and S are Alexandrov spaces with curv ≥ 1, $\operatorname{rad} E \geq \frac{\pi}{2}$, and $\partial S \neq \emptyset$. A special case is when S is a point and $E * S$ becomes the spherical cone over E. However, there are many more examples. They will be classified in dimensions at most four in corollary 5.12 using the Topological Regularity Theorem below.

In general dimensions the following result is very useful for establishing several interesting rigidity theorems. Recall that when X has curv ≥ 1 and and boundary $\partial X \neq \emptyset$, then there is a unique point s at maximal distance $\leq \frac{\pi}{2}$ from ∂X, called the *soul* point of X. Also, a point is called regular when its tangent cone is Euclidean.

Theorem (Inner Regularity). *Let X be an n-dimensional Alexandrov space with* curv ≥ 1 *and $\partial X \neq \emptyset$. If $\operatorname{rad} X = \frac{\pi}{2}$ and the soul of X is a regular point, then X*

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is isometric to a spherical join $\mathbb{S}^k(1) * S$, *where* S *is an* $(n-k-1)$*-dimensional Alexandrov space with* $\operatorname{curv} \geq 1$, *nonempty boundary, and* $\operatorname{rad} S < \frac{\pi}{2}$.

A natural problem raised by Lytchak asks whether an n-dimensional Alexandrov space with curvature ≥ 1 has the property that its boundary has volume $\leq \operatorname{vol} \mathbb{S}^{n-1}(1)$. Petrunin answered this in the affirmative in [26, section 3.3.5]. Lytchak further asked what happens when the boundary has maximal volume [21]. Obviously the hemisphere is an example, but so is the intersection of two hemispheres making an angle $\alpha < \pi$. We refer to this as an *Alexandrov lens* and denote it by L^n_α (note that L^n_π is the hemisphere). The above result can be used to prove that these exhaust all possibilities.

Theorem (Maximal Volume)**.** *Let* X *be an* n*-dimensional Alexandrov space with* $\operatorname{curv} \geq 1$ *and* $\partial X \neq \emptyset$. *If* $\operatorname{vol} \partial X = \operatorname{vol} \mathbb{S}^{n-1}(1)$, *then* X *is isometric to* L^n_α *for some* $0 < \alpha \leq \pi$.

This theorem shows that ∂X with its induced inner metric is isometric to the unit sphere $\mathbb{S}^{n-1}$. In the special case where X is a leaf space, this result was proved in [15] and played a key role in confirming the boundary conjecture for Alexandrov spaces that happen to be leaf spaces as well.

The Inner Regularity Theorem can with the help of [27] be extended in several ways. First we have a version with slightly more flexibility than the above join examples and with purely topological assumptions about the boundary.

Theorem (Topological Regularity)**.** *Let* X^n *be an Alexandrov space with* $\operatorname{curv} \geq 1$, $\partial X \neq \emptyset$, *and* $\operatorname{rad} X = \frac{\pi}{2}$. *If* ∂X *is a topological manifold and a* $\mathbb{Z}_2$*-homology sphere, then the double is a finite quotient of a join:* $D(X) = (A * B)/G$. *Here* G *is a finite group acting effectively and isometrically on both* A *and* B *whose action is extended to the spherical join* $A * B$.

Since ∂X and the space of directions at the soul, $S_s X$, are homeomorphic we could, like in the Inner Regularity case, have made the topological restrictions on $S_s X$ instead. This helps us show that all examples in dimensions ≤ 4, in fact, satisfy the conclusion of this theorem even though not all satisfy the topological assumption. This theorem also suggests that weaker geometric assumptions than those from the Inner Regularity Theorem might be used to obtain rigidity. The next result offers a very general geometric condition that guarantees that the space is a spherical join.

Theorem (Weak Inner Regularity)**.** *Let* X^n *be an Alexandrov space with* $\operatorname{curv} \geq 1$, $\partial X \neq \emptyset$, *and* $\operatorname{rad} X = \frac{\pi}{2}$. *If* $\operatorname{rad} S_s X > \frac{\pi}{2}$, *then* $X = \hat{E} * \hat{S}$, *where* $\operatorname{rad} \hat{E} > \frac{\pi}{2}$ *and* $\operatorname{rad} S_s \hat{S} > \frac{\pi}{2}$.

Remark. Note that the concluding geometric restrictions imply that $\hat{E}$ is topologically a sphere and that $\hat{S}$ is topologically a closed disk.

We now turn to the case where $\operatorname{curv} X \geq k$, in particular allowing $k \leq 0$. As already mentioned this does not yield an upper bound on the radius of X. However, when the boundary is strictly convex in the sense studied [1], such a bound does exist. Specifically, one can *quantify convexity* of the boundary of an Alexandrov by comparing with strictly convex model spaces. The closed r_0-ball $\bar{B}_k(r_0)$ in the simply connected space form of constant curvature k has a boundary that is totally umbilic: $\mathrm{II}_{\partial B} = \lambda_0 g_{\partial B}$, where $\lambda_0 = \lambda_0 (r_0, k)$. Specifically:

$$\begin{aligned} \lambda_0 (r_0, 0) &= \frac{1}{r_0}, \\ \lambda_0 (r_0, 1) &= \cot r_0, \\ \lambda_0 (r_0, -1) &= \coth r_0. \end{aligned}$$

Definition. We say that an Alexandrov space X has λ_0*-convex boundary*, where $\lambda_0 > 0$, provided: for each $x \in \mathrm{int} X$ and $p \in \partial X$ with $|xp| = |x \partial X|$ we have

$$|pq| \cos (\angle (\overrightarrow{px}, \overrightarrow{pq})) - \frac{\lambda_0}{2} |pq|^2 \geq o\left(|pq|^2\right)$$

for all $q \in \partial X$ sufficiently near p.

Here $\overrightarrow{pq} \in S_p X$ denotes the direction at p of a minimal geodesic from p to q and of length $|pq|$. The law of cosines shows that $\bar{B}_k (r_0)$ has λ_0-convex boundary in this sense.

This type of quantified convexity was studied in detail in [1]. We shall use it to solve the analogue of Lytchak's problem for a general lower curvature bound and strictly convex boundary. This was also done with slightly different techniques in [8] assuming that $k \geq 0$.

Theorem. *Let X be an Alexandrov space with curvature $\geq k$ and λ_0-convex boundary, where $\lambda_0^2 > \max\{-k, 0\}$. If r_0 is defined by $\lambda_0 (r_0, k) = \lambda_0$, then $\mathrm{rad}\, X \leq r_0$ and $\mathrm{vol}\, \partial X \leq \mathrm{vol}\, \partial \bar{B}_k (r_0)$. Moreover, if $\mathrm{rad}\, X = r_0$, then X is isometric to a cone $C_k Y$ with constant radial curvature k; and if $\mathrm{vol}\, \partial X = \mathrm{vol}\, \partial \bar{B}_k (r_0)$, then X is isometric to $\bar{B}_k (r_0)$.*

The questions discussed so far are related to another circle of ideas that come from the positive mass conjectures. The original general version first formulated and proved by Miao in [22] is similar to the theorem just mentioned above.

Conjecture. *If (M, g) is a Riemannian n-manifold with $\mathrm{scal} \geq 0$, $\partial M = S^{n-1}(1)$, and $\mathrm{II}_{\partial M} \geq g_{\partial B(0,1)}$, then $M = B(0, 1) \subset \mathbb{R}^n$.*

Min-Oo in [23] established the hyperbolic equivalent and also proposed a version for positive curvature.

Conjecture (Min-Oo [23])**.** *If (M, g) is a Riemannian n-manifold with $\mathrm{scal} \geq n(n-1)$, $\partial M = S^{n-1}(1)$, and $\mathrm{II}_{\partial M} \geq 0$, then M is a hemisphere.*

Brendle, Marques, and Neves in [2], however, found a counter example to this conjecture. But just prior to this example Hang and Wang in [16] proved the following version.

Theorem. *If the scalar curvature assumption is replaced with the stronger condition: $\mathrm{Ric} \geq n - 1$, then the conclusion of Min-Oo's conjecture holds.*

It is worth noting that this theorem is indeed extremely sensitive to the condition that the boundary be smooth. Even with the much stronger condition that $\sec \geq 1$ the Alexandrov lens L^n_α is an example whose boundary is intrinsically isometric to $\mathbb{S}^{n-1}(1)$. Note, however, that the boundary is only convex, not strictly convex in the sense studied above. Nevertheless, it is not clear if something like Lytchak's problem is true for Riemannian n-manifolds with $\mathrm{Ric} \geq n-1$ and nonempty convex boundary.

The proofs employ several important techniques from the theory of Alexandrov spaces that are explained in section 1. It is interesting to note that some of these concepts appear to be necessary even when the interiors of the spaces are Riemannian manifolds. Section 2 contains some preliminary results for Alexandrov spaces with positive curvature and maximal radius. In section 3 we offer several examples that indicate how intricate and complex such spaces can be. Section 4 includes the proof of the Inner Regularity Theorem. This is used to resolve Lytchak's problem as well as another result related to the main focus of [9]. Section 5 is focused on establishing the Topological Regularity Theorem, which in turn is used to establish the Weak Inner Regularity Theorem and a complete classification of positively curved spaces with maximal radius in low dimensions. Section 6 contains a short account of the last theorem about Alexandrov spaces with strictly convex boundary.

It is worth pointing out that it is unknown whether the boundary of an Alexandrov space is a priori an Alexandrov space, so some care must be taken when working with conditions that pertain to the boundary.

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1. Alexandrov Geometry Preliminaries

In this section we establish notation and explain the important constructions from Alexandrov geometry that are needed. The book [3] explains all of the basic notions and the survey article [26] covers a number of more advanced topics including the gradient exponential map.

The distance between points p, q in a metric space X is denoted $|pq|$, or $|pq|_X$ if confusion is possible. In our case this will always be an intrinsic or inner distance that measures the length of a shortest path joining the points. For an Alexandrov space T_pX denotes the tangent cone at $p \in X$ and $S_pX \subset T_pX$ the space of (unit) directions. Our geodesics and quasi-geodesics are always parametrized by arclength. The notation $\overrightarrow{pq} \in S_pX$ refers to the direction of a minimal geodesic from p to q and $\overrightarrow{\overrightarrow{pq}} \subset S_pX$ is the space of all such directions.

If Y is an Alexandrov space with $\operatorname{curv} Y \geq 1$, then the *curvature k cone over Y*, denoted C_kY, is the cone $Y \times [0, \infty)/(Y \times \{0\})$ (with ∞ replaced by $\frac{\pi}{2\sqrt{k}}$ when $k > 0$) equipped with the metric where $|(y,t)(y,s)| = |t-s|$ and $|(y_1,t)(y_2,s)|$ is the distance in the curvature k plane between the end points of a hinge with angle $|y_1y_2|_Y$ and side lengths t and s. C_kY is an Alexandrov space with curvature bounded below by k. Note that when $k > 0$ it is possible to define the cone on $Y \times [0, \frac{\pi}{\sqrt{k}})/(Y \times \{0\})$, but this space is not complete or geodesically convex.

The *spherical suspension* $\Sigma_1 Y$ of Y is simply the double of the spherical cone. It is also the space of directions of $C_0Y \times \mathbb{R}$ (equipped with the product metric) at $(c, 0)$ where c is the cone point of C_0Y.

In general, given two Alexandrov spaces X and Y with curv ≥ 1 the metric product $C_0X \times C_0Y$ is an Alexandrov space with non-negative curvature and its space of directions at the cone point (c_1, c_2) is the *spherical join* $X * Y$. The distances are given by

$$\cos|(x_1,r_1,y_1)(x_2,r_2,y_2)| = \cos(r_1)\cos(r_2)\cos(|x_1x_2|) + \sin(r_1)\sin(r_2)\cos(|y_1y_2|).$$

Note that $\Sigma_1 Y = \{0,\pi\} * Y$.

Let X be a compact Alexandrov space with curv $\geq k$. The *gradient exponential map* at $p \in X$

$$\mathrm{gexp}_p(k;\cdot) : T_pX \to X$$

is defined on the tangent cone when $k \leq 0$ and on the closed ball $\bar{B}\left(o_p, \frac{\pi}{2\sqrt{k}}\right) \subset T_pX$ when $k > 0$. We can identify this domain with the cone $C_k(S_pX)$. With this new metric the gradient curves can be reparametrized so that $\mathrm{gexp}_p(k;\cdot) : C_k(S_pX) \to X$ becomes distance nonincreasing.

Along a radial curve in T_pX the gradient exponential map follows the direction of maximal increase for the distance to p. Thus, it follows minimal geodesics until they hit cut points. In general, it moves in the direction of a point in S_qX that is at maximal distance from $\overrightarrow{\overrightarrow{qp}}$ and at a rate that is specified by both $|pq|$ and how far $\overrightarrow{\overrightarrow{qp}}$ spreads out. Flow lines terminate at critical points q, i.e., when $\overrightarrow{\overrightarrow{qp}}$ forms a $\frac{\pi}{2}$-net in S_qX. Finally, the gradient exponential map is distance nonincreasing and

$$\mathrm{gexp}_p\left(k; \bar{B}(o_p, r)\right) = \bar{B}(p,r)$$

for all r with the caveat that $r \leq \frac{\pi}{2\sqrt{k}}$ when $k > 0$.

We shall also be using *quasi-geodesics*. They have several nice properties. Unlike geodesics they can be defined for all time and there is a quasi-geodesic in each direction of the space.

The left and right derivatives of a unit speed curve, if they exist, are defined as

$$\dot{c}^+(t_0) = \lim_{t\to t_0^+} \overrightarrow{c(t_0)\ c(t)}$$

$$\dot{c}^-(t_0) = \lim_{t\to t_0^-} \overrightarrow{c(t_0)\ c(t)}$$

Note that even for a differentiable curve in a manifold they point in opposite directions. Quasi-geodesics always have right and left derivatives.

Functions similarly can have right and left derivatives when restricted to curves. Moreover, when a distance function is evaluated on a unit speed curve then the left and right derivatives are always defined.

Perel'man's stability theorem also gives us the following result for Alexandrov spaces with boundary.

Theorem 1.1. *Consider a compact Alexandrov space X with $\partial X \neq \emptyset$. If the distance $r(x) = |x\partial X|$ has a unique maximum at a soul $s \in X$ and no other critical points, then S_sX and ∂X are homeomorphic.*

Proof by V. Kapovitch. From [18, lemmas 4.7 and 5.2] one obtains a strictly concave function g near s which has unique maxium at s and whose rescaled level sets are homeomorphic to S_sX. On the other hand by Perel'man's fibration theorem (see [19] and [26, section 8 property 7]) we have that the level sets of r that are near the soul are homeomorphic to ∂X. The interpolated functions $r_\epsilon = (1-\epsilon)r + \epsilon g$ are also strictly concave near s with a unique maximum at s for all $\epsilon \in [0,1]$. This is a continuous family in ϵ so by Perel'man's fibration theorem applied to

$(x, \epsilon) \mapsto (r_\epsilon(x), \epsilon)$ the level sets are homeomorphic for all ϵ. Considering $\epsilon = 0, 1$ we see that the level sets of r near s are homeomorphic to $S_s X$. $\square$

2. Basic structure

Except for section 6 we will only consider Alexandrov spaces X with curvature ≥ 1. In this section we list some basic properties for such spaces when they have $\mathrm{rad} \geq \frac{\pi}{2}$.

Proposition 2.1. *If* $\mathrm{rad}X \geq \frac{\pi}{2}$ *and* $\mathrm{curv}X \geq 1$*, then either* $\mathrm{rad}S_x X \geq \frac{\pi}{2}$ *for all* $x \in X$ *or* $X = \Sigma_1 S_p X$*, where* $\mathrm{rad}S_p X < \frac{\pi}{2}$*.*

Proof. Assume $\mathrm{rad}\, S_p X < \frac{\pi}{2}$ and select a quasi-geodesic $c : [0, \pi] \to X$ such that $c(0) = p$ and $\angle(\dot{c}(0), w) < \frac{\pi}{2}$ for all $w \in S_p X$. Let $q = c\left(\frac{\pi}{2}\right)$. Then $|xq| < \frac{\pi}{2}$ unless $|px| = 0, \pi$. In the latter case we are finished. So if no such x exists then p is the one and only point at distance $\frac{\pi}{2}$ from q. This shows that $X - B(p, t) \subset \bar{B}\left(c(t), \frac{\pi}{2} - \delta(t)\right)$ for $t < \frac{\pi}{2}$ and t near $\frac{\pi}{2}$. As $c : \left[0, \frac{\pi}{2}\right] \to X$ is now a geodesic and $\angle(\dot{c}(0), w) < \frac{\pi}{2}$, for all $w \in S_p X$, it follows from Toponogov comparison that $\bar{B}(p, t) \subset \bar{B}\left(c(t), \frac{\pi}{2} - \epsilon(t)\right)$ for $t < \frac{\pi}{2}$. This shows that $X \subset B\left(c(t), \frac{\pi}{2}\right)$ for some $t < \frac{\pi}{2}$ and contradicts that $\mathrm{rad}X \geq \frac{\pi}{2}$. $\square$

A set $C \subset Y$ in an Alexandrov space is *π-convex* if any geodesic of length $< \pi$ and whose end points lie in C is entirely contained in C. This notion of convexity should not be confused with the definition of *boundary convexity* discussed in the introduction and section 6.

In case X has boundary it will have a unique soul $s \in X$ at maximal distance from the boundary and well known comparison arguments imply that $X \subset \bar{B}\left(s, \frac{\pi}{2}\right)$ (see also [26]). This shows that $\mathrm{gexp}_s(1; \cdot) : C_1(S_s X) \to X$ is onto. Since this map is also distance nonincreasing it follows that $\mathrm{rad}\, S_s X \geq \frac{\pi}{2}$ when $\mathrm{rad}\, X = \frac{\pi}{2}$ (see also proposition 2.1).

In the extreme case where $\mathrm{rad}\, X = \frac{\pi}{2}$ we define the *edge* of X to be the dual set to s:

$$E = \left\{x \in X \mid |xs| = \frac{\pi}{2}\right\}.$$

It has the following important property.

Proposition 2.2. *If X has boundary and* $\mathrm{rad}\, X = \frac{\pi}{2}$*, then s is at maximal distance from E.*

Proof. We first show that when s is a critical point for the distance to E, then s is in fact at maximal distance from E. To see this select $x \in X$ and a geodesic direction $\overrightarrow{sx} \in S_s X$. If s is critical for E, then there is a geodesic direction $\overrightarrow{se} \in S_s X$ with $e \in E$ such that $\angle(\overrightarrow{sx}, \overrightarrow{se}) \leq \frac{\pi}{2}$. Toponogov comparison then implies that $|xe| \leq \frac{\pi}{2}$ as $|xs|, |es| \leq \frac{\pi}{2}$.

Thus we need to show that when s is not critical, then $\mathrm{rad}\, X < \frac{\pi}{2}$. For that choose a unit speed geodesic $c(t)$ such that $\dot{c}(0) \in S_s X$ forms an angle $< \frac{\pi}{2} - \epsilon$ with every direction $\overrightarrow{se} \in S_s X$ and $e \in E$. We can now find an open neighborhood $U \supset E$ such that the directions $U' \subset S_s X$ for minimal geodesics from s to points in U also form an angle $< \frac{\pi}{2} - \epsilon$ with $\dot{c}(0)$. By compactness there exists $\delta > 0$ such that $|sz| \leq \frac{\pi}{2} - \delta$ for all $z \notin U$. So we can in addition fix t such that $|c(t)\, z| \leq \frac{\pi}{2} - \frac{\delta}{2}$

for all $z \notin U$. On the other hand if $\overrightarrow{sy} \in S_x X$ denotes a direction to a $y \in U$, then Toponogov comparison implies

$$\cos|c(t)\,y| \geq \cos t \cos|sy| + \sin t \sin|sy| \cos\left(\frac{\pi}{2} - \epsilon\right).$$

Here the left-hand side is uniformly positive for any fixed small t, so there is an $\epsilon_1 > 0$, such that $|c(t)\,y| \leq \frac{\pi}{2} - \epsilon_1$ for all $y \in U$. This shows that $\operatorname{rad} X < \frac{\pi}{2}$. $\square$

We can now define the *spine* as the dual set to E by

$$S = \left\{x \in X \mid |xE| = \frac{\pi}{2}\right\}.$$

By standard comparison, the complement of any open $\frac{\pi}{2}$ ball in an Alexandrov space X with $\operatorname{curv} X \geq 1$ is π-convex (even relative to ∂X if contained in there, since geodesics in the boundary are quasi-geodesics in X). Moreover, another simple comparison argument shows that the distance functions to E and to S have no critical points in $X - (E \cup S)$. In summary

Proposition 2.3. *Assume that X has* $\operatorname{curv} \geq 1$*, nonempty boundary and* $\operatorname{rad} X = \frac{\pi}{2}$*. It follows that*

(1) *$E \subset X$ is closed and π-convex in both X and ∂X.*
(2) *S is closed, π-convex in X, and* $\operatorname{rad} S < \frac{\pi}{2}$*.*
(3) *E, respectively S, is a deformation retract of $X - S$, respectively of $X - E$*
(4) *E, respectively $S \cap \partial X$, is a deformation retract of $\partial X - S \cap \partial X$, respectively of $\partial X - E$*

Concrete deformations are provided by the gradient flows for the distance functions to S and E respectively, preserving the extremal set ∂X.

We will now see that $E \neq \partial X$ if and only if $\dim S > 0$, in which case $\partial S \neq \emptyset$.

Lemma 2.4. *Assume that X has* $\operatorname{curv} \geq 1$*, nonempty boundary, and* $\operatorname{rad} X = \frac{\pi}{2}$*. If $E \neq \partial X$, $x \in S$, and $q \in \partial X$ is closest to x, then $q \in S$. In particular, $\partial S \neq \emptyset$.*

Proof. By the choice of q, $\angle(\overrightarrow{qx}, \overrightarrow{qe}) \leq \frac{\pi}{2}$ for any $\overrightarrow{qe}$, $e \in E$. As $|xe| = \frac{\pi}{2}$ and $|xq| < \frac{\pi}{2}$, this implies $|eq| = \frac{\pi}{2}$. Thus $q \in S$. In particular, the closest points in ∂X to the soul s lie in S and hence in ∂S. $\square$

Remark 2.5. This shows that $S \cap \partial X$ is nonempty provided $E \neq \partial X$ but not that $\partial S \subset \partial X$. We will construct examples (3.6, 3.9, 3.10) where the soul $s \in \partial S$. In particular, s need not be the soul of S.

In any case, the distance function to ∂X restricted to S agrees with the distance function in S to $S \cap \partial X$. Thus

Proposition 2.6. *The distance function to $S \cap \partial X$ on S is strictly concave and has its maximum at s. In particular, S is homeomorphic to the cone on $S \cap \partial X$. Moreover, s is the soul of S if and only if $S \cap \partial X = \partial S$, and if not $s \in \partial S$.*

Remark 2.7. When $S \cap \partial X \neq \partial S$, then $\partial S = (S \cap \partial X) \cup (\partial S \cap \operatorname{int} X)$. In this case, $S \cap \partial X$ is a *face* of the boundary ∂S, and s is the soul point of S relative to this face.

We have one more simple general observation about S.

Proposition 2.8. *When* $\dim S = 0$*, then $E = \partial X$ and $X = C_1 E = C_1 \partial X$.*

Proof. Proposition 2.3 shows that any gradient curve for E ends in S. If we start in a direction of ∂X, then the gradient curve will stay in the extremal set ∂X. Therefore, it ends in a point $\partial X \cap S$. This is clearly not possible when $S = \{s\}$, so it follows that $E = \partial X$.

Next we give the details of the rigidity statement along the same lines as in the proof of theorem 6.7. Consider $f_\partial(x) = \sin|x\partial X|$ and $f_1(x) = 1 - \cos|xs|$. These functions satisfy $\ddot{f}_\partial(x) \leq -\sin|x\partial X|$ (see [26, theorem 3.3.1]) and $\ddot{f}_1(x) \leq \cos|xs|$. Since every point on the boundary is at distance $\frac{\pi}{2}$ from s, it follows that $|x\partial X| + |xs| \geq \frac{\pi}{2}$. Hence,

$$\begin{aligned} \ddot{f}_\partial(x) + \ddot{f}_1(x) &\leq -\sin|x\partial X| + \cos|xs| \\ &\leq -\sin|x\partial X| + \cos\left(\frac{\pi}{2} - |x\partial X|\right) \\ &= 0. \end{aligned}$$

On the other hand

$$\begin{aligned} f_\partial(x) + f_1(x) &= \sin|x\partial X| + 1 - \cos|xs| \\ &\geq \sin|x\partial X| + 1 - \cos\left(\frac{\pi}{2} - |x\partial X|\right) \\ &= 1 \end{aligned}$$

with equality holding for any x that lies on a geodesic from s to ∂X. The minimum principle then shows that $f_\partial + f_1 = 1$ on all of X. This shows that the gradient exponential map $\mathrm{gexp}_s(1;\cdot) : C_1(S_sX) \to X$ is an isometry. □

Proposition 2.9. *Let X be an Alexandrov space with* curv ≥ 1*, nonempty boundary, and* rad $= \pi/2$*. It follows that*

(1) *the gradient curves for $r(x) = |x\partial X|$ that start in E are minimal geodesics from E to s,*
(2) $\operatorname{rad} E \geq \frac{\pi}{2}$.

Proof. Let $c : [0, L] \to X$ be a gradient curve for r reparametrized by arclength with $c(0) \in \partial X$ and $c(L) = s$. We will use comparison along this gradient curve as in [26, lemma 2.1.3]. To that end consider $f(t) = \sin(r \circ c)$ and note that c is clearly also a gradient curve for f. This implies $\ddot{f} + f \leq 0$ in the support sense. Since $f(0) = 0$ this shows that $f(t) \leq \dot{f}(0)\sin t$. As $f \geq 0$, this implies that $L \leq \pi$. Define $g(\tau) = f(L - \tau)$ and note that also $\ddot{g} + g \leq 0$ in the support sense. This time

$$g(\tau) \leq g(0)\cos\tau + \dot{g}(0)\sin\tau.$$

At $\tau = L$ this becomes

$$0 \leq g(0)\cos(L) + \dot{g}(0)\sin(L),$$

where $g(0) > 0$ and $\dot{g}(0) \leq 0$ as r, f, and hence g are maximal at s. Since $L \leq \pi$, this forces $L \leq \frac{\pi}{2}$. On the other hand we always have $L \geq |s\,c(0)|$ so when $c(0) = e \in E$ this shows that $L = \frac{\pi}{2}$ and that the gradient curve must be a minimal geodesic from e to s. This proves (1).

For (2) we use that E consists of all points at distance $\frac{\pi}{2}$ from s. Moreover, by (1) of Proposition 2.3 intrinsic distances in E are the same as the extrinsic distances in X. Assume that $B(E, \epsilon) \subset B\left(e, \frac{\pi}{2}\right)$. In (1) we saw that there is a minimal geodesic $c : \left[0, \frac{\pi}{2}\right] \to X$ from s to e which is a reparametrized gradient curve for r. We first claim that $\overrightarrow{es} = \dot{c}^{-}\left(\frac{\pi}{2}\right) \in S_eX$ is the soul. In fact, by first variation, $d_e r(\xi)$, $\xi \in S_eX$ is maximal when ξ is the soul of S_eX (see [26, definition 1.3.2]). Thus, $\angle\left(w, \overrightarrow{es}\right) \leq \frac{\pi}{2}$ for all $w \in S_eX$ and by Toponogov comparison $B(E,\epsilon) \subset B\left(c(t), \frac{\pi}{2}\right)$ for all $t > 0$. However, also $X - B(E,\epsilon) \subset B\left(c(t), \frac{\pi}{2}\right)$ for sufficiently small t. This shows that $\operatorname{rad} X < \frac{\pi}{2}$. □

As an immediate consequence we have

Corollary 2.10. *When* $\dim E = 0$*, then* $X = \Sigma_1 S$*.*

Proof. Since E is π-convex and has rad $\geq \frac{\pi}{2}$ it follows that $E = \{0, \pi\}$. Thus $\operatorname{diam} X = \operatorname{diam} E = \pi$ and the result follows. □

Define $E' \subset S_sX$ as the directions $\overrightarrow{se}$, $e \in E$, that correspond to the gradient curves for r as in part (1) of proposition 2.9. Note that we have not excluded the possibility that there might be other minimal geodesics from s to points in E that do not correspond to such gradient curves.

Lemma 2.11. *E' and E are isometric via the spherical gradient exponential map. Consequently, $E' \subset S_sX$ is closed, convex, and* $\operatorname{rad} E' \geq \frac{\pi}{2}$*.*

Proof. The goal is to show that the reparametrized gradient curves for r that start in E and end in s form rigid constant curvature 1 triangles along minimal geodesics in E. As these curves are minimal geodesics it follows that they are also gradient curves for the distance to s. This will show that the gradient exponential map yields an isometry from E' to E.

The proof uses the parallel translation construction from [25]. Consider a minimal geodesic $c(\tau) : [0,b] \to E$ and fix a gradient curve reparametrized by arclength from $c(\tau_0)$ to s for some $\tau_0 \in (0,b)$. As c is at constant distance $\frac{\pi}{2}$ from s, there must be a rigid geodesic triangle $c(\tau,t) : [0,b] \times \left[0, \frac{\pi}{2}\right] \to X$, where $t \mapsto c(\tau,t)$ is a unit speed geodesic from $c(\tau)$ to s and $t \mapsto c(\tau_0,t)$ is the given reparametrized gradient curve for r (see [12]). There are choices involved in the construction of parallel translation, however, inside the rigid triangle the field $\frac{dc}{dt}(\tau,0)$ is intrinsically parallel. Therefore, regardless of other choices, we can always start by declaring that the constructions in [25] map $\frac{dc}{dt}(\tau_1,0)$ to $\frac{dc}{dt}(\tau_2,0)$ for $\tau_{1,2} \in (0,b)$. Thus we can assume that parallel translations $P_\tau : S_{c(\tau_0)}X \to S_{c(\tau)}X$ preserve the rigid triangle for all $\tau \in (0,b)$. As P_τ is also an isometry it follows that it must map the soul $\frac{dc}{dt}(\tau_0,0) \in S_{c(\tau_0)}X$ to the soul of $S_{c(\tau)}X$. This shows that the direction $\frac{dc}{dt}(\tau,0) \in S_{c(\tau)}X$ is the soul for any $\tau \in (0,b)$ and that $t \mapsto c(\tau,t)$ is in fact the gradient curve from $c(\tau)$ to s. By continuity of gradient curves this will also be the case for $\tau = 0, b$. This proves the claim. □

This lemma immediately tells us that

$$\operatorname{diam} E' = \operatorname{diam} E \leq \operatorname{diam} X.$$

This will be crucial for the proof of the Interior Regularity and Maximal Volume Theorems.

Before turning to the proofs of the theorems in the introduction we present examples that illustrate various phenomena discussed up till now.

3. Examples

The first result gives an easy way of checking that that the radius of the examples below indeed are $\frac{\pi}{2}$.

Proposition 3.1. *Assume that $A, B \subset X$ are convex subsets such that*

$$|ab| = \frac{\pi}{2} \text{ and } |Ax| + |xB| = \frac{\pi}{2}$$

for all $a \in A$, $b \in B$, and $x \in X$. If $\operatorname{rad} B \leq \frac{\pi}{2}$ and $\operatorname{rad} A \geq \frac{\pi}{2}$, then $\operatorname{rad} X = \frac{\pi}{2}$.

Proof. Clearly $X \subset \bar{B}\left(b, \frac{\pi}{2}\right)$ for any $b \in B$. We claim that for each $x \in X$, there exists $a \in A$ with $|xa| = \frac{\pi}{2}$. By assumption there is a geodesic $c : \left[0, \frac{\pi}{2}\right] \to X$ such that $c(0) \in A$, $c\left(\frac{\pi}{2}\right) \in B$, and $x = c(t)$ for some t. Let $a \in A$ be chosen so that $|a\, c(0)| = \frac{\pi}{2}$. Since A is at constant distance $\frac{\pi}{2}$ from $c\left(\frac{\pi}{2}\right)$ we obtain totally geodesic triangles that contain c and a. In each of these triangles the intrinsic distance from x to a is $\frac{\pi}{2}$. These triangles are uniquely determined by a minimal geodesic from x to an interior point on a geodesic from $c(0)$ to a (see [12]). If we select the point very close to a, then we obtain a contradiction provided $|xa| < \frac{\pi}{2}$. $\square$

Remark 3.2. Note that any spherical join $A * B$ with $\operatorname{rad} B \leq \frac{\pi}{2}$ and $\operatorname{rad} A \geq \frac{\pi}{2}$ satisfies the conditions of the proposition. Below we give examples which are quotients of such joins but not themselves spherical joins. Moreover, in the special case of a join the conditions are necessary in the sense that when $\operatorname{rad} A, \operatorname{rad} B < \frac{\pi}{2}$, then $\operatorname{rad} A * B < \frac{\pi}{2}$. The sphere of radius $\frac{1}{2}$, $X = \mathbb{S}^n\left(\frac{1}{2}\right)$, with A, B being a pair of antipodal points is an example that has $\operatorname{rad} X = \frac{\pi}{2}$; satisfies the first condition in proposition 3.1; while $\operatorname{rad} A = \operatorname{rad} B = 0$.

We start with a simple example to show that one cannot always expect to obtain a submetry $X \to \left[0, \frac{\pi}{2}\right]$ when $\operatorname{curv} \geq 1$ and $\operatorname{diam} = \frac{\pi}{2}$.

Example 3.3. Consider an ellipse X given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

where $1 > a > b > c > 0$. The smallest curvature is obtained at $z = \pm c$ and is given by $\frac{c^2}{a^2 b^2}$. We select $c = \frac{1}{4}$ and $b = \frac{1}{3}$. In order to have curvature ≥ 1 we then need $a < \frac{3}{4}$. The diameter of the ellipse is the distance between the two points $x = \pm a$. This distance is half the perimeter of the ellipse

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$$

and can be estimated by

$$\frac{\pi}{2}(a + c) < \operatorname{diam} X < 2(a + c).$$

When $a = \frac{3}{4}$ we obtain $\operatorname{diam} X > \frac{\pi}{2}$, while $a = \frac{1}{3}$ gives $\operatorname{diam} X < \frac{7}{6} < \frac{\pi}{2}$. So for some $a \in \left(\frac{1}{3}, \frac{4}{3}\right)$ we obtain an ellipse with $\operatorname{curv} \geq 1$ and $\operatorname{diam} = \frac{\pi}{2}$. This gives an example where there is no submetry $X \to \left[0, \frac{\pi}{2}\right]$.

The remainder of the section contains various examples of Alexandrov spaces with nonempty boundary, $\operatorname{curv} \geq 1$, and $\operatorname{rad} = \frac{\pi}{2}$.

Example 3.4. The most basic construction is a spherical join $S * E$, where S has curv ≥ 1, $\partial S \neq \emptyset$, $\operatorname{rad} S < \frac{\pi}{2}$; and E has curv ≥ 1, $\partial E = \emptyset$, $\operatorname{rad} E \geq \frac{\pi}{2}$. To see how we might obtain such a decomposition consider a spherical join $[0,\pi] * [0,\pi]$ where both factors have boundary and radius $\frac{\pi}{2}$. This space has radius $\frac{\pi}{2}$ and can be rewritten as follows:

$$\begin{aligned}
[0,\pi] * [0,\pi] &= [0,\pi] * \left(\left\{\frac{\pi}{2}\right\} * \{0,\pi\}\right) \\
&= \left([0,\pi] * \left\{\frac{\pi}{2}\right\}\right) * \{0,\pi\} \\
&= \left(\left[0,\frac{\pi}{2}\right] * \{0,\pi\}\right) * \{0,\pi\} \\
&= \left[0,\frac{\pi}{2}\right] * (\{0,\pi\} * \{0,\pi\}) \\
&= \left[0,\frac{\pi}{2}\right] * S^1(1).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
[0,\alpha] * [0,\pi] &= \left([0,\alpha] * \left\{\frac{\pi}{2}\right\}\right) * \{0,\pi\} \\
&= \Sigma_1 \left([0,\alpha] * \left\{\frac{\pi}{2}\right\}\right).
\end{aligned}$$

Example 3.5. Assume we have an example $X = S * E$ as above and a compact group G that acts isometrically and effectively on S and E. This action is naturally extended to $S * E$ in such a way that it preserves the slices $S \times \{t\} \times E$ at constant distance from S and E. On these slices it is the diagonal action by G on $S \times E$. This leads to a new Alexandrov space X/G. Note that G preserves ∂X and ∂S and consequently also the common soul of both spaces. In particular, $E/G \subset (\partial X)/G$ is at maximal distance $\frac{\pi}{2}$ from the soul and E/G and S/G are dual sets in X/G. It follows from proposition 3.1 that $\operatorname{rad} X/G \geq \frac{\pi}{2}$ provided $\operatorname{rad} E/G \geq \frac{\pi}{2}$. Topologically, $S * E$ is a cone over $\partial X = (\partial S) * E$, where the action fixes the soul and preserves the boundary. The quotient is likewise a topological cone over $(\partial X)/G$.

Below we offer some concrete examples of this construction.

Example 3.6 (Projective Lenses)**.** Consider $E = \{0,\pi\}$ and $S = [-\alpha,\alpha]$, $\alpha < \frac{\pi}{2}$. Let $G = \mathbb{Z}_2$ be the natural reflection on both spaces. Note that $S * E = \Sigma_1 [-\alpha,\alpha] = L^2_{2\alpha}$ looks topologically like a hemisphere. The action fixes $0 \in [-\alpha,\alpha]$ and acts like the antipodal map on $\partial (S * E) = \Sigma_1 \{-\alpha,\alpha\} = \mathbb{S}^1(1)$. The quotient looks topologically like a cone with vertex 0. The boundary has one point at distance $\frac{\pi}{2}$ from 0. The issue is that $\operatorname{rad}(E/G) < \frac{\pi}{2}$ and $\operatorname{rad}(X/G) < \frac{\pi}{2}$.

More generally one can consider the $\mathbb{Z}_2$ quotient of the Alexandrov lens $L^n_{2\alpha} = S^{n-2}(1) * [-\alpha,\alpha]$ (cf. also [9]). When $n > 2$, this space will have $\operatorname{rad} = \frac{\pi}{2}$ and boundary isometric to $\mathbb{RP}^{n-1}$.

Example 3.7 (Edge with Nonempty Boundary)**.** Consider $S = (\Sigma_1 [-\alpha,\alpha])/\mathbb{Z}_2$ as above and define $X = \mathbb{S}^1(r) * S$, where $r \in \left[\frac{1}{2}, 1\right]$. The soul of X is the soul of S and $E = C_1 \mathbb{S}^1(r)$. This is an example where E has nonempty boundary.

Further examples that indicate the complexities in trying to classify spaces with maximal radius can be obtained as follows:

Example 3.8 (Higher Dimensional Spines)**.** Select $E = S^1(1)$ and $S = B(p,r) \subset S^2(1)$, where $r < \frac{\pi}{2}$. Let $G = \mathbb{Z}_2$ be a rotation by π on both S and E. This gives a 4-dimensional example where the boundary is homeomorphic to $\mathbb{RP}^3$. The same can be done with $E = S^n(1)$ and $S = B(p,r) \subset S^m(1)$, and $G = \mathbb{Z}_2$ the antipodal map on both S and E, giving an $n+m+1$-dimensional example with boundary homeomorphic to $\mathbb{RP}^{n+m}$.

Example 3.9 (Spines with Soul on Boundary)**.** Select $E = S^1(1)$ and $S = B(p,r) \subset S^2(1)$, where $r < \frac{\pi}{2}$. Let $G = \mathbb{Z}_2$ be a rotation by π on E and a reflection on S. This gives a 4-dimensional example where the boundary of $S/\mathbb{Z}_2$ is connected and contains the soul. The boundary is homeomorphic to a suspension $\Sigma\mathbb{RP}^2$. As in the previous example we can choose $E = S^n(1)$ and $S = B(p,r) \subset S^m(1)$, with $G = \mathbb{Z}_2$ action on E as the antipodal map and on S as a reflection (or any other isometric involution). The resulting example is $n+m+1$-dimensional with boundary homeomorphic to $\Sigma^{m-1}\mathbb{RP}^{n+1}$.

In both of these examples the key is that S is an Alexandrov space with curvature at least 1, non-empty boundary and radius $r < \frac{\pi}{2}$, and with an isometric involution.

In corollary 5.12 we will show that above examples exhaust all the possibilities in dimensions ≤ 4, while the next examples shows that one can have more complex behavior in dimensions ≥ 5.

Example 3.10 (Dual Pairs of Nonmaximal Dimension)**.** Select $E = S^3(1)$ and $S = B(p,r) \subset S^2(1)$, where $r < \frac{\pi}{2}$. Let $G = S^1$ be the Hopf action on E and rotation around p on S. This gives a 5-dimensional example that is not a finite quotient of a spherical join and with boundary homeomorphic to $\mathbb{CP}^2$.

4. Interior regularity and Lytchak's problem

We need the following result for convex subsets of the standard sphere.

Proposition 4.1. *If $A \subset \mathbb{S}^{n-1}(1)$ is closed, convex and has $\operatorname{rad} A \geq \frac{\pi}{2}$, then $\operatorname{diam} A = \pi$.*

Proof. In case $\partial A = \emptyset$, the radius condition is redundant and A is totally geodesic unit sphere or two antipodal points.

In general note that $\operatorname{rad} A < \frac{\pi}{2}$ is equivalent to A lying in an open hemisphere. Since A is convex we have that it lies in a closed hemisphere H. In case $\operatorname{rad}(A \cap \partial H) < \frac{\pi}{2}$ it follows that we can move H so that A lies in an open hemisphere. In this way we obtain a convex subset $A \cap \partial H \subset \partial H$ inside a lower dimensional sphere with radius $\geq \frac{\pi}{2}$. This ultimately reduces the problem to the trivial case: $A \subset \mathbb{S}^0(1)$, where the radius condition forces $A = \mathbb{S}^0(1)$. □

Corollary 4.2. *If $S_sX = \mathbb{S}^{n-1}(1)$, then $\operatorname{diam} E' = \pi$.*

Proof. As $E' \subset \mathbb{S}^{n-1}(1)$ is closed, $\operatorname{rad} E' \geq \frac{\pi}{2}$, and convex, the result follows from the previous proposition. □

We are now ready to complete the proof of the Interior Regularity Theorem from the introduction.

Theorem 4.3. *Let X be an n-dimensional Alexandrov space with $\operatorname{curv} \geq 1$ and $\partial X \neq \emptyset$. If $\operatorname{rad} X = \frac{\pi}{2}$ and the soul of X is a regular point, then X is isometric to a spherical join $\mathbb{S}^k(1) * S$, where S is an $(n-k-1)$-dimensional Alexandrov space with $\operatorname{curv} \geq 1$, $\partial S \neq \emptyset$, and $\operatorname{rad} S < \frac{\pi}{2}$.*

Proof. The assumption that the soul is regular means that $S_sX = \mathbb{S}^{n-1}(1)$. From the preceding corollary we know that $X = \Sigma_1 S_p X$ for some $p \in E \subset \partial X$. Further note that the soul s lies in the slice $\left\{\frac{\pi}{2}\right\} \times S_p X$ and also corresponds to the soul of $S_p X$. Moreover as

$$\mathbb{S}^{n-1}(1) = S_s X = S_s \Sigma_1 S_p X = \Sigma_1 S_s S_p X$$

it follows that $S_p X$ also has the property that its soul is a regular point. When $\operatorname{rad} S_p X < \frac{\pi}{2}$ we have obtained the desired decomposition, otherwise the construction can be iterated until one reaches the desired decomposition. □

From this we deduce the rigidity part of Lytchak's problem by first observing the following reformulation of the results in [26, 3.3.5].

Proposition 4.4. *Let X be an n-dimensional Alexandrov space with* $\operatorname{curv} \geq 1$*, $\partial X \neq \emptyset$, and s the soul of X. If* $\operatorname{vol}_{n-1} \partial X = \operatorname{vol}_{n-1} \mathbb{S}^{n-1}(1)$*, then* $\operatorname{rad} X = \frac{\pi}{2}$*, $X = \bar{B}\left(s, \frac{\pi}{2}\right)$,*

$$\operatorname{gexp}_s(1; S_s X) = \partial X = \partial \bar{B}\left(s, \frac{\pi}{2}\right),$$

and $S_s X$ is isometric to $\mathbb{S}^{n-1}(1)$.

Proof. It follows from Petrunin's solution to Lytchak's problem [26, 3.3.5] that $\operatorname{rad} X \leq \frac{\pi}{2}$. Similarly, if $r \leq \frac{\pi}{2}$ and $X = \bar{B}(p, r)$ for some $p \in M$, then $\partial X \subset \operatorname{gexp}_p\left(1; \partial \bar{B}(p, r)\right)$. In particular, as $\operatorname{gexp}_p(1; \cdot)$ is distance nonincreasing:

$$\begin{aligned}
\operatorname{vol}_{n-1}(\partial X) &\leq \operatorname{vol}_{n-1}\left(\operatorname{gexp}_p\left(1; \partial \bar{B}(o_p, r)\right)\right) \\
&\leq \operatorname{vol}_{n-1}\left(\partial \bar{B}(o_p, r)\right) \\
&\leq \operatorname{vol}_{n-1} \mathbb{S}^{n-1}(1).
\end{aligned}$$

Here equality can only hold when $r = \frac{\pi}{2}$ and in that case we can use $X = \bar{B}\left(s, \frac{\pi}{2}\right)$. Moreover, $\partial X = \operatorname{gexp}_s\left(1; \partial \bar{B}\left(s, \frac{\pi}{2}\right)\right)$ as otherwise $\partial \bar{B}\left(o_s, \frac{\pi}{2}\right) \cap \operatorname{gexp}_s^{-1}(\operatorname{int} X)$ is a nonempty open set and that forces $\operatorname{vol}_{n-1}(\partial X) < \operatorname{vol}_{n-1}\left(\operatorname{gexp}_s\left(1; \partial \bar{B}\left(o_s, \frac{\pi}{2}\right)\right)\right)$. Finally, we know that $S_s X = \partial \bar{B}\left(o_s, \frac{\pi}{2}\right)$ also has maximal volume, showing that $S_s X$ is isometric to $S^{n-1}(1)$. □

From 4.3 it follows that when $\operatorname{vol}_{n-1} \partial X = \operatorname{vol}_{n-1} \mathbb{S}^{n-1}(1)$, then ∂X is isometric to $\mathbb{S}^k(1) * \partial S$, where S is an $(n-k-1)$-dimensional Alexandrov space with $\operatorname{curv} \geq 1$. This is isometric to $\mathbb{S}^{n-1}(1)$ if and only if $S = [0, \alpha]$. Consequently, we have answered Lytchak's problem as in the Maximal Volume Theorem from the introduction.

Corollary 4.5. *Let X be an n-dimensional Alexandrov space with* $\operatorname{curv} \geq 1$ *and nonempty boundary. If* $\operatorname{vol} \partial X = \operatorname{vol} \mathbb{S}^{n-1}(1)$*, then X is isometric to L^n_α for some $0 < \alpha \leq \pi$.*

These results can be used to complement (if not complete) the main theorem in [9].

Proposition 4.6. *If X^n is an Alexandrov space with* $\operatorname{curv} \geq 1$, $\operatorname{rad} X = \frac{\pi}{2}$*, and boundary $\partial X = M$ that is a Riemannian manifold and a topological sphere with* $\sec \geq 1$*, then $M = \mathbb{S}^{n-1}(1)$. Consequently, X is an Alexandrov lens.*

Proof. Suppose $\partial E \neq \emptyset$. Since E is a π-convex subset of M with $\operatorname{rad} E \geq \frac{\pi}{2}$ it must have radius $\frac{\pi}{2}$ and by the Regularity Theorem 4.3 E becomes a join with a unit sphere. In particular,

$$\pi = \operatorname{diam} E \leq \operatorname{diam} M$$

and by Toponogovs maximal diameter theorem $M = \mathbb{S}^{n-1}(1)$.

It remains to consider the case where E is a smooth totally geodesic submanifold of M with rad $\geq \frac{\pi}{2}$. Now $M \cap S$ is a π-convex subset of M and dual to E in the sense of [11]. The arguments in [11] show that $M \cap S$ is a smooth totally geodesic submanifold of M without boundary. Since M is topologically a sphere with nontrivial dual submanifolds it follows again from [11] that all points in $M - (E \cup (S \cap M))$ lie on a unique minimal geodesic of length $\frac{\pi}{2}$ from E to $S \cap M$. Further, for each $x \in S \cap M$ the corresponding map from the normal sphere to $S \cap M$ at x to E is a Riemannian submersion (and likewise for points in E). The classification of Riemannian submersions from spheres (cf. [11, 29]) and the fact that M is a topological sphere implies that $M = \mathbb{S}^{n-1}(1)$. □

5. Rigidity from Topology

In this section we discuss several more general results for Alexandrov spaces X^n with curv ≥ 1, $\partial X \neq \emptyset$, and maximal radius rad $= \frac{\pi}{2}$. These include the generalizations to the inner regularity theorem mentioned in the introduction and lead to a classification in dimensions ≤ 4.

For the purposes of this section we shall need an improved dual set decomposition of X.

Proposition 5.1. *For an Alexandrov space X^n with* curv ≥ 1*, $\partial X \neq \emptyset$, and* $\operatorname{rad} X = \frac{\pi}{2}$ *there exists a dual space decomposition $\hat{E}, \hat{S} \subset X$ with the properties that $\hat{E} \subset E$, $\partial \hat{E} = \emptyset$,* $\operatorname{rad} \hat{E} \geq \frac{\pi}{2}$*, $S \subset \hat{S}$, and $\partial \hat{S} \neq \emptyset$.*

Proof. When $\partial E = \emptyset$ there is nothing to prove. Otherwise we have $\operatorname{rad} E = \frac{\pi}{2}$ and E itself admits a dual space decomposition $E_1 \subset \partial E$, $S_{1/2} \subset E$. Define $S_1 = \left\{x \in X \mid |xE_1| \geq \frac{\pi}{2}\right\}$. Note that $S, S_{1/2} \subset S_1$. Thus any point at distance $\frac{\pi}{2}$ from S_1 must lie in E and hence also in E_1. This shows that $E_1, S_1 \subset X$ are dual to each other. By construction $\operatorname{rad} E_1 \geq \frac{\pi}{2}$ and $S_1 \cap \partial X \neq \emptyset$. We can now continue this procedure until the desired decomposition is reached. □

Remark 5.2. Note that we haven't claimed $\operatorname{rad} \hat{S} < \frac{\pi}{2}$.

Our rigidity results depend on the following version of Lefschetz duality.

Theorem 5.3. *Assume Z is a compact connected ANR that is a $\mathbb{Z}_2$-homology sphere, and $A, B \subset Z$ are disjoint, compact, connected, and ANR. If $B \subset Z - A$ is a deformation retract and $Z - A$ is a topological n-manifold, then*

$$H_q(B; R) \simeq H^{n-1-q}(A; R), \, q = 1, ..., n-2.$$

Proof. It follows from Lefschetz duality and the fact that A and Z are ANRs that for all q

$$H_q(Z - A; \mathbb{Z}_2) \simeq H^{n-q}(Z, A; \mathbb{Z}_2).$$

The fact that $H^p(Z; \mathbb{Z}_2) = 0$ for $p = 1, ..., n-1$ shows, via the long exact sequence for relative cohomology, that

$$H^{n-q}(Z, A; \mathbb{Z}_2) \simeq H^{n-1-q}(A; \mathbb{Z}_2), q = 1, ..., n-2.$$

The fact that $B \subset Z - A$ is a deformation retract then implies the claim. □

We require some extra notation. For a convex subset $A \subset X$ of an Alexandrov space we define the normal space at $a \in A$ as

$$N_a A = \left\{ v \in S_a X \mid \angle (v, w) \geq \frac{\pi}{2} \text{ for all } w \in S_a A \right\}.$$

By first variation any unit speed geodesic that starts in a and minimizes the distance to A has initial velocity that lies in $N_a A$.

Lemma 5.4. *Assume $Z = \bar{Z}/H$ is an Alexandrov space with* curv ≥ 1 *and $\partial Z = \emptyset$, where H is a finite group of isometries, $\bar{Z}$ is a closed topological n-manifold that is a $\mathbb{Z}_2$-homology sphere, and an Alexandrov space with* curv ≥ 1*. If $A, B \subset Z$ form a dual pair and $\partial A = \emptyset$, then there exists $x \in B$, $y \in A$ and finite group, G, that acts effectively and isometrically on both $N_x B$ and $N_y A$, such that $Z = (N_x B * N_y A)/G$.*

Proof. The goal is to prove that

$$\dim A + \dim B = n - 1$$

and that no points in B have distance $> \frac{\pi}{2}$ to A and vice versa. This allows us to use [27, theorem A] when $\partial B = \emptyset$ and otherwise [27, theorem B] to reach the conclusion of the lemma.

We lift the situation to $\bar{A}, \bar{B} \subset \bar{Z}$. In case $\bar{A}$ (or $\bar{B}$) is not connected the components must be distance π apart as $\bar{A}$ is π-convex. This forces $\bar{A}$ to consist of two points and $\bar{Z}$ to be a suspension with $\bar{B} = S_a X$, $a \in \bar{A}$. So we can assume that $\dim \bar{A} = p > 0$. Since $\bar{A}, \bar{B}$ form a dual pair it follows that each of these sets contains the set of critical points for the distance function to the other set. The gradient flow then shows that $\bar{Z} - \bar{A}$ deformation retracts to $\bar{B}$. From $\partial \bar{A} = \emptyset$ we conclude that $H^p(\bar{A}, \mathbb{Z}_2) = \mathbb{Z}_2$. By Alexander duality we can then conclude that $H_q(\bar{B}, \mathbb{Z}_2) = \mathbb{Z}_2$ for $p = n - 1 - q$. This shows that

$$\dim \bar{A} + \dim \bar{B} \geq n - 1.$$

Since $\bar{B} \subset \bar{Z}$ is convex it follows that $\partial \bar{B} = \emptyset$ as it would otherwise be contractible. Frankel's theorem for Alexandrov spaces (see [25]) then shows that

$$\dim \bar{A} + \dim \bar{B} \leq n - 1.$$

Moreover, points in $\bar{B}$ cannot have distance $> \frac{\pi}{2}$ from $\bar{A}$ and vice versa. This finishes the proof. □

Remark 5.5. It is in general not possible to conclude that $\bar{Z}$ in lemma 5.4 is a join. The icosahedral group G acts freely on $\mathbb{S}^3(1)$ and hence on $\mathbb{S}^3(1) * \mathbb{S}^3(1) = \mathbb{S}^7(1)$. While the quotient $\mathbb{S}^7(1)/G$ is clearly a homology sphere it is not a join as it is a space form that is not homeomorphic to a sphere.

Remark 5.6. Note that from the classification obtained in [10] any positively curved Alexandrov space of dimension ≤ 3 and empty boundary is of the form $Z = \bar{Z}/H$ where $\bar{Z}$ is homeomorphic to a sphere. If in addition $\operatorname{diam} Z = \frac{\pi}{2}$, then we obtain two dual sets $A, B \subset Z$. If both of these have boundary, then Z is topologically a suspension and therefore topologically a sphere or $\Sigma\mathbb{RP}^2$. Otherwise we can apply the lemma.

We can now prove the Topological Regularity Theorem from the introduction.

Theorem 5.7. *Let X^n be an Alexandrov space with* $\operatorname{curv} \geq 1$*, $\partial X \neq \emptyset$, and* $\operatorname{rad} X = \frac{\pi}{2}$*. If ∂X is a topological manifold and a $\mathbb{Z}_2$-homology sphere, then $D(X) = (X_1 * X_2)/G$. Here G is a finite group acting effectively and isometrically on both X_1 and X_2 whose action is extended to the spherical join $X_1 * X_2$.*

Proof. The idea is simply to apply lemma 5.4 to the double. This requires a few minor adjustments. We use the dual decomposition for $D(X)$ that consists of $D\left(\hat{S}\right)$ and the copy of $F \subset D(X)$ that corresponds to $\hat{E} \subset \partial X$. Here $D\left(\hat{S}\right)$ will turn out to be the double of $\hat{S}$, but for now it is simply the preimage of $\hat{S}$. Note that inside X the gradient flows for $\hat{S}$ and $\hat{E}$ preserve ∂X. Thus we obtain deformation retractions of $D(X) - D\left(\hat{S}\right)$ to F and $D(X) - F$ to $D\left(\hat{S}\right)$ relative to ∂X as in proposition 2.3. Moreover, as X is homeomorphic to the cone over the boundary it follows that $D(X) - D\left(\hat{S}\right)$ is a topological n-manifold. Additionally, $D(X)$ is a $\mathbb{Z}_2$-homology sphere by Meyer-Vietoris. This again shows that

$$H_q(F;\mathbb{Z}_2) \simeq H^{n-1-q}\left(D\left(\hat{S}\right);\mathbb{Z}_2\right),\ q = 1, ..., n-2.$$

We can then argue as in the proof of lemma 5.4 that $\partial D\left(\hat{S}\right) = \emptyset$ and that we obtain the desired decomposition for $D(X)$. □

Remark 5.8. Note that $N_xF \subset S_xD(X)$ is the double of $N_x\hat{E} \subset S_xX$. This shows that N_xF, and thus also $N_yD\left(\hat{S}\right) * N_xF$, come with a natural reflection whose quotient is $N_x\hat{E}$, respectively, $N_yD\left(\hat{S}\right) * N_x\hat{E}$. Let R be the natural reflection on both N_xF and $D\left(\hat{S}\right) = N_xF/G$. This results in a commutative diagram

$$\begin{array}{ccc} N_xF & \overset{R}{\to} & N_xF \\ \downarrow & & \downarrow \\ D\left(\hat{S}\right) & \overset{R}{\to} & D\left(\hat{S}\right). \end{array}$$

In case G commutes with R on N_xF it follows that G will also act on $N_x\hat{E}$. Consequently, also X becomes the quotient of a join: $X = \left(N_x\hat{E} * N_y\hat{S}\right)/G$. It is not, in general, clear whether R and G will commute. However, in the case where $G = \langle I \rangle$, $I^2 = \mathrm{id}$ this is automatically true. To see this assume $x, Ix \in D(Z)$ are mapped to $s \in D\left(\hat{S}\right)$. Then Rx, RIx are both mapped to Rs, but the preimage of Rs also consists of the two points Rx, IRx, this shows that $IR = RI$.

Next we establish the Weak Inner Regularity Theorem from the introduction.

Theorem 5.9. *Let X^n be an Alexandrov space with* $\operatorname{curv} \geq 1$*, $\partial X \neq \emptyset$, and* $\operatorname{rad} X = \frac{\pi}{2}$*. If* $\operatorname{rad} S_sX > \frac{\pi}{2}$*, then $X = \hat{E} * \hat{S}$, where* $\operatorname{rad} \hat{E} > \frac{\pi}{2}$ *and* $\operatorname{rad} S_s\hat{S} > \frac{\pi}{2}$*.*

Proof. The radius assumption on the space of directions is first used to see that X is a topological manifold near the soul. Since X is homeomorphic to a cone it follows that it is a topological manifold with boundary. Finally, the space of directions is homeomorphic to a sphere and thus ∂X is also homeomorphic to a sphere. This shows that we can apply theorem 5.7.

Next we can use it in combination with the following fact: If G is a compact group acting by isometries on an Alexandrov space Y with $\operatorname{curv} Y \geq 1$, then $\operatorname{rad}(Y/G) \leq$

$\frac{\pi}{2}$ unless the action is trivial. Moreover, if the action is free, then $\operatorname{diam} Y/G \leq \frac{\pi}{2}$. This is obvious when $\dim Y = 1$ and follows in general from an induction argument. To see this, first note that when $\operatorname{diam}(Y/G) > \frac{\pi}{2}$, then there are two orbits that are at distance $> \frac{\pi}{2}$ from each other. Thus one orbit is forced to lie in a π-convex set that is at distance $> \frac{\pi}{2}$ from some point. The action must then have a fixed point y in this set. It now follows from induction that when the action is nontrivial, then $\operatorname{rad}(S_y Y/G) \leq \frac{\pi}{2}$, and hence that $\operatorname{rad}(Y/G) \leq \frac{\pi}{2}$ (see e.g. [26, lemma 5.2.1]).

With notation as in the proof of theorem 5.7 we show that $D(X) = F * D\left(\hat{S}\right)$. Let $s \in D(X)$ be fixed to be one of the two soul points and note that $S_s\hat{S} = S_s D\left(\hat{S}\right)$ and $S_s X = S_s D(X)$.

We first show that $\hat{E}' = N_s\hat{S}$. Observe that as $\hat{E}' \subset N_s\hat{S}$ we have:

$$\dim N_s\hat{S} + \dim S_s\hat{S} \geq \dim \hat{E} + \dim \hat{S} - 1 = n - 2.$$

On the other hand as $\partial S_s\hat{S} = \emptyset$ it follows from [27, theorem A, part (A1)] that

$$\dim N_s\hat{S} + \dim S_s\hat{S} \leq n - 2.$$

Thus $\dim \hat{E}' = \dim N_s\hat{S}$. In case both spaces are 0-dimensional they will both consist of two points at distance π apart. This is because both spaces have $\operatorname{curv} \geq 1$ and $\operatorname{rad} \hat{E}' \geq \frac{\pi}{2}$ while $N_s\hat{S} \subset S_s X$ is π-convex. When both spaces have dimension > 0, we know that $N_s\hat{S}$ is connected as it is π-convex. Since $\dim \hat{E}' = \dim N_s\hat{S}$ and $\partial\hat{E}' = \emptyset$ it follows that $\hat{E}' \subset N_s\hat{S}$ is an open and closed subset and hence that $\hat{E}' = N_s\hat{S}$.

This shows that only one "point" $\bar{s} \in N_x F \subset N_y D\left(\hat{S}\right) * N_x F$ is mapped to $s \in D\left(\hat{S}\right) \subset D(X)$ and hence that $\bar{s}$ is a fixed point for the isometric action of G on $N_y D\left(\hat{S}\right) * N_x F$. In particular, G preserves $S_{\bar{s}}\left(N_y D\left(\hat{S}\right) * N_x F\right)$ and $S_s X = \left(S_{\bar{s}}\left(N_y D\left(\hat{S}\right) * N_x F\right)\right)/G$. However, this can only happen if G acts trivially on $S_{\bar{s}}\left(N_y D\left(\hat{S}\right) * N_x F\right)$ as $\operatorname{rad} S_s X > \frac{\pi}{2}$. In conclusion, G acts trivially on $N_y D\left(\hat{S}\right) * N_x F$ and $D(X)$ is a spherical join. This shows that $S_s X = \hat{E}' * S_s\hat{S}$ and $\operatorname{rad} \hat{E} > \frac{\pi}{2}$ and $\operatorname{rad} S_s\hat{S} > \frac{\pi}{2}$.

Finally note that the natural reflection on $D(X) = D\left(\hat{S}\right) * F$ fixes F and has orbit space X. Thus also $X = \hat{S} * \hat{E}$. $\square$

Remark 5.10. In the context of this result it is worth noting that the icosahedral group I acts on $\mathbb{S}^5(1) = \mathbb{S}^1(1) * \mathbb{S}^3(1)$ with a quotient $\mathbb{S}^1(1) * \left(\mathbb{S}^3(1)/I\right)$ that is a topological sphere (see e.g., [5]). This quotient also shows that one can not expect to use general position or transversality arguments for Alexandrov spaces that are topological manifolds.

Based on lemma 5.4 we obtain the following generalization of theorem 5.7.

Corollary 5.11. *Let X^n be an Alexandrov space with* $\operatorname{curv} \geq 1$, *$\partial X \neq \emptyset$, and* $\operatorname{rad} X = \frac{\pi}{2}$. *If $S_s X$ is a topological manifold that is covered by a $\mathbb{Z}_2$-homology sphere or $S_s X$ is homeomorphic to $\Sigma\mathbb{RP}^2 = \mathbb{S}^3/\mathbb{Z}_2$, then $D(X) = (X_1 * X_2)/G$. Here G is a finite group acting effectively and isometrically on both X_1 and X_2 whose action is extended to the spherical join $X_1 * X_2$.*

Proof. The goal is to find a suitable ramified or branched cover of X. In all cases these will in fact be good orbifold covers. Specifically, for a given Alexandrov space X we seek an Alexandrov space Y, with the same lower curvature bound, and a finite group G acting by isometries on Y such that $X = Y/G$. The relevant results are established in [17, Theorem A] and [6, Subsection 2.2]. The main technical tools for showing that Y has the desired properties are proven in [20]. We only need these covers for positively curved spaces with boundary, i.e., for spaces that are homeomorphic to cones over the space of directions at the soul $X \simeq C_1 S_s X$. This means that the cover $Y \simeq C_1 S_{\bar{s}} Y$ and $(S_{\bar{s}} Y)/G = S_s X$ since G is forced to fix $\bar{s}$. Here $S_{\bar{s}} Y$ is either a covering space over $S_s X$ or a good orbifold cover of $S_s X$.

When $S_s X$ has a covering space there is only one isolated branch point and the complement of the soul is convex. This means we can use exactly the same strategy as in [6, Subsection 2.2] to create the Alexandrov space structure on Y. When $S_s X$ is homeomorphic to $\Sigma \mathbb{RP}^2$ it follows that X is not orientable as it does not have a local orientation at the soul. This means that we can use the orientation covering as in [17, Theorem A] as Y.

The assumptions of the corollary now show that Y exists and is a topological manifold. We can then apply lemma 5.4 to finish the proof. □

Corollary 5.12. *When* $\dim X \leq 4$*, then* X *is either isometric to a join or a* $\mathbb{Z}_2$ *quotient of a join, where* $\mathbb{Z}_2$ *acts effectively on both factors of the join.*

Proof. Note that when $\dim X \leq 2$, then either $\dim E = 0$ or $\dim S = 0$. Thus the result follows from propositions 2.10 and 2.8. Similarly, in higher dimensions we can assume that both $\hat{S}$ and $\hat{E}$ have positive dimensions.

In case $\dim X = 3, 4$ we need to use corollary 5.11 and remark 5.8. First observe that when $\dim X = 3$, it follows that $S_s X$ is homeomorphic to $\mathbb{S}^2$ or $\mathbb{RP}^2$, while when $\dim X = 4$ we can use the classification from [10] to conclude that $S_s X$ is homeomorphic to a spherical space form or the suspension over the real projective plane. This places us in a position where we can use corollary 5.11.

We assume that $D(X) = X_1 * X_2/G$, with $X_1/G = D\left(\hat{S}\right)$ and $X_2/G = F$. When G is trivial there is nothing to prove so we also assume $|G| \geq 2$.

As $\operatorname{rad} F \geq \frac{\pi}{2}$ we note that when $\dim X_2 = 1$ we have $G = \mathbb{Z}_2$ and acting as the antipodal map since $\partial F = \emptyset$.

Assume that $\dim X_2 = 2$. When $\operatorname{diam} F = \pi$ we obtain a dimension reduction.

In case $\operatorname{diam} F \in \left(\frac{\pi}{2}, \pi\right)$ it follows as in the proof of theorem 5.9 that G has two fixed points $x, y \in X_2$ with $|xy| = \operatorname{diam} X_2$. We then have that $S_{x,y} F = (S_{x,y} X_2)/G = \mathbb{S}^1\left(\frac{1}{|G|}\right)$. Let $z \in F$ be any point with $|zx| = |zy| < \frac{\pi}{2}$. The radius assumption shows that there is $\bar{z} \in F$ with $|z\bar{z}| = \frac{\pi}{2}$. Consider a hinge with vertex at x (or y). Since $\operatorname{diam} S_{x,y} F \leq \frac{\pi}{2}$ the angle of the hinge is $\leq \frac{\pi}{2}$. But this leads to a contradiction as either $|x\bar{z}| < \frac{\pi}{2}$ or $|y\bar{z}| < \frac{\pi}{2}$ which by Toponogov comparison implies $|z\bar{z}| < \frac{\pi}{2}$. This shows that this situation is impossible.

Finally we consider the case when $\operatorname{diam} F = \operatorname{rad} F = \frac{\pi}{2}$. This allows us to obtain dual sets $A, B \subset F$ where, say, $\dim A = 0$. By lemma 5.4 $F = (N_x B * N_y A)/\Gamma$, where Γ acts effectively on $N_x B$ and $N_y A$. Since $\dim N_x B = 0$ this implies that either $N_x B$ is a point and Γ is trivial or $N_x B$ consists of two points distance π apart and $\Gamma = \mathbb{Z}_2$. In the former case $F = \{0\} * \mathbb{S}^1\left(\frac{1}{2}\right)$ which is impossible as $\partial F = \emptyset$. In the latter case $F = \left(\Sigma_1 \mathbb{S}^1(r)\right)/\langle I \rangle$, $r \in \left[\frac{1}{2}, 1\right]$, where I interchanges the two suspension points and is an involution on $\mathbb{S}^1(r)$. Such an involution is either the antipodal map or a reflection. When I is a reflection it fixes two points at distance πr apart which implies that $\operatorname{diam} F = \pi r$ and consequently $r = \frac{1}{2}$. In F consider two points e, f that form an angle θ at A. By Toponogov comparison

$$\cos|ef| \geq \cos|Ae| \cos|Af| + \sin|Ae| \sin|Af| \cos\theta.$$

So when $\theta \leq \frac{\pi}{2}$ and $|Ae|, |Af| \in \left(0, \frac{\pi}{2}\right)$ we have $|ef| < \frac{\pi}{2}$. This shows that $\operatorname{rad} F < \frac{\pi}{2}$, when I is a reflection and $r = \frac{1}{2}$. When I is an antipodal map we must have $\theta = \pi r$ to obtain maximal distance between e and f. So when $|Ae|, |Af| \in \left(0, \frac{\pi}{2}\right)$ and $r < 1$ we have $|ef| < |Ae| + |Af|$ as well as $|ef| \leq \pi - |Ae| - |Af|$ when we measure the distance through B. But this also forces $|ef| < \frac{\pi}{2}$. Thus F is isometric to $\mathbb{RP}^2(1)$. This also forces $X_2 = \mathbb{S}^2(1)$ with G acting as the antipodal map.

We can now use remark 5.8 to conclude that either $X = (X_1/\langle R\rangle) * X_2$ or $X = ((X_1/\langle R\rangle) * X_2)/\mathbb{Z}_2$. □

Remark 5.13. If we combine the constructions in the proof and remark 5.6, then it is possible to obtain a classification for Alexandrov spaces with $\operatorname{curv} \geq 1$, $\operatorname{rad} \geq \frac{\pi}{2}$, and dimension $1, 2, 3$. It would be interesting to investigate what results one can obtain for such spaces in higher dimension when they are not finite quotients of spheres. Do they, e.g., admit submetries onto $\left[0, \frac{\pi}{2}\right]$.

6. Quantified Convexity

The main theorem about the rigidity of lenses and hemispheres has counter parts for all convex balls in space forms. To understand this better we introduce some specially tailored modified distance functions for the distance to the boundary. As mentioned in the introduction the results here have appeared in [8] for lower curvature bounds ≥ 0, but as we use slightly different modified distance functions we thought it useful to include our proofs.

Consider the metric ball $\bar{B}_k(r_0)$ of radius r_0 in the space form of curvature k, when $k > 0$ we assume that $r_0 < \frac{\pi}{2\sqrt{k}}$ so as to only consider strictly convex balls. The boundary of this ball is totally umbilic and when using the outward pointing normal the eigenvalue of the shape operator is denoted $\lambda_0 > 0$. However, it is more natural to consider the inward pointing normal as that is also the gradient for the distance to $\partial\bar{B}_k(r_0)$ inside $\bar{B}_k(r_0)$. This means that the eigenvalue becomes $-\lambda_0$. The specific formula for λ_0 in terms of r_0 and k will be given below. Evidently λ_0 is a measure of the convexity of the boundary.

Let $r(x) = \left|x\partial\bar{B}_k(r_0)\right|$ be the distance to the boundary inside $\bar{B}_k(r_0)$ and consider the concentric ball of radius $r_0 - r$ that consists of points at distance $\geq r$ from the boundary. Denote by $\lambda(r) < 0$ the eigenvalue of the shape operator of this smaller ball for the inward pointing normal. It is not hard to see that λ satisfies the Riccati equation:

$$\dot{\lambda} + \lambda^2 = -k,\ \lambda(0) = -\lambda_0.$$

The related function ϕ coming from:

$$\ddot{\phi} + k\phi = -\lambda_0,\ \phi(0) = 0, \dot{\phi}(0) = 1,$$

also satisfies $\lambda\dot{\phi} = \ddot{\phi}$ and can be used to create a suitable modified distance function $f = \phi \circ r$ satisfying

$$\mathrm{Hess} f + kf = -\lambda_0.$$

Below we give explicit formulas for λ and ϕ when $k = 0, \pm 1$. With that information it is fairly easy to prove that λ and f satisfy the above equations:

Example 6.1. When $k = 0$ it follows that $r_0 \lambda_0 = 1$ and

$$\lambda(r) = \frac{1}{r - r_0} = \frac{\lambda_0}{\lambda_0 r - 1},$$
$$\phi(r) = r - \frac{\lambda_0}{2} r^2.$$

Example 6.2. When $k = 1$ it follows that $\lambda_0 = \cot(r_0)$ and

$$\lambda = \cot(r - r_0) = \frac{\lambda_0 \cot r + 1}{\lambda_0 - \cot r},$$
$$\phi = \sin r + \lambda_0 \cos r - \lambda_0.$$

Example 6.3. When $k = -1$ and there are no restrictions on λ_0 we have

$$\lambda = \begin{cases} \tanh(r - r_0), & 1 > \lambda_0 = \tanh r_0, \\ -1, & 1 = \lambda_0, \\ \coth(r - r_0), & 1 < \lambda_0 = \coth r_0. \end{cases}$$

Here only the last case leads to a ball of finite radius. In all three cases

$$\phi = \sinh r - \lambda_0 \cosh r + \lambda_0.$$

We say that the boundary in a Riemannian manifold M is λ_0-convex, when all eigenvalues for the shape operator are $\leq -\lambda_0$ with respect to the inward normal. When $\lambda_0^2 > -k$ there exists a metric ball $\bar{B}_k(r_0)$ as above with λ_0-convex boundary. This will be our comparion model space when M has sectional curvature $\geq k$.

If $r(x) = |x \partial M|$ and the boundary is λ_0-convex, then standard Riccati comparison shows that the super level sets $\{x \in M \mid r(x) \geq r\}$ have $-\lambda(r)$-convex boundary at points where r is smooth, here $\lambda(r)$ is given by

$$\dot{\lambda} + \lambda^2 = -k, \ \lambda(0) = -\lambda_0.$$

This in turn imples that the modified distance function $f = \phi \circ r$ satisfies

$$\mathrm{Hess} f + kf \leq -\lambda_0$$

in the support sense everywhere.

In the case of Alexandrov spaces there are several interesting model spaces aside from the constant curvature balls $\bar{B}_k(r_0)$.

Example 6.4. Let Y be an Alexandrov space with curvature ≥ 1 and consider the cone of radius r_0:

$$C = C_k(Y)(r_0) = \{(t, y) \mid y \in Y, t \in [0, r_0], (0, y_0) \sim (0, y_1)\}$$

with constant radial curvature k. The distance between points (t_0, y_0) and (t_1, y_1) is determined by the law of cosines in constant curvature k with the understanding that (t_i, y_i) is distance t_i from the cone point $(0, y)$ and that the angle between (t_0, y_0) and (t_1, y_1) at the cone point is given by $|y_0 y_1|_Y$. When $k > 0$ we also assume that $r_0 \leq \frac{\pi}{2\sqrt{k}}$ in order for this to become an Alexandrov space with curvature $\geq k$.

These cones have the same convexity properties as our model space $\bar{B}_k(r_0)$. Specifically, if $f(r(x)) = \phi(|x \partial C|)$ and $c(t)$ is any quasi-geodesic, then $f(t) = f \circ c(t)$ satisfies

$$\ddot{f} + kf = -\lambda_0.$$

Further observe that since the distance to the cone point is $d(t,y) = t$, we have $d + r = r_0$ everywhere on C and the standard modified distance $h(x) = \psi(d(x))$, where $\ddot{\psi} + k\psi = 1$, $\psi(0) = 0$, $\dot{\psi}(0) = 0$ satisfies

$$\ddot{h} + kh = 1$$

along all quasi-geodesics.

It follows from [1, Theorem 1.8 (a)] that the metric definition of convexity given in the introduction implies the following theorem.

Theorem 6.5. *Let X be an Alexandrov space with curvature $\geq k$ and boundary that is λ_0-convex. The radius of X is $\leq r_0$ and the modified distance function $f = \phi \circ r \circ \gamma$ satisfies $\ddot{f} + kf \leq -\lambda_0$ in the support sense along all (quasi)-geodesics γ in X.*

This shows in particular that

Corollary 6.6. *For any (quasi)-geodesic $c(t)$ in X with curvature $\geq k$ and λ_0-convex boundary we have that $f(t) = \phi \circ r \circ c(t)$ satisfies*

$$f(t) \leq \bar{f}(t)$$

for all $t \geq 0$ where $\bar{f} \geq 0$ and $\bar{f}$ is the unique solution to:

$$\ddot{\bar{f}} + k\bar{f} = -\lambda_0,\ \bar{f}(0) = f(0),\ \dot{\bar{f}}(0) = \dot{f}^{+}(0).$$

The second theorem in the introduction can now be proven as follows.

Theorem 6.7. *Let X be an Alexandrov space with curvature $\geq k$ and boundary that is λ_0-convex, where $\lambda_0^2 > \max\{-k, 0\}$. The radius satisfies $\operatorname{rad} X \leq r_0$ with equality only when X is isometric to the cone $C_k(S_sX)(r_0)$ for a suitable $s \in X$.*

Proof. It follows from [1, Cor 1.9 (1)] that $r(x) \leq r_0$ for all x, i.e., the inradius is $\leq r_0$. This in turn implies that f is strictly concave (see also [1, Thm 1.8 (a)]). For our modified distance functions this is obvious when $k \geq 0$ as we have $\ddot{f} \leq -kf - \lambda_0$. When $k = -1$ this also shows:

$$\ddot{f} \leq f - \lambda_0 = \sinh r - \coth r_0 \cosh r = -\frac{\cosh(r - r_0)}{\sinh r_0}.$$

This shows that X has a unique point soul $s \in X$ at maximal distance $r_1 \leq r_0$ from the boundary. Consider a quasi-geodesic $c : [0, b] \to X$ with $c(0) = s$ and the function $f(t) = f \circ c$. This function satisfies

$$\ddot{f} + kf \leq -\lambda_0,\ f(0) = \phi(r_1),\ \dot{f}(0) \leq 0.$$

By proposition 6.6 $f \leq \bar{f}$, where

$$\ddot{\bar{f}} + k\bar{f} = -\lambda_0,\ \bar{f}(0) = \phi(r_1),\ \dot{\bar{f}}(0) = 0.$$

The explicit form of $\bar{f}$ is as follows

$$\bar{f} = \begin{cases} -\cot r_0 + (\sin r_1 + \cot r_0 \cos r_1)\cos t, & k = 1, \\ r_1 - \frac{r_1^2}{2r_0} - \frac{t^2}{2r_0}, & k = 0, \\ \coth r_0 - (\sinh r_1 - \coth r_0 \cosh r_1)\cosh t, & k = -1. \end{cases}$$

These expressions imply that $\bar{f}(r_0) \leq 0$, and that $\bar{f}(r_0) = 0$ only occurs when $r_1 = r_0$. This shows that $b \leq r_0$ and consequently that the radius is $\leq r_0$. Moreover, equality can only happen when $r_1 = r_0$, i.e., the boundary is at constant distance from the soul.

The remainder of the proof is along the same lines as the rigidity statement in proposition 2.8. If we assume that $r_1 = r_0$, then this shows more generally that any quasi-geodesic emanating from the soul must hit the boundary precisely when $t = b = r_0$. Since this agrees with the distance to any point on the boundary any such quasi-geodesic is a minimal geodesic. In particular, any point in X lies on a minimal geodesic from the soul to the boundary and $d(x)+r(x) = |xs|+|x\partial X| = r_0$ for all $x \in X$. This shows that $\dot{d} = -\dot{r}$ and $\ddot{r} = -\ddot{d}$ along any geodesic in X. As both d and r have natural upper bounds on these second derivatives they also have the same natural lower bounds. For example when $k = 0$ we have

$$\begin{aligned} \ddot{d} &\leq \frac{1-\dot{d}^2}{d} \text{ and} \\ \ddot{r} &\leq \frac{1-\dot{r}^2}{r-r_0} = -\frac{1-\dot{d}^2}{d}. \end{aligned}$$

In particular, $\ddot{r} = \lambda(r)$ and $\ddot{f} + kf = -\lambda_0$. Moreover, with notation as in example 6.4 the modified distance function $h = \psi \circ d$, satisfies $\ddot{h} + kh = 1$. This shows that X is isometric to the cone $C = C_k(S_s X)(r_0)$ via the gradient exponential map $\mathrm{gexp}_s(k;\cdot) : C \to X$. Specifically, in both cases the gradient flow for h from the center is a flow along minimal geodesics and the distance between points on these radial geodesics is governed by the equation $\ddot{h} + kh = 1$ (see also [26, Section 2, esp. Thm 2.3.1]). □

Remark 6.8. The rigidity aspect of this proof can also be found in [1, Cor 1.10 (1)] where the authors establish rigidity for spaces X with curvature $\geq k$, λ_0-convex boundary, and maximal inradius, i.e., the maximal distance to the boundary is r_0. Note that while the radius of the lens L^n_α is $\frac{\pi}{2}$, its inradius is $\frac{\alpha}{2}$.

Corollary 6.9. *Let X be an Alexandrov space with curvature $\geq k$ and boundary that is λ_0-convex, where $\lambda_0^2 > \max\{-k, 0\}$. If $\mathrm{vol}_{n-1}\partial X = \mathrm{vol}_{n-1}\partial \bar{B}_k(r_0)$, then X is isometric to $\bar{B}_k(r_0)$.*

Proof. First observe that for any Alexandrov space with curvature $\geq k$ and rad $\leq r$ we can use Petrunin's solution of Lytchak's problem (see [26, 3.3.5]) to show that $\mathrm{vol}_{n-1}\partial X \leq \mathrm{vol}\,\partial C_k(S_p X)(r)$. In particular, if $\mathrm{rad}\,X < r_0$, then it follows that $\mathrm{vol}_{n-1}\partial X < \mathrm{vol}_{n-1}\partial\bar{B}_k(r_0)$. In particular, the condition $\mathrm{vol}_{n-1}\partial X = \mathrm{vol}_{n-1}\partial\bar{B}_k(r_0)$ implies that $\mathrm{rad}\,X = r_0$ and the above theorem that $X = C_k(S_s X)(r_0)$. Finally, $\mathrm{vol}_{n-1}(\partial C_k(S_s X)(r_0)) = \mathrm{vol}_{n-1}\partial\bar{B}_k(r_0)$ only when $S_s X = S^{n-1}(1)$. This shows that

$$X = C_k(S_s X)(r_0) = C_k\left(S^{n-1}(1)\right)(r_0) = \bar{B}_k(r_0).$$

□

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Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556
E-mail address: `kgrove2@nd.edu`

520 Portola Plaza, Dept of Math UCLA, Los Angeles, CA 90095
E-mail address: `petersen@math.ucla.edu`