UCLA UCLA Electronic Theses and Dissertations

Title

Network Formation, Information Acquisition and Social Learning

Permalink https://escholarship.org/uc/item/56s9589k

Author Song, Yangbo

Publication Date 2015

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA

Los Angeles

Network Formation, Information Acquisition and Social Learning

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy in Economics

by

Yangbo Song

© Copyright by Yangbo Song 2015

Network Formation, Information Acquisition and Social Learning

by

Yangbo Song

Doctor of Philosophy in Economics University of California, Los Angeles, 2015 Professor Ichiro Obara, Chair

Social and economic networks play an increasingly significant role in people's lives. The formation of such networks do not only provide exchange of material or monetary benefit, but also transmission of information. By creating direct links and indirect connections with others, on one hand the delivery of goods and financial transfers are made such as in trade networks or in buyer-seller networks, and on the other hand agents learn valuable, payoff related information from own observation or peer reviews. For this dissertation I studied network formation and related problems from various aspects. These studies have pointed to overlooked but crucial factors in the information structure among strategic agents and the process of information acquisition, which once taken into account have produced theoretical predictions in stark contrast to the existing literature, and have reconciled long-existing inconsistency between theory and data.

Chapter 1 studies the problem of social learning through observation. Social learning is the study of how dispersed information gets aggregated in a society of strategic agents, and what kind of structure of information acquisition the society needs to facilitate efficient information aggregation. The existing literature always assumes that the observation structure is exogenous, in other words, the structure of who observes whose actions is exogenously given by some deterministic or stochastic process. However, it is more natural or realistic to assume that observation is costly and strategic. My study takes this alternative assumption and shows that in contrast to the condition of expanding observation – meaning to observe a close predecessor – in the literature, a sufficient and necessary condition for the highest level of social learning with costly endogenous observation is infinite observation, i.e. to observe an arbitrarily large

number of predecessor. Endogenous observation also brings about a great difference in terms of individual behavior and social welfare.

Chapter 2 addresses the question of how networks form and what their ultimate topology is, under the much more natural yet hardly adopted assumption of incomplete information: agents do not know in advance – but must learn –the value of linking. This study shows that incomplete information has profound implications for the formation process and the ultimate topology. Under complete information, the network topologies that form and are stable typically consist of agents of relatively high value only. Under incomplete information, a much wider collection of network topologies can emerge and be stable. Moreover, even with the same topology, the locations of agents can be very different: an agent can achieve a central position purely as the result of chance rather than as the result of merit. All of this can occur even in settings where agents eventually learn everything so that information, although initially incomplete, eventually becomes complete. The ultimate network topology depends significantly on the formation history, which is natural and true in practice, and incomplete information makes this phenomenon more prevalent.

Chapter 3 again provides an analysis of the network formation process, but in another understudied strategic environment – one where agents are foresighted and care about both current and future payoffs. The related theoretical literature has often adopted a framework with homogeneous agents, and predicted that in generic cases it is impossible to sustain efficient networks even if such a network generates a positive payoff for every agent. However, these predictions are contradicted by data from existing real-life networks. This study analyzes a dynamic model allowing for agent heterogeneity and foresight, and establishes a Folk Theorem of networks, characterizing the set of sustainable networks in equilibrium for patient agents, and find that efficient networks can be sustained in equilibrium as long as it guarantees every agent a positive payoff. In the widely studied connections model which adopts a particular class of this valuation structure, a full characterization of efficient network is presented, which turn out to bear a striking resemblance in topology to networks observed in real life. The dissertation of Yangbo Song is approved.

William Zame

Alexander Stremitzer

Ichiro Obara, Committee Chair

University of California, Los Angeles 2015

TABLE OF CONTENTS

1	Soci	al Lear	ning with Endogenous Network Formation	1
	1.1	Introdu	uction	1
	1.2	Literat	ure Review	5
	1.3	Model		8
		1.3.1	Private Signal Structure	8
		1.3.2	The Sequential Decision Process	9
		1.3.3	Strong and Weak Private Beliefs	11
	1.4	Equilit	prium and Learning	12
		1.4.1	Perfect Bayesian Equilibrium	12
		1.4.2	Characterization of Individual Behavior	14
		1.4.3	Learning	17
	1.5	Learni	ng with Zero Cost	19
	1.6	Costly	Learning with Strong Private Beliefs	24
		1.6.1	Maximal Learning with Infinite Observations	24
		1.6.2	An Example	34
		1.6.3	Welfare Analysis	35
	1.7	Discus	sion	39
		1.7.1	Observation Preceding Signal	39
		1.7.2	Information Diffusion	42
		1.7.3	Flexible Observations with Non-Negative Marginal Cost	44
	1.8	Costly	Learning with Weak Private Beliefs	46
	1.9	Conclu	usion	51
2	D	omia N	atwork Formation with Incomplete Information	51
4	2 1	Intro d		54
	∠.1	mirodi		34

	2.2	Literati	are Review	6
	2.3	Model		9
		2.3.1	Networks with Incomplete Information	9
		2.3.2	Dynamic Network Formation Game	0
	2.4	Analys	is	2
		2.4.1	Stable Equilibrium and Stable Network	2
		2.4.2	Information Revelation	6
		2.4.3	Contrast between Complete and Incomplete Information 6	8
		2.4.4	Characterizing Topological Differences	6
		2.4.5	Social Welfare	1
	2.5	Bayesia	an Learning	6
	2.6	Conclu	sion and Future Research	9
3	Dvn	Dynamic Network Formation with Foresighted Agents		
-	- J			
-	3.1	Introdu	ction	1
-	3.1 3.2	Introdu Literatu	ction	1 6
-	3.1 3.2 3.3	Introdu Literatu Model	vection 9 ure Review 9	1 6 9
	3.1 3.2 3.3	Introdu Literatu Model 3.3.1	ction 9 ure Review 9	1 6 9 9
	3.1 3.2 3.3	Introdu Literatu Model 3.3.1 3.3.2	ction9ure Review9	1 6 9 9
	3.1 3.2 3.3	Introdu Literatu Model 3.3.1 3.3.2 3.3.3	ction9ure Review9	1 6 9 9
	3.1 3.2 3.3	Introdu Literatu Model 3.3.1 3.3.2 3.3.3 3.3.4	ction9ure Review9	1 6 9 9 1 2
	3.1 3.2 3.3 3.4	Introdu Literatu Model 3.3.1 3.3.2 3.3.3 3.3.4 Folk T	arction 9 are Review 9	1 6 9 9 1 2 3
	3.1 3.2 3.3 3.4	Introdu Literatu Model 3.3.1 3.3.2 3.3.3 3.3.4 Folk TU 3.4.1	action 9 ure Review 9	1 6 9 9 1 2 3 3
	3.1 3.2 3.3 3.4	Introdu Literatu Model 3.3.1 3.3.2 3.3.3 3.3.4 Folk Th 3.4.1 3.4.2	ction9ure Review9	1 6 9 9 9 1 2 3 3 6
	3.1 3.2 3.3 3.4	Introdu Literatu Model 3.3.1 3.3.2 3.3.3 3.3.4 Folk Th 3.4.1 3.4.2 3.4.3	ction 9 ure Review 9	1 6 9 9 9 1 2 3 3 6 7
	3.1 3.2 3.3 3.4	Introdu Literatu Model 3.3.1 3.3.2 3.3.3 3.3.4 Folk Th 3.4.1 3.4.2 3.4.3 3.4.4	are Review9	1 6 9 9 9 1 2 3 3 6 7 1

	3.5.1	Modeling Incomplete Information
	3.5.2	Construction of Equilibrium Strategies
	3.5.3	The Folk Theorem
	3.5.4	Incomplete Information and Inefficiency
3.6	Conne	ctions Model
	3.6.1	Characterization of Strongly Efficient Network
	3.6.2	Strong Efficiency and Stability
3.7	Conclu	usion
Referen	ices	

LIST OF FIGURES

1.1	Illustration of Network Topology	12
1.2	Equilibrium Behavior of Agent <i>n</i>	16
1.3	Equilibrium Learning Probability for Agent <i>n</i>	17
1.4	Learning Probability as a Function of <i>c</i>	36
1.5	Limit Learning Probability and Cost of Observation	38
1.6	Limit Learning Probability under Two Timing Schemes	42
2.1	Not Connected under Complete Info. vs. Connected under Incomplete Info	71
2.2	Empty under Complete Info. vs. Connected under Incomplete Info	71
2.3	Different Connectivity Degree Distributions	72
2.4	Simulations: Rank of "Low-Type" Agent	73
2.5	More Links under Complete Information	74
2.6	Simulations: Expected Difference	82
2.7	Simulations: Fraction of Expected Difference	83
2.8	Simulations: Number of Stable Networks	84
2.9	Simulations: Expected Social Welfare	86
2.10	Connected under Simple Updating Rule vs. Empty under Bayesian Learning	
	by Formation History	88
3.1	Sample core-periphery network	93
3.2	AHEP network	94
3.3	Strongly efficient network in connections model	132

LIST OF TABLES

3.1	Summary statistics of sample networks	93
-----	---------------------------------------	----

ACKNOWLEDGMENTS

First and foremost I would like to thank my advisor Ichiro Obara. It has been a great honor to be one of his Ph.D. students, and I am grateful for all his contributions of time, ideas and teaching to make my Ph.D. experience exciting and fruitful. During smooth times of my degree pursuit, he encouraged me to keep on exploring my academic potential; at tough moments of my research, he offered generous help and motivated me to maintain my effort. He has provided me an excellent example as a economic theorist and professor. I am also deeply indebted to William Zame, who has enlightened me academically on research and has supported me continuously during the job market season. He has shown me how a brilliant mind works and how rigorous and logical a good scholar should be.

The UCLA economic theory group has contributed immensely to my time as a Ph.D. student. The group has been a source of good advice on my research as well as insightful questions during departmental proseminars. In particular, I thank Simon Board and Moritz Meyer-ter-Vehn for my time spent as a research assistant and teaching assistant for them, Sushil Bikhchandani for his significant suggestions on my work, and Marek Pycia and Jernej Copic for their exceptional classes as well as inspiring discussions with me. I would also like to acknowledge the graduate students who attended many of my proseminars and shared their comments.

Most of my work on networks would not have been possible without the help of Mihaela van der Schaar, who is the co-author of Chapter 2 and 3 of this thesis. As a renowned electrical engineer and an expert in network theory, she has inspired me in many ways during our time of collaboration. She has helped me on formalizing research ideas, refining presentation skills and responding to manuscript referees. The joy and enthusiasm she has for her research was contagious and motivational for me. I am also thankful to graduate students in her group for assisting me with Matlab simulation, as well as many productive discussions.

For this dissertation I would like to thank the other members in my committee, Moritz Meyer-ter-Vehn and Alexander Stremitzer, for their time, interest and helpful comments.

My time at UCLA was made enjoyable in large part due to the many friends I made. I am grateful for the time spent with friends in the mountains, on the badminton courts and at the board-game tables. I have also appreciated the concern and support from my friends in China. Their encouragement has brought comfort during many difficult times in the past five years.

Lastly, I would like to thank my parents for all their love and care. They have raised me with determination in all my pursuits and have had faith in me all along. I wish to make them proud in my future academic career and other life endeavors. Thank you.

VITA

2010	B.A. Economics and Finance, the University of Hong Kong.
2010-2014	Teaching Assistant, Department of Economics, UCLA. Taught sec-
	tions of ECON201B (second course in the Ph.D. microeconomic se-
	quence) under supervision of Prof. Simon Board and Prof. Moritz
	Meyer-ter-Vehn.
2012	M.A. Economics, UCLA.
2012–2014	Research Assistant, Department of Electrical Engineering, UCLA.
	Worked under the supervision of Prof. Mihaela van der Schaar.
2014	Summer course instructor, UCLA. Taught an upper-division elec-
	tive course on pricing and strategy.

PUBLICATIONS

"Dynamic Network Formation with Incomplete Information" (with Mihaela van der Schaar), Economic Theory, 2015, DOI: 10.1007/s00199-015-0858-y.

"Sharing in Networks of Strategic Agents" (with Jie Xu and Mihaela van der Schaar), IEEE J. Sel. Topics Signal Process. – Special Issue on "Signal Processing for Social Networks", vol. 8, no. 4, pp. 717-731, Aug. 2014.

"Incentivizing Information Sharing in Networks" (with Jie Xu and Mihaela van der Schaar), 2014 IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP).

CHAPTER 1

Social Learning with Endogenous Network Formation

1.1 Introduction

How do people aggregate dispersed information? Imagine a scenario with a large number of agents, each trying to match her action with some underlying state of the world, e.g., consumers choosing the highest quality product, firms implementing the technology with the highest productivity, etc. On one hand, each agent may have some informative but noisy private signal about the particular state; combining all the signals will yield sufficient information for the entire society to learn the true state. However, such private signals are typically not directly observable to others; in other words, information is decentralized. On the other hand, an agent's action is observable and informative regarding her knowledge; thus, by observing one another, agents can still hope for some level of information aggregation. Therefore, it is of great importance to investigate the relation between the type of observation structure and the type of information aggregation that is achievable.

A large and growing literature has studied this problem of learning via observation of actions. Renowned early research, including the seminal works by Bikhchandani, Hirshleifer and Welch[BHW92], Banerjee[Ban92] and Smith and Sorensen[SS00], demonstrates that efficient information aggregation may fail in an equilibrium of a dynamic game, i.e., when agents act sequentially and observe the actions of all their predecessors, they may eventually "herd" on the wrong action. In a more recent paper, Acemoglu et al.[ADL11] consider a more general, and stochastic observation structure. They point out that society's learning of the true state depends on two factors: the possibility of arbitrarily strong private signals (unbounded private beliefs), and the nonexistence of excessively influential individuals (expanding observations). Expanding observations refer to the condition that in the limit, each agent observes the action of some predecessor whose position in the decision sequence is not particularly far from her own. In particular, Acemoglu et al.[ADL11] show that when private beliefs are unbounded, a necessary and sufficient condition for agents to undertake the correct action almost certainly in a large society is expanding observations.

In the studies discussed above and many other related works, a common modeling assumption is that the network of observation is exogenous: agents are not able to choose whose actions to observe or whether to observe at all. In practice, however, observation is typically both costly and strategic. First, time and resources are required to obtain information regarding others' actions. Second, an agent would naturally choose to observe what are presumably more informative actions based on the positions of individuals in the decision sequence. In this paper, I analyze an endogenous network formation framework where the choice of observation depends on the private signal, and address how it affects the aggregation of equilibrium information.

The outline of the model can be illustrated with the following example. Consider a firm facing the choice between two new technologies, and the productivity of these technologies cannot be perfectly determined until they are implemented. The firm has two sources of information to help guide its decision as to which technology to implement: privately received news regarding the productivity of the two technologies, on the one hand, and observing other firms' choices, on the other. The firm knows the approximate timing of those choices, but there is no direct communication: it is unable to obtain the private information of others and can know only which technology they have chosen. Moreover, observation is costly and is also a part of the firm's decision problem – the firm must decide whether to make an investment to set up a survey group or hire an outside agent to investigate other firms' choices. If it chooses to engage in observation, the firm must decide which of the other firms it would like to observe because there is likely a constraint on how much information can be gathered within a limited time and with limited resources.

More formally, there is an underlying state of the world in the model, which is binary in value. A large number of agents sequentially choose between two actions with the goal of matching their action with the true state. Each agent receives a private signal regarding what the true state is, but the signal is not perfectly revealing. In addition, after receiving her signal, each agent can pay a cost to observe a number of her predecessors, i.e., to connect with a certain neighborhood. Exactly which of the predecessors to observe is the agent's strategic choice, and

the number of others to observe is limited by an exogenous capacity structure. By observing a predecessor, the agent knows the action of the other, but not the other's private signal or which agents that have been observed by the other¹. After this process of information gathering, the agent makes her own choice.

In the present paper, I address the central question in this line of research under the new context of endogenous network formation, i.e., when can agents achieve the highest possible level of learning (taking the right action)? In the literature, this scenario is referred to as *asymptotic learning*, which means that the true state is revealed in the limit, and that information aggregation in equilibrium would be the same as if all private information were public. When observation is endogenous, asymptotic learning may never occur in any equilibrium (e.g., when the cost of observation is too high for a rational agent to acquire information). Hence, asymptotic learning no longer characterizes the upper bound of social learning with endogenous observation. I therefore generalize the notion of the highest equilibrium learning probability to *maximal learning*, which means that, in the limit, information aggregation in equilibrium would be if an agent could pay to access and observe all prior private information. In fact, maximal learning reduces to asymptotic learning when the cost of observation is 0 or when private beliefs are relatively weak with respect to cost.

There are thus two central factors determining the type of learning achievable in equilibrium. The first is the relative precision of the private signal, which is represented by the relation between the likelihood ratio and the cost of observation. Consider a hypothetical scenario in which an agent can pay to directly observe the true state. If a private signal indicates that the costly acquisition of the true state is not worthwhile, then the agents have strong private beliefs; otherwise, they have weak private beliefs. The extreme case of strong private belief is unbounded private belief, i.e., the likelihood ratio may approach infinity and is not bounded away from zero. If the likelihood ratio is always finite and bounded away from zero, then the agents have bounded private beliefs. Note that bounded private belief can also be strong, depending on the cost. The second key factor is the capacity structure, which describes the maximum number of observations for each agent. I say that the capacity structure has infinite observations when the number of observations goes to infinity as the size of the society becomes arbitrarily large; otherwise, the capacity structure has finite observations. Infinite ob-

¹If observing an agent also reveals her observation, there exists information diffusion in the game. In the present paper, I discuss this case after presenting the main results.

servations imply that the influence of any one agent's action on the others becomes trivial as the size of the society grows because that action only accounts for an arbitrarily small part of the observed neighborhood.

The main results of this paper are presented in three theorems. Theorem 1 posits that when the cost of observation is zero and agents have unbounded private beliefs, asymptotic learning occurs in every equilibrium. As discussed above, the previous literature has shown that a necessary and sufficient condition for asymptotic learning under unbounded private beliefs is expanding observations. This theorem implies that when observation can be strategically chosen with zero cost, the condition of expanding observations becomes a property that is automatically satisfied in every equilibrium, i.e., every rational agent will choose to observe at least some action of a close predecessor. This can be regarded as a micro-foundation for the prevalence of expanding observations when observation is free.

Theorem 2 is this paper's most substantive contribution and demonstrates that a sufficient and necessary condition for maximal learning is infinite observations when cost is positive and private beliefs are strong. Multiple implications can be drawn from this result. First, when cost is positive and private beliefs are strong, asymptotic learning is impossible because there is always a positive probability that an agent chooses not to observe. In other words, maximal learning marks the upper bound of social learning. Second, to achieve maximal learning, this theorem implies that no agent can be significantly influential at all, which contrasts sharply with the results in the previous literature. In other words, no matter how large the society is, an agent can no longer know the true state by observing a bounded number of actions (even if they are actions by close predecessors); however, an agent can and only can do so via observing an arbitrarily large neighborhood. Since each agent makes a mistake with positive probability (when he decides not to observe), efficient information aggregation can only occur when the influence of any agent is arbitrarily small. Third, this result leads to a number of interesting comparative statics. For instance, in the limit, the equilibrium learning probability (the probability that an agent's action is correct) may be higher when the cost is positive than when the cost is zero and may be higher when private beliefs become weaker. Fourth, this theorem facilitates several important variations of the model. For instance, it can be shown that the pattern of social learning would be much different if endogenous observation preceded private signal, in contrast to the existing literature where this order of information makes no

difference when observation is exogenous. In addition, a partial characterization of the level of social learning can be obtained under a more general cost structure.

Theorem 3 provides a method to approximate maximal learning when private beliefs are weak. The key idea is to introduce a stochastic capacity constraint: with a probability uniformly bounded away from zero, an agent can only choose her observation within a non-persuasive neighborhood, i.e., a subset of agents such that her private signal may be more informative than any realized observation. In this manner, when an agent can observe freely with infinite observations, he can still almost surely learn the true state by paying the cost. The welfare impact of this general stochastic observation structure is consistent with the discussion above: weak private beliefs may ultimately result in a higher learning probability than strong private beliefs.

The remainder of this paper is organized as follows: Section 2 provides a review of the related literature. Section 3 introduces the model. Section 4 defines the equilibrium and each type of learning that is discussed in this paper and characterizes the equilibrium behavior. Sections 5 to 8 present the main results and their implications, in addition to a number of extensions. Section 9 concludes.

1.2 Literature Review

A large and growing literature studies the problem of social learning by Bayesian agents who can observe others' choices. This literature begins with Bikhchandani, Hirshleifer and Welch[BHW92] and Banerjee[Ban92], who first formalize the problem systematically and concisely and point to information cascades as the cause of herding behavior. In their models, the informativeness of the observed action history outweighs that of any private signal with a positive probability, and herding occurs as a result. Smith and Sorensen[SS00] propose a comprehensive model of a similar environment with a more general signal structure, and show that apart from the usual herding behavior, a new robust possibility of confounded learning occurs when agents have heterogeneous preferences: they neither learn the true state asymptotically nor herd on the same action. Smith and Sorensen[SS00] clearly distinguish "private" belief that is given by private signals and "public" belief that is given by observation, and they also introduce the concepts of bounded and unbounded private beliefs, whose meaning and

importance were discussed above. These seminal papers, along with the general discussion by Bikhchandani, Hirshleifer and Welch[BHW98], assume that agents can observe the entire previous decision history, i.e., the whole ordered set of choices of their predecessors. This assumption can be regarded as an extreme case of exogenous network structure. In related contributions to the literature, such as Lee[Lee93], Banerjee[Ban93] and Celen and Kariv[CK04], agents may not observe the entire decision history, but exogenously given observation remains a common assumption.

A more recent paper, Acemoglu et al.[ADL11], studies the environment where each agent receives a private signal about the underlying state of the world and observes (some of) their predecessors' actions according to a general stochastic network topology. Their main result states that when the private signal structure features unbounded belief, asymptotic learning occurs in each equilibrium if and only if the observation structure exhibits expanding observations. Other recent research in this area include Banerjee and Fudenberg[BF04], Gale and Kariv[GK03], Callander and Horner[CH09] and Smith and Sorensen[SS13], which differ from Acemoglu et al.[ADL11] mainly in making alternative assumption for observation, i.e., that agents only observe the number of others taking each available action but not the positions of the observed agents in the decision sequence. However, all these papers also share the assumption of exogenous observation that is shared in the earlier literature discussed above.

The key difference between my paper and the literature discussed above is that observation is costly and strategic. First, each agent can choose whether to pay to acquire more information about the underlying state via observation. If the private signal is rather strong or the cost of observation is too high, an agent may rationally choose not to observe at all. Second, upon paying the cost, each agent can choose exactly which actions are included in the observation up to an exogenously given capacity constraint. In this way, society's observation network is endogenously formed, and hence we can examine not only the rational choice of action to match the true state but also the rational choice of whether to observe and which actions to observe as a cost-efficient decision regarding the acquisition of additional information.

There have been several recent papers that discuss the impact of costly observation on social learning. In Kultti and Miettinen[KM06][KM07], both the underlying state and the private signal are binary, and an agent pay a cost for each action she observes. In Celen[Cel08], the signal structure is similar to the general one adopted in this paper, but it is assumed that an

agent can pay a cost to observe the entire action history before her. My model can be regarded as a richer treatment as compared to those papers, in the sense that it allows for a wide range of signal strucutures, as well as the possibility that agents would have to strategically choose a proper subset of their predecessors' actions to observe². More importantly, this paper provides necessary and sufficient conditions for social learning to reach its theoretical upper bound under costly observation, which is a central question that remains unanswered in previous research. Some of the major findings in the above cited works, for example the existence of cost may lead to welfare improvement, are confirmed in this paper as well.

Another branch of the literature introduces a costly and strategic choice into the decision process – each agent can pay to acquire an informative signal, or to "search", i.e., sample an available option and know its value. Notable works in this area include Hendricks, Sorensen and Wiseman[HSW12], Mueller-Frank and Pai[MP14] and Ali[Ali14]. My paper differs from this stream of the literature in two aspects. On one hand, in those papers, the observation structure – the neighborhood that each agent observes – remains exogenous. In addition, agents in their models can obtain direct information about the true state such as signal or value of an option, whereas agents in the present paper can only acquire indirect information (others' actions) by paying the applicable cost.

There is also a well-known literature on non-Bayesian learning in social networks. In these models, rather than applying Bayes' update to obtain the posterior belief regarding the underlying state of the world by using all the available information, agents may adopt some intuitive rule of thumb to guide their choices (Ellison and Fudenberg[EF93][EF95]), only update their beliefs according to part of their information (Bala and Goyal[BG98][BG01]), or be subject to a certain bias in interpreting information (DeMarzo, Vayanos and Zwiebel[DVZ03]). Despite the various ways to model non-Bayesian learning, it is still common to assume that the network topology is exogenous. In terms of results, Golub and Jackson[GJ10] utilize a similar implication to that of Theorem 2 in this paper: they assume that agents naively update beliefs by taking weighted averages of their neighbors' beliefs and show that a necessary and sufficient condition for complete social learning (almost certain knowledge of the true state in a large and connected society over time) is that the weight put on each neighbor converges to zero for each

²For the main parts of this paper, agents are assumed to pay a single cost for observation, but in a later section I also present results regarding a more general cost function, which is non-decreasing in the number of actions observed.

agent as the size of the society increases.

Finally, the importance of observational learning via networks has been well documented in both empirical and experimental studies. Conley and Udry[CU01] and Munshi[Mun04] both focus on the adoption of new agricultural technology and not only support the importance of observational learning but also indicate that observation is often constrained because a farmer may not be able, in practice, to receive information regarding the choice of every other farmer in the area. Munshi[Mun03] and Ioannides and Loury[IL04] demonstrate that social networks play an important role in individuals' information acquisition regarding employment. Cai, Chen and Fang[CCF09] conduct a natural field experiment to indicate the empirical significance of observational learning in which consumers obtain information about product quality from the purchasing decisions of others.

1.3 Model

1.3.1 Private Signal Structure

Consider a group of countably infinite agents: $\mathcal{N} = \{1, 2, ...\}$. Let $\theta \in \{0, 1\}$ be the state of the world with equal prior probabilities, i.e., $Prob(\theta = 0) = Prob(\theta = 1) = \frac{1}{2}$. Given θ , each agent observes an i.i.d. private signal $s_n \in S = (-1, 1)$, where *S* is the set of possible signals. The probability distributions regarding the signal conditional on the state are denoted as $F_0(s)$ and $F_1(s)$ (with continuous density functions $f_0(s)$ and $f_1(s)$). The pair of measures (F_0, F_1) are referred to as the *signal structure*, and I assume that the signal structure has the following properties:

- 1. The pdfs $f_0(s)$ and $f_1(s)$ are continuous and non-zero everywhere on the support, which immediately implies that no signal is fully revealing regarding the underlying state.
- 2. Monotone likelihood ratio property (MLRP): \$\frac{f_1(s)}{f_0(s)}\$ is strictly increasing in s. This assumption is made without loss of generality: as long as no two signals generate the same likelihood ratio, the signals can always be re-aligned to form a structure that satisfies the MLRP.
- 3. Symmetry: $f_1(s) = f_0(-s)$ for any *s*. This assumption can be interpreted as indicating that the signal structure is unbiased. In other words, the distribution of an agent's private

belief, which is determined by the likelihood ratio, would be symmetric between the two states.

Assumption 3 is strong compared with the other two assumptions. Nevertheless, many results in this paper can be easily generalized in an environment with an arbitrarily asymmetric signal structure. For those that do rely on symmetry, the requirement is not strict – the results will hold as long as $f_1(s)$ and $f_0(-s)$ do not differ by very much, and an agent's equilibrium behavior is similar when receiving *s* and -s for a large proportion of private signals $s \in (-1, 1)$. Therefore, the symmetry of signal structure serves as a simplification of a more general condition, whose essential elements are similar (in a symmetric sense) private signal distributions and similar equilibrium behavior under the two states.

1.3.2 The Sequential Decision Process

The agents sequentially make a single action each between 0 and 1, where the order of agents is common knowledge. Let $a_n \in \{0, 1\}$ denote agent *n*'s decision. The payoff of agent *n* is

$$u_n(a_n, \theta) = \begin{cases} 1, \text{ if } a_n = \theta; \\ 0, \text{ otherwise.} \end{cases}$$

After receiving her private information and before engaging in the above action, an agent may acquire information about others from a network of observation³. In contrast with much of the literature on social learning, I assume that the network topology is not exogenously given but endogenously formed. Each agent *n* can pay a cost $c \ge 0$ to obtain a *capacity* $K(n) \in \mathbb{N}$; otherwise, he pays nothing and chooses \emptyset . I assume that the number of agents whose capacity is zero is finite, i.e., there exists $N \in \mathbb{N}$ such that K(n) > 0 for all n > N.

With capacity K(n), agent *n* can select a *neighborhood* $B(n) \subset \{1, 2, \dots, n-1\}$ of at most K(n) size, i.e., $|B(n)| \leq K(n)$, and observe the action of each agent in B(n). The actions in B(n) are observed at the same time, and no agent can choose any additional observation based on what she has already observed. Let $\mathscr{B}(n)$ be the set of all possible neighborhoods of at most K(n) size (including the empty neighborhood) for agent *n*. We say that there is a *link* between agent *n* and every agent in the neighborhood that *n* observes. By the definition set forth above,

³In a later section, I will discuss the case involving an alternative order in which observation of actions takes place before a realization of private signal.

a link in the network is *directed*, i.e., unilaterally formed, and without affecting the observed agent. I refer to the set $\{K(n)\}_{n=1}^{\infty}$ as the capacity structure, and I define a useful property for it below.

Definition 1. A capacity structure $\{K(n)\}_{n=1}^{\infty}$ has infinite observations if

$$\lim_{n\to\infty}K(n)=\infty$$

If the capacity structure does not satisfy this property, then we say it has finite observations.

Example 1. Some typical capacity structures are described below:

- 1. K(n) = n 1 for all n: each agent can pay the cost to observe the entire previous decision history, which conforms to the early literature on social learning.
- 2. K(n) = 1 for all n: each agent can observe only one of her predecessors. If observation is concentrated on one agent, the network becomes a star; at the other extreme, if each agent observes her immediate predecessor, the network becomes a line.

In between the above two extreme examples is the general case that $K(n) \in (1, n-1)$ for all n: each agent can, at a cost, observe an ordered sample of her choice among her predecessors. Note that a capacity structure featuring infinite observations requires only that the sample size grows without bound as the society becomes large but does not place any more restrictions on the sample construction. As will be shown in the subsequent analysis, this condition on sample size alone plays a key role in determining the achievable level of social learning.

An agent's strategy in the above sequential game consists of two problems: (1) given her private signal, whether to make costly observation and, if yes, whom to observe; (2) after observation (or not), which action to take between 0 and 1. Let $H_n(B(n)) = \{a_m \in \{0,1\} : m \in B(n)\}$ denote the set of actions that *n* can possibly be observed from B(n) and let $h_n(B(n))$ be a particular action sequence in $H_n(B(n))$. Let $I_n(B(n)) = \{s_n, h_n(B(n))\}$ be *n*'s *information set*, given B(n). Agent *n*'s information set is her private information and cannot be observed by others⁴. The set of all possible information sets of *n* is denoted as $\mathscr{I}_n = \{I_n(B(n)) : B(n) \subset \{1, 2, \dots, n-1\}, |B(n)| \leq K(n)\}$.

⁴In a later section, I will discuss the scenario with information diffusion, i.e., $h_n(B(n))$ can also be observed by creating a link with agent *n*.

A *strategy* for *n* is the set of two mappings $\sigma_n = (\sigma_n^1, \sigma_n^2)$, where $\sigma_n^1 : S \to \mathscr{B}(n)$ selects *n*'s choice of observation for every possible private signal, and $\sigma_n^2 : \mathscr{I}_n \to \{0,1\}$ selects a decision for every possible information set. A *strategy profile* is a sequence of strategies $\sigma = \{\sigma_n\}_{n \in \mathbb{N}}$. I use $\sigma_{-n} = \{\sigma_1, \dots, \sigma_{n-1}, \sigma_{n+1}, \dots\}$ to denote the strategies of all agents other than *n*. Therefore, for any *n*, $\sigma = (\sigma_n, \sigma_{-n})$.

1.3.3 Strong and Weak Private Beliefs

An agent's *private belief* given signal *s* is defined as the conditional probability of the true state being 1, i.e., $\frac{f_1(s)}{f_0(s)+f_1(s)}$. Note that it is a function of *s* only, since it does not depend on the agents' actions. I now define several categories of private beliefs that will be useful in the subsequent analysis. The notions *unbounded and bounded private beliefs* follow from the existing literature; the notions *strong and weak private beliefs* are applicable for costly observation in particular.

• 1. Agents have unbounded private beliefs if

$$\lim_{s \to 1} \frac{f_1(s)}{f_0(s) + f_1(s)} = 1$$
$$\lim_{s \to -1} \frac{f_1(s)}{f_0(s) + f_1(s)} = 0.$$

Agents have bounded private beliefs if

$$\begin{split} &\lim_{s \to 1} \frac{f_1(s)}{f_0(s) + f_1(s)} < 1 \\ &\lim_{s \to -1} \frac{f_1(s)}{f_0(s) + f_1(s)} > 0. \end{split}$$

The above definitions of unbounded and bounded private beliefs are standard in the previous literature and do not depend on the cost of observation c.

• 2. When c > 0, agents have *strong private beliefs* if there is $s^* < 1$ and $s_* > -1$ such that

$$\frac{f_1(s^*)}{f_1(s^*) + f_0(s^*)} = 1 - c$$
$$\frac{f_0(s_*)}{f_1(s_*) + f_0(s_*)} = 1 - c.$$

Given the symmetry assumption in the private signal structure, $s^* = -s_*$. Agents have *weak private beliefs* if the above defined s^* and s_* do not exist.

Strong and weak private beliefs describe the relation between such private beliefs and the cost of observation. When an agent has strong private beliefs, she is not willing to pay the cost *c* for a range of extreme private signals, even if doing so guarantees the knowledge of the true state of the world. In other words, private signals may have sufficiently high informativeness to render costly observation unnecessary. The opposite is weak private beliefs, in which case an agent always prefers observation when it contains enough information about the true state.

It is clear that unbounded private belief implies strong private belief for any positive cost. In the subsequent analysis, we will see that properties of private beliefs play an important role in the type of observational learning that can be achieved. To make the problem interesting, I assume that $c < \frac{1}{2}$; in other words, an agent will never choose not to observe merely because the cost is too high.

1.4 Equilibrium and Learning

1.4.1 Perfect Bayesian Equilibrium

Given a strategy profile, the sequence of decisions $\{a_n\}_{n\in\mathbb{N}}$ and the network topology (i.e., the sequence of the observed neighborhood) $\{B(n)\}_{n\in\mathbb{N}}$ are both stochastic processes. I denote the probability measure generated by these stochastic processes as \mathscr{P}_{σ} and \mathscr{Q}_{σ} correspondingly.



Figure 1.1: Illustration of Network Topology

Figure 1 illustrates the world from the perspective of agent 5, who knows her private signal s_5 , her capacity K(5), and the possible observation behavior of predecessors 1 - 4 (denoted by different colors). If agent 5 knows the strategies of her predecessors, some possible behaviors

may be excluded, e.g., agent 3 may never observe agent 1.

Definition 2. A strategy profile σ^* is a pure strategy **perfect Bayesian equilibrium** if for each $n \in \mathbb{N}$, σ_n^* is such that given σ_{-n}^* , (1) $\sigma_n^{*2}(I_n)$ maximizes the expected payoff of n given every $I_n \in \mathscr{I}_n$; (2) $\sigma_n^{*1}(s_n)$ maximizes the expected payoff of n, given every s_n and given σ_n^{*2} .

For any equilibrium σ^* , agent *n* first solves

$$\max_{\mathbf{y}\in\{0,1\}}\mathscr{P}_{\boldsymbol{\sigma}_{-n}^*}(\mathbf{y}=\boldsymbol{\theta}|s_n,h_n(B(n)))$$

for any $s_n \in (-1, 1)$ and any observed action sequence $h_n(B(n))$ from any $B(n) \subset \{1, \dots, n-1\}$ satisfying $|B(n)| \leq K(n)$. This maximization problem has a solution for each agent *n* because it is a binary choice problem. Denote the solution to this problem as $y_n^*(s_n, h_n(B(n)))$. Then, agent *n* solves

$$\max_{B(n)\subset\{1,\cdots,n-1\}:|B(n)|\leq K(n)}\mathbb{E}[\mathscr{P}_{\sigma_{-n}^*}(y_n^*(s_n,h_n(B(n)))=\theta|s_n,h_n(B(n)))|s_n]$$

This is, once again, a problem of discrete choice. Hence, given an indifference-breaking rule, there is a solution for every s_n . Finally, if the difference between the maximized expected probability of taking the correct action with observation and that without observation exceeds the cost of observation c, then the agent chooses to observe (the observed neighborhood being the solution to the above problem); otherwise, she chooses not to observe. Proceeding inductively for each agent determines an equilibrium.

Note that the existence of a perfect Bayesian equilibrium does not depend on the assumption of a symmetric signal structure. However, this assumption guarantees the existence of a *symmetric* perfect Bayesian equilibrium, i.e., an equilibrium σ^* in which, for each $s_n \in [0, 1)$, $\sigma_n^{*1}(s_n) = \sigma_n^{*1}(-s_n)$. In other words, when the optimal neighborhood to observe (including the empty neighborhood, i.e., not to observe any predecessor) in equilibrium σ^* is the same for every agent *n*, given any pair of private signals, s_n and $-s_n$, then σ^* is a symmetric equilibrium. In fact, if the optimal observed neighborhood is unique for each agent and every private signal, then each perfect Bayesian equilibrium will be symmetric. For instance, if for some private signal $s_4 > 0$, the unique optimal neighborhood for agent 4 to observe is $\{2,3\}$, then due to the symmetric signal structure, it must also be her unique optimal neighborhood to observe when her private signal is $-s_4$. In the remainder of the paper, the analysis focuses on pure strategy

symmetric Bayesian equilibria, and henceforth I simply refer to them as "equilibria". I note the existence of equilibrium below.

Proposition 1. There exists a pure strategy symmetric perfect Bayesian equilibrium.

As discussed briefly above, I will clearly identify which results can be generalized to an environment with asymmetric private signal distributions (and thus with asymmetric equilibria).

1.4.2 Characterization of Individual Behavior

My first results show that equilibrium individual decisions regarding whether to observe can be represented by an interval on the support of private signal.

Proposition 2. When c > 0, then in every equilibrium σ^* , for every $n \in \mathbb{N}$:

- 1. For any $s_n^1 > s_n^2 \ge 0$ (or $s_n^1 < s_n^2 \le 0$), if $\sigma_n^{*1}(s_n^1) \ne \emptyset$, then $\sigma_n^{*1}(s_n^2) \ne \emptyset$.
- 2. $\mathscr{P}_{\sigma^*}(a_n = \theta | s_n)$ is weakly increasing (weakly decreasing) in s_n for all non-negative (non-positive) s_n such that $\sigma_n^{*1}(s_n) \neq \emptyset$.
- 3. There is one and only one signal $s_*^n \in [0,1]$ such that $\sigma_n^{*1}(s_n) \neq \emptyset$ if $s_n \in [0,s_*^n)$ (if $s_n \in (-s_*^n,0]$) and $\sigma_n^{*1}(s_n) = \emptyset$ if $s_n > s_*^n$ (if $s_n < -s_*^n$).

Proof. 1: Consider any $s_n \ge 0$. Let $H_n^1(s_n)$ $(H_n^0(s_n))$ denote the set of observed actions in equilibrium that will induce agent *n* to choose action 1 (0) when her private signal is s_n . We know that

$$\mathcal{P}_{\sigma^*}(a_n = \theta | s_n) = \frac{f_0(s_n)}{f_0(s_n) + f_1(s_n)} \mathcal{P}_{\sigma^*}(h_n(\sigma_n^{*1}(s_n)) \in H_n^0(s_n) | \theta = 0) + (1 - \frac{f_0(s_n)}{f_0(s_n) + f_1(s_n)}) \mathcal{P}_{\sigma^*}(h_n(\sigma_n^{*1}(s_n)) \in H_n^1(s_n) | \theta = 1).$$

Hence, the marginal benefit of observation is

$$\mathcal{P}_{\sigma^*}(a_n = \theta | s_n) - \frac{f_1(s_n)}{f_0(s_n) + f_1(s_n)}$$

= $\frac{f_0(s_n)}{f_0(s_n) + f_1(s_n)} \mathcal{P}_{\sigma^*}(h_n(\sigma_n^{*1}(s_n)) \in H_n^0(s_n) | \theta = 0)$
- $\frac{f_1(s_n)}{f_0(s_n) + f_1(s_n)} \mathcal{P}_{\sigma^*}(h_n(\sigma_n^{*1}(s_n)) \in H_n^0(s_n) | \theta = 1).$

Now, consider any $s_n^1 > s_n^2 \ge 0$, and the following sub-optimal strategy $\sigma'_n(s_n^2)$ for agent n when her private signal is s_n^2 : observe the same neighborhood and given any observation, choose the same action as if her signal were s_n^1 . The marginal benefit of observation under this strategy is

$$\frac{f_0(s_n^2)}{f_0(s_n^2) + f_1(s_n^2)} \mathscr{P}_{\sigma^*}(h_n(\sigma_n^{*1}(s_n^1)) \in H_n^0(s_n^1) | \theta = 0) - \frac{f_1(s_n^2)}{f_0(s_n^2) + f_1(s_n^2)} \mathscr{P}_{\sigma^*}(h_n(\sigma_n^{*1}(s_n^1)) \in H_n^0(s_n^1) | \theta = 1).$$

Because $\sigma_n^{*1}(s_n^1) \neq \emptyset$ by assumption, we know that

$$\frac{f_0(s_n^1)}{f_0(s_n^1) + f_1(s_n^1)} \mathscr{P}_{\sigma^*}(h_n(\sigma_n^{*1}(s_n^1)) \in H_n^0(s_n^1) | \theta = 0) - \frac{f_1(s_n^1)}{f_0(s_n^1) + f_1(s_n^1)} \mathscr{P}_{\sigma^*}(h_n(\sigma_n^{*1}(s_n^1)) \in H_n^0(s_n^1) | \theta = 1) \ge c.$$

By the MLRP, $\frac{f_1(s_n^2)}{f_0(s_n^2) + f_1(s_n^2)} < \frac{f_1(s_n^1)}{f_0(s_n^1) + f_1(s_n^1)}$ and $\frac{f_0(s_n^2)}{f_0(s_n^2) + f_1(s_n^2)} > \frac{f_0(s_n^1)}{f_0(s_n^1) + f_1(s_n^1)}$. Therefore, we have $\frac{f_0(s_n^2)}{f_0(s_n^2) + f_1(s_n^2)} \mathscr{P}_{\sigma^*}(h_n(\sigma_n^{*1}(s_n^1)) \in H_n^0(s_n^1) | \theta = 0)$ $- \frac{f_1(s_n^2)}{f_0(s_n^2) + f_1(s_n^2)} \mathscr{P}_{\sigma^*}(h_n(\sigma_n^{*1}(s_n^1)) \in H_n^0(s_n^1) | \theta = 1) > c,$

which implies that $\sigma_n^{*1}(s_n^2) \neq \emptyset$.

2: Consider any $s_n^1 > s_n^2 \ge 0$, and the following sub-optimal strategy $\sigma'_n(s_n^1)$ for agent *n* when her private signal is s_n^1 : observe the same neighborhood and, given any observation, choose the same action as if her signal were s_n^2 . We have

$$\mathcal{P}_{\sigma^*}(a_n = \theta | s_n^1) \ge \mathcal{P}_{\sigma_{-n}^*, \sigma_n'(s_n^1)}(a_n = \theta | s_n^1)$$

= $\frac{f_0(s_n^1)}{f_0(s_n^1) + f_1(s_n^1)} \mathcal{P}_{\sigma^*}(h_n(\sigma_n^{*1}(s_n^2)) \in H_n^0(s_n^2) | \theta = 0)$
+ $(1 - \frac{f_0(s_n^1)}{f_0(s_n^1) + f_1(s_n^1)}) \mathcal{P}_{\sigma^*}(h_n(\sigma_n^{*1}(s_n^2)) \in H_n^1(s_n^2) | \theta = 1)$

Therefore, we know that

$$\begin{aligned} \mathscr{P}_{\sigma^*}(a_n = \theta | s_n^1) - \mathscr{P}_{\sigma^*}(a_n = \theta | s_n^2) \\ \geq \mathscr{P}_{\sigma^*_{-n}, \sigma'^1_n(s_n^1)}(a_n = \theta | s_n^1) - \mathscr{P}_{\sigma^*}(a_n = \theta | s_n^2) \\ = &(\frac{f_0(s_n^2)}{f_0(s_n^2) + f_1(s_n^2)} - \frac{f_0(s_n^1)}{f_0(s_n^1) + f_1(s_n^1)}) \\ &(\mathscr{P}_{\sigma^*}(h_n(\sigma_n^{*1}(s_n^2)) \in H_n^1(s_n^2) | \theta = 1) - \mathscr{P}_{\sigma^*}(h_n(\sigma_n^{*1}(s_n^2)) \in H_n^0(s_n^2) | \theta = 0)). \end{aligned}$$

Consider any $h \in H_n^0(s_n^2)$, and consider h' from the same neighborhood such that every action 0 (1) in h is replaced by 1 (0) in h'. We know from the definition of $H_n^0(s_n^2)$ that $f_0(s_n^2) \mathscr{P}_{\sigma^*}(h|\theta=0) > f_1(s_n^2) \mathscr{P}_{\sigma^*}(h|\theta=1)$; by the assumption that $s_n^2 \ge 0$, we have $\mathscr{P}_{\sigma^*}(h|\theta=0) > \mathscr{P}_{\sigma^*}(h|\theta=1)$. By symmetry, it follows that

$$\mathscr{P}_{\sigma^*}(h'|\theta=1)=\mathscr{P}_{\sigma^*}(h|\theta=0)>\mathscr{P}_{\sigma^*}(h|\theta=1)=\mathscr{P}_{\sigma^*}(h'|\theta=0).$$

Hence, we have $f_1(s_n^2) \mathscr{P}_{\sigma^*}(h'|\theta=1) > f_0(s_n^2) \mathscr{P}_{\sigma^*}(h'|\theta=0)$, i.e., $h' \in H_n^1(s_n^2)$. It then follows that $\mathscr{P}_{\sigma^*}(h_n(\sigma_n^{*1}(s_n^2)) \in H_n^1(s_n^2)|\theta=1) \ge \mathscr{P}_{\sigma^*}(h_n(\sigma_n^{*1}(s_n^2)) \in H_n^0(s_n^2)|\theta=0)$, which immediately implies that $\mathscr{P}_{\sigma^*}(a_n=\theta|s_n^1) \ge \mathscr{P}_{\sigma^*}(a_n=\theta|s_n^2)$.

3: This result follows directly from 1.

This proposition shows that observation is more favorable for an agent with a weaker signal, which is intuitive because information acquired from observation is relatively more important when an agent is less confident about her private information. The proposition then implies that for any agent *n*, there is one and only one non-negative cut-off signal in [0, 1], which is denoted as s_n^n , such that agent *n* will choose to observe in equilibrium if $s_n \in [0, s_n^n)$ and not to observe if $s_n > s_n^n$. It is also clear that when $s_n^n \in (0, 1)$, agent *n* must be indifferent between observing and not observing at $s_n = s_n^n$. Under the symmetry assumption regarding the signal structure, the case when the private signal is non-positive is analogous. Figure 2 below illustrates the behavior of agent *n* in equilibrium; note that when agent *n* chooses to observe, the exact neighborhood observed may depend on her private signal s_n .



Figure 1.2: Equilibrium Behavior of Agent n

The second implication of this proposition is that the learning probability (i.e., the probability of taking the correct action) has a nice property of monotonicity when the agent observes a non-empty neighborhood. When he chooses not to observe, i.e., when $s_n > s_*^n$ ($s_n < -s_*^n$), the probability of taking the correct action is also increasing (decreasing) in s_n because the probability is simply equal to $\frac{f_1(s_n)}{f_0(s_n)+f_1(s_n)}$ ($\frac{f_0(s_n)}{f_0(s_n)+f_1(s_n)}$). However, this monotonicity is not preserved from observing to not observing because observation is costly and an agent with a stronger signal may be content with a lower learning probability to save on costs. Figure 3 below shows the shape of this probability with respect to s_n . Of the two continuous curves, the top curve depicts the probability of taking the correct action if agent n always observes (denoted $P(a_n = \theta | O)$), whereas the bottom curve illustrates the probability of taking the correct action if agent n never observes (denoted $P(a_n = \theta | NO)$). The solid "broken" curve measures the learning probability in equilibrium. The difference between the two continuous curves is greater than c at $s_n \in [0, s_*^n)$ (and $s_n \in (-s_*^n, 0]$), less than c at $s_n > s_*^n$ (and $s_n < -s_*^n$), and equal to c at $s_n = s_*^n$ (and $s_n = -s_*^n$).



Figure 1.3: Equilibrium Learning Probability for Agent n

1.4.3 Learning

The main focus of this paper is to determine what type of information aggregation will result from equilibrium behavior. First, I define the different types of learning studied in this paper.

Definition 3. Given a signal structure (F_0, F_1) , we say that asymptotic learning occurs in

equilibrium σ^* if a_n converges to θ in probability: $\lim_{n\to\infty} \mathscr{P}_{\sigma^*}(a_n = \theta) = 1$.

Next, I define *maximal learning*, which is a natural extension of asymptotic learning. Before the formal definition, I introduce an intermediate and conceptual term: suppose that a hypothetical agent can learn the true state by paying cost c. Clearly, an optimal strategy of this agent is to pay c and learn the true state if and only if her private signal lies in some interval $(\underline{s}, \overline{s})$ (this interval is equal to (s_*, s^*) when private beliefs are strong and (-1, 1) when private beliefs are weak). Let $P^*(c)$ denote her probability of taking the right action under this strategy.

Definition 4. Given a signal structure (F_0, F_1) and a cost of observation c, we say that **maximal** learning occurs in equilibrium σ^* if the probability of a_n being the correct action converges to $P^*(c)$: $\lim_{n\to\infty} \mathscr{P}_{\sigma^*}(a_n = \theta) = P^*(c)$.

Asymptotic learning requires that the unconditional probability of taking the correct action converges to 1, i.e., the posterior beliefs converge to a degenerate distribution on the true state. In terms of information aggregation, asymptotic learning can be interpreted as equivalent to making all private signals public and thus aggregating information efficiently. It marks the upper bound of social learning with an *exogenous* observation structure. However, when observation becomes *endogenous*, it is notable that asymptotic learning is impossible in certain cases. For instance, consider the case when cost is positive and private beliefs are strong. Indeed, because there is now a range of signals (to be precise, two intervals of extreme signals) such that an agent would not be willing to pay the cost even to know the true state with certainty, there is always the probability of making a mistake when the private signal falls into such a range. Therefore, an alternative notion is needed to characterize a more appropriate upper bound of social learning, which could theoretically be reached in some equilibrium. Maximal learning, as defined above, serves this purpose.

Maximal learning extends the notion of efficient information aggregation to an environment in which information acquisition is costly and means that, in the limit, agents can learn the true state as if they can pay the cost of observation to access all prior private signals, i.e., efficient information aggregation can be achieved at a price. From the perspective of equilibrium behavior, maximal learning occurring in an equilibrium implies that, in the limit, an agent will almost certainly take the right action whenever she chooses to observe. The term $P^*(c)$ is less than 1 when private beliefs are strong⁵ because an agent may choose not to observe when her private signal is highly informative. It is equal to 1 when c = 0 or when private beliefs are weak. In other words, maximal learning reduces to asymptotic learning in these two circumstances. The goal of this paper is then to characterize conditions that lead to maximal learning (or asymptotic learning, as a special case) in equilibrium.

1.5 Learning with Zero Cost

A central question is determining what conditions must be imposed on the capacity structure of observation for asymptotic/maximal learning. The answer to this is closely connected with the relation between the precision of private signals and the cost of observation. I begin by considering the case in which there is zero cost, i.e., observation is free. Even in this case, it is notable that not every agent will always choose to observe in equilibrium: if private signals are sufficiently strong, there may not be any realized action sequence in an observed neighborhood that can alter the agent's action. In other words, an agent may be indifferent between observation and no observation.

The following theorem is one of the main results of the paper, and shows that unbounded private beliefs play a crucial role for learning in a society with no cost of observation. In particular, asymptotic learning, the strongest form of social learning, can be achieved in every equilibrium. This result holds even without the symmetry assumption for the signal structure.

Theorem 1. When c = 0 and agents have unbounded private beliefs, asymptotic learning occurs in every equilibrium.

Proof. First, note that when c = 0, it is always feasible for an agent to observe and imitate her immediate predecessor, which guarantees her the same expected payoff as this immediate predecessor. Hence, we know that $\mathscr{P}_{\sigma^*}(a_n = \theta)$ is weakly increasing in *n*. Since this probability is upper bounded by 1, the sequence $\{\mathscr{P}_{\sigma^*}(a_n = \theta)\}$ converges. Let *r* denote the limit.

Suppose that r < 1. Thus, for any $\varepsilon > 0$, N exists such that for any n > N, we have $\{\mathscr{P}_{\sigma^*}(a_n = \theta)\} \in (r - \varepsilon, r]$. Take one such n, and consider agent n + 1 and her sub-optimal strategy of observing agent n. By the assumption of unbounded private belief, \overline{s}_{n+1} and \underline{s}_{n+1}

⁵To be more precise, when private beliefs are strong, $P^*(c) = \frac{1}{2}F_0(s^*) + \frac{1}{2}(1 - F_1(s_*))$. With a symmetric signal structure, it is equal to $F_0(s^*)$.

exists such that agent n + 1 is indifferent between following her own signal and following agent n's action, i.e., the following two conditions are satisfied:

$$\begin{aligned} \frac{f_1(\bar{s}_{n+1})}{f_0(\bar{s}_{n+1}) + f_1(\bar{s}_{n+1})} &= \frac{f_1(\bar{s}_{n+1})}{f_0(\bar{s}_{n+1}) + f_1(\bar{s}_{n+1})} \mathscr{P}_{\sigma^*}(a_n = 1 | \theta = 1) \\ &+ \frac{f_0(\bar{s}_{n+1})}{f_0(\bar{s}_{n+1}) + f_1(\bar{s}_{n+1})} \mathscr{P}_{\sigma^*}(a_n = 0 | \theta = 0) \\ \frac{f_1(\underline{s}_{n+1})}{f_0(\underline{s}_{n+1}) + f_1(\underline{s}_{n+1})} &= \frac{f_1(\underline{s}_{n+1})}{f_0(\underline{s}_{n+1}) + f_1(\underline{s}_{n+1})} \mathscr{P}_{\sigma^*}(a_n = 1 | \theta = 1) \\ &+ \frac{f_0(\underline{s}_{n+1})}{f_0(\underline{s}_{n+1}) + f_1(\underline{s}_{n+1})} \mathscr{P}_{\sigma^*}(a_n = 0 | \theta = 0). \end{aligned}$$

The above equation can be further simplified as

$$\frac{f_1(\bar{s}_{n+1})}{f_0(\bar{s}_{n+1})} = \frac{\mathscr{P}_{\sigma^*}(a_n = 0|\theta = 0)}{\mathscr{P}_{\sigma^*}(a_n = 0|\theta = 1)}$$
$$\frac{f_0(\underline{s}_{n+1})}{f_0(\underline{s}_{n+1})} = \frac{\mathscr{P}_{\sigma^*}(a_n = 1|\theta = 1)}{\mathscr{P}_{\sigma^*}(a_n = 1|\theta = 0)}.$$

By the previous argument, $\mathscr{P}_{\sigma^*}(a_n = \theta) = \frac{1}{2} \mathscr{P}_{\sigma^*}(a_n = 1 | \theta = 1) + \frac{1}{2} \mathscr{P}_{\sigma^*}(a_n = 0 | \theta = 0) \leq r$, and hence $\min\{\mathscr{P}_{\sigma^*}(a_n = 1 | \theta = 1), \mathscr{P}_{\sigma^*}(a_n = 0 | \theta = 0)\} \leq r$. Without loss of generality, assume that $\mathscr{P}_{\sigma^*}(a_n = 1 | \theta = 1) \leq r$. Then, $\mathscr{P}_{\sigma^*}(a_n = 0 | \theta = 1) \geq 1 - r$, and hence $\frac{f_1(\bar{s}_{n+1})}{f_0(\bar{s}_{n+1})} \leq \frac{1}{1-r}$. Let \hat{s} be the value of the private signal such that $\frac{f_1(\hat{s})}{f_0(\hat{s})} = \frac{2}{1-r}$. We know that

$$\begin{split} \mathscr{P}_{\sigma^*}(a_{n+1} = \theta) - \mathscr{P}_{\sigma^*}(a_n = \theta) \\ \geq \int_{s}^{1} \frac{1}{2} (f_1(s) \mathscr{P}_{\sigma^*}(a_n = 0 | \theta = 1) - f_0(s) \mathscr{P}_{\sigma^*}(a_n = 0 | \theta = 0)) ds \\ \geq \int_{s}^{1} \frac{1}{2} ((1 - r) f_1(s) - f_0(s)) ds \\ \geq \int_{s}^{1} \frac{1}{2} f_0(s) ds. \end{split}$$

Therefore, we have

$$\int_{\hat{s}}^{1} \frac{1}{2} f_0(s) ds \le \mathscr{P}_{\sigma^*}(a_{n+1} = \theta) - \mathscr{P}_{\sigma^*}(a_n = \theta) < \varepsilon$$

For sufficiently small ε , this inequality is violated, and thus we have a contradiction.

Theorem 1 presents an interesting comparison with the existing literature. Most of the above mentioned studies examine conditions on exogenous networks that induce (or do not induce) asymptotic learning; in contrast, Theorem 1 shows that as long as private beliefs are unbounded, a network topology that ensures asymptotic learning will *automatically* form. In

other words, with an endogenous network formation, the individual interest in maximizing the expected payoff and the social interest of inducing the agents' actions to converge to the true state are now aligned in the limit. In every equilibrium, agents with private signals that are not particularly strong will seek to increase the probability that they will take the right action via observation. Because there is no cost for observation, the range of signals given that an agent could choose to observe enlarges unboundedly within the signal support as the society grows. In the limit, each agent almost certainly chooses to observe, and information is thus efficiently aggregated without any particular assumption on the capacity structure.

The argument above also provides a general intuition behind the proof of this theorem. Suppose that asymptotic learning did not occur in some equilibrium, then there must be (1) a limit to the probability of taking the correct action – which is less than one – and (2) two limit thresholds for private signals, one in (0,1) and one in (-1,0), beyond which an agent will be indifferent between observation and no observation. However, for an agent whose behavior is very close to this limit, observing her immediate predecessor (whose behavior is also close to the limit) will produce a strict improvement, i.e., her learning probability will exceed that limit probability and her threshold of signals that imply indifference will exceed the limit signals, which is a contradiction.

Acemoglu et al.[ADL11] note that a necessary condition of network topology that leads to asymptotic learning is expanding observations, i.e., no agent is excessively influential in terms of being observed by others. In other words, no agent (or subset of agents) is the sole source of observational information for infinitely many other agents. This important result leads to the second implication of Theorem 1 regarding the equilibrium network topology: although it is difficult to precisely characterize agents' behavior in each equilibrium, we know that the equilibrium network must feature expanding observations, i.e., agents will always observe a close predecessor. This is an intuitive result because the action of someone later in the decision sequence presumably reveals more information. I formally describe this property of equilibrium network below.

Corollary 1. If c = 0 and agents have unbounded private beliefs, then every equilibrium σ^* exhibits expanding observations:

$$\lim_{n\to\infty}\mathscr{Q}_{\sigma^*}(\max_{b\in B(n)}b< M)=0 \text{ for any } M\in\mathbb{N}.$$

Proof. By Theorem 1 in Acemoglu et al. (2011), any network topology that does not have expanding observations cannot support asymptotic learning. Because asymptotic learning occurs in every equilibrium when c = 0 and agents have unbounded private beliefs, it must follow that \mathcal{Q}_{σ^*} has expanding observation for any σ^* .

A very simple but illustrative example of the foregoing is that of K(n) = 1 for all n. As will be illustrated in details in the next section, the optimal observation of each agent (if any) in equilibrium must be the action of her immediate predecessor. The condition of expanding observations is satisfied exactly according to this description: because observation has no cost, each agent almost certainly chooses to observe in the limit, but no agent excessively influences other agents because each agent only influences her immediate successor. However, we will learn in the next section that when cost is positive and private beliefs are strong, an analogous condition – observing a close predecessor when choosing to observe – would *not* suffice for the highest level of information aggregation achievable in equilibrium, i.e., for maximal learning.

In the other direction, when agents have bounded private beliefs, asymptotic learning does not occur for a number of typical capacity structures and associated equilibria. The following result lists some scenarios.

Proposition 3. If c = 0 and agents have bounded private beliefs, then asymptotic learning does not occur in the following scenarios:

- (*a*) K(n) = n 1.
- (b) Some constant \overline{K} exists such that $K(n) \leq \overline{K}$ for all n.

Proof. Part (*a*) is already proved by Smith and Sorensen[SS00] in their Theorem 1. Acemoglu et al.[ADL11] offer an alternative proof in their Theorem 3. First, note that when the cost of observation is zero and K(n) = n - 1 for all *n*, for every agent with any private signal in any equilibrium, observing her optimal choice of neighborhood generates the same equilibrium behavior as in the equilibrium in which each agent observes all her predecessors' actions regardless of her private signal because any action that the agent chooses not to observe must have no influence on her own action. Second, in the equilibrium where each agent observes all her predecessors' actions with her predecessors' actions regardless of her private signal because of her private signal, the agents' behavior coincides with
that in a model where this observation structure (each agent observing the entire action history before her) is exogenously given. Hence, the proofs above apply directly.

For Part (*b*), assume that c = 0 and private beliefs are bounded, and suppose that there is an equilibrium in which asymptotic learning occurs. I first show that for all $M \in \mathbb{N}$, $N \in \mathbb{N}$ exists such that $\max_{b \in \sigma_n^{*1}(s_n)} b > M$ for every s_n and every n > N.

To prove the above claim, first note that by assumption, $N' \in \mathbb{N}$ exists such that K(n) > 0for all n > N'. It immediately follows from c = 0 that $\mathscr{P}_{\sigma^*}(a_n = \theta)$ is weakly increasing in n for all n > N'. Next, for any $M \in \mathbb{N}$, let $\hat{a}(s)$ denote the action that maximizes the expected payoff for an agent whose private signal is s and who observes the neighborhood $\{1, \dots, M\}$. It is clear that $\sup_{s \in (-1,1)} \mathscr{P}_{\sigma_1^*, \dots, \sigma_M^*}(\hat{a}(s) = \theta) < 1$ because private beliefs are bounded and M is finite. Because asymptotic learning occurs by assumption, $N'' \in \mathbb{N}$ exists such that $\mathscr{P}_{\sigma^*}(a_n =$ $\theta) > \sup_{s \in (-1,1)} \mathscr{P}_{\sigma_1^*, \dots, \sigma_M^*}(\hat{a}(s) = \theta)$ for all n > N''. Let $N = \max\{N', N''\} + 1$. For any agent n > N, by observing agent N and copying agent N's action, she can achieve a strictly higher probability of taking the right action than by observing $\{1, \dots, M\}$. Hence, it must be the case that $\max_{b \in \sigma_n^{*1}(s_n)} b > M$ for every s_n and every n > N. Then, the argument in the proof of Theorem 3 of Acemoglu et al.[ADL11], whose validity requires only the existence of capacity upper bound \overline{K} , can be applied to show that asymptotic learning does not occur in any equilibrium.

The proposition above highlights two scenarios in which bounded private beliefs block asymptotic learning. In the first scenario, which corresponds to part (a), it can be shown that the "social belief" for any agent, i.e., the posterior belief established from observation alone, is bounded away from 1 in either state of the world, 0 and 1, regardless of the true state. As a result, asymptotic learning becomes impossible. With a positive probability, herding behavior occurs in equilibrium: either starting from some particular agent, *all subsequent* agents choose the same (wrong) action (when social belief exceeds private belief at some point); or the equilibrium features longer and longer periods of uniform behavior, punctuated by increasingly rare switches (when social belief converges to but never exceeds private belief).

The second scenario, which corresponds to part (b), posits that under either state, there is a positive probability that *all* the agents choose incorrectly, which is another form of herding behavior. When private beliefs are bounded, an agent's private signal may not be strong enough to "overturn" the implication from a rather informative observation, and the agent would thus ignore her private information and simply follow her observation. This affects not only her own behavior but also the observational learning of her successors because they would also be aware that her action no longer reveals any information about her own private signal. Therefore, efficient information aggregation cannot proceed. For instance, it is clear that under either state of the world, the probability that the first N agents all choose action 1 is bounded away from zero. When N is large, and when agent N+1 observes a large neighborhood such that an action sequence of 1's is more informative than each of her possible private signals, she will then also choose 1 regardless of her own signal, and so will every agent after her. Herding behavior thus ensues as a result.

1.6 Costly Learning with Strong Private Beliefs

1.6.1 Maximal Learning with Infinite Observations

I have already argued before that when observation is costly and private beliefs are strong, asymptotic learning is impossible in any equilibrium. Furthermore, as will be shown below, maximal learning is not guaranteed in equilibrium either. In fact, we can see that whenever agents have finite observations, maximal learning cannot occur in any equilibrium. For an agent to choose to make a costly observation given her private signal, it must be the case that some realized action sequence in her observed neighborhood is so informative that she would rather turn against her signal and choose the other action. When private beliefs are strong, each agent chooses actions 0 and 1 with positive probabilities regardless of the true state; therefore, under either state 0 or 1, the above informative action sequence occurs with a positive probability. As a result, for any agent who chooses to observe, there is always a positive probability of making a mistake. For instance, consider again the example of K(n) = 1 for all n. When private beliefs are strong, the probabilities that any agent would choose 0 when $\theta = 1$ and 1 when $\theta = 0$ are bounded below by $F_1(-s^*)$ and $1 - F_0(s^*)$ correspondingly (with the symmetry assumption regarding the signal structure, the two probabilities are equal); therefore, the probabilities that any agent would take the wrong action when $\theta = 1$ and when $\theta = 0$ have the same lower bounds as well, given that this agent chooses to observe.

The next main result of this paper, Theorem 2, shows that a necessary and sufficient con-

dition for maximal learning with strong private beliefs consists of infinite observations in the capacity structure.

Theorem 2. When agents have strong private beliefs, maximal learning occurs in every equilibrium if and only if the capacity structure has infinite observations.

The implication of Theorem 2 is two-fold. On one hand, the necessity of infinite observations stands in stark contrast to the expanding observations in the previous section, which means that no agent can be excessively influential but an agent may still be significantly influential for infinitely many others. In a world in which the cost of observation is positive and agents may sometimes rationally choose not to observe, for maximal learning to occur, *no agent* can be significantly influential in the sense that any agent's action can only take up an arbitrarily small proportion in any other agent's observation. Indeed, because the probability of any agent making a mistake is now bounded away from zero, infinite observations must be required to suppress the probability of the wrong implication from an observed neighborhood.

On the other hand, Theorem 2 guarantees maximal learning when there are infinite observations. Whenever the size of the observed neighborhood can become arbitrarily large as the society grows, the probability of taking the right action based on observation converges to one. The individual choice of not observing, given some extreme signals – and thus a source for any single agent to make a mistake on her own – actually facilitates social learning by observation: because any agent may choose not to observe with positive probability, her action in turn must reveal some information about her private signal. Thus, by adding sufficiently many observations to a given neighborhood, i.e., by enlarging the neighborhood considerably, the informativeness of the entire observed action sequence can always be improved. Once a neighborhood can be arbitrarily large, information can be aggregated efficiently to reveal the true state.

Following this intuition, I now introduce an outline of the proof of Theorem 2. Several preliminary lemmas are needed. The first lemma below simply formalizes the argument that when private beliefs are strong, each agent will choose not to observe with a probability bounded away from zero.

Lemma 1. When agents have strong private beliefs, in every equilibrium σ^* , for all $n \in \mathbb{N}$, $s_*^n < s^*$.

Proof. The definition of s^* implies that when agent *n* has a private signal of s^* , he is indifferent between paying *c* to know the true state and choosing accordingly, and paying nothing and choosing 1. Note that the largest possible benefit from observing is always strictly less than knowing the true state with certainty. Hence, the (positive) private signal that makes agent *n* indifferent between observing and not observing must be less than s^* .

Next, I show that infinite observations are a necessary condition for maximal learning, in contrast to most existing literature stating that observing some close predecessor's action suffices for knowing the true state, in an exogenously given network of observation.

Lemma 2. Assume that agents have strong private beliefs. If the capacity structure has finite observations, then maximal learning does not occur in any equilibrium.

Proof. Suppose that when the capacity structure has finite observations, maximal learning occurs in some equilibrium σ^* . It follows that $K \in \mathbb{N}$ exists such that, for any $\varepsilon > 0$ and $N \in \mathbb{N}$, n > N exists such that $\mathscr{P}_{\sigma^*}(a_n = \theta | s_n \in (-s_*^n, s_*^n)) > 1 - \varepsilon$ and $0 < K(n) \le K$.

Consider one such agent *n*. For at least one state $\theta \in (0, 1)$, there must be some $s_n \in (-s_*^n, s_*^n)$ such that $\mathscr{P}_{\sigma^*}(a_n = \theta | \theta, s_n) > 1 - \varepsilon$. Without loss of generality, assume that (one) such θ is 1. Since $s_n \in (-s_*^n, s_*^n)$, there must be some realized action sequence in *n*'s observed neighborhood $\sigma_n^{*1}(s_n)$, given which *n* will take action 0. Because, by Lemma 1, $K(n) \leq K$, we know that when the true state is 1, the probability of this action sequence occurring is bounded below by min $\{F_1(s_*), 1 - F_1(s^*)\}^K$. Hence, we have

$$1 - \varepsilon < \mathscr{P}_{\sigma^*}(a_n = 1 | 1, s_n) \le 1 - \min\{F_1(s_*), 1 - F_1(s^*)\}^K$$

However, for sufficiently small ε we have $1 - \varepsilon > 1 - \min\{F_1(s_*), 1 - F_1(s^*)\}^K$, which is a contradiction.

The logic behind the proof of Lemma 2 is rather straightforward. With strong private beliefs, in either state of the world, each agent takes actions 0 and 1 with probabilities bounded away from zero. Thus, when agent n observes a neighborhood of at most K size, the probability that the realized action sequence in this neighborhood would induce agent n to take the wrong action is also bounded away from zero. If infinitely many agents can only observe a neighborhood whose size has the same upper bound, maximal learning can never occur. The following few lemmas contribute to the proof of the sufficiency of infinite observations for maximal learning in every equilibrium. Given an equilibrium σ^* , let $B_k = \{1, \dots, k\}$, and consider any agent who observes B_k . Let $R_{\sigma^*}^{B_k}$ be the random variable of the posterior belief on the true state being 1, given each decision in B_k . For each realized belief $R_{\sigma^*}^{B_k} = r$, we say that a realized private signal *s* and decision sequence *h* in B_k induce *r* if $\mathcal{P}_{\sigma^*}(\theta = 1|h, s) = r$.

Lemma 3. For either state $\theta = 0, 1$ and for any $s \in (s_*, s^*)$, we have

$$\begin{split} &\lim_{\varepsilon \to 0^+} (\limsup_{k \to \infty} \mathscr{P}_{\sigma^*}(R^{B_k}_{\sigma^*} > 1 - \varepsilon | 0, s)) \\ &= \lim_{\varepsilon \to 0^+} (\limsup_{k \to \infty} \mathscr{P}_{\sigma^*}(R^{B_k}_{\sigma^*} < \varepsilon | 1, s)) = 0. \end{split}$$

Proof. I prove here that $\lim_{\varepsilon \to 0^+} (\limsup_{k \to \infty} \mathscr{P}_{\sigma^*}(R^{B_k}_{\sigma^*} > 1 - \varepsilon | 0, s)) = 0$, and the second equality would follow from an analogous argument. Suppose the equality does not hold, then $s \in (s_*, s^*)$ and $\rho > 0$ exist such that for any $\varepsilon > 0$ and any $N \in \mathbb{N}$, k > N exists such that $\mathscr{P}_{\sigma^*}(R^{B_k}_{\sigma^*} > 1 - \varepsilon | 0, s) > \rho$. Consider any realized decision sequence h_{ε} from B_k that, together with *s*, induces some $r > 1 - \varepsilon$, and let H_{ε} denote the set of all such decision sequences; thus, we know that

$$\frac{\mathscr{P}_{\sigma^*}(h_{\varepsilon}|\theta')f_{\theta'}(s)}{\mathscr{P}_{\sigma^*}(h_{\varepsilon}|\theta)f_{\theta}(s) + \mathscr{P}_{\sigma^*}(h_{\varepsilon}|\theta')f_{\theta'}(s)} = r$$
$$\sum_{h_{\varepsilon}\in H_{\varepsilon}}\mathscr{P}_{\sigma^*}(h_{\varepsilon}|\theta) > \rho.$$

The above two conditions imply that

$$1 \geq \sum_{h_{\varepsilon} \in H_{\varepsilon}} \mathscr{P}_{\sigma^*}(h_{\varepsilon}|\theta') > \frac{(1-\varepsilon)\rho f_{\theta}(s)}{\varepsilon f_{\theta'}(s)}.$$

For sufficiently small ε , we have $\frac{(1-\varepsilon)\rho f_{\theta}(s)}{\varepsilon f_{\theta'}(s)} > 1$, which is a contradiction.

Lemma 3 shows that the action sequence in neighborhood B_k cannot induce a degenerate belief on the wrong state of the world with positive probability as k becomes large. This result is necessary because the posterior belief on the wrong state after observing the original neighborhood must be bounded away from 1 if any strict improvement on the learning probability is to occur by expanding a neighborhood. In the next lemma, I demonstrate the feasibility of such strict improvement. **Lemma 4.** Assume that agents have strong private beliefs. Given any realized belief $r \in (0,1)$ on state 1 for an agent observing B_k , for any $\hat{r} \in (0,r)$ ($\hat{r} \in (r,1)$), $N(r,\hat{r}) \in \mathbb{N}$ exists such that a realized belief that is less than \hat{r} (higher than \hat{r}) can be induced by additional $N(r,\hat{r})$ consecutive observations of action 0 (1) in any equilibrium.

Proof. Without loss of generality, assume that $\hat{r} \in (0, r)$. We know that there is a private signal *s* and an action sequence *h* from B_k such that

$$r = \frac{\mathscr{P}_{\sigma^*}(h|1)f_1(s)}{\mathscr{P}_{\sigma^*}(h|1)f_1(s) + \mathscr{P}_{\sigma^*}(h|0)f_0(s)}$$

Consider $h \cup \{a_{k+1}\}$ where $a_{k+1} = 0$. The new belief would then be

$$r_{1} = \frac{\mathscr{P}_{\sigma^{*}}(h|1)f_{1}(s) \times \mathscr{P}_{\sigma^{*}}(a_{k+1}=0|h,1)}{\mathscr{P}_{\sigma^{*}}(h|1)f_{1}(s) \times \mathscr{P}_{\sigma^{*}}(a_{k+1}=0|h,1) + \mathscr{P}_{\sigma^{*}}(h|0)f_{0}(s) \times \mathscr{P}_{\sigma^{*}}(a_{k+1}=0|h,0)}$$

Note that

$$\mathcal{P}_{\sigma^*}(a_{k+1} = 0|h, 1) = F_1(-s_*^{k+1}) + \mathcal{P}_{\sigma^*}(a_{k+1} = 0, \text{ observe}|h, 1)$$
$$\mathcal{P}_{\sigma^*}(a_{k+1} = 0|h, 0) = F_0(-s_*^{k+1}) + \mathcal{P}_{\sigma^*}(a_{k+1} = 0, \text{ observe}|h, 0),$$

and that

$$\mathcal{P}_{\sigma^*}(a_{k+1}=0, \text{ observe}|h, 1) = \int_{-s_*^{k+1}}^{s_*^{k+1}} \mathcal{P}_{\sigma^*}(a_{k+1}=0, |h, 1, s_{k+1}) f_1(s_{k+1}) ds_{k+1}$$
$$\mathcal{P}_{\sigma^*}(a_{k+1}=0, \text{ observe}|h, 0) = \int_{-s_*^{k+1}}^{s_*^{k+1}} \mathcal{P}_{\sigma^*}(a_{k+1}=0, |h, 0, s_{k+1}) f_0(s_{k+1}) ds_{k+1}$$

In any equilibrium, note that $\mathscr{P}_{\sigma^*}(a_{k+1} = 0, |h, 1, s_{k+1}) = \mathscr{P}_{\sigma^*}(a_{k+1} = 0, |h, 0, s_{k+1})$ for any given *h* and $s_{k+1} \in [-s_*^{k+1}, s_*^{k+1}]$. Moreover, given any $s_{k+1} \in [0, s_*^{k+1}]$, $\mathscr{P}_{\sigma^*}(a_{k+1} = 0, |h, 1, s_{k+1})$ and $\mathscr{P}_{\sigma^*}(a_{k+1} = 0, |h, 0, s_{k+1})$ are either 0 or 1.

For any $s_{k+1} \in [0, s_*^{k+1}]$, note that in a symmetric equilibrium, agent k+1 observes the same neighborhood, given private signal s_{k+1} and $-s_{k+1}$. Hence, if k+1 chooses 1 with private signal $-s_{k+1}$, then he will also choose 1 with private signal s_{k+1} ; if k+1 chooses 0 with private signal s_{k+1} , then he will also choose 0 with private signal $-s_{k+1}$. Together with the assumptions of symmetric signal structure and the MLRP, which imply that $f_1(-s_{k+1}) = f_0(s_{k+1}) \le f_1(s_{k+1}) =$ $f_0(-s_{k+1})$, it then follows that

$$\mathcal{P}_{\sigma^*}(a_{k+1}=0,|h,1,s_{k+1})f_1(s_{k+1}) + \mathcal{P}_{\sigma^*}(a_{k+1}=0,|h,1,-s_{k+1})f_1(-s_{k+1})$$

$$\leq \mathcal{P}_{\sigma^*}(a_{k+1}=0,|h,0,s_{k+1})f_0(s_{k+1}) + \mathcal{P}_{\sigma^*}(a_{k+1}=0,|h,0,-s_{k+1})f_0(-s_{k+1}).$$

Therefore, we have $\mathscr{P}_{\sigma^*}(a_{k+1} = 0, \text{ observe}|h, 0) \ge \mathscr{P}_{\sigma^*}(a_{k+1} = 0, \text{ observe}|h, 1)$. Together with Lemma 1, we have

$$\frac{\mathscr{P}_{\sigma^*}(a_{k+1}=0|h,1)}{\mathscr{P}_{\sigma^*}(a_{k+1}=0|h,0)} \le \frac{F_1(s^{k+1})}{F_0(s^{k+1})} < \frac{F_1(s^*)}{F_0(s^*)} < 1.$$

The second inequality is based on the fact that $F_1(s^*) - F_1(-s^*) = F_0(s^*) - F_0(-s^*)$ by the symmetry of the signal structure. Therefore, we have

$$\begin{split} & \frac{r}{r_1} = \frac{1 + \frac{\mathscr{P}_{\sigma^*}(h|0)f_0(s)}{\mathscr{P}_{\sigma^*}(h|1)f_1(s)} \frac{\mathscr{P}_{\sigma^*}(a_{k+1}=0|h,0)}{\mathscr{P}_{\sigma^*}(a_{k+1}=0|h,1)}}{1 + \frac{\mathscr{P}_{\sigma^*}(h|0)f_0(s)}{\mathscr{P}_{\sigma^*}(h|1)f_1(s)}} \\ & = r + (1-r) \frac{\mathscr{P}_{\sigma^*}(a_{k+1}=0|h,0)}{\mathscr{P}_{\sigma^*}(a_{k+1}=0|h,1)} > r + (1-r) \frac{F_0(s^*)}{F_1(s^*)}. \end{split}$$

Note that the expression on the right-hand side above is decreasing in *r*. Let r_m denote the belief induced by $\cup \{a_{k+1}, \dots, a_{k+m}\}$ where $a_{k+1} = \dots = a_{k+m} = 0$. We have

$$r_m = r \times \frac{r_1}{r} \times \cdots \times \frac{r_m}{r_{m-1}} < r \times (\frac{r_1}{r})^m.$$

Because $\frac{r_1}{r} = \frac{1}{r+(1-r)\frac{F_0(s^*)}{F_1(s^*)}} < 1$, we can find the desired $N(r, \hat{r})$ for any $\hat{r} \in (0, r)$, such that a realized belief that is less than \hat{r} can be induced by s and $h \cup \{a_{k+1}, \cdots, a_{k+N(r,\hat{r})}\}$, where $a_{k+1} = \cdots = a_{k+N(r,\hat{r})} = 0$.

Lemma 4 confirms the initial intuition that enlarging a neighborhood can strictly improve the informativeness of the observed action sequence. This improvement is represented by correcting a wrong decision by adding a sufficient number of observed actions. Moreover, the number of observed actions needed, $N(r, \hat{r})$, is independent of equilibrium. In the next lemma, I show that the strict improvement almost surely happens as B_k becomes arbitrarily large, i.e., any posterior belief that leads to the wrong action will almost surely be reversed toward the true state after a sufficiently large number of actions are observed.

In the following lemma, given private signal *s*, let $\mathscr{P}^{B_k}_{\sigma^*}(\hat{a} \neq \theta | s)$ denote the probability of taking the wrong action for an agent observing B_k .

Lemma 5. Assume that agents have strong private beliefs. Given any equilibrium σ^* and any private signal $s \in (s_*, s^*)$, let \hat{a} be the action that a rational agent would take after observing s and every action in B_k . Then we have $\lim_{k\to\infty} \mathscr{P}^{B_k}_{\sigma^*}(\hat{a} \neq \theta | s) = 0$.

Proof. Suppose not, then noting that $\mathscr{P}_{\sigma^*}^{B_k}(\hat{a} \neq \theta | s)$ must be weakly decreasing in k, it follows that $\lim_{k\to\infty} \mathscr{P}_{\sigma^*}^{B_k}(\hat{a} \neq \theta | s) > 0$. Let $\rho > 0$ denote this limit. From Lemma 3, we know that for any $\alpha > 0$ and for either true state $\theta = 0, 1, z \in [\frac{1}{2}, 1)$ exists such that $M \in \mathbb{N}$ exists such that $\max\{\mathscr{P}_{\sigma^*}(R_{\sigma^*}^{B_k} > z | 0, s), \mathscr{P}_{\sigma^*}(1 - R_{\sigma^*}^{B_k} > z | 1, s)\} < \alpha$ for any k > M. Let $\alpha = \frac{1}{2}\rho$, then we have $\max\{\mathscr{P}_{\sigma^*}(R_{\sigma^*}^{B_k} > z | 0, s), \mathscr{P}_{\sigma^*}(1 - R_{\sigma^*}^{B_k} > z | 1, s)\} < \frac{1}{2}\rho$ for any k > M. Then, for any $\delta > 0$, we can find a sufficiently large k such that for any $k' \ge k$, $(1) \mathscr{P}_{\sigma^*}^{B_{k'}}(\hat{a} \neq \theta | s) \in (\rho, \rho + \delta)$ and $(2) \max\{\mathscr{P}_{\sigma^*}(R_{\sigma^*}^{B_{k'}} > z | 0, s), \mathscr{P}_{\sigma^*}(1 - R_{\sigma^*}^{B_{k'}} > z | 1, s)\} < \frac{1}{2}\rho$. Hence, we have

$$\begin{aligned} &\frac{f_0(s)}{f_0(s) + f_1(s)} \mathscr{P}_{\sigma^*}(R^{B_{k'}}_{\sigma^*} \in [\frac{1}{2}, z] | 0, s) + \frac{f_1(s)}{f_0(s) + f_1(s)} \mathscr{P}_{\sigma^*}(1 - R^{B_{k'}}_{\sigma^*} \in [\frac{1}{2}, z] | 1, s) \\ &= \mathscr{P}^{B_{k'}}_{\sigma^*}(\hat{a} \neq \theta | s) - \frac{f_0(s)}{f_0(s) + f_1(s)} \mathscr{P}_{\sigma^*}(R^{B_k}_{\sigma^*} > z | 0, s) \\ &- \frac{f_1(s)}{f_0(s) + f_1(s)} \mathscr{P}_{\sigma^*}(1 - R^{B_k}_{\sigma^*} > z | 1, s) > \frac{1}{2}\rho \end{aligned}$$

By Lemma 3, for any $\pi > 0$, $N(\pi) = \max\{N(z, \frac{1}{2+\pi}), N(1-z, 1-\frac{1}{2+\pi})\} \in \mathbb{N}$ exists such that whenever $\theta = 0$ and $R_{\sigma^*}^{B_k} \in [\frac{1}{2}, z]$ or $\theta = 1$ and $1 - R_{\sigma^*}^{B_k} \in [\frac{1}{2}, z]$, additional $N(\pi)$ observations can reverse an incorrect decision. Consider the following (sub-optimal) updating method for a rational agent who observes $B_{k'} = B_{k+N(\pi)}$: switch her action from 1 to 0 if and only if $R_{\sigma^*}^{B_k} \in [\frac{1}{2}, z]$, and $a_{k+1} = \cdots = a_{k+N(\pi)} = 0$; switch her action from 0 to 1 if and only if $1 - R_{\sigma^*}^{B_k} \in [\frac{1}{2}, z]$, and $a_{k+1} = \cdots = a_{k+N(\pi)} = 1$. Let *h* denote a decision sequence from B_k that, together with *s*, induces such a posterior belief in the former case, and let *h'* denote a decision sequence from B_k that, together with *s*, induces such a posterior belief in the latter case. Let *H* and *H'* denote the sets of such decision sequences correspondingly. We have

$$\begin{aligned} \mathscr{P}_{\sigma^*}^{B_k}(\hat{a} \neq \theta | s) &- \mathscr{P}_{\sigma^*}^{B_{k'}}(\hat{a} \neq \theta | s) \\ \geq \sum_{h \in H} \left(\frac{f_0(s)}{f_0(s) + f_1(s)} \, \mathscr{P}_{\sigma^*}(h, a_{k+1} = \dots = a_{k+N(\pi)} = 0 | 0) \right) \\ &- \frac{f_1(s)}{f_0(s) + f_1(s)} \, \mathscr{P}_{\sigma^*}(h, a_{k+1} = \dots = a_{k+N(\pi)} = 0 | 1)) \\ &+ \sum_{h' \in H'} \left(\frac{f_1(s)}{f_0(s) + f_1(s)} \, \mathscr{P}_{\sigma^*}(h', a_{k+1} = \dots = a_{k+N(\pi)} = 1 | 1) \right) \\ &- \frac{f_0(s)}{f_0(s) + f_1(s)} \, \mathscr{P}_{\sigma^*}(h', a_{k+1} = \dots = a_{k+N(\pi)} = 1 | 0)). \end{aligned}$$

From the proof of Lemma 4, we know that for every *h*,

$$\frac{\mathscr{P}_{\sigma^*}(h, a_{k+1} = \dots = a_{k+N(\pi)} = 0|0)f_0(s)}{A} \ge \frac{1+\pi}{2+\pi},$$

where

$$A = \mathscr{P}_{\sigma^*}(h, a_{k+1} = \dots = a_{k+N(\pi)} = 0|0)f_0(s)$$
$$+ \mathscr{P}_{\sigma^*}(h, a_{k+1} = \dots = a_{k+N(\pi)} = 0|1)f_1(s).$$

This implies that

$$\mathcal{P}_{\sigma^*}(h, a_{k+1} = \dots = a_{k+N(\pi)} = 0|0) f_0(s)$$

- $\mathcal{P}_{\sigma^*}(h, a_{k+1} = \dots = a_{k+N(\pi)} = 0|1) f_1(s)$
 $\geq \pi f_1(s) \mathcal{P}_{\sigma^*}(h, a_{k+1} = \dots = a_{k+N(\pi)} = 0|1)$
 $\geq \pi f_1(s) F_1(-s^*)^{N(\pi)} \mathcal{P}_{\sigma^*}(h|1).$

By the definition of *h*, we have

$$\frac{1}{2} \leq \frac{\mathscr{P}_{\sigma^*}(h|1)f_1(s)}{\mathscr{P}_{\sigma^*}(h|1)f_1(s) + \mathscr{P}_{\sigma^*}(h|0)f_0(s)} \leq z,$$

which implies that

$$\mathscr{P}_{\sigma^*}(h|1)f_1(s) \ge \mathscr{P}_{\sigma^*}(h|0)f_0(s).$$

Similarly, we have

$$\mathcal{P}_{\sigma^*}(h', a_{k+1} = \dots = a_{k+N(\pi)} = 1|1)f_1(s)$$

$$-\mathcal{P}_{\sigma^*}(h', a_{k+1} = \dots = a_{k+N(\pi)} = 1|0)f_0(s)$$

$$\geq \pi f_0(s)(1 - F_0(s^*))^{N(\pi)}\mathcal{P}_{\sigma^*}(h'|0) = \pi f_0(s)F_1(-s^*)^{N(\pi)}\mathcal{P}_{\sigma^*}(h'|0),$$

and

$$\mathscr{P}_{\sigma^*}(h'|0)f_0(s) \ge \mathscr{P}_{\sigma^*}(h'|1)f_1(s).$$

From the previous construction, we know that

$$\begin{split} & \frac{f_0(s)}{f_0(s) + f_1(s)} \sum_{h \in H} \mathscr{P}_{\sigma^*}(h|0) + \frac{f_1(s)}{f_0(s) + f_1(s)} \sum_{h' \in H'} \mathscr{P}_{\sigma^*}(h'|1) \\ & = \frac{f_0(s)}{f_0(s) + f_1(s)} \mathscr{P}_{\sigma^*}(R^{B_{k'}}_{\sigma^*} \in [\frac{1}{2}, z]|0, s) + \frac{f_1(s)}{f_0(s) + f_1(s)} \mathscr{P}_{\sigma^*}(1 - R^{B_{k'}}_{\sigma^*} \in [\frac{1}{2}, z]|1, s) \\ & > \frac{1}{2}\rho. \end{split}$$

Combining the previous inequalities, we have

$$\mathscr{P}^{B_k}_{\sigma^*}(\hat{a} \neq \theta | s) - \mathscr{P}^{B_{k'}}_{\sigma^*}(\hat{a} \neq \theta | s)$$

 $> \pi F_1(-s^*)^{N(\pi)} \frac{1}{2} \rho.$

From the previous construction, we also know that

$$\mathscr{P}^{B_k}_{\sigma^*}(\hat{a} \neq \theta | s) - \mathscr{P}^{B_{k'}}_{\sigma^*}(\hat{a} \neq \theta | s) < \delta.$$

Clearly, for some given $\pi > 0$, a sufficiently small δ exists such that $\pi F_1(-s^*)^{N(\pi)} \frac{1}{2}\rho > \delta$, which implies a contradiction.

Lemma 5 is the most important lemma in the proof. It implies that a sub-optimal strategy – observing the first k agents in the decision sequence – is already sufficient to reveal the true state when k approaches infinity. Moreover, the sufficiency of this condition does not require any assumption regarding equilibrium strategies of the observed agents, which ensures its validity in every equilibrium. It then follows naturally that any agent's equilibrium strategy of choosing the observed neighborhood should generate a weakly higher posterior probability of taking the correct action. This argument is central for the proof of Theorem 2, which is presented below.

Proof. Lemma 2 has already shown that maximal learning occurs in every equilibrium only if the capacity structure has infinite observations. Now I prove the sufficiency of infinite observations for maximal learning in every equilibrium. Note that in any equilibrium σ^* , for any *n* and any $s_n \in (-s_*^n, s_*^n)$,

$$\mathscr{P}_{\sigma^*}(a_n=\theta|s_n) \geq \mathscr{P}^{B_{K(n)}}_{\sigma^*}(a_n=\theta|s_n).$$

When the capacity structure has infinite observations, i.e., $\lim_{n\to\infty} K(n) = \infty$, by Lemma 5 we know that

$$\lim_{n\to\infty}\mathscr{P}^{B_{K(n)}}_{\sigma^*}(a_n=\theta|s_n)=1,$$

and thus

$$\lim_{n\to\infty}\mathscr{P}_{\sigma^*}(a_n=\theta|s_n)=1,$$

which further implies that

$$\lim_{n \to \infty} \mathscr{P}_{\sigma^*}(a_n = \theta) = \frac{1}{2} F_0(s_*) + \frac{1}{2} (1 - F_1(s_*)) = P^*(c).$$

Therefore, maximal learning occurs in every equilibrium.

The key idea in Theorem 2 and its proof is that every observed action adds informativeness to the entire action sequence. The detailed proof shows that the symmetry assumption regarding the signal structure which leads to the existence of a symmetric equilibrium plays an important role in ensuring this condition. It has a natural interpretation: first, given that an agent makes a certain observation, because private signals are generated in an unbiased manner and agents behave similarly (in terms of choosing the observed neighborhood) when receiving symmetric signals, the Bayes' update by an observer of his action must be weakly in favor of the corresponding state as a result of the MLRP. Second, given that an agent chooses not to observe, the Bayes' update by the same observer would clearly strictly favor the corresponding state. These two effects combined show that the observation of every single action contributes a positive amount to information aggregation.

A special case that also ensures the positive information contribution of each observed action is when s^* is sufficiently small (i.e., when s_* is sufficiently large). Intuitively, if an agent chooses not to observe given a relatively large range of private signals, her action should favor the corresponding state from a Bayesian observer's point of view, regardless of her behavior when she chooses to observe. In other words, the second effect mentioned above already suffices for a definitive Bayesian update, *even without the symmetry assumption*. The following result formalizes this argument.

Corollary 2. Given a general (potentially asymmetric) signal structure, if agents have strong private beliefs and $F_0(s_*) > F_1(s^*)$, then maximal learning occurs in every equilibrium if and only if the capacity structure has infinite observations.

Proof. To apply the argument for Theorem 2 without the symmetry assumption, it suffices to show that for any agent *n* and any observed action sequence *h* in any neighborhood, $\mathscr{P}_{\sigma^*}(a_n = 0|h, 0) - \mathscr{P}_{\sigma^*}(a_n = 0, |h, 1)$ and $\mathscr{P}_{\sigma^*}(a_n = 1|h, 1) - \mathscr{P}_{\sigma^*}(a_n = 1, |h, 0)$ are both positive and bounded away from zero.

Note that

$$\begin{aligned} \mathscr{P}_{\sigma^*}(a_n = 0 | h, 0) > F_0(s_*) \\ \mathscr{P}_{\sigma^*}(a_n = 0 | h, 1) < F_1(s^*) \\ \mathscr{P}_{\sigma^*}(a_n = 1 | h, 1) > 1 - F_1(s^*) \\ \mathscr{P}_{\sigma^*}(a_n = 1 | h, 0) < 1 - F_0(s_*). \end{aligned}$$

Hence, we have

$$\mathcal{P}_{\sigma^*}(a_n = 0|h, 0) - \mathcal{P}_{\sigma^*}(a_n = 0, |h, 1) > F_0(s_*) - F_1(s^*)$$
$$\mathcal{P}_{\sigma^*}(a_n = 1|h, 1) - \mathcal{P}_{\sigma^*}(a_n = 1, |h, 0) > F_0(s_*) - F_1(s^*).$$

Therefore, the assumption $F_0(s_*) > F_1(s^*)$ suffices for the above two differences to be positive and bounded away from zero.

1.6.2 An Example

In this subsection, I introduce an example below to illustrate the difference among asymptotic learning, maximal learning and (a typical case of) learning in equilibrium with strong private beliefs and finite observations.

Assume the following signal structure:

$$F_0(s) = \frac{1}{2}(s+1)(\frac{3}{2} - \frac{s}{2})$$

$$F_1(s) = \frac{1}{2}(s+1)(\frac{1}{2} + \frac{s}{2}).$$

This signal structure implies the probability density functions

$$f_0(s) = \frac{1-s}{2} \\ f_1(s) = \frac{1+s}{2}.$$

Hence, it is easy to see that agents have unbounded (thus strong) private beliefs.

In addition, assume that K(n) = 1 for all *n*. Consider the case when the cost of observation is low. When each agent can only observe one of her predecessors, if she chooses to observe then she would rationally choose to observe the agent with the highest probability of taking the

right action. Agent 2 can only observe agent 1; agent 3, in view of this fact, will choose to observe agent 2 since agent 2's action is more informative than agent 1's action. Proceeding inductively, in every equilibrium, each agent will only observe their immediate predecessor when she chooses to observe, which results in a (probabilistic) "line" network. Let $\hat{s}^* = \lim_{n \to \infty} s_*^n$ and let $\hat{P}^* = \lim_{n \to \infty} \mathscr{P}_{\sigma^*}(a_n = \theta)$, it follows that the equations characterizing \hat{s}^* and \hat{P}^* are

$$\begin{split} \hat{P}^* &= F_0(-\hat{s}^*) + (F_0(\hat{s}^*) - F_0(-\hat{s}^*))\hat{P} \\ \hat{P}^* &- \frac{f_1(\hat{s}^*)}{f_0(\hat{s}^*) + f_1(\hat{s}^*)} = c, \end{split}$$

The first condition decomposes the learning probability in the limit, \hat{P}^* , into the probability that an agent correctly follows his own signal without any observation, and the probability that she observes and her immediate predecessor's action is correct. The second condition indicates the indifference (in the limit) of an agent with signal \hat{s}^* between observing and not observing her immediate predecessor, in the sense that the expected marginal benefit from observation must be equal to *c*. They can be further simplified as

$$\frac{1}{f_0(\hat{s}^*) + f_1(\hat{s}^*)} \frac{f_0(\hat{s}^*)F_0(-\hat{s}^*) - f_1(\hat{s}^*)F_1(-\hat{s}^*)}{F_0(-\hat{s}^*) + F_1(-\hat{s}^*)} = c.$$

From the above equation and from the definition of strong private beliefs, we have

$$\hat{s}^* = 1 - 4c$$
, if $c \le \frac{1}{4}$
 $s^* = 1 - 2c$, if $c \le \frac{1}{2}$.

It further implies that

$$\mathscr{P}_{\sigma^*}(a_n = \theta) = 1 - 4c^2, \text{ if } c \le \frac{1}{4}$$

 $P^*(c) = 1 - c^2, \text{ if } c \le \frac{1}{2}.$

The first term $1 - 4c^2$ is the equilibrium probability of learning the true state in the limit when K(n) = 1; the second term $1 - c^2$ is the probability of learning the true state under maximal learning. Figure 4 illustrates the learning probability in the limit under asymptotic learning, under maximal learning and in equilibrium, as a function of the cost of observation *c*.

1.6.3 Welfare Analysis

In this subsection, I analyze the impact of observation cost and signal precision on the limit learning probability in equilibrium, $\lim_{n\to\infty} \mathscr{P}_{\sigma^*}(a_n = \theta)$. This probability represents the ul-



Figure 1.4: Learning Probability as a Function of *c*

timate level of social learning achieved in a growing society. Two sets of parameters are of particular interest in this comparative statics: the cost of observation, c, and the precision of the private signal structure relative to cost. In many practical scenarios, these parameters capture the essential characteristics of a community with respect to how difficult it is to obtain information from others and how confident an agent can be about her private knowledge. The aim of this analysis is to identify the type(s) of environment that facilitate social learning.

In the following analysis, I show that compared to an environment with free observation, having a positive cost may actually improve the level of social learning. In previous sections, Theorem 1 shows that zero cost and unbounded private beliefs imply the highest learning probability, i.e., asymptotic learning; Theorem 2 allows us to obtain a straightforward formula for computing the limit learning probability with strong private beliefs and infinite observations:

$$\lim_{n \to \infty} \mathscr{P}_{\sigma^*}(a_n = \theta) = F_0(s^*) - F_0(-s^*) + F_0(-s^*) = F_0(s^*).$$

Note that $F_0(s^*) - F_0(-s^*)$ is the probability (in the limit) that an agent chooses to observe; Theorem 2 indicates that observation reveals the true state of the world with near certainty when the society gets large. $F_0(-s^*)$ is the probability (in the limit) that an agent chooses not to observe and undertakes the correct action. My first result concerns an in-between case, i.e., under bounded private beliefs and infinite observations, the comparison between an environment with zero cost and one with positive cost. For any single agent, other things equal, it is always beneficial to observe with no cost than with positive cost. However, positive cost may actually be desirable for the society as a whole: for any agent, even though relying on her signal more often is harmful to her own learning, it provides more information for her successors who observe her action, hence raising the informativeness of observation. This argument provides the intuition for the formal result below.

Consider the capacity structure K(n) = n - 1, i.e., any agent is able to observe all her predecessors. Let $\sigma^*(c)$ be an equilibrium under cost c, and let $P^*(\sigma^*(c))$ be the limit probability of learning, given $\sigma^*(c)$, i.e., $P^*(\sigma^*(c)) = \lim_{n \to \infty} \mathscr{P}_{\sigma^*(c)}(a_n = \theta)$.

Proposition 4. Assume that agents have bounded private beliefs. Let $\sigma^*(0)$ be any equilibrium under zero cost. There are positive values \bar{c}, \underline{c} ($\bar{c} > \underline{c}$), such that for any $c \in (\underline{c}, \bar{c})$ and any $\sigma^*(c), P^*(\sigma^*(0)) < P^*(\sigma^*(c)).$

Proof. Let $\bar{\beta} = \lim_{s \to 1} \frac{f_1(s)}{f_0(s) + f_1(s)}$ and $\underline{\beta} = \lim_{s \to -1} \frac{f_1(s)}{f_0(s) + f_1(s)}$. Let $\Delta = \frac{\underline{\beta}(1-\bar{\beta})}{-\underline{\beta}+\overline{\beta}+2\underline{\beta}(1-\bar{\beta})} \in (0,1)$. Accomoglu et al. (2011) have shown that when agents have bounded private beliefs and when each agent observes all her predecessors, the learning probability of any agent is bounded above by max $\{\Delta, 1-\Delta\} < 1$, i.e., $P^*(\sigma^*(0)) \le \max\{\Delta, 1-\Delta\} < 1$.

Note that for any $\varepsilon \in (0,1)$, c' > 0 exists such that agents have strong private beliefs under c', and that $F_0(s^*) - F_0(-s^*) = F_1(s^*) - F_1(-s^*) = 1 - \varepsilon$ (due to symmetry and continuity of signal distributions). The capacity structure K(n) = n - 1 has infinite observations; thus, by Theorem 2, maximal learning occurs in any equilibrium $\sigma^*(c')$, which implies that

$$\lim_{n \to \infty} \mathscr{P}_{\sigma^*(c')}(a_n = \theta) = F_0(s^*) - F_0(-s^*) + F_0(-s^*) = F_0(s^*) > 1 - \varepsilon.$$

Let $\varepsilon = 1 - \max{\{\Delta, 1 - \Delta\}} \in (0, 1)$, then c' exists such that $P^*(\sigma^*(0)) < P^*(\sigma^*(c'))$ in any equilibrium $\sigma^*(c')$. Again, by continuity of the signal distributions, the desired \bar{c}, \underline{c} exist. \Box

With zero cost and bounded private beliefs, herding occurs because at some point in the decision sequence, an agent may abandon *all* her private information, although her observation is not perfectly informative of the true state. Yielding to observation, in turn, causes her own actions to reveal no information about her private signal, and thus information aggregation ends. However, with positive cost and strong private beliefs – and although nothing has changed in the signal structure – now every agent relies on *some* of her possible private signals, which strengthens the informativeness of observation. When the probability of an agent choosing to

observe is sufficiently high (but still bounded away from one), an agent may enjoy a higher chance of taking the right action than when observation is free for everyone. The comparison is illustrated in Figure 5.



Figure 1.5: Limit Learning Probability and Cost of Observation

Next, I consider the effect of signal strength that is measured by the probability of receiving relatively more informative private signals. In most existing literature, the network of observation is exogenously given. In other words, observation is "free" and non-strategic, which is not affected by how accurate an agent's private signal is. However, when observation becomes strategic and costly, there is a trade-off between obtaining a higher probability of taking the right action and saving the cost. As a result, when an agent receives a rather strong signal, she might as well cede the opportunity of observational learning and just act according to her private information. Therefore, in this environment, strong signals can be detrimental to social learning. The next result demonstrates this phenomenon.

With strong private beliefs, denote the *strength* of the private signal relative to cost as $F_0(-s^*) + 1 - F_0(s^*)$, i.e., the probability of not observing even if observing reveals the true state.

Proposition 5. *Given c and two signal structures that both generate strong private beliefs, that with higher strength may lead to lower limit learning probability.*

Proof. Let (F_0, F_1) and (G_0, G_1) denote two (symmetric) signal structures that both generate strong private beliefs, and let s_F^* and s_G^* be the positive private signals such that an agent is indifferent between paying *c* to know the true state and not paying *c* and acting according to the signal. Assume that (F_0, F_1) has higher signal strength: $F_0(-s_F^*) + 1 - F_0(s_F^*) > G_0(-s_G^*) + 1 - G_0(s_G^*)$. We already know that for any equilibrium σ_F^* under the first signal structure and for any equilibrium σ_G^* under the second signal structure,

$$\lim_{n \to \infty} \mathscr{P}_{\sigma_F^*}(a_n = \theta) = F_0(s_F^*)$$
$$\lim_{n \to \infty} \mathscr{P}_{\sigma_G^*}(a_n = \theta) = G_0(s_G^*)$$

Hence, when $G_0(s_G^*) > F_0(s_F^*)$, the signal structure with higher strength will lead to lower limit learning probability.

In summary, we only see clear monotonicity (lower cost or stronger signals are better for social learning) in extreme scenarios (unbounded private beliefs or zero cost). When private beliefs are bounded and cost is positive, two new factors enter the determinant of the limit learning probability. First, the fact that costly observation alone may now provide higher informativeness than free observation implies that positive cost may actually be more favorable for social learning. Second, positive cost signifies a trade-off between two components of an agent's final payoff – the probability of undertaking the right action and the cost of observation – thus, from the perspective of social learning, weaker private signals may be preferred because they incentivize agents to achieve better learning by observation. As a result of these joint effects, the learning process via endogenous networks of observation becomes more complex.

1.7 Discussion

1.7.1 Observation Preceding Signal

In the previous analysis, we see that under strong private beliefs, (1) asymptotic learning is impossible and (2) if the capacity structure has only finite observations and observation is costly, then maximal learning does not occur in any equilibrium even when private beliefs are unbounded. As it turns out, an agent's *timing* of choosing her observation plays an important role: because an agent receives her private signal before choosing observations, it always remains possible that an agent chooses not to observe and bases her action solely on her private signal, which may be rather strong but is nonetheless not perfectly informative. Thus, when observations are finite, there is always a probability bounded away from zero that observations will induce incorrect action.

However, in practical situations, the timing of the arrival of different types of information is often not fixed. For instance, when a firm decides whether to adopt a new production technology, it may well take less time to conduct a survey about which nearby firms have already implemented the technology than to obtain private knowledge about the technology itself via research and trials. It is then interesting to study the different patterns that the social learning process would exhibit under this alternative timing. The next result demonstrates that when observation precedes private signal, learning somehow becomes easier as long as the cost of observation is not too high: asymptotic learning can occur even when cost is positive and observations are finite. Such difference between timing schemes only arises when observation is endogenous and costly – when observation is exogenous or free, the two timing schemes would essentially lead to identiical equilibria.

Consider the alternative dynamic process in which each agent chooses her observed neighborhood before receiving her private signal. Let Y(m) denote the probability that an agent will take the right action if she can observe a total of *m* other agents, each of whose actions are based solely on her own private signal. The result can be easily generalized to the case with an asymmetric signal structure.

Proposition 6. When agents have unbounded private beliefs, asymptotic learning occurs in every equilibrium if and only if there exists n such that $Y(K(n)) - F_0(0) \ge c$.

Proof. "Only if": assume that *n* does not exist. Then no agent will observe and clearly asymptotic learning does not occur in any equilibrium.

"If": the condition implies that agent n will observe K(n) of her predecessors. Then, beginning with n, observing will be weakly better than not observing for each agent because an agent can at least observe agent n and achieve exactly the same expected payoff as agent n. Then, the argument for Theorem 1 can be applied to show that asymptotic learning occurs in every equilibrium.

Proposition 6 shows that a necessary and sufficient condition for asymptotic learning is

the existence of an agent who initiates the information aggregation process by beginning to observe. Because observation precedes the private signal, each of her successors can be at least as well off as she is simply by observing her action. Therefore, after this starting point, observation becomes the optimal choice even for agents with lower capacity. Furthermore, because there is no conflict between a strong signal and costly observation (observation occurs first anyway), there is no blockade of information once observation begins. As in the case with unbounded private beliefs and zero cost, a network topology featuring expanding observations will spontaneously form and asymptotic learning will occur as a result.

For better illustration, consider an environment with unbounded private beliefs and infinite observations in which the limit learning probability can be fully characterized both when a signal precedes observation and when observation precedes a signal. Figure 6 shows the relation between the limit learning probability and the cost of observation under the two timing schemes; this figure also shows that asymptotic learning occurs within a much larger range of cost when observation comes first, while the limit learning probability falls abruptly to that with no observation when cost becomes high because of the lack of an agent to trigger observational learning. When agents receive private signals first, the limit learning probability is continuous against the cost of observation, and the threshold of cost above which observation stops is higher; as a result, agents learn less when cost is relatively low and more when cost is intermediate compared with the other timing scheme.

When private beliefs are bounded, a partial characterization analogous to Proposition 3 can be obtained under this alternative timing: asymptotic learning fails for a number of typical capacity structures and associated equilibria. In each scenario, as with the analysis above, once observation is initiated by some agent, all the successors will choose to observe. Of course, the cost of observation must be bounded by a certain positive value (which can be characterized based on the specified equilibrium behavior) to ensure the existence of the particular equilibrium; otherwise, observation never begins and each agent would simply act in isolation according only to her private signal.



Figure 1.6: Limit Learning Probability under Two Timing Schemes

1.7.2 Information Diffusion

Another important assumption in the renowned herding behavior and information cascades literature is that observing an agent's action does not reveal any additional information regarding the agent's knowledge about the actions of others, which occurs without much loss of generality in earlier models because agents are assumed to observe the entire past action history in any event. In a more general setting, it can be expected that allowing an agent to access the knowledge (still about actions and not about private signals) of agents in their observed neighborhood makes a significant difference because information can now flow not only through direct links in the network but also through indirect paths. In this section, I discuss the impact of such added informational richness on the level of social learning.

Assume the following *information diffusion* in the observation structure: if agent n has observed the actions in neighborhood B(n) before choosing her own action, then any agent observing n knows a_n and each action in B(n). In our model of endogenous network formation, this alternative assumption has a particular implication: if agent n sees that another agent mchose action 1 but m did not know the action of anyone else, n can immediately infer that m must have received a rather strong signal. As a result, when the observed neighborhood becomes large, the observing agent can apply the weak law of large numbers to draw inferences regarding the true state of the world. With this simpler argument, the symmetry assumption regarding the signal structure can be relaxed to obtain the following result.

Proposition 7. With information diffusion, when agents have strong private beliefs, maximal learning occurs in every equilibrium if and only if the capacity structure has infinite observations.

Proof. "Only if": see the proof of Theorem 2.

"If": since the capacity structure has infinite observations, we can construct a sequence of agents $\{i_m\}_{m=1}^{\infty}$ such that $K(i_m) \leq K(i_{m+1})$ for any *m*, and $\lim_{m\to\infty} K(i_m) = \infty$. It thus suffices to show that maximal learning occurs in this sequence of agents.

Denote $\bar{s}_*^{i_m}$ and $\underline{s}_*^{i_m}$ as the positive and negative private signals such that agent i_m is indifferent between observing and not observing (when either of such private signals does not exist, let $\bar{s}_*^{i_m}$ or $\underline{s}_*^{i_m}$ be the private signal such that i_m is indifferent between choosing 1 and 0). Clearly, both $\bar{s}_*^{i_m}$ and $\underline{s}_*^{i_m}$ converge. Denote \bar{s} and \underline{s} as the corresponding limits. We know that for any m, $F_0(\underline{s}_*^{i_m}) - F_1(\underline{s}_*^{i_m}) \ge F_0(\underline{s}) - F_1(\underline{s}) > 0$. Let $\varepsilon = \frac{1}{2}(F_0(\underline{s}) - F_1(\underline{s}))$, and find $M \in \mathbb{N}^+$ such that for any $m \ge M$, $F_1(\underline{s}_*^{i_m}) - F_1(\underline{s}) < \varepsilon$.

For any $m \ge M$, denote a random variable

 $Z(i_m) = 1\{i_m \text{ chooses } 0 \text{ without any observation}\}.$

We know that $Z(i_M), Z(i_{M+1}), \cdots$ are mutually independent, and that for any $m \ge M$, $Z(i_m)$ is equal to 1 with probability greater than $F_0(\underline{s})$ when $\theta = 0$ and with probability less than $F_1(\underline{s}) + \varepsilon = \frac{1}{2}(F_0(\underline{s}) + F_1(\underline{s})) < F_0(\underline{s})$ when $\theta = 1$.

By the weak law of large numbers, for any $\rho > 0$, we have

$$\lim_{n \to \infty} \operatorname{Prob}(\frac{\sum_{m=M}^{M+n} Z(i_m)}{n+1} < F_0(\underline{s}) - \rho | \theta = 0) = 0$$
$$\lim_{n \to \infty} \operatorname{Prob}(\frac{\sum_{m=M}^{M+n} Z(i_m)}{n+1} > \frac{1}{2}(F_0(\underline{s}) + F_1(\underline{s})) + \rho | \theta = 1) = 0.$$

Take $\rho < \frac{1}{2}(F_0(\underline{s}) - F_1(\underline{s}))$, and consider the following sub-optimal strategy for any agent i_m , m > M: observe the neighborhood $\{M, M+1, \dots, M+n\}$ where $n = \min\{m-1-M, K(i_m) - 1\}$. Choose 0 if $\frac{\sum_{m=M}^{M+n} Z(i_m)}{n+1} \ge F_0(\underline{s}) - \rho$, and 1 otherwise. By the above two conditions, we know that the probability of making the right choice converges to 1 under either state when the agent's number of observations approaches infinity. Hence, maximal learning occurs.

Somewhat curiously, introducing information diffusion into the model only leads to relaxing the symmetry assumption but still results in the same necessary and sufficient condition for maximal learning. The underlying reason for this result is that as long as each agent chooses only to observe with a probability bounded away from 1 (i.e., for a range of signals that differs significantly from full support), every agent will only know the finite actions of others with near certainty if observations are finite. Thus, when the capacity structure has finite observations, the additional information acquired via information diffusion will not be sufficient for maximal learning.

We have seen from the above result that even with information diffusion, a capacity structure with finite observations never leads to maximal learning in any equilibrium. An even more surprising observation is that, when the capacity structure has finite observations, information diffusion may not be helpful at all in terms of the limit learning probability. For instance, consider the capacity structure K(n) = 1 for all *n* and symmetric private signals, and consider agent n where n is large. In equilibrium, if any agent chooses to observe, she will observe her immediate predecessor. By choosing to observe some agent m_1 , agent n will know the actions of an almost certainly finite "chain" of agents m_1, m_2, \dots, m_l , such that m_1 observed m_2, \dots, m_{l-1} observed m_l , and m_l chose not to observe. It is first clear that $a_{m_{l-1}}$ must almost surely equal a_{m_l} because the range of signals – given that m_{l-1} chooses to observe – is close to that for m_l (by the assumption that n is large), which induces m_{l-1} to follow m_l 's action that implies a stronger private signal. Hence, m_{l-1} 's action does not reveal any additional information about the true state. Repeating this argument inductively, the only action that is informative to agent n is a_{m_i} . If agent *n* can only observe a single action, she can use the identical argument to deduce that the observation ultimately reflects the action of an agent who chose not to observe. Therefore, the limit learning probability $\lim_{n\to\infty} \mathscr{P}_{\sigma^*}(a_n = \theta)$ is not affected by information diffusion.

1.7.3 Flexible Observations with Non-Negative Marginal Cost

Thus far, I have assumed a single and fixed cost for observing any neighborhood of size up to the capacity constraint. Another interesting setting is to assume that the cost of observation depends on the number of actions observed. It can be easily anticipated that a full characterization of the pattern of social learning is difficult, given an arbitrary cost function; such characterization requires detailed calculations regarding the marginal benefit of any additional observed action,

which varies substantially based on the specific signal distributions. However, in the following typical class of cost functions, the previous results can easily be applied to describe the level of social learning when the cost of observation changes with the number of observed actions.

Consider the following setting: after receiving her private signal, each agent can decide how many actions (up to K(n)) to observe. As in the original model, the actions are observed simultaneously⁶. The cost function of observing *m* actions is denoted with c(m). Assume that c(0) = 0 and that c(m) satisfies the property of non-negative marginal cost: $c(m+1) - c(m) \ge 0$ for all $m \in \mathbb{N}$. Here, maximal learning is defined as to achieve efficient information aggregation in the limit by paying the least cost possible: $\lim_{n\to\infty} \mathscr{P}_{\sigma^*}(a_n = \theta) = P^*(c(1))$. The following result is essentially a corollary of Theorems 1 and 2 and characterizes the pattern of social learning under this class of cost functions.

Proposition 8. Assume that agents have unbounded private beliefs. Under the above class of cost functions, the following propositions are true for the social learning process:

- (a) Asymptotic learning occurs in every equilibrium if and only if c(1) = 0.
- (b) When c(1) > 0, maximal learning occurs in every equilibrium if c(m+1) − c(m) = 0 for all m ≥ 1; otherwise, maximal learning does not occur in any equilibrium.

Proof. Part (a) follows from Theorem 1. When c(1) > 0, it is clear that asymptotic learning does not occur in any equilibrium. When c(1) = 0, we know that in any equilibrium σ^* , the probability of taking the right action $\mathscr{P}_{\sigma^*}(a_n = \theta)$ is weakly increasing in *n*. Suppose that asymptotic learning does not occur in σ^* , then $\mathscr{P}_{\sigma^*}(a_n = \theta)$ must converge to a limit in (0, 1). Then the argument in the proof for Theorem 1 can be applied to derive a contradiction.

Part (b) follows from Theorem 2. By Theorem 2, maximal learning requires infinite observations as *n* grows large. When c(m+1) - c(m) = 0 for all $m \ge 1$, it is clear that maximal learning occurs in every equilibrium. When c(m+1) - c(m) > 0 for some *m*, in any equilibrium, consider the case when an agent receives a private signal that makes her indifferent between not observing and paying c(1) to know the true state. We know that such a private signal exists because by assumption, private beliefs are unbounded, signal distributions are continuous and c(1) > 0. Since c(m+1) - c(m) > 0 for some *m*, the cost of observing infinitely many

⁶Nevertheless, the results below still hold in the context of sequential observation, i.e., an agent can choose whether to observe an additional action by paying the marginal cost, based on what she has already observed.

actions is strictly higher than c(1), and hence the agent will choose not to observe, regardless of her capacity of observation. Because signal distributions are continuous, there exists a positive measure of signals given which the agent will choose not to observe. Therefore, maximal learning does not occur in any equilibrium.

Proposition 8 indicates that the difference in the level of social learning produced by the different costs of observation becomes even larger in the context of a more general cost function. First, the results show that the key factor determining whether asymptotic learning occurs in equilibrium is c(1), i.e., the cost of observing the first action. When c(1) = 0, any agent can at least do as well as any of her predecessors by simply observing the latter and following the observed action; thus, when private beliefs are unbounded, learning does not stop until agents almost certainly undertake the correct action. Secondly, when c(1) > 0, maximal learning occurs if and only if each additional observation is free. We already know that maximal learning requires agents to choose to observe arbitrarily large neighborhoods when the society is large; with non-negative marginal cost, the cost of observing some other action is positive. As a result, even if the capacity structure has infinite observations, an agent will choose not to observe when she receives a signal that makes her more or less indifferent between not observing and paying c(1) to know the true state. Therefore, maximal learning never occurs.

1.8 Costly Learning with Weak Private Beliefs

When private beliefs are weak, asymptotic/maximal learning may not be achieved in the previous model because of the possibility of herding. More specifically, as *n* becomes large, some agents' actions may be so accurate (although not perfect) that successors choose to observe and follow them regardless of their own private signal. Hence, more observations do not necessarily provide more information about the true state when the observation structure is *determinant*, i.e., any agent *n* can choose their neighborhood among $\{1, \dots, n-1\}$. A partial characterization similar to Proposition 3 can be obtained that shows that maximal learning (which is equivalent to asymptotic learning when private beliefs are weak) cannot occur for a number of typical capacity structures and the associated equilibria. However, maximal learning can still be approximated with certain general *stochastic* observation structures. First, I introduce the notion of ε -maximal learning, which is the approximation of the original notion of maximal learning.

Definition 5. Given a signal structure (F_0, F_1) , we say that ε -maximal learning occurs in equilibrium σ^* when the limit inferior of the probability of a_n being the correct action is at least $(1 - \varepsilon)P^*(c)$: $\liminf_{n\to\infty} \mathscr{P}_{\sigma^*}(a_n = \theta) \ge (1 - \varepsilon)P^*(c)$.

 ε -maximal learning describes the situation in which an agent undertakes the correct action with a probability of at least $(1 - \varepsilon)P^*(c)$. Note that $P^*(c) = 1$ when private beliefs are weak. Hence, ε -maximal learning under weak private beliefs implies that the learning probability $\mathscr{P}_{\sigma^*}(a_n = \theta)$ is bounded below by $1 - \varepsilon$ in the limit. In fact, when ε is close to zero, social learning will be almost asymptotic. I will provide sufficient conditions below at an equilibrium for ε -maximal learning to occur. The essential factor that facilitates ε -maximal learning is the existence of a *non-persuasive neighborhood*, which is introduced by Acemoglu et al.[ADL11]. I define this concept below.

Definition 6. When agents have weak private beliefs, let

$$\bar{\beta} = \lim_{s \to 1} \frac{f_1(s)}{f_0(s) + f_1(s)}$$
$$\underline{\beta} = \lim_{s \to -1} \frac{f_0(s)}{f_0(s) + f_1(s)}$$

denote the upper and lower bounds of an agent's private beliefs on the true state being 1. A finite subset of agents B is a non-persuasive neighborhood in equilibrium σ^* if

$$\mathscr{P}_{\sigma^*}(\theta = 1 | a_k = y_k \text{ for all } k \in B) \in (\beta, \beta)$$

for any set of values $y_k \in \{0,1\}$ for each k.

A neighborhood *B* is non-persuasive with respect to equilibrium σ^* if given any possible realized action sequence in this neighborhood, an agent that observes it may still rely on his own private signal. If a neighborhood is non-persuasive, then any agent will choose not to observe it and follow his own private signal with positive probability. In other words, regardless of the realized action sequence, there exist a positive measure of private signals such that the agent takes action 0 and another positive measure of private signals such that the agent takes action 1. Note that the definition of a non-persuasive neighborhood depends on the particular equilibrium σ^* ; however, there is a class of neighborhoods that are non-persuasive in any equilibrium. Suppose that K(n) = 0 for $n = 1, 2, \dots, M$ for some $M \in \mathbb{N}^+$; in other words, the first M agents cannot observe any of the actions of others. In this case, there is a positive $M' \leq M$ such that any subset of $\{1, \dots, M'\}$ is a non-persuasive neighborhood in any equilibrium. To illustrate this, simply note that $\{1\}$ already satisfies the criterion: for any agent who can only observe agent 1's action, there must be a range of strong private signals that are more informative about the true state than the action of agent 1.

Consider the following stochastic observation structure for equilibrium σ^* : there are M non-persuasive neighborhoods C_1, \dots, C_M such that for all n, agent n can only observe within some C_i , $i \in \{1, \dots, M\}$, with probability $\varepsilon_n > 0$; with probability $1 - \varepsilon_n$, agent n can observe within $\{1, \dots, n-1\}$. In both cases, the capacity structure $\{K(n)\}_{n=1}^{\infty}$ stays the same. Theorem 3 below provides a class of stochastic observation structures in which ε -maximal learning occurs for any signal structure. Establishing this result depends on how to interpret an observed action from a Bayesian observer's perspective. Given the positive probability with which an agent can only choose to observe a non-persuasive neighborhood, her final action now may reflect either conformity with her observation or her strong private signal. This property holds even when private beliefs are weak. Hence, we can follow a similar argument to that in the proof of Theorem 2 that shows that the total informativeness of the action sequence in any neighborhood can always be increased by including in it enough additional actions.

Theorem 3. In any equilibrium σ^* with the above stochastic capacity structure, ε -maximal learning occurs if $\lim_{n\to\infty} \varepsilon_n = \varepsilon$ and $\lim_{n\to\infty} K(n) = \infty$.

Proof. For all $i \in \{1, \dots, M\}$, let \hat{s}^i denote the positive private signal that will make an agent indifferent between her private beliefs and the most extreme belief induced by C_i , i.e., \hat{s}^i is characterized by

$$\max_{y_k \text{ for all } k \in C_i} \mathscr{P}_{\sigma^*}(\theta = 0 | a_k = y_k \text{ for all } k \in C_i) = \frac{f_1(\hat{s}^i)}{f_1(\hat{s}^i) + f_0(\hat{s}^i)}$$

Let $\hat{s} = \max_i \hat{s}_i < 1$. Let X_n denote the event that agent *n* can only observe within some C_i , $i \in \{1, \dots, M\}$; let Y_n denote the event that agent *n* can observe within $\{1, \dots, n-1\}$. Consider the term $\frac{\mathscr{P}_{\sigma^*}(a_{k+1}=0|h,1)}{\mathscr{P}_{\sigma^*}(a_{k+1}=0|h,0)}$ in the proof of Lemma 3. It can now be written as

$$\begin{split} & \frac{\mathscr{P}_{\sigma^*}(a_{k+1}=0|h,1)}{\mathscr{P}_{\sigma^*}(a_{k+1}=0|h,0)} \\ & = & \frac{\varepsilon_n \mathscr{P}_{\sigma^*}(a_{k+1}=0|h,1,X_{k+1}) + (1-\varepsilon_n) \mathscr{P}_{\sigma^*}(a_{k+1}=0|h,1,Y_{k+1})}{\varepsilon_n \mathscr{P}_{\sigma^*}(a_{k+1}=0|h,0,X_{k+1}) + (1-\varepsilon_n) \mathscr{P}_{\sigma^*}(a_{k+1}=0|h,0,Y_{k+1})}. \end{split}$$

We know that $\mathscr{P}_{\sigma^*}(a_{k+1} = 0 | h, 1, Y_{k+1}) \le \mathscr{P}_{\sigma^*}(a_{k+1} = 0 | h, 0, Y_{k+1})$. Thus, we have

$$\begin{split} & \frac{\mathscr{P}_{\sigma^*}(a_{k+1}=0|h,1)}{\mathscr{P}_{\sigma^*}(a_{k+1}=0|h,0)} \\ \leq & \frac{\varepsilon_n \mathscr{P}_{\sigma^*}(a_{k+1}=0|h,1,X_{k+1}) + (1-\varepsilon_n) \mathscr{P}_{\sigma^*}(a_{k+1}=0|h,0,Y_{k+1})}{\varepsilon_n \mathscr{P}_{\sigma^*}(a_{k+1}=0|h,0,X_{k+1}) + (1-\varepsilon_n) \mathscr{P}_{\sigma^*}(a_{k+1}=0|h,0,Y_{k+1})} \\ \leq & \frac{\varepsilon_n \mathscr{P}_{\sigma^*}(a_{k+1}=0|h,1,X_{k+1}) + (1-\varepsilon_n)}{\varepsilon_n \mathscr{P}_{\sigma^*}(a_{k+1}=0|h,0,X_{k+1}) + (1-\varepsilon_n)}. \end{split}$$

By an argument similar to that in the proof of Lemma 3, we also have

$$\mathscr{P}_{\sigma^*}(a_{k+1}=0, \text{ observe}|h, 1, X_{k+1}) \leq \mathscr{P}_{\sigma^*}(a_{k+1}=0, \text{ observe}|h, 0, X_{k+1})$$

which implies that

$$\begin{split} & \frac{\mathscr{P}_{\sigma^*}(a_{k+1}=0|h,1)}{\mathscr{P}_{\sigma^*}(a_{k+1}=0|h,0)} \\ \leq & \frac{\varepsilon_n F_1(\hat{s}) + (1-\varepsilon_n)}{\varepsilon_n F_0(\hat{s}) + (1-\varepsilon_n)} \\ \leq & \frac{\varepsilon F_1(\hat{s}) + (1-\varepsilon)}{\varepsilon F_0(\hat{s}) + (1-\varepsilon)} < 1 \end{split}$$

The argument for Theorem 2 can then be applied at this juncture to prove that if $\lim_{n\to\infty} K(n) = \infty$, $\lim_{n\to\infty} \mathscr{P}_{\sigma^*}(a_n = \theta | Y_n) = 1$. Therefore, we have

$$\liminf_{n\to\infty}\mathscr{P}_{\sigma^*}(a_n=\theta)\geq (1-\varepsilon)\lim_{n\to\infty}\mathscr{P}_{\sigma^*}(a_n=\theta|Y_n)=1-\varepsilon.$$

Hence, ε -maximal learning occurs.

The implication of Theorem 3 is rather surprising. In contrast to many existing results in the literature that lead to herding behavior more easily when agents receive weaker signals, this theorem indicates that learning dynamics can be much richer and social learning can be rather accurate in a general network of observation in which links are formed endogenously. In particular, the limit learning probability can even be higher under weak private beliefs than under strong private beliefs. For example, in an environment with infinite observations, consider an equilibrium in which beginning with some agent *n*, each agent chooses to observe regardless of her private signal when she can observe within the whole set of her predecessors (such an equilibrium can exist only under weak private beliefs). Theorem 3 shows that the learning probability $\mathscr{P}_{\sigma^*}(a_n = \theta)$ is bounded below by $1 - \varepsilon$ in the limit because the probability that an agent chooses to observe has the same lower bound in the limit. When ε is arbitrarily small, the agents can almost asymptotically learn the true state of the world, which they cannot do under strong private beliefs and which also relates to the result that weaker private signals may actually imply a higher limit learning probability in the welfare analysis of the previous section.

Comparing Theorem 3 with existing results in the literature on achieving asymptotic learning under bounded private beliefs in an exogenously given and stochastic observation structure (e.g., Theorem 4 in Acemoglu et al.[ADL11]) is also illuminating. The main implication from existing results is that, if agents observe the entire history of actions with some probability (which is uniformly bounded away from zero) and observe some non-persuasive neighborhood with some probability (which converges to zero but when the infinite sum of such probabilities over agents is unbounded) in a stochastic observation structure satisfying expanding observations, then observing some close predecessor reveals the true state in the limit. In Theorem 3, the role of observing a non-persuasive neighborhood with positive probability is similar – to make agents rely on their own signals with positive probability such that their actions become informative – but to know the true state in the limit, the key factor continues to be to observe an arbitrarily large neighborhood. Thus, to approximate maximal learning with endogenous observation, even when $\lim_{n\to\infty} K(n) = \infty$, it is not sufficient for the probability of making observations within $\{1, \dots, n-1\}$ to be bounded away from zero; instead, such probability must be close to one.

As a useful sidenote, the results in this paper also apply in the generalized context in which agents are divided into groups g_1, g_2, \cdots . In period $i \in \mathbb{N}$, agents in group g_i make their choices simultaneously with capacity K(i). In other words, the largest neighborhood that an agent in g_i can observe is $\bigcup_{j=1}^{i-1} g_j$. This generalization allows for the possibility that multiple agents move (choose their observed neighborhood and then their action) simultaneously during each period.

1.9 Conclusion

In this paper, I have studied the problem of sequential learning in a network of observation. A large and growing literature has studied the problem of social learning in exogenously given networks. Bikhchandani, Hirshleifer and Welch[BHW92], Banerjee[Ban92] and Smith and Sorensen[SS00] first studied environments in which each agent can observe the entire past history of actions. Recent contributions, such as Acemoglu et al.[ADL11], generalized the network topology to being stochastic. The central question is whether equilibria lead to asymptotic learning, i.e., efficient aggregation of information, and the results in the literature point to two crucial factors: unbounded private beliefs, and expanding observations (i.e. no agent is excessively influential). When these two criteria are satisfied, asymptotic learning occurs in every equilibrium.

In many relevant situations, individuals do not automatically acquire information from a given network of observation but are instead engaged in strategic and costly observations of others' actions. Such behavior can be understood as forming links with others, which constitutes the ultimate network of observation. This raises the question of how information is aggregated in an endogenously formed network and what level of social learning can be achieved in equilibrium in different scenarios. To address these questions, I have formulated a model of sequential learning in a network of observation constructed by agents' strategic and costly choices of observed neighborhoods.

In the model, agents sequentially make strategic moves. Each agent receives an informative private signal, after which she can pay a cost to observe a neighborhood among her predecessors. The size of the observed neighborhood is limited by a given capacity constraint. Given her signal and the realized action sequence in her chosen neighborhood, each agent then chooses one of two possible actions. I have characterized pure-strategy perfect Bayesian equilibria for arbitrary capacity structures and have also characterized the conditions under which different types of social learning occur. In particular, I focused on asymptotic learning and maximal learning. Asymptotic learning refers to efficient information aggregation, i.e. agents' actions converging in probability to the right action as the society becomes large. Maximal learning refers to the same convergence that is conditional on observation. Maximal learning reduces to asymptotic learning when agents almost certainly choose to observe.

Two concepts are shown to be crucial in determining the level of social learning in equilibrium. The first is the precision of private signals. Apart from whether private beliefs are bounded, the relation between private beliefs and the cost of observation is equally important. When the cost is positive, I say that private beliefs are strong if for some signals an agent would not be willing to pay the cost even to know the true state, and that otherwise private beliefs are weak. The second important concept is that of infinite or finite observations with respect to the capacity structure.

My first main result, Theorem 1, shows that when private beliefs are unbounded and the cost of observation is zero, asymptotic learning occurs in every equilibrium. It further implies that the network topology in every equilibrium will automatically feature expanding observations. It provides a micro-foundation for the above mentioned condition of expanding observation, i.e. agents tend to observe some close predecessor when observation is not costly. In this case, information is always efficiently aggregated unconditionally.

The next main theorem, Theorem 2, characterizes the necessary and sufficient condition for maximal learning when agents have strong (not necessarily unbounded) private beliefs: if and only if the capacity structure has infinite observations, agents will learn the true state with near certainty via observation when the society becomes large. This result stands in stark contrast to the literature in the sense that each agent must be infinitesimally influential to others in any observed neighborhood to ensure maximal learning. As long as there are "influential" agents, even though they may not be excessively influential, there is always a positive probability of taking the wrong action after observation. Conversely, whenever observations are infinite, information can be aggregated efficiently to guarantee the revelation of the true state conditional on observation.

In Theorem 3, I have characterized a class of stochastic observation structures that approximate maximal learning under weak private beliefs. In such observation structures, the agents sometimes face a limited set of available neighborhoods that are non-persuasive. Again, the key factor in approximating maximal learning is that agents can observe an arbitrarily large neighborhood with a high probability. One important welfare implication is that weaker private beliefs may actually be beneficial for the society in an environment with general stochastic observation structures because the limit learning probability can be higher.

I believe that the framework developed in this paper has the potential to facilitate a more

general analysis of sequential learning dynamics in an endogenously formed network of observation. The following questions are among those that can be studied in future work using this framework: (1) equilibrium learning when agents' preferences are heterogeneous; (2) equilibrium learning when the cost of observation is random and is part of an agent's private information; (3) the relation between the cost of observation and the speed (rate of convergence) of sequential learning.

CHAPTER 2

Dynamic Network Formation with Incomplete Information

2.1 Introduction

How do social and economic networks form and what is their ultimate shape (topology)? This important question is addressed in a substantial literature which begins with the seminal paper by Jackson and Wolinsky[JW96] and continues with the works by Bala and Goyal[BG00], Jackson and Watts[JW02], Ballester et al.[BCZ06], and others. The central conclusion of this literature is that only special shapes of networks can occur and persist. However, this literature makes the strong assumption of complete information: agents know in advance the value of linking to any other agents - even agents they have never met and with whom they have had no previous interaction, direct or indirect. (Most of the literature on dynamic network formation, for instance Bala and Goyal[BG00] and Watts[Wat01], also assumes that agents have complete information about the entire history of link formation at every moment in time.) The present paper addresses the same questions under what seems to us to be the much more natural assumption of incomplete information: agents do not know in advance - but must learn - the values of linking to agents they have never met ¹. As is usual in environments of incomplete information, agents begin only with beliefs about the values of linking to other agents, make choices on the basis of their beliefs, and update their beliefs (learn the true values) on the basis of their experience (history).

We show that the assumption of incomplete information has profound implications for the process of network formation, the shape of the networks that ultimately form and persist, and the location of various agents within the network. Much of the literature that assumes complete information shows that the only networks that can form and persist have a star or core-periphery

¹We emphasize that we study learning *during the network formation process*, rather than learning in an exogenously given and fixed network structure. For a study on the latter, see for example Acemoglu et al.[ADL11]

form, with agents of relatively high value in the core. By contrast, when information is incomplete, we show that a much larger variety of networks and network shapes can form and persist: frequently a strict superset of the set of networks that can form and persist when information is complete. This phenomenon occurs because, due to incomplete information, some unattractive ("low-value") agent may get linked and even secure a high connectivity degree in an early stage of the formation process, and then would remain connected thereafter since it offers other agents indirect access to numerous "high-value" agents. Under complete information, this "connection by mistake" never happens. However, these connections by mistake which happen only in the incomplete information setting may be often socially valuable: the ultimate network that forms and persists when information is initially incomplete may often yield higher social welfare than any network that can form and persist when information is initially complete. We stress that all of this can occur even in settings where agents eventually learn *everything* so that information, although initially incomplete, eventually becomes complete. Incompleteness of information may eventually disappear but its influence may persist forever.

To make these points we adopt precisely the same framework as in Watts[Wat01] except that we assume that information is initially incomplete. As usual, agents begin with common prior beliefs about the types of other agents but learn these types over time by forming links (which might later be broken) in a process where they are randomly selected to take action. Moreover, even when the network shapes that form are the same or similar as in the complete information case, the locations of agents within the network can be very different. For instance, when information is incomplete, it is possible for a star network with a "low-value" agent in the center to form and persist indefinitely; thus, an agent can achieve a central position purely as the result of chance rather than as the result of merit. Perhaps even more strikingly, when information is incomplete, a connected network can form and persist even if, when information were complete, *no links* would ever form so that the final form would be a totally disconnected network.

However, the most important consequence of incomplete information is not that a larger variety of network shapes (topologies) might emerge, but that the particular shape that *does* emerge depends on the history of link formation and of link formation opportunities. For instance, when information is incomplete, agents i, j might choose to form a link because each *expects* the value of the link to exceed the cost of forming it. Having formed the link, the agents

may learn that their expectations were wrong and so might wish to sever it. Nevertheless, before the agents have the opportunity to sever the link, each of them may have formed *other* links, so that the *indirect* value of the link between *i* and *j* – the value of the connection to *other* agents – may be sufficiently large that they prefer to maintain the link between them after all; as a result, even when all information is eventually revealed, the link persists². However, whether these other links have formed will depend not only on the values of those links but also on the random opportunities presented to form them.

2.2 Literature Review

The conclusions about network formation and stability that can be found in the existing literature depend both on the process of formation and the notion of stability. Our paper is most closely related to Watts[Wat01], which is an extension of Jackson and Wolinsky[JW96]. The focus of Watts[Wat01] is to analyze the formation of networks in a dynamic framework, where agents are homogeneous (hence information is complete in the strongest form), link formation is undirected and requires bilateral consent, and the pair of agents selected to update their potential link follows a given stochastic process. This model predicts that the network topology that ultimately form and become stable must be either *empty* or *connected*. We adopt exactly the same network formation game and the same notion of pairwise stability (as in Jackson and Wolinsky[JW96] as well), but we assume instead that agents are of different types and thus connecting to them results in *different (heterogeneous)* payoffs; moreover, there is *incomplete* information: an agent does not know the types of agents that he has never connected with, but he is able to form *beliefs* based on which he chooses the optimal action. Networks which result under this, more realistic, assumption are strikingly different from those obtained in the model assuming homogeneous agents and complete information. On one hand, connectedness is no longer a key property of stable non-empty networks - the formation process converges to connected networks in some range of parameters and to non-empty networks with singleton agents in other ranges. On the other hand, even if the network stays empty forever under complete information, a non-empty, even connected network may emerge and be stable with positive

²Indirect linking provides a positive externality when information is complete as well, but the effect is completely different: i, j know exactly the value of the (potential) link between them, so they will never form a link which either may regret forming.

probability under incomplete information.

Jackson and Watts[JW02] study the same dynamic network formation process and applies the same notion of stable network as above, but their focus is to characterize the set of stable networks when there is a tremble in link formation, i.e. there is a small probability that a link is automatically deleted or added after the agents take their action. In contrast, our paper emphasizes the strategic interaction among agents along the network formation process, and focuses on how agents' belief affects their optimal behavior and thus the network topology.

The renowned paper by Bala and Goyal[BG00] is close to our paper in motivation, as they also analyze the dynamics in network formation under the assumption of homogeneous agents. However, their model is based on *directed* link formation, such that the action of one individual to link to another neither requires the second individual's consent nor incurs any cost to the second individual. In other words, new links can be formed unilaterally. Moreover, instead of allowing only one pair of agents to meet and interact in one period as in Watts[Wat01], they assume that all agents move simultaneously in each period and play some Nash equilibrium. Such differences in modelling lead to fundamentally different theoretical results and applications.

In the literature studying how agent heterogeneity affects network formation, such as Haller and Sarangi[HS03], Galeotti[Gal06], Galeotti et al.[GGK06], Galeotti and Goyal[GG10] and Zhang and van der Schaar[ZS12][ZS13], even though the theoretical frameworks vary from one to another, complete information is still a common assumption, and the predictions in the above papers are often restricted to a few, specific, types of network topologies such as stars, wheels or core-periphery networks with "high-value" or low-cost agents enjoying higher connectivity than others. We differ from these works in three aspects. First, as mentioned above, agents have no precise knowledge about their exact payoffs due to incomplete information; instead, they choose an optimal action according to their *beliefs* about the payoffs that they will obtain from connecting to others. Secondly, we show that the interaction of incomplete information and agent heterogeneity produces a *much wider* range of network topologies, which includes stars, wheels, core-periphery networks, etc. Finally, the topology that emerges and becomes stable strongly depends on the *formation history*: an agent may exhibit a high degree of connectivity in equilibrium not necessarily because he is of a special (high) type but also because initially he was fortunate to obtain sufficiently many links by chance, which in turn attracted others to form and maintain links with him due to the large indirect benefits that he can offer. Therefore,

unlike most existing literature that only emphasizes *what* topologies can be formed, we argue that *how* a certain topology comes into being is equally important.

Among the empirical literature on network formation games, Falk and Kosfeld[FK03], Corbae and Duffy[CD08], Goeree et al.[GRU09] and Rong and Houser[RH12] have conducted experimental studies on the types of emerging topologies. The experimental results indicate that (1) typical equilibrium network topologies predicted by the existing theoretical analysis are *not* always consistent with the empirical observations; especially, stars are formed only in a fraction of the total number of experiments conducted (see Corbae and Duffy[CD08], Rong and Houser[RH12]) and such fractions, under some treatments such as a two-way flow of payoffs, are rather low (see Falk and Kosfeld[FK03]); (2) even in the experiments where equilibrium network topologies do emerge with high frequency, such topologies are *developed* rather than *born* (see Goeree et al.[GRU09]), which we believe suggests a dynamic network formation process of a sufficiently long duration as a more appropriate environment for stable networks to emerge, compared with a static one. Moreover, the study of networks which actually get formed in large social communities, as presented for instance by Mele[Mel10] and Leung[Leu13], shows that in environments where agents are heterogeneous and withhold certain private information in their payoffs from links, numerous phenomena which are not predicted by the existing theoretical literature (such as multiple components and clustering of agents with different attributes) can happen. We do not claim to explain all these phenomena in this paper since the way to model network formation and stability makes a big difference to the conclusions and not every model is appropriate in every circumstance. However, we believe that incorporating incomplete information in the dynamic network formation game represents a first and important step towards understanding why several previously seemingly irregular network topologies can emerge and remain stable in practice.

The rest of the paper is organized as follows. Section 3 introduces the model. Section 4 analyzes the model in detail and interprets the results. Section 5 discusses an alternative approach in modeling. Section 6 concludes and introduces relevant future research topics.
2.3 Model

2.3.1 Networks with Incomplete Information

2.3.1.1 Networks and the Agents' Types

Let $I = \{1, 2, ..., N\}$ denote a group of *N* agents. Agents are characterized by their private type $k \in X$, where *X* is a finite set of types³. The probability distribution of types on *X* is *H*. In the actual network there are *N* agents -1, 2, ..., N, whose types are drawn independently from *X* according to distribution *H*. Each agent *i* knows its private type $k_i \in X$. Let $\kappa = \{k_i\}_{i=1}^N$ denote the type vector of the agents.

A *network* is denoted by $\mathbf{g} \subset \{ij : i, j \in I, i \neq j\}$, and the *sub-network* of \mathbf{g} on $I' \subset I$, denoted $\mathbf{g}_{sub}(I')$, is defined as a subset of \mathbf{g} such that $ij \in \mathbf{g}_{sub}(I')$ if and only if $i, j \in I'$ and $ij \in \mathbf{g}$. ij is called a *link* between agents *i* and *j*. We assume throughout that links are *undirected*, in the sense that we do not specify whether link ij points from *i* to *j* or vice versa. A network \mathbf{g} is *empty* if $\mathbf{g} = \emptyset$.

We say that agents *i* and *j* are *connected*, denoted $i \stackrel{\mathbf{g}}{\leftrightarrow} j$, if there exist $j_1, j_2, ..., j_n$ for some *n* such that $ij_1, j_1 j_2, ..., j_n j \in \mathbf{g}$. Let d_{ij} denote the *distance*, or the smallest number of links between *i* and *j*. If *i* and *j* are not connected, define $d_{ij} := \infty$. An agent *i* in a network is a *singleton* if $ij \notin \mathbf{g}$ for any $j \neq i$.

Let $N(\mathbf{g}) = \{i | \exists j \text{ s.t. } ij \in \mathbf{g}\}$. A *component* of network \mathbf{g} is a maximally connected subnetwork, i.e. a set $C \subset \mathbf{g}$ such that for all $i \in N(C)$ and $j \in N(C)$, $i \neq j$, we have $i \stackrel{C}{\leftrightarrow} j$, and for any $i \in N(C)$ and $j \in N(\mathbf{g})$, $ij \in \mathbf{g}$ implies that $ij \in C$. Let C_i denote the component that contains link ij for some $j \neq i$. Unless otherwise specified, in the remaining parts of the paper we use the word "component" to refer to any *non-empty* component.

A network **g** is said to be *empty* if $\mathbf{g} = \emptyset$, and *connected* if **g** has only one component which is itself. **g** is *minimal* if for any component $C \subset \mathbf{g}$ and any link $ij \in C$, the absence of ij would disconnect at least one pair of formerly connected agents. **g** is *minimally connected* if it is minimal and connected.

³A general measure space of types could be accommodated easily but with added technical complications.

2.3.1.2 Payoff Structure

We assume non-local externalities in payoffs: once agents *i* and *j* form a link, *i* not only obtains payoffs from his immediate neighbor *j*, but also from the agents that he is indirectly connected to via that particular link. The payoff to forming a direct link is type dependent and given by the function $f: X \to \mathbb{R}^{++}$; f(k) is the direct payoff to any agent who forms a link to an agent of type *k*. Agents also obtain utilities from indirect links (discounted by distance) and pay costs for maintaining links. Specifically, an agent *i*'s payoff in the network **g** in a given period is given by

$$u_i(k_{-i}, \mathbf{g}) = u_i(k_{-i}, C_i) := \sum_{\substack{j \in i \\ j \leftrightarrow i}} \delta^{d_{ij}-1} f(k_j) - \sum_{j: ij \in C_i} c$$

 $f(k_j) > 0$ denotes the payoff to an agent *i* from the link to an agent *j*, whose value depends on *j*'s type k_j . $\delta \in (0,1)$ denotes a common decay factor, such that the payoff of *i* from *j* with a distance of d_{ij} is $\delta^{d_{ij}-1}f(k_j)$. c > 0 is the cost of maintaining a link, which is assumed to be bilateral and homogeneous across agents. The assumption of *c* being homogeneous across agents is without loss of generality and is made merely to avoid redundant analysis, as the incomplete information and heterogeneity in agents' payoffs have been reflected by the potentially different types of agents. All of our major results can be obtained with slight technical changes in an environment where the cost is heterogeneous (even when the cost is also private information) across agents.

Let $\mathbb{E}[f(x)] = \int_X f(x) dH(x)$ denote the expected benefit from a link to a single agent, under the prior type distribution. As mentioned before, the only assumption we require on X (and functions H and f) is that this expected payoff is well-defined.

2.3.2 Dynamic Network Formation Game

The dynamic game is the same as in Watts[Wat01], except that information is incomplete. Time is discrete and the horizon is infinite: t = 0, 1, 2, ... The game is played as follows: agents start with an empty network \emptyset in period 0. In each following period, a pair of agents (i, j) is randomly selected to update the link between them. As long as each pair of agents is selected with positive probability, the specific probability distribution for the selection process does not affect our main results.

The two agents then play a simultaneous move game, where either agent can choose to sever the link between them if there is one, and if there is not, whether to agree to form a link with the other agent. Let $a_{ij} = 1$ denote the action that *i* agrees to form a link with *j* (if there is no existing link) or not to sever the link (if there is an existing one), and $a_{ij} = 0$ otherwise. A link is formed or maintained after bilateral consent (i.e. $a_{ij} = a_{ji} = 1$). The agents are assumed to be myopic and not forward looking, i.e. they only care about their *current* payoffs when choosing an action.

With incomplete information, agents maximize their *expected* payoffs, rather than actual payoffs, when deciding the optimal action. Therefore, the *belief* of an agent on the types of the other agents plays a crucial role in shaping his behavioral patterns. We assume the following simple updating rule for the agents:

- 1. If two agents were ever connected, they know each other's type.
- 2. Otherwise, their belief on each other's type remains at the prior.

The first part of this updating rule is a simple representation of the realistic situation that people will (sooner or later) find out the true value they receive via connection, and this specification does not affect any technical part of analysis or the implication that follows. The second part of the updating rule can be regarded as a straight forward way of modelling agents' constrained interpretation of the past formation history. In a lot of realistic situations, agents can only observe a very small part or even none of the past formation history due to many different constraints; this is especially true in social networks and business networks where a large number of agents' actions are kept private. In addition, agents may not be able to perform complicated and precise update based on their observations. As a result, when agent *i* meets *j*, *i* may have some idea about how well j is connected to others, but due to the above constraints, high connectivity alone does not imply high quality/value, and vice versa. In fact, our analysis shows exactly this point, i.e. a low quality agent can assume a central position in the network just by chance. As a result, when agents do not know a lot about past history and neither can infer a lot about someone else's value from her connectivity, the prior seems a justifiable belief to hold. A similar assumption appears in McBride[McB06], which is referred to as imperfect monitoring and describes agents' inability to update according to all other agents' strategies in a static network formation game. In the next section, we will highlight the role of this updating rule in

the network formation process.

A plausible alternative updating rule is to perform complete Bayesian update according to the entire formation history, which results in very complicated belief formation. We will discuss this alternative updating rule in Section 5. Of course, many other assumptions can be made about how agents update their beliefs. The two types of updating rules we consider in this paper are two extreme assumptions about observations and beliefs which readily illustrate the general difference that incomplete information makes in the network formation process.

2.4 Analysis

In this section, we analyze the nature of the network formation process and show a clear contrast between the existing results under complete information and our results under incomplete information. We begin by defining the solution concept for the two-player game in each period, and the notion of a stable network.

2.4.1 Stable Equilibrium and Stable Network

2.4.1.1 Strategy

In this section, we formally define a strategy in the network formation game. We first provide a standard and conventional definition, and then introduce a way of simplification for the subsequent analysis.

First, we define a strategy in a standard way. Let (i_{τ}, j_{τ}) denote the pair of agents selected in period τ , and let \mathbf{g}_{τ} denote the network formed in period τ . In the network formation process, the following two conditions must be satisfied:

$$\begin{split} \mathbf{g}_0 &= \varnothing \\ \mathbf{g}_{\tau+1} &\in \{\mathbf{g}_\tau + i_{\tau+1} j_{\tau+1}, \mathbf{g}_\tau - i_{\tau+1} j_{\tau+1}\}. \end{split}$$

The first condition refers to the initially empty network; the second reflects the fact that $i_{\tau+1}j_{\tau+1}$ is the only link that can be potentially changed in period $\tau + 1$. Denote $\sigma_t = \{(i_{\tau}, j_{\tau}), \mathbf{g}_{\tau}\}_{\tau=1}^t$ as the formation history up to period *t*, and let Σ_t denote the set of all possible formation histories up to period *t*, with the initial condition $\Sigma_0 = \emptyset$. Let $\Sigma = \bigcup_{t=0}^{\infty} \Sigma_t$, a (pure) strategy of agent *i*

is then a mapping $s_i : X \times \Sigma \times (I - i) \to \{0, 1\}$, where *X* is the set of possible types and I - i is the set of agents other than *i*. For agent *i*, given the strategy profile of other agents s_{-i} , a *best response* is then a strategy s_i such that given any $k_i \in X$, $\sigma \in \Sigma$ and $j \in I - i$, s_i selects an action that maximizes *i*'s (current) expected payoff according to her belief.

The complete set of strategies for an agent is enormous: a different action can be chosen based on each different history, and in infinite periods of time there will be infinitely many possible histories. However, we argue below that it is sufficient to consider a particular subset of strategies, and these strategies admit a particularly simple description.

First, note that the current-period payoff of an agent does not depend on her own type. In any period, given any strategy profile of other agents, if some action is strictly optimal for agent *i* of type k_i , then it must also be strictly optimal for agent *i* of any other type. (Except in non-generic cases where an agent is indifferent between the two available actions, any best response – any candidate for equilibrium – must be independent of an agent's own type.)

Agents are assumed to be myopic; they choose an action to maximize current expected payoff according to belief. Now for agent i, a weakly best response for i is always choose 1, if her expected payoff according to her belief is non-negative and 0, otherwise. We assume agents link when indifferent. (Again, except for non-generic cases where i is indifferent between her two available actions, this is in fact the strict best response.) Here, the sufficient information for i to determine her optimal action is her belief on the type vector, and the component structure of the other agent selected.

Hence, we do not need to consider all strategies, only those that are ever best responses (optimal) and these admit a simple alternative description as mappings from beliefs and component structures to actions. Such strategies are straightforward to characterize. Formally, let $B_i : \Sigma \to \Delta(X^N)$ be the belief updating function of agent *i*, with the constraint that B_i always assigns probability 1 on *i*'s true type and $B_i(\emptyset)$ is equal to the prior belief, and let $C_j : \Sigma \to G$ (*G* being the set of all possible networks) be the mapping from the formation history to the resulting component containing *j*. Then let $Y_j = \{(B_i(\sigma_t), C_j(\sigma_t)) : \sigma_t \in \Sigma\}$. We can describe a strategy of agent *i* as a mapping $s_i : X \times Y_i \times (I - i) \to \{0, 1\}^4$. With this characterization, we can break down the solution of equilibria with sequential rationality to the much more straight

⁴As mentioned before, in generic cases agent *i*'s best response will not depend on her own type. Nevertheless, we still keep the type of *i* as an argument in *i*'s strategy since it is part of *i*'s knowledge after all.

forward inspection of equilibria in the 2-person link formation game in each period.

2.4.1.2 Stable Equilibrium

Now we define our solution concept, which we refer to as the *stable equilibrium* (*SE*). With a slight abuse of notation, in the subsequent analysis we use B_i to denote the *realized* belief vector of agent *i* based on her belief updating function, and C_i to denote the component containing *i*.

Definition 7. A strategy profile *s* is a stable equilibrium (SE) if for any agent $j \neq i$, s_i is such that $s_i(k_i, (B_i, C_j), j) = 1$ if and only if

$$\mathbb{E}[u_i(k_{-i}, (C_i \cup C_j) + ij)|B_i] \ge u_i(k_{-i}, C_i).$$

We would like to emphasize here that even though the definition of SE does not involve an explicit expression of best response, it essentially represents bilateral best response in the one-period linking game. Note that the above strategy profile does not depend on the agents' types, i.e. for agent *j*, given B_j , she takes the same action no matter what her type is. Hence for agent *i*, her payoff from a link with agent *j* only depends on her belief vector B_i and the component C_j . In other words, *i*'s expected payoff is $\mathbb{E}[u_i(k_{-i}, (C_i \cup C_j) + ij)|B_i]$ if a link is formed, and $u_i(k_{-i}, C_i)$ otherwise. A weakly best response for *i* is then to choose action 1 if $\mathbb{E}[u_i(k_{-i}, (C_i \cup C_j) + ij)|B_i] \ge u_i(k_{-i}, C_i)$ and 0 otherwise, which is exactly as prescribed by the above strategy. Hence the above strategy maximizes *i*'s current expected payoff given s_j , and vice versa.

In words, a SE is a Nash equilibrium where any agent would choose to agree to form a link (if there is no existing link) or choose not to sever the link (if there is an existing link) as long as the expected payoff from the link is non-negative according to her current belief. It is *stable* because the prescribed strategy is robust to small probabilistic changes in the counter party's strategy when the above inequality is strict. More specifically, if some agent j other than i changes her action with a sufficiently small probability ε , i's best response would not change. In particular, it excludes the "pessimistic" or null equilibria in which no link formation occurs even though each agent has a non-negative expected payoff from the potential link. In other words, agents choose to link as long as they are at least indifferent. This equilibrium notion is similar to pairwise stability in Jackson and Wolinsky[JW96], but for the setting of incomplete

information; thus, the conditions that characterize "stability" are based on expected rather than realized payoffs.

The following lemma shows the existence and uniqueness of such an equilibrium.

Lemma 6. SE exists and is unique.

Proof. Consider any period. Assume that agents *i* and *j* are selected in that period. Given the belief vectors B_i and B_j , and the components C_i and C_j , *i*'s (similarly, *j*'s) strategy satisfying the condition in the definition of a SE can be expressed as

$$s_i(k_i, (B_i, C_j), j) = \begin{cases} 1, \text{ if } \mathbb{E}[u_i(k_{-i}, (C_i \cup C_j) + ij)|B_i] \ge u_i(k_{-i}, C_i) \\ 0, \text{ otherwise} \end{cases}$$

Since the terms on both sides of the above inequality are well-defined and agent i's action is binary, agent i's optimal choice in this period is unique. Since the period and agents assumed are arbitrary, SE is unique. (Given that agents link when indifferent, the argument is that agents follow a dominant strategy.)

Lemma 6 together with the above robustness property of a SE ensures that the outcome of the formation process is unique and robust to small perturbation (or tremble) in agents' action. It is also useful to note that Lemma 6 holds for any belief updating function.

2.4.1.3 Stable Network

From now on, we assume that agents play the SE in every period. Let $\gamma_t := \{(i_\tau, j_\tau)\}_{\tau=1}^t$ $(t \ge 1)$ denote a *selection path* up to time *t*, or the set of selected pairs of agents, ordered from 1 to *t*. Since the equilibrium in each period exists and is unique, the realization of the entire network formation process can be fully characterized by κ , the type vector of agents and γ_{∞} , the realization of the random selection process. We use Γ to denote a formation process, and in particular denote Γ_C and Γ_{IC} as network formation processes under complete and incomplete information correspondingly.

Let $\mathbf{g}_C(\gamma_t)$ denote the unique network formed after period *t*, following a selection path of γ_t , under complete information, and $\mathbf{g}_{IC}(\gamma_t)$ the network under incomplete information. Let $B_C(\gamma_t) = \{B_{i,C}(\gamma_t)\}_{i=1}^N$ denote the associated belief vector after period *t* under complete information (where $B_{i,C}(\gamma_t)$ is always equal to the degenerate belief on the true type vector *k*, for

every *i*), and $B_{IC}(\gamma_t)$ the belief vector under incomplete information. By Lemma 6 we know that both $\mathbf{g}_C(\gamma_t)$ ($\mathbf{g}_{IC}(\gamma_t)$) and $B_C(\gamma_t)$ ($B_{IC}(\gamma_t)$) are well-defined. We say that:

- 1. A network **g** *can emerge* under complete information (incomplete information) if there exists a selection path γ_t for some *t*, such that $\mathbf{g} = \mathbf{g}_C(\gamma_t) (= \mathbf{g}_{IC}(\gamma_t))$.
- 2. A network with the associated belief vector, (\mathbf{g}, B) $(B = \{B_i\}_{i=1}^N)$, is a *stable network* under complete information (incomplete information) if no link is formed or broken given *any* subsequent selection path in Γ_C (Γ_{IC}). If *B* refers to the belief vector under complete information (where B_i is the degenerate belief on the true type vector for all *i*), then we just say that **g** is a stable network.
- 3. Γ_C (Γ_{IC}) can converge to **g** if there exists a selection path γ_t for some *t*, such that $\mathbf{g} = \mathbf{g}_C(\gamma_t) (= \mathbf{g}_{IC}(\gamma_t))$ and $(\mathbf{g}_C(\gamma_t), B_C(\gamma_t)) ((\mathbf{g}_{IC}(\gamma_t), B_{IC}(\gamma_t)))$ is stable.

2.4.2 Information Revelation

Before discussing the differences between the network topologies that can emerge and be stable under complete information and incomplete information, we first inspect how long incomplete information can persist. We say that *information is complete* (following a given history) when every agent's belief (following that history) is the degenerate belief on the true type vector κ , i.e. $Prob(\kappa|B_i) = 1$ for any *i*, and that *information is incomplete* otherwise.

Recall that agents update their beliefs based on the simple updating rule: if two agents have ever been connected, they know each other's type; otherwise, their beliefs on each other's type remain at the prior. In other words, for agent *i*, the following is true for any realized belief B_i : the argument in B_i for the type of any other agent *j* is the degenerate belief on *j*'s true type k_j if *i* and *j* have ever been connected, and is equal to the prior belief otherwise. Hence, from the definition of SE, we know that if $\mathbb{E}[f(x)] < c$, no link will ever form from the very beginning; if $\mathbb{E}[f(x)] \ge c$, then every agent is willing to form a link with any other agent that she has not been connected to before. This observation enables us to determine the property of *information revelation* in the network formation process, which is noted in the following proposition.

Proposition 9. *For any κ:*

- *I*. If 𝔼[f(x)] < c, information never becomes complete: agents' beliefs remain forever at the prior.
- If E[f(x)] ≥ c, information becomes complete within finitely many periods almost surely, and information is complete in any stable network.

Proof. If $\mathbb{E}[f(x)] < c$: by inspecting the SE we know that for any pair of agents selected, no link would be formed in any period. Therefore, no agent ever learns the type of any other agent, and the beliefs would stay at the prior.

If $\mathbb{E}[f(x)] \ge c$: we first show that information becomes complete within finitely many periods almost surely. It suffices to show that any two agents are connected for at least one period within finitely many periods almost surely. Since $\mathbb{E}[f(x)] \ge c$, by the definition of SE it further suffices to show that any two agents are selected at least once within finitely many periods almost surely. Consider any two agents *i* and *j*; the probability of the event that they are not selected in one period is $1 - \frac{2}{N(N-1)} < 1$, and thus the probability of this event occurring for infinitely many periods is 0.

Next, we show that information must be complete in any stable network. If **g** is connected, then clearly there is complete information. If **g** is unconnected and information is not complete, then there must exist two unconnected agents such that their beliefs on each other's type remain at the prior. When they are selected they would form a link, which implies that (\mathbf{g}, B) is not stable, a contradiction.

When the expected benefit from a link under the prior weakly exceeds the link formation cost, everyone has the incentive to form links with others whose types are unknown, and thus eventually learns the true type vector with probability 1 over time. Indeed, as agents are always willing to form links with strangers, after sufficiently many periods the probability of pairs of agents who never connected (i.e. pairs of agents that have never met and hence have never learnt about each other) would be arbitrarily small. In other words, only in an early stage of the formation process can incomplete information make a difference and affect the ultimate network topology, as compared with complete information.

2.4.3 Contrast between Complete and Incomplete Information

Even though Proposition 9 may leave the impression that incomplete information is not crucial for the formation process as it only takes effect in the short run, we will emphasize in the following analysis that such short-term influence is actually persistent over time.

Let $G_C(\kappa)$ denote the set of networks that can emerge under complete information given κ , and $G_{IC}(\kappa)$ that under incomplete information. Similarly, let $G_C^*(\kappa)$ denote the set of networks that can emerge and be stable under complete information given κ , and $G_{IC}^*(\kappa)$ that under incomplete information.

Theorem 4. For any κ :

- 1. If $\mathbb{E}[f(x)] < c$, then $G_{IC}(\kappa) = G^*_{IC}(\kappa) = \{\varnothing\}$.
- 2. If $\mathbb{E}[f(x)] \ge c$, then $G_{IC}(\kappa) \supset G_C(\kappa)$, and $G^*_{IC}(\kappa) \supset G^*_C(\kappa)$.

Proof. If $\mathbb{E}[f(x)] < c$: as in the proof of Proposition 9, no link would ever form and thus the only network that can emerge is the empty network. This proves 1.

To prove 2, assume $\mathbb{E}[f(x)] \ge c$ and consider $\mathbf{g} \in G_C(\kappa)$.

If **g** is empty: since $\mathbf{g} \in G_C(\kappa)$, there must exist two agents i, j such that $f(k_i) < c$ or $f(k_j) < c$. Consider the selection path $\gamma_2 = ((i, j), (i, j))$ under incomplete information. It is clear that a link would form between *i* and *j* in period 1, but the link would then be severed in period 2, and thus $\mathbf{g} = \mathbf{g}(\gamma_2)$, which implies that $\mathbf{g} \in G_{IC}(\kappa)$.

If **g** is non-empty: consider any selection path γ_t^C such that **g** emerges for the first time in period *t*. By the definition of $G_C(\kappa)$, we know that such γ_t^C exists. We construct a different selection path under which **g** forms when information is incomplete (but the true type vector is κ). Let γ_{τ}^{IC} be a selection path constructed from γ_t^C such that the pairs of agents in γ_t^C between whom there is no existing link, but a new link is not formed either, are deleted. Note that the two selection paths may take different number of time periods: $\tau \leq t$ by the above construction.

Consider the formation process under incomplete information given γ_{τ}^{IC} . First, it is clear that for any agent *i* with $f(k_i) < c$, no link will be formed between *i* and any other agent under complete information. Hence, if a link is formed between *i* and *j* under complete information, it must be the case that $f(k_i)$ and $f(k_j)$ are both weakly higher than *c*; then according to the simple updating rule, the same link will also be formed under incomplete information regardless

of whether *i* and *j* know each other's type. Also, it is clear that the decision of severing a link by any agent is the same under complete information and incomplete information, since such a decision is based on the realized payoff. Therefore, the formation process yields the same link formation and severance results under complete information given γ_t^C and under incomplete information given γ_τ^{IC} . Hence given γ_τ^{IC} , **g** emerges for the first time in period τ under incomplete information, which implies that $\mathbf{g} \in G_{IC}(\kappa)$. Therefore $G_C(\kappa) \subset G_{IC}(\kappa)$. This proves the first part of 2.

To prove the second part of 2 (stability), we first observe that, by the above argument, we already know that any network that can emerge under complete information can also emerge under incomplete information. Thus it suffices to show that, for any network $\mathbf{g} \in G_C^*(\kappa)$, there exists a subsequent selection path under incomplete information after \mathbf{g} 's first appearance that would make \mathbf{g} stable. We prove the result by construction.

If **g** is empty: consider the selection path $\gamma_{\frac{N(N-1)}{2}}$ such that every pair of agents is selected exactly twice consecutively. Since by assumption $\mathbf{g} \in G_C^*(\kappa)$, we know that for every pair of agents a link would first form and then be severed in the next period. In period $\frac{N(N-1)}{2} + 1$, information is complete and the empty network becomes stable.

If **g** is non-empty: denoting the number of components and singleton agents in **g** as $q(\mathbf{g})$, under incomplete information let the subsequent selection path after **g**'s first appearance be such that in the first $q(\mathbf{g})(q(\mathbf{g}) - 1)$ periods, two agents from different components are selected exactly twice consecutively and every two components (or singleton agents) are involved. By assumption $\mathbf{g} \in G_C^*(\kappa)$, which means that, should two agents from different components know each other's type, no link would be formed between them. Therefore, under incomplete information, when a pair of agents from different components is selected for the second time, either there is no existing link between them and no link would be formed, or an existing link would be severed. In either case, the agents know each other's type as well as the types of agents in the counter party's component. Therefore, after $q(\mathbf{g})(q(\mathbf{g}) - 1)$ periods, every agent knows κ and essentially there is no incomplete information. Again by the assumption that $\mathbf{g} \in G_C^*(\kappa)$, we can conclude that g is such that no link would be formed or severed in any later period given any selection path. Thus $\mathbf{g} \in G_{IC}^*(\kappa)$, which completes the proof.

Theorem 4 states that, when expected benefits are sufficiently high, if some network can

emerge (and be stable) under complete information, then it can also emerge (and be stable) under incomplete information, but the reverse may not be true. Intuitively, if a link could be formed under complete information, then given high expected benefits and the simple updating rule, it can also be formed under incomplete information, whether or not the relevant agents know each other's type. Note that the reverse is not necessarily true: even if a link could be formed under incomplete information, it may not form under complete information because the expected payoffs are sufficiently higher than the realized payoffs, i.e. in the incomplete information setting, were the agents to know each other's type beforehand, the link may never be formed. In other words, under incomplete information, high expected payoffs can initialize link formation such that even though agents would "regret" the links they formed after knowing each other's type. However, if more links have formed before they are selected again to update their initial links, then due to increasing returns to link formation the positive externalities would in turn ensure that the initial links are maintained. The following example illustrates this point.

Example 2. Assume the following: N = 5, $X = \{a, b\}$, and the other parameters are such that f(b) < c < f(a), $\mathbb{E}[f(x)] \ge c$, $(1 + \delta - \delta^2 - \delta^3)f(b) \ge c$, and $(1 - \delta^3)f(a) < c$. In addition, we assume that the realized type distribution is consistent with the agents' prior belief, i.e. the actual number of type a agents = $N \times Prob(k_i = a)$. For illustrative purpose, we consider the case that $k_1 = k_2 = b$, $k_3 = k_4 = k_5 = a$, and $Prob(k_i = a) = 0.6$.

Consider the following selection path: ((1,2),(2,3),(3,4),(4,5),(1,5)). Under complete information, the network is never connected, as in Figure 1(A). Under incomplete information, the SE can be explicitly computed in each period. For example, in period 1, agent 1's expected payoff from the link with agent 2 is $\mathbb{E}[f(x)] \ge c$ and vice versa, and thus the link is formed. The formation process is shown in Figure 1(B).

According to the assumptions on parameters, one can then easily show that the network formed in period 5 is stable. Note that agent 1 prefers to maintain the link to agent 2 even though agent 2 is of a low type because that shortens agent 1's path to agent 3.

Of course, though the agents' beliefs are always consistent with each other and with the prior distribution of types, they may not be always consistent with the realized type vector; when they are not, the difference between complete and incomplete information is even larger. For instance, assume that $k_i = b$ for all i and consider the same selection path as above. Under



Figure 2.1: Not Connected under Complete Info. vs. Connected under Incomplete Info.

complete information, the network stays empty; under incomplete information, the formation process still converges to a wheel network.



Figure 2.2: Empty under Complete Info. vs. Connected under Incomplete Info.

An important factor that enables networks that are never formed under complete information to form and be stable under incomplete information is that agents do not sever undesirable links immediately. In the model, this is an event that happens with significantly high probability, since the probability that the same pair of agents is selected twice consecutively is rather low. It becomes even lower in a larger society. In numerous works on network formation, such as Jackson and Wolinsky[JW96], Watts[Wat01] and Dutta et al.[DGR05], the random selection of a potential link to update is a common modelling assumption. In practical situations, this event can be understood as follows: it takes time for people to make up their minds to disconnect with someone they do not like. Before deciding to sever a link with a person, that person may have built new connection with others, which will change the value of linking with her after all. The reason for such delay in decision may be some legal, geographic or technological barrier. For instance, if a link represents a binary contract, one cannot easily terminate the contractual relationship before the expiration date. Alternatively, it might be the case that knowing the true value of linking with others is a time-consuming process. It usually takes a certain amount of communication and interaction before people/entities really know the value of their connection, especially in social networks.

Another feature of incomplete information is the *history dependence* of the formation process, in the sense that the ultimate network topology depends greatly on the selection path. As a result, even if a type is more valuable or preferable than another, under incomplete information it is not necessary that an agent of that type ends up with a higher connectivity degree. Consider the following example: assume the same parameter values as in Example 1, and consider a group of agents consisting of 4 type a agents and 5 type b agents. There exists a selection path such that: under complete information, the formation process converges to a star network with only type a agents (Figure 3(A)); under incomplete information, the formation, the formation process converges to a "hub-and-spokes" network (Figure 3(B)).



Figure 2.3: Different Connectivity Degree Distributions

Under complete information, the center of the star network has to be a type a agent since no type b agent ever gets linked with anyone else. In fact, regardless of the selection path, no type b agent can *ever* get linked. By contrast, under incomplete information, it first becomes possible for two type b agents to form a link; and then, as it turns out in this particular topology, each type b agent's distance with the type a agent is sufficiently small. Even though the type a agent has a low connectivity degree, the other agents do not find a new link with the type aagent attractive, because it does not offer sufficient indirect benefits. Hence, the agent with the more valuable type – type a – ends up with the lowest connectivity degree in the network. This is in stark contrast with the existing results in the literature (for instance the property of "law of the few" in Galeotti and Goyal[GG10]), which often show that a more valuable agent is better connected. As mentioned in the literature review, violation of such theoretical predictions has been documented in a number of empirical and experimental studies. From this perspective, our result can be regarded as a micro-foundation for the prevalent phenomenon that an agent may obtain a central position in a network by chance instead of merit.

Furthermore, the event that some agent of "low value" ends up with relatively high connectivity under incomplete information is not a rare event. In Figure 4, we show the simulation result of a network formation process under incomplete information with 15 "high-type" (in the sense that $(1 - \delta)f(x) < c < f(x)$) agents and 1 "low-type" (in the sense that f(x) < c) agent⁵. The figure shows the "low-type" agent's rank in terms of connectivity. Under complete information, the "low-type" agent would have never been connected; under incomplete information, the probability that the "low-type" agent obtains the highest connectivity is rather low, the probability that she ranks among the top half (at or above 8th) is more than 0.2.



Figure 2.4: Simulations: Rank of "Low-Type" Agent

The above examples also highlight and clarifies the point made by Theorem 4: incomplete information generates a superset of *networks*, not a superset of *links*, as compared with complete information. In other words, *new and different* networks can be formed under incomplete

⁵We set the "high-type" agent's value to be f(x) = 6 and the "low-type" agent's value to be f(x) = 4. In addition, we assume that c = 5 and $\delta = 0.6$. In each simulation, we let the formation process run for 2500 periods. We use 100 simulations to find the average result.

information, rather than a mere addition of links to networks formed under complete information. Indeed, the network in Figure 3(A) has 3 links and that in Figure 3(B) has 12, but they share *no links in common*.

Moreover, even when $\mathbb{E}[f(x)] \ge c$, incomplete information does *not* imply that more links are always formed in the stable network. For instance, let $X = \{a, b\}$, and consider a group of 8 type *a* agents (indexed 1, 2, ..., 8) and 1 type *b* agent (indexed 9). The payoffs are f(b) < c, $f(a) \ge c$, $(1-\delta)f(a) < c \le (1-\delta^2)f(a)$ and $\mathbb{E}[f(x)] \ge c$. Let the selection path be as follows: first, select 9 once with each of 1,2,...,8. Then select $((1,2),(2,3),\cdots,(7,8),(8,1))$. Finally, select ((1,5),(2,6),(3,7),(4,8)). The resulting stable network is shown in Figure 5 below: under complete information, the network has 12 links, while under incomplete information it has only 8.



Figure 2.5: More Links under Complete Information

The above examples also show that, unlike the literature on network formation with complete information, which proves that in each model only one or two types of network topologies can be stable (see for example Bala and Goyal[BG00], Galeotti et al.[GGK06] and Galeotti and Goyal[GG10]), under incomplete information many *more types* of networks can emerge and be stable. Even when compared with models that allow for more possibilities in network types (see Jackson and Wolinsky[JW96] and Watts[Wat01]), incomplete information again brings about a wider range of stable network topologies. The next theorem formalizes this statement; but first we need to recall some familiar definitions of network structures.

- 1. **g** is *complete* if $ij \in \mathbf{g} \forall i, j$ such that $i \neq j$.
- 2. **g** is a *star* network if there exists $i \in I$ such that $ij \in \mathbf{g} \forall j \neq i, j \in I$ and $i'j \notin \mathbf{g} \forall i', j \neq i$.

- 3. g is a *core-periphery* network if there exists non-empty I' ⊊ I, such that ij ∈ g ∀i, j ∈ I', i ≠ j, and that ∀j' ∈ I \ I', ij' ∈ g for some i ∈ I' and jj' ∉ g ∀j ≠ i. Note that a star network is a special case of a core-periphery network.
- 4. g is a *tree* network if there exists a partition of *I*, *I*₁,...,*I_n*, such that (1)#(*I*₁) = 1;
 (2)∀n' = 2,...,n, each agent in *I_{n'}* has one and only one link with some agent in *I_{n'-1}*;
 (3)no other link exists.⁶
- 5. **g** is a *wheel* network if there exists a bijection $\pi : I \to I$ such that $\mathbf{g} = \{\pi^{-1}(1)\pi^{-1}(2), \pi^{-1}(2)\pi^{-1}(3), ..., \pi^{-1}(N-1)\pi^{-1}(N)\}.$

Theorem 5. Assume that $\mathbb{E}[f(x)] \ge c$. Fix a type vector κ and a network g. If g is stable when information is complete and belongs to any one of the following categories:

- 1. Empty network;
- 2. Minimally connected network (i.e. tree network, including star network);
- 3. Fully connected network;
- 4. Core-periphery network;
- 5. Wheel network.

then g can emerge and be stable when information is incomplete, i.e. $\mathbf{g} \in G^*_{IC}(\kappa)$.

Note that we have assumed only that g is stable when information is complete, not that it can emerge when information is complete.

Proof. For 1, see the proof of Theorem 4. For 2 - 4, since \mathbf{g} is connected and by assumption \mathbf{g} is stable under complete information, it suffices to show that when \mathbf{g} belongs to any of the categories there exists a selection path such that \mathbf{g} can emerge in the formation process. We discuss case by case and prove them by construction below.

2: Let *L* be the total number of links in **g**. Let the selection path be such that the pair of agents for each link in **g** is selected once and only once in the first *L* periods. Since $\mathbb{E}[f(x)] \ge c$, we know that each link will be formed, and thus **g** emerges in period *L*.

⁶Essentially, a tree network is equivalent to a minimally connected network, and a star network is a special case of a tree network.

3: Since **g** is fully connected and stable, we know that for any two agents *i* and *j*, $(1 - \delta)f(k_i) \ge c$. Therefore regardless of the selection path **g** would emerge.

4: Let the selection path be such that: first each periphery agent is selected once and only once with their corresponding core agent, then every two core agents are selected once and only once before any other pair of agents is selected. Since $\mathbb{E}[f(x)] \ge c$ and **g** is stable under complete information, we know that each link will be formed, and thus **g** emerges after the last pair of core agents is selected.

5: Let the selection path be such that the pair of agents for each link in **g** is selected once and only once in the first N-1 periods. Since $\mathbb{E}[f(x)] \ge c$ and **g** is stable under complete information, we know that each link will be formed, and thus **g** emerges in period N-1. \Box

Theorem 5 explicitly characterizes types of connected networks that can emerge and be stable under incomplete information, and most typical networks in both the literature and empirical studies are included. However, note that there may be some stable networks that cannot emerge under complete or incomplete information – e.g. a network **g** with a subset of links **g'** such that (1) within **g'**, the benefit from any one link cannot cover the maintenance cost without the existence of the other links and (2) $\mathbf{g} \setminus \mathbf{g'}$ is still connected. Such network topologies may never be formed since only one pair of agents is selected in each period and the agents are myopic.

2.4.4 Characterizing Topological Differences

In the previous analysis, we have seen that even with the same selection path, very different networks can emerge and be stable under incomplete information; in this section, we formalize a way of describing such topological differences, and characterize the corresponding conditions under which these differences are achieved.

To obtain a clear characterization result on the topological differences, we first categorize the agents based on their types. We say that *i* is a **low-value** agent if $f(k_i) < c$, i.e. a link with this agent is not worthwhile anyway; a **medium-value** agent if $(1 - \delta)f(k_i) < c \le f(k_i)$, i.e. a link with this agent would be beneficial if there exists no indirect path; and a **high-value** agent if $(1 - \delta)f(k_i) \ge c$, i.e. a link would still be beneficial even if there already exists an indirect path with two links (the shortest indirect path). Let n_l , n_m and n_h denote the number of agents in the corresponding category.

We will consider selection paths for which the formation process converges under both complete and incomplete information. The following lemma establishes that such paths exist.

Lemma 7. For every κ , there exists a formation path such that the formation process leads to a stable network under both complete and incomplete information.

Proof. Consider the following selection path:

- 1. Fix a high-value or medium-value agent i^* . In the first N 1 periods, select i^* and every other agent exactly once.
- 2. In the following N-1 periods, select i^* and every other agent exactly once again.
- 3. In the following $\frac{n_h(n_h-1)}{2}$ periods, select every pair of high-value agents.
- 4. In the following $n_l(n_l 1)$ periods, select every pair of low-value agents twice consecutively.

If $n_m + n_h \ge 1$: under complete information: after step 2, a star with i^* as the center and all other high-value or medium-value agents as the periphery would be formed. After step 3, there will be a link between every pair of high-value agents. The network formed after step 3 is stable. Under incomplete information: after step 1, a star with i^* as the center and all other agents as the periphery would be formed. After step 2, every link between i^* and a low-type agent will be severed, and a star with i^* as the center and all other high-value or medium-value agents as the periphery would be formed. After step 3, there will be a link between every pair of high-value agents. In step 4, no link will be formed since information has been complete after step 1. Hence, the network formed after step 3 is stable.

If $n_m + n_h = 0$: under complete information, it is clear that no link ever forms. Under incomplete information, during step 4 a link would be formed and then severed between every pair of low-value agents, after which information would be complete. Therefore, under both complete and incomplete information, the empty network after step 4 is stable.

We say that two networks are *identical* if they are both empty or have the same links, and *entirely different* if at least one of them is non-empty and they share no link in common.

To say that two formation processes can converge to identical networks (or entirely different networks), we mean that there exists a selection path for both formation processes such that identical (or entirely different) networks emerge and be stable.

The following proposition states the topological differences in terms of the resulting stable network topology, between a formation process under complete information Γ_C and one under incomplete information Γ_{IC} .

Proposition 10. For every κ , the following properties hold:

- 1. If $\mathbb{E}[f(x)] < c$, then Γ_C and Γ_{IC} converge to identical networks with probability 1 if $n_m + n_h \leq 1$, and they converge to entirely different networks with probability 1 if $n_m + n_h > 1$.
- 2. If $\mathbb{E}[f(x)] \ge c$, then Γ_C and Γ_{IC} converge to identical networks with positive probability for any values of other parameters, and:
 - a. If $n_h \ge 2$ or $n_l = 0$, then Γ_C and Γ_{IC} never converge to entirely different networks.
 - b. If $n_h < 2$ and $n_l > 0$, then Γ_C and Γ_{IC} converge to entirely different networks with positive probability if (1) $n_m + n_h$ is sufficiently large, or (2) $n_m + n_h \ge 2$, δ is sufficiently close to 1 and n_l is sufficiently large.

Proof. 1: We already know from Theorem 4 that if $\mathbb{E}[f(x)] < c$, the network stays empty under incomplete information for any κ and γ_{∞} . If $n_m + n_h \leq 1$, clearly the network stays empty under complete information for any κ and γ_{∞} , otherwise at least one pair of high-value or medium-value agents will be linked with probability 1.

2: The claim that Γ_C and Γ_{IC} converge to identical networks with positive probability is a direct result from Theorem 4.

If $n_h \ge 2$, in any stable network under complete and incomplete information, any pair of high-value agents must be linked. Thus Γ_C and Γ_{IC} never converge to entirely different networks. If $n_l = 0$, then the formation processes under complete and incomplete information would be the same, so again Γ_C and Γ_{IC} never converge to entirely different networks.

If $n_h < 2$ and $n_l > 0$, first consider the following selection path when the parameter values satisfy $n_m + n_h \ge \frac{c - \max_{i \text{ is low-value } f(k_i)}{\delta c} + 1$:

- 1. Fix an agent j^{*} ∈ arg max_{i is low-value} f(k_i). In the first n_m + n_h periods, select j^{*} and every medium-value or high-value agent.
- 2. In the following $n_l 1$ periods, select j^* and every other low-value agent twice consecutively.
- 3. In the remaining periods, let the selection path be the same as in the proof of Lemma 7.

Under complete information, as in the proof of Lemma 7, the formation process would converge to a network only consisting of links between medium-value or high-value agents. Under incomplete information, after step 1, a star with j^* as the center and all the medium-value or high-value agents as the periphery would be formed. In step 2, a link would be formed and then severed between j^* and every other low-value agent. After that, information becomes complete and no low-value agent except j^* would ever be linked. For every medium-value or high-value agent, since $n_m + n_h \ge \frac{c-\max_i \text{ is low-value } f(k_i)}{\delta c} + 1$, the benefit from the link with j^* is at least max_i is low-value $f(k_i) + \delta(n_m + n_h - 1)c \ge c$, which implies that the agent has incentive to maintain the link. In addition, as $n_h < 2$, no link would be formed between any pair of medium-value or high-value agents, and thus the network is stable. This last fact also shows that there are no common links between the networks converged to under complete and incomplete information, and thus Γ_C and Γ_{IC} converge to entirely different networks.

Secondly, consider the following selection path when $n_m + n_h \ge 2$, δ is sufficiently close to 1 and $n_l \ge n_m + n_h - 1$:

- In the first period, select a low-value agent and a medium-value or high-value agent; in the second period, select a second medium-value or high-value agent and the previous low-value agent; in the third period, select a second low-value agent and the previous medium-value or high-value agent; ...; in the 2(n_m + n_h − 1)th period, select the last medium-value or high-value agent and the previous low-value agent.
- 2. In the following $n_l (n_m + n_h 1)$ periods, select a medium-value or high-value agent and every remaining low-value agent.
- 3. In the remaining periods, let the selection path be the same as in the proof of Lemma 7.

Under complete information, as in the proof of Lemma 7, the formation process would converge to a network only consisting of links between medium-value or high-value agents. Under incomplete information, after step 1, a line network only consisting of links between a low-value agent and a medium-value or high-value agent is formed. After step 2, information becomes complete, and as δ is sufficiently close to 1, the network is stable (note that δ being sufficiently close to 1 is consistent with the condition $n_h < 2$). Therefore there are no common links between the networks converged to under complete and incomplete information, and thus Γ_C and Γ_{IC} converge to entirely different networks.

Case 2(b) in the above proposition is of particular interest, because apart from establishing the property that Γ_C and Γ_{IC} may converge to entirely different networks, it also provides insight on the particular types of network topologies that can result in such a difference. For instance, when $n_m + n_h$ is sufficiently large, a *star* network with a low-value agent can emerge and be stable under incomplete information, which immediately implies that there are no common links with any stable network under complete information.

The above implication points to various applications. One particular case of this scenario occuring is when the selection process exhibits "preferential attachment", i.e. agents with higher connectivity degree are more likely to be selected, as in several well-documented net-works such as that of movie actors and the world wide web (see Barabasi and Albert[BA99]). In this case, when a low-value agent gets "lucky" and obtains a high connectivity degree initially, it is more and more likely over time that agents with higher quality would link to this low-value agent, instead of linking between themselves.

On the other hand, when n_l is sufficiently large, a *line* network (i.e. a tree network with only one agent in each subset in the partition of *I*) under incomplete information, where low-value agents and medium-value or high-value agents are linked alternately, will ensure that Γ_C and Γ_{IC} converge to entirely different networks. Similar scenarios, for example tree networks with only a few but lengthy branches, are more likely to emerge in relatively sparse and sometimes anonymous communities where agents only have the opportunity to make a limited number of links with unknown others, for example technical and biological networks (see Fricke et al.[FFL13]). In other words, disassortativity – in this case, the tendency for agents to have small and similar connectivity degrees – is another source of significant differences between complete and incomplete information.

The above two types of phenomena both result from the interaction of incomplete information, characteristics of the selection process, and agents' myopia.

Figures 6 and 7 provide simulation results on expected (or average) difference, both in its absolute value and as a fraction of the total number of links, between networks under complete and incomplete information⁷. We define difference between two networks as the number of non-common links in the networks. More precisely, the difference between networks \mathbf{g}_1 and \mathbf{g}_2 is given by $|(\mathbf{g}_1 \setminus \mathbf{g}_2) \cup (\mathbf{g}_2 \setminus \mathbf{g}_1)|$, and the total number of links is given by $|\mathbf{g}_1 \cup \mathbf{g}_2|$. The results indicate that there is a significant difference between networks that emerge and are stable under complete and incomplete information; this difference tends to be larger when the total number of agents *N* increases, and when there are more low-value agents in the group.

Figure 8 provides simulation results on the number of stable networks⁸ under complete and incomplete information. By comparing the difference between the curves in each graph, we can see clear monotonicity: the more likely an agent's type is low and the less likely an agent's type is high, the more likely the formation process converges to different networks under complete and incomplete information. This is because low type agents do not connect under complete information and they may under incomplete information, and high type agents link to each other under both scenarios.

2.4.5 Social Welfare

An alternative and very important way of comparing complete and incomplete information is to evaluate the upper bound in social welfare in the two cases. Formally, let $W_C(\kappa)$ and $W_{IC}(\kappa)$ be the maximum social welfare that can be achieved by a network in $G_C^*(\kappa)$ and $G_{IC}^*(\kappa)$ respectively. By Theorem 4, it is clear that under incomplete information, if $\mathbb{E}[f(x)] \ge c$, then more networks are possible and hence, the welfare upper bound under incomplete information

⁷In the simulation, we assume that the payoffs from a high-value, medium-value and low-value agent are 15, 10 and 4 respectively. We assume that c = 5 and $\delta = 0.6$. The probabilities for an agent to be of high-value, medium-value and low-value are (1/3, 1/3, 1/3) for uniform distribution, (4/7, 2/7, 1/7) for high-type environment, (1/7, 2/7, 4/7) for low-type environment and (1/7, 4/7, 2/7) for medium-type environment. In each simulation, we let the formation process run for 2500 periods. We use 100 simulations to find the average difference.

⁸We use the network that forms at the end of the simulation as a proxy for a stable network. In counting the number of networks we keep the agents anonymous, i.e. they only differ by their types, so that the result reflects the number of distinct network topologies rather than the number of permutations.



Figure 2.6: Simulations: Expected Difference

is weakly higher than that under complete information; the following results provide sharper information.

Lemma 8. Under both complete and incomplete information, if some stable network g_1 is a proper superset of some other stable network g_2 , then every agent's payoff is weakly higher in g_1 than in g_2 . As a result, g_1 yields a weakly higher social welfare than g_2 .

Proof. Note that the social welfare is the sum of each agent's payoff. For agents having the same links in \mathbf{g}_1 and \mathbf{g}_2 , it is clear that they are weakly better off in \mathbf{g}_1 .

Now consider an agent *i* whose links in \mathbf{g}_1 is a proper superset of those in \mathbf{g}_2 . Let ij_1, \dots, ij_m denote *i*'s links in \mathbf{g}_1 but not in \mathbf{g}_2 . Suppose that $u_i(k_{-i}, \mathbf{g}_1) < u_i(k_{-i}, \mathbf{g}_2)$. It implies that there must be a permutation of ij_1, \dots, ij_m , denoted ij'_1, \dots, ij'_m , such that for some $m' \in \{1, \dots, m\}$, $u_i(k_{-i}, \mathbf{g}_1 - ij'_1 - \dots - ij'_{m'}) > u_i(k_{-i}, \mathbf{g}_1 - ij'_1 - \dots - ij'_{m'-1})$.

Denote L_i as an arbitrary proper subset of *i*'s links (including the empty set), and observe that for any **g** and any of *i*'s link ij, $u_i(k_{-i}, \mathbf{g} - ij) - u_i(k_{-i}, \mathbf{g}) \ge u_i(k_{-i}, \mathbf{g} \setminus L_i - ij) - u_i(k_{-i}, \mathbf{g} \setminus L_i)$



Figure 2.7: Simulations: Fraction of Expected Difference

L_i). Therefore, $u_i(k_{-i}, \mathbf{g}_1 - ij'_{m'}) - u_i(k_{-i}, \mathbf{g}_1) \ge u_i(k_{-i}, \mathbf{g}_1 - ij'_1 - \cdots - ij'_{m'}) - u_i(k_{-i}, \mathbf{g}_1 - ij'_1 - \cdots - ij'_{m'-1}) > 0$, which implies that in \mathbf{g}_1 , severing $ij'_{m'}$ would strictly increase *i*'s payoff. But this is a contradiction with the assumption of stability, and thus it must be the case that $u_i(k_{-i}, \mathbf{g}_1) \ge u_i(k_{-i}, \mathbf{g}_2)$. Therefore, we can conclude that \mathbf{g}_1 yields a weakly higher social welfare than \mathbf{g}_2 .

This lemma determines the social welfare relation between two stable networks when one contains the other. The following partial characterization can then be shown.

Proposition 11. For any κ , the following properties hold:

- 1. If $\mathbb{E}[f(x)] < c$, then $W_C(\kappa) = W_{IC}(\kappa) = 0$ if $n_m + n_h \le 1$, and $W_C(\kappa) > W_{IC}(\kappa) = 0$ otherwise.
- 2. If $\mathbb{E}[f(x)] \ge c$, then $W_C(\kappa) \le W_{IC}(\kappa)$, and:
 - a. If $n_l = 0$, then $W_C(\kappa) = W_{IC}(\kappa)$.
 - b. If $n_l > 0$ and $n_m + n_h = 1$, then $W_C(\kappa) < W_{IC}(\kappa)$ if there exists a stable wheel network among a subset of the agents.



Figure 2.8: Simulations: Number of Stable Networks

- c. If
$$n_l > 0$$
 and $n_m + n_h > 1$, then $W_C(\kappa) < W_{IC}(\kappa)$ if δ is sufficiently close to 1.

Proof. 1: We already know from Theorem 4 that if $\mathbb{E}[f(x)] < c$, the network stays empty under incomplete information, yielding $W_{IC}(\kappa) = 0$. Therefore, $W_C(\kappa) > W_{IC}(\kappa)$ if and only if there is some non-empty network in $G_C^*(\kappa)$, which is equivalent to the condition $n_m + n_h > 1$.

2: The claim that $W_C(\kappa) \leq W_{IC}(\kappa)$ is a direct result from Theorem 4.

If $n_l = 0$, $G_C^*(\kappa)$ and $G_{IC}^*(\kappa)$ are identical, and thus $W_C(\kappa) = W_{IC}(\kappa)$.

If $n_l > 0$ and $n_m + n_h = 1$, under complete information the network stays empty, yielding a social welfare of 0. By Lemma 8, we know that in the stable wheel network, every agent's payoff is at least 0. In addition, since $n_m + n_h = 1$ the assumption that such a network is stable implies that the total number of agents is at least 5, and that the medium-value or high-value agent must be non-singleton in this network. Thus, the two low-value agents who link with the medium-value or high-value agent must have a strictly positive payoff, which means that the social welfare is strictly positive. Finally, it is easy to see that this network cannot be formed under complete information. Thus $W_C(\kappa) < W_{IC}(\kappa)$.

If $n_l > 0$ and $n_m + n_h > 1$, consider the network $\mathbf{g} \in G^*_C(\kappa)$ which yields the highest social

welfare (this network must exist, since there are only finitely many networks in $G_C^*(\kappa)$). Let δ be sufficiently close to 1 such that **g** is minimal. Thus, there exist medium-value or high-value agents *i* and *j* (in fact, when δ is very close to 1, there is no high-value agent) such that *ij* is the only link *i* has in **g**. Note that since **g** is minimal, **g** is also stable for any larger δ .

Consider a selection path under complete information in which **g** emerges, such that no link is formed or severed after *ij* is formed, and no low-value agent is selected before *ij* is formed. Under incomplete information, consider the following variation of this selection path: before the period in which *ij* is formed, insert two periods: in the first period, select some low-value agent *i'* and *i*; in the second period, select *i'* and *j*. Since $\mathbb{E}[f(x)] \ge c$, we know that *ii'* and *i' j* would both be formed. As δ gets sufficiently close to 1, the payoffs of the medium-value or high-value agents would strictly increase due to the connection to *i'*. Therefore $W_C(\kappa) < W_{IC}(\kappa)$.

Just as in Proposition 10, this result points to particular network topologies (2(b) and 2(c)) that result in a clear welfare comparison. In the presence of low-value agents, when there is only one medium-value or high-value agent, the empty network is the only stable network that can emerge under complete information; under incomplete information, for any other network to be stable, the network must exhibit a "wheel-like" feature, i.e. apart from the medium-value or high-value agent, every agent must have at least two links. Once such a network is stable, it can be immediately shown that it yields a strictly positive social welfare. When there are more than one medium-value or high-value agents, as δ gets sufficiently close to 1 the network that yields the highest social welfare must be minimal; then under incomplete information, there always exists a way to "insert" a low-value agent between two medium-value or high-value agents (since δ is close to 1) but generates a strictly positive payoff for the low-value agent. Therefore social welfare is strictly improved.

Figure 9 generated from simulation shows the expected (or average) social welfare achieved under complete and incomplete information, in various environments. The interpretation of this figure is two-fold. On one hand, the numerical value of the difference in social welfare is mostly an artifact of the particular simulations, since it is rather sensitive to how much the values of different types of agents differ from the cost, as well as how large the discount factor δ is.

On the other hand, the significant "difference in difference" in terms of social welfare across various environments is a general and robust phenomenon. In an environment with more low type agents, the difference in social welfare between complete and incomplete information is larger than than in an environment with less low type agents.



Figure 2.9: Simulations: Expected Social Welfare

2.5 Bayesian Learning

The results we have derived so far are based on the simple updating rule, which assumes that every agent's posterior belief on another agent's type is binary: either it is the degenerate belief on the true type, or the prior. Note again that such an updating rule implicitly assumes that agents can only observe their own formation history. If the agents adopt a different updating rule, which reflects either more or less available information, the formation process can exhibit a much different pattern. We discuss one such alternative in detail, which we call *Bayesian learning by formation history*.

We assume that agents can observe the entire formation history, i.e. the pair of agents

selected and the resulting network structure each period, in addition to knowing the types of agents connected to themselves. However, if a link is severed, they do not observe the identity of the agent that severs the link. They then apply Bayesian updating in forming posterior beliefs. In the literature, for instance Jackson and Wolinsky[JW96], Bala and Goyal[BG00] and Watts[Wat01], since there is no uncertainty on payoffs and agents are myopic, it does not matter whether the entire formation history is observed. However, with incomplete information, what agents can observe and how they update information accordingly are crucial for shaping the ultimate network topology. The following result highlights the key difference between the simple learning rule and this alternative.

Theorem 6. Under Bayesian learning by formation history, assume that in the prior type distribution, the probability of an agent being low-value is positive. Then when there are sufficiently many low-value agents, there is always a positive probability that the formation process converges to an empty network, and that information remains incomplete forever.

Proof. We prove the result by construction. Consider any agent *i*, and consider the selection path γ_i that *i* is selected twice consecutively with a low-value agent, then selected twice consecutively with another low-value agent, and so on. We know that initially a link forms and then breaks each time *i* is selected with a different low-value agent. Let p'_m be the posterior probability that *i* is a medium-value or high-value agent after the link between *i* and the *m*th low-value agent breaks. We know that by Bayesian updating, $p'_{m+1} = \frac{p'_m(1-p'_0)}{1-p'_mp'_0}$ with the initial condition that p'_0 is equal to the prior probability that an agent is medium-value or high-value. By assumption, we know that $p'_0 < 1$. Therefore $\frac{p'_{m+1}}{p'_m} = \frac{1-p'_0}{1-p'_mp'_0} \le \frac{1-p'_0}{1-p'_0^2} < 1$, and thus there exists a sufficiently large n_l such that, following the above described selection path γ_l , for any agent j' who has not been selected with *i* before, $\mathbb{E}[f(k_i)|B_{j'}(\gamma_l)] < c$.

After γ_i , when *i* is selected with any other agent *j*, no link can ever form: if *j* has not been selected with *i* before, *j* is not willing to form a link with *i* since *j* believes *i* to be low-value with a high probability; if *j* has been selected with *i* before, *i* is not willing to form a link with *j* since *i* already knows that *j* is low-value. This process can be replicated for every agent *i*; as a result, no link will be formed between any two agents, and the formation process converges to an empty network. Finally, since not every pair of agents has been connected before the formation process converges, information remains incomplete forever.

A major implication here is that Bayesian learning by formation history makes it possible for the posterior probability of an agent being of high type to fall close to 0. As a result, even if making a link with some agent *i* is incentivized with the simple learning rule, it may no longer be the case under Bayesian learning by formation history, given some particular selection path that would drag posterior beliefs towards *i* being low-value. The following example illustrates this difference.

Example 3. Assume the following: N = 5, $X = \{a, b\}$, $k_i = b \ \forall i = 1, ..., 5$, and the other parameters are such that f(b) < c, $\mathbb{E}[f(x)] \ge c$, $(1 + \delta - \delta^2 - \delta^3)f(b) \ge c$. Let p = h(a), and assume that $\frac{p(1-p)}{1-p^2}f(a) + \frac{p(1-p)+(1-p)^2}{1-p^2}f(b) < c$. Consider the following selection path in period 1-9: ((1,3), (1,3), (2,4), (2,4), (1,2), (3,4), (4,5), (2,3), (1,5)).

Under the simple learning rule, the formation process is shown in Figure 10(A). The network formed in period 9 is stable; under Bayesian learning by formation history, the formation process is shown in Figure 10(B). The network formed in period 4 is stable.



Figure 2.10: Connected under Simple Updating Rule vs. Empty under Bayesian Learning by Formation History

Here with the simple learning rule, agents hold the prior belief each time they are selected with another agent with an unknown type, and thus the given selection path induces a connected network at last. Yet with Bayesian learning by formation history, each agent updates from their observation to conclude that others are low-value with a sufficiently large probability, and thus are unwilling to make any link.

One implication of the above theorem and example is that more learning can sometimes be "bad", i.e. it may lead to inefficient outcomes. Despite the specific differences brought about by an alternative updating rule, our general results still hold under a range of parameters. It can be shown that if any typical network as depicted in Theorem 5 can emerge and be stable under complete information, then it can under incomplete information and Bayesian learning by formation history as well. The topological differences characterized by Proposition 10 also hold except 10(a) - now it becomes possible that Γ_C and Γ_{IC} converge to entirely different networks even when $n_h \ge 2$. Finally, the welfare comparison shown in Proposition 11 stays the same under this updating rule.

2.6 Conclusion and Future Research

In this paper we analyzed the network formation process under agent heterogeneity and incomplete information. Our results are in stark contrast with the existing literature: instead of restricting the equilibrium network topologies to fall into one or two specific categories, our model generates a great variety of network types. Besides *what* networks can emerge as a result of convergence, we argue that it is also important to understand *how* a network gets formed, since we want to know, for instance, *why* some agents become central and others do not. While link formation and belief formation are usually treated as two independent processes to be studied separately, we combine them in our model and show that belief formation is in fact a key factor that could facilitate or deter link formation. Even if incomplete information vanishes in the long run, its impact on shaping the network topology is persistent.

Several future research topics can be built up on the basis of our model. One of these challenges is to pin down the structure of an efficient network and implement it in a game-theoretic setting. The usual definition on efficiency in networks adopted in the literature is *strong efficiency*, i.e. a network is strongly efficient if it maximizes the sum of agents' payoffs. In general, we know that a strongly efficient network must exist (though not necessarily be unique) because the set of possible network structures is finite. However, since payoffs are heterogeneous across agents according to the type vector, the exact topology of an efficient

network becomes difficult to characterize; moreover, the efficient network may not be unique because in an agent-heterogeneous environment there could be multiple ways of generating the same level of social welfare.

Most importantly, we have assumed throughout, as does most of the literature, that agents are myopic rather than forward-looking. If it is assumed otherwise that agents are foresighted and are concerned about both their current and future welfare, then the aim of analysis essentially becomes solving an agent's dynamic optimization problem in the presence of other similarly foresighted agents. One can then easily anticipate a very different evolution pattern of network topologies as well as very different stable network topologies in the limit, for now link formation does not only serve as an action of maximizing the current expected payoff, but also as a way of acquiring information for potential future benefit.

CHAPTER 3

Dynamic Network Formation with Foresighted Agents

3.1 Introduction

In social and economic contexts, a *network* represents the structure of interactions among individual agents. Such structure may have a crucial impact on the ultimate outcome of interaction: on one hand, *whom* an agent is connected to in a network has an effect to be reckoned with in the agent's payoff; on the other hand, *how* the connection is formulated often makes a significant difference. For instance, on Facebook, LinkedIn and other social/professional networking media, a user typically cares about both the individual characteristics of her contact (background, common interests, geographic location, etc.) and her own position in the social circle (intimacy with her contacts, degree of received attention, etc.). Similar examples include trading networks (Tesfatsion[Tes98]), channels of information sharing (Chamley and Gale[CG94]) and buyer-seller networks (Kranton and Minehart[KM01]), where payoffs may be represented in alternative forms but users' general focus remains the same.

In the widely adopted network formalism, the space of interactions is described as a graph, with the set of nodes representing the set of agents, and an arc or link between two nodes indicating bilateral interaction between the corresponding agents. The *formation* of such a network is often understood as the result of a collection of decision rules, describing each agent's decisions on whether to establish or sever links with one another. Starting from Jackson and Wolinsky[JW96] and Bala and Goyal[BG00], a large and growing theoretical literature in economics have studied the network formation problem among self-interested and strategic agents. The research emphasis is placed on two issues. The first issue is the characterization of the network topologies that will form and persist in this game-theoretic environment, while the second issue is the determination of socially efficient network topologies and whether such topologies can be supported in equilibrium.

The typical approach in modeling network formation has been a static framework (e.g. Jackson and Wolinsky[JW96]) where agents take actions only once simultaneously, or a dynamic one (e.g. Watts[Wat01]) where agents meet others randomly over time and choose their actions whenever they are allowed to do so. Even though the dynamic model is already highly consistent with the above mentioned applications in terms of the time line of interaction and agents' feasible behavior, there remain two limitations on its application on realistic situations. On one hand, it is often assumed that agents are *myopic*: their actions at any point in time are solely guided by current payoffs, with no consideration of future consequences. This assumption restricts an agent's ability of accounting for the future, and as a result prohibits even a limited way of cooperation among agents, while both features in behavior are expected to emerge in real social or economic interaction. On the other hand, agents are often taken as *homogeneous* in the sense that an agent's payoff is only affected by the network topology and her position in the network, but not individual characteristics of her peers. Such simplification sets aside the important and realistic possibility that different types of agents generate different valuation structures, even in the same network topology.

One of the key conclusions from this branch of literature states that efficiency cannot be attained generically in equilibrium, in the sense that the set of achievable network topologies and that of efficient network topologies often differ. There have been a limited volume of works on network formation with foresighted agents (see for example Dutta et al.[DGR05]), but in general they share the negative implication, i.e. there are various valuation structures in which no equilibrium can sustain efficient network topologies.

However, in great contrast to the above theoretical findings, data collected from existing real networks seems to suggest otherwise. In our subsequent analysis, we provide a full characterization of the strongly efficient network (the network that yields the largest sum of payoffs) in a standard connections model with *heterogeneous* agents, in which we find that such a network generally exhibit a "core-periphery" pattern: it usually consists of a group of fully connected agents (the core), a group of agents that only link to the core (the periphery), and a group of singleton (disconnected) agents. Prominent features of such network topologies include (1) a large ratio of number of links to number of agents, (2) a large ratio of number of "triangles" (subsets of 2 or 3 links among three connected agents) to number of agents, and (3) a short diameter (the number of links in the longest of shortest paths). We imposed the above criteria

	Facebook	AHEP
No. of users	4039	12008
No. of links	88234	118521
Average clustering coefficient	0.6055	0.6115
Number of triangles	1612010	3358499
Fraction of closed triangles	0.2647	0.3923
Diameter	8	13
90% effective diameter	4.7	5.3

Table 3.1: Summary statistics of sample networks

to data analysis on sample social networks from Facebook and collaboration networks of Arxiv High Energy Physics (AHEP) (whose strategic environment can be analyzed by the connections model)¹ and found that the actual networks recorded in data bear a striking resemblance to the strongly efficient ones in theoretical analysis. Table 1 below provides summary statistics on the networks and Figure 1 and 2 illustrate a sample core-periphery network and the actual network topology in AHEP. As this observation suggests, in reality people may have been considerably successful in achieving a social welfare close to the efficiency boundary, which contradicts the conclusion from existing literature.



Figure 3.1: Sample core-periphery network

This paper aims at providing the first comprehensive analysis on dynamic network formation in absence of the previously mentioned modeling limitations, and reconciling the inconsistency between data and theory. In the model, we adopt a standard dynamic network formation game: agents meet one another randomly on a discrete time line, and they choose individual

¹Source of datasets: SNAP Datasets: Stanford Large Network Dataset Collection [LK14].



Figure 3.2: AHEP network

actions on whether or not to agree to form a link with others. Link formation requires bilateral consent while link severance is unilateral. In contrast to most existing literature, we make two fundamental assumptions on the agents. The first is that agents are *foresighted* in that in each period they choose actions to maximize their discounted sum of payoffs. The second is that agents are *heterogeneous* so that their own characteristics, or types, constitute a potential determinant on payoffs of others as well as of their own. These assumptions bring about the possibility of reasonable cooperation, and allow the analysis to be undertaken in a very general valuation structure. We seek to answer the above mentioned central questions, i.e. what network topologies may arise and persist in equilibrium, what network topologies are efficient and whether they can be sustained in equilibrium.

The dynamic game we analyze is essentially a *stochastic game* where states are determined by the random selection process as well as agents' actions. In the case where types are private knowledge, it is also a *Bayesian game* with a dynamic process of learning, as agents will update on the game fundamentals based on any observation upon the past formation history. As can be expected, our focus is the equilibrium behavior of patient agents in this context; however, instead of characterizing the set of achievable *payoffs* as in the repeated-game literature, we aim
at characterizing the set of achievable *networks*, i.e. the underlying structure regulating agents' strategic interactions, which is in turn shaped by agents' behavior. The type of equilibrium we study is one in which the formation process converges to a particular network (that is, the network is formed and remains for ever). In other words, the payoff-relevant state in the game would be random in an early stage of the dynamics, but will finally rest on a set of states that no agent has the incentive to break away from.

Our main findings are presented in three theorems. Theorem 7 is a Folk Theorem on sustainable networks under complete information on agents' types, which states that a network topology can form and persist forever in equilibrium for patient agents if and only if it yields a positive one-period payoff to every agent. This theorem characterizes the largest possible set of network topologies that can be sustained by any equilibrium. In particular, we construct a Markov strategy profile with public signals that constitutes an equilibrium for patient agents, such that the formation process always converges to a designated network with positive payoff for everyone, regardless of the previous formation history. This "self-fulfilling prophecy" is robust against small independent trembles in action as well as typical group deviation behavior.

Theorem 8 is again a Folk Theorem, but established under incomplete information where agents start with prior beliefs and perform Bayesian update upon observing equilibrium behavior. We establish a condition on the valuation structure, under which there exists an equilibrium where patient agents are incentivized to reveal their types by making connections. As above, we construct a Markov equilibrium strategy profile with public signals. In such an equilibrium, information becomes complete ultimately and again any network yielding a positive payoff for every agent can be sustained. This result not only generalizes the above analysis to include incomplete information, but also points to a tractable equilibrium strategy profile that covers the range of sustainable network topologies but does not involve a complicated updating process.

The above two theorems induce a rather positive result on sustaining efficiency: whenever an efficient network provides every agent with a positive payoff, it *can* be sustained in equilibrium for patient agents whether ever information is complete. Nevertheless, we also find that in typical cases agents need to be more patient under incomplete information than under complete information to achieve efficiency.

A widely studied specification of our model, the connections model, generates tighter results on sustaining efficient networks. In Theorem 9, we provide a full characterization of the strongly efficient network, which extends the result of Jackson and Wolinsky[JW96] to heterogeneous agents for the first time. The topological characteristics of the strongly efficient network have been introduced before. Moreover, we provide sufficient and necessary conditions on agents' type distribution for the strongly efficient network to be sustained in equilibrium, with the same robustness properties established after Theorem 7. Combined with Theorems 7 and 8, this result serves as micro-foundation of numerous real-life networks which fit the connections model, and thus reconciles the discrepancy between theory and data.

The remainder of the paper is organized as follows: Section 2 provides a review of the related literature. Section 3 introduces the model. Section 4 and 5 establish the Folk Theorem of networks under complete and incomplete information. Section 6 provides a full characterization of the strongly efficient networks in the connections model. Section 7 concludes.

3.2 Literature Review

Existing theoretical literature on network formation can be generally categorized by the criterion of whether link formation is bilateral or unilateral, i.e. whether the creation of a link requires bilateral consent of both agents involved, or can be done unilaterally by either agent. In the first category, one representative theoretical framework is the *connections model* by Jackson and Wolinsky[JW96]. In a static strategic environment with homogeneous agents, they posited that the strongly efficient network is either empty, a star or a clique. In addition, they predicted that in generic cases the strongly efficient network topology cannot be sustained via agents' self-interested behavior. This paper belongs to the first category as the applications that we are concerned with, such as Facebook and Google+, can be best described by this approach. As for the second category, Bala and Goyal[BG00] provided a comprehensive analysis which yields quite different predictions on the efficient network topologies and the equilibrium network topologies. We will not elaborate on their results in this paper since we focus on the scenario where links can only be formed after bilateral agreement.

A more recent development in static network formation is to introduce heterogeneity among agents. Heterogeneity takes different forms in different branches in the literature, but it can be summarized into two categories. The first type of heterogeneity is *exogenous*, such as different failure probabilities for different links (Haller and Sarangi[HS03]) and agent-specific

values and costs (Galeotti[Gal06], Galeotti et al.[GGK06]). The second type is *endogenous* heterogeneity, often represented as the amount of valuable resource produced by the agents themselves, as in Galeotti and Goyal[GG10]. In our paper, we adopt the first approach since the heterogeneity we focus on is an agent's endowed individual characteristic. We assume a more general framework than most existing literature by assigning each agent a *type* that may affect others' payoffs as well as her own. We conduct our analysis in both cases where types are common knowledge (complete information) and where they are private knowledge (incomplete information).

Another direction of research built on the early theoretical frameworks is to describe network formation as a interaction process over time, instead of a one-shot, static action profile. Again, methodologies vary across researchers. Johnson and Gilles[JG00] and Deroian[Der03] analyze variations of the connections model in a finite sequential game, and Konig et al.[KTZ14] models network formation as a continuous-time Markov chain with random arrival of link creation opportunities. The convention we follow in this paper is the network formation game introduced by Watts[Wat01], in which pairs of agents are selected randomly on a discrete and infinite time line to update the potential link between them. A link is formed or maintained with bilateral consent, and not formed or severed if either agent chooses to do so. This model is consistent with the above mentioned applications, and its framework as well as variations have been adopted widely in theoretical studies to analyze strategic interactions in social and economic networks (Jackson and Watts[JW02], Skyrms and Pemantle[SP00], Song and van der Schaar[SS15]).

In works adopting this dynamic model, agent myopia is a common assumption, which means that agents only take into account their current payoffs at every point of decision; another prevalent assumption is agent homogeneity, with the exception of Song and van der Schaar[SS15], who analyze a variation of the connections model where an agent's payoff is affected by others' types but not his own. In terms of sustaining efficient networks, predictions made are similar to Jackson and Wolinsky[JW96]: the strongly efficient network cannot be sustained at all times in the formation process. The formation and persistence of the strongly efficient network is random – it depends on the realized selection of agent pairs in the early state – and as a result the probability of sustaining the efficient network decreases as the number of agents increases. There have been a few attempts to introduce foresightedness into dynamic network formation, but overall this topic remains understudied. This paper is related to Dutta et al.[DGR05], who also adopted the model of Watts[Wat01] and assumed that agents take future payoffs into account. Their main result once again points to the impossibility of sustaining efficient networks in equilibrium by constructing a representative example. The major difference made in this paper is that we allow for a *public signal* in the definition of a state in a Markov strategy profile – in this way, the agents may have only limited knowledge about the past formation history but will still be able to cooperate in achieving efficiency. Our positive result on sustaining efficient networks holds for a more general valuation structure than most existing frameworks. Alternative models on foresightedness in network formation include Page Jr. et al.[PWK03] and Herings et al.[HMV09], whose solution concept is pairwise stable network instead of equilibrium. Yet again the efficiency-related results point to cases where the strongly efficient network cannot be sustained even if it provides each player a positive payoff.

Our methodology of analyzing the dynamic game and constructing particular equilibria is related to Dutta[Dut95] and Forges[For12], but we step aside from providing a complete review on the various versions of Folk Theorem because this paper focuses on the network formation game. Nevertheless, we share the general notion of characterizing patient agents' behavior (though in the sense of network topologies formed instead of payoffs attained) and our construction of equilibrium strategy profiles benefit from the existence of a "uniform punishment strategy" as mentioned in Forges[For12].

There are numerous empirical studies on properties of real-world networks. Major identified properties include short diameter (Albert and Barabasi[BA99]), high clustering (Watts and Strogatz[WS98]), positive assortativity (Newman[New02][New03]), inverse relation between clustering coefficient and degree (Goyal et al.[GLM06]), etc. Experimental studies such as Falk and Kosfeld[FK03], Corbae and Duffy[CD08], Goeree et al.[GRU09] and Rong and Houser[RH12] have indicated that typical equilibrium network topologies predicted by the existing theoretical analysis, especially the star network, only emerge in a small fraction of experimental outcomes. Last but not least, Mele[Mel10] and Leung[Leu13] show that networks formed in large social communities, where agents are heterogeneous and withhold certain private information, often exhibit patterns not predicted by existing theoretical literature. In the subsequent analysis, we will discuss most of the above properties and illustrate how they can be accounted for in our framework.

3.3 Model

3.3.1 Network Topology

Consider a group of agents $I = \{1, 2, ..., N\}$. A *network* is denoted by $\mathbf{g} \subset \{ij : i, j \in I, i \neq j\}$. *ij* is called a *link* between agents *i* and *j*. We assume throughout that links are *undirected*, in the sense that we do not specify whether link *ij* points from *i* to *j* or vice versa. A network **g** is *empty* if $\mathbf{g} = \emptyset$. Let *G* denote the set of all possible networks. Given a subset of agents *I'*, let $G_{I'}$ denote a network that is formed within *I'*.

We say that agents *i* and *j* are *connected*, denoted $i \stackrel{\mathbf{g}}{\leftrightarrow} j$, if there exist $j_1, j_2, ..., j_n$ for some *n* such that $ij_1, j_1 j_2, ..., j_n j \in \mathbf{g}$. Let d_{ij} denote the *distance*, or the smallest number of links between *i* and *j*. If *i* and *j* are not connected, define $d_{ij} := \infty$. An agent *i* in a network is a *singleton* if $ij \notin \mathbf{g}$ for any $j \neq i$.

Let $N(\mathbf{g}) = \{i | \exists j \text{ s.t. } ij \in \mathbf{g}\}$, and let $N_i(\mathbf{g}) = \{j : ij \in \mathbf{g}\}$. A *component* of network \mathbf{g} is a maximally connected sub-network, i.e. a set $C \subset \mathbf{g}$ such that for all $i \in N(C)$ and $j \in N(C)$, $i \neq j$, we have $i \stackrel{C}{\leftrightarrow} j$, and for any $i \in N(C)$ and $j \in N(\mathbf{g})$, $ij \in \mathbf{g}$ implies that $ij \in C$. Let C_i denote the component that contains link ij for some $j \neq i$. Unless otherwise specified, in the remaining parts of the paper we use the word "component" to refer to any *non-empty* component.

A network **g** is said to be *empty* if $\mathbf{g} = \emptyset$, and *connected* if **g** has only one component which is itself. **g** is *minimal* if for any component $C \subset \mathbf{g}$ and any link $ij \in C$, the absence of ij would disconnect at least one pair of formerly connected agents. **g** is *minimally connected* if it is minimal and connected.

3.3.2 Dynamic Network Formation Game

We adopt the framework by Watts[Wat01] to formulate the network formation game. Time is discrete and the horizon is infinite: t = 1, 2, ... The initial network among them is denoted g(0). The game is played as follows:

- 1. In each period, a pair of agents (*i*, *j*) is randomly selected with equal probabilities to update the link between them.
- 2. The two selected agents (each knowing the identity of the other) then play a simultaneous move game, where each can choose to sever the link between them if there is one, and if there is not, whether to agree to form a link with the other agent. An existing link can be severed unilaterally, whereas the formation of a link requires mutual consent.
- 3. In addition, in each period every agent (whether or not she is selected in the current period) can choose to sever any of her existing link.

Let $\mathscr{A} = \{0, 1\}$ denote the set of actions towards an agent and let $a_{ij} = 1$ denote the action that *i* agrees to form a link with *j* (if there is no existing link) or not to sever the link (if there is an existing one), and $a_{ij} = 0$ otherwise. A link is formed or maintained after bilateral consent (i.e. $a_{ij} = a_{ji} = 1$).

Let $\phi(t)$ be the pair of agents selected in period *t* and let $\sigma(t) := {\phi(\tau), \mathbf{g}(\tau)}_{\tau=1}^{t}$ denote a *formation history* or a *formation path*² up to time *t*, with the initial condition that $\sigma(0) = (\emptyset, \emptyset)$. Let $\Sigma = {\sigma(t) : t \in \mathbb{N}}$ denote the set of all possible formation histories.

Agents may not observe the entire formation history, but in each period every agent *i* knows the set $N_i(\mathbf{g})$, i.e. the set of agents she links to. In addition, in each period the agents observe a *public signal* which is generated by a signal device $y : \Sigma \to Y$, where *Y* is the set of signal realization. We assume that *y* is common knowledge but the agents cannot memorize the realization of *y* prior to the current period. The signal device determines how precise agents know about the formation history. For instance, $Y = \{0\}$ and $s(\sigma(t)) = 0$ refer to no knowledge of the formation history whatsoever, while $Y = \Sigma$ and $y(\sigma(t)) = \sigma(t)$ refer to full knowledge. The precision of knowledge about the formation history from any signal device varies between these two extreme cases. In the subsequent analysis, we will specify the signal device used in each equilibrium.

In the various applications of this model, especially social networks, the signal device can be interpreted as a news media, e.g. a newspaper, a television program or a website. It will not record everything in the past for its audience, but it broadcast important events that attract

²For the sake of convenience, whenever we mention a formation history or a formation path, we refer to one that occurs with positive probability, unless otherwise specified.

public attention, for instance irregular or inappropriate activities by certain individuals. In the subsequence, we will demonstrate how such incomplete knowledge about the formation history suffice for an equilibrium that sustains an efficient network.

3.3.3 Payoff Structure

Each agent has a type, denoted by θ_i for agent *i*. Let Θ denote the set of possible types. Let $\bar{\theta} = (\theta_1, \theta_2, \cdots)$ denote the type vector for the whole group of agents. Given a subset of agents I', let $\bar{\theta}_{I'}$ denote the associated type vector. The probability distribution of types on Θ is i.i.d. and is denoted by $H(\theta)$.

The one-period payoff of agent *i* depends on the network structure and the type vector. Specifically, this payoff is a function $u_i : \Theta^N \times G \to \mathbb{R}$. We normalize the payoff for *i* to zero in a network where *i* is a singleton: $u_i(\bar{\theta}, \mathbf{g}) = 0$ for all \mathbf{g} in which *i* is a singleton, regardless of $\bar{\theta}$. Also, we assume that each agent's payoff satisfy *component independence*: $u_i(\bar{\theta}, \mathbf{g}) = u_i(\bar{\theta}, \mathbf{g}')$ if \mathbf{g}' is a component of \mathbf{g} .

For each agent, her payoff is realized in *every period*, though payoffs in different periods may well be different according to the network topology. A payoff that realizes t periods from now is discounted by γ^t , where $\gamma \in (0,1)$ is the *time discount factor*. Hence, if the vector of networks that form over time is $\bar{g} = \{\mathbf{g}(\tau)\}_{\tau=1}^{\infty}$, agent *i*'s total (discounted) payoff evaluated at period t is

$$U_i(\bar{\boldsymbol{\theta}}, \bar{\mathbf{g}}, t) = \sum_{\tau=0}^{\infty} \gamma^{\tau} u_i(\bar{\boldsymbol{\theta}}, \mathbf{g}(t+\tau)).$$

In our analysis, we will be discussing the possibility of sustaining efficiency under different payoff structures. Following the convention in the literature, our benchmark for efficiency will be the *strongly efficient network*, i.e. the network that yields the largest sum of one-period payoffs. We provide a formal definition below.

Definition 8. Given $\bar{\theta}$, a network g is strongly efficient if $\sum_{i=1}^{N} u_i(\bar{\theta}, g) \ge \sum_{i=1}^{N} u_i(\bar{\theta}, g')$ for any g'.

Since the number of possible network topologies is finite, the strongly efficient network always exists.

Another type of network we identify is *stable network*. In a later section we will demonstrate that such a network entails important additional properties of the formation process.

Definition 9 (Stable Network). A network g is a stable network if there exists no subgroup of agents $I' \subset I$ and network g' among I' such that

$$u_i(\bar{\boldsymbol{\theta}},\boldsymbol{g}') \geq u_i(\bar{\boldsymbol{\theta}},\boldsymbol{g}),$$

for every agent $i \in I'$, and the inequality is strict for some agent $i \in I'$. If g is not stable, we say that g' blocks g and call I' a blocking group.

We call such a network stable because they discourage any subgroup of agents to break away from it and form a network on their own. Note that this terminology is different from pairwise stability defined in Jackson and Wolinsky[JW96]. Pairwise stability of a network means that between any two agents, forming a new link cannot benefit both and severing an existing link must hurt at least one, *holding other links in the network constant*. It is a widely used solution concept for the static analysis of network formation. Our notion of stability appeals more to the dynamic and foresighted case, in the sense that once a subgroup of agents decide to depart from the current network, at least in the long run they can do no worse than severing every link with the other agents and form a network within themselves. Again this results from the bilateral consent for link formation or maintenance, and the assumption of component independence on the payoff structure.

3.3.4 Example: Connections Model

We use the connections model in Jackson and Wolinsky[JW96] to illustrate how each factor in a network – the network topology itself, the agents' positions and the type vector – affects an agent's payoff. The connections model is widely applied in the theoretical literature on network formation as well as empirical studies.

The payoff structure in this model is described as follows. If agent *i* is connected with agent *j* in **g** (denote this connection as $i \stackrel{\mathbf{g}}{\Leftrightarrow} j$), agent *i* gets payoff $f(\theta_j)$ and agent *j* gets payoff $f(\theta_i)$ from this connection, where *f* is a mapping from Θ to \mathbb{R}^{++} . In other words, payoffs through connection are *bilateral*. In addition, this payoff is discounted by $\delta^{d_{ij}-1}$, where $\delta \in (0,1)$ is the *spatial discount factor*, and d_{ij} is the *distance* between *i* and *j* measured in the number of

links. Meanwhile, agent *i* pays a cost of c > 0 per period for every link that *i* has. Hence, in a single period with network **g**, agent *i*'s current payoff is

$$u_i(ar{m{ heta}}, \mathbf{g}) = \sum_{j: i \stackrel{\mathbf{g}}{\Leftrightarrow} j} \delta^{d_{ij}-1} f(m{ heta}_j) - \sum_{j: ij \in \mathbf{g}} c.$$

It is easy to see that the above payoff structure satisfies the assumptions we made in the previous section. In the original and widely adopted version of the model, agents are assumed to be homogeneous, i.e. each agent yields the same payoff (before spatial discount) to others via connection. In the above formulation, we allow the existence of agent heterogeneity in the sense that an agent's payoff obtained from a connection depends on the type of the other party. Moreover, the payoff structure exhibits non-local externalities: though an agent gets a positive payoff from each agent she connects to, she only pays a cost for each link she maintains. Finally, the network topology as well as an agent's position are also determinants of her payoff. The further away two connected agents are apart from each other, the less payoffs they obtain from the connection. In various applications, this spatial discount can be regarded as the decay of valuable resource or information due to increased noise or risk. In a later section, we will discuss the connections model in more details and present important related results.

3.4 Folk Theorem with Complete Information

In this section, we characterize the set of networks that can persist in equilibrium when agents are patient, assuming that the type vector is commonly known. We start by defining a strategy in this environment and the concept of an equilibrium. In particular, we are interested in the type of equilibrium in which the network formation process converges, i.e. over time the network rests on a specific topology which then persists for ever.

3.4.1 Strategy, Equilibrium and Convergence

Throughout the paper, we will focus on *Markov strategies* and show that Markov strategies alone suffice for construction of an equilibrium with nice properties on sustaining efficiency. In this context, a (pure) *strategy* of agent *i* is a set of mappings that assigns an action in $\{0, 1\}$ to every other agent *j*. The constraint on these mappings is that if *i* and *j* are not linked and the pair (i, j) is not selected in the current period, then agent *i*'s action towards agent *j* has to

be 0. Formally, let $\omega_{ij} \in \Omega = \{0, 1\}$ denote the state whether the pair (i, j) is selected, and let $\zeta_{ij} \in Z = \{0, 1\}$ denote the state whether *i* and *j* are linked in the current period. We have the following definition:

Definition 10 (Strategy). A (pure) strategy of agent *i* is a set of mappings $s_i^y = \{s_{ij}^y\}_{j \neq i}$:

$$s_{ii}^{y}: Y \times \Omega \times Z \to \mathscr{A}$$

such that $s_{ij}^{y}(\cdot, 0, 0) \equiv 0$.

Let S denote the set of all possible strategies.

Associated with the device for public signals, the interpretation of a strategy in this game is rather straight forward. For any agent *i*, the *state* in a Markov strategy at a given time period is basically represented by her knowledge about the game at that period, which is the combination of two elements: her knowledge about every other agent *j*, which includes *j*'s type and whether *j* is linked to herself; and her knowledge about the formation history $\sigma(t)$. *i*'s information on the former is complete since she knows both the identity *j* and *j*'s type. The precision of her information on the latter, on the other hand, may vary according to the public signal generating function *y*. Note that strategies thus defined include strategies that assign actions only based on the network formed in the previous period (so that Y = G) as in some existing literature, for instance Dutta et al.[DGR05].

Now we are ready to define an equilibrium.

Definition 11 (Equilibrium). A (pure strategy) equilibrium is a vector of strategies, denoted $s^{y*} = (s_1^{y*}, \dots, s_N^{y*})$, such that: for each agent *i*, every period *t* and every possible realization of public signal $y(\sigma(t))$ ($\sigma(t) \in \Sigma$), $s_i^*(y)$ maximizes agent *i*'s expected discounted total payoff at period *t* given s_{-i}^{y*} .

The concept of equilibrium here follows the convention of subgame perfection in stochastic games. In other words, an equilibrium strategy profile must ensure the optimality of its assigned action for each agent given *every* public signal (and hence every formation history), including those that may never occur. Also, it is worth noting again that each agent's payoff is realized in every period, so there is no delay in the model before payoffs are assigned. Payoffs are realized at the same time as network formation.

It is easy to see that a pure strategy equilibrium for the game always exists, regardless of the type vector and the specific payoff structure. Indeed, since link formation and maintenance requires bilateral consent, the strategy profile that every agent always chooses action 0 already constitutes an equilibrium. We note the existence of an equilibrium below.

Proposition 12. There exists a pure strategy equilibrium.

Given the potential plethora of equilibria, in this paper we focus on equilibria in which the formation process leads to a persisting network. In such an equilibrium, the network topology evolves according to the randomness in the selection of agent pairs only in an early stage of formation. Once some particular network is formed, the topology remains thereafter. We believe that this type of equilibria provides an appropriate account for what to be expected in the formation process in various applications such as social circles. People tend to form and sever links constantly in the starting phase of building their social milieu, but over time they incline to maintaining a relatively fixed circle of acquaintances (Kossinets and Watts[KW06]). We formally describe such convergence in our model below.

Given a realized formation history $\sigma(t)$ and a realized formation history $\sigma(t)$, we know that the entire network topology thereafter $\{\mathbf{g}(\tau)\}_{\tau=t+1}^{\infty}$ is a stochastic process. We denote the probability measure generated by this stochastic process as $\mathcal{Q}_{s^{y*},\sigma(t)}$.

Definition 12 (Convergence). *Given a realized formation history* $\sigma(t)$ *we say that the network formation process converges to network* g *in equilibrium* s^{y*} *if*

$$\lim_{T\to\infty}\mathscr{Q}_{s^{y*},\boldsymbol{\sigma}(t)}(\boldsymbol{g}(T')=\boldsymbol{g}\;\forall T'\geq T)=1.$$

Since the selection of agent pairs is random with equal probability for each pair, the above definition of convergence is equivalent to no formation of new link and no severance of existing link after network \mathbf{g} is formed (for a finite number of times). In addition, we differentiate between two forms of convergence, which will prove useful in presenting our major results later.

Definition 13 (Strong and Weak Convergence). We say that the network formation process converges to **g** strongly in equilibrium $s^*(y)$ if it does regardless of the formation history, and that it converges to **g** weakly in equilibrium s^{y*} if it does given the initial formation history (\emptyset, \emptyset) .

3.4.2 Construction of Equilibrium Strategies

We explicitly construct a strategy profile that serves as a candidate for an equilibrium in the Folk Theorem established later. Since a strategy is a mapping from the set of public signals and the set of selected counter party to the set of actions, we first need to specify the associated public signal device, denoted y_c . We characterize y_c recursively as follows:

- 1. *Y* = {*C*,*P*}, where *C* represents the *cooperation phase* and *P* represents the *punishment phase*.
- 2. $y_c(\sigma(0)) = C$.
- 3. In period t ≥ 1, if y_c(σ(t − 1)) = C: for some given network g and any pair of agents (i, j), if ij ∈ g and ij was formed (if selected only) or maintained (selected or not) in period t, or if ij ∉ g and ij was severed or not maintained in period t, then y_c(σ(t)) = C. Otherwise, y_c(σ(t)) = P.
- 4. In period $t \ge 1$, if $y_c(\sigma(t-1)) = P$: for some fixed integer *K*, if $y_c = P$ for the last *K* periods, then $y_c(\sigma(t)) = C$. Otherwise, $y_c(\sigma(t)) = P$.

Essentially, y_c divides the formation process into two phases: the cooperation phase which continues for ever if agents choose their actions in order to form or maintain network **g**, and the punishment phase that starts when agents depart from the cooperation phase and continues for *K* periods. From the public signal *C* or *P*, each agent knows what phase she is currently in, but not how long that phase has lasted or how many times the same phase has occurred before. y_c is not the only signal device exhibiting such features: the only requirement is that it represents the two phases with different sets of realized signals.

Now consider the following strategy profile, denoted $\hat{s}_c^{v_c}$:

$$s_{ij}^{y_c} = \begin{cases} 1, \text{ if } ij \in \mathbf{g}, y_c = C, \text{ and } \max\{\omega_{ij}, \zeta_{ij}\} = 1; \\ 0, \text{ otherwise.} \end{cases}$$

The above strategy profile can be interpreted as the following pattern of behavior: the agents start by cooperation towards building a designated network. They form or maintain a link if and only if that link belongs to the particular network \mathbf{g} . Once a "deviation", i.e. a link in

g not formed or a link not in **g** formed, is detected – note that a deviation here is link-wise instead of agent-wise, since it is impossible to observe whose action caused the deviation – all agents leave the social circle for K periods before starting cooperation again. During the punishment phase, there is no further punishment even if some link is formed or maintained as it is not supposed to be; the agents simply resume cooperation when and only when the punishment phase is over. In the context of real social networks, it is as if the agents would not care what happens in the group when they have decided to leave temporarily. Following this strategy profile, the network topology will eventually stay at **g** almost surely, regardless of the formation history to start with. In the next section, this tractable strategy profile will be used in proving the Folk Theorem.

3.4.3 The Folk Theorem

We state our first main result, the Folk Theorem with complete information, as follows.

Theorem 7. For a network **g**, the following are true:

- b. There exists γ
 ∈ (0,1) such that for all γ ∈ [γ,1), there exists an equilibrium in which the formation process converges to g strongly if u_i(θ
 ,g) > 0 for all i.

Part (a) of the theorem states that a network can form and persist forever only if it provides each agent a non-negative payoff. Given the bilateral nature of link formation/maintenance and the assumption that an agent always obtain payoff 0 as a singleton, this result is a first natural implication of the model. We provide the proof of this part below.

Proof of (a). Suppose that there exists an equilibrium where the formation process converges to **g** weakly, and that $u_i(\bar{\theta}, \mathbf{g}) < 0$ for some *i*. Then on the equilibrium path when **g** has been formed and will persist forever, *i* is always strictly better off by deviating to the strategy $s_{ij}^y = 0$ and obtaining payoff 0 thereafter. This is a contradiction to the assumption of an equilibrium.

Part (b) is the more important part of the theorem. It is a partial converse to part (a), showing that when agents are sufficiently patient, any network \mathbf{g} that yields a positive payoff to

every agent can be sustained in equilibrium. Moreover, the convergence is strong: the formation history, even if an off-equilibrium one, does not affect the ultimate convergence to **g**. We prove this part using the particular strategy profile constructed in the previous section.

Proof of (b). Consider the public signal device y_c and the strategy profile $\hat{s}_c^{y_c}$. Given $\bar{\theta}$, let \bar{v} denote the largest marginal benefit that an agent can obtain from forming or severing a link in any network **g**. \bar{v} measures the largest possible marginal benefit that an agent can get from deviating in one period. Since the number of network is finite, we know that \bar{v} exists.

Given $\bar{\theta}$, a formation history $\sigma(t)$ and a strategy profile *s*, let $\bar{\mu}_C(\gamma, M)$ and $\underline{\mu}_C(\gamma, M)$ denote the largest and smallest expected total payoff any agent gets within *M* periods of cooperation phase, starting from any network. Let $\bar{\mu}_P(\gamma, M)$ and $\underline{\mu}_P(\gamma, M)$ denote the largest and smallest expected total payoff any agent gets within *M* periods of punishment phase, starting from any network. We first establish the following lemma.

Lemma 9. If $u_i(\bar{\theta}, g) > 0$ for all *i*, then the following properties hold:

- *a*. $\lim_{\gamma \to 1} \bar{\mu}_C(\gamma, \infty) = \lim_{\gamma \to 1} \underline{\mu}_C(\gamma, \infty) = \infty$.
- b. There exists A > 0 such that $\overline{\mu}_C(\gamma, \infty) \underline{\mu}_C(\gamma, \infty) < A$, regardless of γ .
- c. There exist B and C (B < C) such that for any M, $B < \underline{\mu}_P(\gamma, M) \le \overline{\mu}_P(\gamma, M) < C$, regardless of γ .

Proof. Let $W \in \mathbb{R}$ denote the smallest possible payoff of any agent in any network in one period.

For (*a*), it suffices to show that a lower bound of the two payoffs converges to infinity as γ converges to 1. Consider the following hypothetical payoff structure: agent *i*'s one-period payoff is *W* if the network is different from **g**, and $u_i(\bar{\theta}, \mathbf{g})$ otherwise. Starting from any network $\mathbf{g}(0)$, the probability that $\mathbf{g}(t) \neq \hat{\mathbf{g}}$ is bounded above by min $\{\frac{N(N-1)}{2}(1-\frac{2}{N(N-1)})^t, 1\}$ (this upper bound is constructed by supposing that $\mathbf{g}(0) = \emptyset$ and **g** is the complete network, and calculating the probability that some pair of agents has never been selected during the *t* periods). For all *t* such that $\frac{N(N-1)}{2}(1-\frac{2}{N(N-1)})^t < 1$ (let t^* be the smallest *t* satisfying this condition), *i*'s expected payoff in $\mathbf{g}(t)$ is bounded below by

$$\frac{N(N-1)}{2}\left(1-\frac{2}{N(N-1)}\right)^{t}W+\left(1-\frac{N(N-1)}{2}\left(1-\frac{2}{N(N-1)}\right)^{t}\right)u_{i}(\bar{\theta},\mathbf{g})$$

and agent *i*'s total expected payoff is bounded below by

$$\begin{split} &\sum_{t=1}^{t^*-1} \gamma^{t-1} W + \sum_{t=t^*}^{\infty} \gamma^{t-1} (\frac{N(N-1)}{2} (1 - \frac{2}{N(N-1)})^t W \\ &+ (1 - \frac{N(N-1)}{2} (1 - \frac{2}{N(N-1)})^t) u_i(\bar{\theta}, \mathbf{g})) \\ &= \sum_{t=1}^{t^*-1} \gamma^{t-1} W + \sum_{t=t^*}^{\infty} \gamma^{t-1} u_i(\bar{\theta}, \mathbf{g}) + \sum_{t=t^*}^{\infty} \gamma^{t-1} \frac{N(N-1)}{2} (1 - \frac{2}{N(N-1)})^t (W - u_i(\bar{\theta}, \mathbf{g})) \\ &= \frac{W(1 - \gamma^{t^*})}{1 - \gamma} + \frac{\gamma^{t^*-1} u_i(\bar{\theta}, \mathbf{g})}{1 - \gamma} + \frac{N(N-1)}{2} \frac{\gamma^{t^*-1} (1 - \frac{2}{N(N-1)})^{t^*} (W - u_i(\bar{\theta}, \mathbf{g}))}{1 - \gamma(1 - \frac{2}{N(N-1)})}. \end{split}$$

It is clear that the sum of the first term and the third term above has a lower bound which is independent of γ . In addition, the second term converges to infinity as γ converges to 1 regardless of *i*. Hence part (*a*) is proved. (*b*) and (*c*) can be proved using a similar argument.

Consider agent *i* at period *t* following any formation history. Note that *i* cannot really "deviate" in the punishment phase given that all the agents other than *i* are using their prescribed strategy in $\hat{s}_c^{y_c}$. Hence we only need to consider deviation of agent *i* in the cooperation phase. According to the one-step deviation principle, in order to determine whether $\hat{s}_c^{y_c}$ is an equilibrium we only need to consider *i*'s deviation in one period, after which *i* returns to her prescribed strategy in $\hat{s}_c^{y_c}$. As mentioned before, the largest possible marginal benefit that *i* gets from this deviation in this period is \bar{v} . Starting from the next period, *i*'s expected total payoff is bounded above by

$$\gamma \bar{\mu}_P(\gamma, K) + \gamma^{1+K} \bar{\mu}_C(\gamma, \infty).$$

If i does not deviate, then starting from the next period, i's expected total payoff is bounded below by

$$\gamma \underline{\mu}_C(\gamma, K) + \gamma^{1+K} \underline{\mu}_C(\gamma, \infty).$$

Therefore, we have

Total expected marginal benefit from deviation

$$\leq \bar{v} + \gamma \bar{\mu}_{P}(\gamma, K) + \gamma^{1+K} (\bar{\mu}_{C}(\gamma, \infty) - \underline{\mu}_{C}(\gamma, \infty)) - \gamma \underline{\mu}_{C}(\gamma, K)$$

$$< \bar{v}(\bar{\theta}) + \gamma C + \gamma^{1+K} A - \gamma \underline{\mu}_{C}(\gamma, K),$$

from properties (*b*) and (*c*) above. Then from property (*a*), there exists $\gamma' \in (0, 1)$ and \bar{K} such that $\underline{\mu}_C(\gamma, K) > 2(\bar{\nu} + C + A)$ for any $\gamma \ge \gamma'$ and $K \ge \bar{K}$. Let $\bar{\gamma} = \max\{\gamma', \frac{1}{2}\}$, then for any $\gamma \in [\bar{\gamma}, 1)$, we have

$$\bar{v} + \gamma C + \gamma^{1+K} A - \gamma \underline{\mu}_{C}(\gamma, K) < 0,$$

which implies that deviation is not profitable and hence $\hat{s}_{c}^{y_{c}}$ is an equilibrium where the formation process converges to **g**.

Combining part (a) and part (b) of the theorem affords us a reasonable characterization of the set of network topologies that can be sustained in equilibrium. When agents are patient, it is necessary and sufficient to sustain a network for ever that this network guarantees a positive payoff to every agent. With our proposed strategy profile, the underlying mechanism for convergence to such a network can be described as a "self-fulfilling prophecy": the agents cooperate in order to form a network that is commonly envisioned, and punish any detected deviation (there can be "undetected" deviation such as *i* choosing 1 but *j* still choosing 0 for some link $ij \notin g$) by opting out of the group for *K* periods. For any agent, this punishment is incentive compatible once everyone else complies. Afterwards, the agents opt back in and resume cooperation. In this way, **g** always gets formed and persists no matter what the initial network was and what the formation history has been.

An immediate yet important result from Theorem 7 is a clear criterion on sustaining efficiency. For a strongly efficient network to be sustained in any equilibrium with convergence, it needs to ensure a non-negative payoff for each agent. Conversely, if a strongly efficient network yields every agent a positive payoff, then it can be sustained in equilibrium if the agents are patient enough.

Corollary 3. There exists an equilibrium where the formation process converges to a strongly efficient network \mathbf{g} weakly only if $u_i(\bar{\theta}, \mathbf{g}) \ge 0$ for all *i*. If a strongly efficient network \mathbf{g} is such that $u_i(\bar{\theta}, \mathbf{g}) > 0$ for all *i*, then there exists an equilibrium where the formation process converges to \mathbf{g} strongly when γ is sufficiently large.

This corollary presents a striking contrast to the existing literature on network formation with foresight, which posits that in generic cases efficiency cannot be sustained even if agents are patient and each agent's payoff in the strongly efficient network is positive. We provide an example below, which appears in Dutta et al.[DGR05], to illustrate the difference.

Example 4. This example is taken from Dutta et al.[DGR05], Theorem 2. Consider $I = \{1,2,3\}$ and assume that there is no type difference across agents. The payoff structure is symmetric: for any $i, j, k, u_i(\bar{\theta}, \emptyset) = 0, u_i(\bar{\theta}, \{ij\}) = 2v, u_i(\bar{\theta}, \{ij, ik, jk\}) = v$, while $u_i(\bar{\theta}, \{ij, jk\}) = 0$. The unique strongly efficient network is the complete network $g = \{12, 13, 23\}$.

In the cited paper, there exists $\bar{\gamma} < 1$ such that for all $\gamma \in (\bar{\gamma}, 1)$ there is no pure strategy equilibrium where the formation process strongly converges to the strongly efficient network. This results from the constraint of agents' knowledge on the formation history: in the paper, it is assumed that the agents only know the network formed in the previous period and the pair of agents selected in the current period.

In our model, consider the strategy profile $\hat{s}_c^{y_c}$. In the punishment phase, no unilateral action can change the network formation outcome, so we only need to inspect the cooperation phase for incentive to deviate. Using the same tactics of amplification and minification as in the proof of Theorem 7 and plugging in the values in this example, we can obtain a range of γ and K to make $\hat{s}_c^{y_c}$ an equilibrium: $\gamma \in (0.97, 1)$, $K \ge 60$.

At the end of this section we would like to emphasize again the importance and significance of the public signal device. To sustain cooperation which leads to efficiency over time, the agents need not know *who* committed a deviation or *when* a deviation occurred, but it is vital that they know if *someone* has deviated in the recent past and *whether* they are supposed to carry on punishment. A public signal device (newspaper, TV, website, etc.) conveys such information across the group of agents and ensures a limited but effective form of cooperation. On one hand, the action profile cannot be conditioned on the entire formation history due to a constrained set of realized signals; on the other hand, this limited set of strategies with Markov nature already suffices for sustaining efficiency once the signal device maps history to signal in an appropriate way. As a practical implication, our analysis strongly suggests that modern media, with its function of public broadcast, plays a crucial role in enhancing social welfare.

3.4.4 Robustness of Equilibrium

In some applications of the model such as online social networks, it is sometimes not quite realistic to assume that agents are always rational and abide by the same set of decision rules, or that each agent always chooses her actions independently without consulting others. Fortunately, our model and our equilibrium notion are robust enough to accommodate such alterations. In this section, we first present a result illustrating the robustness of our constructed equilibrium against small trembles in an agent's action.

Proposition 13. Assume that in each period, an agent randomly chooses her action with probability $\varepsilon > 0$. For any g such that $u_i(\bar{\theta}, g) > 0$ for all i, there exists $\bar{\gamma} \in (0, 1)$ and $\bar{K} \in \mathbb{N}$ such that given any $\gamma \in [\bar{\gamma}, 1)$ and $K \ge \bar{K}$, there exists $\varepsilon(\gamma) > 0$ such that $\hat{s}_c^{y_c}$ is an equilibrium for all $\varepsilon \le \varepsilon(\gamma)$.

Proof. Referring to the proof of Theorem 7, it suffices to show that Lemma 9 still holds under this alternative environment with some function $\varepsilon(\gamma)$. For part (*a*) of Lemma 9, let t^* be the smallest *t* satisfying $\frac{N(N-1)}{2}(1-\frac{2}{N(N-1)})^t < 1$, and following a similar argument as in the proof of Theorem 7, we know that *i*'s expected payoff in $\mathbf{g}(t)$ is bounded below by

$$W(1 - (1 - \varepsilon)^{Nt}) + \frac{N(N - 1)}{2} (1 - \frac{2}{N(N - 1)})^{t} W(1 - \varepsilon)^{Nt} + (1 - \frac{N(N - 1)}{2} (1 - \frac{2}{N(N - 1)})^{t}) u_{i}(\bar{\theta}, \mathbf{g}) (1 - \varepsilon)^{Nt}.$$

Agent *i*'s total expected payoff is bounded below by

$$\begin{split} &\sum_{t=1}^{t^*-1} \gamma^{t-1} W + \sum_{t=t^*}^{\infty} \gamma^{t-1} W (1 - (1 - \varepsilon)^{Nt}) \\ &+ \sum_{t=t^*}^{\infty} \gamma^{t-1} \frac{N(N-1)}{2} (1 - \frac{2}{N(N-1)})^t W (1 - \varepsilon)^{Nt} \\ &+ \sum_{t=t^*}^{\infty} (1 - \frac{N(N-1)}{2} (1 - \frac{2}{N(N-1)})^t) u_i(\bar{\theta}, \mathbf{g}) (1 - \varepsilon)^{Nt} \\ &= \frac{W}{1 - \gamma} - \frac{\gamma^{t^*-1} (1 - \varepsilon)^{Nt^*} W}{1 - \gamma(1 - \varepsilon)^N} + \frac{\gamma^{t^*-1} (1 - \varepsilon)^{Nt^*} u_i(\bar{\theta}, \mathbf{g})}{1 - \gamma(1 - \varepsilon)^N} \\ &+ \frac{N(N-1)}{2} \frac{\gamma^{t^*-1} (1 - \varepsilon)^{Nt^*} (1 - \frac{2}{N(N-1)})^{t^*} (W - u_i(\bar{\theta}, \mathbf{g}))}{1 - \gamma(1 - \varepsilon)^N}. \end{split}$$

The last term has a lower bound which is independent of γ . Also, note that as $\varepsilon \to 0$, we have

$$\frac{W}{1-\gamma} - \frac{\gamma^{t^*-1}(1-\varepsilon)^{Nt^*}W}{1-\gamma(1-\varepsilon)^N} \to \frac{W}{1-\gamma} - \frac{\gamma^{t^*-1}W}{1-\gamma}$$
$$\frac{\gamma^{t^*-1}(1-\varepsilon)^{Nt^*}u_i(\bar{\theta},\mathbf{g})}{1-\gamma(1-\varepsilon)^N} \to \frac{\gamma^{t^*-1}u_i(\bar{\theta},\mathbf{g})}{1-\gamma},$$

and as $\gamma \rightarrow 1$, we have

$$\frac{W}{1-\gamma} - \frac{\gamma^{t^*-1}W}{1-\gamma} \to t^* - 1$$
$$\frac{\gamma^{t^*-1}u_i(\bar{\theta}, \mathbf{g})}{1-\gamma} \to \infty.$$

Hence, for any number x > 0, there exists a function $\varepsilon(\gamma)$ such that for some $\gamma' \in (0,1)$, any (γ, ε) such that $\gamma \ge \gamma'$ and $\varepsilon \le \varepsilon(\gamma)$ makes agent *i*'s expected total payoff higher than *x*. This proves part (*a*) of Lemma 9. Part (*b*) and part (*c*) can be proved by a similar argument. \Box

With sufficiently small trembles (the exact range of allowable trembles depends on γ), our constructed strategy profile would still be an equilibrium for patient agents. In other words, such perturbation of randomness cannot drive the formation process elsewhere than finally to **g**. This result reveals a further strong property of the particular equilibrium: under this equilibrium the formation process will not only converge to the designated network **g** regardless of any formation history in the past, but will also exhibit the same convergence even though every agent is aware of the small probability of mistake or irrationality in everyone's behavior in the future.

Next, we discuss how the additional property of stability of a network brings about an equilibrium that prevents typical *group* deviations. Recall that a network is stable if there is no subgroup of agents that can form another network on their own (without linking to any agent not in the subgroup) to provide a higher payoff for everyone in the subgroup. Formally, let $s'(\hat{I}, \hat{g})$ denote the following strategy profile by a sub-group of agents $\hat{I} \subsetneq I$ with respect to a network \hat{g} that can be formed within \hat{I} : form or maintain link ij if and only if $ij \in \hat{g}$. Following a formation history $\sigma(t)$, we say that a strategy profile *s* is **immune to** $s'(\hat{I}, \hat{g})$ if the (discounted expected total) payoff of some agent in \hat{I} is strictly higher in *s* than in $s'(\hat{I}, \hat{g})$.

In addition, we assume here that in $\hat{s}_c^{y_c}$, the public signal device reveals the remaining number of periods for the punishment phase. The next proposition identifies the relation between stability and immunity to the above group deviation.

Proposition 14. Assume that for a stable network \mathbf{g} , $u_i(\bar{\theta}, \mathbf{g}) > 0$ for all i. Then there exist \hat{K} and $\hat{\gamma} \in (0,1)$ such that for all $\gamma \in [\hat{\gamma}, 1)$ and $K \ge \hat{K}$, $\hat{s}_c^{\nu_c}$ is a stable equilibrium where the formation process converges to \mathbf{g} . Moreover, when $\gamma \in [\hat{\gamma}, 1)$, there exists $M(\gamma)$ such that:

- a. Following any formation history with the remaining punishment phase no longer than $M(\gamma)$ periods, $\hat{s}_c^{y_c}$ is immune to $s'(\hat{I}, \hat{g})$ for any $\hat{I} \subsetneq I$ and any \hat{g} that can be formed within \hat{I} .
- *b.* $M(\gamma)$ *is increasing in* γ *and* $\lim_{\gamma \to 1} M(\gamma) = \infty$.

Proof. Consider a formation history with the remaining punishment phase being K' periods (including the current period). By the assumption that **g** is stable, for any $\hat{I} \subsetneq I$ and associated $\hat{\mathbf{g}}$, there exists an agent *i* such that $u_i(\bar{\theta}, \hat{\mathbf{g}}) - u_i(\bar{\theta}, \mathbf{g}) < 0$. Fix one such *i*. From the current period onwards, if the agents follow $\hat{s}_c^{v_c}$, *i*'s total payoff is bounded below by

$$\underline{\mu}_{P}(\gamma, K') + \gamma^{K'} \underline{\mu}_{C}(\gamma, \infty)$$

Let V > 0 denote the largest possible payoff of any agent in any network in one period. With a little abuse of notation, let $\mathbf{g}(t)$ denote the network formed *t* periods from the current period. If the group of agents \hat{I} follow $s'(\hat{I}, \hat{\mathbf{g}})$, *i*'s payoff in $\mathbf{g}(t)$ is bounded above by

$$1\{\mathbf{g}(t)\neq\hat{\mathbf{g}}\}V+(1-1\{\mathbf{g}(t)\neq\hat{\mathbf{g}}\})u_i(\bar{\boldsymbol{\theta}},\hat{\mathbf{g}}).$$

If the group of agents \hat{I} follow $s'(\hat{I}, \hat{g})$, the probability that $g(t) \neq \hat{g}$ is bounded above by

$$\min\{\frac{N(N-1)}{2}(1-\frac{2}{N(N-1)})^{t+1},1\}.$$

For all t such that $\frac{N(N-1)}{2}(1-\frac{2}{N(N-1)})^{t+1} < 1$, i's expected payoff in $\mathbf{g}(t)$ is bounded above by

$$\frac{N(N-1)}{2}\left(1-\frac{2}{N(N-1)}\right)^{t+1}V+\left(1-\frac{N(N-1)}{2}\left(1-\frac{2}{N(N-1)}\right)^{t+1}\right)u_i(\bar{\theta},\hat{\mathbf{g}}).$$

Following a similar argument to the proof of Lemma 9, there exists D > 0 (regardless of γ) such that *i*'s discounted expected total payoff from $s'(\hat{I}, \hat{g})$ is less than $D + \sum_{t=0}^{\infty} \gamma^t u_i(\bar{\theta}, \hat{g})$. Now, the difference in *i*'s payoff between the two strategy profiles is bounded above by

$$D-B+\sum_{t=0}^{\infty}\gamma^{t}u_{i}(\bar{\boldsymbol{\theta}},\hat{\mathbf{g}})-\gamma^{K'}\underline{\mu}_{C}(\gamma,\infty).$$

With a similar argument to above, there exists E > 0 (regardless of γ) such that

$$\underline{\mu}_{\mathcal{C}}(\gamma,\infty) > \sum_{t=0}^{\infty} \gamma^{t} u_{i}(\bar{\theta},\mathbf{g}) - E.$$

Hence, the difference in *i*'s payoff between the two strategy profiles is bounded above by

$$\begin{split} D - B + E + \sum_{t=0}^{\infty} \gamma^t u_i(\bar{\theta}, \hat{\mathbf{g}}) - \gamma^{K'} \sum_{t=0}^{\infty} \gamma^t u_i(\bar{\theta}, \mathbf{g}) \\ = D - B + E + \sum_{t=0}^{K'-1} \gamma^t u_i(\bar{\theta}, \hat{\mathbf{g}}) + \gamma^{K'} (\sum_{t=0}^{\infty} \gamma^t (u_i(\bar{\theta}, \hat{\mathbf{g}}) - u_i(\bar{\theta}, \mathbf{g}))) \\ \leq D - B + E + \sum_{t=0}^{K'-1} \gamma^t u_i(\bar{\theta}, \hat{\mathbf{g}}) + \frac{\gamma^{K'}F}{1 - \gamma} \\ \leq D - B + E + K'V + \frac{\gamma^{K'}F}{1 - \gamma}, \end{split}$$

where $F = \max_{\hat{I}, \hat{g}} \{ u_i(\bar{\theta}, \hat{g}) - u_i(\bar{\theta}, g) \}$. Since the total number of networks is finite, we know that *F* exists and that F < 0.

Let γ'' be such that $\frac{F}{1-\gamma} = -|D-B+E| - 1$, and let \bar{K} and $\bar{\gamma}$ be as derived in the proof of Theorem 7. Let $\hat{K} = \bar{K}$ and let $\hat{\gamma} = \max\{\gamma'', \bar{\gamma}\}$. For every $\gamma \ge \hat{\gamma}$, let $M(\gamma)$ be the largest $K' \in \mathbb{N}$ such that $D - B + E + K'V + \frac{\gamma^{K'}F}{1-\gamma} < 0$. We know that $M(\gamma)$ exists because K' = 0always satisfies the inequality.

Now, given \hat{K} and any $\gamma \ge \hat{\gamma}$, $\hat{s}_c^{y_c}$ is an equilibrium where the formation process converges to **g** by Theorem 7, and hence a stable equilibrium by the assumption that **g** is stable. From the construction of $M(\gamma)$, given any \hat{I} following any formation history with the remaining punishment phase no longer than $M(\gamma)$ periods, there is always some agent in $\hat{I} \subsetneq I$ and associated $\hat{\mathbf{g}}$ whose payoff under strategy profile $s'(\hat{I}, \hat{\mathbf{g}})$ is strictly lower than that under strategy profile $\hat{s}_c^{y_c}$. Hence, $\hat{s}_c^{y_c}$ is immune to $s'(\hat{I}, \hat{\mathbf{g}})$. Finally, since the term $D - B + E + K'V + \frac{\gamma^{K'F}}{1-\gamma}$ is increasing in K' and decreasing in γ , $M(\gamma)$ is increasing in γ ; the fact that $\lim_{\gamma \to 1} \frac{\gamma^{K'F}}{1-\gamma} = -\infty$ for any given K' ensures that $\lim_{\gamma \to 1} M(\gamma) = \infty$. This completes the proof.

The set of stable networks resembles the core in a cooperative game, even though utilities are not transferable in this environment. In the above result, we apply the concept of no improvement by a subset of agents to a dynamic setting. The reason why the equilibrium strategy profile is immune to group deviation only within a bounded number of periods towards the end of the punishment phase (which is also the reason for assuming a richer public signal indicating the remaining time of the punishment phase) is that once the remaining punishment phase is too long, the agents may expect too many periods in the future with an empty network if they follow the equilibrium strategy profile, and hence may improve their payoffs by forming a sub-coalition immediately. It is worth noting that group deviation may not be prevented if it involves *every* agent instead of only a subgroup, even if the designated network **g** is stable. For instance, in some period during the punishment phase when the current network is empty, no matter how many periods of the phase are expected ahead on equilibrium path, each agent may enjoy a higher payoff if the whole group just abandons the punishment phase and start cooperation immediately. On the other hand, for a non-stable network, this result does not suggest that a profitable group deviation would constitute another equilibrium: such deviation is merely one possible way of making every agent in the subgroup better off than in the original network.

3.5 Folk Theorem with Incomplete Information

In real-life applications, scenarios with complete information among agents are extremely rare, if existing at all. As in Song and van der Schaar[SS15], the introduction of incomplete information could produce enormous differences in agents' strategic behavior and equilibrium network topology. In this section, we extend the Folk Theorem to the environment with incomplete information and somewhat surprisingly, we are able to identify an undemanding condition on the payoff structure, under which the formation process will again converge to the strongly efficient network in equilibrium.

3.5.1 Modeling Incomplete Information

At the beginning of t = 1, each agent only knows her own type and holds the prior belief (that types are i.i.d. according to H) on other agents' types. To avoid some technical redundancy while maintaining a general implication of the model, we assume that the type set Θ is finite, and that H places positive probability on each possible type. Also without loss of generalization we assume that any network yields a non-zero payoff to any agent.

Let $\mathscr{B} = \Delta(\Theta^N)$ denote the set of possible beliefs on the type vector. A (pure) **strategy** of agent *i* is now a set of mapping

$$s_i^{\mathbf{y}} = \{s_{ij}^{\mathbf{y}} : \mathscr{B} \times Y \times \Omega \times Z \to \mathscr{A}\}_{j \neq i},$$

with the same constraint $s_{ij}^{y}(\cdot, \cdot, 0, 0) \equiv 0$. An **equilibrium** is similarly defined as before, except for the additional requirement that *i* maximizes her expected discounted total payoff given her

belief at every period.

Let $B_i : Y \to \mathscr{B}$ denote agent *i*'s *belief updating function*, which is a mapping from the set of possible public signals to the set of possible beliefs. We assume that it satisfies the following properties:

- 1. *i* knows her own type: regardless of $\sigma(t)$, she puts probability 1 on her true type.
- 2. *i* knows the type of any agent that she has been connected to: if some g such that *ij* ∈ g has ever been formed in σ(*t*), then *i* always puts probability 1 on *j*'s true type starting from period *t*.
- 3. If *i* and *j* have not been connected, then given a strategy profile *s*, at the beginning of each period *i* performs Bayes' update on the type vector upon observing any realized public signal. In the case where the realized public signal would have never occurred according to *s*, agent *i*'s belief stays the same as in the previous period.

We now define some concepts related to the payoff structure that will be useful in constructing equilibrium strategies later. First we define a *partial equilibrium network* for a subset of agents.

Definition 14. Given $\overline{\theta}$, a network g formed in $I' \subset I$ is a partial equilibrium network for $I'' \subset I'$ if (1) each agent in I'' gets a positive payoff from g; (2) no agent in I'' can increase her payoff by severing any of her links in g.

Given a subset of agents I' and the associated type vector $\bar{\theta}_{I'}$, consider a function $r: \Theta^{|I'|} \to G_{I'}$. We define the following property for r:

Definition 15. We say that r is an admissible function for I' if for every $\bar{\theta}_{I'} \in \Theta^{|I'|}$, $r(\bar{\theta}_{I'})$ is a network such that (1) every non-singleton agent in $r(\bar{\theta}_{I'})$ has a positive payoff; (2) there exists a partial equilibrium network in $G_{I'}$ for the set of singleton agents in $r(\bar{\theta}_{I'})$, denoted as $r'(\bar{\theta}_{I'})$. We say that I' is admissible if there exists an admissible function r for I'.

In a partial equilibrium network, no agent has incentive to unilaterally sever any of her link, hence the name "partial equilibrium". Then (the existence of) an admissible function essentially characterizes a particular type of agent subgroup: it maps every type vector in the subgroup to a network that provides every connected agent a positive payoff, and at the same time guarantees the existence of a partial equilibrium network for the set of singleton agents. Intuitively, the former network can be sustained in the long run when agents are patient, and the latter can be used as a way to reward the future singleton agents for their revelation of private information. We will construct an equilibrium strategy profile following this argument in the next section.

In many cases, the whole agent set *I* is admissible. One of the simplest scenarios is that for every type vector there exists a network yielding a positive payoff for every agent, so that a partial equilibrium network will not even be necessary because the set of singleton agents will be empty. For instance, consider the connections model discussed before, and consider the following two-type scenario: $\Theta = \{\alpha, \beta\}, N = 5, f(\alpha) > c > f(\beta), (1 + \delta)f(\beta) > c$, and $f(\alpha) + 3f(\beta) > 3c$. First, note that when there exists at least one type α agent, a partial equilibrium network for any number of type β agents is a star network with a type α agent as the center. Then the following function *r* is an admissible function for *I*:

$$r(\bar{\theta}) = \begin{cases} \text{Star network with type } \alpha \text{ center, if at least two agents are of type } \alpha \\ \text{Wheel network, otherwise.} \end{cases}$$

It is easy to verify that r is indeed admissible for I. Moreover, it is also straight forward to show that r maps $\bar{\theta}$ to the strongly efficient network whenever the strongly efficient network gives every agent a positive payoff. In this type of payoff structure, the larger N is, the more likely such a simple admissible function for I exists. When N is large, even if $f(\alpha)$ and $f(\beta)$ are both small relative to c, a topology such as a star or a wheel may still ensure a positive payoff for every agent. In the case where r maps a type vector to some unconnected network (a network with singleton agents), a star or a wheel can be used as partial equilibrium networks (in the case of a star, the singleton agents in r would be placed in the periphery).

3.5.2 Construction of Equilibrium Strategies

As with complete information, we explicitly construct a strategy profile that will constitute an equilibrium when agents are sufficiently patient. First we specify the associated public signal device, denoted y_{ic} .

To simplify the description of y_{ic} , we first introduce some additional notations. For a given

time period *t* and a given subgroup of agents $I' \subset I$, we denote as $I''_1(\mathbf{g},t)$ the subset of agents in *I'* that failed to form/maintain a link in **g** when possible, and as $I''_2(t)$ the subset of agents in *I'* that formed/maintained a link between *I'* and $I \setminus I'$. We let $\hat{\mathbf{g}}_{I'}$ denote the clique on *I'*, i.e. the network containing and only containing every link within *I'*. Finally, we say that information is complete within *I'* if every agent knows the type of every other agent in *I'*, and that information is incomplete within *I'* otherwise. Denote these events in period *t* as $O_c(I', t)$ and $O_{ic}(I', t)$.

- 1. $Y = \{X_0, X_1, T, E_C, E_P\} \times 2^I$. The subset of *I* in the second argument represents the subgroup of *non-solitary agents*, and X_0, X_1, T, E_C, E_P represent five phases with respect to this subgroup: the *experimentation phase with incomplete information, experimentation phase with complete information, transition phase, exploitation phase with cooperation* and *exploitation phase with punishment* correspondingly. We will define and explain these concepts later.
- 2. $y_{ic}(\sigma(0)) = (X_0, I)$.
- 3. In period $t \ge 1$, for any pair of agents $i, j \in I' \subset I$:

- a. If
$$y_{ic}(\sigma(t-1)) = (X_0, I')$$
:

$$O_c(I' \setminus I_1''(\hat{\mathbf{g}}_{I'}, t), t) \to y_{ic}(\sigma(t)) = (T, I' \setminus I_1''(\hat{\mathbf{g}}_{I'}), t)$$
$$O_{ic}(I' \setminus I_1''(\hat{\mathbf{g}}_{I'}, t), t) \to y_{ic}(\sigma(t)) = (X_0, I' \setminus I_1''(\hat{\mathbf{g}}_{I'}), t).$$

- b. If
$$y_{ic}(\sigma(t-1)) = (X_1, I')$$
: $y_{ic}(\sigma(t)) = (T, I' \setminus (I''_1(r'(\bar{\theta}_{I'}), t) \cup I''_2(t)))$.

c. If y_{ic}(σ(t − 1)) = (T, I'): y_{ic}(σ(t)) = (E_C, I') if r'(θ
{I'}) has been the network topology within I' (including no link between I' and I \ I') for a fixed number of J consecutive periods. Otherwise, y{ic}(σ(t)) = (T, I' \ (I''₁(r'(θ
_{I'}), t) ∪ I''₂(t))).

- d. If
$$y_{ic}(\sigma(t-1)) = (E_C, I')$$
:

$$I_1''(r(\bar{\theta}_{I'}),t) \cup I_2''(t) = \emptyset \to y_{ic}(\sigma(t)) = (E_C, I')$$
$$I_1''(r(\bar{\theta}_{I'}),t) \cup I_2''(t) \neq \emptyset \to y_{ic}(\sigma(t)) = (E_P, I')$$

- e. If $y_{ic}(\sigma(t-1)) = (E_P, I')$: if $y_{ic} = (E_P, I')$ for a fixed number of *K* consecutive periods, then $y_{ic}(\sigma(t)) = (E_C, I')$. Otherwise, $y_{ic}(\sigma(t)) = (E_P, I')$.

Essentially, the realization of y_{ic} reveals publicly the current phase of the game and whether agents in a certain subgroup I' are cooperating in every phase. The meaning of cooperation is phase-specific. In the experimentation phase, agents are supposed to form and maintain every link within I' whenever possible, until information becomes complete in I' which brings the game into the transition phase. Then cooperation among agents becomes forming the network $r'(\bar{\theta}_{I'})$ and keeping it for J periods, with no link with $I \setminus I'$ at the same time. In these two phases, anyone who fails to cooperate will be marked as a solitary agent. Afterwards, the game enters the exploitation phase and the public signal works in the same way as y_c in the previous section, with $r(\bar{\theta}_{I'})$ as the designated network.

Now we characterize the strategy profile based on y_{ic} , denoted $\hat{s}_{ic}^{y_{ic}}$. For any $i, j \in I$:

- 1. If $\max{\{\omega_{ij}, \zeta_{ij}\}} = 1$, the set of non-solitary agents I' is admissible and $i, j \in I'$, then $a_{ij} = 1$ if any of the following is true:
 - a. $y_{ic} = (X_0, I')$.
 - b. $y_{ic} = (X_1, I')$ and $ij \in r'(\bar{\theta}_{I'})$.
 - c. $y_{ic} = (T, I')$ and $ij \in r'(\bar{\theta}_{I'})$.

- d.
$$y_{ic} = (E_C, I')$$
 and $ij \in r(\bar{\theta}_{I'})$.

• 2. $a_{ij} = 0$ in all the other cases.

In this strategy profile, agents in an admissible set cooperate as much as they can according to the public signal. First, they reveal their types by forming and maintaining links in the experimentation phase until information becomes complete. In the transition phase that follows, they form a partial equilibrium network $r'(\bar{\theta}_{I'})$ to provide positive payoffs to the singleton agents in network $r(\bar{\theta}_{I'})$, the network that will persist in the long run. After $r'(\bar{\theta}_{I'})$ has existed for a specified length of time, the agents enter the exploitation phase in which the formation process ultimately converges to $r(\bar{\theta}_{I'})$. Agents who do not conform before the exploitation phase are categorized as solitary agents and are left as singletons for ever, and those who deviate during the exploitation phase only get temporary punishment. Same as before, the exact deviating agent(s) cannot be identified, so any punishment would be placed on pairs of agents rather than individual ones.

3.5.3 The Folk Theorem

In an environment with incomplete information, the Folk Theorem still holds in an admissible set of agents, but our constructed equilibrium strategy profile leads to weak convergence only. The proof below shows that when agents are sufficiently patient, (1) there exists a length of punishment *K* in the exploitation phase to ensure cooperation and (2) there exists a length of reward *J* in the transition phase to ensure information revelation for the singleton agents in $r(\bar{\theta})$ in the experimentation phase.

Theorem 8. If I is admissible, then for any admissible function r for I, there exists $\bar{\gamma} \in (0, 1)$ such that for all $\gamma \in [\bar{\gamma}, 1)$, there exists an equilibrium where the formation process converges to $r(\bar{\theta})$ weakly.

Proof. Consider the public signal device y_{ic} and the strategy profile $\hat{s}_{ic}^{y_{ic}}$. It suffices to show that for some *J* and *K*, there exists $\bar{\gamma} \in (0, 1)$ such that for all $\gamma \in [\bar{\gamma}, 1)$, $\hat{s}_{ic}^{y_{ic}}$ is an equilibrium. We need to check for sequential rationality given any possible formation history. We do it in the following order:

In the exploitation phase: since information is complete within I'', sequential rationality is given by Theorem 7.

Given any formation history, for any solitary agent: given that no other agent will agree to form a link with her, her subsequent action will not affect her payoff, and hence sequential rationality is satisfied.

Given any formation history such that the set of non-solitary agents is not admissible, for any non-solitary agent: given that every other agent is choosing action 0, her subsequent action will not affect her payoff, and hence sequential rationality is satisfied.

For any non-solitary agent in an admissible set I' in the transition phase: we need to discuss two cases. Following the proof of Theorem 7, we argue as follows:

• 1. For any singleton agent *i* in $r(\bar{\theta}_{I'})$: since *I'* is admissible, *i*'s payoff in this phase is positive (and bounded away from zero, from the assumptions that Θ is finite and *I* is finite) for *J* periods in $r'(\bar{\theta}_{I'})$ if she follows the prescribed strategy. Also, her maximum expected loss (negative payoff) before $r'(\bar{\theta}_{I'})$ is formed for the first time and in the exploitation phase is bounded above regardless of γ . Hence, given a sufficiently large *J*

and a sufficiently large γ , *i* does not have the incentive to deviate before $r'(\bar{\theta}_{I'})$ is formed for the first time and become a solitary agent (the payoff from which is bounded above regardless of γ). After $r'(\bar{\theta}_{I'})$ is formed, since it is a partial equilibrium network by assumption, there is no incentive for *i* to deviate and sever any of her link either.

2. For any non-singleton agent j in r(θ_{l'}): given J, j's maximum expected loss in this phase and before r(θ_{l'}) is formed in the exploitation phase is bounded above regardless of γ. By the definition of r, j's payoff in r(θ_{l'}) is positive (and bounded away from zero, by the same argument as above), which realizes every period after r(θ_{l'}) is formed in the exploitation phase. Hence, given a sufficiently large γ, j does not have the incentive to deviate in this phase and become a solitary agent.

For any non-solitary agent in an admissible set I' in the experimentation phase: we need to discuss two cases. Following the proof of Theorem 7, we argue as follows:

- 1. For any singleton agent *i* in *r*(θ_{I'}): *i*'s maximum expected loss in this phase is bounded above regardless of γ since according to the strategy profile, information becomes complete within finitely many periods almost surely. Then with a similar argument to part (1) above, given a sufficiently large *J* and a sufficiently large γ, *i* does not have the incentive to deviate and become a solitary agent.
- 2. For any non-singleton agent j in $r(\bar{\theta}_{I'})$: again, j's maximum expected loss in this phase is bounded above regardless of γ . Then with a similar argument to part (2) above, given a sufficiently large γ , j does not have the incentive to deviate and become a solitary agent.

This completes the proof. Note that there needs to be an upper bound on *J* in generic cases: given γ , the larger *J* gets, the less incentive a non-solitary and non-singleton agent in $r(\bar{\theta}_{I'})$ may have for following the prescribed strategy in the transition phase.

The reason why we are not able to show strong convergence directly relates to incomplete information. Under complete information, agents' beliefs on the type vector stay constant (on the true types) over time despite the possibly changing public signals, which guarantees unanimous knowledge on the payoff structure. Under incomplete information, however, the beliefs can be heterogeneous and can evolve over time according to the realization of public

signals. The evolution of beliefs in turn leads to each agent forming a belief on others' beliefs, and hence it is difficult for them to agree on cooperation towards one network topology. No matter how precise the public signal is (the least precise being one constant signal, and the most precise being equal to the formation history), this potential obstacle to coordination exists as long as there is incomplete information on the type vector among the agents.

Similar to Corollary 1, we obtain a result on sustaining a strongly efficient network in equilibrium.

Corollary 4. Assume that I is admissible. If $r(\bar{\theta})$ is strongly efficient for any $\bar{\theta}$, there exists $\bar{\gamma} \in (0,1)$ such that for all $\gamma \in [\bar{\gamma},1)$, there exists an equilibrium where the formation process always converges to a strongly efficient network weakly.

As for robustness, we obtain a curious result about independent trembles in action. With incomplete information, accidental formation of links has additional merit compared with complete information. Apart from the payoff flow created, it also enables information transmission in the sense that at least the two agents directly involved would know each other's type from the link. In our result, we consider the cases of trembles in finite periods and in infinite periods. In the former situation, as long as the ultimate network topology is concerned, there is always the same equilibrium with weak convergence as with no tremble – the agents can simply ignore anything but the revealed information on types during the time periods with trembles. In the latter situation, the difference between incomplete information and complete information vanishes, because almost surely information will become complete due to accidental link formation. We state the formal result below.

Suppose that each agent *trembles* with independent probability ε in the first \hat{T} periods: whenever she is selected, with probability ε she chooses action 0 and 1 with equal probability. We have the following corollary:

Corollary 5. Regardless of \hat{T} , if I is admissible, then for sufficiently large γ there exists an equilibrium where the formation process converges to $r(\bar{\theta})$ weakly. When $\hat{T} = \infty$, for sufficiently large γ there exists an equilibrium where information becomes complete almost surely, after which the agents play the strategy profile $\hat{s}_c(\boldsymbol{g}, K)$.

3.5.4 Incomplete Information and Inefficiency

Comparing Theorem 8 with Theorem 7, we may conclude that incomplete information hinders the possibility of sustaining a strongly efficient network in equilibrium. In this section, we provide a direct comparison between complete and incomplete information based on the same payoff structure, which confirms that incomplete information is indeed a factor causing potential inefficiency.

Consider the following example: $\Theta = \{\alpha, \beta\}$, and the payoff from any network is described as follows: the only network in which an agent can get a positive payoff is some connected network \mathbf{g}^* . In \mathbf{g}^* , each agent gets payoff A > 0 if at least half of the agents are of type α , and 0 otherwise. Regardless of the network topology, if an agent is connected in any period, she pays a cost c < A. The prior distribution of types is (p, 1 - p) with $p \in (0, 1)$.

Proposition 15. Let γ_C and γ_{IC} be the infimum of γ such that for all higher γ there exists an equilibrium where the formation process always converges weakly to the strongly efficient network for any $\overline{\theta}$, under complete and incomplete information respectively. We have $\gamma_C \leq \gamma_{IC}$.

Proof. Let \mathscr{P} denote the stochastic process of network topology in each period, given the following strategy profile: for selected agents i, j, they choose $a_{ij} = a_{ji} = 1$ if $ij \in \mathbf{g}^*$, and $a_{ij} = a_{ji} = 0$ otherwise. \mathscr{P} denotes the stochastic process that the network topology will reach \mathbf{g}^* in the shortest amount of time.

Under incomplete information, consider any equilibrium such that the formation process always converges to the strongly efficient network and that at least half of the agents are of type α . Let $\overline{B} \leq 1$ be the highest possible belief on equilibrium path for any agent that at least half of the agents are of type α . Let $n^* = |\mathbf{g}^*|$, in equilibrium there must be some link *ij* that is maintained for at least $n^* - 1$ periods before \mathbf{g}^* is formed for the first time. Consider agent *i*. For *i* to form *ij* for the first time in equilibrium, it must be the case that

$$\sum_{t=n^*}^{\infty} \gamma^{t-1} \bar{B} \mathscr{P}(\mathbf{g}(t) = \mathbf{g}^*) A - \sum_{t=1}^{\infty} \gamma^{t-1} c \ge 0$$

The left-hand side is the largest possible expected payoff that *i* gets from following her equilibrium action, and the right-hand side is the payoff from one feasible way of deviation (staying singleton thereafter). Let γ' be the infimum of γ such that for all higher γ the above inequality is satisfied. We know that $\gamma_{IC} \geq \gamma'$.

Now it suffices to show that for any $\gamma > \gamma'$, there exists an equilibrium under complete information where the formation process always converges to the strongly efficient network. Consider the following strategy profile:

- If at least half of the agents are of type α, the agents start with cooperation: for selected agents *i*, *j*, they choose a_{ij} = a_{ji} = 1 if ij ∈ g^{*}, and a_{ij} = a_{ji} = 0 otherwise.
- 2. Upon observing any deviation, every agent chooses action 0 thereafter.
- 3. If less than half of the agents are of type α , each agent always chooses action 0.

It is clear that this strategy profile constitutes an equilibrium when less than half of the agents are of type α , regardless of γ . The formation process converges to the empty network which is also the strongly efficient network. When at least half of the agents are of type α , upon deviation it is clear that the above strategy profile constitutes an equilibrium in the following subgame as well. During cooperation, consider the first time a pair of agents (i, j) such that $ij \in \mathbf{g}^*$ is selected. For *i* to follow the above strategy profile, it must be the case that

$$\sum_{t=n^*}^{\infty} \gamma^{t-1} \mathscr{P}(\mathbf{g}(t) = \mathbf{g}^*) A - \sum_{t=1}^{\infty} \gamma^{t-1} c \ge 0$$

Since $\overline{B} \leq 1$, this inequality is satisfied for any γ that satisfies the previous inequality. Therefore whenever $\gamma > \gamma'$, the above strategy profile constitutes an equilibrium where the formation process converges to the strongly efficient network. We can then conclude that $\gamma_C \leq \gamma' \leq \gamma_{IC}$.

Under incomplete information, an agent is unsure whether at least half of the agents are of type α , i.e. whether she will get a positive payoff even if all the agents cooperate to form the connected network \mathbf{g}^* . Such uncertainty diminishes her incentive to form new links which incur current and certain cost. As a result, the patience level required under incomplete information for always sustaining the strongly efficient network becomes higher than that under complete information.

3.6 Connections Model

So far we have shown that as long as a strongly efficient network secures a positive payoff for every agent, it can be sustained in an equilibrium with an appropriate public signal device. To evaluate whether agents are indeed cooperating for efficiency in practical situations, the question that remains is what type of topology the strongly efficient network belongs to for different type vectors. For a general payoff structure, the full characterization of strongly efficient network is extremely complicated, if not impossible, due to the plethora of possible allocation rules of payoff based on topologies. However, we are able to provide a full characterization for one widely studied class of payoff structure – the connections model first introduced by Jackson and Wolinsky[JW96].

In the original model, agents are homogeneous and the strongly efficient network is either empty, a star or a clique. Our characterization is the first to consider heterogeneous agents and we find a richer set of strongly efficient networks. As mentioned in the introduction, our findings suggest that the observed network topologies in real life are considerably close to the strongly efficient ones. This corroborates the previous theoretical result that efficient network can be sustained in generic cases.

In the analysis that follows, we introduce a two-type environment to avoid technical redundancy. We will also demonstrate how the analysis can be extended to multiple types.

3.6.1 Characterization of Strongly Efficient Network

Assume that each agent can be one of two types, α or β . Let n_{α}, n_{β} denote the number of type α agents and that of type β agents correspondingly. Let \mathbf{g}^{e} denote the strongly efficient network.

Theorem 9. g^e can be described as follows:

- a. If $(1 \delta)f(\beta) > c$, g^e is a clique encompassing every agent.
- b. If (1-δ) f(α)+f(β)/2 > c > (1-δ)f(β), g^e is such that every two type α agents are linked, and every type α agent is linked with every type β agent, but no type β agent is linked with another type β agent.
- c. If (1-δ)f(α) > c > (1-δ) f(α)+f(β)/2, and (1+δ(nα-1))f(α) + (1+δ(nα + nβ 2))f(β) > 2c, g^e is such that every two type α agents are linked, and every type β agent is linked with the same type α agent, but no type β agent is linked with another type β agent.

- *d.* If $(1-\delta)f(\alpha) > c > (1-\delta)\frac{f(\alpha)+f(\beta)}{2}$, and $(1+\delta(n_{\alpha}-1))f(\alpha) + (1+\delta(n_{\alpha}+n_{\beta}-2))f(\beta) < 2c$, g^e is a clique encompassing every type α agent but no type β agent.
- *e*. If $(1 \delta)f(\alpha) < c$, $f(\alpha) + f(\beta) + \delta[(n_{\beta} 1)f(\beta) + (n_{\alpha} 1)(f(\alpha) + f(\beta))] > 2c$, and

$$\begin{split} &2(n_{\alpha}-1)f(\alpha)+n_{\beta}(f(\alpha)+f(\beta))+\delta[(n_{\alpha}-1)(n_{\alpha}-2)f(\alpha)+n_{\beta}(n_{\beta}-1)f(\beta)\\ &+n_{\beta}(n_{\alpha}-1)(f(\alpha)+f(\beta))]-2(n_{\alpha}+n_{\beta}-1)c>0, \end{split}$$

 g^e is a star encompassing every agent, with a type α agent as the center.

• f. If $(1 - \delta) f(\alpha) < c$, and

$$(1+\delta(n_{\alpha}-1))f(\alpha)+(1+\delta(n_{\alpha}+n_{\beta}-2))f(\beta)<2c< f(\alpha)(2+\delta(n_{\alpha}-2)),$$

 g^e is a star encompassing every type α agent but no type β agent.

• g. If
$$(1 - \delta) f(\alpha) < c$$
, and

$$\max\{2(n_{\alpha}-1)f(\alpha) + n_{\beta}(f(\alpha) + f(\beta)) + \delta[(n_{\alpha}-1)(n_{\alpha}-2)f(\alpha) + n_{\beta}(n_{\beta}-1)f(\beta) + n_{\beta}(n_{\alpha}-1)(f(\alpha) + f(\beta))] - 2(n_{\alpha}+n_{\beta}-1)c,$$
$$f(\alpha)(2 + \delta(n_{\alpha}-2)) - 2c\} < 0,$$

 g^e is the empty network.

Proof. (a) The result is clear since $(1 - \delta)f(\beta) > c$ implies that the benefit from any link (bounded below by $2(1 - \delta)f(\beta)$) is greater than the associated cost (2c).

(b) If $(1-\delta)\frac{f(\alpha)+f(\beta)}{2} > c$, a link between two type α agents or one type α agent and one type β agent always increases the total payoff in the network. Given that there is a link between these agents, $c > (1-\delta)f(\beta)$ implies that a link between two type β agents always decreases the total payoff in the network. Hence, \mathbf{g}^e is as described in the result.

(c) The first condition implies that any pair of type α agents are linked in \mathbf{g}^e . Furthermore, for *n* type β agents with m_1, m_2, \dots, m_n links respectively, the largest possible contribution to total payoff is

$$\sum_{k=1}^{n} [m_k(f(\alpha) + f(\beta)) + \delta((n_\alpha - m_k)(f(\alpha) + f(\beta)) + (n_\beta - 1)2f(\beta)) - 2m_k c].$$

Given the condition $c > (1 - \delta) \frac{f(\alpha) + f(\beta)}{2}$, this value is maximized at $m_k = 1$ for $k = 1, 2, \dots, n$. This upper bound is reached when all n type β agents are linked to the same type α agent. It is not difficult to see that if the contribution by n type β agents is positive, then that by n + 1 connected type β agents is also positive and larger. Hence, in \mathbf{g}^e , either no type β agent is connected, or every type β agent is linked to the same type α agent. The condition $(1 + \delta(n_\alpha - 1))f(\alpha) + (1 + \delta(n_\alpha + n_\beta - 2))f(\beta) > 2c$ implies that the contribution by n_β type β agents is positive, and hence \mathbf{g}^e is as described in the result.

- (d) It follows from the proof of (c).
- (e) First, we establish the following lemma.

Lemma 10. Any strongly efficient network has at most one non-empty component.

Proof. Suppose that there exists some strongly efficient network that has two non-empty components C_1 and C_2 . For component m (m = 1, 2), let i_m denote (one of) the agent who has the highest payoff in component C_m . Since the network is strongly efficient, we know that this payoff is non-negative for m = 1, 2. Let D_m be the set of links that i_m has in component C_m . For any $i_m j \in D_m$ and any $D'_m \subset D_m - i_m j$, define

$$\Delta u_m(i_m j, D'_m) = u_{i_m}(\bar{\theta}_{-i_m}, C_m \setminus D_m \cup D'_m) - u_{i_m}(\bar{\theta}_{-i_m}, C_m \setminus D_m \cup D'_m - i_m j).$$

This term denotes the marginal payoff of *i* from link $i_m j$ given D'_m . It is not difficult to see that for $D''_m \subset D'_m$, $\Delta u_m(i_m j, D''_m) \ge \Delta u_m(i_m j, D'_m)$. Since agent i_m 's payoff is non-negative, it follows that there must exist j_m such that $\Delta u_m(i_m j, \emptyset) \ge 0$.

Consider the network $C_1 \cup C_2 + j_1 j_2$. For j_1 , the marginal payoff from link $j_1 j_2$ is strictly larger than $\Delta u_2(i_2 j_2, \emptyset)$; similarly for j_2 , the marginal payoff from link $j_1 j_2$ is strictly larger than $\Delta u_1(i_1 j_1, \emptyset)$. Hence, the total payoff in this network is strictly higher than that in the original network $C_1 \cup C_2$, which contradicts the assumption that $C_1 \cup C_2$ is strongly efficient.

We then show that if $(1 - \delta)f(\alpha) < c$, for any non-empty component **g**, the total payoff in this component is weakly less than that in a star component with a type α agent as the center, denoted **g**^{*}. Let k_{α}, k_{β} denote the number of type α and type β agents respectively. Let $k_{\alpha\alpha}^d, k_{\alpha\beta}^d, k_{\beta\beta}^d$ denote the number of links between two type α agents, one type α agent and one type β agent, and two type β agents respectively. Let $k_{\alpha\alpha}^{ind}, k_{\alpha\beta}^{ind}, k_{\beta\beta}^{ind}$ denote the number of shortest indirect paths between two agents in the same three cases above. Let $v(\mathbf{g})$ and $v(\mathbf{g}^*)$ denote the total payoff in the original component and the star component respectively. Note that the length of any indirect path is at least 2, and thus we have

$$\begin{split} \mathbf{v}(\mathbf{g}) \leq & (k_{\alpha\alpha}^d + \delta k_{\alpha\alpha}^{ind}) 2f(\alpha) + (k_{\alpha\beta}^d + \delta k_{\alpha\beta}^{ind})(f(\alpha) + f(\beta)) + (k_{\beta\beta}^d + \delta k_{\beta\beta}^{ind}) 2f(\beta) \\ & - 2(k_{\alpha\alpha}^d + k_{\alpha\beta}^d + k_{\beta\beta}^d)c \\ & \mathbf{v}(\mathbf{g}^*) = & (k_{\alpha} - 1 + \delta \frac{(k_{\alpha} - 1)(k_{\alpha} - 2)}{2}) 2f(\alpha) + (k_{\beta} + \delta k_{\beta}(k_{\alpha} - 1))(f(\alpha) + f(\beta)) \\ & + \delta \frac{k_{\beta}(k_{\beta} - 1)}{2} 2f(\beta) - 2(k_{\alpha} + k_{\beta} - 1)c. \end{split}$$

Note that

$$\begin{aligned} k^{d}_{\alpha\alpha} + k^{ind}_{\alpha\alpha} &= \frac{k_{\alpha}(k_{\alpha}-1)}{2} \\ k^{d}_{\alpha\beta} + k^{ind}_{\alpha\beta} &= k_{\alpha}k_{\beta} \\ k^{d}_{\beta\beta} + k^{ind}_{\beta\beta} &= \frac{k_{\beta}(k_{\beta}-1)}{2}. \end{aligned}$$

Then we have

$$\mathbf{v}(\mathbf{g}) - \mathbf{v}(\mathbf{g}^*) \leq (1 - \delta)((k_{\alpha\alpha}^d - (k_{\alpha} - 1))2f(\alpha) + (k_{\alpha\beta}^d - k_{\beta})(f(\alpha) + f(\beta))) + k_{\beta\beta}^d 2f(\beta)) - 2(k_{\alpha\alpha}^d + k_{\alpha\beta}^d + k_{\beta\beta}^d - (k_{\alpha} + k_{\beta} - 1))c \leq 2(k_{\alpha\alpha}^d + k_{\alpha\beta}^d + k_{\beta\beta}^d - (k_{\alpha} + k_{\beta} - 1))((1 - \delta)f(\alpha) - c) \leq 0.$$

Finally, note that equality is achieved if and only if (1) $k_{\alpha\beta}^d = k_\beta$, $k_{\beta\beta}^d = 0$, $k_{\alpha\alpha}^d + k_{\alpha\beta}^d + k_{\beta\beta}^d = k_\alpha + k_\beta - 1$ and (2) there is no (shortest) indirect path with length greater than two between any two type α agents. These two conditions are satisfied if and only if **g** is also a star component with a type α agent as the center.

It is clear that if a star component with a type α agent as the center results in a positive total payoff, then adding an agent of type α as the periphery will increase the total payoff; moreover, if the marginal payoff brought about by all the type β agents in the star component is positive, then adding an agent of type β as the periphery will increase the total payoff. Hence, \mathbf{g}^e can only be one of the following three: (1) the empty network, (2) a star encompassing only type α agents, and (3) a star encompassing all agents, with a type α agent as the center. The second

and the third conditions in the result ensure that the marginal payoff brought about by all the type β agents is positive, and that (3) has a positive total payoff. Hence, (3) is g^e .

- (f) It follows from the proof of (e).
- (g) It follows from the proof of (e).

The characterization can be extended to an environment with more than two types. Consider the case with $q \ge 2$ types, and label the types according to value (high to low) as $\theta^1, \dots, \theta^q$, with θ^1 being of the highest value. Without loss of generality, denote agent 1 as a type 1 agent. Define the following terms:

- a. $q_1 = \max\{q' : (1 \delta) \frac{f(\theta^{q'}) + f(\theta^{q''})}{2} > c \text{ for some } q''\}$. If such q_1 does not exist, denote $q_1 = 0$.
- b. Assume that agents of type $\theta^1, \dots, \theta^{q_1}$ form a clique, and assume that all the type $\theta^{q_1+1}, \dots, \theta^{q'}$ agents link to agent 1.

 $q_2 = \max\{q' > q_1: \text{ the marginal benefit from type } \theta^{q_1+1}, \cdots, \theta^{q'} \text{ agents is positive}\}.$

If such q_2 does not exist, denote $q_2 = q_1$.

Connect the agents according to types in the following way:

• A. $\theta^1, \dots, \theta^{q_1}$: every agent of type q' and every agent of type q'' link if

$$(1-\delta)\frac{f(\theta^{q'})+f(\theta^{q''})}{2} > c.$$

- B. $\theta^{q_1+1}, \cdots, \theta^{q_2}$: link to agent 1.
- C. $\theta^{q_2+1}, \dots, \theta^q$: remain disconnected.

Despite the lengthy conditions on the payoffs from different types, the underlying argument for the above characterization is straight forward. According to how much benefit each type can provide via connection, we can categorize types in a systematic way and assign linkage correspondingly to maximize the sum of total payoffs. For the highest types among which a direct link always brings higher benefit than maintenance cost, a clique must be formed among them in any strongly efficient network. The next class contains the types for which any single
link to one of the highest types is socially beneficial, but links among themselves are not. As a result, these types will not link directly to themselves but will form every possible link to the first category. These two categories constitute (A) in the list above. In (B), when the benefit from a type gets even lower, such types can only add to social welfare by having only one link to one agent of the strictly highest type, thus minimizing the cost and receiving/providing most of the benefit via indirect connection. The last class (C) consists of the lowest types, which cannot increase the social welfare in any way and will remain singletons in a strongly efficient network.

The connected component of the efficient network exhibits a "core-periphery" pattern in topology (illustrated in Figure 3 below). The core, which corresponds to (A), consists of agents with high connectivity degree and relatively large clustering coefficient. The periphery agents in (B) have only one link each and all their links are formed with the same agent in (A) so that the length of indirect paths can be minimized. As the number of agent grows, classes (B) and (C) tend to contain less types because a larger number of agents implies higher indirect payoffs created via connection, and hence more types will be considered beneficial to the whole society by remaining connected. Overall distinguishing features of the strongly efficient network, especially when the number of agents gets large, include a small diameter, a large ratio between number of links and number of agents. All of these features have been observed in available data.

3.6.2 Strong Efficiency and Stability

The connections model also allows us to investigate the relation between a strongly efficient network and a stable network. Note that these two concepts do not imply one another. In a strongly efficient network, there may be a subgroup of agents that can improve the payoff of each member by "local autonomy": for instance, the center of a star will be strictly better of staying a singleton if all periphery agents are of low type. On the other hand, even though the agents cannot improve everyone's payoff in a stable network, there may be a way to improve the total payoff. An example would be to switch the center of a star from a low type agent to a high type one.



Figure 3.3: Strongly efficient network in connections model

In this section, we establish necessary and sufficient conditions for a strongly efficient network to be stable. It is helpful to first compute the largest possible one-period payoff an agent of each type can get in any network.

Proposition 16. Let $V(\alpha)$, $V(\beta)$ denote the maximum payoff that an agent of the corresponding type can obtain in any network in a single period. We have:

• a. If $(1 - \delta)f(\beta) > c$, then

$$V(\alpha) = (n_{\alpha} - 1)f(\alpha) + n_{\beta}f(\beta) - (n_{\alpha} + n_{\beta} - 1)c$$
$$V(\beta) = n_{\alpha}f(\alpha) + (n_{\beta} - 1)f(\beta) - (n_{\alpha} + n_{\beta} - 1)c.$$

• b. If $(1 - \delta)f(\alpha) > c > (1 - \delta)f(\beta)$, then

$$V(\alpha) = (n_{\alpha} - 1)f(\alpha) + \delta n_{\beta}f(\beta) - (n_{\alpha} - 1)c$$
$$V(\beta) = n_{\alpha}f(\alpha) + \delta(n_{\beta} - 1)f(\beta) - n_{\alpha}c.$$

• c. If $\min\{f(\alpha) + \delta((n_{\alpha}-2)f(\alpha) + n_{\beta}f(\beta)), f(\alpha) + \delta((n_{\alpha}-1)f(\alpha) + (n_{\beta}-1)f(\beta))\} > c > (1-\delta)f(\alpha)$, then

$$V(\alpha) = f(\alpha) + \delta((n_{\alpha} - 2)f(\alpha) + n_{\beta}f(\beta)) - c$$
$$V(\beta) = f(\alpha) + \delta((n_{\alpha} - 1)f(\alpha) + (n_{\beta} - 1)f(\beta)) - c$$

• *d. If*

$$f(\alpha) + \delta((n_{\alpha} - 1)f(\alpha) + (n_{\beta} - 1)f(\beta)) > c$$

> max{ $f(\alpha) + \delta((n_{\alpha} - 2)f(\alpha) + n_{\beta}f(\beta)), (1 - \delta)f(\alpha)$ },

then

$$V(\alpha) = 0$$
$$V(\beta) = f(\alpha) + \delta((n_{\alpha} - 1)f(\alpha) + (n_{\beta} - 1)f(\beta)) - c.$$

• e. If
$$\max\{f(\alpha) + \delta((n_{\alpha}-1)f(\alpha) + (n_{\beta}-1)f(\beta)), f(\alpha) + \delta((n_{\alpha}-2)f(\alpha) + n_{\beta}f(\beta)), (1-\delta)f(\alpha)\} < c$$
, then

$$V(\boldsymbol{\alpha}) = 0$$
$$V(\boldsymbol{\beta}) = 0.$$

Proof. (a) Let $\hat{V}(\theta, m)$ be the largest possible payoff of an agent of type θ with m links. We have

$$\begin{split} \hat{V}(\alpha,m) &= \\ 0, \text{ if } m = 0 \\ mf(\alpha) + \delta((n_{\alpha} - 1 - m)f(\alpha) + n_{\beta}f(\beta)) - mc, \text{ if } 0 < m \le n_{\alpha} - 1 \\ (n_{\alpha} - 1)f(\alpha) + (m - (n_{\alpha} - 1))f(\beta) + \delta(n_{\alpha} + n_{\beta} - 1 - m)f(\beta) - mc, \text{ otherwise} \\ 0, \text{ if } m = 0 \\ 0, \text{ if } m = 0 \\ mf(\alpha) + \delta((n_{\alpha} - m)f(\alpha) + n_{\beta}f(\beta)) - mc, \text{ if } m \le n_{\alpha} \\ n_{\alpha}f(\alpha) + (m - n_{\alpha})f(\beta) + \delta(n_{\alpha} + n_{\beta} - 1 - m)f(\beta) - mc, \text{ otherwise.} \end{split}$$

It is clear that these largest possible payoffs are achievable (for instance, let the agent be a periphery agent in a star with a type α agent as the center, and form the rest of the *m* links first with type α agents, then with type β agents if she is already linked with every type α agent.). If $(1 - \delta)f(\beta) > c$, for an agent of either type her payoff is maximized when she makes every possible link, hence $V(\alpha)$ and $V(\beta)$ are as shown in the result.

(b) If $(1 - \delta)f(\alpha) > c > (1 - \delta)f(\beta)$, for an agent of either type her payoff is maximized when she links with every type α agent but with no type β agent, hence $V(\alpha)$ and $V(\beta)$ are as shown in the result.

(c) If $c > (1 - \delta)f(\alpha)$, given that an agent is connected (not a singleton), her largest payoff is higher when she has fewer links. Hence, her payoff is $\hat{V}(\theta, 1)$ if $\hat{V}(\theta, 1) \ge 0$ and 0 otherwise. Hence $V(\alpha)$ and $V(\beta)$ are as shown in the result.

- (d) It follows from the proof of (c).
- (e) It follows from the proof of (c). \Box

We can perceive from the proof above that the largest payoff an agent obtains from a network is closely related to the network topology. Hence, if a network offers the largest possible payoff to most of the agents, it is very likely to be stable since those agents' payoffs cannot be improved further. The final criterion of stability then rests on whether the few agents that do not get the highest possible payoff can form a beneficial coalition. Using this argument, we inspect the strongly efficient network in every possible type vector and present the result below.

Proposition 17. Consider cases (a) - (g) in Theorem 9. We have:

- $a, d, g. g^e$ is stable.
- *b.* g^e is stable if and only if $f(\beta) \ge c$.
- c. g^e is stable if and only if the type α agent linking with all the type β agents has a non-negative payoff.
- *e.* g^e is stable if and only if the type α agent at the center has a non-negative payoff.
- *f.* g^e is stable if and only if $f(\alpha) \ge c$.

Proof. (a)(d)(g) The cases (a) and (g) are clear. For (d), suppose that \mathbf{g}^e is not stable, it then follows that any blocking group I' (with network \mathbf{g}') must contain at least one type α agent, but not all type α agents. Consider the network \mathbf{g}' that blocks \mathbf{g}^e . Since every agent in I' has a weakly higher payoff in \mathbf{g}' than in \mathbf{g}^e and some agent in I' has a strictly higher payoff in \mathbf{g}' than in \mathbf{g}^e and some agent in I' has a strictly higher payoff in \mathbf{g}' than in \mathbf{g}^e , the total payoff of agents in I' must be strictly higher than that in \mathbf{g}^e . By Theorem 9, we can re-organize \mathbf{g}' into a clique with only type α agents to yield an even higher total payoff. However, if a clique with only type α agents has a positive total payoff, then it is impossible for a proper subset of these agents to form a clique with a higher total payoff than their total payoff in the original clique. Hence we have a contradiction.

(b) If $f(\beta) < c$, then \mathbf{g}^e is not stable because a clique formed by all type α agents blocks \mathbf{g}^e . Suppose that $f(\beta) \ge c$ and that \mathbf{g}^e is not stable, then any blocking group I' either only contains type α agents, or contains all the agents because each type β agent in \mathbf{g}^e gets payoff $V(\beta)$, and getting $V(\beta)$ requires connection to every other agent. Both cases contradict the fact that \mathbf{g}^e is strongly efficient.

(c) Let *i* denote the type α agent linking with all the type β agents. If *i* has a negative payoff, then \mathbf{g}^e is not stable because $\{i\}$ blocks \mathbf{g}^e . Suppose that *i* has a non-negative payoff and that \mathbf{g}^e is not stable, then any blocking group I' either only contains no type α agent other than *i* and at least one type β agent, or contains all the agents because each type α agent other than *i* in \mathbf{g}^e gets payoff $V(\alpha)$, and getting $V(\alpha)$ requires connection to every other agent. The second case contradicts the fact that \mathbf{g}^e is strongly efficient. In the first case, note that the largest payoff of any type β agent in I' is strictly less than $\hat{V}(\beta, 1)$ (since $c > (1 - \delta)f(\beta)$), which is the payoff of every type β agent in \mathbf{g}^e . Hence we again have a contradiction.

(e) Let j denote the type α agent at the center. If j has a negative payoff, then \mathbf{g}^e is not stable because $\{j\}$ blocks \mathbf{g}^e . Suppose that j has a non-negative payoff and that \mathbf{g}^e is not stable, then any blocking group I' either only contains j, or contains all the agents because each agent other than j in \mathbf{g}^e gets payoff $V(\alpha)$ (or $V(\beta)$), and getting $V(\alpha)$ (or $V(\beta)$) requires connection to every other agent. The first case contradicts the assumptions that I' blocks \mathbf{g}^e and j has a non-negative payoff in \mathbf{g}^e , and the second case contradicts the fact that \mathbf{g}^e is strongly efficient.

(f) If $f(\alpha) < c$, then \mathbf{g}^e is not stable because the center agent in \mathbf{g}^e blocks \mathbf{g}^e . Suppose that $f(\alpha) \ge c$ and that \mathbf{g}^e is not stable, it then follows that any blocking group I' (with network \mathbf{g}') must contain at least one type α agent, but not all type α agents. Consider the network \mathbf{g}' that blocks \mathbf{g}^e . Since every agent in I' has a weakly higher payoff in \mathbf{g}' than in \mathbf{g}^e and some agent in I' has a strictly higher payoff in \mathbf{g}' that in \mathbf{g}^e , the total payoff of agents in I' must be strictly higher than that in \mathbf{g}^e . By Theorem 9, we can re-organize \mathbf{g}' into a star with only type α agents to yield an even higher total payoff. However, if $f(\alpha) \ge c$, it is impossible for a proper subset of the type α agents to form a star with a higher total payoff than their total payoff in \mathbf{g}^e . Hence we have a contradiction.

As it turns out, the key to making a strongly efficient network stable is its "weakest link": the agent maintaining the most links but enjoying the least net payoff. Moreover, once this

particular agent gets a negative payoff, it does not even require a coalition for improvement: she can simply sever all her links and stay a singleton. It implies that to retain efficiency over time, instead of keeping track of agents' payoffs globally, it is more effective to focus locally and ensure that those with more responsibilities are sufficiently rewarded.

3.7 Conclusion

In this paper, we have studied the problem of dynamic network formation with heterogeneous and foresighted agents. A large and growing literature has examined the network formation process from various aspects, but the impact of agent heterogeneity and foresight has been understudied. The existing works point to a limited set of strongly efficient network topologies with homogeneous agents, and the inability to sustain strongly efficient networks in equilibrium. We question these results based on two grounds. On one hand, the assumption of agent homogeneity is hard to justify in most real-life applications. On the other hand, according to our characterization of strongly efficient networks with heterogeneous agents and observations in data collected from existing large networks, networks formed in practical scenarios appear very similar to our theoretical prediction. Therefore, we establish a dynamic network formation model to analyze the network formation process and explain our findings.

In our model, agents meet randomly over time and voluntarily form or sever links with each other. Link formation requires bilateral consent but severance is unilateral. An agent's payoff in a single period is determined by the network topology, her position in the network, and the individual characteristics, also referred to as types, of all agents she connects to (including herself). The agents are foresighted in the sense that their final payoff is a discounted sum of payoffs from each period. In every period, the agents observe the set of their direct neighbors (the agents they link to) and a public signal which is an indicator of the formation history.

We establish a Folk Theorem of networks under both complete and incomplete information on the type vector, which characterizes the set of sustainable networks in equilibrium for patient agents. Under each environment, we show that a network can be sustained in equilibrium as long as it provides each agent a positive payoff. As a corollary, a strongly efficient network is sustainable when every agent's payoff is positive, which presents a great contrast to the existing literature. The difference between the two information structures is that in the corresponding equilibria we construct, the formation process converges to the designated network regardless of formation history under complete information, but only does so on the equilibrium path under incomplete information. We hence argue that incomplete information is an important potential source of inefficiency, which is corroborated by evaluating the lower bound on agents' patience to sustain strongly efficient networks. Finally, we use the connections model to fully characterize the set of strongly efficient networks, whose topologies bear striking resemblance to networks observed in data. This finding again confirms our theoretical prediction that strongly efficient networks can be sustained in equilibrium.

We believe that many more problems regarding dynamic network formation with foresightedness can be analyzed with the framework developed in this paper. Questions that can be studied in future work include: (1) how different stochastic meeting processes affect the level of patience needed for sustaining efficient networks; (2) whether the signal device can be generalized to transmit information only locally; (3) in a connections model, how the spatial discount factor affects the set of sustainable networks and the stability of efficient networks.

REFERENCES

- [ADL11] Daron Acemoglu, Munther A. Dahleh, Ilan Lobel, and Asuman Ozdaglar. "Bayesian Learning in Social Networks." *Review of Economic Studies*, 78(4):1201– 1236, 2011.
- [Ali14] S. Nageeb Ali. "Social Learning with Endogenous Information." *Working Paper*, 2014.
- [BA99] Albert-Laszlo Barabasi and Reka Albert. "Emergence of Scaling in Random Networks." *Science*, **286**:509–512, 1999.
- [Ban92] Abhijit V. Banerjee. "A Simple Model of Herd Behavior." *The Quarterly Journal* of Economics, **107**(3):797–817, 1992.
- [Ban93] Abhijit V. Banerjee. "The Economics of Rumours." *The Review of Economic Studies*, **60**(2):309–327, 1993.
- [BCZ06] Coralio Ballester, Antoni Calvo-Armengol, and Yves Zenou. "Who's Who in Networks. Wanted: The Key Player." *Econometrica*, **74**(5):1403–1417, 2006.
- [BF04] Abhijit Banerjee and Drew Fudenberg. "Word-of-mouth Learning." *Games and Economic Behavior*, **46**(1):1–22, 2004.
- [BG98] Venkatesh Bala and Sanjeev Goyal. "Learning from Neighbours." *The Review of Economic Studies*, **65**(3):595–621, 1998.
- [BG00] Venkatesh Bala and Sanjeev Goyal. "A Noncoorperative Model of Network Formation." *Econometrica*, **68**(5):1181–1229, 2000.
- [BG01] Venkatesh Bala and Sanjeev Goyal. "Conformism and Diversity under Social Learning." *Economic Theory*, **17**(1):101–120, 2001.
- [BHW92] Sushil Bikhchandani, David Hirshleifer, and Ivo Welch. "A Theory of Fads, Fashion, Custom, and Cultural Change as Informational Cascades." *Journal of Political Economy*, **100**(5):992–1026, 1992.
- [BHW98] Sushil Bikhchandani, David Hirshleifer, and Ivo Welch. "Learning from the Behavior of Others: Conformity, Fads, and Informational Cascades." *Journal of Economic Perspectives*, 12(3):151–170, 1998.
- [CCF09] Hongbin Cai, Yuyu Chen, and Hanming Fang. "Observational Learning: Evidence from a Randomized Natural Field Experiment." *American Economic Review*, 99(3):864–882, 2009.
- [CD08] Dean Corbae and John Duffy. "Experiments with Network Formation." *Games and Economic Behavior*, **64**(1):81–120, 2008.
- [Cel08] Bogachan Celen. "Does Costly Information Acquisition Improve Welfare in a Social Learning Environment?" *Working Paper*, 2008.
- [CG94] Christophe Chamley and Douglas Gale. "Information Revelation and Strategic Delay in a Model of Investment." *Econometrica*, **62**(5):1065–1085, 1994.

- [CH09] Steven Callander and Johannes Horner. "The Wisdom of the Minority." *Journal of Economic Theory*, **144**(4):1421–1439, 2009.
- [CK04] Bogachan Celen and Shachar Kariv. "Observational learning under imperfect information." *Games and Economic Behavior*, **47**(1):72–86, 2004.
- [CU01] Timothy Conley and Christopher Udry. "Social Learning through Networks: The Adoption of New Agricultural Technologies in Ghana." *American Journal of Agricultural Economics*, **83**(3):668–673, 2001.
- [Der03] Frederic Deroian. "Farsighted Strategies in the Formation of a Communication Network." *Economic Letters*, 80:343–349, 2003.
- [DGR05] Bhaskar Dutta, Sayantan Ghosal, and Debraj Ray. "Farsighted Network Formation." *Journal of Economic Theory*, **122**:143–164, 2005.
- [Dut95] Prajit K. Dutta. "A Folk Theorem for Stochastic Games." *Journal of Economic Theory*, **66**:1–32, 1995.
- [DVZ03] Peter M. DeMarzo, Dimitri Vayanos, and Jeffrey Zwiebel. "Persuasion Bias, Social Influence, and Unidimensional Opinions." *The Quarterly Journal of Economics*, 118(3):909–968, 2003.
- [EF93] Glenn Ellison and Drew Fudenberg. "Rules of Thumb for Social Learning." *Journal of Political Economy*, **101**(4):612–643, 1993.
- [EF95] Glenn Ellison and Drew Fudenberg. "Word-of-mouth Communication and Social Learning." *The Quarterly Journal of Economics*, **110**(1):93–125, 1995.
- [FFL13] Daniel Fricke, Karl Finger, and Thomas Lux. "On Assortative and Dissasortative Mixing in Scale-Free Networks: The Case of Interbank Credit Networks." *Kiel Working Paper*, p. 1830, 2013.
- [FK03] Armin Falk and Michael Kosfeld. "It's All About Connections: Evidence on Network Formation." *CEPR Discussion Papers*, (3970), 2003.
- [For12] Francoise Forges. "Folk Theorems for Bayesian (Public Good) Games." *Lecture Notes for PET International Meeting*, 2012.
- [Gal06] Andrea Galeotti. "One-Way Flow Networks: The Role of Heterogeneity." *Economic Theory*, **29**(1):163–179, 2006.
- [GG10] Andrea Galeotti and Sanjeev Goyal. "The Law of the Few." *The American Economic Review*, **100**(4):1468–1492, 2010.
- [GGK06] Andrea Galeotti, Sanjeev Goyal, and Jurjen Kamphorst. "Network Formation with Heterogeneous Players." *Games and Economic Behavior*, **54**:353–372, 2006.
- [GJ10] Benjamin Golub and Matthew O. Jackson. "Naive Learning in Social Networks and the Wisdom of Crowds." *American Economic Journal: Microeconomics*, **2**(1):112–149, 2010.
- [GK03] Douglas Gale and Shachar Kariv. "Bayesian Learning in Social Networks." *Games and Economic Behavior*, **45**(2):329–346, 2003.

- [GLM06] Sanjeev Goyal, Marco J. van der Leij, and Jose Luis Moraga-Gonzalez. "Economics: An Emerging Small World." *Journal of Political Economy*, 114(2):403– 412, 2006.
- [GRU09] Jacob K. Goeree, Arno Riedl, and Aljaz Ule. "In Search of Stars: Network Formation among Heterogeneous Agents." *Games and Economic Behavior*, 67(2):455– 466, 2009.
- [HMV09] Jean-Jacques Herings, Ana Mauleon, and Vincent Vannetelbosch. "Farsightedly Stable Networks." *Games and Economic Behavior*, **67**:526–541, 2009.
- [HS03] Hans Haller and Sudipta Sarangi. "Nash Networks with Heterogeneous Agents." *Discussion Papers of DIW Berlin*, p. 337, 2003.
- [HSW12] Kenneth Hendricks, Alan Sorensen, and Thomas Wiseman. "Observational Learning and Demand for Search Goods." American Economic Journal: Microeconomics, 4(1):1–31, 2012.
- [IL04] Yannis M. Ioannides and Linda Datcher Loury. "Job Information Networks, Neighborhood Effects, and Inequality." *Journal of Economic Literature*, 42(4):1056–1093, 2004.
- [JG00] Cathleen Johnson and Robert P. Gilles. "Spatial Social Networks." *Review of Economic Design*, **5**(3):273–299, 2000.
- [JW96] Matthew O. Jackson and Asher Wolinsky. "A Strategic Model of Social and Economic Networks." *Journal of Economic Theory*, **71**:44–74, 1996.
- [JW02] Matthew O. Jackson and Alison Watts. "The Evolution of Social and Economic Networks." *Journal of Economic Theory*, **106**:265–295, 2002.
- [KM01] Rachel E. Kranton and Deborah F. Minehart. "A Theory of Buyer-Seller Networks." *The American Economic Review*, **91**(3):485–508, 2001.
- [KM06] Klaus K. Kultti and Paavo A. Miettinen. "Herding with Costly Information." *International Game Theory Review*, **8**(2):21–33, 2006.
- [KM07] Klaus K. Kultti and Paavo A. Miettinen. "Herding with Costly Observation." *The B.E. Journal of Theoretical Economics*, **7**(1 (Topics)):Article 28, 2007.
- [KTZ14] Michael D. Konig, Claudio J. Tessone, and Yves Zenou. "Nestedness in Networks: A Theoretical Model and Some Applications." *Theoretical Economics*, 9(3):695– 752, 2014.
- [KW06] Gueorgi Kossinets and Duncan J. Watts. "Empirical Analysis of an Evolving Social Network." Science, 311(5757):88–90, 2006.
- [Lee93] In Ho Lee. "On the Convergence of Informational Cascades." *Journal of Economic Theory*, **61**(2):395–411, 1993.
- [Leu13] Michael Leung. "Two-Step Estimation of Network-Formation Models with Incomplete Information." *Working Paper*, 2013.

- [LK14] Jure Leskovec and Andrej Krevl. "SNAP Datasets: Stanford Large Network Dataset Collection." http://snap.stanford.edu/data, Jun 2014.
- [McB06] Michael McBride. "Imperfect Monitoring in Communication Networks." *Journal* of Economic Theory, **126**:97–119, 2006.
- [Mel10] Angelo Mele. "A Structural Model of Segregation in Social Networks." *NET Institute Working Papers*, (10-16), 2010.
- [MP14] Manuel Mueller-Frank and Mallesh M. Pai. "Social Learning with Costly Search." *Working Paper*, 2014.
- [Mun03] Kaivan Munshi. "Networks in the Modern Economy: Mexican Migrants in the U.S. Labor Market." *The Quarterly Journal of Economics*, **118**(2):549–597, 2003.
- [Mun04] Kaivan Munshi. "Social Learning in a Heterogeneous Population: Technology Diffusion in the Indian Green Revolution." *Journal of Development Economics*, **73**(1):185–213, 2004.
- [New02] Mark E. J. Newman. "Assortive Mixing in Networks." *Phisics Review Letters*, **89**(20):208701, 2002.
- [New03] Mark E. J. Newman. "Mixing Patterns in Networks." *Phisics Review E*, **67**(2):026126, 2003.
- [PWK03] Frank H. Page Jr., Myrna H. Wooders, and Samir Kamat. "Networks and Farsighted Stability." *Warwick Ecomomic Research Papers*, (689), 2003.
- [RH12] Rong Rong and Daniel Houser. "Emergent Star Networks with Ex Ante Homogeneous Agents." *GMU Working Paper in Economics*, (12-33), 2012.
- [SP00] Brian Skyrms and Robin Pemantle. "A Dynamic Model of Social Network Formation." *Proceedings of the National Academy of Sciences*, **97**(16):9340–9346, 2000.
- [SS00] Lones Smith and Peter Sorensen. "Pathological Outcomes of Observational Learning." *Econometrica*, **68**(2):371–398, 2000.
- [SS13] Lones Smith and Peter Sorensen. "Rational Social Learning with Random Sampling." *Working Paper*, 2013.
- [SS15] Yangbo Song and Mihaela van der Schaar. "Dynamic Network Formation with Incomplete Information." *Economic Theory*, pp. DOI: 10.1007/s00199–015–0858– y, 2015.
- [Tes98] Leigh Tesfatsion. "Gale-Shapley Matching in an Evolutionary Trade Network Game." *Iowa State University Economic Report NO. 43*, 1998.
- [Wat01] Alison Watts. "A Dynamic Model of Network Formation." *Games and Economic Behavior*, **34**:331–341, 2001.
- [WS98] Duncan J. Watts and Steven H. Strogatz. "Collective Dynamics of 'Small World' Networks." *Nature*, **393**:440–442, 1998.

- [ZS12] Yu Zhang and Mihaela van der Schaar. "Information Production and Link Formation in Social Computing Systems." *IEEE Journal on Selected Areas in Communications*, **30**(11):2136–2145, 2012.
- [ZS13] Yu Zhang and Mihaela van der Schaar. "Strategic Networks: Information Dissemination and Link Formation Among Self-interested Agents." *IEEE Journal on Selected Areas in Communications*, **31**(6):1115–1123, 2013.