Actions of Lie groups and Lie algebras on manifolds

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Dedicated to the memory of Raoul Bott

Contents

Introduction

Lie algebras were introduced by Sophus Lie under the name "infinitesimal group," meaning the germ of a finite dimensional, locally transitive Lie algebra of analytic vector fields in $\mathbf{R}^n$. In his 1880 paper *Theorie der Transformationsgruppen* [20, 14] and his later book with F. Engel [21], Lie classified infinitesimal groups acting in dimensions 1 and 2 up to analytic coordinate changes. This work stimulated much research, but attention soon shifted to the classification and representation of abstract Lie algebras and Lie groups. Later the topology of Lie groups was studied, with fundamental contributions by Bott.

In 1950 G. D. Mostow [23] completed Lie's program of classifying effective transitive surface actions.[1] One of his major results is:

[1]For each equivalence class of transitive surface actions, Mostow describes a representative Lie algebra by formulas for a basis of vector fields. Determining whether one of these representatives is isomorphic to a given Lie algebra can be nontrivial. Here the succinct summary of the classification in M. Belliart [4] is useful.

Theorem 1 (MOSTOW). *A surface M without boundary admits a transitive Lie group action if and only if M is a plane, sphere, cylinder, torus, projective plane, Möbius strip or Klein bottle.*

By a curious coincidence these are the only surfaces without boundary admitting nontrivial compact Lie group actions (folk theorem).

The following nontrivial extension of Theorem 1 deserves to be better known:

Theorem 2. *Let G be a Lie group and H a closed subgroup such that the manifold $M = G/H$ is compact. Then $\chi(M) \geq 0$, and if $\chi(M) > 0$ then M has finite fundamental group.*

This is due to Gorbatsevich *et al.* [11, Corollary 1, p. 174]. See also Felix *et al.* [10, Prop. 32.10], Halperin [12], Mostow [24].

While much is known about the topology of compact group actions, there has been comparatively little progress on classification of actions of Lie algebras and noncompact groups, an exception being D. Stowe's classification [28] of analytic actions of $SL(2, \mathbf{R})$ on compact surfaces. The present article addresses the easier tasks of deciding whether a group or algebra acts nontrivially on a given manifold, determining the possible smoothness of such actions, and investigating their orbit structure. Most proofs are omitted or merely outlined, with details to appear elsewhere.

The low state of current knowledge is illustrated by the lack of both counterexamples and proofs for the following

Conjectures. Let $\mathfrak{g}$ denote a real, finite dimensional Lie algebra.

   (C1): *If $\mathfrak{g}$ has effective actions on M^n, then $\mathfrak{g}$ also has smooth effective actions on M^n.*

   (C2): *If $\mathfrak{g}$ is semisimple and has effective smooth actions on M^n, $n \geq 2$, then $\mathfrak{g}$ also has effective analytic actions on M^n.*

But however plausible these statements may appear, they can't both be true:

   • (C1) *or* (C2) *is false for $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{R})$.*

For $\mathfrak{sl}(2, \mathbf{R})$ has effective actions on every M^2 (Theorem 7), but no effective analytic action if M^2 is compact with Euler characteristic $\chi(M^2) < 0$ (Corollary 16(b)).

It is unknown whether such a surface can support a smooth effective action β of $\mathfrak{sl}(2, \mathbf{R})$. If it does, Theorem 15(ii) implies that the vector fields X^β are infinitely flat at the fixed points of $\mathfrak{so}(2, \mathbf{R})^\beta$.

The analog of (C2) for nilpotent algebras is false. If $\mathfrak{n}$ denotes the Lie algebra of 3×3 niltriangular real matrices, by Theorem 3 and Example 13:

   • *On every connected surface $\mathfrak{n} \times \mathfrak{n}$ has effective C^∞ actions, but no effective analytic actions.*

Further conjectures and questions are given below.

Terminology. $\mathbf{F}$ stands for the real field $\mathbf{R}$, or the complex field $\mathbf{C}$. The complex conjugate of $\lambda := a + \imath b$ is $\bar{\lambda} := a - \imath b$. The sets of integers, positive integers and natural numbers are $\mathbf{Z}$, $\mathbf{N}_+ = \{1, 2, \dots\}$ and $\mathbf{N} = 0 \cup \mathbf{N}_+$ respectively. i, j, k, l, m, n, r denote natural numbers, assumed positive unless the contrary is indicated. $\lfloor s \rfloor$ denotes the largest integer $\leq s$.

M or M^n denotes an n-dimensional analytic manifold, perhaps with boundary; its tangent space at p is $T_p M$. $\mathfrak{v}^s(M)$ denotes the vector space of C^s vector fields

on M, with the weak C^s topology ($1 \leq s \leq \infty$). The Lie bracket makes $\mathfrak{v}^\infty$ a Lie algebra, with analytic vector fields forming a subalgebra. The value of $Y \in \mathfrak{v}^1(M)$ at $p \in M$ is Y_p. The derivative of Y at p is a linear operator on dY_p on T_pM.

Except as otherwise indicated, manifolds, Lie groups and Lie algebras are real and finite dimensional; manifolds and Lie groups are connected; and maps between manifolds are C^∞.

G denotes a Lie group with Lie algebra $\mathfrak{g}$ and universal covering group $\widetilde{G}$. The k-fold direct product $G \times \cdots \times G$ is G^k and similarly for $\mathfrak{g}$. $SL(m, \mathbf{F})$ is the group of $m \times m$ matrices over $\mathbf{F}$ of determinant 1, and $ST(m, \mathbf{F})$ is the subgroup of upper triangular matrices. The corresponding identity components and Lie algebras are denoted by $SL_\circ(m, \mathbf{F})$, $\mathfrak{st}(m, \mathbf{F})$ and so forth.

An *action* α of G on M, indicated by (α, G, M), is a homomorphism $g \mapsto g^\alpha$ from G to the group of homeomorphisms of M with a continuous *evaluation map* $\mathsf{ev}_\alpha \colon G \times M \to M$, $(g, x) \mapsto g^\alpha(x)$. We call α *smooth*, or *analytic*, when ev_α has the corresponding property.[2]

Small gothic letters denote linear subspaces of Lie algebras, with $\mathfrak{g}$ and $\mathfrak{h}$ reserved for Lie algebras. Recursively define $\mathfrak{g}^{(0)} = \mathfrak{g}$ and $\mathfrak{g}^{(j+1)} = \mathfrak{g}^{(j)}{}' = [\mathfrak{g}^{(j)}, \mathfrak{g}^{(j)}]$ = commutator ideal of $\mathfrak{g}^{(j)}$. Recall that $\mathfrak{g}$ (and also G) is *solvable* of *derived length* $l = \ell(\mathfrak{g}) = \ell(G)$ if $l \in \mathbf{N}_+$ is the smallest number satisfying $\mathfrak{g}^{(l)} = 0$. For example, $\ell(\mathfrak{st}(m, \mathbf{F})) = m$.

$\mathfrak{g}$ is *nilpotent* if there exists $k \in \mathbf{N}$ such that $\mathfrak{g}_{(k)} = \{0\}$, where $\mathfrak{g}_{(0)} = \mathfrak{g}$ and $\mathfrak{g}_{(j+1)} := [\mathfrak{g}, \mathfrak{g}_{(j)}]$. It is known that $\mathfrak{g}$ is solvable if and only $\mathfrak{g}'$ is nilpotent.

$\mathfrak{g}$ is *supersoluble* if the spectrum of $\mathsf{ad}\, X$ is real for all $X \in \mathfrak{g}$, where $\mathsf{ad} := \mathsf{ad}_\mathfrak{g}$ denotes the adjoint representation of $\mathfrak{g}$ on itself defined by $(\mathsf{ad}\, X)Y = [X, Y]$. Equivalently: $\mathfrak{g}$ is solvable and faithfully represented by upper triangular real matrices.

An *action* β of $\mathfrak{g}$ on M, recorded as $(\beta, \mathfrak{g}, M)$, is a continuous homomorphism $X \mapsto X^\beta$ from $\mathfrak{g}$ to $\mathfrak{v}^\infty(M)$. An *n-action* means an action on an n-dimensional manifold.

A smooth action (α, G, M) determines a smooth action $(\hat{\alpha}, \mathfrak{g}, M)$. Conversely, if G is simply connected and $(\beta, \mathfrak{g}, M)$ is such that each vector field X^β is complete (as when M is compact), then there exists (α, G, M) such that $\beta = \hat{\alpha}$.

The *orbit* of $p \in M$ under (α, G, M) is $\{g^\alpha(p) \colon g \in G\}$, and the orbit of p under a Lie algebra action $(\beta, \mathfrak{g}, M)$ is the union over $X \in \mathfrak{g}$ of the integral curves of p for X^β. An action is *transitive* if it has only one orbit.

The *fixed point set* of (α, G, M) is

$$\mathsf{Fix}(\alpha) = \{x \in M \colon g^\alpha(x) = x, \ (g \in G)\},$$

and that of $(\beta, \mathfrak{g}, M)$ is

$$\mathsf{Fix}(\beta) := \{p \in M \colon X_p^\beta = 0, \ (X \in \mathfrak{g})\}$$

The *support* of any action γ on M is the closure of $M \setminus \mathsf{Fix}(\gamma)$.

An action is *effective* if its kernel is trivial, and *nondegenerate* if the fixed point set of every nontrivial element has empty interior. Effective analytic actions are nondegenerate. A group action is *almost effective* if its kernel is discrete.

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[2]Most of the results here can be adapted to C^r actions and local actions

Construction of actions

Every G acts effectively and analytically on itself by translation. Every $\mathfrak{g}$ admits a faithful finite dimensional representation $\mathsf{R}\colon \mathfrak{g} \to \mathfrak{gl}(n,\mathbf{R})$ by Ado's theorem (Jacobson [19]). If $\mathsf{R}(\mathfrak{g})$ has trivial center, it induces effective analytic action by $\mathfrak{g}$ on the projective space $\mathbf{P}^{n-1}$ and the sphere $\mathbf{S}^{n-1}$.

An action gives rise to actions on other manifolds by blowing up invariant submanifolds in various ways; this preserves effectiveness and analyticity. Blowing up fixed points of standard actions of $ST_{\circ}(3,\mathbf{R})$ on $\mathbf{P}^2, \mathbf{S}^2$ and $\mathbf{D}^2$ yields:

Theorem 3. $ST_{\circ}(3,\mathbf{R})$ *has effective analytic actions on all compact surfaces.*
(F. Turiel [30])

Conjecture. $ST_{\circ}(3,\mathbf{R})$ *has effective analytic actions on all surfaces.*

Analytic approximation theory is used to prove:

Theorem 4. *The vector group* $\mathbf{R}^m$ *has effective analytic actions on* M^n *if* $m \geq 1, \ n \geq 2$. (M. Hirsch & J. Robbin [16])

On open manifolds it is comparatively easy to produce effective Lie algebra actions:

Theorem 5. *Assume there is an effective action* $(\alpha, \mathfrak{g}, W^n)$. *Then a noncompact* M^n *admits an effective action* $(\beta, \mathfrak{g}, M^n)$ *in the following cases:*

(a): M^n *is parallelizable*
(b): $n = 2$ *and* W^2 *is nonorientable.*

Moreover β *can be chosen nondegenerate, analytic, transitive or fixed-point free provided* α *has the same property.*

PROOF. Define β as the pullback of α through an immersion $M^n \to W^n$ (for immersion theory see Hirsch [15], Poenaru [26], Adachi [1]). $\quad\square$

Corollary 6. *Every noncompact* M^2 *supports effective analytic actions by* $\mathfrak{sl}(3,\mathbf{R})$ *and* $\mathfrak{sl}(2,\mathbf{C})$. *Every parallelizable noncompact* M^n *has effective analytic actions by* $\mathfrak{sl}(n+1,\mathbf{R})$, *by* $\mathfrak{sl}(\frac{n}{2},\mathbf{C})$ *if* n *is even, and by* $\mathfrak{sl}(\lfloor \frac{n}{2} \rfloor + 1, \mathbf{C})$ *if* n *is odd.*

Actions of G on the circle $\mathbf{S}^1$ lift to actions of $\widetilde{G}$ on $\mathbf{R}$, and by compactification to actions on $[0,1]$. Such actions can be concatenated to get effective actions of $\widetilde{G_1} \times \cdots \times \widetilde{G_m}$ on $[0,1]$. Further topological constructions lead to effective actions on closed n-disks, trivial on the boundary. Embedding such disks disjointly into an n-manifold leads to:

Theorem 7. $\widetilde{SL}_{\circ}(2,\mathbf{R})^j \times ST_{\circ}(2,\mathbf{R})^k \times \mathbf{R}^m$ *acts effectively on every manifold of positive dimension* $(j,k,m \geq 0)$.

In many cases such actions cannot be analytic and their smoothness is unknown; but see Theorem 9.

Algebraically contractible groups

The actions constructed above are either analytic or merely continuous. Next we exhibit a large class of solvable groups having effective actions— often smooth— on manifolds of moderately low dimensions. In many case these are smooth but cannot be analytic.

Let $\mathcal{E}(G)$ denote the space of endomorphisms of G, topologized as a subset of the continuous maps $G \to G$. We call G and $\mathfrak{g}$ *algebraically contractible* (AC) if there is a path $\phi = \{\phi_t\}$ in $\mathcal{E}(G)$ joining the the identity endomorphism ϕ_0 of G to the trivial endomorphism ϕ_1. Equivalently: G is solvable and simply connected, and the identity and trivial endomorphisms of $\mathfrak{g}$ are joined by a path $\psi = \{\psi_t\}$ in the affine variety $\mathcal{E}(\mathfrak{g})$ of Lie algebra endomorphisms of $\mathfrak{g}$. Every path ψ comes from a unique path ϕ.

The class of AC groups contains the vector group $\mathbf{R}^n$, the matrix groups $ST_\circ(n, \mathbf{R})$, $\widetilde{ST}_\circ(n, \mathbf{C})$, and many of their subgroups and quotient groups. It is closed under direct products. If $\mathfrak{g}$ is AC and an ideal $\mathfrak{h}$ is mapped into itself by every endomorphism of $\mathfrak{g}$, then $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are AC.

However, some nilpotent Lie algebras are not AC (DeKimpe [7]): The derivation algebra of an AC Lie algebra cannot be unipotent, but there are 8-dimensional nilpotent Lie algebras having unipotent derivation algebras (Dixmier & Lister [8], Ancochea & Campoamor [2]).

Proposition 8. *Assume G is algebraically contractible and (α, G, M) is almost effective. There is an effective action $(\beta, G, M \times \mathbf{R})$ with the following properties:*

   (a): $g^\beta(x, 0) = (g^\alpha(x), 0)$
   (b): $g^\beta(x, t) = (x, t)$ *if $|t| \geq 1$.*
   (c): *If α is smooth so is β.*

PROOF. We can choose the path $\psi \colon [0, 1] \to \mathcal{E}(\mathfrak{g})$ in the definition of AC to be C^∞ and constant in a neighborhood of $\{0, 1\}$. The corresponding path $\phi \colon [0, 1] \to \mathcal{E}(G)$ has the same properties. Extend ϕ over $\mathbf{R}$ by setting $\phi_t = \phi_1$ (= the trivial endomorphism) for $t \geq 1$, and $\phi_t = \phi_{-t}$ for $t \leq 0$. Now define β by

$$g^\beta(x, t) := \phi_t(g)^\alpha(x), \qquad (g \in G, \ (x, t) \in M \times \mathbf{R}).$$

$\square$

Theorem 9. *Assume G_i is AC and $(\alpha_i, G, \mathbf{S}^{n-1})$ is almost effective, $(i = 1, \ldots, k)$. For every M^n there exists an effective action $(\delta, G_1 \times \cdots \times G_k, M^n)$ that is smooth provided the α_i are smooth.*

PROOF. Let $(\beta_i, G_i, \mathbf{S}^{n-1} \times \mathbf{R})$ obtained from α_i as in Proposition 8. Through an identification $\mathbf{S}^{n-1} \times \mathbf{R} = \mathbf{D}^n \setminus (\mathbf{S}^{n-1} \cup 0)$, extend β_i to an action $(\gamma_i, G_i, \mathbf{D}^n)$ with compact support in $\mathbf{D}^n \setminus \mathbf{S}^{n-1}$. (Here $\mathbf{D}^n$ is the unit n-disk with boundary $\mathbf{S}^{n-1}$.) Transfer the γ_i to actions δ_i in k disjoint coordinate disks $D_i^n \subset M^n$. Define δ to coincide with δ_i in D_i^n and to be trivial outside $\cup_i D_i^n$. $\square$

Corollary 10. *Assume $G_i \subset GL(n, \mathbf{R})$ is algebraically contractible and contains no scalar multiple of the identity matrix, $(i = 1, \ldots, k)$. Then $G_1 \times \cdots \times G_k$ has effective smooth actions on all n-manifolds.*

PROOF. The natural actions of G_i on $\mathbf{P}^{n-1}$ and $\mathbf{S}^{n-1}$ are smooth and effective. Apply Theorems 9 and 5. $\square$

The Epstein-Thurston theorem

D.B.A. Epstein and W.P. Thurston [9, Theorem 1.1] discovered fundamental lower bounds on the dimensions in which solvable Lie algebras can act effectively:

Theorem 11 (EPSTEIN-THURSTON)**.** *Assume $\mathfrak{g}$ is solvable and has an effective n-action. Then $n \geq \ell(\mathfrak{g}) - 1$, and $n \geq \ell(\mathfrak{g})$ if $\mathfrak{g}$ is nilpotent.*

In the critical dimensions there is further information on orbit structure:

Theorem 12. *Let α be an effective n-action of a solvable Lie algebra $\mathfrak{g}$. Assume $n = \ell(\mathfrak{g}) - 1$, or $\mathfrak{g}$ is nilpotent and $n = \ell(\mathfrak{g})$.*

(i): *There is an open orbit. If α is nondegenerate the union of the open orbits is dense.*

(ii): *Assume $\mathfrak{g}^{(n-1)} \subset \mathfrak{c} = $ the center of $\mathfrak{g}$. Then:*

(a): *each nontrivial orbit of $\mathfrak{g}^{(n-1)}$ lies in an open orbit of $\mathfrak{g}$ and has dimension 1,*

(b): *the number of open orbits is $\geq \dim \mathfrak{g}^{(n-1)}$*

(c): *if α is nondegenerate then $\dim \mathfrak{c} = 1$.*

PROOF. The union of orbits of dimensions $< n$ is a closed set L in which $\mathfrak{g}^{(\ell(\mathfrak{g})-1)}$ acts trivially by Epstein-Thurston. Therefore $M^n \backslash L$, the union of the open orbits, is nonempty because α is effective, and dense if α is nondegenerate. This proves (i). Next we prove (ii).

(a) Let L be a nontrivial orbit of $\mathfrak{g}^{(n-1)}$ and let O be the orbit of $\mathfrak{g}$ containing L. Then O is an open set because $\dim(O) = n$ by Epstein-Thurston. This proves the first assertion of (a). To prove the second we can assume the action is transitive. Fix a 1-dimensional subspace $\mathfrak{z} \subset \mathfrak{c}$ having a 1-dimensional orbit $L_1 \subset L$. After replacing O by a suitably small open subset, we can assume the domain of the action is $O = \mathbf{R}^{n-1} \times \mathbf{R}$ with the slices $x \times \mathbf{R}$ being the orbits of $\mathfrak{z}$. The induced action of $\mathfrak{g}$ on the n-dimensional space of $\mathfrak{z}$-orbits kills $\mathfrak{g}^{(n-1)}$ by Epstein-Thurston. This implies $L_1 = L$, which implies (a).

(b) Suppose $\dim \mathfrak{g}^{(n-1)} = s \geq 1$ and there are exactly r open orbits O_i, $i = 1, \ldots r$. As $\mathfrak{g}$ acts transitively in O_i and $\mathfrak{g}^{(n-1)}$ is central, there is a codimension-one subalgebra $\mathfrak{k}_i \subseteq \mathfrak{g}^{(n-1)}$ acting trivially in O_i. If $1 \leq r < s$ then $\cap_i \mathfrak{k}_i$ has positive dimension and acts trivially in each open orbit, and also in all other orbits by Epstein-Thurston. This implies (b).

(c) Assume α is nondegenerate. By (a) there is an open orbit O, which we can assume is the only orbit. Let O, L, $\mathfrak{z}$ be as in the proof of (a). If (c) is false we choose $\mathfrak{z}$ so that the central ideal $\mathfrak{j} := \mathfrak{g}^{(n-1)} + \mathfrak{z}$ has dimension ≥ 2. In the proof of (a) we saw that every nontrivial orbit of $\mathfrak{j}$ is 1-dimensional, hence every orbit of $\mathfrak{j}$ is 1-dimensional because α is transitive and $\mathfrak{j}$ is central. Therefore for every $p \in O$ there is a maximal nontrivial linear subspace $\mathfrak{k}_p \subset \mathfrak{j}$ annihilated by α. As α is transitive and $\mathfrak{j}$ is central, all the $\mathfrak{k}_p$ coincides with an ideal that acts trivially in O. This contradicts the assumption that α is nondegenerate. $\square$

Example 13. The nilpotent algebra $\mathfrak{n} = \mathfrak{st}(n+1, \mathbf{R})' \times \mathbf{R}$ has derived length n and 2-dimensional center $\mathfrak{st}(n+1, \mathbf{R})^{(n-1)} \times \mathbf{R}$. Being algebraically contractible, $\mathfrak{n}$ acts effectively on all n-manifolds by Corollary 10. On the other hand, Theorem 12 implies:

- *Every n-action of $\mathfrak{n}$ is degenerate and hence nonanalytic.*

Weight spaces and spectral rank

Let $T\colon \mathfrak{g} \to \mathfrak{g}$ be linear. For λ in the spectrum $\mathsf{spec}\,(T) \subset \mathbf{C}$ define the (generalized) *weight space* $\mathfrak{w}(T, \lambda) \subset \mathfrak{g}$ to be the largest T-invariant subspace in which T has spectrum $\{\lambda, \bar{\lambda}\}$. The largest subspace of $\mathfrak{w}(T, \lambda)$ in which T acts semisimply is

$$\mathfrak{m}(T, \lambda) := \begin{cases} \text{kernel of } T - \lambda I & \text{if } \lambda \in \mathbf{R} \\ \text{kernel of } T^2 - 2(\Re\mathfrak{e}\,\lambda)T + |\lambda|^2 I & \text{if } \lambda \notin \mathbf{R} \end{cases}$$

For any set $S \subset \mathbf{C}$ let $\Gamma(S)$ denote the additive free abelian subgroup of $\mathbf{C}$ generated by S. The rank of $\Gamma(\mathsf{spec}\,(T))$ is the *spectral rank* $\boldsymbol{r}(T)$. The rank of $\Gamma(\mathsf{spec}\,(T) \setminus \mathbf{R})$ is the *nonreal spectral rank* $\boldsymbol{r}_{\mathsf{NR}}(T)$. For a Lie algebra $\mathfrak{g}$ define

$$\boldsymbol{r}(\mathfrak{g}) = \max_{X \in \mathfrak{g}} \boldsymbol{r}(\mathsf{ad}\,X), \quad \boldsymbol{r}_{\mathsf{NR}}(\mathfrak{g}) = \max_{X \in \mathfrak{g}} \boldsymbol{r}_{\mathsf{NR}}(\mathsf{ad}\,X)$$

For example, if $X \in \mathfrak{st}(m, \mathbf{R})$ is a sufficiently irrational diagonal matrix then

$$\boldsymbol{r}(\mathfrak{st}(m, \mathbf{R})) = \boldsymbol{r}(\mathsf{ad}\,X) \quad = m - 1,$$
$$\boldsymbol{r}(\mathfrak{st}(m, \mathbf{C})) = \boldsymbol{r}(\mathsf{ad}\,\imath X) \quad = m - 1,$$
$$\boldsymbol{r}_{\mathsf{NR}}(\mathfrak{st}(m, \mathbf{C})) = \boldsymbol{r}_{\mathsf{NR}}(\mathsf{ad}\,\imath X) = m - 1.$$

If $\mathfrak{s}$ is semisimple of rank r with a Cartan decomposition $\mathfrak{k} + \mathfrak{p}$, almost every X in the Cartan subalgebra $\mathfrak{k}$ satisfies

$$\boldsymbol{r}(\mathfrak{s}) = \boldsymbol{r}(\mathsf{ad}_\mathfrak{s}\,X) = r, \qquad \boldsymbol{r}_{\mathsf{NR}}(\mathfrak{s}) = \boldsymbol{r}_{\mathsf{NR}}(\mathsf{ad}_\mathfrak{s}\,X) = r$$

(see Helgason [13, Prop. III.7.4]).

$Y \in \mathfrak{v}^\infty(M)$ is *flat at* $p \in M$ when its Taylor series vanishes in local coordinates centered at p. If such a Y is analytic it is trivial. Given $(\alpha, \mathfrak{g}, M)$ and $p \in M$, define $\mathfrak{f}_p(\alpha) \subset \mathfrak{g}$ as the set of $Y \in \mathfrak{g}$ such that Y^α is flat at p. This is an ideal.

Proposition 14. *Assume* $(\alpha, \mathfrak{g}, M^n)$ *is smooth,* $X \in \mathfrak{g}$ *and* $p \in \mathsf{Fix}(X^\alpha)$. *Suppose* $\mathfrak{m}(\mathsf{ad}\,X, \lambda) \cap \mathfrak{f}_p(\alpha) = 0$ *for all* $\lambda \in \mathsf{spec}\,(\mathsf{ad}\,X)\setminus 0$. *Then*

$$\mathsf{spec}\,(\mathsf{ad}\,X) \subset \Gamma(\mathsf{spec}\,(dX_p^\alpha))$$

and therefore

$$n \geq \max\{\boldsymbol{r}(X),\ 2\boldsymbol{r}_{\mathsf{NR}}(\mathsf{ad}\,X)\}.$$

PROOF. We can assume $M^n = \mathbf{R}^n$, $p = 0$. Write every $Z \in \mathfrak{v}^\infty(\mathbf{R}^n)$ as the formal sum $\sum_{r \in \mathbf{N}} Z_{(r)}$ where the components of the vector field $Z_{(r)}$ are homogeneous polynomial functions of degree r. Then $X_{(0)}^\alpha = 0$, $X_{(1)}^\alpha = dX_p^\alpha$. The *order* of Z is the smallest r for which $Z_{(r)} \neq 0$ if Z is not flat at 0, otherwise the order is ∞. Suppose $Y \in \mathfrak{k}(\mathsf{ad}\,X, \lambda)$ is not flat at 0 and has finite order r. Then $(\mathsf{ad}_{\mathbf{C} \otimes \mathfrak{g}}\,X - \lambda I)Y = 0$, implying $[X_{(1)}^\alpha, Y_{(r)}^\alpha] = \lambda Y_{(r)}^\alpha$. Hence $\lambda \in \mathsf{spec}\,(\mathsf{ad}_{\mathfrak{v}^\infty(\mathbf{R}^n)}\,dX_p^\alpha)$. A calculation shows that $\mathsf{spec}\,(\mathsf{ad}_{\mathfrak{v}^\infty(\mathbf{R}^n)}\,Z) \subset \Gamma(\mathsf{spec}\,(Z))$ for every linear vector field $Z\colon \mathbf{R}^n \to \mathbf{R}^n$. Apply this to $Z := dX_p^\alpha$. $\qquad\square$

The following result is derived from Proposition 14:

Theorem 15. *Suppose* $(\alpha, \mathfrak{g}, M^n)$ *is smooth,* $X \in \mathfrak{g}$ *and* $p \in \mathsf{Fix}(X^\alpha)$.

   (i): *Assume* $\boldsymbol{r}(\mathsf{ad}\,X) = n + k > n$. *Then* $\mathsf{ad}\,X$ *has* k *different eigenvalues* $\lambda \neq 0$ *such that* $\mathfrak{w}(\mathsf{ad}\,X, \lambda) \subset \mathfrak{f}_p(\alpha)$.

(ii): *Assume* $2r_{\mathsf{NR}}(\mathsf{ad}\,X) = n$, α *is effective, and* $\mathfrak{m}(\mathsf{ad}\,X, \lambda) \cap \mathfrak{f}_p(\alpha) = 0$ *for all* $\lambda \in \mathsf{spec}\,(\mathsf{ad}\,X) \setminus \mathbf{R}$. *Then* dX_p^α *has only nonreal eigenvalues,* X^α *has index* 1 *at* p, *and if* M^n *is compact then* $\chi(M^n) > 0$.

This has powerful consequences for analytic actions:

Corollary 16. *Assume* $(\alpha, \mathfrak{g}, M^n)$ *is effective and analytic and* $X \in \mathfrak{g}$.

(a): *If* $\mathsf{Fix}(X^\alpha) \neq \varnothing$ *then* $n \geq \max\{r(\mathsf{ad}\,X), 2r_{\mathsf{NR}}(\mathsf{ad}\,X)\}$.

(b): *Suppose* M^n *is compact and* $n = 2r_{\mathsf{NR}}(\mathsf{ad}\,X)$. *Then*

$$\chi(M^n) = \#\,\mathsf{Fix}(X^\alpha) \geq \#\,\mathsf{Fix}(\alpha).$$

Therefore $\chi(M^n) \geq 0$, *and* $\mathsf{Fix}(\alpha) = \varnothing$ *if* $\chi(M^n) = 0$.

For surface actions, (b) is due to Turiel [30].

Corollary 17. *Assume* M^n *is compact and* $\chi(M^n) \neq 0$. *If* $(\alpha, \mathfrak{g}, M^n)$ *is analytic with kernel* $\mathfrak{k}$, *then* $\dim \mathfrak{k} \geq \max\left\{r(\mathfrak{g}) - n,\ r_{\mathsf{NR}}(\mathfrak{g}) - \left\lfloor \frac{n}{2} \right\rfloor \right\}$.

Example 18. Assume $\mathfrak{s}$ is semisimple of rank r with a Cartan decomposition $\mathfrak{k} + \mathfrak{p}$ where $\mathfrak{k}$ is a Cartan subalgebra. The set $U := \{X \in \mathfrak{k} : r_{\mathsf{NR}}(\mathsf{ad}\,X) = r\}$ is dense and open in $\mathfrak{k}$. Let $(\alpha, \mathfrak{s}, M^n)$ be effective and analytic, with $\mathsf{Fix}(X^\alpha) \neq \varnothing$ for some $X \in U$. Then Corollary 16 implies:

- $n \geq 2r$. *If* $n = 2r$ *and* M^n *is compact then* $\chi(M^n) = \#\,\mathsf{Fix}(Y^\alpha) > 0$ *for all* $Y \in \mathfrak{k}$.

Example 19. Assume $m, n, k \in \mathbf{N}_+$ with $m \leq n$. Theorem 9 shows that every n-manifold supports a smooth effective action of $\mathfrak{st}(m + 1, \mathbf{R})^k$. Because $r(\mathfrak{st}(m + 1, \mathbf{R})^k) = mk$, Corollary 17 implies:

- *Assume* M^n *is compact and* $\chi(M^n) \neq 0$. *If* $(\alpha, \mathfrak{st}(m + 1, \mathbf{R})^k, M^n)$ *is analytic and effective then* $k \leq \left\lfloor \frac{n}{m} \right\rfloor$.

To take a specific example:

- $\mathfrak{st}(n + 1, \mathbf{R}) \times \mathfrak{st}(n + 1, \mathbf{R})$ *does not have an effective analytic action on any compact* n-manifold.

Fixed points

For actions of G on compact surfaces M^2 the following results are known:

Proposition 20.

(a): $ST_\circ(2, \mathbf{R})$ *has effective, fixed-point free* C^∞ *actions on all compact surfaces.* (Lima [22], Plante [25], Belliart & Liousse [3], Turiel [29, 31])

(b): *If* G *acts without fixed point and* $\chi(M^2) < 0$ *then* $ST_\circ(2, \mathbf{R})$ *is a quotient group of* G. (Belliart [4])

(c): *If* G *acts analytically without fixed point,* $\chi(M^2) \geq 0$. (Turiel [30])

(d): *If* G *is nilpotent and acts without fixed point,* $\chi(M^2) = 0$. (Lima [22], Plante [25])

(e): *If* G *is supersoluble and acts analytically without fixed point,* $\chi(M^2) = 0$. (Hirsch & Weinstein [17])

Careful use of the blowup construction shows that some supersoluble groups have effective analytic surface actions with arbitrarily large numbers of fixed points:

Theorem 21. *Let M_g^2 denote a closed surface of genus $g \geq 0$. For every $k \in \mathbf{N}$ there is an effective analytic action $(\beta, ST_\circ(3, \mathbf{R}), M_g^2)$ such that*

$$\# \, \mathsf{Fix}(\beta) = \begin{cases} 2(g + k + 1) & \text{if } M_g^2 \text{ is orientable,} \\ g + k & \text{if } M_g^2 \text{ is nonorientable and } g \geq 1. \end{cases}$$

On the other hand:

- *Suppose G is not supersoluble. If M^2 is compact and (α, G, M^2) is effective and analytic, then $0 \leq \# \, \mathsf{Fix}(\alpha) \leq \chi(M^2) \leq 2$.*

This follow from Corollary 16(b), because G is not supersoluble if and only if $\boldsymbol{r}_{\mathsf{NR}}(G) \geq 1$.

Questions. Is the analog of Proposition 20(a) true for $ST_\circ(3, \mathbf{R})$? Does this group have an effective analytic action with a unique fixed point on some orientable closed surface? Can $ST(3, \mathbf{R})$ act effectively on $\mathbf{S}^2$ with a unique fixed point? Can a smooth effective action of $SL(2, \mathbf{R})$ on $\mathbf{S}^1 \times \mathbf{S}^1$ have a fixed point?

For noncompact group actions in higher dimensions the following are known:

- *$\mathbf{R}$ acts effectively without fixed point on a compact $M^n \iff \chi(M^n) = 0$.*
  (Poincaré [27], Hopf [18])

- *An algebraic action of a solvable complex algebraic group on a complete complex algebraic variety has a fixed point.* (Borel [5])

- *If M^n is compact, $n = 3$ or 4, and $\chi(M^n) \neq 0$, then every analytic action of $\mathbf{R}^2$ on M^n has a fixed point.* (Bonatti [6])

Spectral rigidity

$\mathcal{A}_1(\mathfrak{g}, M)$ denotes the space of C^∞ actions of $\mathfrak{g}$ on M under the the smallest topology making the maps the map $\mathcal{A}_1(\mathfrak{g}, M) \to \mathfrak{v}^1(M)$, $\alpha \mapsto X^\alpha$, continuous for all $X \in \mathfrak{g}$. An action $(\alpha, \mathfrak{g}, M)$ is *spectrally rigid* at (X, p) if $X \in \mathfrak{g}$, $p \in \mathsf{Fix}(X^\alpha)$, and there exist arbitrarily small neighborhoods $\mathcal{N} \subset \mathcal{A}_1(\mathfrak{g}, M^n)$ of α and $W \subset M$ of p such that for all $\beta \in \mathcal{N}$:

(SR1): $\mathsf{Fix}(X^\beta) \cap W \neq \varnothing$

(SR2): $q \in \mathsf{Fix}(X^\beta) \cap W \implies dX_q^\beta$ and dX_p^α have the same nonzero eigenvalues.

While spectral rigidity is impossible for nontrivial abelian algebras and dubious for nilpotent algebras, many solvable and semisimple algebras exhibit it:

Theorem 22. *Assume $(\alpha, \mathfrak{g}, M^n)$ is effective and analytic, $X \in \mathfrak{g}$ and $\boldsymbol{r}(\mathsf{ad}\, X) = n$. Then α is spectrally rigid at (X, p) for all $p \in \mathsf{Fix}(X^\alpha)$.*

The proof is based on Proposition 14.

Conjecture. *An analytic action α of a semisimple Lie algebra $\mathfrak{s}$ is spectrally rigid at (X, p) for all $X \in \mathfrak{s}$, $p \in \mathsf{Fix}(\alpha)$.*

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