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Lagrangian and Eulerian Forms of Finite Plasticity

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Abstract

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Over the past half century, much work has been published on the theory of elastic-plastic materials undergoing large deformations. However, there is still disagreement about certain basic issues. In this dissertation, the strain-based Lagrangian theory developed by Green, Naghdi and co-workers is re-assessed. Its basic structure is found to be satisfactory. For the purpose of applications, the theory is recast in Eulerian form. Additionally, a novel three-factor multiplicative decomposition of the deformation gradient is employed to define a unique intermediate configuration. The resulting theory of finite plasticity contains an elastic strain tensor measured from the intermediate stress-free configuration. The constitutive equations involve the objective stress rate of the rotated Cauchy stress, which can be expressed in terms of the rate of deformation tensor. In their general forms, the Lagrangian theory can be converted into the Eulerian theory and vice versa. Because the Green-Naghdi theory has a strain measure that represents the difference between total strain and plastic strain, rather than representing elastic strain, it does not lend itself to a physically realistic linearization for the case of small elastic deformations accompanied by large plastic deformations. The proposed theory is well suited to describe this case as it can be linearized about the intermediate configuration while allowing the plastic deformations to be large. Differences between the linearized Green-Naghdi theory and the new theory are illustrated for uniaxial tension and compression tests.

For Jimmy. Thank you for saving my life.

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List of Symbols

$\mathbf{A}$, $\mathbf{B}$	general notation for second-order tensors
$\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathscr{A}}$	general notation for fourth-order tensors
$\boldsymbol{\mathscr{A}}^{a,b}$, $\boldsymbol{\mathscr{B}}^{a,b}$	fourth-order tensors that relates objective rates of stress
$\mathbf{a}$	acceleration vector
$\mathscr{B}$	a continuous body in space
$\mathbf{b}$	body force per unit mass
$\mathbf{C}$	Cauchy-Green tensor
$\mathbf{D}$	rate of deformation
$\mathbf{E}$	Lagrangian strain
$\mathbb{E}^3$	3D Euclidean space
$\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$	basis vectors in the reference configuration
$\mathbf{E}_e$	elastic strain
$\mathbf{E}_p$	plastic strain
$\mathbf{E}_*$	strain tensor in the intermediate configuration
$\mathbf{F}$	deformation gradient
$\mathbf{H}$	displacement gradient
$\mathbf{I}$	second-order identity tensor
$\boldsymbol{\mathscr{I}}$, $\boldsymbol{\mathscr{J}}$	fourth-order identity tensors
J	Jacobian
$\boldsymbol{\mathscr{K}}$	fourth-order tensor relating stress and strain
$\mathbf{L}$	velocity gradient
$\boldsymbol{\mathscr{L}}$	fourth-order elasticity tensor
$\mathscr{N}(X)$	neighborhood of X
$\mathbf{P}$	first Piola-Kirchhoff tensor
$\mathbf{Q}$	rotation tensor
$\mathbf{R}$	rotation tensor
$\mathbf{S}$	second Piola-Kirchhoff tensor
$\mathbf{T}$	Cauchy stress tensor
$\widetilde{\mathbf{T}}$, $\mathbf{T}_*$	rotated Cauchy stress tensors
$\mathbf{U}$	right stretch tensor

$\mathscr{U}$	plastic parameters $\mathbf{E}_p$, $\boldsymbol{\alpha}_R$, and κ
$\mathbf{V}$	left stretch tensor
$\mathbf{v}$	velocity vector
$\mathbf{W}$	vorticity tensor
X	material point of a body
$\mathbf{X}$, $\mathbf{x}$	position vectors of a material point X
$\boldsymbol{\mathcal{Z}}$	fourth-order tensor involving the derivative of $\mathbf{U}_p$ with respect to $\mathbf{E}_p$

$\mathbf{a}$, $\mathbf{b}$	general vector quantities
$\mathbf{e}$	Eulerian strain tensor
$\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$	basis vectors in the current configuration
f	yield function in stress space
$\hat{f}$	loading index in stress space
g	yield function in strain space
$\hat{g}$	loading index in strain space
k	bulk modulus of elasticity
$o(\epsilon)$	order of epsilon

$\boldsymbol{\alpha}$	back stress or shift tensor
$\boldsymbol{\beta}$	constitutive function for the rate of back stress
$\boldsymbol{\Gamma}$	push-forward of the normal to the yield surface in strain space
$\boldsymbol{\gamma}$	deviatoric part of a strain tensor
δ_{ij}	Kronecker delta
ϵ	an infinitesimal or small quantity
$\boldsymbol{\kappa}$, $\boldsymbol{\kappa}_*$, $\boldsymbol{\kappa}_0$	configurations of the body $\mathscr{B}$
κ	strain hardening measure
λ	constitutive function for $\dot{\kappa}$
λ, λ_p, λ_*	stretches
μ	shear modulus of elasticity
π	loading index constant
$\boldsymbol{\pi}$	transformation of a kinetical tensor
ρ	density
$\boldsymbol{\rho}$	constitutive function for the rate of plastic strain
$\boldsymbol{\Sigma}$	representation of a kinetical tensor
$\boldsymbol{\sigma}$	a rate-type tensor involving plastic parameters
$\boldsymbol{\tau}$	deviatoric part of a stress tensor
Φ	strain hardening index
$\boldsymbol{\chi}$	mapping between material points
ψ	plastic flow rule constant
$\boldsymbol{\Omega}$	rate of rotation tensor

Chapter 1

Introduction

1.1 The Classical Theory of Plasticity: Infinitesimal Deformation

The origins of classical plasticity can be traced back to the experiments of Henri Tresca, first published in 1868. He established that when metals reach a critical value of shear stress, they begin to flow like a fluid. Thus, he introduced the first yield condition, now called the Tresca yield condition. In 1870, Barré Saint-Venant used Tresca's criterion to present a set of equations for the problem of plane deformations. These are now identified as a special finitely deforming rigid-perfectly plastic incompressible material. In the same year, Saint-Venant's student Maurice Lévy extended these results to the three-dimensional case. Because of inherent mathematical difficulties, the Saint-Venant-Lévy theory had no application for many years. In 1913, Richard von Mises, who accepted Tresca's condition as an experimental fact, presented a similar set of equations with a simpler yield criterion, which involves an analytic function of the stress. The Mises yield criterion was used in the development of equations for rigid-perfectly plastic materials, now known as Saint-Venant-Lévy-Mises equations.

In 1924, Ludwig Prandtl established a connection between the Saint-Venant-Lévy equations for the plane problem and the description of incremental elastic behavior. In 1930, András (Endre) Reuss extended the theory to three dimensions. Thus were born the celebrated Prandtl-Reuss equations, which give a relationship between the rate of deviatoric stress $\boldsymbol{\tau}$ and the rate of deviatoric strain $\boldsymbol{\gamma}$. In modern notation, these equations can be written as

$$\dot{\boldsymbol{\tau}} = 2\mu \left(\boldsymbol{\mathscr{I}} - \frac{1}{2K^2} \boldsymbol{\tau} \otimes \boldsymbol{\tau} \right) \dot{\boldsymbol{\gamma}}, \tag{1.1}$$

where μ is the shear modulus, $\mathscr{I}$ is a fourth order identity tensor, and K^2 is a measure of work hardening, which is obtained from the yield criteria. The Prandtl-Reuss equations are generally accepted over the entire range of purely elastic response to initial loading to plastic response, provided that the deformation gradients are small. References to the original works of Prandtl and Reuss, as well as those of Tresca, Saint-Venant, Lévy, and Mises, can be found in [36].

Beginning in the late 1940s, William Prager provided extensions of the Prandtl-Reuss theory. Adopting the additive decomposition of strain into elastic and plastic parts, he used arbitrary yield functions, various flow rules and hardening rules to extend the Prandtl-Reuss theory for infinitesimal deformations [35]. Also, in an effort to provide realistic methods of determining safety factors and technological forming processes, Prager and P.G. Hodge applied the Prandtl-Reuss theory and the Saint-Venant-Lévy-Mises theory to perfectly-plastic materials and solved various statically determinate problems such as the torsion of cylindrical and prismatic bars and problems with plane plastic strain [36].

1.2 Finite Plasticity: Theories and Issues from the '60s

A major movement of plasticity theory occurred in the 1960s. As P.M. Naghdi mentioned in his review paper [28] on stress-strain relations in plasticity in 1960, the foundation of the theory of plasticity was not yet firmly in place. The scope of the various survey papers at the time were different and the points of view of different authors did not agree with each other.

Then in 1965, with the theory of nonlinear elasticity completely available to them, Green and Naghdi formulated a general thermodynamical theory of finite plasticity [18, 19]. They used total strain $\mathbf{E}$, plastic strain $\mathbf{E}_p$ and a work hardening parameter κ as primitive kinematical variables. The constitutive laws are of the rate-type, and the formulations are in stress space. Influenced by the development of infinitesimal plasticity, they took a geometrical point of view. They used yield surfaces, loading criteria, flow rules, and hardening rules for stress rates. The Green-Naghdi theory has been expanded upon by the same authors and their collaborators during the past 50 years. A Lagrangian strain space formulation was introduced by Naghdi and Trapp in 1975, showing that there are disadvantages in using the stress space formulation [33]. A natural characterization of hardening, softening, and perfectly plastic responses was then demonstrated by Casey and Naghdi in 1981 [9].

Today, there are still many areas of disagreement about specific aspects of finite plasticity. Among these are the prescriptions of strain measures, admissibility of strain hardening variables, invariance requirements, yield criteria, stress space and strain space formulations, flow rules and hardening rules, and Lagrangian and Eulerian descriptions [29].

1. Prescription of strain measures

There are several approaches to defining a measure of plastic strain in finite plasticity. Green and Naghdi regarded the plastic strain $\mathbf{E}_p$ as a primitive quantity, not defining it explicitly, but relegating its identification to special assumptions. For example, $\mathbf{E}_p$ can be specified as the value of total strain upon removal of all stress components [19]. Another prescription is the process of maximal unloading, which involves taking the stress point closest to the origin in stress space but within the yield surface and measuring the corresponding plastic strain [15].

Thus, Green and Naghdi used the total strain $\mathbf{E}$ and the plastic strain $\mathbf{E}_p$ as primitive variables in finite plasticity. Although an "elastic" tensor might be thought of as the difference between $\mathbf{E}$ and $\mathbf{E}_p$, it is not the usual elastic strain tensor except in some special cases.

On the other hand, some authors defined plastic strain using an intermediate stress-free configuration. This configuration is associated with the multiplicative decomposition of the deformation gradient into an elastic part and a plastic part:

$$\mathbf{F} = \mathbf{F}_e \mathbf{F}_p. \tag{1.2}$$

(See Figure 1.1.) The corresponding elastic strain and plastic strain are then defined by

$$\mathbf{E}_e = \frac{1}{2}\left(\mathbf{F}_e^T \mathbf{F}_e - \mathbf{I}\right), \qquad \mathbf{E}_p = \frac{1}{2}\left(\mathbf{F}_p^T \mathbf{F}_p - \mathbf{I}\right). \tag{1.3}$$

Introduced by Kröner in 1960, the decomposition (1.2) was popularized by E. H. Lee in the late 1960s [23]. It was also used in Lee and Liu in the context of plane wave analysis [25]. This convenient concept of the separation of the reversible elastic part and irreversible plastic part is perhaps why this decomposition is popular.

There are some limitations, however, to using the multiplicative decomposition (1.2) and an intermediate stress-free configuration. One is the fact that the stress at a point can be reduced to zero without changing the plastic strain only if the origin in stress space remains in the region enclosed by the yield surface. For some special hardening rules such as isotropic hardening, the origin does remain enclosed by the yield surface (Figure 1.2). However, for kinematic hardening, the decomposition will not work since the yield surface can move around in a general manner (Figure 1.3). This limitation does not exist if the plastic strain is defined as a primitive variable, as in the Green-Naghdi theory, since no special kinematical relation is used to obtain $\mathbf{E}_p$ [20].

Further, even if the stress can be reduced to zero at each point of the body, the resulting configuration, in general, cannot be pieced together to form a global configuration, rather it will only be a collection of local configurations.

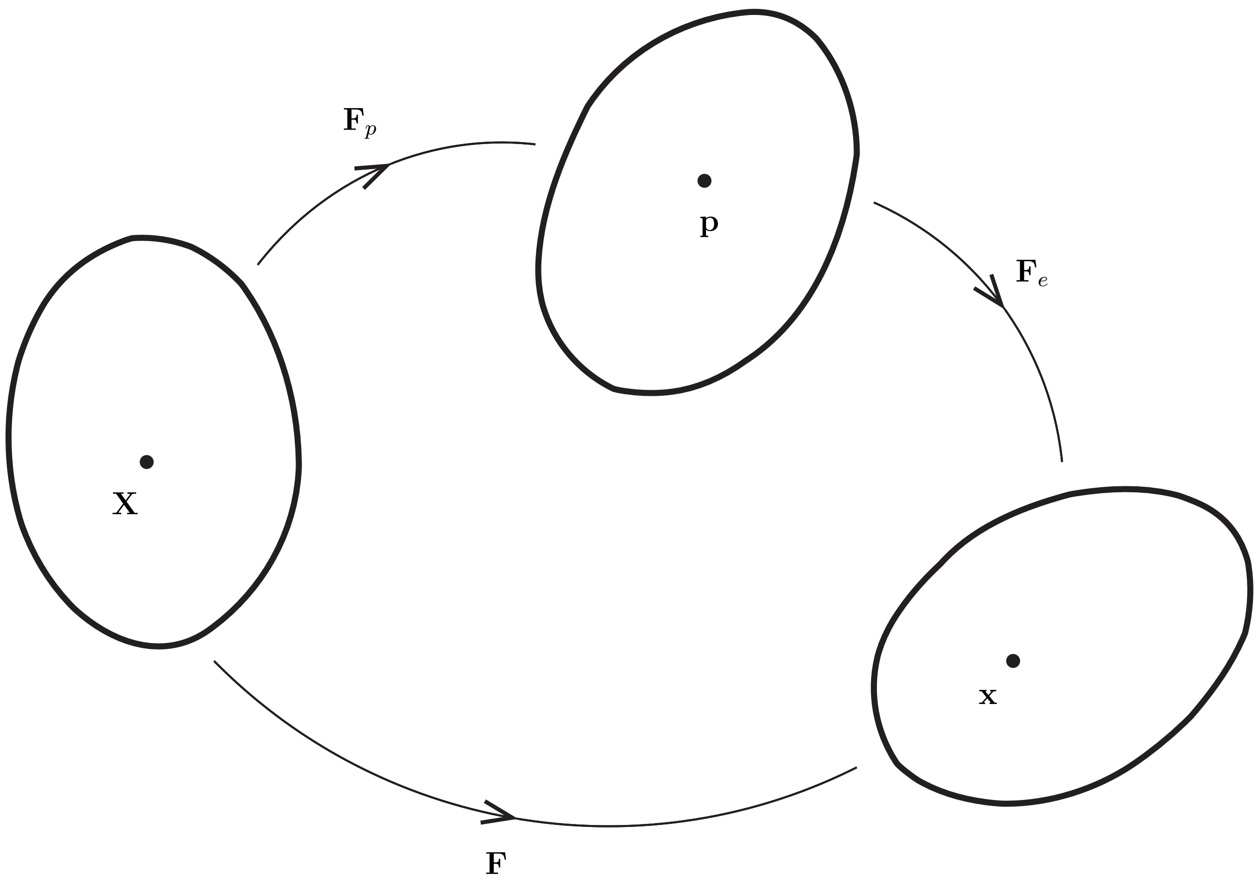

Figure 1.1: Multiplicative decomposition of the deformation gradient $\mathbf{F}$. $\mathbf{F}_p$ denotes the plastic deformation, and $\mathbf{F}_e$ is the elastic deformation.

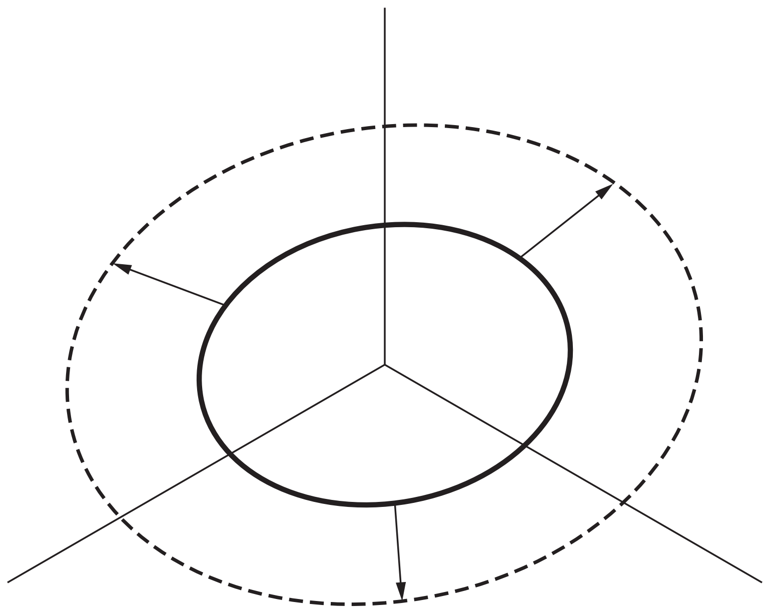

Figure 1.2: Isotropic hardening. The yield surface in $\mathbb{E}^3$ expands equally in all directions.

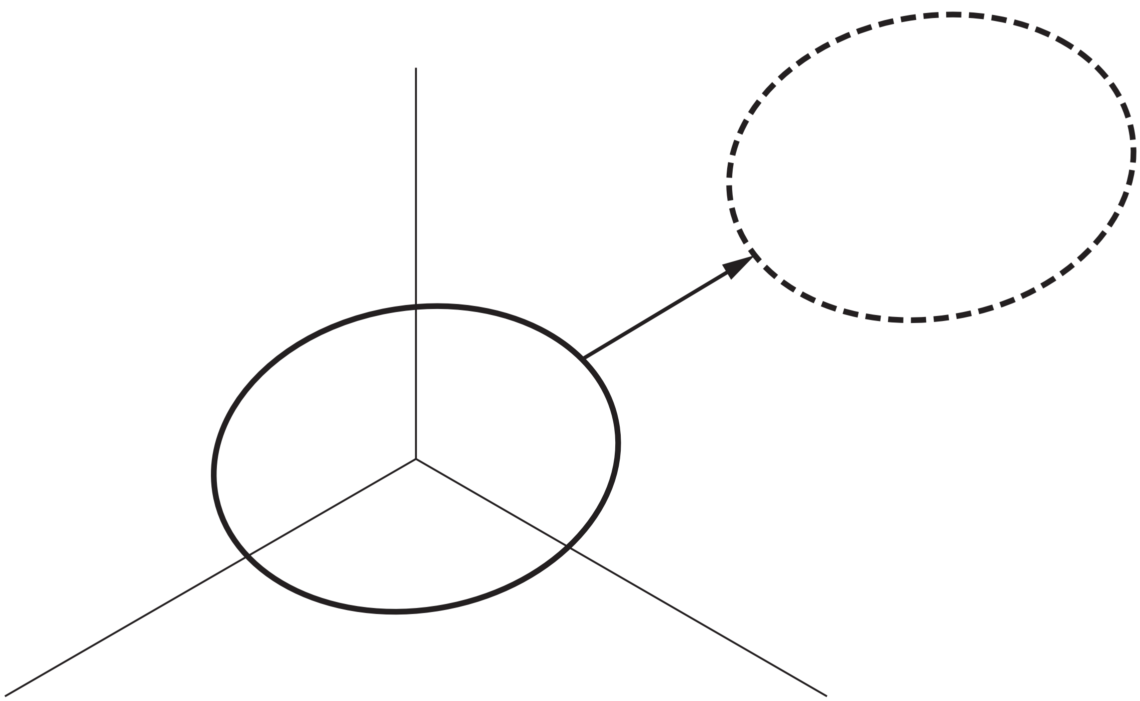

Figure 1.3: Kinematic hardening. The yield surface keeps the same size but translates. A general hardening rule allows for distortion and a general motion of the yield surface.

2. Invariance requirements

Physical considerations dictate that variables such as $\mathbf{E}_p$ must satisfy invariance properties under superposed rigid body motions. Since the intermediate stress-free configuration is just another configuration, then, by the same physical reasoning, it must satisfy invariance requirements. However, many authors who use Lee's decomposition introduce additional assumptions that violate invariance requirements. For Lee, the unstressed state is only used as a thought experiment to devise convenient variables for the elastic-plastic theory, and demanding invariance on the intermediate configuration is regarded by him as redundant [24].

3. Lagrangian and Eulerian descriptions

The choice between Lagrangian and Eulerian descriptions of finite plasticity is still in question. The arguments usually begin with the choice of stress tensor for the constitutive equations. Lagrangian formulations usually involve the symmetric Piola-Kirchhoff stress $\mathbf{S}$. Some Eulerian formulations involve the Cauchy stress tensor $\mathbf{T}$.

For expressions of flow rules, some authors who prefer the Eulerian formulation first consider the additive decomposition of the velocity gradient $\mathbf{L}$ or the rate of deformation tensor $\mathbf{D}$ into elastic and plastic parts. A constitutive equation for the plastic part $\mathbf{D}_p$ is then prescribed in terms of an objective rate of $\mathbf{T}$. The partition of the rate of deformation $\mathbf{D}$ into elastic and plastic parts is still in question today. It is also of note that the choice of objective rates is another area of disagreement.

Casey and Naghdi (1988) examined the Lagrangian and Eulerian formulations of finite plasticity with respect to rigid plastic materials. They concluded that the Lagrangian and Eulerian formulations are equivalent for such materials, and that the choice of objective rate is immaterial [14].

Lubarda (2004) presented a review article of the applications of the multiplicative decomposition [26]. These include thermoplasticity, elastoplasticity, crystal plasticity, and surface growth in biomechanics. Lubarda concluded that to a large extent, the use of the multiplicative decomposition is optional in thermoplasticity and crystal plasticity. However, when the plastic deformation affects the initial elastic properties of a material, as in damage elastoplasticity, the multiplicative decomposition can be employed.

Bammann and Johnson (1987) sought a way to unify the rate-type theory of Green and Naghdi with that of an intermediate stress-free configuration and the multiplicative decomposition. Thus, they introduced a three-factor decomposition, in which the subfactors: a left stretch tensor, a right stretch tensor, and a rotation tensor, are all uniquely determined by destressing at a material point [1]. Associated with this decomposition are two stress-free configurations, which differ by a unique rotation.

An alternative form of the three-factor decomposition of Bammann and Johnson was recently proposed by Casey [7]. It employs a right stretch tensor $\mathbf{U}_*$, which does not change under superposed rigid motions. It introduces a unique intermediate configuration, which can be used as an evolving reference configuration.

This dissertation employs this new multiplicative decomposition and the resulting unique intermediate configuration $\boldsymbol{\kappa}_*$ as a basis for expressing finite plasticity in the Eulerian form. Using the structure in the strain-based Lagrangian theory developed by Green and Naghdi and an elastic strain tensor furnished from the intermediate configuration, constitutive equations involving the objective rate of the rotated Cauchy stress are formulated to present a recast version of the Lagrangian theory into an Eulerian form.

All relevant elements of $\boldsymbol{\kappa}_*$, including stress measures, elastic and plastic strains, yield functions and yield criteria, are first established. Then a set of constitutive equations (6.60) involving stress rates and strain rates are formulated. In theory, all four equations in (6.60) could be used to express finite plasticity. That is, in their general forms, the Lagrangian stress measures can be converted into their Eulerian versions and vice versa. While the Lagrangian theory is expressed using the Piola-Kirchhoff stress tensor $\mathbf{S}$, the proposed Eulerian theory involves the rotated Cauchy stress $\widetilde{\mathbf{T}}$.

Because the Lagrangian theory employs a strain measure that takes the difference $\mathbf{E} - \mathbf{E}_p$ between total strain and plastic strain, rather than representing elastic strain, it does not lend itself to a physically realistic linearization for the case of small elastic deformations accompanied by large plastic deformations. The proposed Eulerian theory is well suited to describe this case as it can be linearized about $\boldsymbol{\kappa}_*$ while allowing the plastic deformations to be large. This difference is illustrated when the two theories are applied to a uniaxial stress test, both in tension and compression.

The relevant fundamental features of a general three-dimensional continuum are presented in Chapter 2. The Green-Naghdi rate-type rate-independent plasticity theory for finite deformations are presented in Chapter 3. Three multiplicative decompositions of the deformation gradient, including that of Lee, are summarized in Chapter 4. In Chapter 5, different forms and variables involving the stress response are developed. These equations are based on the intermediate configuration $\boldsymbol{\kappa}_*$. They are then used in Chapter 6 to formulate Eulerian forms of the constitutive equations. We find that the Cauchy stress rate can be written as a linear function of the rate of deformation tensor. General objective rates of the rotated Cauchy stress tensor and the back stress tensor are also presented. Finally, Chapter 7 features some special cases, including linear responses between the stress and strain measures and their rates. That is followed by a simple example to demonstrate the differences between the Green-Naghdi theory and this new Eulerian theory of finite plasticity.

Chapter 2

Deformable Continua

After some mathematical preliminaries, the mechanics of a three-dimensional deformable continuum are summarized. The content is based on the works of Naghdi [28–33] and Truesdell and Noll [39]. In addition, the basic features of a nonlinear elastic solid is presented.

2.1 Mathematical Preliminaries

A second-order tensor $\mathbf{A}$ is a linear transformation on a vector space $\mathcal{V}$ that assigns to each vector $\mathbf{u}$ a vector $\mathbf{Au}$. It is invariant under a change of coordinate system. The set of 2-tensors is called Lin. The tensor product of two vectors $\mathbf{a}$ and $\mathbf{b} \in \mathcal{V}$ is denoted by $\mathbf{a} \otimes \mathbf{b}$ and is defined by

$$(\mathbf{a} \otimes \mathbf{b})\, \mathbf{u} = (\mathbf{b} \cdot \mathbf{u})\, \mathbf{a}, \tag{2.1}$$

for any vector $\mathbf{u} \in \mathcal{V}$. If the inverse of a tensor $\mathbf{A}$ exists, it is denoted $\mathbf{A}^{-1}$. The transpose of $\mathbf{A}$ is $\mathbf{A}^T$, and $(\mathbf{a} \otimes \mathbf{b})^T = (\mathbf{b} \otimes \mathbf{a})$. A symmetric tensor is equal to its transpose. A skew-symmetric tensor is equal to the negative of its transpose. The trace and determinant of $\mathbf{A}$ are $\operatorname{tr}\mathbf{A}$ and $\det\mathbf{A}$, respectively. The inner product between two tensors $\mathbf{A}$ and $\mathbf{B}$ is defined by $\mathbf{A} \cdot \mathbf{B} = \operatorname{tr}(\mathbf{A}\mathbf{B}^T) = \operatorname{tr}(\mathbf{A}^T\mathbf{B})$. We recall that $(\mathbf{a} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{d}) = (\mathbf{b} \cdot \mathbf{c})\mathbf{a} \otimes \mathbf{d}$, $(\mathbf{a} \otimes \mathbf{b}) \cdot (\mathbf{c} \otimes \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d})$, $\mathbf{A}(\mathbf{c} \otimes \mathbf{d}) = (\mathbf{Ac}) \otimes \mathbf{d}$, and $(\mathbf{c} \cdot \mathbf{d})\mathbf{A} = \mathbf{c} \otimes (\mathbf{A}^T\mathbf{d})$, where $\mathbf{c}$ and $\mathbf{d}$ are also vectors. If $\mathbf{e}_i$, $(i = 1, 2, 3)$, is an orthonormal basis for $\mathcal{V}$, then $\mathbf{e}_i \otimes \mathbf{e}_j$, $(i, j = 1, 2, 3)$, is an orthonormal basis for Lin. Thus, $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$, where δ_{ij} is the Kronecker delta and

$$(\mathbf{e}_i \otimes \mathbf{e}_j) \cdot (\mathbf{e}_k \otimes \mathbf{e}_\ell) = \delta_{ik}\delta_{j\ell}, \tag{2.2}$$

where all indices range from 1 to 3. Thus we can write $\mathbf{A}$ as

$$\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \qquad A_{ij} = \mathbf{e}_i \cdot \mathbf{A}\mathbf{e}_j = \mathbf{A} \cdot (\mathbf{e}_i \otimes \mathbf{e}_j), \qquad (i, j = 1, 2, 3), \tag{2.3}$$

where the summation convention is used for i and j. The expression $\mathbf{a} = \mathbf{A}\mathbf{b}$ can be written in index notation as $a_i = A_{ij}b_j$. The identity tensor $\mathbf{I}$ transforms a vector $\mathbf{u}$ into itself ($\mathbf{u} = \mathbf{I}\mathbf{u}$), and has a representation

$$\mathbf{I} = \delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{e}_i \otimes \mathbf{e}_i. \tag{2.4}$$

The scalar triple product of three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, denoted by $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$, is defined by

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}). \tag{2.5}$$

The determinant of a tensor $\mathbf{A}$ can be written as

$$\det \mathbf{A} = \frac{[\mathbf{A}\mathbf{a}, \mathbf{A}\mathbf{b}, \mathbf{A}, \mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}, \tag{2.6}$$

where $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is an arbitrary set of basis vectors, and $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \neq 0$. Suppose that $\mathbf{A}$ is invertible. The time derivative of $\det \mathbf{A}$ is given by

$$\dot{\overline{\det \mathbf{A}}} = \operatorname{tr}\left(\dot{\mathbf{A}}\mathbf{A}^{-1}\right) \det \mathbf{A}. \tag{2.7}$$

A fourth-order tensor $\boldsymbol{\mathcal{L}}$ is a linear mapping that assigns to each second-order tensor $\mathbf{A}$ a second-order tensor $\mathbf{B} = \boldsymbol{\mathcal{L}}[\mathbf{A}]$, with $\mathbf{A}, \mathbf{B} \in \mathsf{Lin}$. This can be written in index notation as

$$B_{KL} = \mathcal{L}_{KLMN}A_{MN}. \tag{2.8}$$

The tensor product $\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} \otimes \mathbf{d} = (\mathbf{a} \otimes \mathbf{b}) \otimes (\mathbf{c} \otimes \mathbf{d})$ is defined by

$$(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} \otimes \mathbf{d})[\mathbf{u} \otimes \mathbf{v}] = \mathbf{a} \otimes \mathbf{b}(\mathbf{c} \otimes \mathbf{d}) \cdot (\mathbf{u} \otimes \mathbf{v}) \tag{2.9}$$

for all $\mathbf{u}, \mathbf{v} \in \mathcal{V}$. The 4-tensor $\boldsymbol{\mathcal{L}}$ has a representation

$$\boldsymbol{\mathcal{L}} = \mathcal{L}_{ijk\ell}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_\ell, \tag{2.10}$$

where

$$\mathcal{L}_{ijk\ell} = (\mathbf{e}_i \otimes \mathbf{e}_j) \cdot \boldsymbol{\mathcal{L}}[\mathbf{e}_k \otimes \mathbf{e}_\ell] = \mathbf{e}_i \cdot \boldsymbol{\mathcal{L}}[\mathbf{e}_k \otimes \mathbf{e}_\ell]\mathbf{e}_j, \qquad (i, j, k, \ell = 1, 2, 3). \tag{2.11}$$

The fourth-order identity tensor $\boldsymbol{\mathcal{I}}$ has a representation

$$\boldsymbol{\mathcal{I}} = \delta_{ik}\delta_{j\ell}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_\ell = \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_i \otimes \mathbf{e}_j. \tag{2.12}$$

It can further be decomposed into

$$\boldsymbol{\mathcal{I}} = \boldsymbol{\mathscr{I}} + \boldsymbol{\mathscr{J}}, \tag{2.13}$$

where, in component form, $\boldsymbol{\mathscr{I}}$ and $\boldsymbol{\mathscr{J}}$ are

$$\mathscr{I}_{ijk\ell} = \frac{1}{2}(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}), \qquad \mathscr{J}_{ijk\ell} = \frac{1}{2}(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}). \tag{2.14}$$

The transpose $\boldsymbol{\mathcal{L}}^T$ of $\boldsymbol{\mathcal{L}}$ is defined by its action on two $\mathbf{A}, \mathbf{B} \in \mathsf{Lin}$:

$$\mathbf{A} \cdot \boldsymbol{\mathcal{L}}[\mathbf{B}] = \mathbf{B} \cdot \boldsymbol{\mathcal{L}}^T[\mathbf{A}]. \tag{2.15}$$

Also,

$$(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} \otimes \mathbf{d})^T = \mathbf{c} \otimes \mathbf{d} \otimes \mathbf{a} \otimes \mathbf{b}. \tag{2.16}$$

We define an operation $\odot$ between two fourth-order tensors by

$$(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} \otimes \mathbf{d}) \odot (\mathbf{e} \otimes \mathbf{f} \otimes \mathbf{g} \otimes \mathbf{h}) = (\mathbf{c} \cdot \mathbf{f})(\mathbf{d} \cdot \mathbf{h})(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{e} \otimes \mathbf{g}). \tag{2.17}$$

Applied to a tensor product $\mathbf{u} \otimes \mathbf{v}$, it gives

$$\begin{aligned}(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} \otimes \mathbf{d}) \odot (\mathbf{e} \otimes \mathbf{f} \otimes \mathbf{g} \otimes \mathbf{h})[\mathbf{u} \otimes \mathbf{v}] &= (\mathbf{c} \cdot \mathbf{f})(\mathbf{d} \cdot \mathbf{h})(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{e} \otimes \mathbf{g})[\mathbf{u} \otimes \mathbf{v}] \\ &= (\mathbf{c} \cdot \mathbf{f})(\mathbf{d} \cdot \mathbf{h})(\mathbf{e} \cdot \mathbf{u})(\mathbf{g} \cdot \mathbf{v})(\mathbf{a} \otimes \mathbf{b}).\end{aligned} \tag{2.18}$$

We also define another operation $\boxdot$ between two fourth-order tensors by

$$(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} \otimes \mathbf{d}) \boxdot (\mathbf{e} \otimes \mathbf{f} \otimes \mathbf{g} \otimes \mathbf{h}) = (\mathbf{b} \cdot \mathbf{e})(\mathbf{d} \cdot \mathbf{f})(\mathbf{a} \otimes \mathbf{c} \otimes \mathbf{g} \otimes \mathbf{h}) \tag{2.19}$$

Applied to $\mathbf{u} \otimes \mathbf{v}$,

$$\begin{aligned}(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} \otimes \mathbf{d}) \boxdot (\mathbf{e} \otimes \mathbf{f} \otimes \mathbf{g} \otimes \mathbf{h})[\mathbf{u} \otimes \mathbf{v}] &= (\mathbf{b} \cdot \mathbf{e})(\mathbf{d} \cdot \mathbf{f})(\mathbf{a} \otimes \mathbf{c} \otimes \mathbf{g} \otimes \mathbf{h})[\mathbf{u} \otimes \mathbf{v}] \\ &= (\mathbf{b} \cdot \mathbf{e})(\mathbf{d} \cdot \mathbf{f})(\mathbf{g} \cdot \mathbf{u})(\mathbf{h} \cdot \mathbf{v})(\mathbf{a} \otimes \mathbf{b}).\end{aligned} \tag{2.20}$$

Define a fourth-order tensor $\boldsymbol{\mathcal{J}}$ by

$$\boldsymbol{\mathcal{J}} = \mathbf{e}_i \otimes \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_j = \mathbf{I} \otimes \mathbf{I}. \tag{2.21}$$

We then have the following properties of $\odot$ and $\boxdot$:

$$\boldsymbol{\mathcal{A}} \odot \boldsymbol{\mathcal{J}} = \boldsymbol{\mathcal{A}}, \qquad \boldsymbol{\mathcal{J}} \boxdot \boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{A}}, \tag{2.22}$$

where $\boldsymbol{\mathcal{A}}$ is any fourth-order tensor.

2.2 Kinematics and Field Equations

Consider a deformable body $\mathscr{B}$. A configuration of $\mathscr{B}$ is the region of the three-dimensional Euclidean space $\mathbb{E}^3$ that the body occupies at time t. Let X be any particle of

$\mathscr{B}$. Choose a fixed origin in $\mathbb{E}^3$, and denote by $\mathbf{X}$ the position vector of X in a fixed reference configuration $\boldsymbol{\kappa}_0$. Let $\mathbf{x}$ be the position vector of X in the configuration $\boldsymbol{\kappa}$ at time t. We can use the rectangular Cartesian components X_A $(A = 1, 2, 3)$ and x_i $(i = 1, 2, 3)$ as the coordinates of $\mathbf{X}$ and $\mathbf{x}$, respectively.

A motion of $\mathscr{B}$ is a smooth one-parameter family of configurations such that

$$\mathbf{x} = \bar{\boldsymbol{\chi}}(X, t) = \boldsymbol{\chi}(\mathbf{X}, t) = \boldsymbol{\chi}_t(\mathbf{X}), \tag{2.23}$$

where the mapping $\boldsymbol{\chi}$ depends on the choice of reference configuration, while $\bar{\boldsymbol{\chi}}$ does not. For fixed t, the mapping $\boldsymbol{\chi}_t$ is invertible. Its inverse is given by

$$\mathbf{X} = \boldsymbol{\chi}_t^{-1}(\mathbf{x}, t). \tag{2.24}$$

The velocity and acceleration of the particle are

$$\mathbf{v} = \dot{\mathbf{x}} = \frac{\partial \boldsymbol{\chi}}{\partial t}, \qquad \mathbf{a} = \dot{\mathbf{v}} = \frac{\partial^2 \boldsymbol{\chi}}{\partial t^2}, \tag{2.25}$$

where the superposed dot indicates a material time derivative. The deformation gradient relative to the reference and its determinant are

$$\mathbf{F} = \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}}(\mathbf{X}, t) = \frac{\partial x_i}{\partial X_A}\, \mathbf{e}_i \otimes \mathbf{E}_A, \qquad J = \det \mathbf{F} > 0. \tag{2.26}$$

where $\mathbf{e}_i$ and $\mathbf{E}_A$ are fixed orthonormal bases in $\mathbb{E}^3$. The polar decomposition theorem provides two decompositions of $\mathbf{F}$ into unique factors,

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}, \tag{2.27}$$

where $\mathbf{U}$ and $\mathbf{V}$ are symmetric positive definite tensors called the right and left stretch tensor, respectively, and the rotation tensor $\mathbf{R}$ is proper orthogonal. The right and left Cauchy-Green tensors $\mathbf{C}$ and $\mathbf{B}$ are

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2, \qquad \mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2 = \mathbf{R}\mathbf{C}\mathbf{R}^T. \tag{2.28}$$

The Lagrangian strain tensor is given by

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{C} - \mathbf{I}\right), \tag{2.29}$$

where $\mathbf{I}$ is the identity tensor.

The material time derivative of $\mathbf{F}$ is

$$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}, \tag{2.30}$$

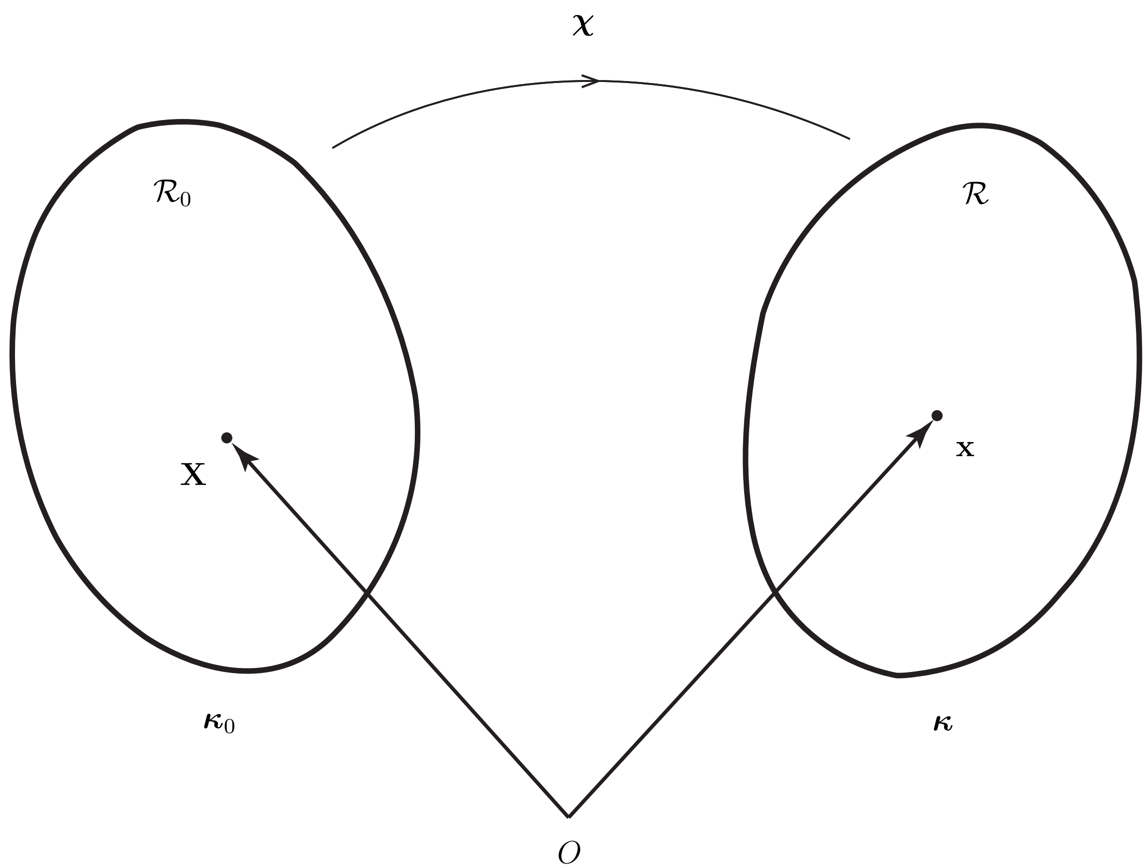

Figure 2.1: Reference and current configurations of a deformable continuum

where

$$\mathbf{L} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \frac{\partial v_i}{\partial x_j}\, \mathbf{e}_i \otimes \mathbf{e}_j \tag{2.31}$$

is the velocity gradient of the body in its current configuration. The tensor $\mathbf{L}$ can be decomposed into its symmetric part and skew-symmetric part:

$$\mathbf{L} = \mathbf{D} + \mathbf{W}, \tag{2.32}$$

where

$$\mathbf{D} = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^T\right) = \mathbf{D}^T \tag{2.33}$$

is the rate of deformation tensor and

$$\mathbf{W} = \frac{1}{2}\left(\mathbf{L} - \mathbf{L}^T\right) = -\mathbf{W}^T \tag{2.34}$$

is the vorticy tensor. The Lagrangian strain rate can be expressed in terms of the rate of deformation tensor and the velocity gradient by

$$\dot{\mathbf{E}} = \mathbf{F}^T \mathbf{D} \mathbf{F}. \tag{2.35}$$

The Cauchy stress tensor is denoted by $\mathbf{T}$, the non-symmetric Piola-Kirchhoff tensor is $\mathbf{P}$, and the symmetric Piola-Kirchhoff tensor is $\mathbf{S}$. They are related by

$$J\mathbf{T} = \mathbf{P}\mathbf{F}^T = \mathbf{F}\mathbf{S}\mathbf{F}^T. \tag{2.36}$$

We also introduce the rotated Cauchy stress tensor $\widetilde{\mathbf{T}}$:

$$\widetilde{\mathbf{T}} = \mathbf{R}^T \mathbf{T} \mathbf{R} = \frac{1}{J}\mathbf{U}\mathbf{S}\mathbf{U}, \tag{2.37}$$

and a rotated form $\widetilde{\mathbf{D}}$ of the rate of deformation tensor:

$$\widetilde{\mathbf{D}} = \mathbf{R}^T \mathbf{D} \mathbf{R}. \tag{2.38}$$

Note that

$$\mathbf{T} \cdot \mathbf{D} = \widetilde{\mathbf{T}} \cdot \widetilde{\mathbf{D}}. \tag{2.39}$$

Let ρ_0 and ρ denote the mass density of $\mathscr{B}$ in the reference and current configurations, respectively. Also, let $\mathbf{b}$ be the body force field per unit mass acting on the current configuration. The local forms of the conservation of mass and balance of linear momentum may be expressed by

$$\begin{aligned} \rho_0 &= \rho J, \\ \operatorname{div} \mathbf{T} + \rho \mathbf{b} &= \rho \dot{\mathbf{v}}. \end{aligned} \tag{2.40}$$

The balance of angular momentum implies the symmetry of $\mathbf{T}$.

2.3 Invariance Requirements

Consider a motion $\boldsymbol{\chi}^+$ of $\mathscr{B}$ which differs from $\boldsymbol{\chi}$ by a rigid motion and maps particles into a configuration $\boldsymbol{\kappa}^+$ at time $t^+ = t + a$, where a =constant:

$$\mathbf{x}^+ = \boldsymbol{\chi}^+(\mathbf{X}, t) = \mathbf{Q}(t)\mathbf{x} + \mathbf{c}(t), \tag{2.41}$$

where $\mathbf{Q}(t)$ is a proper orthogonal tensor that represents rigid rotation and $\mathbf{c}(t)$ is a vector that represents rigid body translation. Thus the deformation gradient $\mathbf{F}^+$ and its determinant J^+ are

$$\mathbf{F}^+ = \frac{\partial \boldsymbol{\chi}^+}{\partial \mathbf{X}} = \mathbf{Q}\mathbf{F}, \qquad J^+ = \det \mathbf{F}^+ = J. \tag{2.42}$$

It follows that

$$\mathbf{C}^+ = \mathbf{C}, \quad \mathbf{U}^+ = \mathbf{U}, \quad \mathbf{R}^+ = \mathbf{QR}, \quad \mathbf{V}^+ = \mathbf{QVQ}^T, \quad \mathbf{B}^+ = \mathbf{QBQ}^T, \tag{2.43}$$

and

$$\mathbf{E}^+ = \mathbf{E}, \qquad \dot{\mathbf{E}}^+ = \dot{\mathbf{E}}. \tag{2.44}$$

Also, $\mathbf{L}$, $\mathbf{D}$, and $\mathbf{W}$ become

$$\mathbf{L}^+ = \mathbf{QLQ}^T + \boldsymbol{\Omega}, \qquad \mathbf{D}^+ = \mathbf{QDQ}^T, \qquad \mathbf{W}^+ = \mathbf{QWQ}^T + \boldsymbol{\Omega}, \tag{2.45}$$

where $\boldsymbol{\Omega} = \dot{\mathbf{Q}}\mathbf{Q}^T$ is the angular velocity tensor due to the superposed rigid motion. Further, in the absence of internal constraints, the stress measures $\mathbf{T}$, $\widetilde{\mathbf{T}}$, $\mathbf{P}$ and $\mathbf{S}$ become

$$\mathbf{T}^+ = \mathbf{QTQ}^T, \qquad (\widetilde{\mathbf{T}})^+ = \widetilde{\mathbf{T}} \qquad \mathbf{P}^+ = \mathbf{QP}, \qquad \mathbf{S}^+ = \mathbf{S} \tag{2.46}$$

under superposed rigid body motions.

2.4 Elastic Solids

A nonlinearly elastic solid possesses a strain energy function $\psi = \psi(\mathbf{F})$ such that

$$\rho\dot{\psi} = \mathbf{T} \cdot \mathbf{D}. \tag{2.47}$$

Invariance requirements imply that ψ is a function of $\mathbf{E}$. The Cauchy stress tensor and the symmetric Piola-Kirchhoff stress tensor may be expressed as

$$\mathbf{T} = \frac{1}{2}\rho\mathbf{F}\frac{\partial\psi}{\partial\mathbf{E}}\mathbf{F}^T, \qquad \mathbf{S} = \rho_0\frac{\partial\psi}{\partial\mathbf{E}}, \tag{2.48}$$

where the tensor $\partial\psi/\partial\mathbf{E}$ is understood to be symmetric, i.e.,

$$\frac{\partial\psi}{\partial\mathbf{E}} = \frac{1}{2}\left[\frac{\partial\psi}{\partial\mathbf{E}} + \left(\frac{\partial\psi}{\partial\mathbf{E}}\right)^T\right]. \tag{2.49}$$

We note that since $\mathbf{S} = \hat{\mathbf{S}}(\mathbf{E})$, the stress rate is

$$\dot{\mathbf{S}} = \frac{\partial\mathbf{S}}{\partial\mathbf{E}}[\dot{\mathbf{E}}] = \mathscr{L}[\dot{\mathbf{E}}], \tag{2.50}$$

where $\mathscr{L}$ is a symmetric fourth-order tensor. The Lagrangian form of the stress rate $\dot{\mathbf{S}}$ is a linear function of the strain rate, and the rate itself is objective. The Eulerian form $\dot{\mathbf{T}}$ is not objective. A commonly used objective rate for the Cauchy stress is the Jaumann stress rate:

$$\overset{o}{\mathbf{T}} = \dot{\mathbf{T}} - \mathbf{W}\mathbf{T} + \mathbf{T}\mathbf{W}. \tag{2.51}$$

In addition, the upper and lower convected rates are

$$\begin{aligned} \overset{\triangledown}{\mathbf{T}} &= \dot{\mathbf{T}} - \mathbf{L}^T\mathbf{T} - \mathbf{T}\mathbf{L} \\ \overset{\vartriangle}{\mathbf{T}} &= \dot{\mathbf{T}} + \mathbf{L}\mathbf{T} + \mathbf{T}\mathbf{L}^T = \overset{o}{\mathbf{T}} + \mathbf{T}\mathbf{D} + \mathbf{D}\mathbf{T} \end{aligned} \tag{2.52}$$

Also, the "box" rate, $\overset{\square}{\mathbf{T}}$, is given by

$$\begin{aligned} \overset{\square}{\mathbf{T}} &= \dot{\mathbf{T}} + \mathbf{T}\mathbf{W}_R - \mathbf{W}_R\mathbf{T} \\ &= \mathbf{R}\dot{\tilde{\mathbf{T}}}\mathbf{R}^T, \end{aligned} \tag{2.53}$$

where

$$\mathbf{W}_R = \dot{\mathbf{R}}\mathbf{R}^T = -\mathbf{W}_R^T \tag{2.54}$$

is the spin associated with the local rotation $\mathbf{R}$. Finally, the Truesdell rate is

$$\overset{t}{\mathbf{T}} = \dot{\mathbf{T}} - \mathbf{L}\mathbf{T} - \mathbf{T}\mathbf{L}^T + (\operatorname{tr}\mathbf{D})\mathbf{T} = \frac{1}{J}\mathbf{F}\dot{\mathbf{S}}\mathbf{F}^T. \tag{2.55}$$

Also, we have the transformation laws

$$\begin{aligned} \overset{o}{\mathbf{T}}{}^+ = \mathbf{Q}\overset{o}{\mathbf{T}}\mathbf{Q}^T, \qquad \overset{\triangledown}{\mathbf{T}}{}^+ = \mathbf{Q}\overset{\triangledown}{\mathbf{T}}\mathbf{Q}^T, \qquad \overset{\vartriangle}{\mathbf{T}}{}^+ = \mathbf{Q}\overset{\vartriangle}{\mathbf{T}}\mathbf{Q}^T, \\ \overset{\square}{\mathbf{T}}{}^+ = \mathbf{Q}\overset{\square}{\mathbf{T}}\mathbf{Q}^T, \qquad \overset{t}{\mathbf{T}}{}^+ = \mathbf{Q}\overset{t}{\mathbf{T}}\mathbf{Q}^T. \end{aligned} \tag{2.56}$$

Chapter 3

The Green-Naghdi Theory of Elastic-Plastic Materials

The main elements of a purely mechanical rate-type theory of a finitely deforming elastic-plastic solid are summarized in this chapter. This constitutive theory of classical plasticity was first proposed by Green and Naghdi in 1965 [18] and expanded upon by them and their various collaborators during the ensuing years.

In addition to the Lagrangian strain tensor $\mathbf{E}$, we assume the existence of the plastic strain $\mathbf{E}_p$, which is a symmetric second-order tensor, a scalar measure of work hardening κ, and a symmetric shift or back stress tensor $\boldsymbol{\alpha}_R$. The constitutive equation for the stress $\mathbf{S}$ is defined in terms of a stress response function given by

$$\mathbf{S} = \hat{\mathbf{S}}(\mathbf{E}, \mathscr{U}), \tag{3.1}$$

where $\mathscr{U}$ is shorthand notation for the plastic parameters

$$\mathscr{U} = (\mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa). \tag{3.2}$$

The response function $\hat{\mathbf{S}}$ depends upon the choice of reference configuration, $\boldsymbol{\kappa}_0$, which can be taken arbitrarily. Under superposed rigid body motions, the variables in (3.2) are assumed to be unaltered, i.e.,

$$(\mathbf{E}_p)^+ = \mathbf{E}_p, \qquad (\boldsymbol{\alpha}_R)^+ = \boldsymbol{\alpha}_R, \qquad \kappa^+ = \kappa. \tag{3.3}$$

Alternatively, the stress response may also be expressed mathematically in the form

$$\mathbf{S} = \bar{\mathbf{S}}(\mathbf{E} - \mathbf{E}_p, \mathscr{U}). \tag{3.4}$$

The elasticity of the elastic-plastic material is described by the fourth-order tensor

$$\mathscr{L} = \frac{\partial \hat{\mathbf{S}}}{\partial \mathbf{E}}(\mathbf{E}, \mathscr{U}) = \frac{\partial \bar{\mathbf{S}}}{\partial (\mathbf{E} - \mathbf{E}_p)}(\mathbf{E} - \mathbf{E}_p, \mathscr{U}). \tag{3.5}$$

More details about this alternative version are presented in Section 3.4 in page 22.

Note that for each fixed value of $\mathscr{U}$, the stress response $\hat{\mathbf{S}}$ has the same form as an elastic material. Thus, an elastic-plastic material can be viewed as a parametrized family of elastic materials. For fixed values of $\mathscr{U}$, it is also assumed that the stress response is smoothly invertible to give

$$\mathbf{E} = \hat{\mathbf{E}}(\mathbf{S}, \mathscr{U}). \tag{3.6}$$

Similarly, (3.4) can be smoothly inverted to

$$\mathbf{E} - \mathbf{E}_p = \hat{\mathbf{E}}(\mathbf{S}, \mathscr{U}) - \mathbf{E}_p = \bar{\mathbf{E}}(\mathbf{S}, \mathscr{U}). \tag{3.7}$$

We also define a fourth-order compliance tensor by

$$\mathscr{M} = \frac{\partial \hat{\mathbf{E}}}{\partial \mathbf{S}}(\mathbf{S}, \mathscr{U}). \tag{3.8}$$

3.1 The elastic regions in strain space and stress space

Now we admit the existence of a scalar-valued function $g(\mathbf{E}, \mathscr{U})$, called a yield or loading function. For fixed values of $\mathscr{U}$, the equation

$$g(\mathbf{E}, \mathscr{U}) = 0 \tag{3.9}$$

represents a smooth orientable five-dimensional hypersurface $\partial\mathscr{E}$ enlosing an open region $\mathscr{E}$ in strain space (Figure 3.1). The hypersurface $\partial\mathscr{E}$ is called the yield surface in strain space. The points on the surface are called elastic-plastic points. The open region $\mathscr{E}$ is the elastic region. The yield function $g < 0$ for all points in $\mathscr{E}$. Corresponding to a motion $\boldsymbol{\chi}$, for each particle of the body, we can associate a smooth orientable curve C_e, which is parametrized by time t. The curve C_e is called a strain trajectory. The strain trajectories in strain space are restricted to lie initially in the elastic region or on its surface. The notation

$$\hat{g} = \frac{\partial g}{\partial \mathbf{E}} \cdot \dot{\mathbf{E}} \tag{3.10}$$

is used for the inner product of the outward normal vector $\partial g / \partial \mathbf{E}$ to the yield surface and the tangent vector $\dot{\mathbf{E}}$ to the strain trajectory.

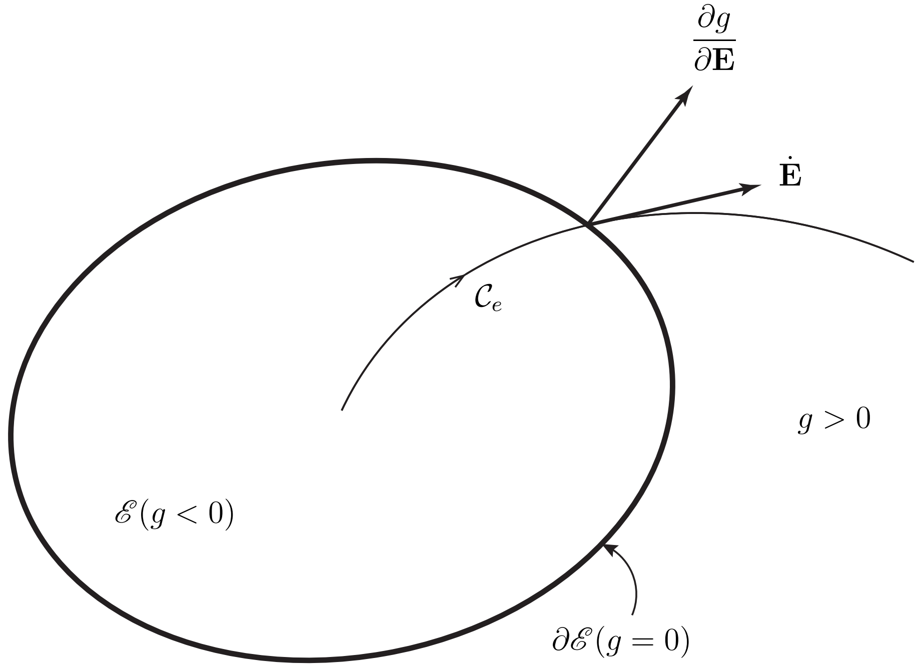

Figure 3.1: The yield surface in strain space

The yield function $f(\mathbf{S}, \mathscr{U})$ in stress space can be constructed from g using the response functions for stress and strain:

$$g(\mathbf{E}, \mathscr{U}) = g(\hat{\mathbf{E}}(\mathbf{S}, \mathscr{U}), \mathscr{U}) = f(\mathbf{S}, \mathscr{U}). \tag{3.11}$$

For fixed values of $\mathscr{U}$, the equation

$$f(\mathbf{S}, \mathscr{U}) = 0 \tag{3.12}$$

represents a five-dimensional hypersurface $\partial\mathscr{S}$, called the yield surface, delineating the elastic region $\mathscr{S}$ in stress space (Figure 3.2). For all points in $\mathscr{S}$, $f < 0$. The yield surface and the elastic region in stress space have the same topological properties as those in strain space. Clearly, from (3.11), a point in strain space belongs to the elastic region $\mathscr{E}$ if, and only if, the corresponding point in stress space belongs to the elastic region $\mathscr{S}$. Likewise, a point in strain space lies on the yield surface $\partial\mathscr{E}$ if, and only if, the corresponding point in stress space lies on the yield surface $\partial\mathscr{S}$. We use the notation

$$\hat{f} = \frac{\partial f}{\partial \mathbf{S}} \cdot \dot{\mathbf{S}} \tag{3.13}$$

for the inner product of the normal to the yield surface in stress space and the tangent vector to the stress trajectory C_s. Recalling (3.5), the normal vectors to the yield surfaces are related by

$$\frac{\partial g}{\partial \mathbf{E}} = \boldsymbol{\mathscr{L}}^T \left[\frac{\partial f}{\partial \mathbf{S}}\right]. \tag{3.14}$$

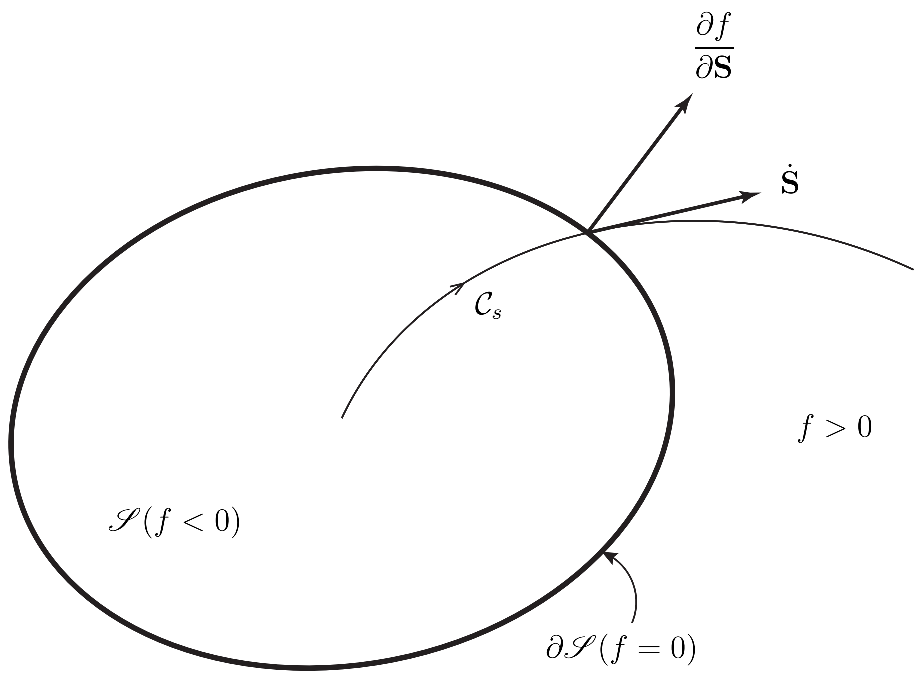

Figure 3.2: The yield surface in stress space

3.2 Loading criteria

Adopting the strain space formulation as primary, we state the loading criteria defined relative to the yield surface $\partial\mathscr{E}$ in strain space as

$$\begin{array}{lll} (a) & g < 0 & \text{Elastic state} \\ (b) & g = 0,\ \hat{g} < 0 & \text{Unloading from an elastic plastic state} \\ (c) & g = 0,\ \hat{g} = 0 & \text{Neutral loading from an elastic plastic state} \\ (d) & g = 0,\ \hat{g} > 0 & \text{Loading from an elastic plastic state.} \end{array} \tag{3.15}$$

In an elastic state, the strain trajectory lies entirely in $\mathscr{E}$, and the yield surface $\partial\mathscr{E}$ remains stationary. During unloading, the strain trajectory intersects the yield surface and is pointing towards the elastic region. The function g is decreasing, and $\partial\mathscr{E}$ remains stationary. During neutral loading, the strain trajectory lies on the yield surface and stays on the yield surface. The yield function remains zero, and $\partial\mathscr{E}$ remains stationary. During loading, the strain trajectory intersects the yield surface and is directed outwards.

The implied conditions in stress space for the elastic state, unloading and neutral loading can be deduced from those in strain space. However, the strain space criterion for loading is not equivalent to the loading criterion in stress space. Thus, $\hat{f}$ cannot be used alone in the loading criteria.

3.3 Strain hardening response, flow rules and hardening rules

The material derivative of the stress tensor $\mathbf{S}$ is given by

$$\dot{\mathbf{S}} = \boldsymbol{\mathscr{L}}[\dot{\mathbf{E}}] + \frac{\partial \hat{\mathbf{S}}}{\partial \mathbf{E}_p}[\dot{\mathbf{E}}_p] + \frac{\partial \hat{\mathbf{S}}}{\partial \boldsymbol{\alpha}_R}[\dot{\boldsymbol{\alpha}}_R] + \frac{\partial \hat{\mathbf{S}}}{\partial \kappa}\dot{\kappa}. \tag{3.16}$$

If the strain trajectory lies in the elastic region, or if unloading or neutral loading occurs, it is assumed that

$$\dot{\mathbf{E}}_p = \mathbf{0}, \qquad \dot{\boldsymbol{\alpha}}_R = \mathbf{0}, \qquad \dot{\kappa} = 0. \tag{3.17}$$

In other words, the plastic parameters are assumed to remain constant. This means that the elastic regions in stress space and strain space remain the same, and the material behaves like an elastic solid.

If loading occurs, we assume that the material time derivatives of the plastic parameters are linear to the strain rate $\dot{\mathbf{E}}$. Following a procedure used by Green and Naghdi [18], the flow and hardening rules are given by the rate-type constitutive equations

$$\dot{\mathbf{E}}_p = \pi\hat{g}\boldsymbol{\rho}, \qquad \dot{\boldsymbol{\alpha}}_R = \pi\hat{g}\boldsymbol{\beta}, \qquad \dot{\kappa} = \pi\hat{g}\lambda, \tag{3.18}$$

where $\boldsymbol{\rho}$, $\boldsymbol{\beta}$ and λ are constitutive functions of the parameters $\mathbf{E}$ and $\mathscr{U}$, and π is a scalar multiplier.

Thus, during loading, the stress rate can be written as

$$\dot{\mathbf{S}} = \boldsymbol{\mathscr{L}}[\dot{\mathbf{E}}] + \pi\hat{g}\boldsymbol{\sigma}, \tag{3.19}$$

where

$$\boldsymbol{\sigma} = \frac{\partial \hat{\mathbf{S}}}{\partial \mathbf{E}_p}[\boldsymbol{\rho}] + \frac{\partial \hat{\mathbf{S}}}{\partial \boldsymbol{\alpha}_R}[\boldsymbol{\beta}] + \frac{\partial \hat{\mathbf{S}}}{\partial \kappa}\lambda. \tag{3.20}$$

Further, with the use of (3.10) and (3.14), the stress rate can be written as

$$\dot{\mathbf{S}} = \boldsymbol{\mathscr{K}}\boldsymbol{\mathscr{L}}[\dot{\mathbf{E}}], \tag{3.21}$$

where

$$\boldsymbol{\mathscr{K}} = \boldsymbol{\mathscr{I}} + \pi\boldsymbol{\sigma} \otimes \frac{\partial f}{\partial \mathbf{S}}, \tag{3.22}$$

is a fourth-order tensor, and the fourth-order identity tensor $\boldsymbol{\mathscr{I}}$ is defined by

$$\boldsymbol{\mathscr{I}} = \frac{1}{2}\left(\delta_{KM}\delta_{LN} + \delta_{KN}\delta_{LM}\right)\mathbf{e}_K \otimes \mathbf{e}_L \otimes \mathbf{e}_M \otimes \mathbf{e}_N. \tag{3.23}$$

The yield surfaces $\partial\mathscr{E}$ and $\partial\mathscr{S}$ do not move if $g < 0$ or if ($g = 0$, $\hat{g} \leq 0$). However, during loading ($g = 0$, $\hat{g} > 0$), it is assumed that as the strain trajectory tries to cross the yield surface $\partial\mathscr{E}$, the surface is locally carried along with the trajectory. That is, loading from an elastic-plastic state always leads to another elastic-plastic state. This is Prager's consistency condition. Hence,

$$g = 0,\ \hat{g} > 0 \Rightarrow \dot{g} = 0. \tag{3.24}$$

And by virtue of (3.11), this also implies that $\dot{f} = 0$. Taking the material derivative of g and using (3.18),

$$\begin{aligned} 0 = \dot{g} &= \hat{g} + \frac{\partial g}{\partial \mathbf{E}_p} \cdot \dot{\mathbf{E}}_p + \frac{\partial g}{\partial \boldsymbol{\alpha}_R} \cdot \dot{\boldsymbol{\alpha}}_R + \frac{\partial g}{\partial \kappa} \cdot \dot{\kappa} \\ &= \hat{g}\left[1 + \pi\left(\frac{\partial g}{\partial \mathbf{E}_p} \cdot \boldsymbol{\rho} + \frac{\partial g}{\partial \boldsymbol{\alpha}_R} \cdot \boldsymbol{\beta} + \frac{\partial g}{\partial \kappa} \cdot \lambda\right)\right]. \end{aligned} \tag{3.25}$$

Thus, π cannot vanish, and it is taken to be positive, without loss of generality. Then we have

$$\frac{1}{\pi} = -\left(\frac{\partial g}{\partial \mathbf{E}_p} \cdot \boldsymbol{\rho} + \frac{\partial g}{\partial \boldsymbol{\alpha}_R} \cdot \boldsymbol{\beta} + \frac{\partial g}{\partial \kappa} \cdot \lambda\right) > 0. \tag{3.26}$$

The multiplier π can be calculated when the constitutive functions are specified. Similarly, taking the material derivative of (3.11), and employing the consistency condition,

$$\begin{aligned} 0 = \dot{f} &= \hat{f} + \frac{\partial f}{\partial \mathbf{E}_p} \cdot \dot{\mathbf{E}}_p + \frac{\partial f}{\partial \boldsymbol{\alpha}_R} \cdot \dot{\boldsymbol{\alpha}}_R + \frac{\partial f}{\partial \kappa} \cdot \dot{\kappa}, \\ &= \hat{f} + \pi\hat{g}\left(\frac{\partial f}{\partial \mathbf{E}_p} \cdot \boldsymbol{\rho} + \frac{\partial f}{\partial \boldsymbol{\alpha}_R} \cdot \boldsymbol{\beta} + \frac{\partial f}{\partial \kappa} \cdot \lambda\right). \end{aligned} \tag{3.27}$$

It follows from (3.27) and (3.26) that during loading,

$$\begin{aligned} \frac{\hat{f}}{\hat{g}} &= -\pi\left(\frac{\partial f}{\partial \mathbf{E}_p} \cdot \boldsymbol{\rho} + \frac{\partial f}{\partial \boldsymbol{\alpha}_R} \cdot \boldsymbol{\beta} + \frac{\partial f}{\partial \kappa} \cdot \lambda\right) \\ &= \frac{\dfrac{\partial f}{\partial \mathbf{E}_p} \cdot \boldsymbol{\rho} + \dfrac{\partial f}{\partial \boldsymbol{\alpha}_R} \cdot \boldsymbol{\beta} + \dfrac{\partial f}{\partial \kappa} \cdot \lambda}{\dfrac{\partial g}{\partial \mathbf{E}_p} \cdot \boldsymbol{\rho} + \dfrac{\partial g}{\partial \boldsymbol{\alpha}_R} \cdot \boldsymbol{\beta} + \dfrac{\partial g}{\partial \kappa} \cdot \lambda} \\ &= \Phi. \end{aligned} \tag{3.28}$$

The quotient $\hat{f}/\hat{g}$ can be defined by a single dimensionless function Φ, which can also be expressed as

$$\begin{aligned} \Phi &= 1 + \pi\left[\left(\frac{\partial g}{\partial \mathbf{E}_p} - \frac{\partial f}{\partial \mathbf{E}_p}\right) \cdot \boldsymbol{\rho} + \left(\frac{\partial g}{\partial \boldsymbol{\alpha}_R} - \frac{\partial f}{\partial \boldsymbol{\alpha}_R}\right) \cdot \boldsymbol{\beta} + \lambda\left(\frac{\partial g}{\partial \kappa} - \frac{\partial f}{\partial \boldsymbol{\kappa}}\right)\right] \\ &= 1 + \pi\frac{\partial f}{\partial \mathbf{S}} \cdot \boldsymbol{\sigma} \end{aligned} \tag{3.29}$$

with the use of $(3.27)_3$, $(3.28)_1$, (3.20), and (3.11).

The yield function g is positive during loading, while f can be positive, negative, or zero. Thus, the function Φ may be positive, negative or zero. To distinguish between the three cases, Casey and Naghdi [9] proposed a classification of strain-hardening behavior into three distinct types:

$$\begin{array}{lll} (a) & \Phi > 0 & \text{hardening} \\ (b) & \Phi < 0 & \text{softening} \\ (c) & \Phi = 0 & \text{perfectly plastic} \end{array} \tag{3.30}$$

The yield surface in stress space moves outward locally if $\Phi > 0$, inward if $\Phi < 0$, and remains stationary if $\Phi = 0$. The loading criteria in strain space and stress space are equivalent when the material is hardening.

It can be shown that the fourth-order tensor $\boldsymbol{\mathscr{K}}$ is related to Φ by [13]

$$\det \boldsymbol{\mathscr{K}} = \Phi. \tag{3.31}$$

During perfectly plastic behavior, $\det \boldsymbol{\mathscr{K}} = 0$, meaning the stress rate in the form (3.21) is not invertible. Thus, perfectly plastic behavior can only be described in strain space.

3.4 An equivalent set of kinematical measures

In the Green-Naghdi theory, we may also represent the constitutive equation for $\mathbf{S}$ in terms of an equivalent set of kinematical measures in the form

$$\mathbf{S} = \bar{\mathbf{S}}(\mathbf{E} - \mathbf{E}_p, \mathscr{U}), \qquad S_{MN} = \bar{S}_{MN}(E_{KL} - E_{pKL}, E_{pKL}, \alpha^R_{KL}, \kappa). \tag{3.32}$$

Both direct notation and index notation are included here for convenience.

In this section we explore the results of the previous section in terms of these new variables. Thus, using the chain rule of differentiation and from (3.1), (3.6), and (3.32), we have

$$\begin{gathered} \frac{\partial \bar{\mathbf{S}}}{\partial(\mathbf{E} - \mathbf{E}_p)} = \boldsymbol{\mathscr{L}}, \qquad \frac{\partial \bar{\mathbf{S}}}{\partial \mathbf{E}_p} = \boldsymbol{\mathscr{L}} + \frac{\partial \hat{\mathbf{S}}}{\partial \mathbf{E}_p}, \qquad \frac{\partial \bar{\mathbf{S}}}{\partial \boldsymbol{\alpha}_R} = \frac{\partial \hat{\mathbf{S}}}{\partial \boldsymbol{\alpha}_R}, \qquad \frac{\partial \bar{\mathbf{S}}}{\partial \kappa} = \frac{\partial \hat{\mathbf{S}}}{\partial \kappa}, \\ \frac{\partial \bar{S}_{MN}}{\partial(E_{KL} - E_{pKL})} = \mathscr{L}_{MNKL}, \qquad \frac{\partial \bar{S}_{MN}}{\partial E_{pKL}} = \mathscr{L}_{MNKL} + \frac{\partial \hat{S}_{MN}}{\partial E_{pKL}}, \\ \frac{\partial \bar{S}_{MN}}{\partial \alpha^R_{KL}} = \frac{\partial \hat{S}_{MN}}{\partial \alpha^R_{KL}}, \qquad \frac{\partial \bar{S}_{MN}}{\partial \kappa} = \frac{\partial \hat{S}_{MN}}{\partial \kappa}. \end{gathered} \tag{3.33}$$

We introduce a second-order tensor by

$$\begin{aligned} \bar{\boldsymbol{\sigma}} &= \frac{\partial \bar{\mathbf{S}}}{\partial \mathbf{E}_p}[\boldsymbol{\rho}] + \frac{\partial \bar{\mathbf{S}}}{\partial \boldsymbol{\alpha}_R}[\boldsymbol{\beta}] + \frac{\partial \bar{\mathbf{S}}}{\partial \kappa}\lambda = \boldsymbol{\sigma} + \boldsymbol{\mathscr{L}}[\boldsymbol{\rho}], \\ \bar{\sigma}_{MN} &= \frac{\partial \bar{S}_{MN}}{\partial E_{pKL}}\rho_{KL} + \frac{\partial \bar{S}_{MN}}{\partial \alpha^R_{KL}}\beta_{KL} + \lambda\frac{\partial \bar{S}_{MN}}{\partial \kappa} = \sigma_{MN} + \mathscr{L}_{MNKL}\rho_{KL}. \end{aligned} \tag{3.34}$$

Then with the use of $(3.34)_2$ and (3.14), the dimensionless function Φ from $(3.29)_2$ becomes

$$\Phi = 1 + \pi\frac{\partial f}{\partial \mathbf{S}}\cdot\bar{\boldsymbol{\sigma}} - \pi\frac{\partial g}{\partial \mathbf{E}}\cdot\boldsymbol{\rho}. \tag{3.35}$$

The stress rate $\dot{\mathbf{S}}$ can be written as

$$\begin{aligned} \dot{\mathbf{S}} &= \boldsymbol{\mathscr{L}}[\dot{\mathbf{E}} - \dot{\mathbf{E}}_p] + \frac{\partial \bar{\mathbf{S}}}{\partial \mathbf{E}_p}[\dot{\mathbf{E}}_p] + \frac{\partial \bar{\mathbf{S}}}{\partial \boldsymbol{\alpha}_R}[\dot{\boldsymbol{\alpha}}_R] + \frac{\partial \bar{\mathbf{S}}}{\partial \kappa}\dot{\kappa}, \\ \dot{S}_{MN} &= \mathscr{L}_{MNKL}(\dot{E}_{KL} - \dot{E}_{pKL}) + \frac{\partial \bar{S}_{MN}}{\partial E_{pKL}}\dot{E}_{pKL} + \frac{\partial \bar{S}_{MN}}{\partial \alpha^R_{KL}}\dot{\alpha}^R_{KL} + \frac{\partial \bar{S}_{MN}}{\partial \kappa}\dot{\kappa}. \end{aligned} \tag{3.36}$$

In an elastic state, during unloading or during neutral loading, the flow rules (3.17) apply, and the stress rate reduces to

$$\dot{\mathbf{S}} = \boldsymbol{\mathscr{L}}[\dot{\mathbf{E}}], \qquad \dot{S}_{MN} = \mathscr{L}_{MNKL}\dot{E}_{KL}. \tag{3.37}$$

During loading, the flow rules (3.18) are assumed and

$$\begin{aligned} \dot{\mathbf{S}} &= \boldsymbol{\mathscr{L}}[\dot{\mathbf{E}} - \pi\hat{g}\boldsymbol{\rho}] + \frac{\partial \bar{\mathbf{S}}}{\partial \mathbf{E}_p}[\pi\hat{g}\boldsymbol{\rho}] + \frac{\partial \bar{\mathbf{S}}}{\partial \boldsymbol{\alpha}_R}[\pi\hat{g}\boldsymbol{\beta}] + \frac{\partial \bar{\mathbf{S}}}{\partial \kappa}\pi\hat{g}\lambda \\ &= \boldsymbol{\mathscr{L}}\left(\boldsymbol{\mathscr{I}} - \pi\boldsymbol{\rho}\otimes\frac{\partial g}{\partial \mathbf{E}}\right)[\dot{\mathbf{E}}] + \left(\pi\bar{\boldsymbol{\sigma}}\otimes\frac{\partial g}{\partial \mathbf{E}}\right)[\dot{\mathbf{E}}], \end{aligned} \tag{3.38}$$

where we have used (3.10) and $(3.34)_1$. Introducing a new fourth-order tensor

$$\boldsymbol{\mathscr{L}}' = \boldsymbol{\mathscr{L}}\left(\boldsymbol{\mathscr{I}} - \pi\boldsymbol{\rho}\otimes\frac{\partial g}{\partial \mathbf{E}}\right), \qquad \mathscr{L}'_{MNPQ} = \mathscr{L}_{MNKL}\left(\mathscr{I}_{KLPQ} - \pi\frac{\partial g}{\partial E_{PQ}}\rho_{KL}\right), \tag{3.39}$$

then the stress rate becomes

$$\begin{aligned} \dot{\mathbf{S}} &= \left(\boldsymbol{\mathscr{L}}' + \pi\bar{\boldsymbol{\sigma}}\otimes\frac{\partial g}{\partial \mathbf{E}}\right)[\dot{\mathbf{E}}], \\ \dot{S}_{MN} &= \left(\mathscr{L}'_{MNPQ} + \bar{\sigma}_{MN}\frac{\partial g}{\partial E_{PQ}}\right)\dot{E}_{PQ}. \end{aligned} \tag{3.40}$$

It is easy to see that by substituting $(3.34)_2$ into (3.22) and also by using (3.14), that equation (3.21) would generate the same coefficient of strain rate in (3.40). That is, (3.21) and (3.40) are equivalent representations of the stress rate $\dot{\mathbf{S}}$.

3.5 A class of elastic-perfectly plastic materials

Consider the special constitutive equations

$$\mathbf{S} = \boldsymbol{\mathscr{L}}[\mathbf{E} - \mathbf{E}_p], \qquad f = \bar{f}(\mathbf{S}) - \kappa, \qquad \kappa = K^2, \qquad \lambda = 0, \tag{3.41}$$

where K is a positive constant and $\boldsymbol{\mathscr{L}}$ is a constant fourth-order tensor. By $(3.41)_1$ and $(3.34)_1$, we have

$$\bar{\boldsymbol{\sigma}} = \mathbf{0}. \tag{3.42}$$

Also, from $(3.29)_1$, and $(3.41)_{2,4}$, it is clear that

$$\Phi = 0, \tag{3.43}$$

By the classification of strain-hardening behavior given in (3.30), this means that the material only exhibits perfectly plastic behavior. Thus the constitutive equations (3.41) describe a class of elastic-perfectly plastic materials with a stress response that is linear in $\mathbf{E} - \mathbf{E}_p$.

The stress rate (3.40) reduces to

$$\begin{gathered} \dot{\mathbf{S}} = \boldsymbol{\mathscr{L}}[\dot{\mathbf{E}} - \dot{\mathbf{E}}_p] = \boldsymbol{\mathscr{L}}'[\dot{\mathbf{E}}], \\ \dot{S}_{MN} = \mathscr{L}_{MNKL}(\dot{E}_{KL} - \dot{E}_{pKL}) = \mathscr{L}'_{MNPQ}\dot{E}_{PQ}, \end{gathered} \tag{3.44}$$

where (3.39) has been used. We will further discuss this special case in Chapter 7.

Chapter 4

Multiplicative Decompositions of the Deformation Gradient

In this chapter, we discuss three useful multiplicative decompositions of the deformation gradient. The first was presented by Lee [23]. The second was formulated by Bammann and Johnson [1]. The third and the most recent was introduced by Casey [7]. The last two decompositions contain three unique factors of $\mathbf{F}$.

4.1 Lee's decomposition

By the mid 1960s, the scope of classical plasticity had not included the treatment of elastic-plastic deformation with finite strains. Two limiting cases were assumed: infinitesimal strains when both the elastic and plastic strains are of the same small order, and rigid-plastic analysis when the elastic strains are considered negligible in comparison to large plastic strains. However, there are many applications that call for the treatment of finite strains with no inherent assumptions. For example, plane waves in metal plates occur due to detonations of contact explosives. The pressure in these metals reach a sufficiently high value that finite elastic and plastic strains are produced [25]. In 1969, Lee presented a theory of elastic-plastic deformation at finite strains, which modified the previous classical plasticity theory to provide a more general form [23]. It replaced the usual assumption that the total strain is the sum of its elastic and plastic parts, as was used in infinitesimal theory, by the multiplicative decomposition $\mathbf{F} = \mathbf{F}_e\mathbf{F}_p$. He considered the rate of expenditure of work in elastic deformation completely uncoupled from the rate of plastic work, which led to a form of yield condition for finite elastic strain. In addition, emphasizing that the elastic characteristics are insensitive to plastic flow, he expressed $\mathbf{T}$ as a function of the elastic deformation gradient only. He also presented a decoupling of the rate of strain tensor, in order to define a law governing the rate of plastic flow, which demands irreversibility. Lee described this finite strain theory as follows.

After an elastic-plastic deformation as shown in Figure 4.1, the configuration of the body is defined by the mapping

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t), \tag{4.1}$$

where $\mathbf{X}$ and $\mathbf{x}$ are the position vectors of a material point X in the reference $\boldsymbol{\kappa}_0$ and the deformed configuration $\boldsymbol{\kappa}$. The local deformation in the neighborhood of X is expressed in terms of the deformation gradient $\mathbf{F}$ ($\det \mathbf{F} = 0$). If $\mathbf{Z}$ ($\det \mathbf{Z} = 0$) is any tensor function of X and t, then $\mathbf{F} = (\mathbf{FZ})\mathbf{Z}^{-1}$ with $\det(\mathbf{FZ}) > 0$. Thus $\mathbf{F}$ can always be decomposed into two factors with positive determinants. In his plasticity theory, Lee used the decomposition

$$\mathbf{F} = \mathbf{F}_e \mathbf{F}_p, \tag{4.2}$$

with $\det \mathbf{F}_e > 0$ and $\det \mathbf{F}_p > 0$. Let dX be an arbitrary material line element in a neighborhood $\mathscr{N}(X)$ of X. Let $\mathrm{d}\mathbf{X}$ and $\mathrm{d}\mathbf{x} = \mathbf{F}\mathrm{d}\mathbf{X}$ be the corresponding line elements in the configurations $\boldsymbol{\kappa}_0$ and $\boldsymbol{\kappa}$. We have

$$\mathrm{d}\mathbf{p} = \mathbf{F}_p \mathrm{d}\mathbf{X}. \tag{4.3}$$

Then, by Lee's decomposition (4.2),

$$\mathrm{d}\mathbf{x} = \mathbf{F}_e \mathrm{d}\mathbf{p}. \tag{4.4}$$

By considering all material elements that are in the neighborhood $\mathscr{N}(X)$, a local configuration can be formed from the elements $\mathrm{d}\mathbf{p}$. For some range of deformations from the current configuration $\boldsymbol{\kappa}$, the material in $\mathscr{N}(X)$ behaves purely elastically. The collection of these local configurations is the intermediate stress-free configuration $\boldsymbol{\kappa}_p$ (Figure 4.2). In general, $\mathbf{F}_e$ and $\mathbf{F}_p$ do not satisfy the compatibility conditions $F_{iA,B} = F_{iB,A}$, and the collection of intermediate local configurations cannot be used to create a compatible configuration.

As part of the definition of $\boldsymbol{\kappa}_p$, it must be required that for each $\mathbf{x}$, the portion of the body that corresponds to the neighborhood $\mathscr{N}(X)$ be reduced to a state of zero stress. That is, there is some value $\mathbf{F}_p$ of $\mathbf{F}$ such that at X, $\mathbf{F} = \mathbf{F}_p$ means $\mathbf{T} = \mathbf{0}$. In general, $\mathbf{T}$ does not vanish for particles other than X that belong to the neighborhood $\mathscr{N}(X)$.

Substituting the decomposition (4.2) into the equation for the Lagrangian strain, we obtain

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right) = \mathbf{F}_p^T \mathbf{E}_e \mathbf{F}_p + \mathbf{E}_p, \tag{4.5}$$

where $\mathbf{E}_e$ and $\mathbf{E}_p$ are the Lagrangian strain measures in terms of the respective deformation gradients:

$$\mathbf{E}_e = \frac{1}{2}\left(\mathbf{F}_e^T\mathbf{F}_e - \mathbf{I}\right), \qquad \mathbf{E}_p = \frac{1}{2}\left(\mathbf{F}_p^T\mathbf{F}_p - \mathbf{I}\right). \tag{4.6}$$

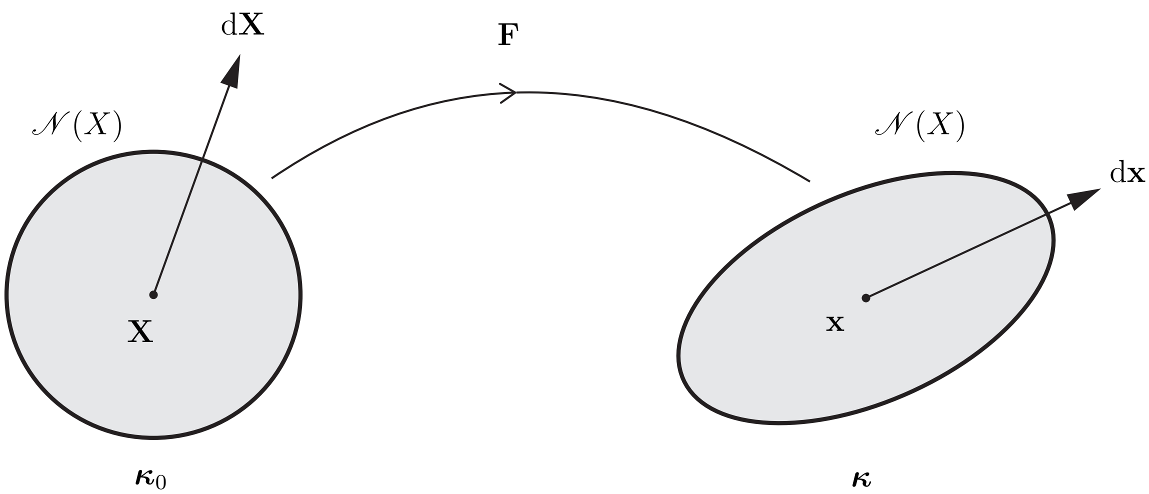

Figure 4.1: The neighborhood $\mathscr{N}(X)$ in the reference $\boldsymbol{\kappa}_0$ is mapped by $\mathbf{F}$ into a neighborhood in the current configuration $\boldsymbol{\kappa}$.

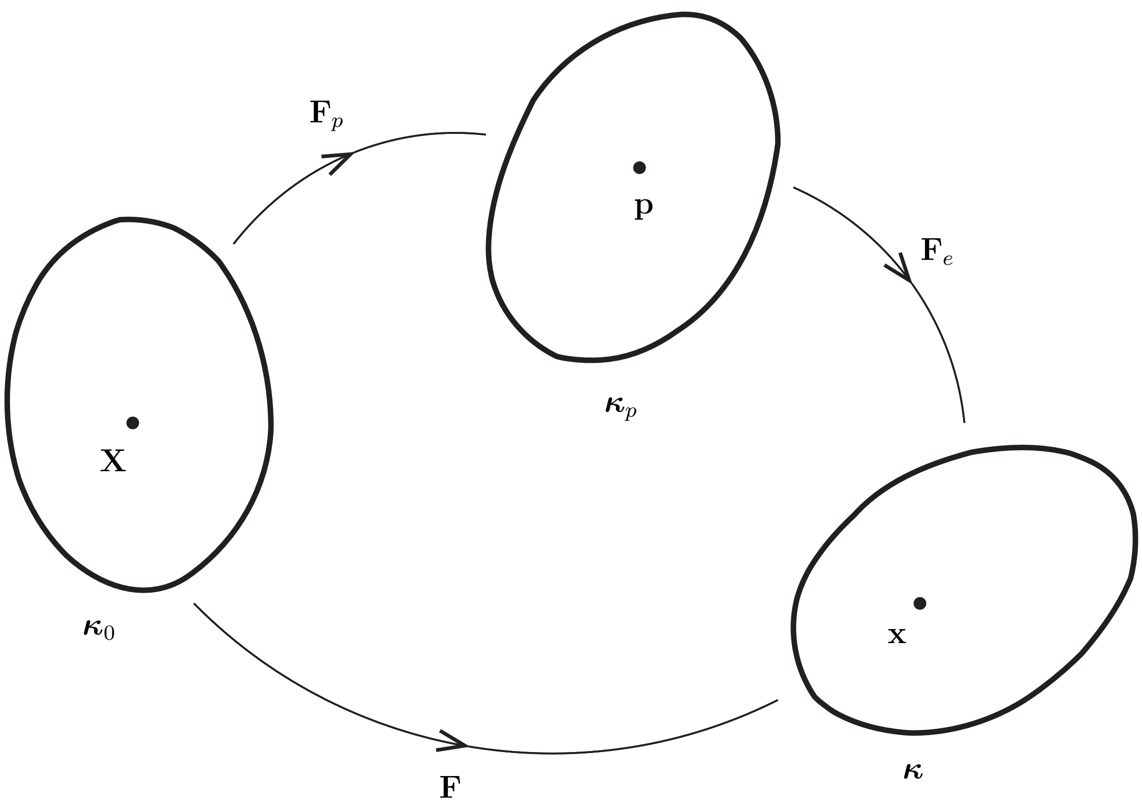

Figure 4.2: Lee's decomposition: $\mathbf{F} = \mathbf{F}_e\mathbf{F}_p$.

It is clear from (4.5) that

$$\mathbf{E} - \mathbf{E}_p = \mathbf{F}_p^T \mathbf{E}_e \mathbf{F}_p. \tag{4.7}$$

We see that the assumption in infinitesimal plasticity that the total strain is the sum of elastic and plastic parts is not valid for this finite strain theory. Also, as part of the definition of $\boldsymbol{\kappa}_p$, we note that the plastic strain $\mathbf{E}_p$ has the same value for X in $\boldsymbol{\kappa}_p$ and $\boldsymbol{\kappa}$.

Substituting the decomposition into the velocity gradient $\mathbf{L}$,

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \dot{\mathbf{F}}_e\mathbf{F}_e^{-1} + \mathbf{F}_e\dot{\mathbf{F}}_p\mathbf{F}_p^{-1}\mathbf{F}_e^{-1}. \tag{4.8}$$

Using the definitions

$$\mathbf{L}_e = \dot{\mathbf{F}}_e\mathbf{F}_e^{-1}, \qquad \mathbf{L}_p = \dot{\mathbf{F}}_p\mathbf{F}_p^{-1}, \tag{4.9}$$

Lee defines the rate of elastic strain $\mathbf{D}_e$ and the rate of plastic strain $\mathbf{D}_p$ as the symmetric parts of $\mathbf{L}_e$ and $\mathbf{L}_p$ respectively.

Furthermore, emphasizing that the elastic properties of the material are not appreciably influenced by the plastic flow, Lee assumed the form of the stress as

$$\mathbf{T} = 2\rho_0\mathbf{F}_e\frac{\partial\psi}{\partial\mathbf{C}_e}\mathbf{F}_e^T / \det\mathbf{F}_e, \tag{4.10}$$

where $\psi = \psi(\mathbf{C}_e)$ is the Helmholtz free energy.

Lee's decomposition includes several further restrictions that limit its use to initially isotropic materials, lead to the non-uniqueness of $\bar{\boldsymbol{\kappa}}$ due to a rotational arbitrariness, and even result in the possible non-existence of the decomposition. Green and Naghdi [20] and Casey and Naghdi [8] discuss some issues concerning the use of Lee's decomposition as follows:

(1) Existence. The stresses can only be reduced to zero, without changing the plastic strain, if, and only if, the origin in stress space stays inside the yield surface. Since this is not true in general, a restriction must be imposed on possible deformations and constitutive relations. The stress-free configuration and thus the decomposition (4.2) will not always exist if there are no imposed restrictions.

(2) Uniqueness. Two stress-free intermediate configurations that correspond to the same current configuration $\boldsymbol{\kappa}$ are unique only to within a proper orthogonal tensor $\mathbf{Q}$, so that $\mathbf{F}_e\mathbf{Q}^T$ and $\mathbf{Q}\mathbf{F}_p$ also satisfy the decomposition (4.2). Thus the intermediate configuration is unique only to within an arbitrary rigid rotation.

(3) Invariance. Physical considerations require that fields and functions be invariant to superposed rigid body motions that take the configuration $\boldsymbol{\kappa}$ into another configuration $\boldsymbol{\kappa}^+$. Since the intermediate configuration $\boldsymbol{\kappa}_p$ is locally another configuration, independent rigid body motions that take $\boldsymbol{\kappa}_p$ into another stress-free configuration $\boldsymbol{\kappa}_p^+$ similarly require the invariance of fields and functions. The transformations are then

$$\mathbf{F}^+ = \mathbf{Q}(t)\mathbf{F} = \mathbf{F}_e^+\mathbf{F}_p^+, \qquad \mathbf{F}_e^+ = \mathbf{Q}(t)\mathbf{F}_e\mathbf{Q}_p^+(t), \qquad \mathbf{F}_p^+ = \mathbf{Q}_p^+(t)\mathbf{F}_p, \tag{4.11}$$

where the proper orthogonal tensors $\mathbf{Q}(t)$ and $\mathbf{Q}_p(t)$ represent the rigid motions that take $\boldsymbol{\kappa} \to \boldsymbol{\kappa}^+$ and $\boldsymbol{\kappa}_p \to \boldsymbol{\kappa}_p^+$, respectively. The superposed rigid motions are shown in Figure 4.3. Under these superposed rigid motions we have the transformations

$$\mathbf{E}^+ = \mathbf{E}, \qquad \mathbf{E}_p^+ = \mathbf{E}_p, \qquad \kappa^+ = \kappa, \qquad \mathbf{S}^+ = \mathbf{S}. \tag{4.12}$$

The invariance will not be satisfied unless there is nonuniqueness in the rotation $\mathbf{Q}_p$. That is, a unique $\boldsymbol{\kappa}_p$ cannot be chosen simply among the possible stress-free configurations. For example, if $\mathbf{F}_e$ is chosen to be symmetric positive definite, then the transformation $(4.11)_2$ does not result in a symmetric positive definite tensor $\mathbf{F}_e^+$ unless $\mathbf{Q}(t) = \mathbf{Q}_p(t)$. Similarly, choosing $\mathbf{F}_p$ to be symmetric positive definite and assuming $\mathbf{F}_p^+$ is also symmetric positive definite would, in general, violate the invariance requirements. In [23], the consequences of choosing $\mathbf{Q}_p(t) = \mathbf{I}$ were avoided because of the additional limitation that the material is isotropic.

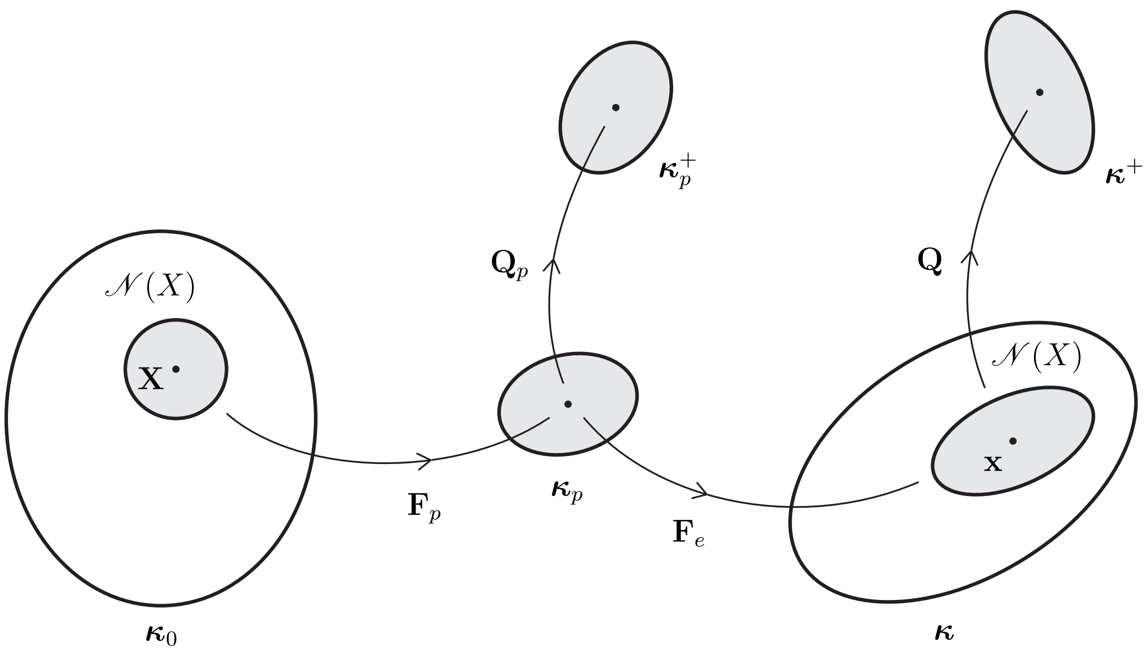

Figure 4.3: Invariance under superposed rigid body motions. The configuration $\boldsymbol{\kappa}$ is rotated by the tensor $\mathbf{Q}$ into $\boldsymbol{\kappa}^+$. Similarly, the rotation tensor $\mathbf{Q}_p$ transforms the stress-free configuration $\boldsymbol{\kappa}_p$ into another stress-free configuration $\boldsymbol{\kappa}_p^+$. In general, $\mathbf{Q}$ and $\mathbf{Q}_p$ are independent rotations.

4.2 A three-factor decomposition of F

It has been shown that $\mathbf{F}_e$ and $\mathbf{F}_p$ and the stress-free configurations associated with them are not unique. That is, there is a rotational arbitrariness in the factors of Lee's decomposition. An alternate approach is to consider a configuration within the entire family of stress-free configurations that is uniquely defined, while giving a convenient multiplicative decomposition of $\mathbf{F}$.

Consider the polar decompositions of $\mathbf{F}_e$ and $\mathbf{F}_p$:

$$\begin{aligned} \mathbf{F}_e &= \mathbf{R}_e\mathbf{U}_e = \mathbf{V}_e\mathbf{R}_e, \\ \mathbf{F}_p &= \mathbf{R}_p\mathbf{U}_p = \mathbf{V}_p\mathbf{R}_p, \end{aligned} \tag{4.13}$$

where $\mathbf{R}_e$ and $\mathbf{R}_p$ are proper orthogonal tensors and $\mathbf{U}_e$, $\mathbf{U}_p$, $\mathbf{V}_e$ and $\mathbf{V}_p$ are symmetric, positive definite tensors. The corresponding Cauchy-Green tensors are

$$\begin{aligned} \mathbf{C}_e &= \mathbf{F}_e^T\mathbf{F}_e = \mathbf{U}_e^2, \qquad & \mathbf{C}_p &= \mathbf{F}_p^T\mathbf{F}_p = \mathbf{U}_p^2, \\ \mathbf{B}_e &= \mathbf{F}_e\mathbf{F}_e^T = \mathbf{V}_e^2, \qquad & \mathbf{B}_p &= \mathbf{F}_p\mathbf{F}_p^T = \mathbf{V}_p^2, \end{aligned} \tag{4.14}$$

We may superpose a rigid body rotation $\mathbf{Q}_p$ on the neighborhood $\mathscr{N}(X)$ in $\boldsymbol{\kappa}_p$, independent of any rotation that may be superposed on $\boldsymbol{\kappa}$. In accordance with invariance requirements, we can transform $\mathbf{F}_p$ into $\mathbf{F}_p' = \mathbf{Q}_p\mathbf{F}_p$. The corresponding elastic factor in the multiplicative decomposition is $\mathbf{F}_e' = \mathbf{F}_e\mathbf{Q}_p^T$. In addition, the stress becomes $\mathbf{T}' = \mathbf{Q}_p\mathbf{0}\mathbf{Q}_p^T = \mathbf{0}$. Thus $\mathbf{F}_e$ and $\mathbf{F}_p$ and the stress-free configurations associated with them are not unique. In order to make sure that $\mathbf{F}_p$ only has this rotational arbitrariness, we impose a condition that the right stretch tensor $\mathbf{U}_p$ be uniquely determined at X in $\boldsymbol{\kappa}_p(X)$. The other factors in the polar decompositions of $\mathbf{F}_e'$ and $\mathbf{F}_p'$ are

$$\begin{aligned} &\mathbf{R}_p' = \mathbf{Q}_p\mathbf{R}_p, \qquad \mathbf{R}_e' = \mathbf{R}_e\mathbf{Q}_p^T, \\ \mathbf{U}_e' = \mathbf{Q}_p\mathbf{U}_e\mathbf{Q}_p^T, \qquad &\mathbf{V}_p' = \mathbf{Q}_p\mathbf{V}_p\mathbf{Q}_p^T, \qquad \mathbf{V}_e' = \mathbf{V}_e. \end{aligned} \tag{4.15}$$

We note that the left stretch tensor $\mathbf{V}_e$ and the product of the rotations

$$\mathbf{R}_* = \mathbf{R}_e\mathbf{R}_p \tag{4.16}$$

are uniquely determined:

$$\mathbf{R}_*' = \mathbf{R}_e'\mathbf{R}_p' = \mathbf{R}_*. \tag{4.17}$$

Also,

$$\mathbf{R}_*^+ = \mathbf{Q}\mathbf{R}_*. \tag{4.18}$$

Thus, the deformation gradient $\mathbf{F}$ possesses a unique decomposition

$$\mathbf{F} = \mathbf{V}_e \mathbf{R}_* \mathbf{U}_p, \tag{4.19}$$

which was first formulated by Bammann and Johnson [1]. This decomposition is illustrated in Fig. 4.4. The local stress-free intermediate configuration that is obtained by mapping the neighborhood $\mathscr{N}(X)$ using $\mathbf{U}_p$ is denoted by $\boldsymbol{\kappa}_*$. The rotation $\mathbf{R}_*$ maps $\boldsymbol{\kappa}_*$ into another stress-free configuration $\tilde{\boldsymbol{\kappa}}$.

For a neighborhood around a material point X, the line element $\mathrm{d}\mathbf{X}$ in $\boldsymbol{\kappa}_0$ is mapped successively as follows:

$$\mathrm{d}\mathbf{x}_* = \mathbf{U}_p \mathrm{d}\mathbf{X}, \qquad \mathrm{d}\tilde{\mathbf{x}} = \mathbf{R}_* \mathrm{d}\mathbf{x}_*, \qquad \mathrm{d}\mathbf{x} = \mathbf{V}_e \mathrm{d}\tilde{\mathbf{x}}, \tag{4.20}$$

where $\mathrm{d}\mathbf{x}_*$ and $\mathrm{d}\tilde{\mathbf{x}}$ are the line elements in $\boldsymbol{\kappa}_*$ and $\tilde{\boldsymbol{\kappa}}$ respectively.

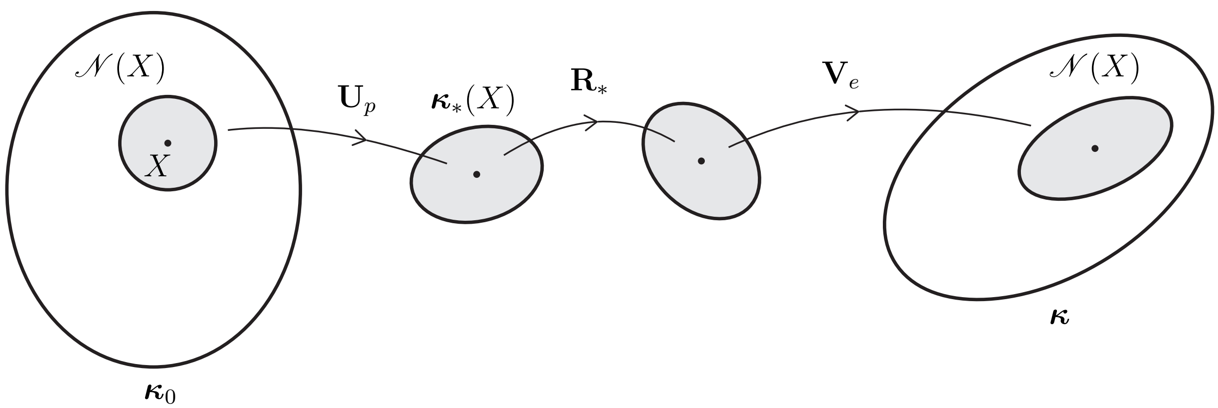

Figure 4.4: The decomposition of the deformation gradient $\mathbf{F}$ due to Bammann and Johnson. A right stretch tensor $\mathbf{U}_p$ is followed by a rotation tensor $\mathbf{R}_*$ and a left stretch tensor $\mathbf{V}_e$. Each factor is uniquely determined.

Invoking invariance under superposed rigid motions of the configurations $\boldsymbol{\kappa}_p$ and $\boldsymbol{\kappa}$,

$$\begin{gathered}
\mathbf{F}_e^+ = \mathbf{Q}\mathbf{F}_e\mathbf{Q}_p^T, \qquad \mathbf{R}_e^+ = \mathbf{Q}\mathbf{R}_e\mathbf{Q}_p^T, \qquad \mathbf{U}_e^+ = \mathbf{Q}_p\mathbf{U}_e\mathbf{Q}_p^T, \qquad \mathbf{V}_e^+ = \mathbf{Q}\mathbf{V}_e\mathbf{Q}^T \\
\mathbf{C}_e^+ = \mathbf{Q}_p\mathbf{C}_e\mathbf{Q}_p^T, \qquad \mathbf{B}_e^+ = \mathbf{Q}\mathbf{B}_e\mathbf{Q}^T \\
\mathbf{F}_p^+ = \mathbf{Q}_p\mathbf{F}_p, \qquad \mathbf{R}_p^+ = \mathbf{Q}_p\mathbf{R}_p, \qquad \mathbf{U}_p^+ = \mathbf{U}_p, \qquad \mathbf{V}_p^+ = \mathbf{Q}_p\mathbf{V}_p\mathbf{Q}_p^T \\
\mathbf{C}_p^+ = \mathbf{C}_p, \qquad \mathbf{B}_p^+ = \mathbf{Q}_p\mathbf{B}_p\mathbf{Q}_p^T.
\end{gathered} \tag{4.21}$$

With regards to the configuration $\boldsymbol{\kappa}_*$, we can rewrite the plastic strain as

$$\mathbf{E}_p = \frac{1}{2}(\mathbf{U}_p^2 - \mathbf{I}). \tag{4.22}$$

Clearly, this still has the same value as the plastic strain $(4.6)_2$ in $\boldsymbol{\kappa}$. The strain associated with the configuration $\boldsymbol{\kappa}_*$ is taken by Bammann and Johnson [1] to be

$$\bar{\mathbf{E}}_e = \frac{1}{2}\left(\mathbf{R}_*^T\mathbf{V}_e^2\mathbf{R}_* - \mathbf{I}\right) = \mathbf{U}_p^{-1}\left(\mathbf{E} - \mathbf{E}_p\right)\mathbf{U}_p^{-1} = \mathbf{R}_p^T\mathbf{E}_e\mathbf{R}_p. \tag{4.23}$$

Further, using the same thermodynamic procedure as Green and Naghdi, the symmetric Piola-Kirchhoff tensor associated with $\boldsymbol{\kappa}_*$ is

$$\bar{\mathbf{S}} = \frac{1}{\det \mathbf{U}_p}\mathbf{U}_p\mathbf{S}\mathbf{U}_p = \bar{\rho}\frac{\partial\bar{\psi}}{\partial\bar{\mathbf{E}}_e}, \tag{4.24}$$

where $\bar{\rho} = \rho_0/\det\mathbf{U}_p$ and $\bar{\psi} = \bar{\psi}(\bar{\mathbf{E}}_e)$.

4.3 An alternate three-factor decomposition

An alternative decomposition was recently presented by Casey [7]. It employs a unique right stretch tensor $\mathbf{U}_*$, which is invariant under superposed rigid body motions. This subfactor of $\mathbf{F}$ serves as a convenient variable for describing the elastic response of a material from its evolving stress-free configuration $\boldsymbol{\kappa}_*$.

Rewriting the decomposition (4.19) as

$$\mathbf{F} = \mathbf{F}_*\mathbf{U}_p, \tag{4.25}$$

where

$$\mathbf{F}_* = \mathbf{V}_e\mathbf{R}_* = \mathbf{R}_*\mathbf{U}_*, \qquad \det\mathbf{F}_* > 0, \tag{4.26}$$

another unique decomposition arises:

$$\mathbf{F} = \mathbf{R}_*\mathbf{U}_*\mathbf{U}_p, \tag{4.27}$$

which is illustrated in Fig. 4.5. The Cauchy-Green tensors associated with $\mathbf{F}_*$ are

$$\mathbf{C}_* = \mathbf{F}_*^T\mathbf{F}_* = \mathbf{U}_*^2 = \mathbf{R}_p^T\mathbf{C}_e\mathbf{R}_p, \qquad \mathbf{B}_* = \mathbf{F}_*\mathbf{F}_*^T = \mathbf{V}_e^2 = \mathbf{B}_e. \tag{4.28}$$

It follows from $(4.14)_1$ and $(4.28)_1$ that

$$\mathbf{U}_*^2 = \left(\mathbf{R}_p^T\mathbf{U}_e\mathbf{R}_p\right)\left(\mathbf{R}_p^T\mathbf{U}_e\mathbf{R}_p\right), \tag{4.29}$$

where the factor $\mathbf{R}_p^T\mathbf{U}_e\mathbf{R}_p$ is symmetric and positive definite. By the uniqueness of a positive definite square root,

$$\mathbf{U}_* = \mathbf{R}_p^T\mathbf{U}_e\mathbf{R}_p, \tag{4.30}$$

which has an equivalent form

$$\mathbf{U}_* = \mathbf{R}_*^T \mathbf{V}_e \mathbf{R}_*, \tag{4.31}$$

where use has been made of $(4.13)_1$ and (4.16).

Under superposed rigid motions, $\mathbf{F}_*$ transforms by

$$\mathbf{F}_*^+ = \mathbf{F}^+ (\mathbf{U}_p^+)^{-1} = \mathbf{Q}\mathbf{F}_*. \tag{4.32}$$

It then follows that

$$\mathbf{C}_*^+ = \mathbf{C}_*, \qquad \mathbf{U}_*^+ = \mathbf{U}_*. \tag{4.33}$$

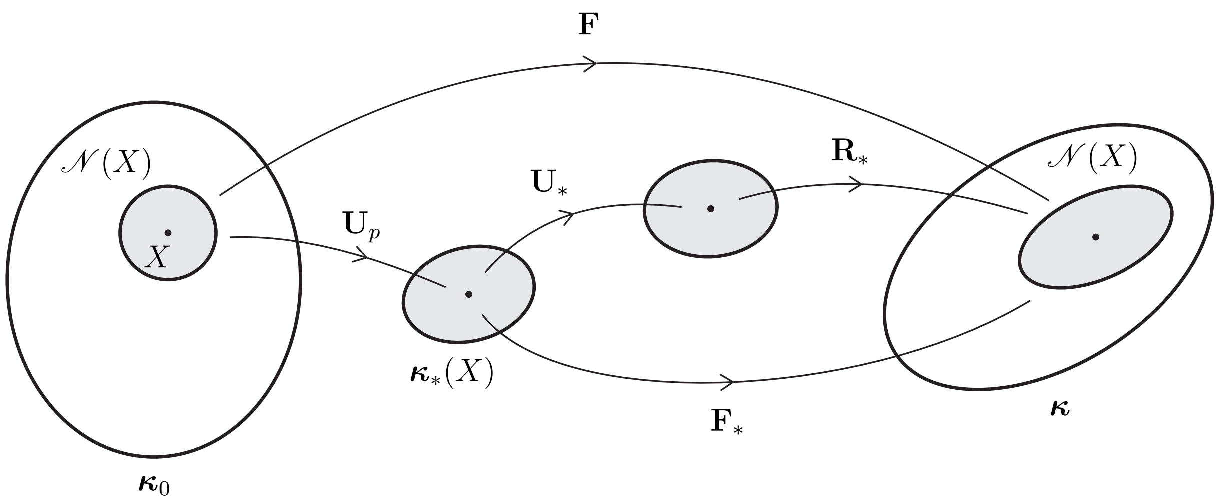

Figure 4.5: Decomposition of $\mathbf{F}$ into a right stretch tensor $\mathbf{U}_p$ followed by a right stretch tensor $\mathbf{U}_*$, followed by a rotation $\mathbf{R}_*$. Each factor is uniquely determined by $\mathbf{F}$.

With the decomposition (4.27) in mind, we want to consider how the neighborhood $\mathscr{N}(X)$ is brought elastically from an intermediate local configuration to its current configuration $\boldsymbol{\kappa}$. We choose the unique configuration $\boldsymbol{\kappa}_*(X)$ as a fixed reference configuration.

Consider a one-parameter family of deformations

$$\mathbf{F}(X, \tau) = \mathbf{F}_*(X, \tau)\mathbf{U}_p(X), \tag{4.34}$$

where τ can be taken to have the physical dimension of time. We may express the strain energy per unit mass as

$$\psi = \psi_*(\mathbf{F}_*(X, \tau)). \tag{4.35}$$

Applying invariance requirements, the strain energy can also be expressed as a function of $\mathbf{U}_*$ or $\mathbf{C}_*$.

$$\psi = \psi_*(\mathbf{U}_*) = \bar{\psi}_*(\mathbf{C}_*). \tag{4.36}$$

Differentiating (4.34) by τ, we have

$$\frac{\partial \mathbf{F}}{\partial \tau}(X,\tau) = \frac{\partial \mathbf{F}_*}{\partial \tau}(X,\tau)\mathbf{U}_p(X). \tag{4.37}$$

The corresponding velocity gradient is

$$\mathbf{L}(X,\tau) = \frac{\partial \mathbf{F}}{\partial \tau}(X,\tau)[\mathbf{F}(X,\tau)]^{-1} = \frac{\partial \mathbf{F}_*}{\partial \tau}(X,\tau)[\mathbf{F}_*(X,\tau)]^{-1}, \tag{4.38}$$

whose symmetric part we denote by $\mathbf{D}(X,\tau)$. In view of $(4.28)_1$, we can differentiate

$$\mathbf{C}_*(X,\tau) = \mathbf{F}_*^T(X,\tau)\mathbf{F}_*(X,\tau) \tag{4.39}$$

to get

$$\frac{\partial \mathbf{C}_*}{\partial \tau}(X,\tau) = 2\mathbf{F}_*^T(X,\tau)\mathbf{D}(X,\tau)\mathbf{F}_*(X,\tau). \tag{4.40}$$

We know that for a Green-elastic material,

$$\rho(X,\tau)\frac{\partial \psi_*}{\partial \tau} = \mathbf{T}(X,\tau)\cdot\mathbf{D}(X,\tau). \tag{4.41}$$

We can then deduce that the Cauchy stress tensor at (X,τ) is given by

$$\mathbf{T}(X,\tau) = 2\rho(X,\tau)\mathbf{F}_*(X,\tau)\frac{\partial \bar{\psi}_*}{\partial \mathbf{C}_*}\mathbf{F}_*^T(X,\tau). \tag{4.42}$$

Employing a Piola-Kirchhoff stress tensor $\mathbf{S}_*$ that is referred back to the configuration $\boldsymbol{\kappa}_*$, we have

$$\mathbf{T}(X,\tau)\det\mathbf{F}_*(X,\tau) = \mathbf{F}_*(X,\tau)\mathbf{S}_*(X,\tau)\mathbf{F}_*^T(X,\tau). \tag{4.43}$$

Then, (4.42) can be written in the simpler form

$$\mathbf{S}_*(X,\tau) = 2\rho_*(X)\frac{\partial \bar{\psi}_*}{\partial \mathbf{C}_*}, \tag{4.44}$$

where

$$\rho_*(X) = \rho(X,\tau)\det\mathbf{F}_*(X,\tau). \tag{4.45}$$

We can deduce from (4.25) and $(4.28)_1$ that the Cauchy-Green tensor can be written as

$$\mathbf{C} = \mathbf{U}_p\mathbf{C}_*\mathbf{U}_p. \tag{4.46}$$

Then by $(4.14)_2$,

$$\mathbf{C} - \mathbf{C}_p = \mathbf{U}_p(\mathbf{C}_* - \mathbf{I})\mathbf{U}_p. \tag{4.47}$$

Using the definition of the Lagrangian strain measure

$$\mathbf{E}_* = \frac{1}{2}(\mathbf{C}_* - \mathbf{I}), \tag{4.48}$$

we can rewrite (4.47) as

$$\mathbf{E}_* = \mathbf{U}_p^{-1}(\mathbf{E} - \mathbf{E}_p)\mathbf{U}_p^{-1}. \tag{4.49}$$

Equation (4.49) is used by Naghdi and Trapp [30] as a strain tensor with no reference to a multiplicative decomposition. Specifically, $\mathbf{E}_*$ is considered as a variable of the Helmholtz free energy function, which is used to develop constitutive equations for ductile metals. Also, $\mathbf{E}_*$ is equal to the strain tensor $\bar{\mathbf{E}}_e$ employed by Bammann and Johnson [1]. See Equation (4.23).

Substituting the form of $\mathbf{F}$ in (4.25) into the velocity gradient $\mathbf{L}$,

$$\begin{aligned} \mathbf{L} &= \dot{\mathbf{F}}_*\mathbf{F}_*^{-1} + \mathbf{F}_*(\dot{\mathbf{U}}_p\mathbf{U}_p^{-1})\mathbf{F}_*^{-1} \\ &= \mathbf{L}_* + \mathbf{F}_*\bar{\mathbf{L}}_p\mathbf{F}_*^{-1}, \end{aligned} \tag{4.50}$$

where we have set

$$\mathbf{L}_* = \dot{\mathbf{F}}_*\mathbf{F}_*^{-1}, \qquad \bar{\mathbf{L}}_p = \dot{\mathbf{U}}_p\mathbf{U}_p^{-1}. \tag{4.51}$$

The symmetric part of $\mathbf{L}$ is

$$\mathbf{D} = \mathbf{D}_* + \frac{1}{2}\left(\mathbf{F}_*\dot{\mathbf{U}}_p\mathbf{U}_p^{-1}\mathbf{F}_*^{-1} + \mathbf{F}_*^{-T}\mathbf{U}_p^{-1}\dot{\mathbf{U}}_p\mathbf{F}_*^{T}\right), \tag{4.52}$$

where $\mathbf{D}_*$ is the symmetric part of $\mathbf{L}_*$. We will also consider the symmetric part of $\bar{\mathbf{L}}_p$, denoted $\bar{\mathbf{D}}_p$:

$$\bar{\mathbf{D}}_p = \frac{1}{2}(\dot{\mathbf{U}}_p\mathbf{U}_p^{-1} + \mathbf{U}_p^{-1}\dot{\mathbf{U}}_p). \tag{4.53}$$

Then from $(4.28)_1$ and $(4.51)_1$,

$$\dot{\mathbf{C}}_* = 2\mathbf{F}_*^{T}\mathbf{D}_*\mathbf{F}_*. \tag{4.54}$$

We have discussed three multiplicative decompositions of the deformation gradient:

$$\begin{aligned} \mathbf{F} &= \mathbf{F}_e\mathbf{F}_p, \\ \mathbf{F} &= \mathbf{V}_e\mathbf{R}_*\mathbf{U}_p, \\ \mathbf{F} &= \mathbf{R}_*\mathbf{U}_*\mathbf{U}_p. \end{aligned} \tag{4.55}$$

We will use the third decomposition and the elements associated with the intermediate configuration $\boldsymbol{\kappa}_*$ and combine them with the strain-space formulation of the Lagrangian theory to construct an Eulerian form of finite plasticity.

Chapter 5

Fields and Functions on the Intermediate Configuration $\boldsymbol{\kappa}_*$

With the intermediate configuration $\boldsymbol{\kappa}_*$ properly defined, we can explore the fields and functions that depend on this intermediate configuration. There are now several versions of the yield function, each of them dependent on a different stress measure. Thus, there are different versions of the yield criteria. The stress measures now take on a distinct dependence on new variables. These elements will help us develop new Prandtl-Reuss type equations for the Eulerian form of plasticity.

5.1 Yield functions and loading criteria

Recall that the constitutive equation for the second Piola-Kirchhoff stress tensor $\mathbf{S}$ is defined in terms of a response function $\hat{\mathbf{S}}$:

$$\mathbf{S} = \hat{\mathbf{S}}(\mathbf{E}, \mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa). \tag{5.1}$$

The Piola-Kirchhoff stress $\mathbf{S}_*$ defined in (4.43) is

$$\mathbf{S}_* = \bar{\mathbf{S}}_*(\mathbf{E}_*, \mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa) \tag{5.2}$$

Now recall the yield function g in strain space

$$g(\mathbf{E}, \mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa) = 0. \tag{5.3}$$

It can be transformed into a another yield function that depends on $\mathbf{E}_*$ by using the relation (4.49):

$$\begin{aligned} g &= g(\mathbf{E}(\mathbf{E}_*, \mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa), \mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa) \\ &= g_*(\mathbf{E}_*, \mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa). \end{aligned} \tag{5.4}$$

Also, using the inverse of (5.2),

$$\begin{aligned} g &= g_*(\mathbf{E}_*(\mathbf{S}_*, \mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa), \mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa) \\ &= f_*(\mathbf{S}_*, \mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa). \end{aligned} \tag{5.5}$$

We have now also transformed g into a yield function f_* that depends on $\mathbf{S}_*$.

Recall the loading index

$$\hat{g} = \frac{\partial g}{\partial \mathbf{E}} \cdot \dot{\mathbf{E}}. \tag{5.6}$$

For elastic states and during unloading and neutral loading, we assume the flow rules (3.17):

$$\dot{\mathbf{E}}_p = \mathbf{0}, \qquad \dot{\boldsymbol{\alpha}}_R = \mathbf{0}, \qquad \dot{\kappa} = 0. \tag{5.7}$$

During loading, we assume that the rates are each linear in $\dot{\mathbf{E}}$ with coefficients that are functions of $\mathbf{E}$ and $\mathscr{U}$:

$$\dot{\mathbf{E}}_p = \pi\hat{g}\boldsymbol{\rho}(\mathbf{E}, \mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa), \qquad \dot{\boldsymbol{\alpha}}_R = \pi\hat{g}\boldsymbol{\beta}(\mathbf{E}, \mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa), \qquad \dot{\kappa} = \pi\hat{g}\lambda(\mathbf{E}, \mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa). \tag{5.8}$$

The constitutive functions $\boldsymbol{\rho}$, $\boldsymbol{\beta}$, and λ can be expressed in Eulerian form, using the relationships involving $\mathbf{E}$, $\mathbf{E}_*$, or $\mathbf{S}_*$ if expressed in stress space:

$$\begin{aligned} \boldsymbol{\rho} &= \boldsymbol{\rho}(\mathbf{E}, \mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa) = \boldsymbol{\rho}(\mathbf{E}(\mathbf{E}_*, \mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa), \mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa) \\ &= \boldsymbol{\rho}_*(\mathbf{E}_*, \mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa) = \boldsymbol{\rho}_*(\mathbf{E}_*(\mathbf{S}_*), \mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa) \\ &= \hat{\boldsymbol{\rho}}_*(\mathbf{S}_*, \mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa). \end{aligned} \tag{5.9}$$

Similar relationships apply to $\boldsymbol{\beta}$ and λ.

Also, the plastic strain rate can be expressed using the velocity gradient $\bar{\mathbf{L}}_p$ in $(4.51)_2$ or its symmetric part $\bar{\mathbf{D}}_p$ given in (4.53).

$$\begin{aligned} \dot{\mathbf{E}}_p &= \frac{1}{2}\left(\dot{\mathbf{U}}_p\mathbf{U}_p + \mathbf{U}_p\dot{\mathbf{U}}_p\right) \\ &= \mathbf{U}_p\left[\frac{1}{2}\left(\dot{\mathbf{U}}_p\mathbf{U}_p^{-1} + \mathbf{U}_p^{-1}\dot{\mathbf{U}}_p\right)\right]\mathbf{U}_p \\ &= \mathbf{U}_p\bar{\mathbf{D}}_p\mathbf{U}_p. \end{aligned} \tag{5.10}$$

In $\mathbf{S}_*$ space, we consider the yield function f_* as obtained in $(5.5)_2$:

$$\begin{aligned}\dot{f}_* &= \frac{\partial f_*}{\partial \mathbf{S}_*}\cdot\dot{\mathbf{S}}_* + \frac{\partial f_*}{\partial \mathbf{E}_p}\cdot\dot{\mathbf{E}}_p + \frac{\partial f_*}{\partial \boldsymbol{\alpha}_R}\cdot\dot{\boldsymbol{\alpha}}_R + \frac{\partial f_*}{\partial \kappa}\dot{\kappa} = 0 \\ &= \hat{f}_* + \frac{\partial f_*}{\partial \mathbf{E}_p}\cdot\dot{\mathbf{E}}_p + \frac{\partial f_*}{\partial \boldsymbol{\alpha}_R}\cdot\dot{\boldsymbol{\alpha}}_R + \frac{\partial f_*}{\partial \kappa}\dot{\kappa},\end{aligned} \tag{5.11}$$

where we have used the consistency condition $\dot{g} = \dot{f}_* = 0$, and a loading index

$$\hat{f}_* = \frac{\partial f_*}{\partial \mathbf{S}_*}\cdot\dot{\mathbf{S}}_*. \tag{5.12}$$

During loading, we assume

$$\dot{\mathbf{E}}_p = \pi\hat{g}\boldsymbol{\rho}, \qquad \dot{\boldsymbol{\alpha}}_R = \pi\hat{g}\boldsymbol{\beta}, \qquad \dot{\kappa} = \pi\hat{g}\lambda, \tag{5.13}$$

the first of which we can express as

$$\bar{\mathbf{D}}_p = \pi\hat{g}\mathbf{U}_p^{-1}\boldsymbol{\rho}\mathbf{U}_p^{-1} = \pi\hat{g}\bar{\boldsymbol{\rho}}. \tag{5.14}$$

There are similar expressions for $\dot{\boldsymbol{\alpha}}_R$ and $\dot{\kappa}$ that employ the Eulerian form of $\boldsymbol{\beta}$ and λ. Thus,

$$\begin{aligned}0 = \dot{f}_* &= \hat{f}_* + \frac{\partial f_*}{\partial \mathbf{E}_p}\cdot\pi\hat{g}\boldsymbol{\rho} + \frac{\partial f_*}{\partial \boldsymbol{\alpha}_R}\cdot\pi\hat{g}\boldsymbol{\beta} + \frac{\partial f_*}{\partial \kappa}\pi\hat{g}\lambda \\ &= \hat{f}_* + \frac{\partial f_*}{\partial \mathbf{E}_p}\cdot\mathbf{U}_p\bar{\mathbf{D}}_p\mathbf{U}_p + \frac{\partial f_*}{\partial \boldsymbol{\alpha}_R}\cdot\mathbf{U}_p\dot{\bar{\boldsymbol{\alpha}}}_R\mathbf{U}_p + \frac{\partial f_*}{\partial \kappa}\pi\hat{g}\lambda \\ &= \hat{f}_* + \mathbf{U}_p\frac{\partial f_*}{\partial \mathbf{E}_p}\mathbf{U}_p\cdot\pi\hat{g}\bar{\boldsymbol{\rho}} + \mathbf{U}_p\frac{\partial f_*}{\partial \boldsymbol{\alpha}_R}\mathbf{U}_p\cdot\pi\hat{g}\bar{\boldsymbol{\beta}} + \frac{\partial f_*}{\partial \kappa}\pi\hat{g}\lambda.\end{aligned} \tag{5.15}$$

Solving for $\hat{f}_*$,

$$\hat{f}_* = -\pi\hat{g}\left(\mathbf{U}_p\frac{\partial f_*}{\partial \mathbf{E}_p}\mathbf{U}_p\cdot\bar{\boldsymbol{\rho}} + \mathbf{U}_p\frac{\partial f_*}{\partial \boldsymbol{\alpha}_R}\mathbf{U}_p\cdot\bar{\boldsymbol{\beta}} + \frac{\partial f_*}{\partial \kappa}\lambda\right). \tag{5.16}$$

Dividing by $\hat{g}$ and using (3.26),

$$\frac{\hat{f}_*}{\hat{g}} = \bar{\Phi}_* = \frac{\mathbf{U}_p\dfrac{\partial f_*}{\partial \mathbf{E}_p}\mathbf{U}_p\cdot\bar{\boldsymbol{\rho}} + \mathbf{U}_p\dfrac{\partial f_*}{\partial \boldsymbol{\alpha}_R}\mathbf{U}_p\cdot\bar{\boldsymbol{\beta}} + \dfrac{\partial f_*}{\partial \kappa}\lambda}{\dfrac{\partial g}{\partial \mathbf{E}_p}\cdot\boldsymbol{\rho} + \dfrac{\partial g}{\partial \boldsymbol{\alpha}_R}\cdot\boldsymbol{\beta} + \dfrac{\partial g}{\partial \kappa}\cdot\lambda} \tag{5.17}$$

Note that $\hat{g}$ is always positive during loading, and $\hat{f}_*$ can be positive, negative or zero. We propose the classification of strain-hardening behavior:

$$\begin{aligned} &\bar{\Phi}_* > 0 \quad \text{hardening} \\ &\bar{\Phi}_* < 0 \quad \text{softening} \\ &\bar{\Phi}_* = 0 \quad \text{perfectly plastic.} \end{aligned} \tag{5.18}$$

Note that this classification is not the same as the one presented in Equation (3.30) on page 22.

We propose a similar criterion for the yield function g_*. It will later be explained that in general, $\hat{g}_*$ cannot be used as a loading index. As such, during loading,

$$\begin{aligned} 0 = \dot{g}_* &= \frac{\partial g_*}{\partial \mathbf{E}_*} \cdot \dot{\mathbf{E}}_* + \frac{\partial g_*}{\partial \mathbf{E}_p} \cdot \dot{\mathbf{E}}_p + \frac{\partial g_*}{\partial \boldsymbol{\alpha}_R} \cdot \dot{\boldsymbol{\alpha}}_R + \frac{\partial g_*}{\partial \kappa}\dot{\kappa} \\ &= \hat{g}_* + \frac{\partial g_*}{\partial \mathbf{E}_p} \cdot \pi\hat{g}\boldsymbol{\rho} + \frac{\partial g_*}{\partial \boldsymbol{\alpha}_R} \cdot \pi\hat{g}\boldsymbol{\beta} + \frac{\partial g_*}{\partial \kappa}\pi\hat{g}\lambda \\ &= \hat{g}_* + \frac{\partial g_*}{\partial \mathbf{E}_p} \cdot \mathbf{U}_p\bar{\mathbf{D}}_p\mathbf{U}_p + \frac{\partial g_*}{\partial \boldsymbol{\alpha}_R} \cdot \mathbf{U}_p\dot{\bar{\boldsymbol{\alpha}}}_R\mathbf{U}_p + \frac{\partial g_*}{\partial \kappa}\pi\hat{g}\lambda \\ &= \hat{g}_* + \mathbf{U}_p\frac{\partial g_*}{\partial \mathbf{E}_p}\mathbf{U}_p \cdot \pi\hat{g}\bar{\boldsymbol{\rho}} + \mathbf{U}_p\frac{\partial g_*}{\partial \boldsymbol{\alpha}_R}\mathbf{U}_p \cdot \pi\hat{g}\bar{\boldsymbol{\beta}} + \frac{\partial g_*}{\partial \kappa}\pi\hat{g}\lambda \\ &= \hat{g}_* + \pi\hat{g}\left(\mathbf{U}_p\frac{\partial g_*}{\partial \mathbf{E}_p}\mathbf{U}_p \cdot \bar{\boldsymbol{\rho}} + \mathbf{U}_p\frac{\partial g_*}{\partial \boldsymbol{\alpha}_R}\mathbf{U}_p \cdot \bar{\boldsymbol{\beta}} + \frac{\partial g_*}{\partial \kappa}\lambda\right). \end{aligned} \tag{5.19}$$

Dividing by $\hat{g}$ and using (3.26), we get

$$\frac{\hat{g}_*}{\hat{g}} = \Phi_* = \frac{\mathbf{U}_p\frac{\partial g_*}{\partial \mathbf{E}_p}\mathbf{U}_p \cdot \bar{\boldsymbol{\rho}} + \mathbf{U}_p\frac{\partial g_*}{\partial \boldsymbol{\alpha}_R}\mathbf{U}_p \cdot \bar{\boldsymbol{\beta}} + \frac{\partial g_*}{\partial \kappa}\lambda}{\frac{\partial g}{\partial \mathbf{E}_p} \cdot \boldsymbol{\rho} + \frac{\partial g}{\partial \boldsymbol{\alpha}_R} \cdot \boldsymbol{\beta} + \frac{\partial g}{\partial \kappa} \cdot \lambda}. \tag{5.20}$$

In general, the elasticity of an elastic-plastic material is described by the fourth order tensor

$$\mathscr{L}_* = \frac{\partial \mathbf{S}_*}{\partial \mathbf{E}_*}(\mathbf{E}_*). \tag{5.21}$$

Then

$$\frac{\partial g_*}{\partial \mathbf{E}_*} = \frac{\partial f_*}{\partial \mathbf{S}_*}\frac{\partial \mathbf{S}_*}{\partial \mathbf{E}_*} = \mathscr{L}_*^T\left[\frac{\partial f_*}{\partial \mathbf{S}_*}\right], \tag{5.22}$$

and

$$\hat{g}_* = \frac{\partial g_*}{\partial \mathbf{E}_*} \cdot \dot{\mathbf{E}}_* = \boldsymbol{\mathscr{L}}_*^T \left[\frac{\partial f_*}{\partial \mathbf{S}_*} \right] \cdot \dot{\mathbf{E}}_*$$
$$= \frac{\partial f_*}{\partial \mathbf{S}_*} \cdot \boldsymbol{\mathscr{L}}_* [\dot{\mathbf{E}}_*] \tag{5.23}$$
$$= \frac{\partial f_*}{\partial \mathbf{S}_*} \cdot \frac{\partial \mathbf{S}_*}{\partial \mathbf{E}_*} [\dot{\mathbf{E}}_*]$$

As a special case, consider $\boldsymbol{\mathscr{L}}_*$ to be a constant tensor. We can then write

$$\hat{g}_* = \frac{\partial f_*}{\partial \mathbf{S}_*} \cdot \boldsymbol{\mathscr{L}}_* [\dot{\mathbf{E}}_*] = \frac{\partial f_*}{\partial \mathbf{S}_*} \cdot \dot{\mathbf{S}}_* = \hat{f}_*. \tag{5.24}$$

We can also express the yield function as a function of the rotated stress as follows.

$$f_*(\mathbf{S}_*, \mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa) = f_*(\hat{\mathbf{S}}_*(\mathbf{E}_*, \mathbf{E}_p, \boldsymbol{\alpha}, \kappa), \mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa)$$
$$= f_*(\mathbf{S}_*(\mathbf{F}_*, \mathbf{T}, \mathbf{E}_p, \boldsymbol{\alpha}, \kappa), \mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa) \tag{5.25}$$
$$= \bar{f}_*(\mathbf{T}, \mathbf{F}_*, \mathbf{E}_p, \boldsymbol{\alpha}, \kappa),$$

where we have used (5.2) and the relations

$$J_*\mathbf{T} = \det \mathbf{F}_* \mathbf{T} = \mathbf{F}_* \mathbf{S}_* \mathbf{F}_*^T = J_* \mathbf{R} \widetilde{\mathbf{T}} \mathbf{R}^T. \tag{5.26}$$

Invoking invariance, we get the yield function $\tilde{f}$:

$$\bar{f}_*(\mathbf{T}, \mathbf{F}_*, \mathbf{E}_p, \boldsymbol{\alpha}, \kappa) = \bar{f}_*(\mathbf{Q}\mathbf{T}\mathbf{Q}^T, \mathbf{Q}\mathbf{F}_*, \mathbf{E}_p, \mathbf{Q}\boldsymbol{\alpha}\mathbf{Q}^T, \kappa)$$
$$= \bar{f}_*(\mathbf{R}^T\mathbf{T}\mathbf{R}, \mathbf{R}^T\mathbf{F}_*, \mathbf{E}_p, \mathbf{R}^T\boldsymbol{\alpha}\mathbf{R}, \kappa)$$
$$= \bar{f}_*(\widetilde{\mathbf{T}}, \mathbf{U}, \mathbf{U}_p, \mathbf{E}_p, \widetilde{\boldsymbol{\alpha}}, \kappa) \tag{5.27}$$
$$= \tilde{f}(\widetilde{\mathbf{T}}, \mathbf{E}_p, \widetilde{\boldsymbol{\alpha}}, \kappa).$$

Alternatively, using the inverse of $(5.48)_3$,

$$g(\mathbf{E}, \mathbf{E}_p, \boldsymbol{\alpha}, \kappa) = \tilde{f}(\widetilde{\mathbf{T}}, \mathbf{E}_p, \widetilde{\boldsymbol{\alpha}}, \kappa). \tag{5.28}$$

By the consistency condition,

$$0 = \dot{g} = \dot{\tilde{f}} = \frac{\partial \tilde{f}}{\partial \widetilde{\mathbf{T}}} \cdot \dot{\widetilde{\mathbf{T}}} + \frac{\partial \tilde{f}}{\partial \mathbf{E}_p} \cdot \dot{\mathbf{E}}_p + \frac{\partial \tilde{f}}{\partial \widetilde{\boldsymbol{\alpha}}} \cdot \dot{\widetilde{\boldsymbol{\alpha}}} + \frac{\partial \tilde{f}}{\partial \kappa} \dot{\kappa}$$
$$= \hat{\tilde{f}} + \frac{\partial \tilde{f}}{\partial \mathbf{E}_p} \cdot \dot{\mathbf{E}}_p + \frac{\partial \tilde{f}}{\partial \widetilde{\boldsymbol{\alpha}}} \cdot \dot{\widetilde{\boldsymbol{\alpha}}} + \frac{\partial \tilde{f}}{\partial \kappa} \dot{\kappa}. \tag{5.29}$$

Again, during loading, we use the constitutive relations

$$\dot{\mathbf{E}}_p = \pi\hat{g}\boldsymbol{\rho}, \qquad \dot{\tilde{\boldsymbol{\alpha}}} = \pi\hat{g}\tilde{\boldsymbol{\beta}}, \qquad \dot{\kappa} = \pi\hat{g}\lambda. \tag{5.30}$$

We can express the previous flow and hardening rules using the Eulerian forms of $\boldsymbol{\rho}$ and $\tilde{\boldsymbol{\beta}}$.

$$\begin{aligned}\bar{\mathbf{D}}_p &= \pi\hat{g}\mathbf{U}_p^{-1}\boldsymbol{\rho}_*\mathbf{U}_p^{-1} = \pi\hat{g}\bar{\boldsymbol{\rho}} = \mathbf{U}_p^{-1}\dot{\mathbf{E}}_p\mathbf{U}_p^{-1},\\ \dot{\bar{\boldsymbol{\alpha}}} &= \pi\hat{g}\mathbf{U}_p^{-1}\tilde{\boldsymbol{\beta}}_*\mathbf{U}_p^{-1} = \pi\hat{g}\bar{\boldsymbol{\beta}} = \mathbf{U}_p^{-1}\dot{\tilde{\boldsymbol{\alpha}}}\mathbf{U}_p^{-1}.\end{aligned} \tag{5.31}$$

Thus (5.29) becomes

$$\begin{aligned}0 = \dot{\tilde{f}} &= \hat{\tilde{f}} + \frac{\partial\tilde{f}}{\partial\mathbf{E}_p}\cdot\pi\hat{g}\boldsymbol{\rho} + \frac{\partial\tilde{f}}{\partial\tilde{\boldsymbol{\alpha}}}\cdot\pi\hat{g}\tilde{\boldsymbol{\beta}} + \frac{\partial\tilde{f}}{\partial\kappa}\pi\hat{g}\lambda\\ &= \hat{\tilde{f}} + \frac{\partial\tilde{f}}{\partial\mathbf{E}_p}\cdot\mathbf{U}_p\bar{\mathbf{D}}_p\mathbf{U}_p + \frac{\partial\tilde{f}}{\partial\tilde{\boldsymbol{\alpha}}}\cdot\mathbf{U}_p\dot{\bar{\boldsymbol{\alpha}}}\mathbf{U}_p + \frac{\partial\tilde{f}}{\partial\kappa}\pi\hat{g}\lambda\\ &= \hat{\tilde{f}} + \mathbf{U}_p\frac{\partial\tilde{f}}{\partial\mathbf{E}_p}\mathbf{U}_p\cdot\bar{\mathbf{D}}_p + \mathbf{U}_p\frac{\partial\tilde{f}}{\partial\tilde{\boldsymbol{\alpha}}}\mathbf{U}_p\cdot\dot{\bar{\boldsymbol{\alpha}}} + \frac{\partial\tilde{f}}{\partial\kappa}\pi\hat{g}\lambda\end{aligned} \tag{5.32}$$

Solving for $\hat{\tilde{f}}$,

$$\hat{\tilde{f}} = -\pi\hat{g}\left(\frac{\partial\tilde{f}}{\partial\mathbf{E}_p}\cdot\boldsymbol{\rho} + \frac{\partial\tilde{f}}{\partial\tilde{\boldsymbol{\alpha}}}\cdot\tilde{\boldsymbol{\beta}} + \frac{\partial\tilde{f}}{\partial\kappa}\lambda\right). \tag{5.33}$$

Dividing by $\hat{g}$ and substituting (3.26):

$$\frac{\hat{\tilde{f}}}{\hat{g}} = \tilde{\Phi}_* = \frac{\dfrac{\partial\tilde{f}}{\partial\mathbf{E}_p}\cdot\boldsymbol{\rho} + \dfrac{\partial\tilde{f}}{\partial\tilde{\boldsymbol{\alpha}}}\cdot\tilde{\boldsymbol{\beta}} + \dfrac{\partial\tilde{f}}{\partial\kappa}\lambda}{\dfrac{\partial g}{\partial\mathbf{E}_p}\cdot\boldsymbol{\rho} + \dfrac{\partial g}{\partial\boldsymbol{\alpha}_R}\cdot\boldsymbol{\beta} + \dfrac{\partial g}{\partial\kappa}\cdot\lambda}. \tag{5.34}$$

A different classification of strain-hardening behavior can then be proposed for the function $\tilde{\Phi}_*$, analogous to (5.18).

Next we show that for yield functions of the form

$$f(\mathbf{S},\mathscr{U}) = f_1(\mathbf{T},\mathscr{U}) = \tilde{f}(\widetilde{\mathbf{T}},\mathscr{U}), \tag{5.35}$$

we have

$$\hat{f} = \frac{\partial f}{\partial\mathbf{S}}\cdot\dot{\mathbf{S}} = \frac{\partial f_1}{\partial\mathbf{T}}\cdot\overset{t}{\mathbf{T}} = \frac{\partial\tilde{f}}{\partial\widetilde{\mathbf{T}}}\cdot\overset{t}{\widetilde{\mathbf{T}}}, \tag{5.36}$$

such that for criteria where $\hat{f}$ is featured, we have some freedom as to the stress measures and stress rates that we use in the theory. First, in index notation,

$$\frac{\partial f_1}{\partial T_{ij}} = \frac{\partial f}{\partial S_{AB}} \frac{\partial S_{AB}}{\partial T_{ij}}. \tag{5.37}$$

With the use of $(2.36)_2$,

$$S_{AB} = J X_{A,m} X_{B,n} T_{mn} \qquad \rightarrow \qquad \frac{\partial S_{AB}}{\partial T_{ij}} = J X_{A,i} X_{B,j}. \tag{5.38}$$

Thus,

$$\frac{\partial f_1}{\partial \mathbf{T}} = J \mathbf{F}^{-T} \frac{\partial f}{\partial \mathbf{S}} \mathbf{F}^{-1} \qquad \rightarrow \qquad \frac{\partial f}{\partial \mathbf{S}} = \frac{1}{J} \mathbf{F}^T \frac{\partial f_1}{\partial \mathbf{T}} \mathbf{F}. \tag{5.39}$$

Then,

$$\begin{aligned} \hat{f} &= \operatorname{tr}\left[\left(\frac{\partial f}{\partial \mathbf{S}} \right)^T \dot{\mathbf{S}} \right] \\ &= \operatorname{tr}\left[\frac{1}{J} \mathbf{F}^T \left(\frac{\partial f_1}{\partial \mathbf{T}} \right)^T \mathbf{F} \; J \mathbf{F}^{-1} \overset{t}{\mathbf{T}} \mathbf{F}^{-T} \right] \\ &= \frac{\partial f_1}{\partial \mathbf{T}} \cdot \overset{t}{\mathbf{T}}, \end{aligned} \tag{5.40}$$

where we have used $(2.55)_2$. Also, since

$$\begin{aligned} \frac{\partial f_1}{\partial \mathbf{T}} &= \frac{\partial \tilde{f}}{\partial \widetilde{\mathbf{T}}} \frac{\partial \widetilde{\mathbf{T}}}{\partial \mathbf{T}} \\ &= \mathbf{R} \frac{\partial \tilde{f}}{\partial \widetilde{\mathbf{T}}} \mathbf{R}^T, \end{aligned} \tag{5.41}$$

we have

$$\begin{aligned} \frac{\partial f_1}{\partial \mathbf{T}} \cdot \overset{t}{\mathbf{T}} &= \mathbf{R} \frac{\partial \tilde{f}}{\partial \widetilde{\mathbf{T}}} \mathbf{R}^T \cdot \overset{t}{\mathbf{T}} \\ &= \frac{\partial \tilde{f}}{\partial \widetilde{\mathbf{T}}} \cdot \mathbf{R}^T \overset{t}{\mathbf{T}} \mathbf{R} \\ &= \frac{\partial \tilde{f}}{\partial \widetilde{\mathbf{T}}} \cdot \overset{t}{\widetilde{\mathbf{T}}}, \end{aligned} \tag{5.42}$$

where

$$\overset{t}{\widetilde{\mathbf{T}}} = \mathbf{R}^T \overset{t}{\mathbf{T}} \mathbf{R} = \frac{1}{J} \mathbf{U} \dot{\mathbf{S}} \mathbf{U}. \tag{5.43}$$

Similarly, if we were to take the rotated stress as our primary stress measure, the loading index would be

$$\hat{\tilde{f}} = \frac{\partial \tilde{f}}{\partial \widetilde{\mathbf{T}}} \cdot \dot{\widetilde{\mathbf{T}}} = \frac{\partial f_1}{\partial \mathbf{T}} \cdot \overset{\square}{\mathbf{T}}. \tag{5.44}$$

Of course, the loading index formulated here is not related to the criteria presented in (3.30). Different strain hardening criteria, for example, would have to be constructed for $\hat{\hat{f}}$.

5.2 The rotated Cauchy stress $\widetilde{\mathbf{T}}$ and its rate

Using (3.1) and (2.36), we can write the Cauchy stress tensor $\mathbf{T}$ in terms of a response function $\hat{\mathbf{T}}$:

$$\mathbf{T} = \frac{1}{J}\mathbf{F}\hat{\mathbf{S}}(\mathbf{E},\mathscr{U})\mathbf{F}^T = \hat{\mathbf{T}}(\mathbf{F},\mathbf{F}_p,\boldsymbol{\alpha},\kappa), \tag{5.45}$$

which satisfies the invariance requirement

$$\begin{aligned}\hat{\mathbf{T}}(\mathbf{F},\mathbf{F}_p,\boldsymbol{\alpha},\kappa) &= \mathbf{Q}^T\hat{\mathbf{T}}(\mathbf{QF},\mathbf{Q}_p\mathbf{F}_p,\mathbf{Q}\boldsymbol{\alpha}\mathbf{Q}^T,\kappa)\mathbf{Q}\\ &= \mathbf{R}\hat{\mathbf{T}}(\mathbf{U},\mathbf{U}_p,\frac{1}{J}\mathbf{U}\boldsymbol{\alpha}_R\mathbf{U},\kappa)\mathbf{R}^T\\ &= \mathbf{R}\widetilde{\mathbf{T}}(\mathbf{E},\mathbf{E}_p,\boldsymbol{\alpha}_R,\kappa)\mathbf{R}^T\\ &= \mathbf{R}\widetilde{\mathbf{T}}(\mathbf{E},\mathscr{U})\mathbf{R}^T,\end{aligned} \tag{5.46}$$

where we have used $(4.11)_1$, $(4.21)_7$, and

$$\boldsymbol{\alpha} = \frac{1}{J}\mathbf{F}\boldsymbol{\alpha}_R\mathbf{F}^T, \qquad \boldsymbol{\alpha}^+ = \mathbf{Q}\boldsymbol{\alpha}\mathbf{Q}^T. \tag{5.47}$$

Thus, as in (2.37), we define a rotated Cauchy stress $\widetilde{\mathbf{T}}$:

$$\widetilde{\mathbf{T}}(\mathbf{E},\mathscr{U}) = \mathbf{R}^T\mathbf{T}\mathbf{R} = \frac{1}{J}\mathbf{USU}. \tag{5.48}$$

The material derivative of the rotated stress is

$$\dot{\widetilde{\mathbf{T}}} = \frac{\partial\widetilde{\mathbf{T}}}{\partial\mathbf{E}}[\dot{\mathbf{E}}] + \frac{\partial\widetilde{\mathbf{T}}}{\partial\mathbf{E}_p}[\dot{\mathbf{E}}_p] + \frac{\partial\widetilde{\mathbf{T}}}{\partial\boldsymbol{\alpha}_R}[\dot{\boldsymbol{\alpha}}_R] + \frac{\partial\widetilde{\mathbf{T}}}{\partial\kappa}\dot{\kappa}. \tag{5.49}$$

Define a fourth-order tensor

$$\tilde{\boldsymbol{\mathscr{L}}} = \frac{\partial\widetilde{\mathbf{T}}}{\partial\mathbf{E}}. \tag{5.50}$$

Then the first term on the right hand side of (5.49) can be written as

$$\begin{aligned}\frac{\partial\widetilde{\mathbf{T}}}{\partial\mathbf{E}}[\dot{\mathbf{E}}] &= \tilde{\boldsymbol{\mathscr{L}}}[\dot{\mathbf{E}}], &\qquad \tilde{\mathscr{L}}_{MNKL}\dot{E}_{KL}\\ &= \tilde{\boldsymbol{\mathscr{L}}}[\mathbf{F}^T\mathbf{D}\mathbf{F}], &\qquad \tilde{\mathscr{L}}_{MNKL}F_{iK}F_{jL}D_{ij}.\end{aligned} \tag{5.51}$$

For the second term, which is the term that involves $\mathbf{E}_p$, we know that

$$\dot{\mathbf{E}}_p = \mathbf{U}_p \bar{\mathbf{D}}_p \mathbf{U}_p. \tag{5.52}$$

During loading, we assume

$$\dot{\mathbf{E}}_p = \pi \hat{g} \boldsymbol{\rho}, \tag{5.53}$$

so that

$$\bar{\mathbf{D}}_p = \mathbf{U}_p^{-1} \dot{\mathbf{E}}_p \mathbf{U}_p^{-1} = \pi \hat{g} \mathbf{U}_p^{-1} \boldsymbol{\rho} \mathbf{U}_p^{-1} = \pi \hat{g} \bar{\boldsymbol{\rho}}. \tag{5.54}$$

Then

$$\frac{\partial \widetilde{\mathbf{T}}}{\partial \mathbf{E}_p}\big[\dot{\mathbf{E}}_p\big] = \frac{\partial \widetilde{\mathbf{T}}}{\partial \mathbf{E}_p}\big[\mathbf{U}_p \bar{\mathbf{D}}_p \mathbf{U}_p\big] = \frac{\partial \widetilde{\mathbf{T}}}{\partial \mathbf{E}_p}\big[\pi \hat{g} \boldsymbol{\rho}\big]. \tag{5.55}$$

Similarly, for the third and fourth terms of (5.49),

$$\begin{aligned} \frac{\partial \widetilde{\mathbf{T}}}{\partial \boldsymbol{\alpha}_R}\big[\dot{\boldsymbol{\alpha}}_R\big] &= \frac{\partial \widetilde{\mathbf{T}}}{\partial \boldsymbol{\alpha}_R}\big[\pi \hat{g} \boldsymbol{\beta}\big], \\ \frac{\partial \widetilde{\mathbf{T}}}{\partial \kappa}\,\dot{\kappa} &= \frac{\partial \widetilde{\mathbf{T}}}{\partial \kappa}\,\pi \hat{g} \lambda. \end{aligned} \tag{5.56}$$

Recalling the form (3.19) introduced in Chapter 3, we can write the rate of change of $\widetilde{\mathbf{T}}$ in a similar way:

$$\dot{\widetilde{\mathbf{T}}} = \tilde{\boldsymbol{\mathscr{L}}}\big[\dot{\mathbf{E}}\big] + \pi \hat{g} \tilde{\boldsymbol{\sigma}}, \tag{5.57}$$

where

$$\tilde{\boldsymbol{\sigma}} = \frac{\partial \widetilde{\mathbf{T}}}{\partial \mathbf{E}_p}\big[\boldsymbol{\rho}\big] + \frac{\partial \widetilde{\mathbf{T}}}{\partial \boldsymbol{\alpha}}\big[\boldsymbol{\beta}\big] + \frac{\partial \widetilde{\mathbf{T}}}{\partial \kappa}\,\lambda. \tag{5.58}$$

We now attempt to extract a linear dependence of the rotated stress rate to the rate of deformation $\mathbf{D}$. We can write (5.57) as follows:

$$\begin{aligned} \dot{\widetilde{\mathbf{T}}} &= \tilde{\boldsymbol{\mathscr{L}}}\big[\dot{\mathbf{E}}\big] + \pi \hat{g} \tilde{\boldsymbol{\sigma}}, \\ &= \tilde{\boldsymbol{\mathscr{L}}}\big[\mathbf{F}^T \mathbf{D} \mathbf{F}\big] + \pi \left(\frac{\partial g}{\partial \mathbf{E}} \cdot \dot{\mathbf{E}} \right) \tilde{\boldsymbol{\sigma}}. \end{aligned} \tag{5.59}$$

In the first term, the transformation $\tilde{\boldsymbol{\mathscr{L}}}$ is linear. Thus, we can define a new fourth-order tensor $\tilde{\boldsymbol{\mathscr{M}}}$. Let

$$\tilde{\mathscr{L}}_{MNKL} F_{iK} F_{jL} = \tilde{\mathscr{M}}_{MNij} \tag{5.60}$$

so that

$$\tilde{\mathscr{L}}_{MNKL} F_{iK} F_{jL} D_{ij} = \tilde{\mathscr{M}}_{MNij} D_{ij}. \tag{5.61}$$

There is symmetry in the indices MN from $\tilde{\boldsymbol{\mathscr{L}}}$ and symmetry in ij from $\mathbf{D}$. In direct notation,

$$\tilde{\boldsymbol{\mathscr{L}}}\big[\mathbf{F}^T \mathbf{D} \mathbf{F}\big] = \tilde{\boldsymbol{\mathscr{M}}}\big[\mathbf{D}\big]. \tag{5.62}$$

We observe that

$$\begin{aligned}
\hat{g} &= \frac{\partial g}{\partial \mathbf{E}} \cdot \dot{\mathbf{E}} \\
&= \frac{\partial g}{\partial \mathbf{E}} \cdot \mathbf{F}^T\mathbf{D}\mathbf{F} \\
&= \mathbf{F}\frac{\partial g}{\partial \mathbf{E}}\mathbf{F}^T \cdot \mathbf{D} \\
&= \mathbf{F}\frac{\partial \tilde{f}}{\partial \widetilde{\mathbf{T}}}\frac{\partial \widetilde{\mathbf{T}}}{\partial \mathbf{E}}\mathbf{F}^T \cdot \mathbf{D} \\
&= \mathbf{F}\tilde{\boldsymbol{\mathscr{L}}}^T\left[\frac{\partial \tilde{f}}{\partial \widetilde{\mathbf{T}}}\right]\mathbf{F}^T \cdot \mathbf{D}, \qquad \tilde{\mathscr{L}}_{MNKL}\frac{\partial \tilde{f}}{\partial \tilde{T}_{KL}}F_{iK}F_{jL}D_{ij}.
\end{aligned} \tag{5.63}$$

Then the second term in (5.57) becomes

$$\begin{aligned}
\pi\hat{g}\tilde{\boldsymbol{\sigma}} &= \pi\mathbf{F}\tilde{\boldsymbol{\mathscr{L}}}^T\left[\frac{\partial \tilde{f}}{\partial \widetilde{\mathbf{T}}}\right]\mathbf{F}^T \cdot \mathbf{D}\ \tilde{\boldsymbol{\sigma}} \\
&= \left(\pi\tilde{\boldsymbol{\sigma}} \otimes \mathbf{F}\tilde{\boldsymbol{\mathscr{L}}}^T\left[\frac{\partial \tilde{f}}{\partial \widetilde{\mathbf{T}}}\right]\mathbf{F}^T\right)[\mathbf{D}] \\
&= \left(\pi\tilde{\boldsymbol{\sigma}} \otimes \tilde{\boldsymbol{\mathscr{L}}}^T\left[\frac{\partial \tilde{f}}{\partial \widetilde{\mathbf{T}}}\right]\right)[\mathbf{F}^T\mathbf{D}\mathbf{F}] \\
&= \left(\pi\tilde{\boldsymbol{\sigma}} \otimes \frac{\partial \tilde{f}}{\partial \widetilde{\mathbf{T}}}\right)\tilde{\boldsymbol{\mathscr{L}}}[\mathbf{F}^T\mathbf{D}\mathbf{F}] \\
&= \left(\pi\tilde{\boldsymbol{\sigma}} \otimes \frac{\partial \tilde{f}}{\partial \widetilde{\mathbf{T}}}\right)\tilde{\boldsymbol{\mathscr{M}}}[\mathbf{D}].
\end{aligned} \tag{5.64}$$

Putting it all together, we can rewrite (5.57) as

$$\dot{\tilde{\mathbf{T}}} = \tilde{\boldsymbol{\mathscr{M}}}[\mathbf{D}] + \left(\pi\tilde{\boldsymbol{\sigma}} \otimes \frac{\partial \tilde{f}}{\partial \widetilde{\mathbf{T}}}\right)\tilde{\boldsymbol{\mathscr{M}}}[\mathbf{D}]. \tag{5.65}$$

Further, we define a new fourth-order tensor $\tilde{\boldsymbol{\mathscr{N}}}$:

$$\tilde{\boldsymbol{\mathscr{N}}} = \mathscr{I} + \pi\tilde{\boldsymbol{\sigma}} \otimes \frac{\partial \tilde{f}}{\partial \widetilde{\mathbf{T}}}. \tag{5.66}$$

Then,

$$\dot{\tilde{\mathbf{T}}} = \tilde{\boldsymbol{\mathscr{N}}}\tilde{\boldsymbol{\mathscr{M}}}[\mathbf{D}]. \tag{5.67}$$

Note that the form of (5.67) is analogous to (3.21) on page 20.

5.3 The second Piola-Kirchhoff stress tensor $\mathbf{S}_*$ and its rate

Consider the rate of the second Piola-Kirchhoff stress tensor $\mathbf{S}_*$ expressed in (5.2),

$$\dot{\mathbf{S}}_* = \frac{\partial \mathbf{S}_*}{\partial \mathbf{E}_*}[\dot{\mathbf{E}}_*] + \frac{\partial \mathbf{S}_*}{\partial \mathbf{E}_p}[\dot{\mathbf{E}}_p] + \frac{\partial \mathbf{S}_*}{\partial \boldsymbol{\alpha}}[\dot{\boldsymbol{\alpha}}] + \frac{\partial \mathbf{S}_*}{\partial \kappa}\dot{\kappa}. \tag{5.68}$$

Define a fourth order tensor $\mathcal{L}_*$ by

$$\boldsymbol{\mathscr{L}}_* = \frac{\partial \mathbf{S}_*}{\partial \mathbf{E}_*}, \tag{5.69}$$

so that the first term in (5.68) is

$$\frac{\partial \mathbf{S}_*}{\partial \mathbf{E}_*}[\dot{\mathbf{E}}_*] = \boldsymbol{\mathscr{L}}_*[\dot{\mathbf{E}}_*]. \tag{5.70}$$

During loading, we have the relations

$$\dot{\mathbf{E}}_p = \pi \hat{g} \boldsymbol{\rho}, \qquad \dot{\boldsymbol{\alpha}}_R = \pi \hat{g} \boldsymbol{\beta}, \qquad \dot{\kappa} = \pi \hat{g} \lambda. \tag{5.71}$$

Thus, the last three terms in (5.68) can simplify to $\pi \hat{g} \boldsymbol{\sigma}_*$, where

$$\boldsymbol{\sigma}_* = \frac{\partial \mathbf{S}_*}{\partial \mathbf{E}_p}[\boldsymbol{\rho}_*] + \frac{\partial \mathbf{S}_*}{\partial \boldsymbol{\alpha}}[\boldsymbol{\beta}] + \frac{\partial \mathbf{S}_*}{\partial \kappa}\lambda. \tag{5.72}$$

In an effort to obtain a nice form of the stress rate analogous to (5.67), recall that

$$\mathbf{E}_* = \mathbf{U}_p^{-1}(\mathbf{E} - \mathbf{E}_p)\mathbf{U}_p^{-1}. \tag{5.73}$$

In index form,

$$E_{*AB} = U_{pKA}^{-1}(E_{KL} - E_{pKL})U_{pBL}^{-1}. \tag{5.74}$$

Differentiating with respect to $\mathbf{E}$,

$$\begin{aligned} \frac{\partial E_{*AB}}{\partial E_{MN}} &= \frac{1}{2}\left(\frac{\partial E_{*AB}}{\partial E_{MN}} + \frac{\partial E_{*AB}}{\partial E_{NM}}\right) \\ &= \frac{1}{2}\left[\left(U_{pKA}^{-1}\delta_{KM}\delta_{LN}U_{pBL}^{-1} + U_{pKA}^{-1}\delta_{KN}\delta_{LM}U_{pBL}^{-1}\right)\right] \\ &= \frac{1}{2}\left(U_{pMA}^{-1}U_{pBN}^{-1} + U_{pAN}^{-1}U_{pBM}^{-1}\right). \end{aligned} \tag{5.75}$$

Then

$$\begin{aligned}\frac{\partial g}{\partial E_{MN}} &= \frac{\partial g_*}{\partial E_{*AB}}\frac{\partial E_{*AB}}{\partial E_{MN}} \\ &= \frac{1}{2}\left(U_{pMA}^{-1}\frac{\partial g_*}{\partial E_{*AB}}U_{pBN}^{-1} + U_{pAN}^{-1}\frac{\partial g_*}{\partial E_{*AB}}U_{pBM}^{-1}\right) \\ &= U_{pAM}^{-1}\ \frac{1}{2}\left(\frac{\partial g_*}{\partial E_{*AB}} + \frac{\partial g_*}{\partial E_{*AB}}\right)\ U_{pBN}^{-1}.\end{aligned} \tag{5.76}$$

In direct notation,

$$\frac{\partial g}{\partial \mathbf{E}} = \mathbf{U}_p^{-1}\frac{\partial g_*}{\partial \mathbf{E}_*}\mathbf{U}_p^{-1}. \tag{5.77}$$

Then

$$\begin{aligned}\hat{g} &= \frac{\partial g}{\partial \mathbf{E}}\cdot\dot{\mathbf{E}} \\ &= \mathbf{U}_p^{-1}\frac{\partial g_*}{\partial \mathbf{E}_*}\mathbf{U}_p^{-1}\cdot\mathbf{F}^T\mathbf{D}\mathbf{F}.\end{aligned} \tag{5.78}$$

With the use of (4.25) and (4.52), we can write

$$\begin{aligned}\mathbf{F}^T\mathbf{D}\mathbf{F} &= \mathbf{F}^T\left[\mathbf{D}_* + \frac{1}{2}\left(\mathbf{F}_*\dot{\mathbf{U}}_p\mathbf{U}_p^{-1}\mathbf{F}_*^{-1} + \mathbf{F}_*^{-T}\mathbf{U}_p^{-1}\dot{\mathbf{U}}_p\mathbf{F}_*^T\right)\right]\mathbf{F} \\ &= \mathbf{U}_p\mathbf{F}_*^T\mathbf{D}_*\mathbf{F}_*\mathbf{U}_p + \frac{1}{2}\left(\mathbf{U}_p\mathbf{F}_*^T\mathbf{F}_*\dot{\mathbf{U}}_p\mathbf{U}_p^{-1}\mathbf{F}_*^{-1}\mathbf{F}_*\mathbf{U}_p + \mathbf{U}_p\mathbf{F}_*^T\mathbf{F}_*^{-T}\mathbf{U}_p^{-1}\dot{\mathbf{U}}_p\mathbf{F}_*^T\mathbf{F}_*\mathbf{U}_p\right) \\ &= \mathbf{U}_p\dot{\mathbf{E}}_*\mathbf{U}_p + \frac{1}{2}\left(\mathbf{U}_p\mathbf{C}_*\dot{\mathbf{U}}_p + \dot{\mathbf{U}}_p\mathbf{C}_*\mathbf{U}_p\right) \\ &= \mathbf{U}_p\dot{\mathbf{E}}_*\mathbf{U}_p + \frac{1}{2}\left(\dot{\mathbf{C}} - \mathbf{U}_p\dot{\mathbf{C}}_*\mathbf{U}_p\right).\end{aligned} \tag{5.79}$$

Multiplying both sides by $\mathbf{U}_p^{-1}$,

$$\begin{aligned}\mathbf{F}_*^T\mathbf{D}\mathbf{F}_* &= \dot{\mathbf{E}}_* + \frac{1}{2}\left(\mathbf{C}_*\dot{\mathbf{U}}_p\mathbf{U}_p^{-1} + \mathbf{U}_p^{-1}\dot{\mathbf{U}}_p\mathbf{C}_*\right) \\ &= \dot{\mathbf{E}}_* + \frac{1}{2}\left(\mathbf{U}_p^{-1}\dot{\mathbf{C}}\mathbf{U}_p^{-1} - \dot{\mathbf{C}}_*\right).\end{aligned} \tag{5.80}$$

Thus, from (5.78),

$$\hat{g} = \frac{\partial g_*}{\partial \mathbf{E}_*}\cdot\mathbf{F}_*^T\mathbf{D}\mathbf{F}_*. \tag{5.81}$$

In the case of no plastic loading, $\dot{\mathbf{U}}_p = \mathbf{0}$, and $\hat{g}$ reduces to

$$\hat{g} = \frac{\partial g_*}{\partial \mathbf{E}_*}\cdot\dot{\mathbf{E}}_*. \tag{5.82}$$

In this case, we can write

$$\begin{aligned}\dot{\mathbf{S}}_* &= \boldsymbol{\mathscr{L}}_*[\dot{\mathbf{E}}_*] + \left(\pi\boldsymbol{\sigma}_* \otimes \frac{\partial g_*}{\partial \mathbf{E}_*}\right)[\dot{\mathbf{E}}_*] \\ &= \boldsymbol{\mathscr{L}}_*[\dot{\mathbf{E}}_*] + \left(\pi\boldsymbol{\sigma}_* \otimes \frac{\partial f_*}{\partial \mathbf{S}_*}\frac{\partial \mathbf{S}_*}{\partial \mathbf{E}_*}\right)[\dot{\mathbf{E}}_*] \\ &= \left(\mathscr{I} + \pi\boldsymbol{\sigma}_* \otimes \frac{\partial f_*}{\partial \mathbf{S}_*}\right)\boldsymbol{\mathscr{L}}_*[\dot{\mathbf{E}}_*] \\ &= \boldsymbol{\mathscr{K}}_*\boldsymbol{\mathscr{L}}_*[\dot{\mathbf{E}}_*].\end{aligned} \tag{5.83}$$

The complete general form for the stress rate $\dot{\mathbf{S}}_*$ can be obtained directly from (5.68), without making any assumptions. Thus,

$$\begin{aligned}\dot{\mathbf{S}}_* &= \boldsymbol{\mathscr{L}}_*[\dot{\mathbf{E}}_*] + \left(\pi\boldsymbol{\sigma}_* \otimes \frac{\partial g_*}{\partial \mathbf{E}_*}\right)[\mathbf{F}_*^T\mathbf{D}\mathbf{F}_*] \\ &= \boldsymbol{\mathscr{L}}_*[\dot{\mathbf{E}}_*] + \left(\pi\boldsymbol{\sigma}_* \otimes \frac{\partial g_*}{\partial \mathbf{E}_*}\right)\left[\dot{\mathbf{E}}_* + \frac{1}{2}\left(\mathbf{C}_*\dot{\mathbf{U}}_p\mathbf{U}_p^{-1} + \mathbf{U}_p^{-1}\dot{\mathbf{U}}_p\mathbf{C}_*\right)\right] \\ &= \boldsymbol{\mathscr{L}}_*[\dot{\mathbf{E}}_*] + \left(\pi\boldsymbol{\sigma}_* \otimes \frac{\partial g_*}{\partial \mathbf{E}_*}\right)[\dot{\mathbf{E}}_*] + \left(\pi\boldsymbol{\sigma}_* \otimes \frac{\partial g_*}{\partial \mathbf{E}_*}\right)\left[\frac{1}{2}\left(\mathbf{C}_*\dot{\mathbf{U}}_p\mathbf{U}_p^{-1} + \mathbf{U}_p^{-1}\dot{\mathbf{U}}_p\mathbf{C}_*\right)\right] \\ &= \left(\mathscr{I} + \pi\boldsymbol{\sigma}_* \otimes \frac{\partial f_*}{\partial \mathbf{S}_*}\right)\boldsymbol{\mathscr{L}}_*[\dot{\mathbf{E}}_*] + \left(\pi\boldsymbol{\sigma}_* \otimes \frac{\partial g_*}{\partial \mathbf{E}_*}\right)\left[\frac{1}{2}\left(\mathbf{C}_*\dot{\mathbf{U}}_p\mathbf{U}_p^{-1} + \mathbf{U}_p^{-1}\dot{\mathbf{U}}_p\mathbf{C}_*\right)\right] \\ &= \boldsymbol{\mathscr{K}}_*\boldsymbol{\mathscr{L}}_*[\dot{\mathbf{E}}_*] + \left(\pi\boldsymbol{\sigma}_* \otimes \frac{\partial g_*}{\partial \mathbf{E}_*}\right)\left[\frac{1}{2}\left(\mathbf{C}_*\dot{\mathbf{U}}_p\mathbf{U}_p^{-1} + \mathbf{U}_p^{-1}\dot{\mathbf{U}}_p\mathbf{C}_*\right)\right].\end{aligned} \tag{5.84}$$

5.4 Application of the von Mises criterion to the expression of stress rates

The von Mises yield criterion can be represented using the different yield functions as follows:

$$\begin{aligned} f_1(\mathbf{T}) &= \frac{1}{2}\mathbf{T}\cdot\mathbf{T} - \kappa^2 \\ &= \frac{1}{2}\mathbf{R}\widetilde{\mathbf{T}}\mathbf{R}^T\cdot\mathbf{R}\widetilde{\mathbf{T}}\mathbf{R}^T - \kappa^2, \\ \tilde{f}(\widetilde{\mathbf{T}}) &= \frac{1}{2}\widetilde{\mathbf{T}}\cdot\widetilde{\mathbf{T}} - \kappa^2 \\ &= \frac{1}{2}\frac{1}{J}\mathbf{USU}\cdot\frac{1}{J}\mathbf{USU} - \kappa^2, \\ f(\mathbf{S}) &= \frac{1}{2J^2}\mathbf{S}\cdot\mathbf{CSC} - \kappa^2. \end{aligned} \tag{5.85}$$

To express the von Mises criterion in terms of the yield function $f_*(\mathbf{S}_*)$, we start with the relation (5.26) to get an expression for $\widetilde{\mathbf{T}}$ as a function of $\mathbf{S}_*$.

$$\begin{aligned} J_*\mathbf{T} &= \mathbf{F}_*\mathbf{S}_*\mathbf{F}_*^T \\ J_*\mathbf{R}\widetilde{\mathbf{T}}\mathbf{R}^T &= \mathbf{F}_*\mathbf{S}_*\mathbf{F}_*^T \\ \widetilde{\mathbf{T}} &= \frac{1}{J_*}\mathbf{R}^T\mathbf{F}_*\mathbf{S}_*\mathbf{F}_*^T\mathbf{R} \\ &= \frac{1}{J_*}\mathbf{R}^T\mathbf{F}\mathbf{U}_p^{-1}\mathbf{S}_*\mathbf{U}_p^{-1}\mathbf{F}^T\mathbf{R} \\ &= \frac{1}{J_*}\mathbf{U}\mathbf{U}_p^{-1}\mathbf{S}_*\mathbf{U}_p^{-1}\mathbf{U}. \end{aligned} \tag{5.86}$$

We then have

$$\begin{aligned} f_*(\mathbf{S}_*) &= \frac{1}{2J_*^2}\mathbf{U}\mathbf{U}_p^{-1}\mathbf{S}_*\mathbf{U}_p^{-1}\mathbf{U}\cdot\mathbf{U}\mathbf{U}_p^{-1}\mathbf{S}_*\mathbf{U}_p^{-1}\mathbf{U} - \kappa^2 \\ &= \frac{1}{2J_*^2}\mathbf{U}_p^{-1}\mathbf{S}_*\mathbf{U}_p^{-1}\cdot\mathbf{C}\mathbf{U}_p^{-1}\mathbf{S}_*\mathbf{U}_p^{-1}\mathbf{C} - \kappa^2. \end{aligned} \tag{5.87}$$

To apply the von Mises criterion to the stress rates $\dot{\mathbf{S}}$, $\dot{\widetilde{\mathbf{T}}}$, and $\dot{\mathbf{S}}_*$, we need the partial derivatives of f, $\tilde{f}$, and f_*. We formulate them below using index notation.

$$
\begin{aligned}
\frac{\partial \tilde{f}}{\partial \widetilde{T}_{mn}} &= \frac{1}{2}\left(\frac{\partial \tilde{f}}{\partial \widetilde{T}_{mn}} + \frac{\partial \tilde{f}}{\partial \widetilde{T}_{nm}}\right) \\
&= \frac{1}{2}\left[\frac{\partial}{\partial T_{mn}}\left(\frac{1}{2}\widetilde{T}_{ij}\widetilde{T}_{ij}\right) + \frac{\partial}{\partial T_{nm}}\left(\frac{1}{2}\widetilde{T}_{ij}\widetilde{T}_{ij}\right)\right] \\
&= \frac{1}{2}\left[\frac{1}{2}\left(\delta_{im}\delta_{jn}\widetilde{T}_{ij} + \widetilde{T}_{ij}\delta_{im}\delta_{jn}\right) + \frac{1}{2}\left(\delta_{in}\delta_{jm}\widetilde{T}_{ij} + \widetilde{T}_{ij}\delta_{in}\delta_{jm}\right)\right] \\
&= \frac{1}{2}\left[\frac{1}{2}\left(\widetilde{T}_{mn} + \widetilde{T}_{mn}\right) + \frac{1}{2}\left(\widetilde{T}_{mn} + \widetilde{T}_{mn}\right)\right] \\
&= \widetilde{T}_{mn}.
\end{aligned}
\tag{5.88}
$$

Thus,

$$
\frac{\partial \tilde{f}}{\partial \widetilde{\mathbf{T}}} = \widetilde{\mathbf{T}}. \tag{5.89}
$$

Next,

$$
\begin{aligned}
\frac{\partial f}{\partial S_{MN}} &= \frac{1}{2}\left(\frac{\partial f}{\partial S_{MN}} + \frac{\partial f}{\partial S_{NM}}\right) \\
&= \frac{1}{2}\left[\frac{\partial}{\partial S_{MN}}\left(\frac{1}{2J^2}S_{AB}C_{AC}S_{CD}C_{BD}\right) + \frac{\partial}{\partial S_{NM}}\left(\frac{1}{2J^2}S_{AB}C_{AC}S_{CD}C_{BD}\right)\right] \\
&= \frac{1}{4J^2}\Big[\left(\delta_{AM}\delta_{BN}C_{AC}S_{CD}C_{BD} + C_{AC}S_{AB}C_{BD}\delta_{CM}\delta_{DN}\right) \\
&\qquad + \left(\delta_{AN}\delta_{BM}C_{AC}S_{CD}C_{BD} + C_{AC}S_{AB}C_{BD}\delta_{CN}\delta_{DM}\right)\Big] \\
&= \frac{1}{4J^2}\left[C_{MC}S_{CD}C_{ND} + C_{AM}S_{AB}C_{BN} + C_{NC}S_{CD}C_{MD} + C_{AN}S_{AB}C_{BM}\right] \\
&= \frac{1}{J^2}\left(C_{AM}S_{AB}C_{BN}\right).
\end{aligned}
\tag{5.90}
$$

Thus,

$$
\frac{\partial f}{\partial \mathbf{S}} = \frac{1}{J^2}\mathbf{CSC}. \tag{5.91}
$$

Finally,

$$
\begin{aligned}
\frac{\partial f_*}{\partial \mathbf{S}_*} &= \frac{\partial}{\partial \mathbf{S}_*}\left(\frac{1}{2J_*^2}\mathbf{U}_p^{-1}\mathbf{S}_*\mathbf{U}_p^{-1}\cdot \mathbf{C}\mathbf{U}_p^{-1}\mathbf{S}_*\mathbf{U}_p^{-1}\mathbf{C}\right),\\
\frac{\partial f_*}{\partial S_{*MN}} &= \frac{1}{2}\left(\frac{\partial f_*}{\partial S_{*MN}}+\frac{\partial f_*}{\partial S_{*NM}}\right)\\
&= \frac{1}{2}\Bigg[\frac{\partial}{\partial S_{*MN}}\left(\frac{1}{2J_*^2}U_{pAK}^{-1}S_{*AB}U_{pBL}^{-1}C_{RK}U_{pCR}^{-1}S_{*CD}U_{pDQ}^{-1}C_{LQ}\right)\\
&\qquad + \frac{\partial}{\partial S_{*MN}}\left(\frac{1}{2J_*^2}U_{pAK}^{-1}S_{*AB}U_{pBL}^{-1}C_{RK}U_{pCR}^{-1}S_{*CD}U_{pDQ}^{-1}C_{LQ}\right)\Bigg]\\
&= \frac{1}{4J_*^2}\Big(\delta_{AM}\delta_{BN}U_{pAK}^{-1}U_{pBL}^{-1}C_{RK}U_{pCR}^{-1}S_{*CD}U_{pDQ}^{-1}C_{LQ}\\
&\qquad + \delta_{CM}\delta_{DN}U_{pAK}^{-1}S_{*AB}U_{pBL}^{-1}C_{RK}U_{pCR}^{-1}U_{pDQ}^{-1}C_{LQ}\\
&\qquad + \delta_{AN}\delta_{BM}U_{pAK}^{-1}U_{pBL}^{-1}C_{RK}U_{pCR}^{-1}S_{*CD}U_{pDQ}^{-1}C_{LQ}\\
&\qquad + \delta_{CN}\delta_{DM}U_{pAK}^{-1}S_{*AB}U_{pBL}^{-1}C_{RK}U_{pCR}^{-1}U_{pDQ}^{-1}C_{LQ}\Big)\\
&= \frac{1}{4J_*^2}\Big(U_{pMK}^{-1}U_{pNL}^{-1}C_{RK}U_{pCR}^{-1}S_{*CD}U_{pDQ}^{-1}C_{LQ}\\
&\qquad + U_{pAK}^{-1}S_{*AB}U_{pBL}^{-1}C_{RK}U_{pMR}^{-1}U_{pNQ}^{-1}C_{LQ}\\
&\qquad + U_{pNK}^{-1}U_{pML}^{-1}C_{RK}U_{pCR}^{-1}S_{*CD}U_{pDQ}^{-1}C_{LQ}\\
&\qquad + U_{pAK}^{-1}S_{*AB}U_{pBL}^{-1}C_{RK}U_{pNR}^{-1}U_{pMQ}^{-1}C_{LQ}\Big).
\end{aligned}
\tag{5.92}
$$

Taking advantage of the symmetries, we find that all four terms are equal. Simplifying, we have

$$
\frac{\partial f_*}{\partial \mathbf{S}_*} = \frac{1}{J_*^2}\left(\mathbf{U}_p^{-1}\mathbf{C}\mathbf{U}_p^{-1}\mathbf{S}_*\mathbf{U}_p^{-1}\mathbf{C}\mathbf{U}_p^{-1}\right). \tag{5.93}
$$

Further, in view of (4.46),

$$
\frac{\partial f_*}{\partial \mathbf{S}_*} = \frac{1}{J_*^2}\left(\mathbf{C}_*\mathbf{S}_*\mathbf{C}_*\right). \tag{5.94}
$$

Thus, after application of the von Mises criterion, the stress rates $\dot{\mathbf{S}}$, $\dot{\tilde{\mathbf{T}}}$, and $\dot{\mathbf{S}}_*$, respectively, are

$$
\begin{aligned}
\dot{\mathbf{S}} &= \left(\mathscr{I} + \pi\boldsymbol{\sigma}\otimes\frac{\partial f}{\partial \mathbf{S}}\right)\frac{\partial \mathbf{S}}{\partial \mathbf{E}}[\dot{\mathbf{E}}]\\
&= \left(\mathscr{I} + \pi\boldsymbol{\sigma}\otimes\frac{1}{J^2}\mathbf{CSC}\right)\boldsymbol{\mathscr{L}}[\dot{\mathbf{E}}],
\end{aligned}
\tag{5.95}
$$

$$
\begin{aligned}
\dot{\widetilde{\mathbf{T}}} &= \left(\mathscr{I} + \pi\widetilde{\boldsymbol{\sigma}} \otimes \frac{\partial \tilde{f}}{\partial \widetilde{\mathbf{T}}}\right) \frac{\partial \widetilde{\mathbf{T}}}{\partial \mathbf{E}}\left[\mathbf{F}^T\mathbf{D}\mathbf{F}\right] \\
&= \left(\mathscr{I} + \pi\widetilde{\boldsymbol{\sigma}} \otimes \widetilde{\mathbf{T}}\right) \tilde{\boldsymbol{\mathscr{M}}}\left[\mathbf{D}\right],
\end{aligned}
\tag{5.96}
$$

$$
\begin{aligned}
\dot{\mathbf{S}}_* &= \left(\mathscr{I} + \pi\boldsymbol{\sigma}_* \otimes \frac{\partial f_*}{\partial \mathbf{S}_*}\right) \frac{\partial \mathbf{S}_*}{\partial \mathbf{E}_*}\left[\dot{\mathbf{E}}_*\right] + \frac{\partial \mathbf{S}_*}{\partial \mathbf{E}_*}\left[\frac{1}{2}\left(\mathbf{C}_*\dot{\mathbf{U}}_p\mathbf{U}_p^{-1} + \mathbf{U}_p^{-1}\dot{\mathbf{U}}_p\mathbf{C}_*\right)\right] \\
&= \left(\mathscr{I} + \pi\boldsymbol{\sigma}_* \otimes \frac{1}{J_*^2}\mathbf{C}_*\mathbf{S}_*\mathbf{C}_*\right)\boldsymbol{\mathscr{L}}_*\left[\dot{\mathbf{E}}_*\right] + \boldsymbol{\mathscr{L}}_*\left[\frac{1}{2}\left(\mathbf{C}_*\dot{\mathbf{U}}_p\mathbf{U}_p^{-1} + \mathbf{U}_p^{-1}\dot{\mathbf{U}}_p\mathbf{C}_*\right)\right] \\
&= \left(\mathscr{I} + \pi\boldsymbol{\sigma}_* \otimes \frac{1}{J_*^2}\mathbf{C}_*\mathbf{S}_*\mathbf{C}_*\right)\boldsymbol{\mathscr{L}}_*\left[\dot{\mathbf{E}}_*\right] + \boldsymbol{\mathscr{L}}_*\left[\bar{\mathbf{D}}_p + \mathbf{E}_*\dot{\mathbf{U}}_p\mathbf{U}_p^{-1} + \mathbf{U}_p^{-1}\dot{\mathbf{U}}_p\mathbf{E}_*\right].
\end{aligned}
\tag{5.97}
$$

5.5 The yield function in $\mathbf{E}_*$ space

In this section, we make the assumption that the Piola-Kirchhoff stress $\mathbf{S}_*$ depends only on one intermediate variable, namely $\mathbf{E}_*$. This helps us obtain the relationship between the original Lagrangian strain space that depends on $\mathbf{E}$ and this new strain space that depends on $\mathbf{E}_*$. Thus, the stress is given by

$$
\mathbf{S}_* = \hat{\mathbf{S}}_*(\mathbf{E}_*), \tag{5.98}
$$

and its rate is

$$
\dot{\mathbf{S}}_* = \boldsymbol{\mathscr{L}}_*[\dot{\mathbf{E}}_*], \tag{5.99}
$$

where

$$
\begin{aligned}
\dot{\mathbf{E}}_* &= \mathbf{F}_*^T\mathbf{D}\mathbf{F}_* - \frac{1}{2}\left(\mathbf{C}_*\dot{\mathbf{U}}_p\mathbf{U}_p^{-1} + \mathbf{U}_p^{-1}\dot{\mathbf{U}}_p\mathbf{C}_*\right) \\
&= \mathbf{F}_*^T\mathbf{D}\mathbf{F}_* - \left(\mathbf{C}_*\bar{\mathbf{L}}_p\right)_{\text{sym}}.
\end{aligned}
\tag{5.100}
$$

The subscript "sym" denotes the symmetric part of the tensor inside the parentheses. The equation for $\dot{\mathbf{E}}_*$ is helpful for our purposes since the first term on the right hand side gives us the rate of deformation $\mathbf{D}$, and the $\bar{\mathbf{L}}_p$ gives $\bar{\mathbf{D}}_p$, which we can get from a flow rule that only involves $\mathbf{D}$. And $\dot{\mathbf{E}}_*$, of course is the rate of the strain measured from the stress-free configuration $\boldsymbol{\kappa}_*$. So in this sense, we do not have to refer back to the reference configuration to get a Prandtl-Reuss type equation for $\dot{\mathbf{S}}_*$ or $\dot{\mathbf{T}}_*$. The ultimate goal is to formulate a Prandtl-Reuss type equation for the Eulerian theory.

Now, the loading index $\hat{g}$ that is used in the Green and Naghdi flow rule for $\dot{\mathbf{E}}_p$ (and ultimately for $\bar{\mathbf{D}}_p$) still requires us to refer back to the reference configuration. We can

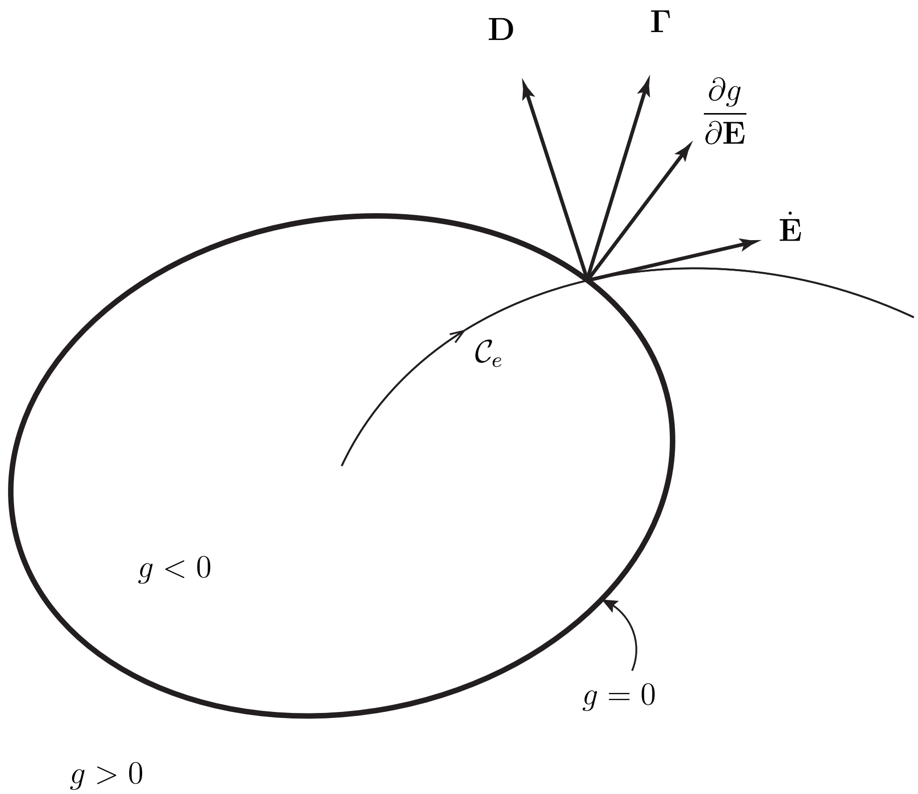

Figure 5.1: The yield surface in $\mathbf{E}$ space. The loading index $\hat{g} = \mathbf{\Gamma} \cdot \mathbf{D}$.

rewrite $\hat{g}$ as

$$\begin{aligned}\hat{g} &= \frac{\partial g}{\partial \mathbf{E}} \cdot \dot{\mathbf{E}} \\ &= \frac{\partial g}{\partial \mathbf{E}} \cdot \mathbf{F}^T \mathbf{D} \mathbf{F} \\ &= \mathbf{F} \frac{\partial g}{\partial \mathbf{E}} \mathbf{F}^T \cdot \mathbf{D} \\ &= \mathbf{\Gamma} \cdot \mathbf{D}.\end{aligned} \tag{5.101}$$

See Figure 5.1. At first glance, we can see that this may give us a loading index that depends on $\mathbf{D}$. However, the reference configuration is still prominent in the tensor

$$\mathbf{\Gamma} = \mathbf{F} \frac{\partial g}{\partial \mathbf{E}} \mathbf{F}^T. \tag{5.102}$$

If we look closely, however, we can interpret $\bar{\mathbf{\Gamma}}$ as a push-forward of $\dfrac{\partial g}{\partial \mathbf{E}}$. Thus, with the help of (4.49), we can transform the tensor $\bar{\mathbf{\Gamma}}$, which depends on the reference configuration, to a new tensor $\mathbf{\Gamma}$, which depends on the stress-free configuration $\boldsymbol{\kappa}_*$:

$$\mathbf{\Gamma} = \bar{\mathbf{\Gamma}}(\mathbf{E}, \mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa) = \mathbf{\Gamma}_*(\mathbf{E}_*, \mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa). \tag{5.103}$$

The tensor $\mathbf{\Gamma}$ is the push-forward of the normal to the yield surface $\dfrac{\partial g}{\partial \mathbf{E}}$. Now we can consider the loading index $\hat{g}$ as

$$\hat{g} = \mathbf{\Gamma}_*(\mathbf{E}_*, \mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa) \cdot \mathbf{D}. \tag{5.104}$$

We note that the invariance requirement for $\mathbf{\Gamma}$ is

$$\mathbf{\Gamma}^+ = \mathbf{F}^+ \frac{\partial g^+}{\partial \mathbf{E}^+} \left(\mathbf{F}^T\right)^+ = \mathbf{Q}\mathbf{F}\frac{\partial g}{\partial \mathbf{E}}\mathbf{F}^T\mathbf{Q}^T = \mathbf{Q}\mathbf{\Gamma}\mathbf{Q}^T. \tag{5.105}$$

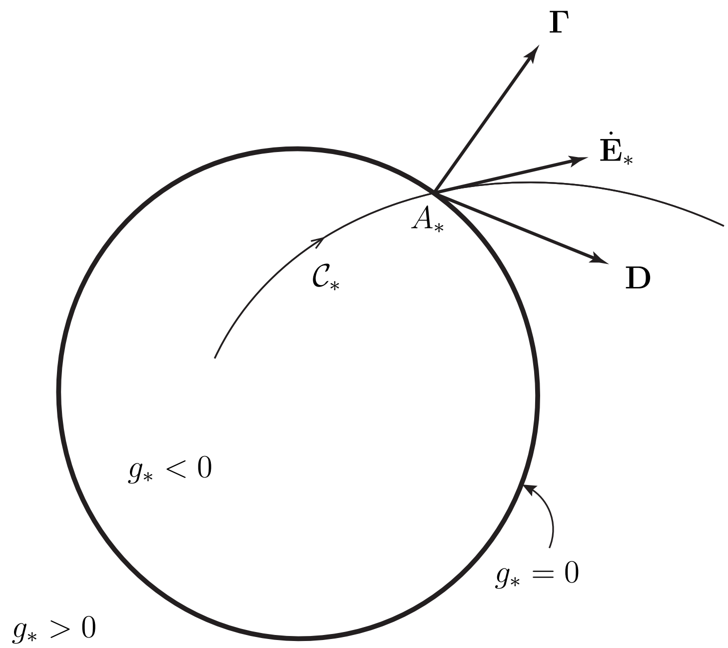

Figure 5.2: The yield surface in $\mathbf{E}_*$ space.

Recall the yield surface in Lagrangian strain space. The normal to this surface is $\dfrac{\partial g}{\partial \mathbf{E}}$ and the tangent to the strain path is $\dot{\mathbf{E}}$. The loading index is the dot product of the two. Now we have

$$\mathbf{\Gamma} = \mathbf{F}\frac{\partial g}{\partial \mathbf{E}}\mathbf{F}^T, \qquad \mathbf{D} = \mathbf{F}^{-T}\dot{\mathbf{E}}\mathbf{F}^{-1}. \tag{5.106}$$

So in strain space, we can transform $\dfrac{\partial g}{\partial \mathbf{E}}$ and $\dot{\mathbf{E}}$ into $\mathbf{\Gamma}$ and $\mathbf{D}$ and still get the same loading index $\hat{g}$. This transformation can involve rotating and stretching, depending on $\mathbf{F}$.

If we then consider the strain space of $\mathbf{E}_*$, we can map certain things from the strain space of $\mathbf{E}$. Certainly, at a fixed value of $\mathbf{E}_p$, we can map the yield surface $\partial\mathscr{E}$ into another surface $\partial\mathscr{E}_*$ in $\mathbf{E}_*$ space. This new surface would thus describe a yield surface in $\mathbf{E}_*$ space, with an elastic region enclosed by its boundary and a plastic region outside of its boundary. The elastic region is given by $g_* < 0$. See Figure 5.2. With the aid of (4.49), this new surface can be described by a function $g_* = 0$:

$$g(\mathbf{E}, \mathbf{E}_p, \boldsymbol{\alpha}, \kappa) = g_*(\mathbf{E}_*, \mathbf{E}_p, \boldsymbol{\alpha}, \kappa) = 0, \tag{5.107}$$

with the usual consistency condition ($\dot{g} = \dot{g}_* = 0$) in place. Note that in general,

$$\boldsymbol{\Gamma} \cdot \mathbf{D} \neq \frac{\partial g_*}{\partial \mathbf{E}_*} \cdot \dot{\mathbf{E}}_*. \tag{5.108}$$

Consider a point A on the strain path $\mathcal{C}$ in $\mathscr{E}$ that coincides with the yield surface. At fixed $\mathbf{E}_p$, that point can be mapped to another point A_* in $\mathscr{E}_*$. As the material continues to yield, we can follow the strain path $\mathcal{C}$, and for each of those points, we can map a series of points in $\mathscr{E}_*$, keeping $\mathbf{E}_p$ fixed. This would give us a path $\mathcal{C}_*$ in $\mathbf{E}_*$ space. It has a tangent vector $\dot{\mathbf{E}}_*$.

The yield surface in $\mathbf{E}$ can be mapped into a yield surface in $\mathbf{E}_*$, but this new surface will not in general give us suitable loading criteria in terms of $\dfrac{\partial g_*}{\partial \mathbf{E}_*}$ and $\dot{\mathbf{E}}_*$. Further, the tensors $\boldsymbol{\Gamma}$ and $\mathbf{D}$ can be mapped into the $\mathbf{E}_*$ space, keeping its inner product $\hat{g}$ intact.

The behavior of the yield surface in $\mathbf{E}_*$ space is analogous to its equivalent surface in stress space. A strain path and a yield surface would only go as far as defining the elastic region, but yield criteria cannot be implemented.

Note that because we are considering the space $\mathscr{E}_*$, it can be useful to express the loading index in terms of $\mathbf{E}_*$ or in terms of $\mathbf{D}_*$:

$$\begin{aligned} 0 < \hat{g} &= \boldsymbol{\Gamma} \cdot \mathbf{D} \\ &= \boldsymbol{\Gamma}_*^O \cdot \dot{\mathbf{E}}_* + \boldsymbol{\Gamma}_*^O \cdot \mathbf{C}_* \dot{\mathbf{U}}_p \mathbf{U}_p^{-1}. \\ &= \boldsymbol{\Gamma}_*^O \cdot \left(\dot{\mathbf{E}}_* + \mathbf{C}_* \dot{\mathbf{U}}_p \mathbf{U}_p^{-1} \right), \end{aligned} \tag{5.109}$$

where

$$\boldsymbol{\Gamma}_*^O = \mathbf{F}_*^{-1} \boldsymbol{\Gamma} \mathbf{F}^{-T} = \mathbf{U}_p \frac{\partial g}{\partial \mathbf{E}} \mathbf{U}_p. \tag{5.110}$$

5.6 The loading index in other spaces

In $\mathbf{E}$ space, we define the rotated rate of deformation tensor

$$\widetilde{\mathbf{D}} = \widetilde{\mathbf{D}}(\mathbf{E}, \mathscr{U}) = \mathbf{U}^{-1} \dot{\mathbf{E}} \mathbf{U}^{-1} = \mathbf{R}^T \mathbf{D} \mathbf{R}. \tag{5.111}$$

We also define a rotated form of $\boldsymbol{\Gamma}$:

$$\widetilde{\boldsymbol{\Gamma}} = \widetilde{\boldsymbol{\Gamma}}(\mathbf{E}, \mathscr{U}) = \mathbf{U}\frac{\partial g}{\partial \mathbf{E}}\mathbf{U} = \mathbf{R}^T\boldsymbol{\Gamma}\mathbf{R}. \tag{5.112}$$

We thus have another representation for the loading index:

$$\hat{g} = \widetilde{\boldsymbol{\Gamma}} \cdot \widetilde{\mathbf{D}}. \tag{5.113}$$

For a particular yield point, we can then construct the two tensors $\widetilde{\boldsymbol{\Gamma}}$ and $\widetilde{\mathbf{D}}$ without reference to a deformation gradient, while keeping the value of the inner product $\hat{g}$. Using appropriate stress-strain relations, $\widetilde{\boldsymbol{\Gamma}}$ and $\widetilde{\mathbf{D}}$ can then be mapped into various stress and strain spaces while still retaining the value of $\hat{g}$.

Using the inverse relation of the rotated Cauchy stress $\widetilde{\mathbf{T}}(\mathbf{E}, \mathscr{U})$, we can write

$$\widetilde{\mathbf{D}} = \widetilde{\mathbf{D}}(\mathbf{E}(\widetilde{\mathbf{T}}, \mathscr{U}), \mathscr{U}) = \widetilde{\mathbf{D}}(\widetilde{\mathbf{T}}, \mathscr{U}), \tag{5.114}$$

and

$$\widetilde{\boldsymbol{\Gamma}} = \widetilde{\boldsymbol{\Gamma}}(\mathbf{E}(\widetilde{\mathbf{T}}, \mathscr{U}), \mathscr{U}) = \widetilde{\boldsymbol{\Gamma}}(\widetilde{\mathbf{T}}, \mathscr{U}), \tag{5.115}$$

which gives us another representation of the loading index, this time in $\widetilde{\mathbf{T}}$ space:

$$\hat{g} = \widetilde{\boldsymbol{\Gamma}} \cdot \widetilde{\mathbf{D}}. \tag{5.116}$$

Further, using the expression (4.49), we can write $\mathbf{E}$ as a function of $\mathbf{E}_*$. Thus,

$$\begin{aligned} \widetilde{\mathbf{D}} &= \widetilde{\mathbf{D}}(\mathbf{E}(\mathbf{E}_*, \mathscr{U}), \mathscr{U}) = \widetilde{\mathbf{D}}_*(\mathbf{E}_*, \mathscr{U}), \\ \widetilde{\boldsymbol{\Gamma}} &= \widetilde{\boldsymbol{\Gamma}}(\mathbf{E}(\mathbf{E}_*, \mathscr{U}), \mathscr{U}) = \widetilde{\boldsymbol{\Gamma}}_*(\mathbf{E}_*, \mathscr{U}). \end{aligned} \tag{5.117}$$

In this chapter, we have introduced the yield functions and stress measures associated with the intermediate configuration $\boldsymbol{\kappa}_*$. We have also mentioned the various forms of the von Mises yield condition and the loading index. These elements will be used in the formulation of stress rates in the next chapter.

Chapter 6

Stress Measures, Their Rates, and Constitutive Equations

With the $\boldsymbol{\kappa}_*$ configuration established, the choice of an appropriate stress measure to describe the elastic-plastic material still remains. A selection of stress tensors and their dependent parameters are further explored in this chapter. For each of these stress measures, an objective stress rate is formulated, and each is expressed in terms of the strain rate or the rate of deformation tensor $\mathbf{D}$. We also develop a class of objective rates for the rotated Cauchy stress tensor.

6.1 The symmetric Piola-Kirchhoff stress tensor S

Recall that the material derivative of $\mathbf{S}$, which is given in (3.1), is

$$\dot{\mathbf{S}} = \boldsymbol{\mathscr{L}}[\dot{\mathbf{E}}] + \frac{\partial \hat{\mathbf{S}}}{\partial \mathbf{E}_p}[\dot{\mathbf{E}}_p] + \frac{\partial \hat{\mathbf{S}}}{\partial \boldsymbol{\alpha}_R}[\dot{\boldsymbol{\alpha}}_R] + \frac{\partial \hat{\mathbf{S}}}{\partial \kappa}\dot{\kappa}, \tag{6.1}$$

where

$$\boldsymbol{\mathscr{L}} = \frac{\partial \hat{\mathbf{S}}}{\partial \mathbf{E}}(\mathbf{E}, \mathbf{E}_p, \boldsymbol{\alpha}, \boldsymbol{\kappa}). \tag{6.2}$$

Since $\dot{\mathbf{E}} = \mathbf{F}^T\mathbf{D}\mathbf{F}$, we can rewrite the first term of (6.1) in index notation as

$$\mathscr{L}_{KLAB}\left(F_{iA}D_{ij}F_{jB}\right) = F_{iA}\mathscr{L}_{KLAB}F_{jB}\left(D_{ij}\right). \tag{6.3}$$

Using (2.17), we can rewrite (6.3) as

$$\begin{aligned}\left(\mathscr{L}_{KLAB}\mathbf{E}_K \otimes \mathbf{E}_L \otimes \mathbf{E}_A \otimes \mathbf{E}_B\right) \odot \left(F_{iM}\mathbf{e}_i \otimes \mathbf{E}_M \otimes F_{jN}\mathbf{e}_j \otimes \mathbf{E}_N\right)[\mathbf{D}] \\ = \mathscr{L}_{KLAB}F_{iM}F_{jN}\delta_{AM}\delta_{BN}\left(\mathbf{E}_K \otimes \mathbf{E}_L \otimes \mathbf{e}_i \otimes \mathbf{e}_j\right)[\mathbf{D}] \\ = \mathscr{L}_{KLAB}F_{iA}F_{jB}\left(\mathbf{E}_K \otimes \mathbf{E}_L \otimes \mathbf{e}_i \otimes \mathbf{e}_j\right)[\mathbf{D}] \\ = \mathscr{F}_{KLij}\left(\mathbf{E}_K \otimes \mathbf{E}_L \otimes \mathbf{e}_i \otimes \mathbf{e}_j\right)[\mathbf{D}] \\ = \boldsymbol{\mathscr{F}}[\mathbf{D}].\end{aligned} \tag{6.4}$$

In direct notation, the fourth-order tensor $\boldsymbol{\mathscr{F}}$ is

$$\boldsymbol{\mathscr{F}} = \boldsymbol{\mathscr{L}} \odot (\mathbf{F} \otimes \mathbf{F}) = \boldsymbol{\mathscr{F}}(\mathbf{F}, \mathbf{E}, \mathbf{E}_p, \boldsymbol{\alpha}, \boldsymbol{\kappa}) \tag{6.5}$$

so that the first term of (6.1) can be rewritten as

$$\boldsymbol{\mathscr{L}}[\mathbf{F}^T\mathbf{D}\mathbf{F}] = \boldsymbol{\mathscr{L}} \odot (\mathbf{F} \otimes \mathbf{F})[\mathbf{D}] = \boldsymbol{\mathscr{F}}[\mathbf{D}]. \tag{6.6}$$

Finally, using (3.21), we write

$$\begin{aligned}\dot{\mathbf{S}} &= \left(\boldsymbol{\mathscr{I}} + \pi\boldsymbol{\sigma} \otimes \frac{\partial f}{\partial \mathbf{S}}\right)\boldsymbol{\mathscr{L}} \odot (\mathbf{F} \otimes \mathbf{F})[\mathbf{D}] \\ &= \boldsymbol{\mathscr{K}}\boldsymbol{\mathscr{L}} \odot (\mathbf{F} \otimes \mathbf{F})[\mathbf{D}].\end{aligned} \tag{6.7}$$

Alternatively, we can simply write

$$\dot{\mathbf{S}} = \left(\frac{\partial \hat{\mathbf{S}}}{\partial \mathbf{E}} + \pi\boldsymbol{\sigma} \otimes \frac{\partial g}{\partial \mathbf{E}}\right) \odot (\mathbf{F} \otimes \mathbf{F})[\mathbf{D}]. \tag{6.8}$$

6.2 The rotated Cauchy stress tensor $\widetilde{\mathbf{T}}$

Recall that the material derivative of $\widetilde{\mathbf{T}}$, which is given in (5.48), is

$$\dot{\widetilde{\mathbf{T}}} = \frac{\partial \widetilde{\mathbf{T}}}{\partial \mathbf{E}}[\dot{\mathbf{E}}] + \frac{\partial \widetilde{\mathbf{T}}}{\partial \mathbf{E}_p}[\dot{\mathbf{E}}_p] + \frac{\partial \widetilde{\mathbf{T}}}{\partial \boldsymbol{\alpha}_R}[\dot{\boldsymbol{\alpha}}_R] + \frac{\partial \widetilde{\mathbf{T}}}{\partial \kappa}\dot{\kappa}. \tag{6.9}$$

Using

$$\dot{\mathbf{E}} = \mathbf{F}^T\mathbf{D}\mathbf{F}, \qquad \dot{E}_{AB} = F_{iA}D_{ij}F_{jB}, \tag{6.10}$$

and the flow rules (during loading)

$$\dot{\mathbf{E}}_p = \pi\hat{g}\boldsymbol{\rho}, \qquad \dot{\boldsymbol{\alpha}}_R = \pi\hat{g}\boldsymbol{\beta}, \qquad \dot{\kappa} = \pi\hat{g}\lambda, \tag{6.11}$$

we have

$$\dot{\widetilde{\mathbf{T}}} = \frac{\partial \widetilde{\mathbf{T}}}{\partial \mathbf{E}}[\mathbf{F}^T \mathbf{D}\mathbf{F}] + \frac{\partial \widetilde{\mathbf{T}}}{\partial \mathbf{E}_p}\big[\pi \hat{g} \boldsymbol{\rho}\big] + \frac{\partial \widetilde{\mathbf{T}}}{\partial \boldsymbol{\alpha}_R}\big[\pi \hat{g} \boldsymbol{\beta}\big] + \frac{\partial \widetilde{\mathbf{T}}}{\partial \kappa}\, \pi \hat{g} \lambda. \tag{6.12}$$

We observe that the first term of (6.12) can be written in index form as

$$\frac{\partial \widetilde{T}_{KL}}{\partial E_{AB}} \left(F_{iA} D_{ij} F_{jB}\right) = F_{iA} \frac{\partial \widetilde{T}_{KL}}{\partial E_{AB}} F_{jB} \left(D_{ij}\right), \tag{6.13}$$

which, with the use of (2.17), can be rewritten as

$$\begin{aligned}
&\left(\frac{\partial \widetilde{T}_{KL}}{\partial E_{AB}} \mathbf{E}_K \otimes \mathbf{E}_L \otimes \mathbf{E}_A \otimes \mathbf{E}_B\right) \odot \left(F_{iM} \mathbf{e}_i \otimes \mathbf{E}_M \otimes F_{jN} \mathbf{e}_j \otimes \mathbf{E}_N\right)[\mathbf{D}] \\
&\qquad = \frac{\partial \widetilde{T}_{KL}}{\partial E_{AB}} F_{iM} F_{jN} \delta_{AM} \delta_{BN} \left(\mathbf{E}_K \otimes \mathbf{E}_L \otimes \mathbf{e}_i \otimes \mathbf{e}_j\right)[\mathbf{D}] \\
&\qquad = \frac{\partial \widetilde{T}_{KL}}{\partial E_{AB}} F_{iA} F_{jB} \left(\mathbf{E}_K \otimes \mathbf{E}_L \otimes \mathbf{e}_i \otimes \mathbf{e}_j\right)[\mathbf{D}] \\
&\qquad = \tilde{\mathscr{F}}_{KLij} \left(\mathbf{E}_K \otimes \mathbf{E}_L \otimes \mathbf{e}_i \otimes \mathbf{e}_j\right)[\mathbf{D}] \\
&\qquad = \tilde{\boldsymbol{\mathscr{F}}}[\mathbf{D}].
\end{aligned} \tag{6.14}$$

In direct notation, the fourth-order tensor $\tilde{\boldsymbol{\mathscr{F}}}$ is

$$\tilde{\boldsymbol{\mathscr{F}}} = \frac{\partial \widetilde{\mathbf{T}}}{\partial \mathbf{E}} \odot \left(\mathbf{F} \otimes \mathbf{F}\right) \tag{6.15}$$

so that the first term of (6.12) can be written as

$$\frac{\partial \widetilde{\mathbf{T}}}{\partial \mathbf{E}}[\mathbf{F}^T \mathbf{D}\mathbf{F}] = \frac{\partial \widetilde{\mathbf{T}}}{\partial \mathbf{E}} \odot \left(\mathbf{F} \otimes \mathbf{F}\right)[\mathbf{D}] = \tilde{\boldsymbol{\mathscr{F}}}[\mathbf{D}]. \tag{6.16}$$

The last three terms of (6.12) can be expressed in the compact form

$$\begin{aligned}
\pi \hat{g} \left\{\frac{\partial \widetilde{\mathbf{T}}}{\partial \mathbf{E}_p}\big[\boldsymbol{\rho}\big] + \frac{\partial \widetilde{\mathbf{T}}}{\partial \boldsymbol{\alpha}_R}\big[\boldsymbol{\beta}\big] + \frac{\partial \widetilde{\mathbf{T}}}{\partial \kappa}\, \lambda\right\} &= \pi \left(\boldsymbol{\Gamma} \cdot \mathbf{D}\right) \tilde{\boldsymbol{\sigma}} \\
&= \pi \left(\tilde{\boldsymbol{\sigma}} \otimes \boldsymbol{\Gamma}\right)[\mathbf{D}],
\end{aligned} \tag{6.17}$$

where we have used (5.104), (5.102), and the definition

$$\tilde{\boldsymbol{\sigma}} = \frac{\partial \widetilde{\mathbf{T}}}{\partial \mathbf{E}_p}\big[\boldsymbol{\rho}\big] + \frac{\partial \widetilde{\mathbf{T}}}{\partial \boldsymbol{\alpha}_R}\big[\boldsymbol{\beta}\big] + \frac{\partial \widetilde{\mathbf{T}}}{\partial \kappa}\, \lambda. \tag{6.18}$$

Recall that

$$
\begin{aligned}
\mathbf{E}_* = \hat{\mathbf{E}}_*(\mathbf{E}, \mathbf{E}_p) = \mathbf{U}_p^{-1}(\mathbf{E} - \mathbf{E}_p)\mathbf{U}_p^{-1}, \\
E_{*MN} = \hat{E}_{*MN}(E_{KL}, E_{pKL}) = U_{pKM}^{-1}(E_{KL} - E_{pKL})U_{pLN}^{-1}.
\end{aligned}
\tag{6.19}
$$

Keeping the parameters $\mathbf{E}$ and $\mathbf{E}_p$ independent of each other, we can differentiate $\mathbf{E}_*$ with respect to $\mathbf{E}$. In index form,

$$
\begin{aligned}
\frac{\partial \hat{E}_{*MN}}{\partial E_{AB}} &= \frac{1}{2}\left(\frac{\partial \hat{E}_{*MN}}{\partial E_{AB}} + \frac{\partial \hat{E}_{*MN}}{\partial E_{BA}}\right) \\
&= \frac{1}{2}\left[\left(U_{pKM}^{-1}\delta_{AK}\delta_{BL}U_{pLN}^{-1} + U_{pKM}^{-1}\delta_{BK}\delta_{AL}U_{pLN}^{-1}\right)\right] \\
&= \frac{1}{2}\left(U_{pAM}^{-1}U_{pBN}^{-1} + U_{pBM}^{-1}U_{pAN}^{-1}\right).
\end{aligned}
\tag{6.20}
$$

Also, using (6.19), we can recast the rotated stress as

$$
\widetilde{\mathbf{T}}(\mathbf{E}, \mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa) = \bar{\mathbf{T}}(\mathbf{E}_*, \mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa) = \bar{\mathbf{T}}(\mathbf{E}_*, \mathscr{U}). \tag{6.21}
$$

Applying $(6.20)_3$ to $\dfrac{\partial \widetilde{\mathbf{T}}}{\partial \mathbf{E}}$, we get

$$
\begin{aligned}
\frac{\partial \widetilde{T}_{KL}}{\partial E_{AB}} &= \frac{\partial \bar{T}_{KL}}{\partial E_{*MN}}\frac{\partial \hat{E}_{*MN}}{\partial E_{AB}} \\
&= \frac{1}{2}\left(U_{pAM}^{-1}\frac{\partial \bar{T}_{KL}}{\partial E_{*MN}}U_{pBN}^{-1} + U_{pBM}^{-1}\frac{\partial \bar{T}_{KL}}{\partial E_{*NM}}U_{pAN}^{-1}\right) \\
&= U_{pAM}^{-1}\frac{\partial \bar{T}_{KL}}{\partial E_{*MN}}U_{pBN}^{-1},
\end{aligned}
\tag{6.22}
$$

where the symmetry of $\mathbf{E}$ has been utilized. From (6.13) and using (6.22) and

$$
\mathbf{F}_* = \mathbf{F}\mathbf{U}_p^{-1}, \qquad F_{*iM} = F_{iA}U_{pAM}^{-1}, \tag{6.23}
$$

we get

$$
\begin{aligned}
F_{iA}\frac{\partial \widetilde{T}_{KL}}{\partial E_{AB}}F_{jB}(D_{ij}) &= F_{iA}U_{pAM}^{-1}\frac{\partial \bar{T}_{KL}}{\partial E_{*MN}}U_{pBN}^{-1}F_{jB}\,(D_{ij}) \\
&= F_{*iM}\frac{\partial \bar{T}_{KL}}{\partial E_{*MN}}F_{*jN}(D_{ij}),
\end{aligned}
\tag{6.24}
$$

or in direct notation,

$$
\frac{\partial \widetilde{\mathbf{T}}}{\partial \mathbf{E}}[\mathbf{F}^T\mathbf{D}\mathbf{F}] = \frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*} \odot (\mathbf{F}_* \otimes \mathbf{F}_*)[\mathbf{D}] = \tilde{\boldsymbol{\mathscr{F}}}[\mathbf{D}]. \tag{6.25}
$$

Recall that the yield function g can be expressed in either of the two strain spaces as

$$g = g(\mathbf{E}, \mathscr{U}) = g_*(\mathbf{E}_*, \mathscr{U}). \tag{6.26}$$

Thus, by the chain rule,

$$\frac{\partial g}{\partial \mathbf{E}} = \frac{\partial g_*}{\partial \mathbf{E}_*}\frac{\partial \mathbf{E}_*}{\partial \mathbf{E}}, \qquad \frac{\partial g}{\partial E_{AB}} = \frac{\partial g_*}{\partial E_{*MN}}\frac{\partial \hat{E}_{*MN}}{\partial E_{AB}}. \tag{6.27}$$

Applying $(6.20)_3$ we find that

$$\frac{\partial g}{\partial E_{AB}} = U^{-1}_{pAM}\frac{\partial g_*}{\partial E_{*MN}}U^{-1}_{pBN}, \tag{6.28}$$

or, in direct notation,

$$\frac{\partial g}{\partial \mathbf{E}} = \mathbf{U}_p^{-1}\frac{\partial g_*}{\partial \mathbf{E}_*}\mathbf{U}_p^{-1}. \tag{6.29}$$

Substituting (6.29) to the tensor $\mathbf{\Gamma}$ in (5.102),

$$\begin{aligned} \mathbf{\Gamma} &= \mathbf{F}\mathbf{U}_p^{-1}\frac{\partial g_*}{\partial \mathbf{E}_*}\mathbf{U}_p^{-1}\mathbf{F}^T = \mathbf{F}_*\frac{\partial g_*}{\partial \mathbf{E}_*}\mathbf{F}_*^T, \\ \Gamma_{ij} &= F_{iA}U^{-1}_{pAM}\frac{\partial g_*}{\partial E_{*MN}}U^{-1}_{pBN}F_{jB} = F_{*iM}\frac{\partial g_*}{\partial E_{*MN}}F_{*jN}. \end{aligned} \tag{6.30}$$

Thus, after combining all terms, the rate of rotated stress is

$$\begin{aligned} \dot{\tilde{\mathbf{T}}} &= \frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*} \odot (\mathbf{F}_* \otimes \mathbf{F}_*)[\mathbf{D}] + \pi\,(\tilde{\boldsymbol{\sigma}} \otimes \mathbf{\Gamma})\,[\mathbf{D}] \\ &= \left\{\frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*} \odot (\mathbf{F}_* \otimes \mathbf{F}_*) + \pi\tilde{\boldsymbol{\sigma}} \otimes \mathbf{\Gamma}\right\}[\mathbf{D}] \\ &= \left\{\frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*} \odot (\mathbf{F}_* \otimes \mathbf{F}_*) + \pi\tilde{\boldsymbol{\sigma}} \otimes \mathbf{F}_*\frac{\partial g_*}{\partial \mathbf{E}_*}\mathbf{F}_*^T\right\}[\mathbf{D}] \\ &= \left\{\frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*} \odot (\mathbf{F}_* \otimes \mathbf{F}_*) + \pi\left(\tilde{\boldsymbol{\sigma}} \otimes \frac{\partial g_*}{\partial \mathbf{E}_*}\right) \odot (\mathbf{F}_* \otimes \mathbf{F}_*)\right\}[\mathbf{D}] \\ &= \left(\frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*} + \pi\tilde{\boldsymbol{\sigma}} \otimes \frac{\partial g_*}{\partial \mathbf{E}_*}\right) \odot (\mathbf{F}_* \otimes \mathbf{F}_*)\,[\mathbf{D}]. \end{aligned} \tag{6.31}$$

Rewriting (6.31) in index form,

$$
\begin{aligned}
\dot{\tilde{\bar{T}}}_{KL} &= F_{*iM}\frac{\partial \bar{T}_{KL}}{\partial E_{*MN}}F_{*jN}D_{ij} + \pi\tilde{\sigma}_{KL}\Gamma_{ij}D_{ij} \\
&= F_{*iM}\frac{\partial \bar{T}_{KL}}{\partial E_{*MN}}F_{*jN}D_{ij} + \pi\tilde{\sigma}_{KL}F_{*iM}\frac{\partial g_*}{\partial E_{*MN}}F_{*jN}D_{ij} \\
&= \left(F_{*iM}\frac{\partial \bar{T}_{KL}}{\partial E_{*MN}}F_{*jN} + \pi\tilde{\sigma}_{KL}F_{*iM}\frac{\partial g_*}{\partial E_{*MN}}F_{*jN}\right)D_{ij} \\
&= \left(\frac{\partial \bar{T}_{KL}}{\partial E_{*MN}} + \pi\tilde{\sigma}_{KL}\frac{\partial g_*}{\partial E_{*MN}}\right)F_{*iM}F_{*jN}D_{ij}.
\end{aligned}
\tag{6.32}
$$

We now verify the invariance of this rotated stress rate under superposed rigid body motions. First, in index form,

$$
\begin{aligned}
\dot{\tilde{\bar{T}}}{}^{+}_{KL} &= \left(\frac{\partial \bar{T}^{+}_{KL}}{\partial E^{+}_{*MN}} + \pi^{+}\tilde{\sigma}^{+}_{KL}\frac{\partial g^{+}_*}{\partial E^{+}_{*MN}}\right)F^{+}_{*iM}F^{+}_{*jN}D^{+}_{ij} \\
&= \left(\frac{\partial \bar{T}_{KL}}{\partial E_{*MN}} + \pi\tilde{\sigma}_{KL}\frac{\partial g_*}{\partial E_{*MN}}\right)Q_{ik}F_{*kM}Q_{j\ell}F_{*\ell N}Q_{ir}D_{rs}Q_{js} \\
&= \left(\frac{\partial \bar{T}_{KL}}{\partial E_{*MN}} + \pi\tilde{\sigma}_{KL}\frac{\partial g_*}{\partial E_{*MN}}\right)Q_{ir}Q_{ik}F_{*kM}Q_{js}Q_{j\ell}F_{*\ell N}D_{rs} \\
&= \left(\frac{\partial \bar{T}_{KL}}{\partial E_{*MN}} + \pi\tilde{\sigma}_{KL}\frac{\partial g_*}{\partial E_{*MN}}\right)\delta_{kr}F_{*kM}\delta_{\ell s}F_{*\ell N}D_{rs} \\
&= \left(\frac{\partial \bar{T}_{KL}}{\partial E_{*MN}} + \pi\tilde{\sigma}_{KL}\frac{\partial g_*}{\partial E_{*MN}}\right)F_{*rM}F_{*sN}D_{rs} \\
&= \left(\frac{\partial \bar{T}_{KL}}{\partial E_{*MN}} + \pi\tilde{\sigma}_{KL}\frac{\partial g_*}{\partial E_{*MN}}\right)F_{*iM}F_{*jN}D_{ij} \\
&= \dot{\tilde{\bar{T}}}_{KL}.
\end{aligned}
\tag{6.33}
$$

Thus, in direct notation,

$$
\begin{aligned}
\dot{\tilde{\bar{\mathbf{T}}}}{}^{+} &= \left(\frac{\partial \bar{\mathbf{T}}^{+}}{\partial \mathbf{E}^{+}_*} + \pi^{+}\tilde{\boldsymbol{\sigma}}^{+}\otimes\frac{\partial g^{+}_*}{\partial \mathbf{E}^{+}_*}\right)\odot\left(\mathbf{F}^{+}_*\otimes\mathbf{F}^{+}_*\right)[\mathbf{D}^{+}] \\
&= \left(\frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*} + \pi\tilde{\boldsymbol{\sigma}}\otimes\frac{\partial g_*}{\partial \mathbf{E}_*}\right)\odot\left(\mathbf{Q}\mathbf{F}_*\otimes\mathbf{Q}\mathbf{F}_*\right)[\mathbf{Q}\mathbf{D}\mathbf{Q}^{T}] \\
&= \left(\frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*} + \pi\tilde{\boldsymbol{\sigma}}\otimes\frac{\partial g_*}{\partial \mathbf{E}_*}\right)\odot\left(\mathbf{Q}^{T}\mathbf{Q}\mathbf{F}_*\otimes\mathbf{Q}^{T}\mathbf{Q}\mathbf{F}_*\right)[\mathbf{D}] \\
&= \left(\frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*} + \pi\tilde{\boldsymbol{\sigma}}\otimes\frac{\partial g_*}{\partial \mathbf{E}_*}\right)\odot\left(\mathbf{F}_*\otimes\mathbf{F}^{T}_*\right)[\mathbf{D}] \\
&= \dot{\tilde{\bar{\mathbf{T}}}}.
\end{aligned}
\tag{6.34}
$$

In addition, we have the representation of the box rate of Cauchy stress, which was also used by Green and McInnis [17]:

$$
\begin{aligned}
\overset{\square}{\mathbf{T}} &= \mathbf{R}\dot{\tilde{\bar{\mathbf{T}}}}\mathbf{R}^T \\
&= \mathbf{R}\left\{ \left(\frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*} + \pi \tilde{\boldsymbol{\sigma}} \otimes \frac{\partial g_*}{\partial \mathbf{E}_*} \right) \odot \left(\mathbf{F}_* \otimes \mathbf{F}_*^T \right) [\mathbf{D}] \right\} \mathbf{R}^T.
\end{aligned}
\tag{6.35}
$$

This satisfies the invariance requirement

$$
\overset{\square}{\mathbf{T}}{}^{+} = \mathbf{Q}\overset{\square}{\mathbf{T}}\mathbf{Q}^T. \tag{6.36}
$$

It is therefore an objective rate. In index form,

$$
\begin{aligned}
\overset{\square}{T}_{mn} &= R_{mK}\dot{\tilde{\bar{T}}}_{KL}R_{nL} \\
&= R_{mK}\left\{ \left(\frac{\partial \bar{T}_{KL}}{\partial E_{*MN}} + \pi \tilde{\sigma}_{KL} \frac{\partial g_*}{\partial E_{*MN}} \right) F_{*iM} F_{*jN} D_{ij} \right\} R_{nL}.
\end{aligned}
\tag{6.37}
$$

6.3 An alternate rotated Cauchy stress $\mathbf{T}_*$ as a function of $\mathbf{E}_*$

Recall that we can write the Cauchy stress in the form

$$
\mathbf{T} = \frac{1}{J}\mathbf{F}\hat{\mathbf{S}}(\mathbf{E}, \mathscr{U})\mathbf{F}^T = \hat{\mathbf{T}}(\mathbf{F}, \mathbf{F}_p, \boldsymbol{\alpha}, \kappa). \tag{6.38}
$$

The function $\hat{\mathbf{T}}$ must satisfy the invariance requirement (2.46). Choosing $\mathbf{Q}^T = \mathbf{R}_*$, we have

$$
\begin{aligned}
\hat{\mathbf{T}}(\mathbf{F}, \mathbf{F}_p, \boldsymbol{\alpha}, \kappa) &= \check{\mathbf{T}}(\mathbf{F}_*, \mathbf{U}_p, \boldsymbol{\alpha}, \kappa) \\
&= \mathbf{Q}^T\check{\mathbf{T}}(\mathbf{Q}\mathbf{F}_*, \mathbf{U}_p, \mathbf{Q}\boldsymbol{\alpha}\mathbf{Q}^T, \kappa)\mathbf{Q} \\
&= \mathbf{R}_*\check{\mathbf{T}}(\mathbf{R}_*^T\mathbf{F}_*, \mathbf{U}_p, \mathbf{R}_*^T\boldsymbol{\alpha}\mathbf{R}_*, \kappa)\mathbf{R}_*^T \\
&= \mathbf{R}_*\check{\mathbf{T}}(\mathbf{U}_*, \mathbf{U}_p, \boldsymbol{\alpha}_*, \kappa)\mathbf{R}_*^T \\
&= \mathbf{R}_*\bar{\mathbf{T}}(\mathbf{E}_*, \mathbf{E}_p, \boldsymbol{\alpha}_*, \kappa)\mathbf{R}_*^T,
\end{aligned}
\tag{6.39}
$$

where we have used $(4.11)_1$, $(4.21)_9$, $(5.47)_2$, and defined

$$
\boldsymbol{\alpha}_* = \mathbf{R}_*^T\boldsymbol{\alpha}\mathbf{R}_*. \tag{6.40}
$$

We can then introduce an alternate rotated stress field

$$
\mathbf{T}_* = \mathbf{R}_*^T\mathbf{T}\mathbf{R}_* = \bar{\mathbf{T}}_*(\mathbf{E}_*, \mathbf{E}_p, \boldsymbol{\alpha}_*, \kappa). \tag{6.41}
$$

which is related to $\mathbf{S}_*$ by

$$J_*\mathbf{T}_* = \mathbf{R}_*^T\mathbf{F}_*\mathbf{S}_*\mathbf{F}_*^T\mathbf{R}_* = \mathbf{U}_*\mathbf{U}_p\mathbf{S}_*\mathbf{U}_p\mathbf{U}_*. \tag{6.42}$$

We can treat $\mathbf{T}_*$ as the recast version of the stress field $\widetilde{\mathbf{T}}$. It is invariant under superposed rigid body motions, because

$$\begin{aligned} \mathbf{T}_*^+ &= \left(\mathbf{R}_*^T\right)^+ \mathbf{T}^+\mathbf{R}_*^+ \\ &= \mathbf{R}_*^T\mathbf{Q}^T\mathbf{Q}\mathbf{T}\mathbf{Q}^T\mathbf{Q}\mathbf{R}_* \\ &= \mathbf{R}_*^T\mathbf{T}\mathbf{R}_* \\ &= \mathbf{T}_*, \end{aligned} \tag{6.43}$$

and thus its rate is also invariant. This material derivative is

$$\dot{\mathbf{T}}_* = \frac{\partial\bar{\mathbf{T}}_*}{\partial\mathbf{E}_*}[\dot{\mathbf{E}}_*] + \frac{\partial\bar{\mathbf{T}}_*}{\partial\mathbf{E}_p}\left[\dot{\mathbf{E}}_p\right] + \frac{\partial\bar{\mathbf{T}}_*}{\partial\boldsymbol{\alpha}_*}\left[\dot{\boldsymbol{\alpha}}_*\right] + \frac{\partial\bar{\mathbf{T}}}{\partial\kappa}\,\dot{\kappa}. \tag{6.44}$$

We first examine the strain rate $\dot{\mathbf{E}}_*$. From (4.48), (4.52), and (4.54),

$$\dot{\mathbf{E}}_* = \mathbf{F}_*^T\mathbf{D}\mathbf{F}_* - \frac{1}{2}\left(\mathbf{C}_*\dot{\mathbf{U}}_p\mathbf{U}_p^{-1} + \mathbf{U}_p^{-1}\dot{\mathbf{U}}_p\mathbf{C}_*\right). \tag{6.45}$$

Note that we can write $\mathbf{U}_p$ as a function of $\mathbf{E}_p$:

$$\mathbf{U}_p = \hat{\mathbf{U}}_p(\mathbf{E}_p), \qquad U_{pRS} = \hat{U}_{pRS}(E_{pKL}), \tag{6.46}$$

such that

$$\dot{\mathbf{U}}_p = \frac{\partial\hat{\mathbf{U}}_p}{\partial\mathbf{E}_p}[\dot{\mathbf{E}}_p], \qquad \dot{U}_{pRS} = \frac{\partial\hat{U}_{pRS}}{\partial E_{pKL}}\dot{E}_{pKL}. \tag{6.47}$$

Then, in index form, the strain rate $\dot{\mathbf{E}}_*$ is

$$\begin{aligned} \dot{E}_{*MN} &= F_{*iM}D_{ij}F_{*jN} - \frac{1}{2}\left(C_{*MR}\frac{\partial\hat{U}_{pRS}}{\partial E_{pKL}}\dot{E}_{pKL}U_{pSN}^{-1} + U_{pMR}^{-1}\frac{\partial\hat{U}_{pRS}}{\partial E_{pKL}}\dot{E}_{pKL}C_{*SN}\right) \\ &= F_{*iM}D_{ij}F_{*jN} - \frac{1}{2}\left(C_{*MR}\frac{\partial\hat{U}_{pRS}}{\partial E_{pKL}}U_{pSN}^{-1} + U_{pMR}^{-1}\frac{\partial\hat{U}_{pRS}}{\partial E_{pKL}}C_{*SN}\right)\dot{E}_{pKL} \\ &= F_{*iM}D_{ij}F_{*jN} - \mathcal{Z}_{MNKL}\dot{E}_{pKL}, \end{aligned} \tag{6.48}$$

where we have defined the components

$$\mathcal{Z}_{MNKL} = \frac{1}{2}\left(C_{*MR}\frac{\partial\hat{U}_{pRS}}{\partial E_{pKL}}U_{pSN}^{-1} + U_{pMR}^{-1}\frac{\partial\hat{U}_{pRS}}{\partial E_{pKL}}C_{*SN}\right). \tag{6.49}$$

In direct notation,

$$\dot{\mathbf{E}}_* = \mathbf{F}_*^T\mathbf{D}\mathbf{F}_* - \boldsymbol{\mathcal{Z}}[\dot{\mathbf{E}}_p]. \tag{6.50}$$

Substituting into the stress rate (6.44) and rearranging,

$$
\begin{aligned}
\dot{\mathbf{T}}_* &= \frac{\partial \bar{\mathbf{T}}_*}{\partial \mathbf{E}_*}\left[\mathbf{F}_*^T \mathbf{D} \mathbf{F}_* - \boldsymbol{\mathcal{Z}}[\dot{\mathbf{E}}_p]\right] + \frac{\partial \bar{\mathbf{T}}_*}{\partial \mathbf{E}_p}[\dot{\mathbf{E}}_p] + \frac{\partial \bar{\mathbf{T}}_*}{\partial \boldsymbol{\alpha}}_R[\dot{\boldsymbol{\alpha}}_R] + \frac{\partial \bar{\mathbf{T}}_*}{\partial \kappa}\dot{\kappa} \\
&= \frac{\partial \bar{\mathbf{T}}_*}{\partial \mathbf{E}_*}\left[\mathbf{F}_*^T \mathbf{D} \mathbf{F}_*\right] - \frac{\partial \bar{\mathbf{T}}_*}{\partial \mathbf{E}_*}\boldsymbol{\mathcal{Z}}[\dot{\mathbf{E}}_p] + \frac{\partial \bar{\mathbf{T}}_*}{\partial \mathbf{E}_p}[\dot{\mathbf{E}}_p] + \frac{\partial \bar{\mathbf{T}}_*}{\partial \boldsymbol{\alpha}}_R[\dot{\boldsymbol{\alpha}}_R] + \frac{\partial \bar{\mathbf{T}}_*}{\partial \kappa}\dot{\kappa} \qquad (6.51) \\
&= \frac{\partial \bar{\mathbf{T}}_*}{\partial \mathbf{E}_*}\left[\mathbf{F}_*^T \mathbf{D} \mathbf{F}_*\right] + \left(\frac{\partial \bar{\mathbf{T}}_*}{\partial \mathbf{E}_p} - \frac{\partial \bar{\mathbf{T}}_*}{\partial \mathbf{E}_*}\boldsymbol{\mathcal{Z}}\right)[\dot{\mathbf{E}}_p] + \frac{\partial \bar{\mathbf{T}}_*}{\partial \boldsymbol{\alpha}}_R[\dot{\boldsymbol{\alpha}}_R] + \frac{\partial \bar{\mathbf{T}}_*}{\partial \kappa}\dot{\kappa}.
\end{aligned}
$$

Recalling the flow rules (3.18) during loading,

$$
\begin{aligned}
\dot{\mathbf{T}}_* &= \frac{\partial \bar{\mathbf{T}}_*}{\partial \mathbf{E}_*}\left[\mathbf{F}_*^T \mathbf{D} \mathbf{F}_*\right] + \left(\frac{\partial \bar{\mathbf{T}}_*}{\partial \mathbf{E}_p} - \frac{\partial \bar{\mathbf{T}}_*}{\partial \mathbf{E}_*}\boldsymbol{\mathcal{Z}}\right)[\pi \hat{g}\boldsymbol{\rho}] + \frac{\partial \bar{\mathbf{T}}_*}{\partial \boldsymbol{\alpha}}_R[\pi \hat{g}\boldsymbol{\beta}] + \frac{\partial \bar{\mathbf{T}}_*}{\partial \kappa}\pi \hat{g}\lambda \\
&= \frac{\partial \bar{\mathbf{T}}_*}{\partial \mathbf{E}_*}\left[\mathbf{F}_*^T \mathbf{D} \mathbf{F}_*\right] + \pi \hat{g}\left\{\left(\frac{\partial \bar{\mathbf{T}}_*}{\partial \mathbf{E}_p} - \frac{\partial \bar{\mathbf{T}}_*}{\partial \mathbf{E}_*}\boldsymbol{\mathcal{Z}}\right)[\boldsymbol{\rho}] + \frac{\partial \bar{\mathbf{T}}_*}{\partial \boldsymbol{\alpha}}_R[\boldsymbol{\beta}] + \frac{\partial \bar{\mathbf{T}}_*}{\partial \kappa}\lambda\right\} \qquad (6.52) \\
&= \frac{\partial \bar{\mathbf{T}}_*}{\partial \mathbf{E}_*}\left[\mathbf{F}_*^T \mathbf{D} \mathbf{F}_*\right] + \pi \hat{g}\widetilde{\boldsymbol{\sigma}}_*,
\end{aligned}
$$

where

$$
\widetilde{\boldsymbol{\sigma}}_* = \left(\frac{\partial \bar{\mathbf{T}}_*}{\partial \mathbf{E}_p} - \frac{\partial \bar{\mathbf{T}}_*}{\partial \mathbf{E}_*}\boldsymbol{\mathcal{Z}}\right)[\boldsymbol{\rho}] + \frac{\partial \bar{\mathbf{T}}_*}{\partial \boldsymbol{\alpha}}_R[\boldsymbol{\beta}] + \frac{\partial \bar{\mathbf{T}}_*}{\partial \kappa}\lambda. \qquad (6.53)
$$

Recalling the loading index

$$
\hat{g} = \boldsymbol{\Gamma}(\mathbf{E}_*, \mathbf{E}_p) \cdot \mathbf{D} = \mathbf{F}_* \frac{\partial g_*}{\partial \mathbf{E}_*}\mathbf{F}_*^T \cdot \mathbf{D}. \qquad (6.54)
$$

Then

$$
\begin{aligned}
\dot{\mathbf{T}}_* &= \frac{\partial \bar{\mathbf{T}}_*}{\partial \mathbf{E}_*}\left[\mathbf{F}_*^T \mathbf{D} \mathbf{F}_*\right] + \pi\left(\mathbf{F}_* \frac{\partial g_*}{\partial \mathbf{E}_*}\mathbf{F}_*^T \cdot \mathbf{D}\right)\widetilde{\boldsymbol{\sigma}}_* \\
&= \frac{\partial \bar{\mathbf{T}}_*}{\partial \mathbf{E}_*}\left[\mathbf{F}_*^T \mathbf{D} \mathbf{F}_*\right] + \pi \widetilde{\boldsymbol{\sigma}}_* \frac{\partial g_*}{\partial \mathbf{E}_*} \cdot \mathbf{F}_*^T \mathbf{D} \mathbf{F}_* \\
&= \frac{\partial \bar{\mathbf{T}}_*}{\partial \mathbf{E}_*}\left[\mathbf{F}_*^T \mathbf{D} \mathbf{F}_*\right] + \left(\pi \widetilde{\boldsymbol{\sigma}}_* \otimes \frac{\partial g_*}{\partial \mathbf{E}_*}\right)\left[\mathbf{F}_*^T \mathbf{D} \mathbf{F}_*\right] \qquad (6.55) \\
&= \left(\frac{\partial \bar{\mathbf{T}}_*}{\partial \mathbf{E}_*} + \pi \widetilde{\boldsymbol{\sigma}}_* \otimes \frac{\partial g_*}{\partial \mathbf{E}_*}\right)\left[\mathbf{F}_*^T \mathbf{D} \mathbf{F}_*\right] \\
&= \left(\frac{\partial \bar{\mathbf{T}}_*}{\partial \mathbf{E}_*} + \pi \widetilde{\boldsymbol{\sigma}}_* \otimes \frac{\partial g_*}{\partial \mathbf{E}_*}\right) \odot (\mathbf{F}_* \otimes \mathbf{F}_*)[\mathbf{D}].
\end{aligned}
$$

The invariance requirement for $\dot{\mathbf{T}}_*$ can be shown in a similar way as (6.34).

6.4 The Piola-Kirchhoff stress $\mathbf{S}_*$ as a function of $\mathbf{E}_*$

Recall the Piola-Kirchhoff stress

$$\mathbf{S}_* = \bar{\mathbf{S}}_*(\mathbf{E}_*, \mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa). \tag{6.56}$$

In a similar fashion to the previous section, we can formulate the stress rate $\dot{\mathbf{S}}_*$ in terms of $\mathbf{D}$. Thus,

$$\begin{aligned}
\dot{\mathbf{S}}_* &= \frac{\partial \bar{\mathbf{S}}_*}{\partial \mathbf{E}_*}[\dot{\mathbf{E}}_*] + \frac{\partial \bar{\mathbf{S}}_*}{\partial \mathbf{E}_p}\left[\dot{\mathbf{E}}_p\right] + \frac{\partial \bar{\mathbf{S}}_*}{\partial \boldsymbol{\alpha}_R}\left[\dot{\boldsymbol{\alpha}}_R\right] + \frac{\partial \bar{\mathbf{S}}_*}{\partial \kappa}\,\dot{\kappa} \\
&= \frac{\partial \bar{\mathbf{S}}_*}{\partial \mathbf{E}_*}\left[\mathbf{F}_*^T\mathbf{D}\mathbf{F}_* - \boldsymbol{\mathcal{Z}}[\dot{\mathbf{E}}_p]\right] + \frac{\partial \bar{\mathbf{S}}_*}{\partial \mathbf{E}_p}\left[\dot{\mathbf{E}}_p\right] + \frac{\partial \bar{\mathbf{S}}_*}{\partial \boldsymbol{\alpha}_R}\left[\dot{\boldsymbol{\alpha}}_R\right] + \frac{\partial \bar{\mathbf{S}}_*}{\partial \kappa}\,\dot{\kappa} \\
&= \frac{\partial \bar{\mathbf{S}}_*}{\partial \mathbf{E}_*}\left[\mathbf{F}_*^T\mathbf{D}\mathbf{F}_*\right] - \frac{\partial \bar{\mathbf{S}}_*}{\partial \mathbf{E}_*}\boldsymbol{\mathcal{Z}}[\dot{\mathbf{E}}_p] + \frac{\partial \bar{\mathbf{S}}_*}{\partial \mathbf{E}_p}\left[\dot{\mathbf{E}}_p\right] + \frac{\partial \bar{\mathbf{S}}_*}{\partial \boldsymbol{\alpha}_R}\left[\dot{\boldsymbol{\alpha}}_R\right] + \frac{\partial \bar{\mathbf{S}}_*}{\partial \kappa}\,\dot{\kappa} \\
&= \frac{\partial \bar{\mathbf{S}}_*}{\partial \mathbf{E}_*}\left[\mathbf{F}_*^T\mathbf{D}\mathbf{F}_*\right] + \left(\frac{\partial \bar{\mathbf{S}}_*}{\partial \mathbf{E}_p} - \frac{\partial \bar{\mathbf{S}}_*}{\partial \mathbf{E}_*}\boldsymbol{\mathcal{Z}}\right)\left[\dot{\mathbf{E}}_p\right] + \frac{\partial \bar{\mathbf{S}}_*}{\partial \boldsymbol{\alpha}_R}\left[\dot{\boldsymbol{\alpha}}_R\right] + \frac{\partial \bar{\mathbf{S}}_*}{\partial \kappa}\,\dot{\kappa} \\
&= \frac{\partial \bar{\mathbf{S}}_*}{\partial \mathbf{E}_*}\left[\mathbf{F}_*^T\mathbf{D}\mathbf{F}_*\right] + \left(\frac{\partial \bar{\mathbf{S}}_*}{\partial \mathbf{E}_p} - \frac{\partial \bar{\mathbf{S}}_*}{\partial \mathbf{E}_*}\boldsymbol{\mathcal{Z}}\right)\left[\pi\hat{g}\boldsymbol{\rho}\right] + \frac{\partial \bar{\mathbf{S}}_*}{\partial \boldsymbol{\alpha}_R}\left[\pi\hat{g}\boldsymbol{\beta}\right] + \frac{\partial \bar{\mathbf{T}}}{\partial \kappa}\,\pi\hat{g}\lambda \\
&= \frac{\partial \bar{\mathbf{S}}_*}{\partial \mathbf{E}_*}\left[\mathbf{F}_*^T\mathbf{D}\mathbf{F}_*\right] + \pi\hat{g}\left\{\left(\frac{\partial \bar{\mathbf{S}}_*}{\partial \mathbf{E}_p} - \frac{\partial \bar{\mathbf{S}}_*}{\partial \mathbf{E}_*}\boldsymbol{\mathcal{Z}}\right)[\boldsymbol{\rho}] + \frac{\partial \bar{\mathbf{S}}_*}{\partial \boldsymbol{\alpha}_R}[\boldsymbol{\beta}] + \frac{\partial \bar{\mathbf{S}}_*}{\partial \kappa}\,\lambda\right\} \\
&= \frac{\partial \bar{\mathbf{S}}_*}{\partial \mathbf{E}_*}\left[\mathbf{F}_*^T\mathbf{D}\mathbf{F}_*\right] + \pi\hat{g}\boldsymbol{\sigma}_*,
\end{aligned} \tag{6.57}$$

where we have used (6.50), (3.18), and defined

$$\boldsymbol{\sigma}_* = \left(\frac{\partial \bar{\mathbf{S}}_*}{\partial \mathbf{E}_p} - \frac{\partial \bar{\mathbf{S}}_*}{\partial \mathbf{E}_*}\boldsymbol{\mathcal{Z}}\right)[\boldsymbol{\rho}] + \frac{\partial \bar{\mathbf{S}}_*}{\partial \boldsymbol{\alpha}_R}[\boldsymbol{\beta}] + \frac{\partial \bar{\mathbf{S}}_*}{\partial \kappa}\,\lambda. \tag{6.58}$$

Further, using (5.104) and (2.17),

$$\begin{aligned}
\dot{\mathbf{S}}_* &= \frac{\partial \bar{\mathbf{S}}_*}{\partial \mathbf{E}_*}\left[\mathbf{F}_*^T\mathbf{D}\mathbf{F}_*\right] + \pi\left(\mathbf{F}_*\frac{\partial g_*}{\partial \mathbf{E}_*}\mathbf{F}_*^T \cdot \mathbf{D}\right)\boldsymbol{\sigma}_* \\
&= \frac{\partial \bar{\mathbf{S}}_*}{\partial \mathbf{E}_*}\left[\mathbf{F}_*^T\mathbf{D}\mathbf{F}_*\right] + \pi\boldsymbol{\sigma}_*\frac{\partial g_*}{\partial \mathbf{E}_*} \cdot \mathbf{F}_*^T\mathbf{D}\mathbf{F}_* \\
&= \frac{\partial \bar{\mathbf{S}}_*}{\partial \mathbf{E}_*}\left[\mathbf{F}_*^T\mathbf{D}\mathbf{F}_*\right] + \left(\pi\boldsymbol{\sigma}_* \otimes \frac{\partial g_*}{\partial \mathbf{E}_*}\right)\left[\mathbf{F}_*^T\mathbf{D}\mathbf{F}_*\right] \\
&= \left(\frac{\partial \bar{\mathbf{S}}_*}{\partial \mathbf{E}_*} + \pi\boldsymbol{\sigma}_* \otimes \frac{\partial g_*}{\partial \mathbf{E}_*}\right)\left[\mathbf{F}_*^T\mathbf{D}\mathbf{F}_*\right] \\
&= \left(\frac{\partial \bar{\mathbf{S}}_*}{\partial \mathbf{E}_*} + \pi\boldsymbol{\sigma}_* \otimes \frac{\partial g_*}{\partial \mathbf{E}_*}\right) \odot (\mathbf{F}_* \otimes \mathbf{F}_*)\,[\mathbf{D}]\,.
\end{aligned} \tag{6.59}$$

This stress rate follows the invariance requirement $\dot{\mathbf{S}}_*^+ = \dot{\mathbf{S}}_*$.

In summary, we have the following four objective stress rates:

$$\begin{aligned}
\dot{\mathbf{S}} &= \left(\frac{\partial \hat{\mathbf{S}}}{\partial \mathbf{E}} + \pi \boldsymbol{\sigma} \otimes \frac{\partial g}{\partial \mathbf{E}}\right) \odot (\mathbf{F} \otimes \mathbf{F})\,[\mathbf{D}], \\
\dot{\tilde{\mathbf{T}}} &= \left(\frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*} + \pi \widetilde{\boldsymbol{\sigma}} \otimes \frac{\partial g_*}{\partial \mathbf{E}_*}\right) \odot (\mathbf{F}_* \otimes \mathbf{F}_*)\,[\mathbf{D}], \\
\dot{\mathbf{T}}_* &= \left(\frac{\partial \bar{\mathbf{T}}_*}{\partial \mathbf{E}_*} + \pi \widetilde{\boldsymbol{\sigma}}_* \otimes \frac{\partial g_*}{\partial \mathbf{E}_*}\right) \odot (\mathbf{F}_* \otimes \mathbf{F}_*)\,[\mathbf{D}], \\
\dot{\mathbf{S}}_* &= \left(\frac{\partial \bar{\mathbf{S}}_*}{\partial \mathbf{E}_*} + \pi \boldsymbol{\sigma}_* \otimes \frac{\partial g_*}{\partial \mathbf{E}_*}\right) \odot (\mathbf{F}_* \otimes \mathbf{F}_*)\,[\mathbf{D}].
\end{aligned} \tag{6.60}$$

Each of the stress rates above have the same form: a fourth-order tensor acting on the rate of deformation tensor $\mathbf{D}$. All of them can be used for finite elastic-plastic deformations.

We can rewrite the stress rates $(6.60)_{1,2}$ using their work conjugates as follows:

$$\begin{aligned}
\dot{\mathbf{S}} &= \left(\frac{\partial \hat{\mathbf{S}}}{\partial \mathbf{E}} + \pi \boldsymbol{\sigma} \otimes \frac{\partial g}{\partial \mathbf{E}}\right) [\dot{\mathbf{E}}] \\
&= \boldsymbol{\mathscr{A}}(\mathbf{E}, \mathscr{U})[\dot{\mathbf{E}}],
\end{aligned} \tag{6.61}$$

$$\begin{aligned}
\dot{\tilde{\mathbf{T}}} &= \left(\frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*} \odot (\mathbf{U}_* \otimes \mathbf{U}_*) + \pi \widetilde{\boldsymbol{\sigma}} \otimes \widetilde{\boldsymbol{\Gamma}}\right) [\widetilde{\mathbf{D}}] \\
&= \boldsymbol{\mathscr{A}}_*(\mathbf{E}_*, \mathscr{U})[\widetilde{\mathbf{D}}]
\end{aligned} \tag{6.62}$$

6.5 General objective rates of the rotated Cauchy stress

In a 1988 paper, Casey and Naghdi [14] developed a relation between two objective rates to demonstrate that the Eulerian and Lagrangian descriptions of finite rigid plasticity is form-invariant under arbitrary transformations of any objective stress rate. We develop a similar form in this section for objective rates of the rotated stress tensor.

Let $\boldsymbol{\Sigma}$ be a kinetical tensor associated with the present configuration of the body. We can define a transform $\tilde{\boldsymbol{\pi}}$ that takes $\boldsymbol{\Sigma}$ into its rotated form $\tilde{\boldsymbol{\Sigma}}$ by

$$\tilde{\boldsymbol{\Sigma}} = \tilde{\boldsymbol{\pi}}\{\boldsymbol{\Sigma}\} = \mathbf{R}^T \boldsymbol{\Sigma} \mathbf{R}. \tag{6.63}$$

Conversely, we can define $\boldsymbol{\Sigma}$ in terms of its rotated form by

$$\boldsymbol{\Sigma} = \tilde{\boldsymbol{\pi}}^{-1}\{\tilde{\boldsymbol{\Sigma}}\} = \mathbf{R}\tilde{\boldsymbol{\Sigma}}\mathbf{R}^T. \tag{6.64}$$

Under superposed rigid body motions, both $\boldsymbol{\Sigma}$ and $\tilde{\boldsymbol{\Sigma}}$ are objective because

$$\boldsymbol{\Sigma}^+ = \mathbf{Q}\boldsymbol{\Sigma}\mathbf{Q}^T, \qquad \tilde{\boldsymbol{\Sigma}}^+ = \tilde{\boldsymbol{\Sigma}}. \tag{6.65}$$

Clearly, the material time derivative of $\tilde{\boldsymbol{\Sigma}}$ is objective, while that of $\boldsymbol{\Sigma}$ is not.

For any two objective rates $\overset{b}{\boldsymbol{\Sigma}}$ and $\overset{a}{\boldsymbol{\Sigma}}$ of a kinetical tensor $\boldsymbol{\Sigma}$, we suppose that

$$\overset{b}{\boldsymbol{\Sigma}} = \overset{a}{\boldsymbol{\Sigma}} + \mathscr{B}^{b,a}(\boldsymbol{\Sigma})[\mathbf{L}], \tag{6.66}$$

where $\mathscr{B}^{b,a}$ is a fourth-order tensor function of $\mathbf{T}, \mathbf{F}_*, \kappa$ and $\boldsymbol{\Sigma}$ acting on the velocity gradient. If we take $\boldsymbol{\Sigma}$ to be the Cauchy stress tensor, we then have the relation

$$\overset{b}{\mathbf{T}} = \overset{a}{\mathbf{T}} + \mathscr{B}^{b,a}(\mathbf{T})[\mathbf{D}], \tag{6.67}$$

where, again, $\mathscr{B}^{b,a}$ is a function of $\mathbf{T}$, $\mathbf{F}_*$, and κ. By invariance requirements under superposed rigid body motions, the fourth-order tensor now acts on the symmetric part of the velocity gradient. Let's take $\overset{b}{\mathbf{T}}$ to be the box rate $\overset{\square}{\mathbf{T}}$, given in (6.35). Thus,

$$\overset{\square}{\mathbf{T}} = \overset{a}{\mathbf{T}} + \mathscr{B}^{\square,a}(\mathbf{T})[\mathbf{D}]. \tag{6.68}$$

From here, we can obtain a similar relation to establish a class of objective rates for the rotated stress $\widetilde{\mathbf{T}}$. Using $(6.35)_1$ and (6.68),

$$\begin{aligned} \dot{\tilde{\mathbf{T}}} &= \mathbf{R}^T\overset{\square}{\mathbf{T}}\mathbf{R} \\ &= \mathbf{R}^T\big\{\overset{a}{\mathbf{T}} + \mathscr{B}^{\square,a}(\mathbf{T})[\mathbf{D}]\big\}\mathbf{R} \\ &= \mathbf{R}^T\overset{a}{\mathbf{T}}\mathbf{R} + \mathbf{R}^T\big\{\mathscr{B}^{\square,a}(\mathbf{T})[\mathbf{D}]\big\}\mathbf{R}. \end{aligned} \tag{6.69}$$

In component form,

$$\begin{aligned} \dot{\tilde{T}}_{KL} &= R_{mK}\overset{\square}{T}_{mn}R_{nL}. \\ &= R_{mK}\left(\overset{a}{T}_{mn} + \mathscr{B}^{\square,a}_{KLij}D_{ij}\right)R_{nL} \\ &= R_{mK}\overset{a}{T}_{mn}R_{nL} + R_{mK}\mathscr{B}^{\square,a}_{KLij}D_{ij}R_{nL}. \end{aligned} \tag{6.70}$$

The last term in $(6.69)_3$ can be rewritten using a fourth-order tensor

$$\begin{aligned} \mathscr{A}^{\square,a}(\widetilde{\mathbf{T}})[\mathbf{D}] &= \mathbf{R}^T\big\{\mathscr{B}^{\square,a}(\mathbf{T})[\mathbf{D}]\big\}\mathbf{R} \\ &= \{(\mathbf{R}\otimes\mathbf{R})\boxdot\mathscr{B}^{\square,a}(\mathbf{T})\}[\mathbf{D}], \end{aligned} \tag{6.71}$$

which has components

$$\mathscr{A}^{\square,a}_{mnij} = R_{mA} R_{nB} \mathscr{B}^{\square,a}_{KLij} \delta_{AK} \delta_{BL}. \tag{6.72}$$

We have used the operation $\boxdot$ in formulating the tensor $\mathscr{A}^{\square,a}$. Thus,

$$\dot{\widetilde{\mathbf{T}}} = \mathbf{R}^T \overset{a}{\mathbf{T}} \mathbf{R} + \boldsymbol{\mathscr{A}}^{\square,a}(\widetilde{\mathbf{T}})[\mathbf{D}]. \tag{6.73}$$

Let

$$\overset{a}{\widetilde{\mathbf{T}}} = \mathbf{R}^T \overset{a}{\mathbf{T}} \mathbf{R}. \tag{6.74}$$

This objective rate of the rotated Cauchy stress is the $\widetilde{\boldsymbol{\pi}}$ transform of any objective rate $\overset{a}{\mathbf{T}}$ of Cauchy stress. Using (6.63), we then have the following:

$$\begin{aligned}
\widetilde{\mathbf{T}} &= \widetilde{\boldsymbol{\pi}}\{\mathbf{T}\} = \mathbf{R}^T \mathbf{T} \mathbf{R}, \\
\overset{\square}{\mathbf{T}} &= \widetilde{\boldsymbol{\pi}}\{\dot{\mathbf{T}}\} = \mathbf{R}^T \dot{\mathbf{T}} \mathbf{R}, \\
\overset{a}{\widetilde{\mathbf{T}}} &= \widetilde{\boldsymbol{\pi}}\{\overset{a}{\mathbf{T}}\} = \mathbf{R}^T \overset{a}{\mathbf{T}} \mathbf{R}.
\end{aligned} \tag{6.75}$$

Also, under superposed rigid body motions,

$$\begin{aligned}
\left(\overset{a}{\widetilde{\mathbf{T}}}\right)^+ &= \left(\mathbf{R}^T \overset{a}{\mathbf{T}} \mathbf{R}\right)^+ \\
&= \left(\mathbf{R}^T \mathbf{Q}^T\right) \left(\mathbf{Q} \overset{a}{\mathbf{T}} \mathbf{Q}^T\right) (\mathbf{Q}\mathbf{R}) \\
&= \mathbf{R}^T \overset{a}{\mathbf{T}} \mathbf{R} \\
&= \overset{a}{\widetilde{\mathbf{T}}}.
\end{aligned} \tag{6.76}$$

Thus,

$$\dot{\widetilde{\mathbf{T}}} = \overset{a}{\widetilde{\mathbf{T}}} + \boldsymbol{\mathscr{A}}^{\square,a}(\widetilde{\mathbf{T}})[\mathbf{D}]. \tag{6.77}$$

Also, for any two objective rates $\overset{a}{\widetilde{\mathbf{T}}}$ and $\overset{b}{\widetilde{\mathbf{T}}}$ of the rotated stress,

$$\overset{b}{\widetilde{\mathbf{T}}} = \overset{a}{\widetilde{\mathbf{T}}} + \boldsymbol{\mathscr{A}}^{b,a}(\widetilde{\mathbf{T}})[\mathbf{D}], \tag{6.78}$$

where

$$\begin{aligned}
\boldsymbol{\mathscr{A}}^{b,a}(\widetilde{\mathbf{T}}) &= \boldsymbol{\mathscr{A}}^{\square,a}(\widetilde{\mathbf{T}}) - \boldsymbol{\mathscr{A}}^{\square,b}(\widetilde{\mathbf{T}}) \\
&= \boldsymbol{\mathscr{A}}^{b,\square}(\widetilde{\mathbf{T}}) - \boldsymbol{\mathscr{A}}^{a,\square}(\widetilde{\mathbf{T}}).
\end{aligned} \tag{6.79}$$

We can therefore adopt an equation of the form

$$\overset{a}{\widetilde{\mathbf{T}}} = \boldsymbol{\mathscr{G}}^a[\mathbf{D}], \tag{6.80}$$

where $\mathscr{G}^a$ is a fourth-order tensor which in addition to depending on $\widetilde{\mathbf{T}}$, $\mathbf{F}_*$, $\mathscr{U}$, also depends on the choice of objective rate. This dependence is indicated by the superscript a. Thus, under transformation of objective rate, the equation above is form-invariant because we can easily transform it into

$$\overset{b}{\widetilde{\mathbf{T}}} = \mathscr{G}^b[\mathbf{D}], \tag{6.81}$$

where

$$\mathscr{G}^b = \mathscr{G}^a + \mathscr{A}^{b,a}(\widetilde{\mathbf{T}}). \tag{6.82}$$

Some examples are presented here to demonstrate this class of objective rates using some common rates of $\mathbf{T}$.

(i) Rotated Jaumann rate

$$\begin{aligned} \overset{j}{\widetilde{\mathbf{T}}} &= \mathbf{R}^T\overset{O}{\mathbf{T}}\mathbf{R} \\ &= \mathbf{R}^T\{\dot{\mathbf{T}} - \mathbf{W}\mathbf{T} + \mathbf{T}\mathbf{W}\}\mathbf{R} \\ &= \mathbf{R}^T\dot{\mathbf{T}}\mathbf{R} - \mathbf{R}^T\mathbf{W}\mathbf{T}\mathbf{R} + \mathbf{R}^T\mathbf{T}\mathbf{W}\mathbf{R}. \end{aligned} \tag{6.83}$$

(ii) Rotated convected rate of stress

$$\begin{aligned} \overset{c}{\widetilde{\mathbf{T}}} &= \mathbf{R}^T\overset{\triangle}{\mathbf{T}}\mathbf{R} \\ &= \mathbf{R}^T\{\dot{\mathbf{T}} + \mathbf{L}^T\mathbf{T} + \mathbf{T}\mathbf{L}\}\mathbf{R} \\ &= \mathbf{R}^T\dot{\mathbf{T}}\mathbf{R} + \mathbf{R}^T\mathbf{L}^T\mathbf{T}\mathbf{R} + \mathbf{R}^T\mathbf{T}\mathbf{L}\mathbf{R}. \end{aligned} \tag{6.84}$$

(iii) Material time derivative of rotated stress

$$\begin{aligned} \dot{\widetilde{\mathbf{T}}} &= \mathbf{R}^T\overset{\square}{\mathbf{T}}\mathbf{R} \\ &= \mathbf{R}^T\{\dot{\mathbf{T}} - \boldsymbol{\Omega}\mathbf{T} + \mathbf{T}\boldsymbol{\Omega}\}\mathbf{R} \\ &= \mathbf{R}^T\dot{\mathbf{T}}\mathbf{R} - \mathbf{R}^T\boldsymbol{\Omega}\mathbf{T}\mathbf{R} + \mathbf{R}^T\mathbf{T}\boldsymbol{\Omega}\mathbf{R} \\ &= \mathbf{R}^T\dot{\mathbf{T}}\mathbf{R} + \dot{\mathbf{R}}^T\mathbf{T}\mathbf{R} + \mathbf{R}^T\mathbf{T}\dot{\mathbf{R}}. \end{aligned} \tag{6.85}$$

We can relate the three rates above using (6.78) or (6.77). Thus,

$$\begin{aligned} \mathscr{A}^{j,c}(\widetilde{\mathbf{T}})[\mathbf{D}] &= \mathbf{R}^T\mathbf{W}^T\mathbf{T}\mathbf{R} + \mathbf{R}^T\mathbf{T}\mathbf{W}\mathbf{R} - \mathbf{R}^T\mathbf{L}^T\mathbf{T}\mathbf{R} - \mathbf{R}^T\mathbf{T}\mathbf{L}\mathbf{R} \\ &= \mathbf{R}^T\{\left(\mathbf{W}^T - \mathbf{L}^T\right)\mathbf{T} + \mathbf{T}\left(\mathbf{W} - \mathbf{L}\right)\}\mathbf{R} \\ &= \mathbf{R}^T\{-\mathbf{D}\mathbf{T} - \mathbf{T}\mathbf{D}\}\mathbf{R}, \end{aligned} \tag{6.86}$$

$$\begin{aligned} \mathscr{A}^{j,\square}(\widetilde{\mathbf{T}})[\mathbf{D}] &= \mathbf{R}^T\mathbf{W}^T\mathbf{T}\mathbf{R} + \mathbf{R}^T\mathbf{T}\mathbf{W}\mathbf{R} - \mathbf{R}^T\boldsymbol{\Omega}^T\mathbf{T}\mathbf{R} - \mathbf{R}^T\mathbf{T}\boldsymbol{\Omega}\mathbf{R} \\ &= \mathbf{R}^T\{\left(\mathbf{W}^T - \boldsymbol{\Omega}^T\right)\mathbf{T} + \mathbf{T}\left(\mathbf{W} - \boldsymbol{\Omega}\right)\}\mathbf{R}, \end{aligned} \tag{6.87}$$

$$\mathscr{A}^{c,\square}(\widetilde{\mathbf{T}})[\mathbf{D}] = \mathbf{R}^T\mathbf{L}^T\mathbf{T}\mathbf{R} + \mathbf{R}^T\mathbf{T}\mathbf{L}\mathbf{R} - \mathbf{R}^T\mathbf{\Omega}^T\mathbf{T}\mathbf{R} - \mathbf{R}^T\mathbf{T}\mathbf{\Omega}\mathbf{R}$$
$$= \mathbf{R}^T\{\left(\mathbf{L}^T - \mathbf{\Omega}^T\right)\mathbf{T} + \mathbf{T}\left(\mathbf{L} - \mathbf{\Omega}\right)\}\mathbf{R}. \tag{6.88}$$

6.6 Strain hardening criteria and objective rates

When using a general objective rate, however, we note that certain quantities depend upon the choice of rate. An important example is the strain hardening criteria, which is defined using the quotient $\Phi = \hat{f}/\hat{g}$. Recall that the yield index is defined by

$$\hat{f} = \frac{\partial f}{\partial \mathbf{S}} \cdot \dot{\mathbf{S}}, \tag{6.89}$$

and that the strain hardening classification was defined in (3.30).

Using a different objective rate for $\mathbf{S}$, the yield index would be

$$\hat{f}^a = \frac{\partial f}{\partial \mathbf{S}} \cdot \mathbf{S}^a. \tag{6.90}$$

If we were to characterize all our stress measures using the objective rate denoted by the superscript a, we will need to define a new strain hardening classification using a function

$$\Phi^a = \frac{\hat{f}^a}{\hat{g}}. \tag{6.91}$$

We can then propose three distinct types of strain hardening behavior:

$$\begin{array}{lll} (a) & \Phi^a > 0 & \text{hardening} \\ (b) & \Phi^a < 0 & \text{softening} \\ (c) & \Phi^a = 0 & \text{perfectly plastic} \end{array} \tag{6.92}$$

This classification is clearly not equivalent to the one given in (3.30). In fact, even if the index $\hat{g}$ is also modified into say, $\hat{g}^a$, using the same objective rate as in $\hat{f}^a$, the strain hardening classification would still be different in general.

6.7 The back stress tensor $\boldsymbol{\alpha}$

In a similar manner as the general objective rates for the rotated Cauchy stress, we can describe general objective rates for the back stress tensor $\boldsymbol{\alpha}$, which has its own form of evolution equation. Let the kinetical tensor $\mathbf{\Sigma}$ be that back stress tensor $\boldsymbol{\alpha}$ in the current configuration. Its corresponding form in the reference configuration is $\boldsymbol{\alpha}_R$. Thus,

$$\boldsymbol{\alpha} = \boldsymbol{\pi}\{\boldsymbol{\alpha}_R\} = \frac{1}{J}\mathbf{F}\boldsymbol{\alpha}_R\mathbf{F}^T. \tag{6.93}$$

If we adopt an evolution equation of the form

$$\overset{a}{\mathring{\boldsymbol{\alpha}}} = \boldsymbol{\mathscr{H}}^a[\mathbf{D}], \tag{6.94}$$

where $\boldsymbol{\mathscr{H}}^a$ is a fourth-order tensor which, in addition to depending on $\widetilde{\mathbf{T}}$, $\mathbf{F}_*$, $\boldsymbol{\alpha}$, and κ, also depends on the choice of objective rate. The latter dependence is again denoted by the subscript a. Thus, under transformation of the objective rate, we have

$$\overset{b}{\mathring{\boldsymbol{\alpha}}} = \boldsymbol{\mathscr{H}}^b[\mathbf{D}], \tag{6.95}$$

with

$$\boldsymbol{\mathscr{H}}^b = \boldsymbol{\mathscr{H}}^a + \boldsymbol{\mathscr{A}}^{b,a}(\boldsymbol{\alpha})[\mathbf{D}], \tag{6.96}$$

where we have used

$$\overset{b}{\mathring{\boldsymbol{\alpha}}} = \overset{a}{\mathring{\boldsymbol{\alpha}}} + \boldsymbol{\mathscr{A}}^{b,a}(\boldsymbol{\alpha})[\mathbf{D}], \tag{6.97}$$

which is analogous to (6.67). Also, (6.96) is analogous to (6.66).

Chapter 7

Special Constitutive Equations

Different classes of materials can be described through special assumptions of the constitutive equations presented in the previous chapter. In this chapter we introduce some of these specialized forms of the stress measures and their rates. The viability of these constitutive equations and the assumptions that led up to them are also discussed. Further, we suggest which stress measures are applicable when smallness assumptions are made. This leads us to conclude that after linearization, both the Green and Naghdi theory and the $\mathbf{E}_*$ theory work very well for small strains, while the new theory is a better choice for large plastic strains.

7.1 The Piola-Kirchhoff stress tensor S and a special class of materials

Recall the symmetric Piola-Kirchhoff stress tensor given as

$$\mathbf{S} = \hat{\mathbf{S}}(\mathbf{E}, \mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa). \tag{7.1}$$

Also recall that $\mathbf{S}$ can be recast as a function of an equivalent set of kinematical measures in the form

$$\mathbf{S} = \bar{\mathbf{S}}(\mathbf{E} - \mathbf{E}_p, \mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa), \tag{7.2}$$

whose rate is

$$\dot{\mathbf{S}} = \frac{\partial \bar{\mathbf{S}}}{\partial (\mathbf{E} - \mathbf{E}_p)}[\dot{\mathbf{E}} - \dot{\mathbf{E}}_p] + \frac{\partial \bar{\mathbf{S}}}{\partial \mathbf{E}_p}[\dot{\mathbf{E}}_p] + \frac{\partial \bar{\mathbf{S}}}{\partial \boldsymbol{\alpha}_R}[\dot{\boldsymbol{\alpha}}_R] + \frac{\partial \bar{\mathbf{S}}}{\partial \kappa}\dot{\kappa}. \tag{7.3}$$

Now consider the special case where $\mathbf{S}$ is independent of the last three arguments $\mathbf{E}_p$,

$\boldsymbol{\alpha}_R$, and κ, such that

$$\mathbf{S} = \bar{\mathbf{S}}(\mathbf{E} - \mathbf{E}_p). \tag{7.4}$$

The time rate of stress can then be written as

$$\dot{\mathbf{S}} = \frac{\partial \bar{\mathbf{S}}}{\partial (\mathbf{E} - \mathbf{E}_p)}[\dot{\mathbf{E}} - \dot{\mathbf{E}}_p] = \boldsymbol{\mathscr{L}}[\dot{\mathbf{E}} - \dot{\mathbf{E}}_p], \tag{7.5}$$

where $(3.33)_1$ has been used.

Let us further suppose that the stress response (7.4) takes the form

$$\mathbf{S} = \boldsymbol{\mathscr{L}}\left[\mathbf{E} - \mathbf{E}_p\right], \tag{7.6}$$

and that $\boldsymbol{\mathscr{L}}$ is a constant tensor. Also consider the special constitutive equations

$$f = \bar{f}(\mathbf{S}) - \kappa, \qquad \kappa = K^2 = const, \tag{7.7}$$

where f is the yield function in stress space, and κ is a measure of work hardening. The stress and strain measures can be decomposed into their spherical and deviatoric parts. For convenience, we use index notation. Thus,

$$\begin{aligned} S_{KL} &= \tau_{KL} + \bar{S}\delta_{KL}, \qquad \bar{S} = \frac{1}{3}S_{MM}, \\ E_{MN} &= \gamma_{MN} + \bar{E}\delta_{MN}, \qquad \bar{E} = \frac{1}{3}E_{KK}, \\ E_{pMN} &= \gamma_{pMN} + \bar{E}_p\delta_{MN}, \qquad \bar{E}_p = \frac{1}{3}E_{pKK}. \end{aligned} \tag{7.8}$$

Assuming an isotropic elastic-plastic material, we write the components of $\boldsymbol{\mathscr{L}}$ as

$$\mathscr{L}_{KLMN} = 2\mu \mathscr{I}_{KLMN} + \left(k - \frac{2}{3}\mu\right)\delta_{KL}\delta_{MN}, \tag{7.9}$$

where μ is the shear modulus and k is the bulk modulus of elasticity. We can then write the stress equation (7.6) as

$$\begin{aligned} S_{KL} &= \mathscr{L}_{KLMN}\left(E_{MN} - E_{pMN}\right), \\ \tau_{KL} + \bar{S}\delta_{KL} &= \left\{2\mu\mathscr{I}_{KLMN} + \left(k - \frac{2}{3}\mu\right)\delta_{KL}\delta_{MN}\right\}\left(\gamma_{MN} + \bar{E}\delta_{MN} - \gamma_{pMN} - \bar{E}_p\delta_{MN}\right), \end{aligned} \tag{7.10}$$

which results in the two equations

$$\tau_{KL} = 2\mu\left(\gamma_{KL} - \gamma_{pKL}\right), \qquad \bar{S} = 3k\left(\bar{E} - \bar{E}_p\right). \tag{7.11}$$

We now use the von Mises yield function

$$f(S_{KL}, \mathscr{U}) = \frac{1}{2}\tau_{KL}\tau_{KL} - K^2. \tag{7.12}$$

In strain space, using $(7.11)_1$, this is expressed as

$$g(E_{KL}, \mathscr{U}) = 2\mu^2 \left(\gamma_{KL} - \gamma_{pKL}\right)\left(\gamma_{KL} - \gamma_{pKL}\right) - K^2. \tag{7.13}$$

The appropriate partial derivatives for f and g are

$$\begin{aligned} \frac{\partial f}{\partial S_{MN}} &= \frac{\partial f}{\partial \tau_{MN}} = \tau_{MN} \\ \frac{\partial g}{\partial E_{MN}} &= \frac{\partial g}{\partial \gamma_{MN}} = 4\mu^2 \left(\gamma_{MN} - \gamma_{pMN}\right) = 2\mu\tau_{MN}. \end{aligned} \tag{7.14}$$

At yield, $f = g = 0$, and from (7.12),

$$\tau_{KL}\tau_{KL} = 2K^2. \tag{7.15}$$

During loading, we have

$$\begin{aligned} \hat{f} = \dot{f} &= \frac{\partial f}{\partial \tau_{MN}}\dot{\tau}_{MN} = \tau_{MN}\dot{\tau}_{MN} = 0, \\ \hat{g} &= \frac{\partial g}{\partial \gamma_{MN}}\dot{\gamma}_{MN} = 2\mu\tau_{MN}\dot{\gamma}_{MN} > 0. \end{aligned} \tag{7.16}$$

Also, we assume the flow rule

$$\dot{\gamma}_{pMN} = \psi\frac{\partial f}{\partial \tau_{MN}} = \psi\tau_{MN}, \qquad \psi = \frac{\tau_{KL}\dot{\gamma}_{KL}}{2K^2}, \tag{7.17}$$

such that during loading, the time derivative of $(7.11)_1$ becomes

$$\begin{aligned} \dot{\tau}_{KL} &= 2\mu\left(\dot{\gamma}_{KL} - \frac{\tau_{MN}\dot{\gamma}_{MN}}{2K^2}\tau_{KL}\right) \\ &= 2\mu\left(\mathscr{I}_{KLMN} - \frac{\tau_{KL}\tau_{MN}}{2K^2}\right)\dot{\gamma}_{MN}. \end{aligned} \tag{7.18}$$

Equation (7.18) are the Prandtl-Reuss equations for elastic-perfectly plastic materials.

As mentioned in Section 3.5, for the specialized constitutive equations presented in this section, since $\hat{f} = 0$, the scalar function $\Phi = 0$. According to the strain hardening criteria (3.29), this means the material only exhibits perfectly-plastic behavior. Thus, the constitutive equations in (7.6) and (7.7), along with the special assumptions used in this section, describe a class of elastic-perfectly plastic materials with a stress response linear in $\mathbf{E} - \mathbf{E}_p$. While similar to the infinitesimal theory of plasticity, no assumption regarding the smallness of plastic deformation is made here. Thus, theoretically, these equations still apply to large deformations.

7.2 The rotated Cauchy stress $\widetilde{\mathbf{T}}$

Now recall the rotated stress given as a function of $\mathbf{E}_*$:

$$\widetilde{\mathbf{T}} = \bar{\mathbf{T}}(\mathbf{E}_*, \mathbf{E}_p, \boldsymbol{\alpha}, \kappa). \tag{7.19}$$

Consider the special case where $\bar{\mathbf{T}}$ does not depend on $\mathbf{E}_p$, $\boldsymbol{\alpha}$ and κ, such that

$$\widetilde{\mathbf{T}} = \bar{\mathbf{T}}(\mathbf{E}_*). \tag{7.20}$$

We can write the stress rate as either

$$\dot{\widetilde{\mathbf{T}}} = \frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*}[\dot{\mathbf{E}}_*] \tag{7.21}$$

or

$$\begin{aligned} \dot{\widetilde{\mathbf{T}}} &= \left(\frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*} - \pi \frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*} \boldsymbol{\mathcal{Z}}[\boldsymbol{\rho}] \otimes \frac{\partial g_*}{\partial \mathbf{E}_*} \right) [\mathbf{F}_*^T \mathbf{D} \mathbf{F}_*] \\ &= \left(\frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*} - \pi \frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*} \boldsymbol{\mathcal{Z}}[\boldsymbol{\rho}] \otimes \frac{\partial g_*}{\partial \mathbf{E}_*} \right) \odot (\mathbf{F}_* \otimes \mathbf{F}_*)\, [\mathbf{D}], \end{aligned} \tag{7.22}$$

depending on which strain rate argument is required. We have used equation (6.45) to express (7.22) from (7.21).

Using (5.5),

$$\begin{aligned} \dot{\widetilde{\mathbf{T}}} &= \left(\frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*} - \pi \frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*} \boldsymbol{\mathcal{Z}}[\boldsymbol{\rho}] \otimes \frac{\partial \tilde{f}}{\partial \widetilde{\mathbf{T}}} \frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*} \right) \odot (\mathbf{F}_* \otimes \mathbf{F}_*)\, [\mathbf{D}] \\ &= \left(\boldsymbol{\mathscr{I}} - \pi \frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*} \boldsymbol{\mathcal{Z}}[\boldsymbol{\rho}] \otimes \frac{\partial \tilde{f}}{\partial \widetilde{\mathbf{T}}} \right) \frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*} \odot (\mathbf{F}_* \otimes \mathbf{F}_*)\, [\mathbf{D}] \\ &= \tilde{\boldsymbol{\mathscr{K}}} \tilde{\boldsymbol{\mathscr{L}}} [\mathbf{F}_*^T \mathbf{D} \mathbf{F}_*], \end{aligned} \tag{7.23}$$

where

$$\tilde{\boldsymbol{\mathscr{K}}} = \left(\boldsymbol{\mathscr{I}} - \pi \frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*} \boldsymbol{\mathcal{Z}}[\boldsymbol{\rho}] \otimes \frac{\partial \tilde{f}}{\partial \widetilde{\mathbf{T}}} \right), \qquad \tilde{\boldsymbol{\mathscr{L}}} = \frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*}. \tag{7.24}$$

While equation $(7.23)_3$ is analogous to (3.21), they do not come from the same assumptions.

It is obvious from $(7.22)_1$ that $\dot{\widetilde{\mathbf{T}}}$ can also be written as

$$\dot{\widetilde{\mathbf{T}}} = \frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*} \left(\boldsymbol{\mathscr{I}} - \pi \boldsymbol{\mathcal{Z}}[\boldsymbol{\rho}] \otimes \frac{\partial g_*}{\partial \mathbf{E}_*} \right) [\mathbf{F}_*^T \mathbf{D} \mathbf{F}_*]. \tag{7.25}$$

If the von Mises criterion were applied to the stress rate $(7.23)_2$, and using (5.89),

$$\dot{\widetilde{\mathbf{T}}} = \left(\boldsymbol{\mathscr{I}} - \pi \frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*} \boldsymbol{\mathcal{Z}}[\boldsymbol{\rho}] \otimes \widetilde{\mathbf{T}} \right) \frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*} \odot (\mathbf{F}_* \otimes \mathbf{F}_*) [\mathbf{D}]. \tag{7.26}$$

This specialization of the constitutive equation still applies to non-linear elastic materials. No smallness approximation has been made yet.

It is also helpful to decompose the rotated stress $\widetilde{\mathbf{T}}$ and $\mathbf{E}_*$ into their spherical and deviatoric parts:

$$\begin{aligned} \tilde{T}_{KL} &= \tilde{\tau}_{KL} + \tilde{T}\delta_{KL}, \qquad \tilde{T} = \frac{1}{3}\tilde{T}_{MM}, \\ E_{*MN} &= \gamma_{*MN} + \bar{\gamma}_*\delta_{MN}, \qquad \bar{\gamma}_* = \frac{1}{3}E_{*KK}. \end{aligned} \tag{7.27}$$

We assume a stress response of the form

$$\widetilde{\mathbf{T}} = \tilde{\boldsymbol{\mathscr{L}}}[\mathbf{E}_*], \tag{7.28}$$

where $\tilde{\boldsymbol{\mathscr{L}}}$ is a constant tensor, and an isotropic elastic-plastic material, so that

$$\tilde{\mathscr{L}}_{KLMN} = \frac{\partial \widetilde{T}_{KL}}{\partial E_{*MN}} = 2\mu \mathscr{I}_{KLMN} + \left(k - \frac{2}{3}\mu \right) \delta_{KL}\delta_{MN}, \tag{7.29}$$

This results in the two equations

$$\tilde{\tau}_{KL} = 2\mu\gamma_{*KL}, \qquad \tilde{T} = 3k\bar{\gamma}_*. \tag{7.30}$$

7.3 The Piola-Kirchhoff stress tensor $\mathbf{S}_*$

Next, we recall the Piola-Kirchhoff tensor in the $\boldsymbol{\kappa}_*$ configuration:

$$\mathbf{S}_* = \bar{\mathbf{S}}_*(\mathbf{E}_*, \mathbf{E}_p, \boldsymbol{\alpha}_R, \kappa) = \bar{\mathbf{S}}_*(\mathbf{E}_*, \mathscr{U}). \tag{7.31}$$

Again, we consider the special case where $\mathbf{S}_*$ does not depend on $\mathbf{E}_p$, $\boldsymbol{\alpha}_R$ and κ. Thus,

$$\mathbf{S}_* = \bar{\mathbf{S}}_*(\mathbf{E}_*). \tag{7.32}$$

We further suppose that $\mathbf{S}_*$ is a linear function of $\mathbf{E}_*$. That is,

$$\mathbf{S}_* = \frac{\partial \bar{\mathbf{S}}_*}{\partial \mathbf{E}_*}[\mathbf{E}_*] = \boldsymbol{\mathscr{L}}_*[\mathbf{E}_*]. \tag{7.33}$$

Are we able to assume that the fourth-order tensor $\boldsymbol{\mathscr{L}}_*$ is a constant tensor? To answer this, first we differentiate with time to get

$$\dot{\mathbf{S}}_* = \boldsymbol{\mathscr{L}}_*[\dot{\mathbf{E}}_*]. \tag{7.34}$$

By (4.49), we know that $\dot{\mathbf{E}}_*$ involves the time derivative of $\mathbf{U}_p$. That is, we must take the rate of plastic deformation into account. Suppose we fix the value of $\mathbf{E}_*$ and let $\mathbf{U}_p$ change. According to the assumption that $\boldsymbol{\mathscr{L}}_*$ is constant, if $\dot{\mathbf{E}}_*$ is zero, then $\dot{\mathbf{S}}_*$ must zero. However, $\dot{\mathbf{S}}_*$ cannot be zero in general because the configuration $\boldsymbol{\kappa}_*$ changes as dictated by the change in $\mathbf{U}_p$. If $\boldsymbol{\kappa}_*$ changes, then the stress $\mathbf{S}_*$ must change as well. Therefore, from a physical standpoint, the tensor $\boldsymbol{\mathscr{L}}_*$ cannot be assumed as constant. This limits the use of $\mathbf{S}_*$ as a stress measure for Eulerian plasticity, even when the deformation is small. Note that this complication does not affect the use of the rotated stress $\widetilde{\mathbf{T}}$ as a function of $\mathbf{E}_*$ because the configuration $\boldsymbol{\kappa}_*$ has no effect on $\widetilde{\mathbf{T}}$.

7.4 Linearization and its physical validity

In finite plasticity, all three strain variables: total strain $\mathbf{E}$, plastic strain $\mathbf{E}_p$, and elastic strain $\mathbf{E}_*$, play important roles, even though only two of them can be independently specified. The strain difference $\mathbf{E} - \mathbf{E}_p$ involves both elastic strain and plastic strain.

In this section, we will observe what happens when the constitutive equations for $\mathbf{S}$ and $\widetilde{\mathbf{T}}$ are linearized. We further discuss which of these linearized equations make sense for different situations in which small deformations are applicable. We will first suppose that $\|\mathbf{E}_*\|$ is small, linearizing the rotated stress about $\mathbf{E}_* = \mathbf{0}$, while allowing for $\|\mathbf{E}_p\|$ to be large. Then we will assume that $\|\mathbf{E} - \mathbf{E}_p\|$ is small, while allowing for $\|\mathbf{E}_p\|$ to be large. We keep in mind that small $\|\mathbf{E} - \mathbf{E}_p\|$ does not necessarily imply that $\|\mathbf{E}_*\|$ is small, so the two assumptions are not equivalent. This begs the question: which stress measure is a better representative for Eulerian plasticity? In the general case, any stress measure would be a good option. However, when linearity is considered, the better stress tensor is not so obvious.

7.4.1 Rotated stress $\widetilde{\mathbf{T}}$

First, consider the constitutive equation for $\widetilde{\mathbf{T}}$. Using a Taylor expansion about $\mathbf{E}_* = \mathbf{0}$, the rotated stress $\widetilde{\mathbf{T}}$ becomes

$$\widetilde{\mathbf{T}} = \bar{\mathbf{T}}(\mathbf{E}_* = \mathbf{0}) + \frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*}(\mathbf{0})\,[\mathbf{E}_*] + \cdots . \tag{7.35}$$

Assuming $\bar{\mathbf{T}}(\mathbf{0}) = \mathbf{0}$, and neglecting the higher order terms,

$$\widetilde{\mathbf{T}} = \frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*}(\mathbf{0})\,[\mathbf{E}_*] = \tilde{\boldsymbol{\mathscr{L}}}\,[\mathbf{E}_*] . \tag{7.36}$$

We now attempt to make the same assumptions as in section 7.1, in order to obtain a Prandtl-Reuss type equation. Thus, suppose the coefficients $\dfrac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*}$ are constant and that the material is isotropic. We have, in index notation, the constant coefficients

$$\frac{\partial \tilde{T}_{KL}}{\partial E_{*MN}} = 2\mu \mathscr{I}_{KLMN} + \left(k - \frac{2}{3}\mu\right) \delta_{KL}\delta_{MN}. \tag{7.37}$$

The von Mises yield criterion is

$$\tilde{f} = \frac{1}{2}\widetilde{\mathbf{T}} \cdot \widetilde{\mathbf{T}} - K^2. \tag{7.38}$$

We also assume that

$$\mathbf{F}_* = \mathbf{I} + \mathbf{H}_* \tag{7.39}$$

with $\|\mathbf{H}_*\|^2 \simeq 0$. Thus,

$$\begin{aligned}
\mathbf{F}_*^T \mathbf{D} \mathbf{F}_* &= (\mathbf{I} + \mathbf{H}_*)^T \,\mathbf{D}\,(\mathbf{I} + \mathbf{H}_*) \\
&= \mathbf{D} + \mathbf{H}_*^T \mathbf{D} + \mathbf{D}\mathbf{H}_* + \mathbf{H}_*^T \mathbf{D}\mathbf{H}_* \\
&\simeq \mathbf{D} + \mathbf{H}_*^T \mathbf{D} + \mathbf{D}\mathbf{H}_*.
\end{aligned} \tag{7.40}$$

Thus, using $(7.23)_3$,

$$\dot{\widetilde{\mathbf{T}}} = \tilde{\boldsymbol{\mathscr{K}}}\tilde{\boldsymbol{\mathscr{L}}}[\mathbf{D} + \mathbf{H}_*^T \mathbf{D} + \mathbf{D}\mathbf{H}_*]. \tag{7.41}$$

Also, using $(7.24)_{1,2}$, (7.40), the von Mises criterion, and the isotropic material assumption, we can write in index notation

$$\begin{aligned}
\tilde{\mathscr{K}}_{KLCD} &= \mathscr{I}_{KLCD} - \pi \left(2\mu \mathscr{I}_{KLPQ} + \left(k - \frac{2}{3}\mu\right) \delta_{KL}\delta_{PQ} \right) \mathcal{Z}_{PQRS}\rho_{RS}\widetilde{T}_{CD}, \\
\tilde{\mathscr{L}}_{CDAB} &= 2\mu \mathscr{I}_{CDAB} + \left(k - \frac{2}{3}\mu\right) \delta_{CD}\delta_{AB}, \\
F_{*iA} D_{ij} F_{*jB} &= \delta_{iA} D_{ij} \delta_{jB} + H_{*iA} D_{ij} \delta_{jB} + \delta_{iA} D_{ij} H_{*jB}.
\end{aligned} \tag{7.42}$$

Let $\widetilde{\mathbf{T}}$ be decomposed into its spherical and deviatoric parts:

$$\widetilde{T}_{KL} = \widetilde{\tau}_{KL} + \widetilde{T}\delta_{KL}, \qquad \widetilde{T} = \frac{1}{3}\widetilde{T}_{MM}. \tag{7.43}$$

From (7.41), $(7.42)_{1,2,3}$ and (7.43), the deviatoric part of $\dot{\widetilde{\mathbf{T}}}$ is

$$\begin{aligned}\dot{\widetilde{\tau}}_{KL} = 2\mu\mathscr{I}_{KLAB}\left(\delta_{iA}D_{ij}\delta_{jB} + H_{*iA}D_{ij}\delta_{jB} + \delta_{iA}D_{ij}H_{*jB}\right) \\ - \pi 4\mu^2 \mathcal{Z}_{KLRS}\rho_{RS}\widetilde{T}_{AB}\left(\delta_{iA}D_{ij}\delta_{jB} + H_{*iA}D_{ij}\delta_{jB} + \delta_{iA}D_{ij}H_{*jB}\right) \\ - \pi 2\mu\left(k - \frac{2}{3}\mu\right)\mathcal{Z}_{KLRS}\rho_{RS}\widetilde{T}_{CC}\left(\delta_{iA}D_{ij}\delta_{jB} + H_{*iA}D_{ij}\delta_{jB} + \delta_{iA}D_{ij}H_{*jB}\right).\end{aligned} \tag{7.44}$$

If we were to further assume that the rotated stress is purely deviatoric, we are left with a Prandtl-Reuss type equation:

$$\dot{\widetilde{\tau}}_{KL} = 2\mu\left(\mathscr{I}_{KLAB} - 2\pi\mu\mathcal{Z}_{KLRS}\rho_{RS}\widetilde{\tau}_{AB}\right)d_{AB}, \tag{7.45}$$

where d_{AB} is the deviatoric part of $(7.40)_3$. In direct notation,

$$\dot{\widetilde{\boldsymbol{\tau}}} = 2\mu\left(\boldsymbol{\mathscr{I}} - 2\pi\mu\boldsymbol{\mathcal{Z}}\left[\boldsymbol{\rho}\right] \otimes \widetilde{\boldsymbol{\tau}}\right)[\mathbf{d}]. \tag{7.46}$$

For the rest of the chapter, we will retain the form of the rotated stress given in (7.36). Decomposing $\widetilde{\mathbf{T}}$ and $\mathbf{E}_*$ into spherical and deviatoric components, and using (7.37), we have

$$\tilde{\tau}_{KL} = 2\mu\gamma_{*KL}, \qquad \tilde{T} = 3k\bar{\gamma}_*, \tag{7.47}$$

which are similar to the forms given in (7.30), except that they are now linearized.

7.4.2 Piola-Kirchhoff stress tensor S

Now consider the constitutive equation for $\mathbf{S}$, given in (7.2). Using a Taylor expansion about $\mathbf{E} = \mathbf{E}_p$,

$$\mathbf{S} = \hat{\mathbf{S}}(\mathbf{E} = \mathbf{E}_p) + \frac{\partial\hat{\mathbf{S}}}{\partial\mathbf{E}}\left[\mathbf{E} - \mathbf{E}_p\right] + \cdots. \tag{7.48}$$

If we assume that $\hat{\mathbf{S}}(\mathbf{E}_p) = \mathbf{0}$ and neglect the higher order terms,

$$\begin{aligned}\mathbf{S} &= \frac{\partial\hat{\mathbf{S}}}{\partial\mathbf{E}}\left[\mathbf{E} - \mathbf{E}_p\right] \\ &= \boldsymbol{\mathscr{L}}\left[\mathbf{E} - \mathbf{E}_p\right].\end{aligned} \tag{7.49}$$

In this case, we are assuming that $\|\mathbf{E}-\mathbf{E}_p\|$ is close to zero, but we are allowing for the plastic strain $\mathbf{E}_p$ to be large. We assume that the tensor $\mathscr{L}$ is constant and that the material is isotropic. We also use the Mises criterion (7.12) and the flow rule (7.17). The Prandtl-Reuss equations can then be derived for small $\|\mathbf{E}-\mathbf{E}_p\|$ in the same manner that it was derived in Section 7.1, where the Prandtl-Reuss equations apply for the more general non-linear case. Thus,

$$\dot{\tau}_{KL} = 2\mu \left(\mathscr{I}_{KLMN} - \frac{\tau_{KL}\tau_{MN}}{2K^2} \right) \dot{\gamma}_{MN}. \tag{7.50}$$

The form is similar to the non-linear version in Equation (7.18), but they are not necessarily the same theory since we have made smallness assumptions here.

7.5 Comparison of stress measures

Using either $\mathbf{S}$ or $\widetilde{\mathbf{T}}$ as the stress measure is valid for any loading situation. In finite plasticity, one stress measure can be converted into the other. The linearized form of $\widetilde{\mathbf{T}}$ and the linearized form of $\mathbf{S}$ do not result in the same theory. That is, due to the smallness assumptions on the two different strain measures, $\mathbf{E}_*$ and $\mathbf{E}-\mathbf{E}_p$, the linearized theory involving the rotated stress $\widetilde{\mathbf{T}}$ is not equivalent to the linearized theory involving $\mathbf{S}$. To see this more explicitly, recall from (4.49) that

$$\mathbf{E}_* = \mathbf{U}_p^{-1} \left(\mathbf{E} - \mathbf{E}_p \right) \mathbf{U}_p^{-1}. \tag{7.51}$$

With the use of the Green-Saint-Venant strain tensors

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{U}^2 - \mathbf{I}\right), \qquad \mathbf{E}_p = \frac{1}{2}\left(\mathbf{U}_p^2 - \mathbf{I}\right), \qquad \mathbf{E}_* = \frac{1}{2}\left(\mathbf{U}_*^2 - \mathbf{I}\right), \tag{7.52}$$

we can rewrite $\mathbf{E}_*$ as

$$\begin{aligned} \mathbf{E}_* &= \mathbf{U}_p^{-1} \left[\frac{1}{2}\left(\mathbf{U}^2 - \mathbf{I}\right) - \frac{1}{2}\left(\mathbf{U}_p^2 - \mathbf{I}\right) \right] \mathbf{U}_p^{-1} \\ &= \mathbf{U}_p^{-1} \left[\frac{1}{2}\left(\mathbf{U}^2 - \mathbf{U}_p^2\right) \right] \mathbf{U}_p^{-1} \\ &= \frac{1}{2}\left(\mathbf{U}_p^{-1}\mathbf{U}^2\mathbf{U}_p^{-1} - \mathbf{I} \right). \end{aligned} \tag{7.53}$$

Now consider a bar in uniaxial stress. Let the stretches in the three principal directions be $\{\lambda_1, \lambda_2, \lambda_3\}$. Let the plastic stretch in the 1-direction be λ_p and the stretch associated with $\mathbf{E}_*$ in the 1-direction be λ_*. Also, there is no plastic volume change, meaning $\det \mathbf{U}_p = 1$. Expressed on the eigenbasis, we have the corresponding stretch tensors in matrix notation as

$$\mathbf{U} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \qquad \mathbf{U}_p = \begin{pmatrix} \lambda_p & 0 & 0 \\ 0 & \frac{1}{\sqrt{\lambda_p}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\lambda_p}} \end{pmatrix}, \qquad \mathbf{U}_* = \begin{pmatrix} \lambda_{*1} & 0 & 0 \\ 0 & \lambda_{*2} & 0 \\ 0 & 0 & \lambda_{*2} \end{pmatrix}, \tag{7.54}$$

such that $(7.53)_3$ becomes

$$\begin{aligned} \mathbf{E}_* &= \frac{1}{2}\left[\begin{pmatrix} \frac{1}{\lambda_p} & 0 & 0 \\ 0 & \sqrt{\lambda_p} & 0 \\ 0 & 0 & \sqrt{\lambda_p} \end{pmatrix} \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_2^2 \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda_p} & 0 & 0 \\ 0 & \sqrt{\lambda_p} & 0 \\ 0 & 0 & \sqrt{\lambda_p} \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \\ &= \frac{1}{2}\begin{pmatrix} \lambda_1^2/\lambda_p^2 - 1 & 0 & 0 \\ 0 & \lambda_2^2\lambda_p - 1 & 0 \\ 0 & 0 & \lambda_2^2\lambda_p - 1 \end{pmatrix}. \end{aligned} \tag{7.55}$$

In the first principal direction, we have

$$\begin{aligned} \frac{1}{2}\left(\lambda_*^2 - 1\right) &= \frac{1}{2}\left(\frac{\lambda_1^2}{\lambda_p^2} - 1\right), \\ \lambda_*^2 &= \frac{\lambda_1^2}{\lambda_p^2}, \\ \lambda_* &= \frac{\lambda_1}{\lambda_p}. \end{aligned} \tag{7.56}$$

Consider first a simple tension test of a bar where all the stretches in the first principal direction are greater than 1. While allowing for $\lambda_p \gg 0$, assume that $\lambda_1 - \lambda_p = o(\epsilon)$. Thus, from $(7.56)_3$

$$\begin{aligned} \lambda_*\lambda_p &= \lambda_1, \\ \lambda_*\lambda_p - \lambda_p &= \lambda_1 - \lambda_p, \\ \lambda_p\left(\lambda_* - 1\right) &= o(\epsilon), \\ \lambda_* - 1 &= o(\epsilon), \\ \lambda_* &= 1 + o(\epsilon). \end{aligned} \tag{7.57}$$

So a small $\lambda_1 - \lambda_p$ implies a small λ_* in a simple tension test. The reverse, however, is generally not true. Let's assume that $\lambda_* = 1 + o(\epsilon)$. Thus,

$$\begin{aligned} 1 + o(\epsilon) &= \frac{\lambda_1}{\lambda_p} \\ o(\epsilon) &= \frac{\lambda_1 - \lambda_p}{\lambda_p}. \end{aligned} \tag{7.58}$$

Since we are allowing for λ_p to be large, the size of $\lambda_1 - \lambda_p$ is actually arbitrary. The two assumptions are thus not equivalent.

On the other side of the spectrum, consider the case of large compressive loading. This means that we are allowing λ_p to tend to zero. First, assume that $\lambda_1 - \lambda_p = o(\epsilon)$. Then using $(7.56)_3$,

$$o(\epsilon) = \lambda_p \left(\lambda_* - 1\right). \tag{7.59}$$

Because $\lambda_p \to 0$, λ_* could be as large as possible. Now assume that $\lambda_* = 1 - o(\epsilon)$. By using $(7.56)_3$, this results in

$$o(\epsilon) = \frac{\lambda_p - \lambda_1}{\lambda_p}. \tag{7.60}$$

Again, $\lambda_1 - \lambda_p$ can have an arbitrary size.

We can then conclude that if $\|\mathbf{E}_*\|$ is small, which is a good physical assumption, then $\|\mathbf{E} - \mathbf{E}_p\|$ is not necessarily small for large $\mathbf{U}_p$. The converse is also true. An assumption that $\|\mathbf{E} - \mathbf{E}_p\|$ is small does not mean that $\|\mathbf{E}_*\|$ is small because we are allowing for $\mathbf{U}_p$ to be large. Niether of the small approximations implies the other for any general loading situation. However, if further linearization is done by assuming $\|\mathbf{E}_p\|$ to be small, then the two theories would coincide (i.e., for classical infinitesimal plasticity).

7.5.1 Uniaxial tension and compression using the new Eulerian theory

We will use the example of a uniaxial tension test to further demonstrate the differences between the Green-Naghdi theory and the new theory involving $\mathbf{E}_*$. First, we apply the new Eulerian theory and observe the behavior of the stress and strain measures. Consider a homogeneous deformation of a steel bar under uniaxial tension, where the rotated Cauchy tensor $\widetilde{\mathbf{T}}$ (= $\mathbf{T}$, now) is given by

$$(\widetilde{T}_{KL}) = (T_{KL}) = \begin{pmatrix} \widetilde{\sigma} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7.61}$$

The spherical and deviatoric parts give

$$\tilde{T} = \frac{1}{3}\widetilde{\sigma} > 0, \qquad \widetilde{\tau}_{KL} = \frac{1}{3}\widetilde{\sigma} c_{KL}, \tag{7.62}$$

where c_{KL} are components of the constant tensor

$$(c_{KL}) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{7.63}$$

Let the stretch tensor be

$$(U_{KL}) = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \tag{7.64}$$

where $\lambda_1(t)$ is the axial stretch and $\lambda_2(t)$ is the transverse stretch. Since there is no rotation, $U_{KL} = F_{KL}$ and $J = \det \mathbf{F} = \det \mathbf{U} = \lambda_1 \lambda_2^2$. Thus, using the relation $J\mathbf{T} = \mathbf{F}\mathbf{S}\mathbf{F}^T$, the 11-component of $\mathbf{S}$ and its deviatoric part are

$$S_{11} = \frac{\lambda_2^2}{\lambda_1} \widetilde{\sigma}, \qquad \tau_{11} = \frac{2}{3} \frac{\lambda_2^2}{\lambda_1} \widetilde{\sigma}. \tag{7.65}$$

The Lagrangian strain is

$$(E_{KL}) = \frac{1}{2} \begin{pmatrix} \lambda_1^2 - 1 & 0 & 0 \\ 0 & \lambda_2^2 - 1 & 0 \\ 0 & 0 & \lambda_2^2 - 1 \end{pmatrix}, \tag{7.66}$$

whose deviatoric part is

$$\gamma_{KL} = \frac{1}{6} \left(\lambda_1^2 - \lambda_2^2 \right) c_{KL}. \tag{7.67}$$

The rate of γ_{KL} is

$$\dot{\gamma}_{KL} = \frac{1}{3} \left(\lambda_1 \dot{\lambda}_1 - \lambda_2 \dot{\lambda}_2 \right) c_{KL}. \tag{7.68}$$

Similarly, let the stretch tensor $\mathbf{U}_*$ be

$$(U_{*KL}) = \begin{pmatrix} \lambda_{*1} & 0 & 0 \\ 0 & \lambda_{*2} & 0 \\ 0 & 0 & \lambda_{*2} \end{pmatrix}. \tag{7.69}$$

The corresponding strain tensor $\mathbf{E}_*$ is

$$(E_{*KL}) = \frac{1}{2} \begin{pmatrix} \lambda_{*1}^2 - 1 & 0 & 0 \\ 0 & \lambda_{*2}^2 - 1 & 0 \\ 0 & 0 & \lambda_{*2}^2 - 1 \end{pmatrix}, \tag{7.70}$$

whose deviatoric part is

$$\gamma_{*KL} = \frac{1}{6} \left(\lambda_{*1}^2 - \lambda_{*2}^2 \right) c_{KL}. \tag{7.71}$$

Also, from $(6.19)_4$,

$$E_{*11} = \frac{1}{2} \left(\frac{\lambda_1^2}{\lambda_p^2} - 1 \right), \qquad E_{*22} = \frac{1}{2} \left(\lambda_2^2 \lambda_p - 1 \right). \tag{7.72}$$

Thus,

$$\gamma_{*11} = \frac{1}{3}\left(\frac{\lambda_1^2}{\lambda_p^2} - \lambda_2^2\lambda_p\right), \qquad \gamma_{*22} = -\frac{1}{6}\left(\frac{\lambda_1^2}{\lambda_p^2} - \lambda_2^2\lambda_p\right) = -\frac{1}{2}\gamma_{*11}. \tag{7.73}$$

In this problem, we will assume that the uniaxial stress $\bar{\sigma}(t)$ is accompanied by a homogoneous deformation specified by the axial strain $E_{11}(t)$.

In the elastic region, we assume that

$$\widetilde{\tau}_{KL} = 2\mu\gamma_{*KL} = 2\mu\gamma_{KL}, \qquad \tilde{T} = 3k\bar{E} = 3k\bar{E}_*. \tag{7.74}$$

We also have the equations from linear elasticity

$$\begin{aligned} \sigma = EE_{11}, \qquad E_{22} = -\nu E_{11}, \\ \widetilde{\tau}_{11} = \frac{2}{3}\widetilde{\sigma} = 2\mu\gamma_{11}, \qquad \tilde{T} = 3k\bar{E} = k\left(E_{11} + 2E_{22}\right), \end{aligned} \tag{7.75}$$

where E is Young's modulus, ν is Poisson's ratio, μ is the shear modulus, and k is the bulk modulus. We assume that initial yield occurs at time $t = t_0$ when the tensile stress $\widetilde{\sigma}$ reaches the yield strength $\tilde{\sigma}_0$. We use the von Mises criterion in the form

$$\tilde{f} = \frac{1}{2}\widetilde{\tau}_{KL}\widetilde{\tau}_{KL} - K^2 = 0, \tag{7.76}$$

which in strain space becomes

$$g_* = 2\mu^2\gamma_{*KL}\gamma_{*KL} - K^2, \tag{7.77}$$

where $(7.74)_1$, (5.5), (5.4) has been used. At initial yield, $\tilde{f} = g_* = 0$. Thus, using (7.76), (7.62), (7.77), (7.71), and $(7.47)_1$,

$$\widetilde{\sigma}^2 = 3K^2 = \widetilde{\sigma}_0, \qquad \widetilde{\tau}_{KL} = 2\mu\gamma_{*KL}. \tag{7.78}$$

During loading, $(\tilde{f} = g_* = 0,\ \hat{g} > 0)$,

$$\widetilde{\sigma}(t) = \widetilde{\sigma}_0 = const., \qquad \widetilde{\tau}_{KL}(t) = 2\mu\gamma_{*KL}(t) = const, \qquad (t \geq t_0). \tag{7.79}$$

Thus, the rotated Cauchy stress, and therefore the strain tensor $\mathbf{E}_*$ are both constant tensors during loading. Specifically, (7.61), (7.62), (7.72), and (7.73) are all constant during loading. Differentiating $(7.72)_{1,2}$ results in two differential equations:

$$\frac{\dot{\lambda}_1}{\lambda_1} = \frac{\dot{\lambda}_p}{\lambda_p}, \qquad -2\frac{\dot{\lambda}_2}{\lambda_2} = \frac{\dot{\lambda}_p}{\lambda_p}. \tag{7.80}$$

Therefore,

$$\frac{\dot{\lambda}_1}{\lambda_1} = -2\frac{\dot{\lambda}_2}{\lambda_2}. \tag{7.81}$$

From (7.76) and (7.77), we obtain

$$\begin{aligned} \widetilde{\tau}_{KL}\widetilde{\tau}_{KL} = 2K^2, \qquad \frac{\partial \tilde{f}}{\partial \widetilde{T}_{MN}} = \widetilde{\tau}_{MN}, \\ \frac{\partial g_*}{\partial E_{*MN}} = \frac{\partial g_*}{\partial \gamma_{*MN}} = 4\mu^2 \gamma_{*MN} = 2\mu \widetilde{\tau}_{MN}. \end{aligned} \tag{7.82}$$

Recall from (3.10), and using (5.102) and $(6.30)_2$, that

$$\begin{aligned} \hat{g} &= \mathbf{\Gamma}(\mathbf{E}_*, \mathscr{U}) \cdot \mathbf{D} \\ &= \mathbf{F}_* \frac{\partial g_*}{\partial \mathbf{E}_*} \mathbf{F}_*^T \cdot \mathbf{D} \end{aligned} \tag{7.83}$$

In our example of uniaxial tension, $\mathbf{F} = \mathbf{U}$, $\mathbf{F}_* = \mathbf{U}_*$ and $\mathbf{L} = \mathbf{D} = \dot{\mathbf{U}}\mathbf{U}^{-1}$. Thus,

$$\begin{aligned} \hat{g} &= \mathbf{U}_* \frac{\partial g_*}{\partial \boldsymbol{\gamma}_*} \mathbf{U}_* \cdot \dot{\mathbf{U}}\mathbf{U}^{-1} \\ &= 2\mu \; \frac{2}{3}\bar{\sigma} \left(\lambda_{*1}^2 \frac{\dot{\lambda}_1}{\lambda_1} - \lambda_{*2}^2 \frac{\dot{\lambda}_2}{\lambda_2} \right) \\ &= 2\mu \; \frac{2}{3}\bar{\sigma} \frac{\dot{\lambda}_1}{\lambda_1} \left(\lambda_{*1}^2 + \frac{1}{2}\lambda_{*2}^2 \right). \end{aligned} \tag{7.84}$$

Recall the constitutive equation

$$\dot{E}_{pKL} = \pi \hat{g} \rho_{KL}. \tag{7.85}$$

By the normality condition,

$$\rho_{KL} = \frac{\partial \tilde{f}}{\partial \widetilde{\tau}_{KL}} = \widetilde{\tau}_{KL}. \tag{7.86}$$

Ensuring an isochoric plastic deformation, we take $\det \mathbf{F}_p = 1$. The stretch tensor $\mathbf{U}_p$ is given by

$$(U_{pKL}) = \begin{pmatrix} \lambda_p & 0 & 0 \\ 0 & 1/\sqrt{\lambda_p} & 0 \\ 0 & 0 & 1/\sqrt{\lambda_p} \end{pmatrix}, \tag{7.87}$$

where λ_p is the plastic stretch. The plastic strain is

$$(E_{pKL}) = \frac{1}{2} \begin{pmatrix} \lambda_p^2 - 1 & 0 & 0 \\ 0 & \frac{1}{\lambda_p} - 1 & 0 \\ 0 & 0 & \frac{1}{\lambda_p} - 1 \end{pmatrix}. \tag{7.88}$$

Since there is no plastic volume change, the spherical part of $\mathbf{E}_p$ retains its zero value at initial yield. Therefore,

$$\gamma_{p11} = E_{p11} = \frac{1}{2}\left(\lambda_p^2 - 1\right). \tag{7.89}$$

Also, the flow rule (7.85) becomes

$$\dot{\gamma}_{pKL} = \pi \hat{g} \rho_{KL}. \tag{7.90}$$

Differentiating (7.89) gives

$$\dot{\gamma}_{p11} = \lambda_p \dot{\lambda}_p. \tag{7.91}$$

Using (7.91), (7.84), and (7.86), the 11-component of the flow rule (7.90) gives a differential equation for λ_p:

$$\lambda_p \dot{\lambda}_p = \pi \left[2\mu \, \frac{2}{3}\widetilde{\sigma} \left(\lambda_{*1}^2 \frac{\dot{\lambda}_1}{\lambda_1} - \lambda_{*2}^2 \frac{\dot{\lambda}_2}{\lambda_2} \right) \right] \frac{2}{3}\widetilde{\sigma}. \tag{7.92}$$

Using $(7.73)_{1,3}$ we can write the strain space yield criterion as

$$g_* = 3\mu^2 \gamma_{*11}^2 - K^2, \tag{7.93}$$

which can be converted into a yield function of the stretches as

$$g(\lambda_1, \lambda_2, \lambda_p) = \frac{1}{3}\mu^2 \left(\frac{\lambda_1^2}{\lambda_p^2} - \lambda_2^2 \lambda_p \right)^2 - K^2. \tag{7.94}$$

The partial derivatives are

$$\begin{gathered} \frac{\partial g}{\partial \lambda_1} = \frac{2}{3}\mu \left(\frac{\lambda_1^2}{\lambda_p^2} - \lambda_2^2 \lambda_p \right) \frac{2\lambda_1}{\lambda_p^2}, \qquad \frac{\partial g}{\partial \lambda_2} = -\frac{\lambda_2 \lambda_p^3}{\lambda_1} \frac{\partial g}{\partial \lambda_1}, \\ \frac{\partial g}{\partial \lambda_p} = -\frac{\lambda_p^2}{2\lambda_1} \frac{\partial g}{\partial \lambda_1} \left(\frac{2\lambda_1}{\lambda_p^3} - \lambda_2^2 \right). \end{gathered} \tag{7.95}$$

During loading ($g = 0, \ \hat{g} > 0$), the consistency condition applies:

$$\dot{g} = 0 = \frac{\partial g}{\partial \lambda_1}\dot{\lambda}_1 + \frac{\partial g}{\partial \lambda_2}\dot{\lambda}_2 + \frac{\partial g}{\partial \lambda_p}\dot{\lambda}_p. \tag{7.96}$$

Using $(7.95)_{1,2,3}$, (7.81), and (7.92), and the consistency condition, we solve for the multiplier π:

$$\pi = \frac{\lambda_p^2}{\left[2\mu \frac{4}{9}\widetilde{\sigma}^2 \left(\lambda_{*1}^2 + \frac{1}{2}\lambda_{*2}^2 \right) \right]}. \tag{7.97}$$

Substituting back into the flow rule (7.92) results in

$$\frac{\dot{\lambda}_p}{\lambda_p} = \frac{\dot{\lambda}_1}{\lambda_1}. \tag{7.98}$$

Using the initial condition ($\lambda_p(t_0) = 1$), we have an equation for $\lambda_p(t)$:

$$\lambda_p(t) = \frac{\lambda_1(t)}{\lambda_1(t_0)}, \tag{7.99}$$

which allows us to solve for the plastic strain $E_{p11}(t)$. It also means that

$$\frac{E_{11} - E_{p11}}{\lambda_1^2} = \frac{1}{2}\left(1 - \frac{1}{\lambda_1^2(t_0)}\right) = const. \tag{7.100}$$

In other words, $\mathbf{E} - \mathbf{E}_p$ is not constant. Further, the solution to the differential equation (7.81), namely

$$\lambda_1(t)\lambda_2^2(t) = \lambda_1(t_0)\lambda_2^2(t_0), \tag{7.101}$$

allows us to solve for the transverse stretch $\lambda_2(t)$ and tells us that $\det \mathbf{U}$ is constant during loading, retaining its value at initial yield.

For a steel bar, we use the elastic constants

$$E = 200 \text{ GPa}, \qquad \nu = 0.32, \qquad \widetilde{\sigma}_0 = 260 \text{ GPa}, \tag{7.102}$$

and calculate the shear modulus and bulk modulus using

$$\mu = \frac{E}{2(1+\nu)}, \qquad k = \frac{E}{3(1-2\nu)}. \tag{7.103}$$

Let the total axial strain be linear in time:

$$E_{11}(t) = 0.01t. \tag{7.104}$$

Using MATLAB, a tension test and a compression test are run for $t = 20$ seconds each. Initial yield occurs at $t_0 = 0.13$ seconds, when $E_{11} = 0.013$. During loading, $\widetilde{T}_{11}$ and E_{*11} remain constant, retaining the values they had at initial yield.

The results are shown in Figure 7.1 to Figure 7.8. The elastic region is nearly identical in both theories. During loading, S_{11} decreases. For small plastic strains, since the stretches are close to 1, the two stress measures are close to each other. That is, for strains of the order of the elastic strain, the stresses are essentially the same. The Cauchy stress remains within 10% of the yield strength up to a strain of $E_{11} = 0.0364$. S_{11} further decreases until reaching a value of $S_{11}(20) = 157$ GPa when $E_{p11} = 0.1982$. Therefore, the Green-Naghdi theory agrees with the new theory for small $\mathbf{E}_p$ and for small $\mathbf{E}_*$, but they are in disagreement when plastic strains are large. As can be seen from Figure 7.8, the Eulerian theory allows for the plastic deformation to be large, while keeping the elastic deformation small.

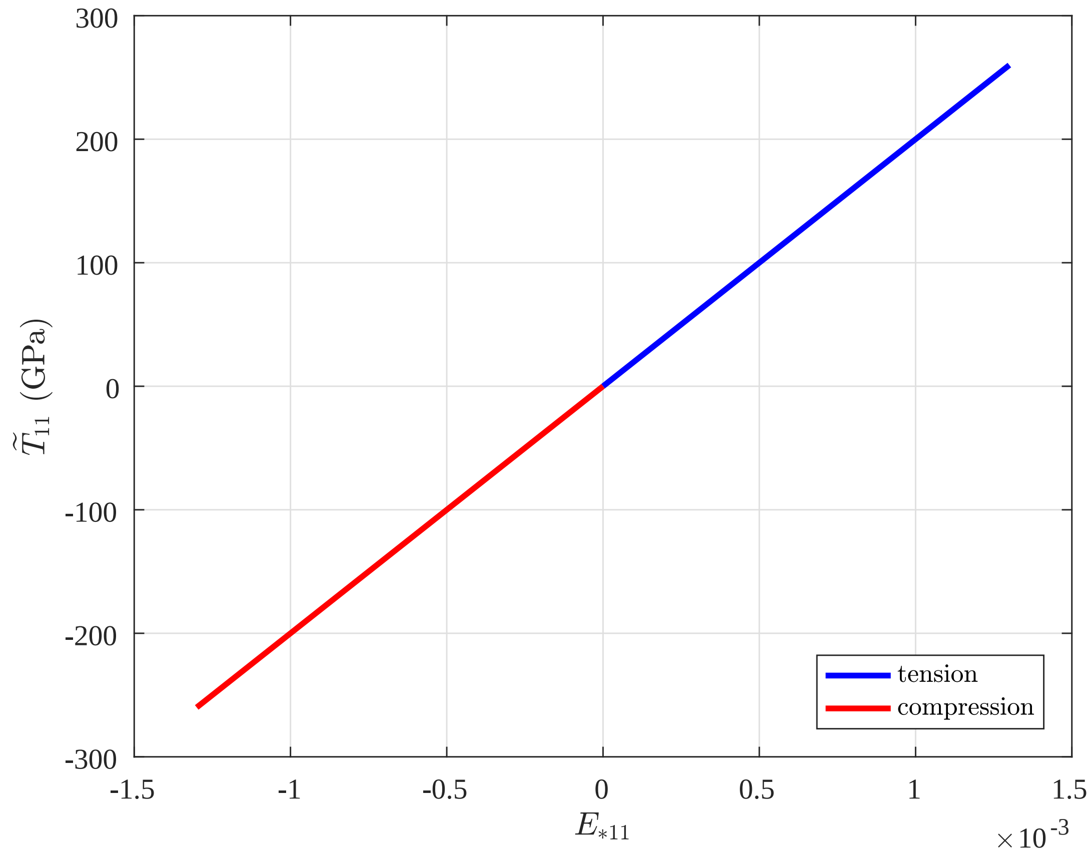

Figure 7.1: Stress and strain for the uniaxial stress test. The elastic part for both tension and compression is linear. Initial yield occurs at $\widetilde{T}_{11} = 260$ GPa and $E_{*11} = 0.0013$. Both $\widetilde{T}_{11}$ and E_{*11} become constant during loading, retaining their values at initial yield.

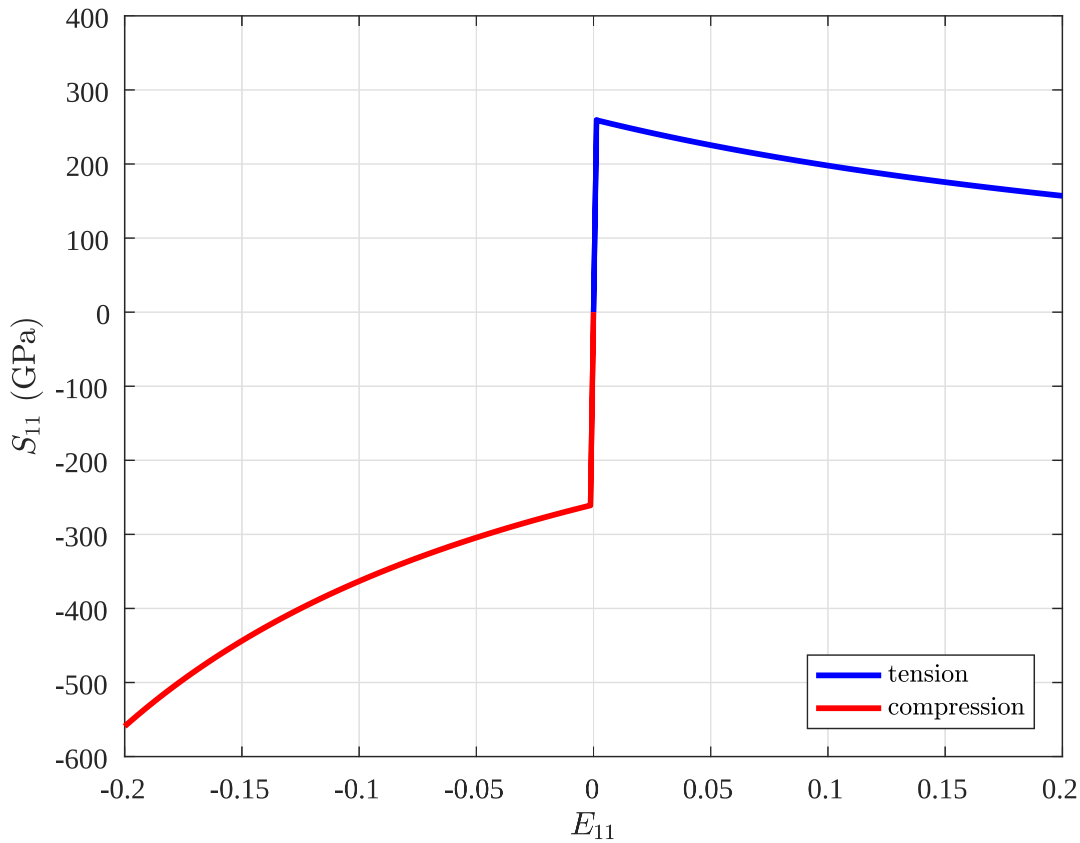

Figure 7.2: The resulting Piola-Kirchoff stress and total strain. The elastic part is linear. During loading, S_{11} decreases in both tension and compression.

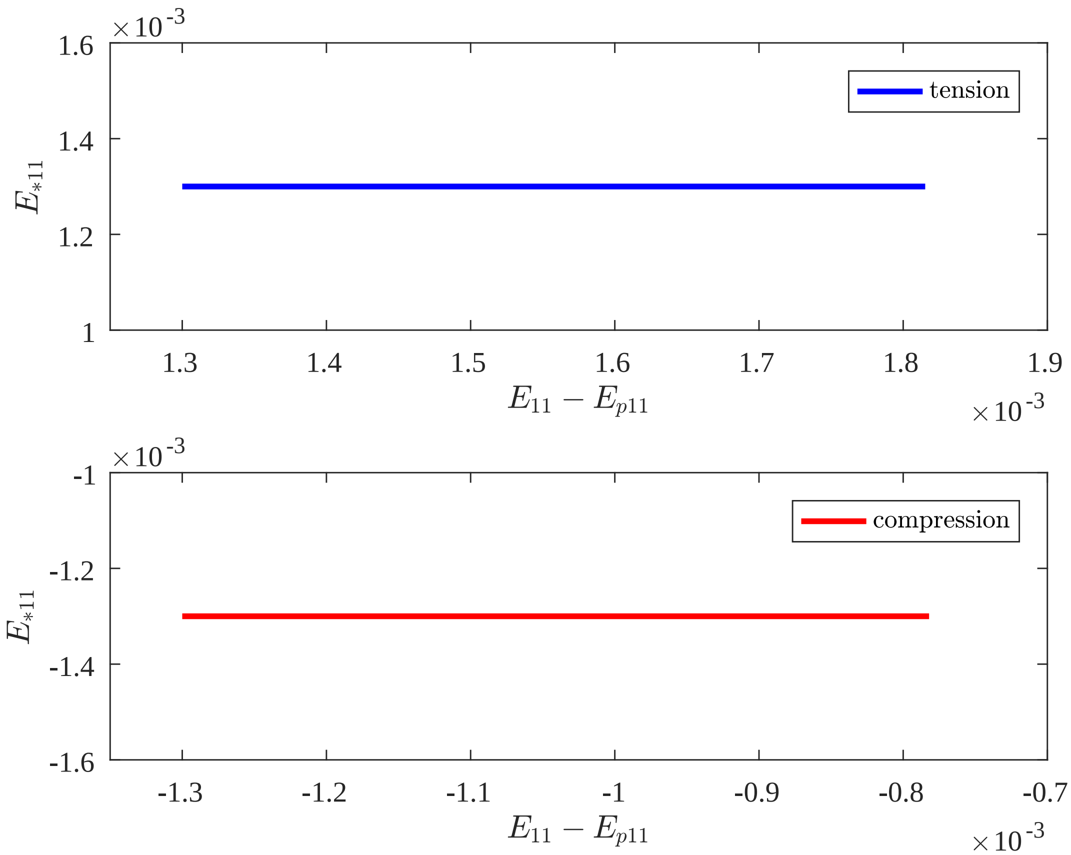

Figure 7.3: Comparison of strain measures. E_{*11} remains constant while the strain difference $E_{11} - E_{p11}$ increases with time both in tension and compression.

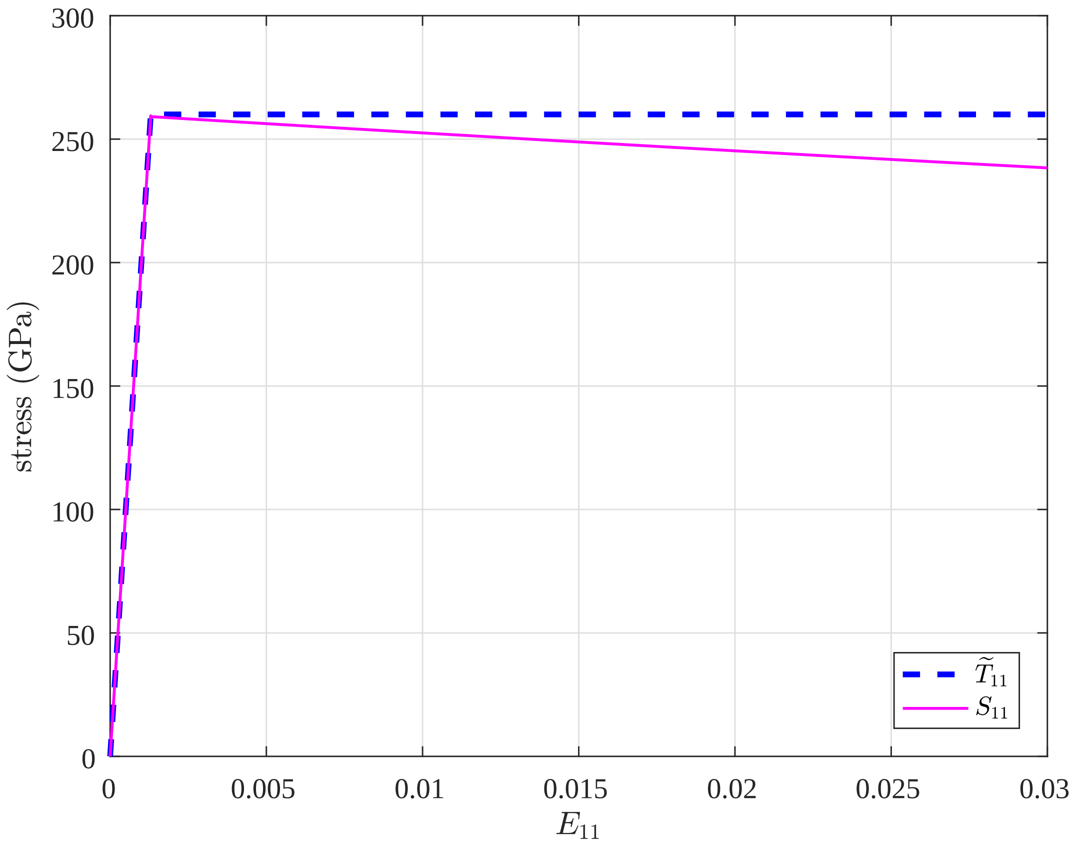

Figure 7.4: A comparison of stress measures in tension for small E_{11}. For strains of the order of the elastic strain, the stresses are essentially the same. The Cauchy stress remains within 10% of the yield strength up to a strain of $E_{11} = 0.0364$.

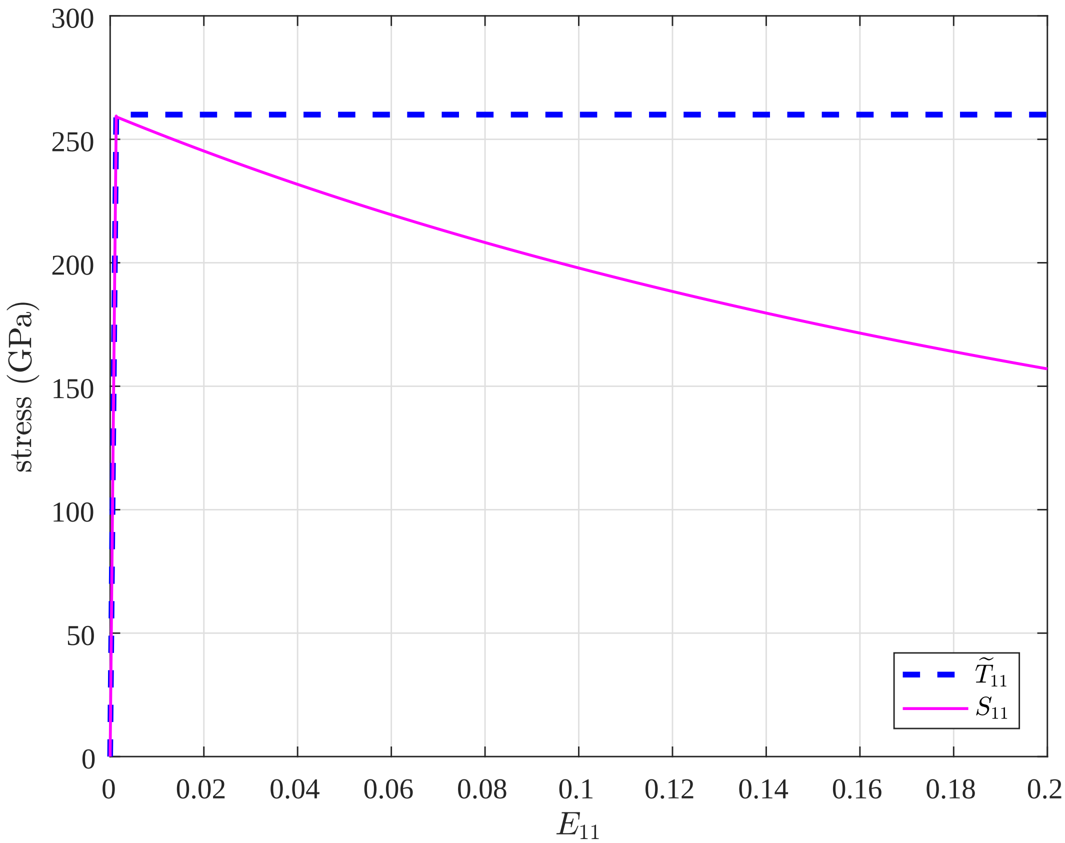

Figure 7.5: Comparison of stresses in tension for large strains. The difference between S_{11} and $\widetilde{T}_{11}$ becomes significant at large strains.

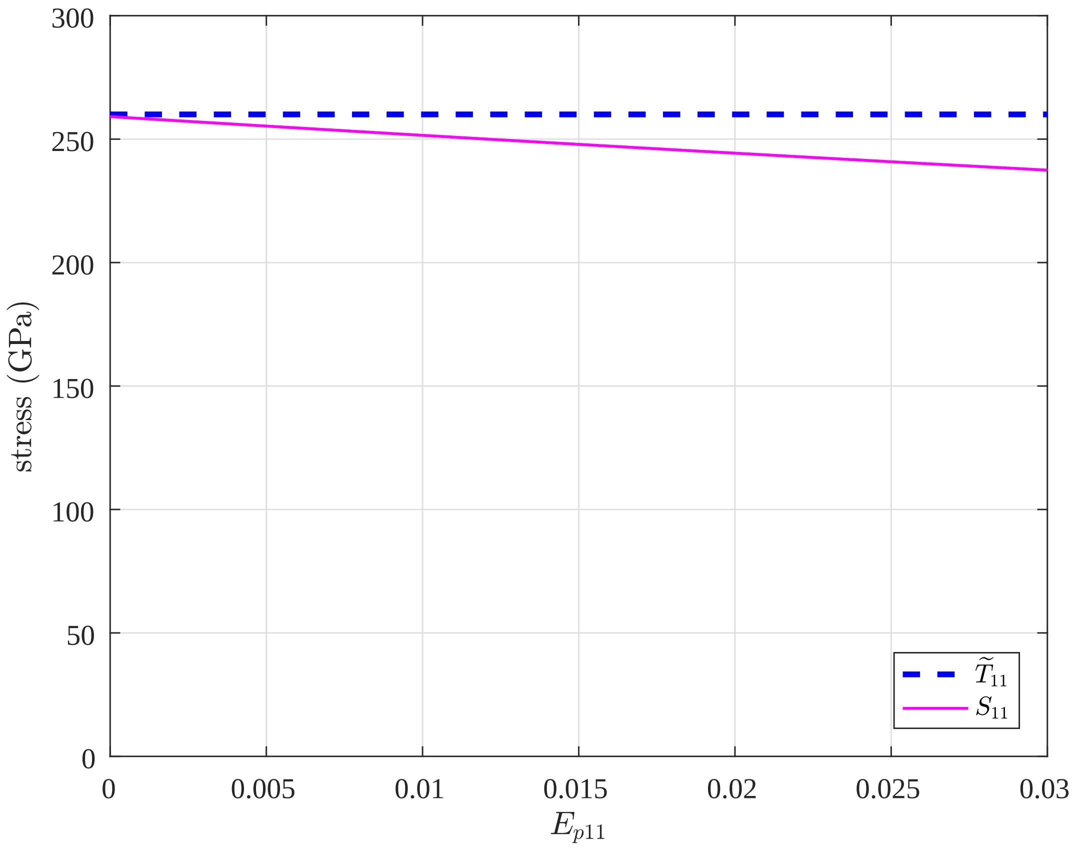

Figure 7.6: Comparison of stresses in tension for small plastic strain. For plastic strains of the order of the elastic strain, the stresses are essentially the same. The Cauchy stress remains within 10% of the yield strength up to a strain of $E_{11} = 0.035$.

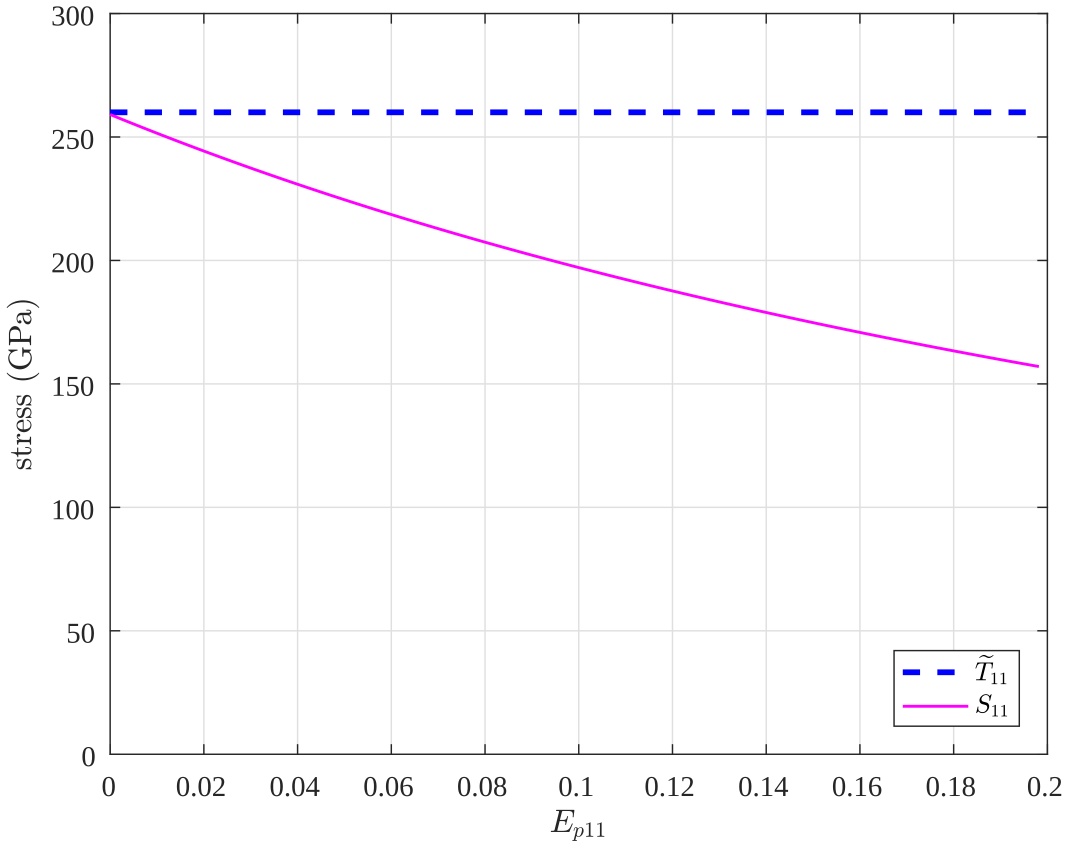

Figure 7.7: Comparison of stresses in tension for large plastic strains. The difference between S_{11} and $\widetilde{T}_{11}$ becomes significant at large plastic strains. For $E_{p11} = 0.01982$, the stress S_{11} has decreased to 157 GPa.

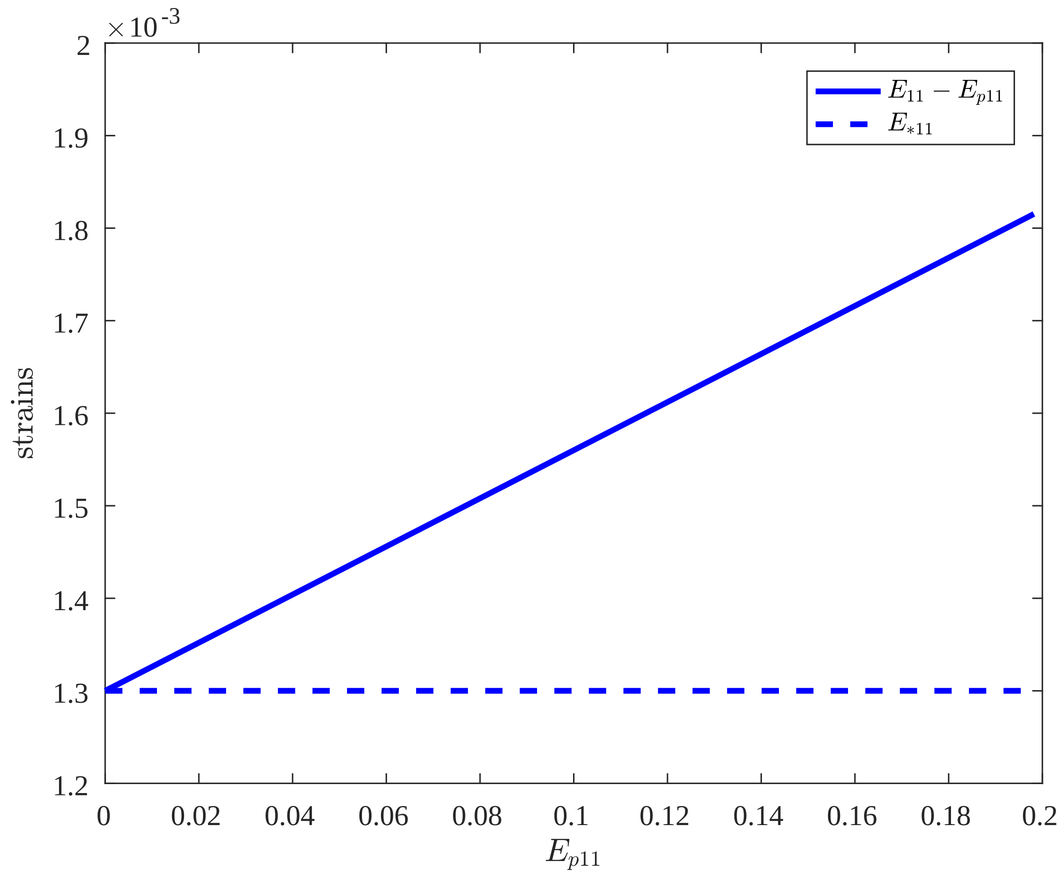

Figure 7.8: Comparison of strains in tension for large plastic strains. The elastic strain E_{*11} remains constant as E_{p11} increases while the strain difference $E_{11} - E_{p11}$ increases with E_{p11}.

7.5.2 Uniaxial tension and compression using the Green-Naghdi theory

For comparison, we also applied the Lagrangian theory to the uniaxial stress example. Again, consider a homogeneous deformation of a steel bar under uniaxial stress. The Piola-Kirchhoff stress tensor is given by

$$(S_{KL}) = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7.105}$$

The rotated Cauchy stress can be calculated from the Piola-Kirchhoff stress using (2.36). Thus,

$$\widetilde{T}_{11} = \frac{\lambda_1}{\lambda_2^2}\,\sigma, \qquad \widetilde{\tau}_{11} = \frac{2}{3}\frac{\lambda_1}{\lambda_2^2}\,\sigma. \tag{7.106}$$

Decomposing S_{KL} into its spherical and deviatoric parts, we have

$$\bar{S} = \frac{1}{3}\,\sigma > 0, \qquad \tau_{KL} = \frac{1}{3}\,\sigma c_{KL}, \tag{7.107}$$

where c_{KL} are the components of the constant tensor $\mathbf{c}$, given in (7.63). The stretch and strain tensors, $\mathbf{U}$ and $\mathbf{E}$, are

$$(U_{KL}) = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \qquad (E_{KL}) = \frac{1}{2}\begin{pmatrix} \lambda_1^2 - 1 & 0 & 0 \\ 0 & \lambda_2^2 - 1 & 0 \\ 0 & 0 & \lambda_2^2 - 1 \end{pmatrix}. \tag{7.108}$$

The deviatoric part of $\mathbf{E}$ is

$$\gamma_{KL} = \frac{1}{6}\left(\lambda_1^2 - \lambda_2^2\right) c_{KL}, \tag{7.109}$$

and its rate is

$$\dot{\gamma}_{KL} = \frac{1}{3}\left(\lambda_1\dot{\lambda}_1 - \lambda_2\dot{\lambda}_2\right) c_{KL}. \tag{7.110}$$

In the elastic region, the stress response is given by Hooke's Law, which furnishes

$$\tau_{KL} = 2\mu\gamma_{KL}, \qquad \bar{S} = 3k\bar{E}. \tag{7.111}$$

Analogous equations from linear elasticity are given in (7.75). We assume that initial yield occurs at time $t = t_0$ when the tensile stress σ reaches the yield strength σ_0. We use the von Mises yield function given in (7.12) and (7.13). At yield, $f = g = 0$ and

$$\tau_{KL}\tau_{KL} = \frac{2}{3}\sigma^2 = 2K^2. \tag{7.112}$$

During loading ($g = 0$, $\hat{g} > 0$), we assume the flow rule (7.17). The 11-component of $\dot{\boldsymbol{\gamma}}_p$ is

$$\begin{aligned} \dot{\gamma}_{p11} &= \frac{\tau_{KL}\dot{\gamma}_{KL}}{2K^2}\tau_{11} \\ &= \frac{2}{9}\frac{\sigma^2}{K^2}\left(\lambda_1\dot{\lambda}_1 - \lambda_2\dot{\lambda}_2\right) \\ &= \frac{2}{3}\left(\lambda_1\dot{\lambda}_1 - \lambda_2\dot{\lambda}_2\right), \end{aligned} \tag{7.113}$$

From (7.110) and $(7.113)_3$, we note that

$$\dot{\gamma}_{p11} = \dot{\gamma}_{11}. \tag{7.114}$$

From the Prandtl-Reuss equations (7.18), this means that τ_{11} is constant during loading. Integrating $\dot{\gamma}_{p11}$ and using the initial condition $\gamma_{p11} = 0$ at yield, we obtain

$$\gamma_{p11} = \frac{1}{3}\left[\left(\lambda_1^2 - \lambda_2^2\right) - \left(\lambda_{10}^2 - \lambda_{20}^2\right)\right], \tag{7.115}$$

where λ_{10} and λ_{20} are the values of λ_1 and λ_2, respectively, at initial yield. The strain difference is therefore

$$\gamma_{11} - \gamma_{p11} = \frac{1}{3}\left(\lambda_{10}^2 - \lambda_{20}^2\right) = \gamma_{11}(t_0), \tag{7.116}$$

which is a constant. Given explicitly as a function of time, we write

$$\gamma_{p11}(t) = \gamma_{11}(t) - \gamma_{11}(t_0). \tag{7.117}$$

From $(7.11)_1$, (7.116), and since the stress remains constant during loading, we also have

$$\tau_{11}(t) = 2\mu\left(\gamma_{11}(t) - \gamma_{p11}(t)\right) = 2\mu\gamma_{11}(t_0). \tag{7.118}$$

Since there is no plastic volume change, $E_{pMM} = 0$, $E_{pKL} = \gamma_{pKL}$, and

$$\gamma_{p11} = E_{p11} = \frac{1}{2}\left(\lambda_p^2 - 1\right). \tag{7.119}$$

Equating (7.119) with (7.115), and using (7.117), results in an equation for the plastic stretch λ_p:

$$\lambda_p^2 = 1 + 2\left[\gamma_{11}(t) - \gamma_{11}(t_0)\right]. \tag{7.120}$$

Further, $\bar{S}$ retains its constant value at initial yield:

$$\bar{S} = 3k\bar{E} = k\left(E_{11}(t_0) + 2E_{22}(t_0)\right). \tag{7.121}$$

We therefore have an equation relating the axial and transverse strains:

$$E_{22}(t) = \frac{1}{2}\left(3\bar{E} - E_{11}(t)\right). \tag{7.122}$$

Using (6.19), (7.88), (7.120), (7.122), and the specified axial strain component E_{11}, we can then calculate the components of the strain $\mathbf{E}_*$, namely

$$E_{*11} = \frac{1}{\lambda_p^2}\left(E_{11} - E_{p11}\right), \qquad E_{*22} = \lambda_p\left(E_{22} - E_{p22}\right). \tag{7.123}$$

The 11-component of $\boldsymbol{\gamma}_*$ is thus

$$\gamma_{*11} = \frac{2}{3}\left(E_{*11} - E_{*22}\right). \tag{7.124}$$

Let the axial strain be linear in time:

$$E_{11}(t) = \dot{E}_{11}\, t, \qquad \dot{E}_{11} = 0.01\ \mathrm{s}^{-1}, \tag{7.125}$$

and run the tension test for $t = 20$ s. The yield strength σ_0 is reached at $t_0 = 0.13$ seconds, at which point, the flow rule (7.113) goes into effect. Thus, during loading, S_{11} remains constant, retaining the value it had at initial yield. A uniaxial compression test is also run for $t = 20$ s. Initial yield also occurs at $t_0 = 0.13$ seconds, and S_{11} remains constant afterwards.

The result of the calculations for the simple tension and compression tests are shown in Figure 7.9 to Figure 7.15. In the elastic region, the stretches λ_1 and λ_2 remain close to 1. Thus the difference between $\widetilde{T}_{11}$ and S_{11} in the elastic region remains small (with a difference of less than 0.1%). During loading, while S_{11} is constant, $\widetilde{T}_{11}$ increases. For plastic strains on the order of the elastic strain, the stress measures are essentially the same. $\widetilde{T}_{11}$ stays within 10% of S_{11} up to $E_{11} = 0.03$. It further increases until it reaches a value of $\widetilde{T}_{11} = 384$ GPa at $t = 20$ s. For both tension and compression, the difference $E_{11} - E_{p11}$ remains constant, while E_{*11} decreases, but not by a significant amount.

We can therefore see that when both $\mathbf{E}_*$ and $\mathbf{E}_p$ are small, the Green-Naghdi theory agrees with the new theory. For large plastic strains, the two theories do not agree. In this case, while the Prandtl-Reuss equations are satisfied, the equation $\widetilde{\boldsymbol{\tau}} = 2\mu\boldsymbol{\gamma}_*$ is not. Also, as can be seen from Figure 7.15, for large plastic strains, the Green-Naghdi theory allows for the elastic measure E_{*11} to decrease. That is, for the case of small elastic deformations accompanied by large plastic deformations, the Green-Naghdi theory does not furnish a realistic linearization.

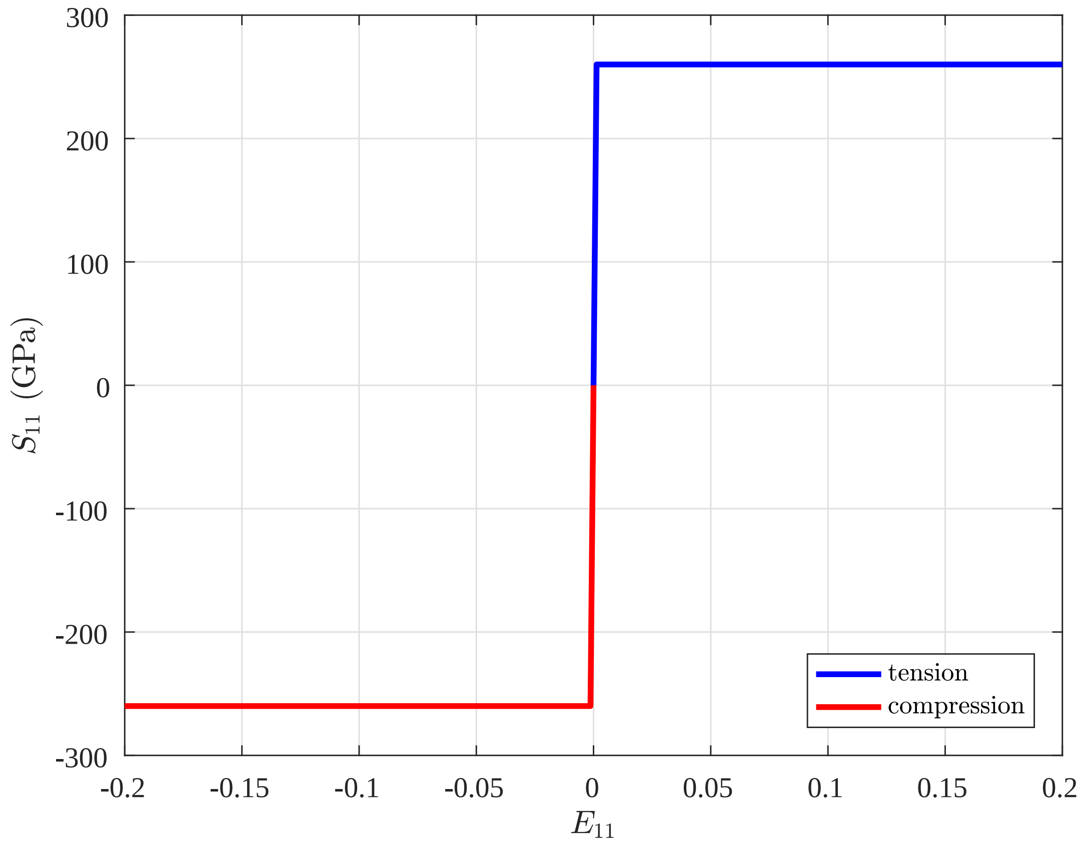

Figure 7.9: Stress and strain for the uniaxial stress test. The elastic part is linear and small. In tension, initial yield occurs at $S_{11} = 260$ GPa, which corresponds to $E_{11} = 0.0013$. In compression, initial yield occurs at $S_{11} = -260$ GPa, when $E_{11} = -0.0013$. From the flow rule and thus the Prandtl-Reuss equations, S_{11} is constant during loading.

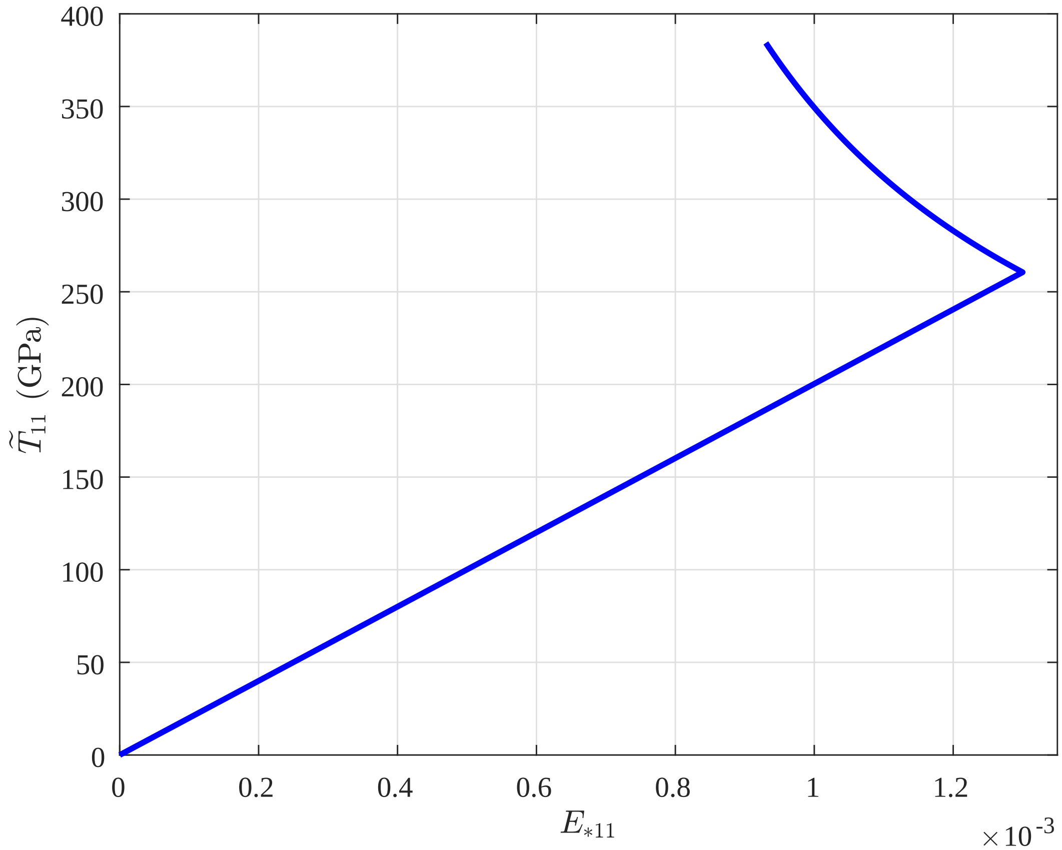

Figure 7.10: The resulting rotated Cauchy stress $\widetilde{T}_{11}$ and strain E_{*11} for the uniaxial tension test during loading. The elastic part is linear and is the same as in the elastic part in Figure 7.9. Both the stress $\widetilde{T}_{11}$ and strain E_{*11} are not constant during loading. Stress increases and strain decreases with time.

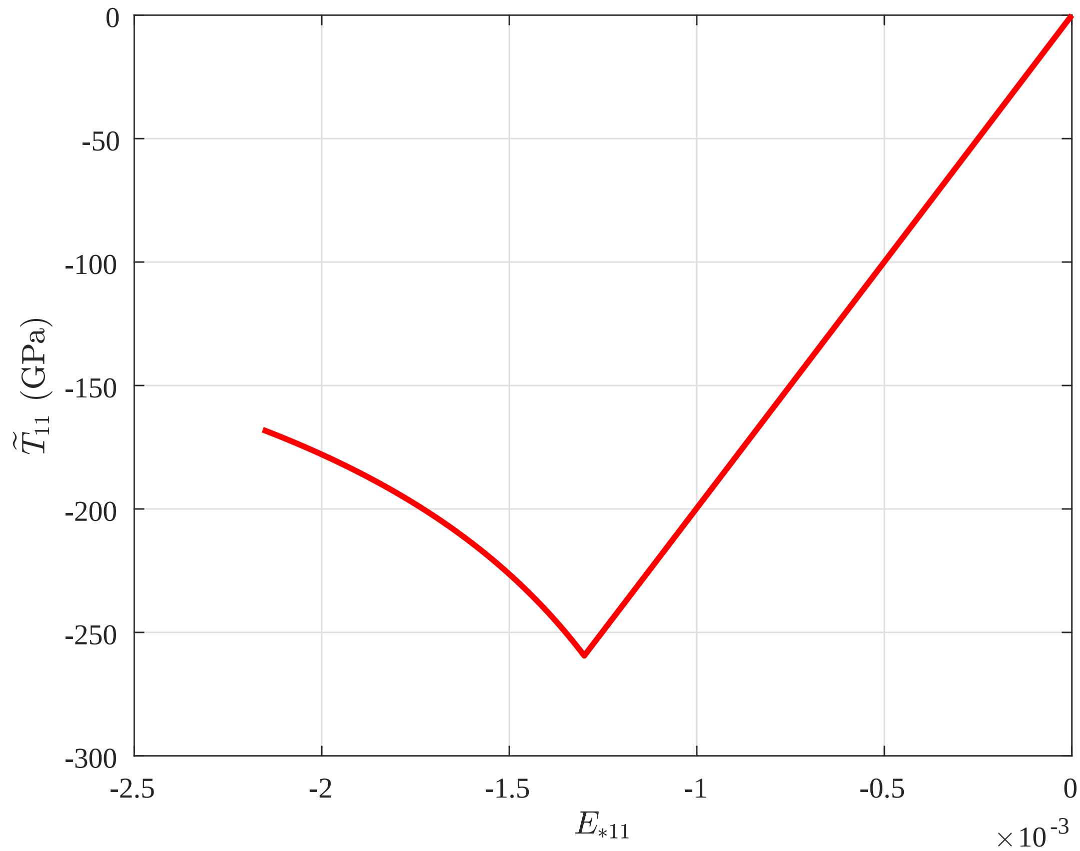

Figure 7.11: The rotated Cauchy stress $\widetilde{T}_{11}$ and strain E_{*11} for the uniaxial compression test during loading. Both the stress $\widetilde{T}_{11}$ and strain E_{*11} are not constant during loading. The strain decreases with time. The elastic part is linear and is the same as in the elastic part in compression in Figure 7.9.

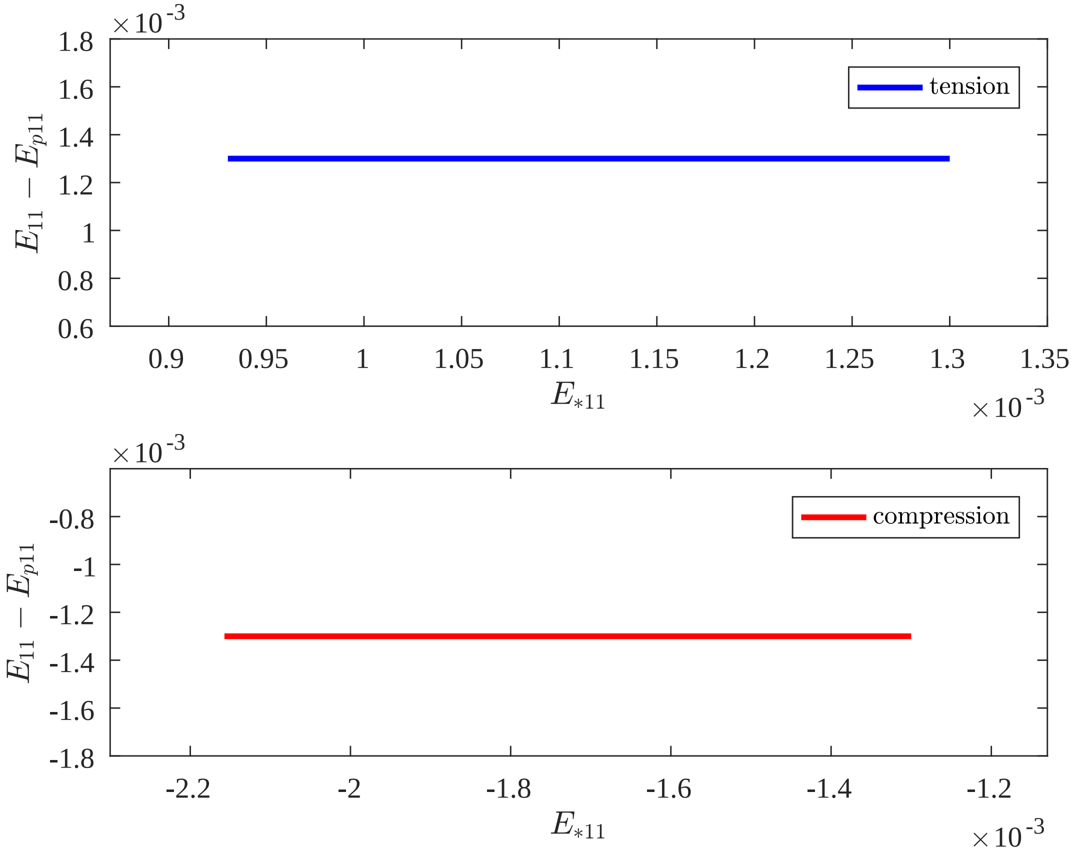

Figure 7.12: The strain difference $E_{11} - E_{p11}$ remain constant for both tension and compression tests. In the tension test, the strain E_{*11} decreases with time, but not by a significant amount, going from $E_{*11}(t_0) = 0.0013$ to $E_{*11}(20) = 0.00093$. In compression, E_{*11} decreases, becoming more negative.

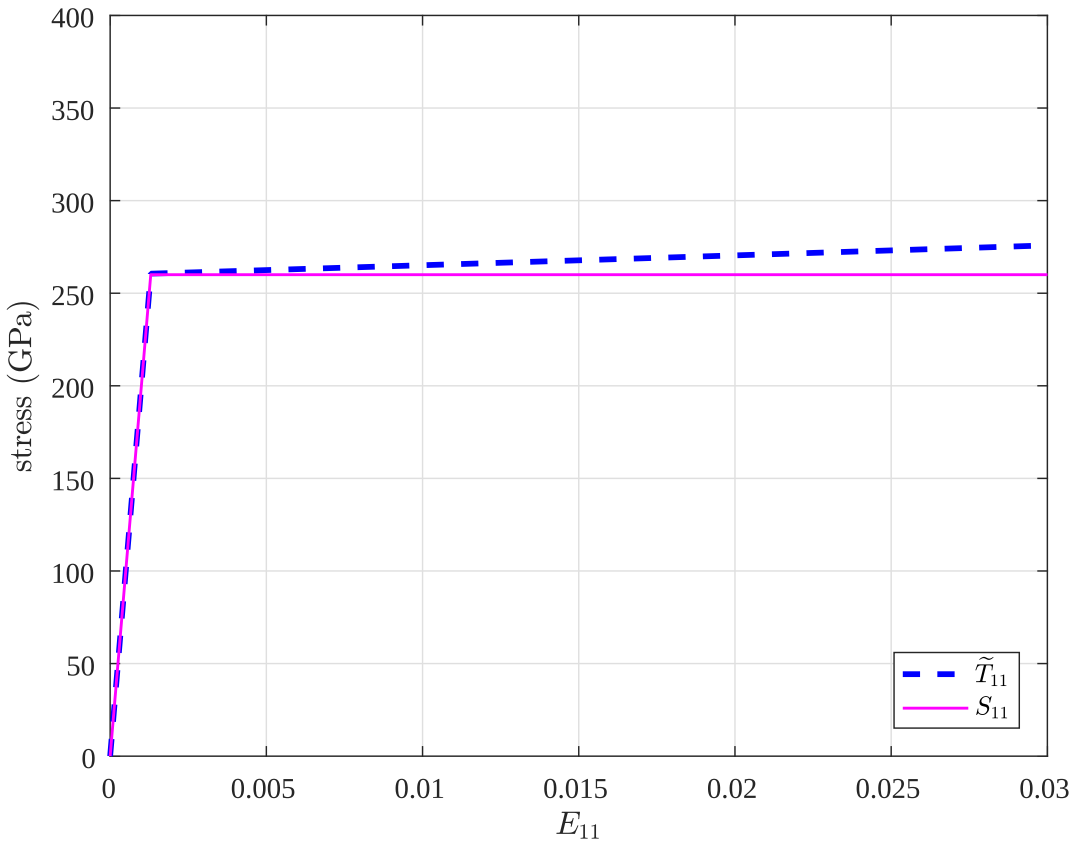

Figure 7.13: A comparison of stress measures in tension for small E_{11}. Since the difference $E_{11} - E_{p11}$ is constant. This plot also corresponds to small E_{p11}. For plastic strains on the order of the elastic strain, the stresses are essentially the same. The Cauchy stress remains within 10% of the yield strength up to a strain of $E_{11} = 0.03$. A similar behavior, with negative values, occurs for the compression test.

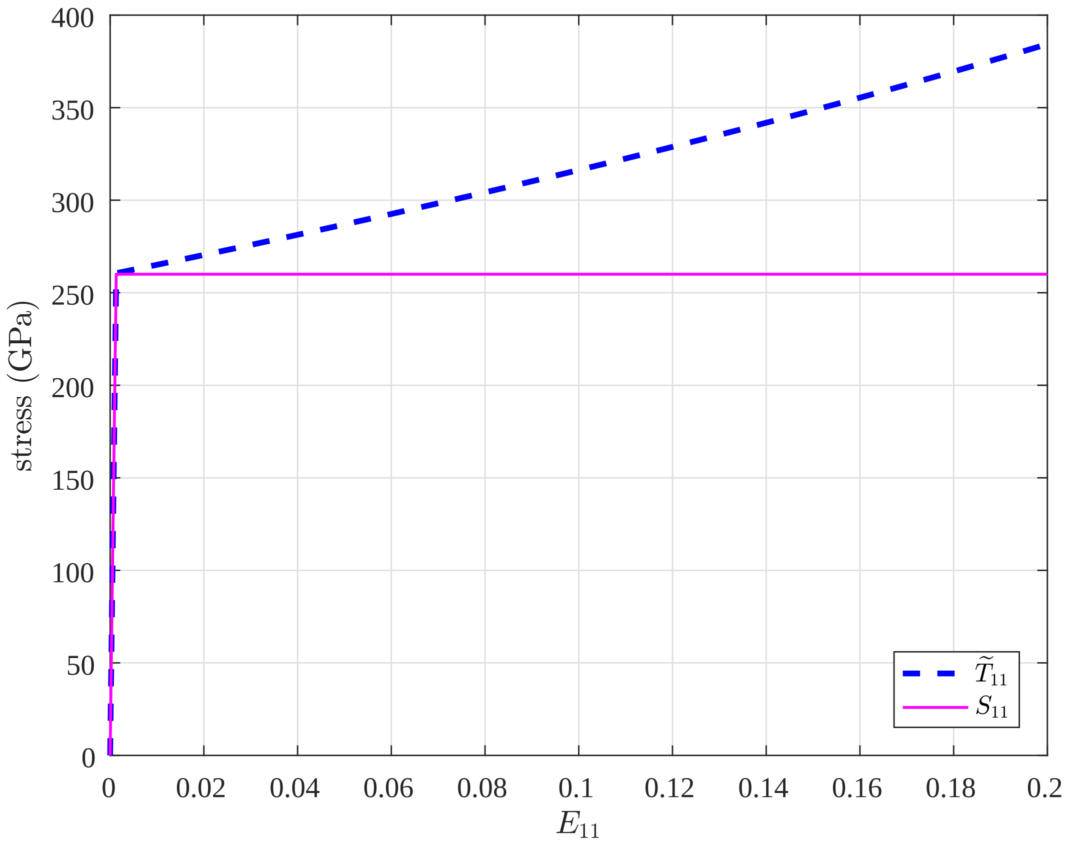

Figure 7.14: A comparison of stress measures in tension for large strains. The Cauchy stress increases until it reaches a value of $\widetilde{T}_{11} = 384$ GPa at $t = 20$ s. Clearly, the two theories do not agree for large plastic strains. A similar behavior occurs for the compression test, with $\widetilde{T}_{11}$ tending towards zero, while S_{11} remains constant.

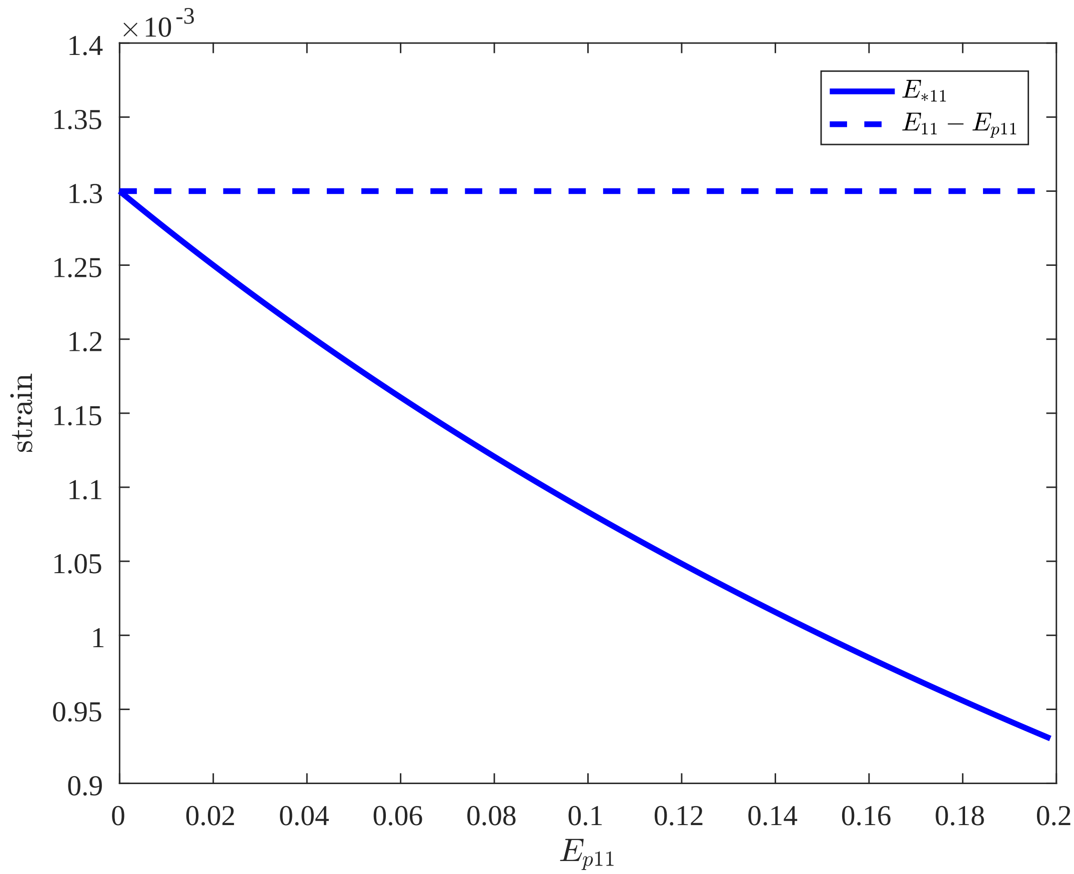

Figure 7.15: A comparison of strain measures in tension for large plastic strains. The strain difference $E_{11} - E_{p11}$ remains constant, and the elastic strain E_{*11} decreases as E_{p11} increases.

Chapter 8

Summary and Conclusions

Starting with the Green-Naghdi Lagrangian theory of finite plasticity, and using a novel multiplicative decomposition of the deformation gradient, a new Eulerian form of finite plasticity has been constructed. The combination of strain space formulation in the Lagrangian theory and the decomposition of $\mathbf{F}$ in a three-factor form ($\mathbf{F} = \mathbf{R}_*\mathbf{U}_*\mathbf{U}_p$) leads to this powerful Eulerian theory. The decomposition is presented in Section 4.3 and again illustrated below.

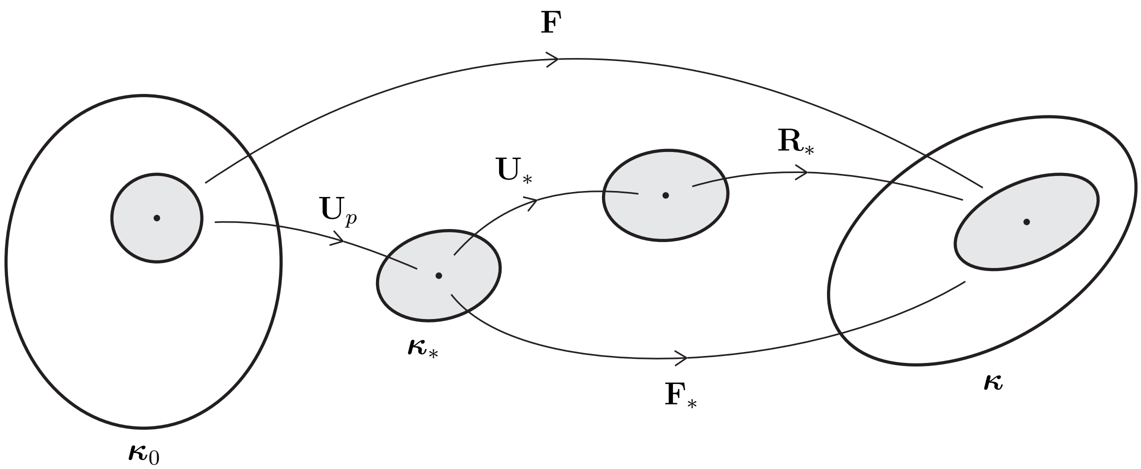

Figure 8.1: The decomposition $\mathbf{F} = \mathbf{R}_*\mathbf{U}_*\mathbf{U}_p$.

The new Eulerian theory involves a new strain tensor $\mathbf{E}_*$, which is related to the strain difference $\mathbf{E} - \mathbf{E}_p$ by the inverse of the plastic stretch tensor $\mathbf{U}_p$.:

$$\mathbf{E}_* = \mathbf{U}_p^{-1}(\mathbf{E} - \mathbf{E}_p)\mathbf{U}_p^{-1}.$$

This tensor is measured from the unique intermediate configuration $\boldsymbol{\kappa}_*$, which is mapped from the reference configuration by $\mathbf{U}_p$. An appropriate stress measure for this new theory is the rotated Cauchy stress $\widetilde{\mathbf{T}} = \mathbf{R}^T\mathbf{T}\mathbf{R}$, which is expressed as a function of $\mathbf{E}_*$. The constitutive equation involves the objective stress rate $\dot{\widetilde{\mathbf{T}}}$, which can be expressed in terms of the rate of deformation tensor $\mathbf{D}$ or its rotated form $\widetilde{\mathbf{D}}$.

$$\begin{aligned}
\dot{\widetilde{\mathbf{T}}} &= \left(\frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*} + \pi \widetilde{\boldsymbol{\sigma}} \otimes \frac{\partial g_*}{\partial \mathbf{E}_*}\right) \odot (\mathbf{F}_* \otimes \mathbf{F}_*)\,[\mathbf{D}], \\
&= \left(\frac{\partial \bar{\mathbf{T}}}{\partial \mathbf{E}_*} \odot (\mathbf{U}_* \otimes \mathbf{U}_*) + \pi \widetilde{\boldsymbol{\sigma}} \otimes \widetilde{\boldsymbol{\Gamma}}\right)[\widetilde{\mathbf{D}}] \\
&= \boldsymbol{\mathscr{A}}_*(\mathbf{E}_*, \mathscr{U})[\widetilde{\mathbf{D}}].
\end{aligned}$$

The Eulerian theory provides physically reasonable results for small elastic strains and for both small and large plastic strains. For comparison, we have linearized the stress measures $\widetilde{\mathbf{T}}$ and $\mathbf{S}$ about $\mathbf{E}_* = \mathbf{0}$ and $\mathbf{E} - \mathbf{E}_p = \mathbf{0}$, respectively. After linearization, we found that the Green-Naghdi theory agrees with the new theory when both $\mathbf{E}_*$ and $\mathbf{E}_p$ are small. However, when the plastic strains are large, the Green-Naghdi theory and the new theory do not agree. Since the Green-Naghdi theory has a strain measure that represents the difference between total strain and plastic strain, rather than representing elastic strain, it does not furnish a physically realistic linearization for the case of small elastic deformations accompanied by large plastic deformations. The proposed theory is better suited to describe this case as it can be linearized about the intermediate configuration while allowing the plastic deformations to be large.

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