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Essays On The Competitive Commodity Storage Model

by

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Abstract

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This dissertation consists of three essays on the competitive commodity storage model. This model provides a basis for rationalizing many of the observed qualitative features of the behavior of prices of storable commodities. I attempt to make a contribution to this model in three dimensions: empirical (chapter 1), numerical (chapter 2), and theoretical (chapter 3).

In the first chapter, I analyze the ability of the standard commodity storage model to replicate serial correlation in annual prices. Calendar year averages of prices induce spurious smoothing of price spikes, a fact that has been surprisingly overlooked in several empirical studies of the annual commodity storage model for agricultural commodities. I present an application of a maximum likelihood estimator of the storage model for maize prices, correcting for the spurious smoothing. My results, using this data set, imply serious differences in magnitudes of interest. These differences include the location and skewness of the empirical distribution of prices relative to the cutoff price of zero stocks, the likelihood of stockouts, and the fit to data on stocks-to-use ratios.

In the second chapter, I propose an alternative numerical strategy for solving nonlinear rational expectation models with inequality constraints. It addresses three problems observed in the standard solution method: lack of robustness to scaling transformation of the stationary rational expectation function, errors of approximation due to extrapolation within the ergodic set, and interpolation around the kink implied by the inequality constraint. In comparison with the standard solution method, my findings suggest that the numerical strategy I propose is robust to scaling transformation, removes the approximation errors due to extrapolation, and avoids interpolation above the kink.

Finally in the third chapter, I present a critique of a theoretical version of the competitive commodity storage model that assumes a support for the speculative storage that is bounded from below at zero, and above at a exogenous predetermined maximum capacity. By proposing a counter-example, I show that the fixed point iteration operator proposed by Oglend and Kleppe (2017) to solve this version of the model does not converge in general, as they claim.

To my parents
Ernesto and Cleria,
and to my sister Magda,
and my brother Sebastian.

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Chapter 1

Empirical Commodity Storage Model: The Challenge of Matching Data and Theory

1.1 Introduction

The importance of proper empirical estimations of key parameters in agricultural commodity markets is evident in the face of international concerns over price volatility for major food commodities.

The commodity storage model, as originally described by Gustafson (1958) and discussed in for example Scheinkman and Schechtman (1983), Wright and Williams (1982, 1984), Williams and Wright (1991), Deaton and Laroque (1992, 1995, 1996), Carter et al. (2011), and Wright (2011), recognizes the role of storage, and provides a basis for rationalizing many of the observed qualitative features of the behavior of prices of storable commodities.

The discrete time annual storage model assumes that in each year price is formed after the realization of a stochastic harvest, when decisions on how much to store out of the available supply are made. Price series that are appropriate for testing such a model are therefore annual price series. The evidence on the empirical validity of the annual competitive storage model is still mixed. Based on a Pseudo Maximum Likelihood (PML) econometric procedure, Deaton and Laroque (1995, 1996) reject the practical relevance of storage arbitrage in explaining annual prices. They conclude that the price serial correlations implied by the annual models they estimate are significantly lower than those measured on the series of price indices they use. Cafiero et al. (2011a, 2015) present more positive evidence for the role of storage arbitrage. Cafiero et al. (2011a) estimate the storage model using the same data, model specification, and PML econometric approach as Deaton and Laroque (1995, 1996), but using a much finer grid to approximate the equilibrium price function. Based on their econometric estimations, they find that, contrary to Deaton and Laroque's claim, the competitive storage model generates the high degree of price autocorrelation for five of the twelve commodities considered by Deaton and Laroque (1995, 1996), and for seven commodities when a marginal storage cost parameter is added into the model.

The series of real annual prices used in (for example) Deaton and Laroque (1992, 1995, 1996) and Cafiero et al. (2011a, 2011b, 2015) have been formed by taking the simple average of prices over the calendar year, that is, from January through December, with no explicit recognition of the fact that the actual span of the marketing season may not coincide with the calendar year, therefore smoothing the most prominent feature of the price series in the storage model: its price spikes.[1] Cafiero and Wright (2006) discuss this spurious smoothing problem.[2] This data issue is particularly delicate in this literature, in which a main focus of the discussion is in the ability of the storage model to explain observed price correlation.

The question of what annual price to use best dealing with the spurious smoothing of price spikes leaves room for various choices. An annual price data set constructed as the average of daily prices over the marketing year is a candidate. Alternatively, the use of a single month per year (as in Roberts and Schlenker, 2013) avoids the complications of inter-seasonal anticipation of information.

In this chapter I illustrate this issue using the case of US maize prices. (Maize is a major agricultural commodity, considered in the influential papers of Deaton and Laroque 1992, 1995, 1996, and in Cafiero et al. 2011a). In the northern hemisphere, where most of maize production is obtained, harvesting occurs from September through November (FAO, 2006, Table 2, p. 5). I form an index of annual real prices in four different ways: as an average of market days prices over the calendar year, over the marketing year (from September through the next August), over a quarter (a single quarter per year), and over a month (a single month per year). The first order serial correlation of the price series data I use is actually highest when the price index is constructed by averaging the daily prices over the calendar year.

I focus on the price of maize in the US because there is a long series of prices for US maize, consistently referring to the same commercial grade (US No.2). This price series is widely considered as the traditional representative price for maize produced in the United States, and it is also accepted to be the world's most representative price (FAO, 2006, p. 4).

In my estimations I implement the Maximum Likelihood (ML) approach of Cafiero et al. (2015). This estimation procedure allows for the estimation of the structural parameters of the storage model using only price data. Cafiero et al. (2015) show that while their ML estimator imposes no additional assumptions on the model, it has small sample properties significantly superior to those of the PML estimator of Deaton and Laroque (1995, 1996).

[1] The empirical models of Miranda and Glauber (1993), Chambers and Bailey (1996), Osborne (2004), and Roberts and Schlenker (2013) are exceptions. In particular, Osborne (2004) generalizes the standard storage model to incorporate information on future harvests, while allowing for seasonal production, two features important to African and other developing countries. Also, Lowry et al. (1987) present a quarterly model that considers the allocative role of storage both within and between crop years in markets for annually harvested field crop.

[2] The discussion of the challenges involved in matching data and theory for agricultural prices is not new. For prices defined by random chains, Working (1960) noted that the use of averages induces spurious correlation in first differences of agricultural prices.

1.2 The model

In this section I present the model. I model a simple competitive commodity market in which storers are risk neutral, face a constant discount rate $r > 0$, and have no other costs of storage.[3] Supply shocks, ω_t, are i.i.d.. The state variable is the total available supply at time t, defined as $z_t \equiv \omega_t + (1-d)x_{t-1}$, where x_{t-1} is storage at time $t-1$, and $0 \leq d < 1$ is the physical deterioration rate of stocks. Price is formed as $p_t = F(c_t)$, where consumption at time t is given by $c_t \equiv z_t - x_t$. The inverse consumption demand, $F : \mathbb{R} \to \mathbb{R}$, is continuous, strictly decreasing, with $EF(\omega_t) > 0$, where E denotes the expectation taken with respect to the random variable ω_t.[4]

A stationary rational expectations equilibrium (SREE) in this model is a price function f which describes the current price p_t as a function of the state z_t, and which satisfies, for all z_t,

$$p_t = f(z_t) = \max\left\{F(z_t), \left(\frac{1-d}{1+r}\right) E_t f\big(\omega_{t+1} + (1-d)\big[z_t - F^{-1}\big(f(z_t)\big)\big]\big)\right\}, \tag{1.1}$$

where E_t denotes expectation conditional on information at time t.

Since the ω_t's are i.i.d., f is the solution to the following functional equation:

$$f(z) = \max\left\{F(z), \left(\frac{1-d}{1+r}\right) Ef\big(\omega + (1-d)\big[z - F^{-1}\big(f(z)\big)\big]\big)\right\}. \tag{1.2}$$

Existence and uniqueness of the SREE, f, as well as some of its properties are given by the following Theorem:

Theorem 1. *There is a unique stationary rational expectations equilibrium f in the class of continuous non-negative, non-increasing functions. Furthermore, if $p^* \equiv \left(\frac{1-d}{1+r}\right) Ef(\omega)$, then:*

$$\begin{aligned} f(z) &= F(z), \quad \textit{for } z \leq F^{-1}(p^*), \\ f(z) &> F(z), \quad \textit{for } z > F^{-1}(p^*). \end{aligned} \tag{1.3}$$

f is strictly decreasing whenever it is strictly positive. The equilibrium level of inventories, is strictly increasing for $z > F^{-1}(p^*)$.

Proof of the Theorem: Deaton and Laroque (1992), Theorem 1.

[3] Deaton and Laroque (1992, 1995, 1996) assume zero additive physical storage cost, while Cafiero et al. (2011a) provide non-zero (but low) estimates for marginal additive storage cost. I set such cost at zero in my model. Since I am implementing my empirical model using detrended prices with a non-negligible trend, fitting a (limit) stationary storage model with non-zero additive marginal storage cost would imply the restriction that prices and storage costs share the same trend.

[4] This assumption implies that the model admits positive storage for a range of positive prices.

1.3 Econometric procedure

I estimate the model described in section 1.2 assuming a linear inverse demand function, $F(c) = a + bc$, with $b < 0$, and normal harvests. The discount rate r is set at 5%. I follow the approach of Deaton and Laroque (1992, 1995, 1996) in using only price data. I use the Maximum Likelihood (ML) procedure introduced by Cafiero et al. (2015). I now provide a general overview of the estimation procedure; a detailed discussion is available in Cafiero et al. (2015).

Given the SREE function f, for positive prices the model implicitly defines a mapping from harvests ω_t to prices p_t, conditional on the previous price p_{t-1}:

$$\begin{aligned} p_t &= f\left(z_t\right) \\ &= f\left[\omega_t + (1-d)\, x_{t-1}\right] \\ &= f\left[\omega_t + (1-d)\left(z_{t-1} - F^{-1}\left(f\left(z_{t-1}\right)\right)\right)\right] \\ &= f\left[\omega_t + (1-d)\left(f^{-1}\left(p_{t-1}\right) - F^{-1}\left(p_{t-1}\right)\right)\right]. \end{aligned}$$

For a vector of parameters θ and a sample of positive prices $p_t,\ t = 0, 1, \cdots, T$, the likelihood function is:

$$L(\theta|p_0, \cdots, p_T) = \prod_{t=1}^{T} \phi(\omega_t)|J_t| = \prod_{t=1}^{T} \phi[f^{-1}(p_t) - (1-d)(f^{-1}(p_{t-1}) - F^{-1}(p_{t-1}))]|J_t|, \quad (1.4)$$

where ϕ is the density of ω_t, and $J_t = \dfrac{df^{-1}}{dp_t}(p_t)$ is the Jacobian of the mapping $p_t \mapsto \omega_t$.

To identify the parameters, I adopt the procedure of Deaton and Laroque (1995, 1996) and set the mean and standard deviation of the unobserved harvests at 0 and 1, respectively (see Proposition 1 in Deaton and Laroque 1996). The equilibrium price function f is approximated using a cubic spline, f^{sp}.[5] The search for f^{sp} follows an iterative procedure based on (1.2). This requires approximating the expectations with respect to the distribution of the harvests. Assuming that the shocks ω have a normal standard distribution, I use a Gauss-Hermite quadrature formula with 10 nodes $\{\omega_s\}_{s=1}^{10}$ and weights $\{\pi_s\}_{s=1}^{10}$.[6] The n-th iteration is:

$$f^{\text{sp}}_{<n+1>}(z) = \max\left\{F(z), \left(\frac{1-d}{1+r}\right)\sum_{s=1}^{10} f^{\text{sp}}_{<n>}\left(\omega_s + (1-d)\left[z - F^{-1}\left(f^{\text{sp}}_{<n+1>}(z)\right)\right]\right)\cdot \pi_s\right\}. \quad (1.5)$$

The first iteration uses a guess $f^{\text{sp}}_{<1>}$ on the right hand side of (1.5). Conditional on $f^{\text{sp}}_{<1>}$, I compute $f^{\text{sp}}_{<2>}$ on an equally spaced grid of 1,000 points over a range of available supply z

[5] For a discussion of function approximation, see Judd (1998, Chapter 6) and Miranda and Fackler (2002, Chapter 6). For applications to the storage model, see Miranda (1985, 1997) and Gouel (2013).

[6] The nodes and weights are $\omega_s = \{\pm 4.8595, \pm 3.5818, \pm 2.4843, \pm 1.4660, \pm 0.4849\}$ and $\pi_s = \{4.3107 \times 10^{-6}, 7.5807 \times 10^{-4}, 1.9112 \times 10^{-3}, 0.1355, 0.3446\}$, respectively.

from -5 to 45. Iterations continue until the maximum difference between $f^{\text{sp}}_{<n+1>}$ and $f^{\text{sp}}_{<n>}$ evaluated at each grid point is less than the preset tolerance of 10^{-13}, in absolute value.

I first use a grid-search routine to locate a candidate maximum for the log of the likelihood function, and then use a gradient-based constrained maximization algorithm to search for a maximum in the neighborhood of the candidate. To approximate the solution function f and the derivatives needed to calculate J_t I use the Matlab® Spline Toolbox™. To maximize the function (1.4) I first use the Matlab® routine `fminsearch`, to locate a preliminary maximizer, and then the routine `fmincon`, both included in the Optimization Toolbox™. The inner-loop tolerance is fixed at 10^{-13}, while the outer-loop tolerances are fixed at 10^{-4} and 10^{-6} for `fminsearch` and `fmincon`, respectively. A grid of 64 vectors distributed uniformly on the set $\{a = [0.3; 3] \times b = [-7; -0.3] \times d = [0; 0.3]\}$ is fixed as the set of initial conditions for each sample. I checked that my parameter estimates are robust to the use of two alternative algorithms: `fminunc` and `ktrlink` from KNITRO® optimization package on MATLAB®

I impose the constraints $b < 0$ and $d > 0$, by programming the likelihood maximization routine in terms of the set of transformed parameters $\eta \equiv \{\eta_1, \eta_2, \eta_3\}$ where: $\eta_1 = a$, $\eta_2 = \ln(-b)$, and $\eta_3 = \ln(d)$. Having identified a maximum, the asymptotic variance-covariance matrix of the estimated parameters, $\mathbf{W}$, is computed as the inverse of the outer product of score vectors, evaluated at the estimated values $\hat{\eta}$. A consistent estimate of the variance covariance matrix $\mathbf{V}$ of the original parameters is obtained using the delta method, as:

$$\mathbf{V} = \mathbf{DWD}',$$

where $\mathbf{D}$ is a diagonal matrix of the derivatives of the transformation functions:

$$\mathbf{D} = \left\{ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & -e^{\hat{\eta}_2} & 0 \\ 0 & 0 & e^{\hat{\eta}_3} \end{array} \right\}.$$

1.4 Data used in the econometric estimation

I use the series of maize prices obtained from Global Financial Data described as "Corn (US), No. 2, yellow, Chicago Board of Trade" from January 1949 to December 2012. From the daily prices I first form monthly averages, which I divide by the January 1977 - December 1979 average, consistent with the description in Pfaffenzeller et al. (2007), to form a series of nominal monthly price indices. I next deflate the nominal values by dividing them by the corresponding United States Monthly Consumer Price Index reported by the US Bureau of Labor Statistics.

The deflated monthly price index (plotted in Figure 1.1) exhibits a downward trend over the sample period. I detrend the price index assuming a log-linear trend.[7] The resulting

[7]Detrending price series without adjusting the estimator for the trend may lead to an estimation bias. Most papers on the estimation of the storage model do not detrend the price series. Others address the interaction of stocks and prices using detrended prices without adjustment for the bias in the structural

series is plotted in Figure 1.2. In my estimations using quarterly and monthly data I take the months and quarters included in the September-December period. Calendar year averages, marketing year averages and the December prices are plotted in Figure 1.3.[8]

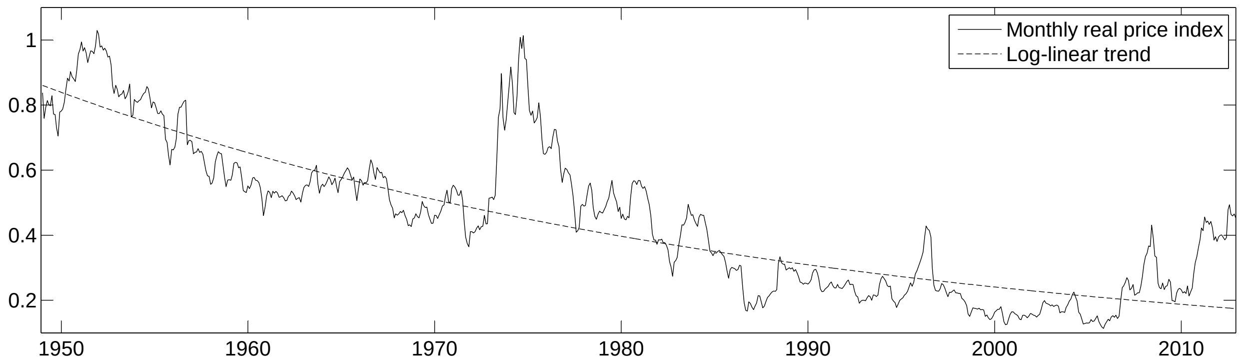

Figure 1.1: Monthly real price index for maize.

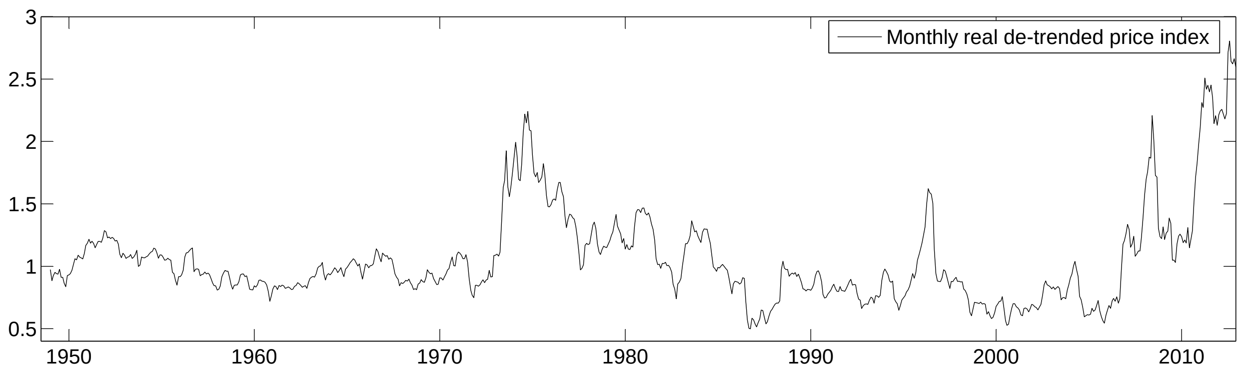

Figure 1.2: Monthly detrended real price index for maize.

1.5 Results

I estimate the annual storage model using eight different annual price indices, formed by averaging prices over the calendar year, the marketing year, quarters and single months.

model (for example Cafiero et al. 2011b, Gospodinov and Ng 2013). To make my work comparable to the literature (excluding Zeng 2012), I do not adjust my structural model of detrended prices.

[8]For calendar year, quarters and months, the data samples have 64 observations while for the marketing year the data sample has 63 observations, included in the period January 1949-December 2012.

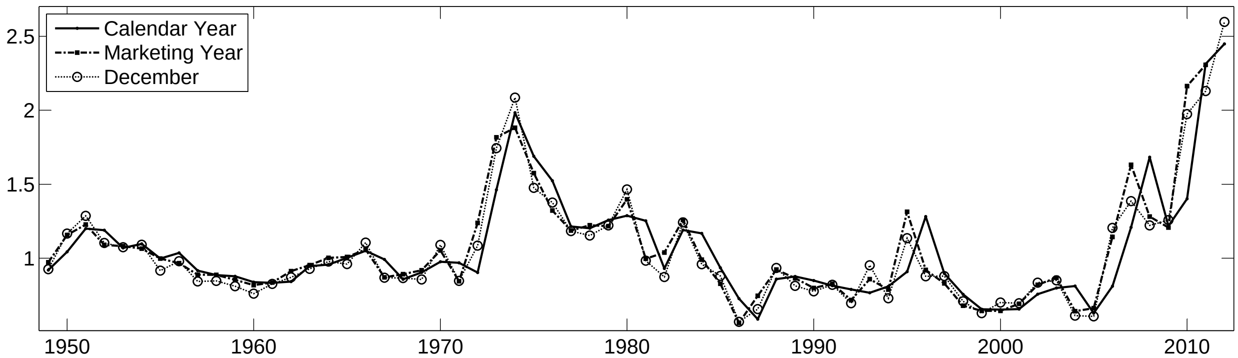

Figure 1.3: Calendar Year, Marketing Year, and December detrended real price indices for maize.

The estimated parameters are reported in Table 1.1 along with the value of the maximized likelihood, and the implied threshold price, p^*.

To evaluate the models' fit, I follow the method presented in Cafiero et al. (2011a), using the estimated parameters to generate a series of 300,000 prices, and then extract from it all possible consecutive subsamples of the same length as the observed data. On each extracted subsample I measure various moments thus generating simulated distributions of implied mean, median, coefficient of variation, first and second order autocorrelation, skewness and kurtosis. I then identify, in each of the simulated distributions, the percentiles corresponding to the values of the corresponding moments observed in the detrended price data. In other words, from the 300.000 simulated price series: a) I extract all the possible consecutive subsamples of 64 observations, which is the sample size of my prices data; b) for each subsample I compute the mean, median, first and second coefficient of autocorrelation, coefficient of variation, skewness, and kurtosis; c) the last step allows me to build an empirical distribution for each of those moments using all the simulated subsamples; d) I measure the same moments in the observed price data. e) Finally, I identify in which percentile of the empirical simulated distributions the moments of the sample are located. Table 1.2 shows the observed moments of the sample and the corresponding percentile of their location in the simulated distributions. The moments measured on the price series lie, in all cases considered, within symmetric ninety percent central confidence regions.

Table 1.1: Parameter estimates.

	a	b	d	$ln(L)$	p^*
Year					
Calendar	1.3210	-2.7104	0.0002	20.8921	2.5542
	(0.1479)†	(0.3893)	(0.0218)		
Marketing	1.2343	-2.8595	0.0069	13.0368	2.5299
	(0.1424)	(0.4887)	(0.0265)		
Quarter					
Sept.-Nov.	1.1110	-3.4210	0.0095	8.6427	2.7103
	(0.1533)	(0.9274)	(0.0292)		
Oct.-Dec.	1.3555	-5.9308	0.0000‡	8.1657	4.2933
	(0.1935)	(1.1797)	n.a.		
Month					
Sept.	1.0799	-3.8902	0.0204	2.9154	2.8822
	(0.1249)	(1.1251)	(0.0320)		
Oct.	1.0496	-3.6785	0.0081	6.6681	2.8012
	(0.1467)	(1.0359)	(0.0301)		
Nov.	1.3193	-6.1934	0.0023	6.8834	4.3881
	(0.2755)	(2.1386)	(0.0357)		
Dec.	1.1874	-3.6100	0.0186	7.3842	2.8313
	(0.1376)	(0.6310)	(0.0257)		

† Asymptotic standard errors in parentheses.

‡ For the Oct.-Dec. quarter, the estimate of $ln(d)$ tends to a large negative number as d approaches zero. I stop the procedure when the slope of the objective function with respect to the estimate falls below the preset tolerance of 10^{-13}. In that case I set $d = 0$, and re-run the estimation. See Cafiero et al. 2011a, p. 50, footnote 13, for a similar procedure.

Table 1.2: Comparison of data features and model predictions.

Period	Mean	Median	1st order a.c.	2nd order a.c.	Coefficient of Variation	Skewness	Kurtosis
Year							
Calendar							
Observed values	*1.0529*	*0.941*	*0.7894*	*0.4748*	*0.3431*	*1.9196*	*4.3092*
Percentiles †	20.39	21.76	78.11	46.68	17.51	47.15	46.11
Marketing							
Observed values	*1.0458*	*0.9667*	*0.7482*	*0.4284*	*0.3313*	*1.6654*	*3.1468*
Percentiles †	22.83	29.91	72.59	41.88	10.73	36.28	35.87
Quarter							
Sept.-Nov.							
Observed values	*1.0212*	*0.9414*	*0.7808*	*0.4746*	*0.3841*	*2.0168*	*4.8905*
Percentiles †	31.01	41.48	75.17	46.00	13.98	47.67	47.97
Oct.-Dec.							
Observed values	*1.0234*	*0.9098*	*0.7812*	*0.5161*	*0.3775*	*2.0181*	*4.8297*
Percentiles †	33.18	38.89	53.13	34.03	9.73	60.00	60.16
Month							
Sept.							
Observed values	*1.0402*	*0.9458*	*0.7390*	*0.3853*	*0.3928*	*1.8539*	*4.0607*
Percentiles †	24.62	34.46	72.8	36.72	10.54	37.28	37.95
Oct.							
Observed values	*1.0072*	*0.9261*	*0.7649*	*0.4642*	*0.3893*	*2.0254*	*4.9299*
Percentiles †	37.13	48.18	66.76	39.90	13.16	50.12	50.19
Nov.							
Observed values	*1.0164*	*0.8937*	*0.7799*	*0.5139*	*0.3819*	*2.0960*	*5.2551*
Percentiles †	31.22	36.58	54.60	35.18	9.36	60.51	60.72
Dec.							
Observed values	*1.0467*	*0.9315*	*0.7778*	*0.5469*	*0.3670*	*1.9012*	*4.1578*
Percentiles †	18.87	23.91	82.81	68.85	9.20	38.29	37.88

† Percentiles of the location of the of the observed value in the relevant simulated distribution. I simulate a series of 300,000 prices using the parameters estimated in Table 1.1. From the simulated series I select the set all possible consecutive sequences of the same size as the observed sample. I then obtain the empirical distribution of each of the relevant moments calculated for each member of this set of subsamples.

1.6 What data to use: does it matter?

The slope of the consumption demand b is a key parameter of the storage model, related to the sensitivity of the market price to a negative supply shock. As shown in Table 1.1, different ways to form the index of annual prices for the series of detrended real US maize imply large differences in b. The slope of the consumption demand estimated using calendar year averages is -2.71, lower in absolute value than the slope estimated using marketing year averages, -2.86, and lower in absolute value than the slopes obtained using either quarterly or monthly data.

The different values for b imply different values for the threshold price p^*. To illustrate the implications of these differences in p^*, I divide each price series by its corresponding p^*, thus providing normalized price series that measure the relative distance of prices to each corresponding threshold price (Figure 1.4). It is striking that the two series with the steepest slope parameters, the Oct.-Dec. quarter and the Nov. month series, exhibit empirical histograms of normalized prices quite distinct from the other series, with most of their probability mass well below 0.25, while most of the probability mass is above 0.25 for the other series (the value of normalized price that corresponds to the threshold price p^*).

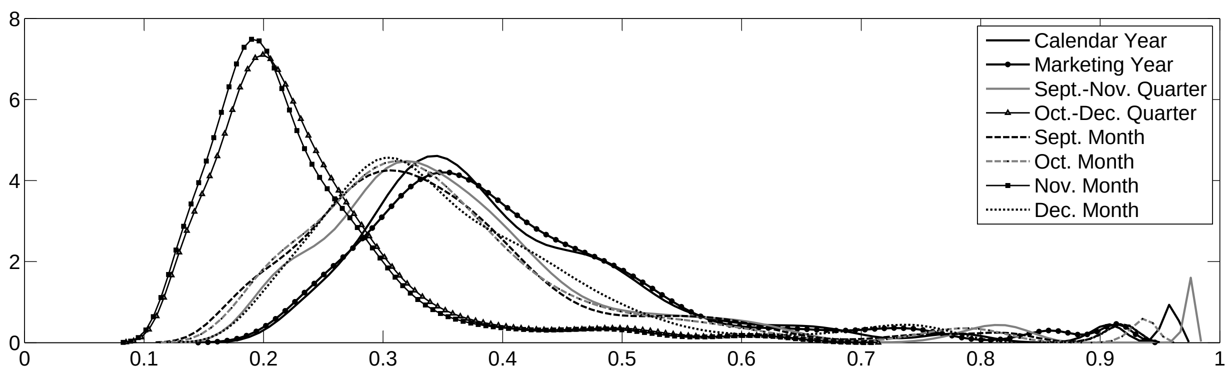

Figure 1.4: Empirical kernels of normalized price series (based on a normal kernel).

Figure 1.5 presents the histograms of stocks implied by the parameter estimates, for each price in the sample (divided by the maximum stock level for each series), for each of the price samples. In symmetry with the price histograms, both the Oct.-Dec. quarter and the Nov. month normalized stocks series exhibit empirical histograms quite distinct from the histograms of the other series, with more of their probability mass near one (their maximum level of stocks, given the normalization).

Although my estimates imply no stockouts in the sample data (because observed prices are always lower than estimated p^*), the histograms for normalized prices and normalized implied stocks are coherent with the implied probabilities of stockouts in samples of the same size as the data, drawn from the simulated series of 300,000 observations: for the Oct.-Dec. quarter and the Nov. month series, the implied probabilities of at least 1, 5, or 10 stockouts

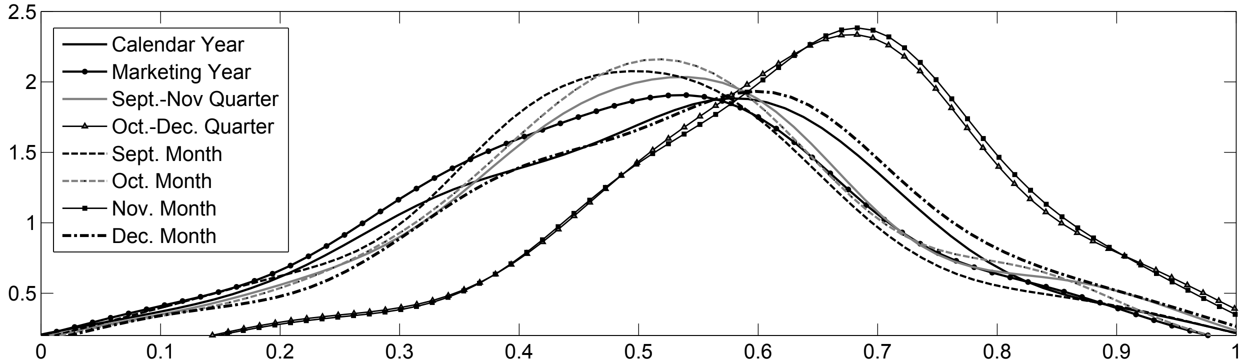

Figure 1.5: Empirical kernels of implied normalized stocks (based on a normal kernel).

Table 1.3: Implied probabilities of at least n stockout, in samples of the same size as the data.

Period	$n = 1$	$n = 5$	$n = 10$
Year			
Calendar	0.7291	0.2333	0.0300
Marketing	0.7443	0.2442	0.0308
Quarter			
Sept.-Nov.	0.6836	0.1954	0.0209
Oct.-Dec.	0.4534	0.0869	0.0061
Month			
Sept.	0.7316	0.2262	0.0268
Oct.	0.6318	0.1597	0.0148
Nov.	0.4645	0.0890	0.0062
Dec.	0.7611	0.2558	0.0333

in the same sample periods used in my estimations, are much lower than for the other price series (Table 1.3).

I calculate the price elasticity (at the sample mean) of consumption demand for maize implied by each of the data sets. Table 1.4 shows that the price elasticities are lower (in absolute value) than those implied by the estimates of Roberts and Schlenker (2013) for maize, comparable to the values of elasticities for maize implied by the estimates of Deaton and Laroque (1995, 1996)[9] and Cafiero et al. (2011a), within the range of values of elasticities of export demand for maize reported by Reimer et al. (2012), and comparable to the elasticities of demand for aggregate calories from maize, rice and soybeans and wheat in Roberts and Schlenker (2013). Appendix A reports the procedure I use to calculate the elasticities, from the parameter estimates.

[9]Replicated in Cafiero et al. (2011a).

Table 1.4: Implied price elasticities of consumption demand for maize.

Literature	Elasticity	Data Interval
Deaton and Laroque (1995, 1996)†	-0.046	1900-1987
Cafiero et al. (2011a)†	-0.018	1900-1987
Reimer et al. (2012)	From -0.251 to -0.003	2001-2011
Roberts and Schlenker (2013)‡	From -0.532 to -0.244	1961-2010
Implied by my estimations		
Year		
Calendar	-0.024	1949-2012
Marketing	-0.023	1949-2012
Quarter		
Sept.-Nov.	-0.019	1949-2012
Oct.-Dec.	-0.011	1949-2012
Month		
Sept.	-0.017	1949-2012
Oct.	-0.017	1949-2012
Nov.	-0.010	1949-2012
Dec.	-0.018	1949-2012

† Deaton and Laroque (1995, 1996) and Cafiero et al. (2011a) do not report elasticities. For this table I calculate the elasticities implied by the estimated parameters, using the values reported in tables 2 and 6 of Cafiero et al. (2011a), for maize.
‡This range includes the values reported in the Online Appendix of Roberts and Schlenker (2013). They also report demand elasticities for aggregate calories from maize, rice, soybeans and wheat, in the range -0.066 to -0.028.

Bobenrieth et al. (2013) show that although quantity data might be unreliable, data on stocks-to-consumption can be a valuable complement to price, as warning of price spikes for maize, rice and wheat. Following their encouraging results, I use the estimated model to predict stock-to-use ratios (SURs), and compare them with the SURs constructed from maize marketing-year ending stocks and consumption from USDA/PSD data, for the overlapping period 1961-2012.[10] I adjust for essential stock following the procedure in Bobenrieth et al. (2013, pp. 5-6). More specifically, essential stocks are calculated as a fixed proportion of the consumption matching the minimum of observed SURs.[11] Figure 1.6 shows observed SURs and price-implied SURs, for calendar year averages and the December price series. It is encouraging that the dynamics of my predicted SURs follow the dynamics implied in

[10]For calendar year averages, I compare the price-implied SURs with the observed SURs constructed using consumption and ending stocks for the same calendar year. For each of the other price series, the comparison is with SURs constructed using consumption and ending stocks for the same marketing year of the price data.

[11]Observed SURs are constructed using re-scaled consumption and stocks, following the procedure described in Appendix A.

PSD data on SURs. However the goodness of fit is not homogeneous. Table 1.5 reports the Root Mean Square Error of the difference between the price-implied SURs and the observed SURs, for each of the price series considered. Price data constructed by taking the month of December offers the best fit. In contrast, model-implied SURs using calendar year price averages yield the worst fit.

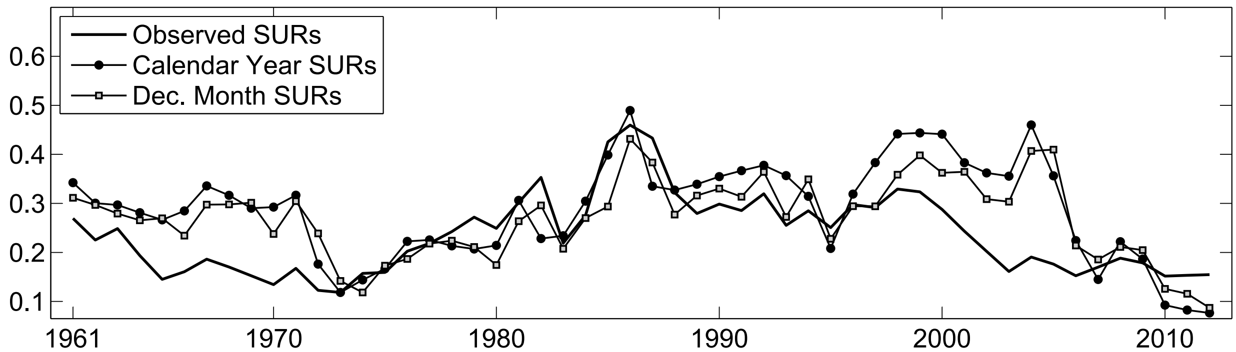

Figure 1.6: Observed SURs and the price-implied SURs for calendar year and December.

Table 1.5: Root Mean Square Error of the difference between the price-implied SURs and the observed SURs.

Period	Root Mean Square Error
Year	
Calendar	0.1734
Marketing	0.1702
Quarter	
Sept.-Nov.	0.0911
Oct.-Dec.	0.1024
Month	
Sept.	0.0864
Oct.	0.0910
Nov.	0.1000
Dec.	0.0819

1.7 Conclusions

I present the results of application of a ML estimator of the standard annual storage model, comparing the use of calendar and marketing year averages with quarterly and monthly averages (one quarter and one month per year, respectively), to form annual price indices.

The results indicate serious differences in magnitudes of practical interest, including the location of the empirical distribution of prices relative to the cutoff price of zero stocks, the likelihood of stockouts, and the fit to data on stocks-to-use ratios.

This chapter explores the limits of econometric estimations of the standard commodity storage model, using annual price data. It is clear that calendar year averages are not appropriate to test the storage model, due the bias induced by averaging two consecutive agricultural years. Although the use of marketing year averages, quarters or months can imply serious differences in magnitudes of policy interest, the theory of the storage model does not provide an answer to the question of what data best represents annual prices. However, in terms of the ability of the estimated model to fit data on SURs, price data constructed by taking the month of December to represent the annual price provides the best fit. In contrast, model-implied SURs using calendar year averages yields the worst fit.

Chapter 2

The Method of Equilibrium Outcome Grid-Points for Solving Nonlinear Rational Expectation Models

2.1 Introduction

Numerical approximation of stationary equilibrium functions derived from Euler conditions is often implemented to solve and estimate rational expectations models in microeconomic and macroeconomic dynamic problems. As widely recognized within the literature, the use of numerical techniques produces approximation errors which are one of the central problems in computational economics. There is an increasing number of papers that explore how these errors might arise, how they propagate and what their impact is on the accuracy of the function being approximated. I consider some of these error sources in this chapter, dividing them into three categories: (i) errors due to stopping rules, (ii) errors due to extrapolation, (iii) errors due to interpolation.[1] The first two sources of error constitute the main subject of this chapter. To illustrate how these errors are generated, I present a version of the competitive commodity storage model. This dynamic model assumes that in each time period the commodity price is formed after the realization of a stochastic harvest, when decisions on how much to store out of the available supply are made. In this context, the consumption and storage are the control variables, the total available supply is the state variable, the harvest is the stochastic shock, and the price is defined as the equilibrium outcome variable.[2] As next section shows, in this rational expectation model the stationary equilibrium function derived from Euler conditions has no known closed-form. Nevertheless, it can be approximated using numerical techniques. I use this approximation to addresses the three categories of error described above.

[1]Some of these error sources definitions follow those used and described by Judd (1998, pp. 39-44).

[2]In equilibrium, supply, including inventories from the previous period, must equal demand, including demand for inventories to carry forward into the next period and consumption for the current period.

Consider first errors due to stopping rules. Euler equations are commonly approximated using iterative algorithms that generate a sequence of functions that converge to a fixed point. Such iterative algorithms terminate after some finite number of iterations, when the sequence of functions is within some pre-defined distance of its limit. Stopping rules provide criteria that define this distance. The literature provides a bound for the approximation error due to stopping rules for cases where the iterative algorithm is generated by a contractive operator. For the family of models studied in this chapter, section 2.3.1 provides such bound and shows that it is not robust to scaling transformations of the equilibrium outcome variable (the price).[3] Scaling transformations are relevant since the data used for the econometric estimation of these models have a scale that is arbitrarily fixed. For example, the econometric estimation approach of the commodity storage model introduced by Deaton and Laroque (1995, 1996), also implemented by Chambers and Bailey (1996), Cafiero et al. (2011a, 2015), Bobenrieth et al. (2013), Guerra et al. (2015), and Gouel and Legrand (2017), uses only real price index data. Such indices are arbitrarily scaled by, for example, some nominal price average, and/or some base-year deflator.[4] Section 2.3.2 introduces a new numerical method and provides a stopping rule that is robust to scaling transformation of the equilibrium outcome variable.

The second category of errors includes those that arise in many empirical and numerical rational expectations models where Euler conditions are approximated over a truncated domain of the state variable (the total available supply) that does not include the entire ergodic region. Extrapolation techniques are required to evaluate Euler conditions over the remainder of the interval in the domain of the ergodic set. This truncation can be either a consequence of a poor choice of the range for such domain, or simply because the ergodic region for the state variable is unbounded while its numerical approximation has to be implemented over a bounded support. In the first case extrapolation can be avoided by increasing the support of the domain of the state variable, while in the second extrapolation is inevitable. For the family of models described in the next section, the ergodic support for the state variable is bounded but its upper bound cannot be computed until the equilibrium function has been approximated. However the equilibrium outcome variable lies in an ergodic set that is bounded and implicitly defined by the support of the shocks and the inverse consumption demand function. The numerical strategy proposed in this chapter avoids extrapolation, suppressing the error due to extrapolation by defining a grid for the equilibrium outcome variable (the price) instead of using a grid for the state variable as in the standard method.

[3]By a scaling transformation I mean the following: if y is a real-valued variable, then its scaling transformation is given by: $\widetilde{y} \equiv \frac{y}{\lambda}$, where λ is a positive constant. If this real variable is a price series, a change of numeraire has the same effect as a scaling transformation.

[4]As for example the real prices indices from Grilli and Yang (1988) and Pfaffenzeller (2007).

The third category, error of interpolation is largely beyond the scope and purpose of the present chapter. An extensive literature provides upper error bounds for different interpolation methods. See for example Daniel (1976), Judd (1992, 1998), Cai and Judd (2012, 2013). Nevertheless, the derivation of error bounds for interpolation in Euler equations is not straightforward (see Santos, 1999, p. 351). Alternative methods have been developed to study the accuracy of numerical approximation of Euler conditions based on Euler equation residuals, see for example Judd (1992), Christiano and Fisher (2000), de Haan and Marcet (1994) and Santos (2000). For a detailed discussion (based on numerical experiments) comparing different interpolation methods see for example, Judd (1992, 1998), Christiano and Fisher (2000), Miranda and Fackler (2002) and Gouel (2013). In particular, Miranda (1997) compares various interpolation methods for the storage model without liquidity constraints concluding that a cubic spline interpolation method is preferable to the other approaches. The version of the commodity storage model I consider in this chapter has a liquidity constraint because storage cannot be negative. This liquidity constraint generates a kink in stationary equilibrium function and its approximation. To avoid the use of interpolation above the kink, following the method proposed by Brumm and Grill (2014) I implement an adaptive grid for prices, as is described in section 2.3.3. Thus, an interpolation method is implemented only below the kink when the inequality constraint is not active.

2.2 The theoretical model

In this section I present the theoretical model. Consider a simple competitive commodity market in which storers are risk neutral, and face a constant discount rate $r > 0$. Supply shocks, ω_t, are i.i.d., with compact support $[\underline{\omega}, \overline{\omega}] \in \mathbb{R}$. The state variable is the total available supply at time t defined as $z_t \equiv \omega_t + (1-d)x_{t-1}$, where x_{t-1} is storage at time $t-1$, and $d \in [0, 1)$ is the physical deterioration rate of stocks. Price is formed as $p_t = F(c_t)$, where consumption at time t is given by $c_t \equiv z_t - x_t$. The inverse consumer demand: $F : \mathbb{R} \to \mathbb{R}$, is strictly decreasing. The inverse consumer demand can be interpreted as the derivative of a HARA utility function, i.e., $F(c) = (A + Bc)^{\frac{1}{1-K}}$, for A, B, and K, real constants, such that $F(\overline{\omega}) > 0$.[5]

A stationary rational expectations equilibrium (SREE) in this model is a price function f which describes the current price p_t as a function of the state z_t, and which satisfies, for all z_t,

$$p_t = f(z_t) = \max\left\{F(z_t), \left(\frac{1-d}{1+r}\right) E_t f\big(\omega_{t+1} + (1-d)\big[z_t - F^{-1}\big(f(z_t)\big)\big]\big)\right\}. \qquad (2.1)$$

[5]Note that particular cases are the iso-elastic, linear, and log-linear F functions. The ergodic region for prices is bounded below and above by $[F(\overline{\omega}), F(\underline{\omega})]$. Then, the condition $F(\overline{\omega}) > 0$ implies that the minimum price in the ergodic set is positive, which is a relevant assumption to avoid negative prices for example in cases with linear demand.

Since the ω_t's are i.i.d., f is the solution to the following functional equation:[6]

$$f(z) = \max\left\{F(z), \left(\frac{1-d}{1+r}\right) Ef\big(\omega + (1-d)\big[z - F^{-1}\big(f(z)\big)\big]\big)\right\}. \tag{2.2}$$

Existence and uniqueness of the SREE, $f(z)$, as well as some of its properties are given by the following Theorem:

Theorem 2. *There is a unique stationary rational expectations equilibrium f in the class of continuous non-negative, non-increasing functions. Furthermore, if $p^* \equiv \left(\frac{1-d}{1+r}\right) Ef(\omega)$, then:*

$$f(z) = F(z), \quad for \quad z \leq F^{-1}(p^*),$$
$$f(z) > F(z), \quad for \quad z > F^{-1}(p^*).$$

f is strictly decreasing whenever it is strictly positive. The equilibrium level of inventories, $x(z)$, is strictly increasing for $z > F^{-1}(p^*)$.

See Deaton and Laroque (1992), Theorem 1, p. 6, for a proof of this theorem.

The assumption that harvest has a bounded support and that $F(\overline{\omega}) > 0$ are crucial for characterizing the ergodic distribution of the prices and total available supply. The fact that the ergodic set of prices is bounded by $[F(\overline{\omega}), F(\underline{\omega})]$, is proved by Scheinkman and Schechtman (1983) for the case of strictly convex storage cost. Bobenrieth et al. (2012) prove boundedness in a model with positive supply response and, as here, a zero storage cost apart from the interest rate.

Denote the total available supply at time t as z_t, and the support of the harvest shocks $[\underline{\omega}, \overline{\omega}]$, with $\underline{\omega} < \overline{\omega}$. Given an initial z_0, the sequence $\{z_t\}_{t\geq 0}$ is given by the recursive relation $z_{t+1} = \omega_{t+1} + (1-d)x_t$. For any given $\omega \in [\underline{\omega}, \overline{\omega}]$, denote by h_ω the function $h_\omega(z) \equiv \omega + (1-d)[z - F^{-1}(f(z))]$. Let $\overline{z}$ be the first fixed point of the function $h_{\overline{\omega}}$, that is $\overline{z} \equiv \min\{z : h_{\overline{\omega}}(z) = z\}$. Next claim show that a suitable state space for available resources is the compact set $Z \equiv [\underline{\omega}, \overline{z}]$. Define z_0 as the initial total available supply.

Proposition 1: *$Prob[z_t \leq \overline{z}$ for some $t \in \mathbb{N} | z_0 = z] = 1$, $\forall z \geq \underline{\omega}$. Furthermore, $z_0 \in Z \Rightarrow z_t \in Z$ with probability one, for all $t \in \mathbb{N}$, and the process $\Phi = \{z_t\}_{t\geq 0}$ is ergodic.*

See Bobenrieth and Bobenrieth (2010) for a proof of this proposition.

The second part of this claim establishes that if the initial total available supply (z_0) belongs to the ergodic set Z, then all the elements in the sequence $\{z_t\}_{t\in\mathbb{N}}$ remain in the ergodic set Z. If the lower and upper bounds of the ergodic set of total available supply are given by $\underline{\omega}$

[6] In the models of saving of Deaton (1991) d is the real interest rate, r is the rate of time preferences.

and $\overline{z}$, them the lower and upper bounds of the ergodic set of prices are given by $[F(\overline{\omega}), F(\underline{\omega})]$.

The lower bound is obtained using the first fixed point condition:

$\overline{z} = \overline{\omega} + (1-d)[\overline{z} - F^{-1}(f(\overline{z}))]$,
$\Rightarrow \overline{z} - (1-d)[\overline{z} - F^{-1}(f(\overline{z}))] = \overline{\omega}$,
$\Rightarrow \overline{z} - [\overline{z} - F^{-1}(f(\overline{z}))] \leq \overline{\omega}$,
$\Rightarrow F^{-1}(f(\overline{z})) \leq \overline{\omega}$,
$\Rightarrow f(\overline{z}) \geq F(\overline{\omega})$.

This result implies that at any finite time the maximum consumption in the ergodic set cannot be larger than the maximum harvest shock $\overline{\omega}$, and therefore the minimum price of the ergodic set cannot be below $F(\overline{\omega})$.
Consider a model exactly equal to the one presented at the beginning of this section, except that the inverse demand function is $\widetilde{F} = \dfrac{F}{\lambda}$, where λ is a positive constant (a scaling transformation).

Proposition 2: *The corresponding SREE functions f and $\widetilde{f}$ satisfy:* $\widetilde{f}(z) = \dfrac{f(z)}{\lambda}$, $\forall z \in [\underline{\omega}, \overline{z}]$.

See Bobenrieth and Bobenrieth (2010) for a proof of this proposition.

In other words a scaling transformation of the inverse consumption demand implies the the same scaling transformation in the SREE function.

2.3 Computation of the Stationary Rational Expectations Equilibrium (SREE)

Since there is no known closed-form for the equilibrium price function, it is computed by using numerical approximation. Theorem 2 of Deaton and Laroque (1992) addresses the operator T which for some $n \in \mathbb{N}$ associates with a functional iterate $f_{<n>}$ the subsequent function $f_{<n+1>}$. It is defined by:

$$f_{<n+1>}(z) = \max\left\{F(z), \left(\frac{1-d}{1+r}\right) Ef_{<n>}\big(\omega + (1-d)\big[z - F^{-1}\big(f_{<n+1>}(z)\big)\big]\big)\right\}, \tag{2.3}$$

and shows that the operator defines a contraction mapping with modulus $\beta \equiv \frac{1-d}{1+r}$.

If G is a space of functions defined as:

$$G \equiv \{g : [\underline{\omega}, +\infty[\to \mathbb{R},\ g \geq 0,\ g \text{ continuous}, g \text{ non-increasing}, g(\underline{\omega}) = F(\underline{\omega})\},$$

then, given a choice of some suitable $f_{<0>} \in G$, the sequence $f_{<0>}, f_{<1>} = Tf_{<0>}, ..., f_{<n+1>} = Tf_{<n>}$, converges to the SREE f.

The recursive equation (2.3) is updated until: $||f_{<n+1>}(z) - f_{<n>}(z)||_\infty < \epsilon$, for some $n \in \mathbb{N}$, where $|| \cdot ||_\infty$ is the supremum norm, and $\epsilon > 0$ is an arbitrarily fixed error bound. Given the contraction property of equation (2.3) the iterates of the SREE iteration satisfy the inequality:

$||f(z) - f_{<n>}(z)||_\infty \leq \left(\frac{1}{1-\beta}\right) ||f_{<n+1>}(z) - f_{<n>}(z)||_\infty$,[7] implying that:

$||f(z) - f_{<n>}(z)||_\infty < \frac{\epsilon}{1-\beta}$.

Therefore the upper bound for the approximation error of the n-iteration is given by $\frac{\epsilon}{1-\beta}$. As before, consider a model exactly equal to the one presented at the beginning of this section, except that the inverse demand function is $\widetilde{F} = \dfrac{F}{\lambda}$, and $\widetilde{f}_{<0>} = \dfrac{f_{<0>}}{\lambda}$, with $\lambda > 0$ (a scaling transformation in the inverse consumption demand and also in the first guess of the SREE).

Proposition 3: *The functions* $\widetilde{f}_{<n+1>}$ *and* $f_{<n+1>}$ *satisfy:* $\widetilde{f}_{<n+1>}(z) = \dfrac{f_{<n+1>}(z)}{\lambda}$, $\forall n \in \mathbb{N}$.

See Appendix B for a proof of this proposition.

Hence, the scaling transformation of the inverse consumption demand and the first guess of the SREE implies the same scaling transformation in the sequence of iterates that converge to the scaled SREE function.

2.3.1 The standard solution method

Given a fixed inverse consumer demand function F, a probability density distribution for ω, parameters $\{d, r\}$, and an initial guess $f_{<0>} \in G$, for all $n \geq 0$, $f_{<n+1>}(z)$ is constructed by calculating its values at a finite grid of possible values of the state variable z.

The Algorithm:

(i) Define some monotone grid of points $z_i \in \overrightarrow{z} \equiv \{z_1, z_2, ..., z_{I-1}, z_I\}$.

(ii) For each element z_i, equation (2.3) is solved for some price $p_{i,<n+1>}$ that implicitly satisfies:

[7] Stokey and Lucas (1989) propose this inequality.

$$p_{i,<n+1>} = \max\left\{F(z_i), \left(\frac{1-d}{1+r}\right) Ef_{<n>}\big(\omega + (1-d)\big[z_i - F^{-1}\big(p_{i,<n+1>}\big)\big]\big)\right\}. \quad (2.4)$$

(iii) Points $\{z_i, p_{i,<n+1>}\}$ are then used to construct $f_{<n+1>}(z)$ using some interpolating approximation.

(iv) The recursive equation (2.4) is updated until $||f_{<n+1>}(z) - f_{<n>}(z)||_\infty < \epsilon$, for some $n \in \mathbb{N}$, where $\epsilon > 0$ is a predefined error bound.

This approximation method has three problems related with the sources of error described in the introduction of this chapter.

Problem 1: *This method is not robust to scaling transformations due to the stopping rule error.*

Without loss of generality, assume that the model is solved for a scale parameter $\lambda = 1$ and the recursive equation is stopped when $||f_{<n+1>}(z) - f_{<n>}(z)||_\infty < \epsilon$ as described in the step (iv). Then, $\forall\epsilon > 0$, $\exists n \in \mathbb{N}$ such that $||f(z) - f_{<n>}(z)||_\infty < \frac{\epsilon}{1-\beta}$. However, if the underlying economy has a scale parameter $\lambda > 1$, and the approximation of the SREE $f_n(z)$ is re-scaled by λ, and used as the approximation of the SREE for the underlying economy, then proposition Proposition 3 implies that the approximation error due to the stopping rule after scaling $f_n(z)$ is given by $\lambda\epsilon > \epsilon$. In other words, the n-iteration at which the recursive equation is stopped will be different for different scales, implying a problem because the scale is fixed arbitrarily. Changing the scale parameter has the same effect as changing ϵ.

Problem 2: *Since the choice of the range for $\overrightarrow{z}$ is arbitrary fixed a priori there is no guarantee that extrapolation outside the ergodic region is avoided.*

Although the lower bound of the support of z is given by the lower bound of the support of ω (equal to $\underline{\omega}$), its upper bound $\overline{z}$ cannot be computed until the SRRE function f has been approximated since the upper bound on stocks is endogenous.[8] If the maximum point of the grid, z_I, is smaller than $\overline{z}$, then extrapolation is required for evaluation of $f_{<n>}$ in the complement interval of the domain of the erogdic set that is not included in the grid, i.e. $[z_I, \overline{z}]$. The use of the extrapolation will inevitably induce a new source of approximation error in the next stage iterates.

This issue is noted by Deaton and Laroque (1995, p. S20) *"... Although it would be possible in principle to extrapolate the values of $f(z)$ beyond the end points in either directions, we*

[8]Since $f(\overline{z}) \geq F(\overline{\omega})$, then $\overline{z} \leq f^{-1}(F(\overline{\omega}))$.

regarded such extrapolation as dangerous and took pains to avoid it.".

When the function $f_{<n>}$ is evaluated using extrapolation, the computation of the prices $p_{i,<n+1>}$ can contain an error $\vartheta_{i,<n+1>}$. For such prices $p_{i,<n+1>}$, the approximation of the function f at the next stage, is constructed interpolating over the points $\{z_i, p_{i,<n+1>} + \vartheta_{i,<n+1>}\}$. Define such interpolation $\widehat{f}_{<n+1>}(z)$, which in general is not equal to $f_{<n+1>}$. Therefore, it is not guaranteed that this sequence converges to the SREE, i.e., for some $n \in \mathbb{N}$, $||\widehat{f}_{<n+1>}(z) - \widehat{f}_{<n>}(z)||_\infty < \epsilon$, does not imply that $||f_{<n+1>}(z) - f_{<n>}(z)||_\infty < \epsilon$. If the maximum grid point of the state variable is large enough to include the ergodic region (i.e. $z_I \geq \overline{z}$), it has to be true that $F(\overline{\omega}) \geq f(z_I)$. The only way to avoid the use of extrapolation is to fix $z_I \geq \overline{z}$. In the standard solution method this is done by trial and error.

Deaton and Laroque (1992, p. 9) observe *"...on some occasions it was necessary to run trials to discover the range."*

In the empirical and numerical literature on the storage model, including one paper of which I am coauthor (Guerra et al., 2015), very often the SREE function is approximated over a grid of points that does not include the entire ergodic region, i.e., where $z_I < \overline{z}$. Therefore, extrapolation induces some error which is not taken into account in such approximations. For example, in Deaton and Laroque (1995) only for 4 of the 12 commodities considered, the parameter estimates of inverse demand function and d imply that $F(\overline{\omega}) \geq f(z_I)$. In Cafiero et al. (2011), and for the version of the model that coincides with the one presented in section 2,[9] only for 1 of the 9 commodities considered, the parameter estimates of inverse demand function and d imply that $F(\overline{\omega}) \geq f(z_I)$. Also, extrapolation is required in the set of parameter estimates for the commodities rice, tea, and maize, presented in p. 13 Table 2 of Gouel and Legrand (2017). Finally, this is also an issue in a numerical example presented by Michaelides and Ng (2000).

Problem 3: *Since the choice of the grid points of $\overrightarrow{z}$ is arbitrary fixed a priori there is no guarantee that interpolation around the kink $z^* \equiv F^{-1}(p^*)$ is avoided.*

In order to know the the kink z^* the value of p^* must be know first. To compute p^* it is necessary to know the SREE first. Again, since the elements of the vector $\overrightarrow{z}$ are arbitrary fixed a priori, there is no guarantee that z^* will belong to $\overrightarrow{z}$. If it does not, interpolation around the kink is necessary. This issue is illustrated in the left hand side panel of Figure 2.1. Bobenrieth et al. (2011) addresses this problem and its impact on econometric estimations of the standard competitive storage model. They compare two approximations of the SREE for the same model. In the first approximation of the SREE they consider a sparse grid of 20

[9] Cafiero et al. (2011) also estimate a model with cost of storage which is not considered in this analysis. The analysis presented here is only based on the parameter estimates reported in Cafiero et al., 2011, p. 50, Table 4.

equally spaced points while in the second they consider a dense grid of 1.000 equally spaced points. Both grids are defined over the same predefined domain for the total available supply z. By considering a more dense grid they attempt to reduce the interpolation error around the kink z^*. Figure 3 of their paper shows that the first approximation is smother than the second. As they recognize: *"...the fine grid of 1.000 points allows for clear identification of the kink in the price function, which occurs at a price equal to p^*, and that the inaccuracy of the approximation of the price function with a sparse grid is especially large around that point"*.

The next subsection proposed a new numerical method that addresses these problems.

2.3.2 Equilibrium outcome grid-points method

The main contribution of this chapter is to introduce an alternative solution method that avoids the three accuracy problems in the approximation of the SREE described in the previous section.

The standard solution method proceeds as follows: given an initial guess for the SREE, the next iteration is computed by interpolation over a given grid of points z_i for the state variable and a grid of prices (the equilibrium outcome of the model) which is obtained by solving the Euler conditions for each z_i. The method presented in this section proceeds as follows: given an initial guess for the SREE, the next iteration is computed as the interpolation over a given grid of prices (the equilibrium outcome of the model) and a grid of points z_i for the state variable, where each z_i is obtained by solving the Euler conditions for each point in the grid of prices.

A key point of this method is that as soon as the inverse consumer demand function and the shock distribution support are known, then the ergodic region for the prices is known beforehand, and equal to $[F(\underline{\omega}), F(\overline{\omega})]$.

Before describing this method in detail, I define a new space of functions $\widetilde{G} \subset G$:

$$\widetilde{G} \equiv \{g : [\underline{\omega}, +\infty[\to \mathbb{R},\ g \geq 0,\ g \text{ continuous},\ g \text{ strictly decreasing},\ g(\underline{\omega}) = F(\underline{\omega})\}\,.$$

Proposition 4: *if $g \in \widetilde{G}$ then $Tg \in \widetilde{G}$, where T is the operator defined by equation (2.3).*

Proof: See Appendix C for a proof of this proposition.

If the initial iterate $f_{<0>}$ has an inverse, then all the functions of the sequences $\{f_{<n+1>}\}_{n\in\mathbb{N}}$ will have an inverse as well.

Given a fixed inverse consumer demand function F, a probability distribution for ω, parameters $\{d, r\}$, and an initial iterate $f_{<0>} \in \widetilde{G}$, for all $n \geq 0$, $f_{<n+1>}(z)$ is constructed by calculating the values of z at a finite grid of possible values of the equilibrium outcome.

The Algorithm:

(i) Define some monotone grid of points $p_i \in \overrightarrow{p} \equiv \{p_1, p_2, ..., p_I\}$, fixing $p_1 = F(\underline{\omega})$ and $p_I = F(\overline{\omega})$.[10]

(ii) For each element p_i, equation (2.3) is solved for some total available supply $z_{i,<n+1>}$ that implicitly satisfies:

$$p_i = \max\left\{F(z_{i,<n+1>}), \left(\frac{1-d}{1+r}\right) Ef_{<n>}\big(\omega + (1-d)\big[z_{i,<n+1>} - F^{-1}\big(p_i\big)\big]\big)\right\}. \quad (2.5)$$

(iii) Points $\{z_{i,<n+1>}, p_i\}_{i=1,2,...I}$ are then used to construct $f_{<n+1>}$ using some interpolating approximation.

(iv) The recursive equation (2.5) is updated until $||\overrightarrow{z}_{<n+1>} - \overrightarrow{z}_{<n>}||_\infty < \epsilon$, for some $n \in \mathbb{N}$, where $\epsilon > 0$ is a predefined error bound.[11]

Solution to Problem 1: *The first advantage of this method with respect to the one presented in the previous section is its robustness to scaling transformations.*

As before and without loss of generality, assume that the model is solved for a scale parameter $\lambda = 1$ and the recursive equation is stopped when $||\overrightarrow{z}_{<n+1>} - \overrightarrow{z}_{<n>}||_\infty < \epsilon$. Then, if the underlying economy has a scale parameter $\lambda > 1$, and the approximation of the SREE $f_n(z)$ is re-scaled by λ, then the sequence of $\{\overrightarrow{z}_{<n>}\}_{n\in\mathbb{N}}$, will be exactly the same as the one that would be gotten by solving the model for $\lambda > 1$. In other words, the n-iteration at which the recursive equation is stopped will be the same for different scales, removing the problem of an arbitrarily fixed scale.

Solution to Problem 2: *The ergodic region of prices is know before hand and extrapolation is avoided.*

In this method the grid of points for prices defined in step (i), includes all the ergodic region $[F(\overline{\omega}), F(\underline{\omega})]$. Trial and error procedure to discover the ergodic region is not longer an issue.

[10]Notice that $p_I = F(\overline{\omega})$ is the minimum price in the ergodic region when $d = 0$, and therefore $F(\overline{\omega})$ is a lower bound for the minimum price in the ergodic region for all $d \in [0, 1)$.

[11]This stopping rule implies that $||f_n(z) - f(z)||_\infty < \frac{\widetilde{\epsilon}}{1-\beta}$, where $\widetilde{\epsilon} \equiv F'(\underline{\omega})\epsilon$, and $\beta \equiv \frac{1-d}{1+r}$.

Solution to Problem 3: *Interpolation above the kink is avoided by using an adaptive grid for interpolation.*

This procedure is described in the next section.

2.3.3 Adaptive grid for interpolation

Brumm and Grill (2014) propose an alternative method for computing equilibria in dynamic models with several continuous state variables and occasionally binding constraints. Their method addresses the interpolation problem induced by the non-differentiabilities in policy functions when constraints bind, locating the non-differentiabilities and adding interpolation nodes there. This idea can be easily implemented in the equilibrium outcome grid method introduced in the previous section. To do this, it is necessary to compute the kink at each iteration, which is equal to $p^*_{<n+1>} \equiv \left(\frac{1-d}{1+r}\right) Ef_{<n>}(\omega)$ before step (i) and re-define an iteration-dependent grid price vector as $\overrightarrow{p}_{<n+1>} = p^*_{<n+1>} \cup \{p_i \in \overrightarrow{p}_{<n>} : p_i < p^*_{<n+1>}\}$. Since the sequence of functions $\{f_{<n>}\}_{n\in\mathbb{N}}$ converges to f, then the sequence of kinks $\{p^*_{<n>}\}_{n\in\mathbb{N}}$ converges to p^*. Now, at each iteration, interpolation is needed only for prices below $p{*}_{<n+1>}$, avoiding interpolation above the kink. Figure 2.1 illustrates how this method works. The dashed line displays a simple one-dimensional policy function with a kink. Suppose this function is approximated by linear interpolation between equidistant grid points. The resulting interpolated policy is displayed as a solid line on the left-hand side of Figure 2.1. Clearly, the approximation error is comparatively large around the kink, and this is just because there is no interpolation node near the kink. If one knew the location of the kink and put a node there, then the approximation would be much better, as the right-hand side of Figure 2.1 shows.

2.4 A numerical example

To illustrate the advantages of the equilibrium outcome grid-point method described in the previous section, consider the heuristic speculative storage model described by Michaelides and Ng (2000). Following the pioneering work of Deaton and Laroque (1992, 1995, 1996) they consider a linear inverse consumer demand fixing the set of parameter $\{A, B, K, d, r\}$ at $\{0.6, -0.3, 0, 0.1, 0.05\}$. The shock distribution is approximated by a 10 point standard normal distribution. Based on the standard solution method they discretize the support of the state variable z, using an equally spaced grid of 50 points in the interval $[\underline{\omega}, \overline{\omega}/d]$. Assuming this setup for the underlying economy, and fixing a predefined tolerance for the stopping rule at 10^{-7}, the SRRE is approximated using the standard method and the one proposed in this chapter. Then, the same model is solved for a scaled transformation of the inverse consumption demand, fixing $\lambda = 100000$. This last approximation is re-scaled back,

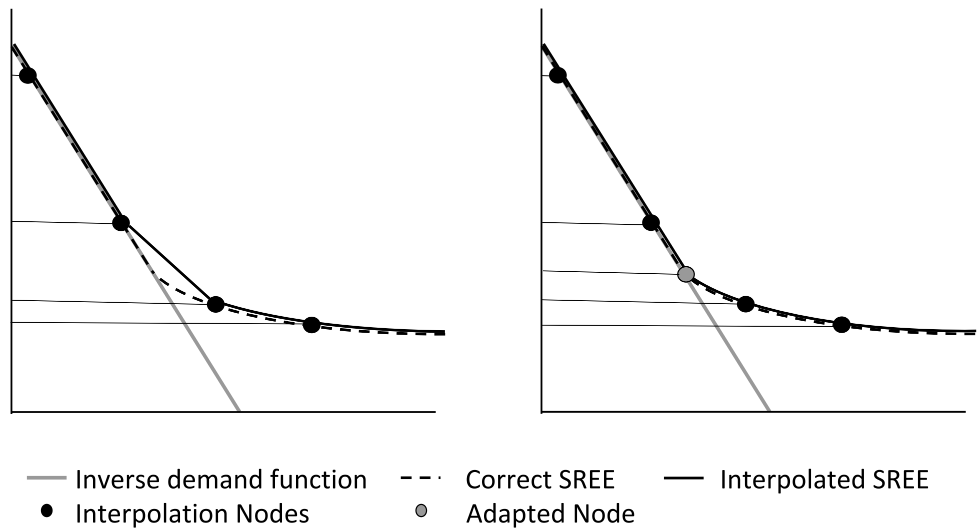

Figure 2.1: Adaptive Grid principle.

using Proposition 2, in order to compare it with the approximated SREE associated to the underlaying economy.

Figure 2.2 compares the standard solution method with the equilibrium outcome grid-points method proposed in this chapter. In both panels the black line represents the inverse consumption demand, the grey line represents the approximated SREE for the underlying economy, and the black dashed line represents the approximated SREE for the scaled transformation of the underlying economy. Panel (a) shows that the both approximations do not coincide when the standard solution method is applied, reflecting the different approximated SREEs at different scales. In contrast to panel (a), panel (b) shows that approximations of the SRRE coincide when the new method is implemented (grey line and black dashed line are coincident).

Figure 2.3 shows the magnitude of the errors due to extrapolation (the black dashed line) implied by the choice of the range for the total available supply considered by Michaelides and Ng (2000). When the standard method is applied errors due to extrapolation lie in the range of 0 and 18×10^{-7}. In contrast with these values, errors due to extrapolation are equal to zero over all the domain when the new method is implemented. Unfortunately, Michaelides and Ng (2000) does not report the magnitude of the errors due to their stopping rule. Nevertheless, the next figure illustrates a comparison between the magnitude of the errors due to extrapolation and the errors due to the stopping rule that I have chosen.

Most of the attention in the numerical literature focuses on the error due to the stopping rule, while error due to extrapolation is rarely reported. Figure 2.4 compares the magnitude of the errors due to extrapolation (the black dashed line) implied by Michaelides and Ng (2000) parametrization against the magnitude of the errors due to the stopping rule (the gray line) which I fixed arbitrarily at 10^{-7}.[12] Figure 2.4 shows that the errors due to extrapolation can be far larger than the errors due to stopping rule.

Finally, to have an approximate measure of the adaptive grid performance, I compute the Euler equation residuals implied by the approximated SREE for the two methods. I calculate the Euler equation residual as the absolute value of the difference between the right and left hand side of equation (2.2) over a grid of one millon points for the total available supply z, and using the approximations of SREE f obtained by each method. Figure 2.5 shows how the adaptive grid for interpolation removes the error due to interpolation for values of z that are lower than z^* and reduces the magnitude of this error below the kink to about one third of its previous value.[13]

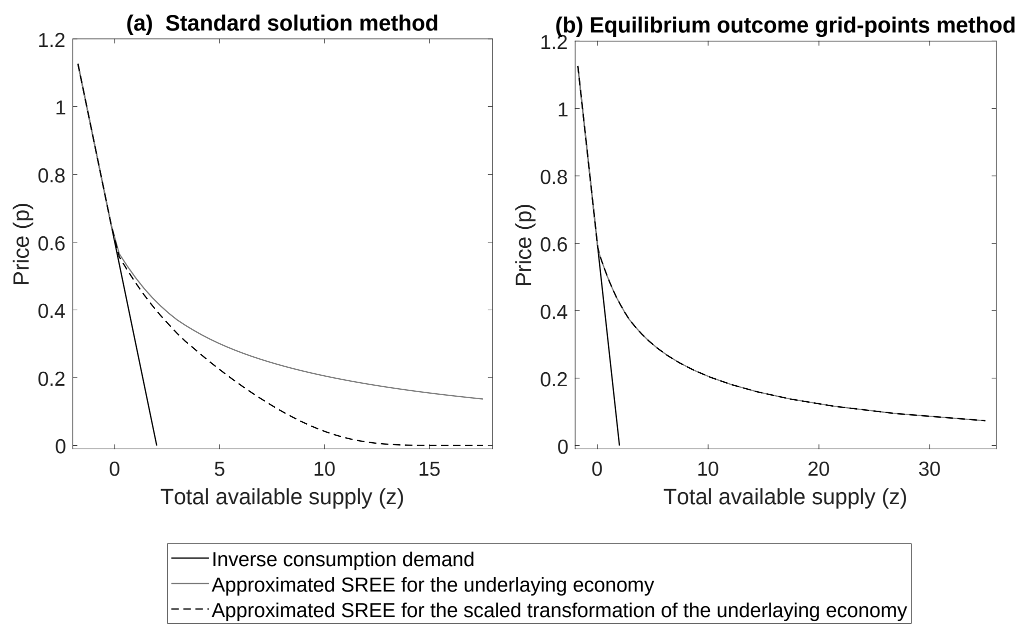

Figure 2.2: Comparison between the standard method and the equilibrium outcome grid-points method.

[12]Michaelides and Ng (2000) do not report the error bound they use as a stopping rule.
[13]In this figure I reduce the domain of z to $[-2, 6]$ for a better visualization of the error around the kink.

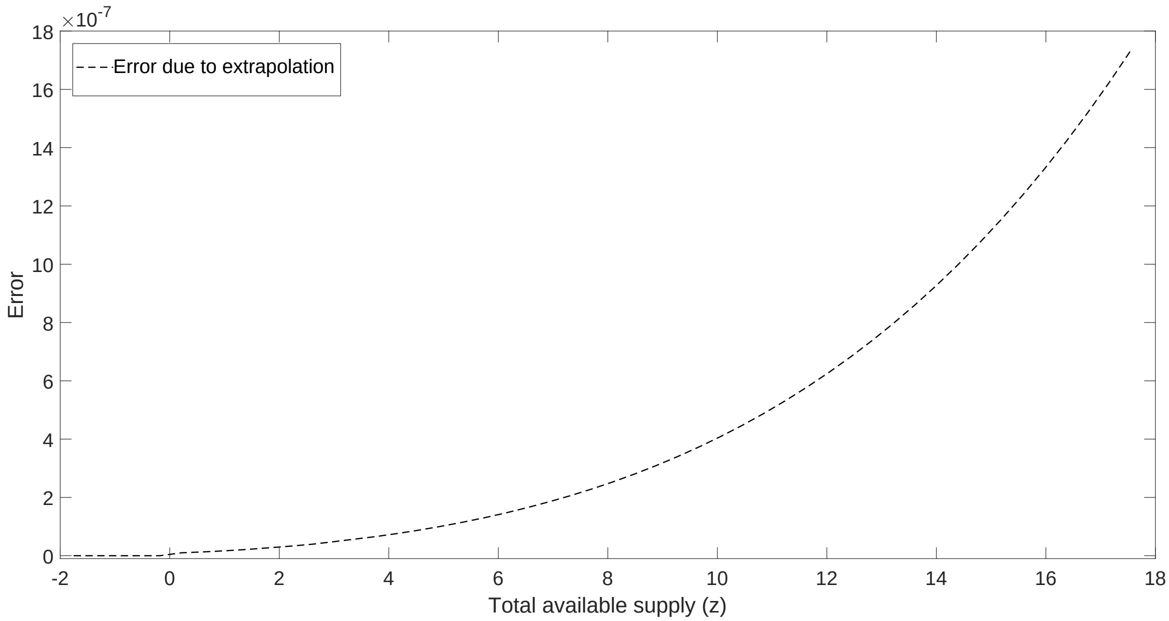

Figure 2.3: Error due to extrapolation for the parametrization used by Michaelides and Ng (2000).

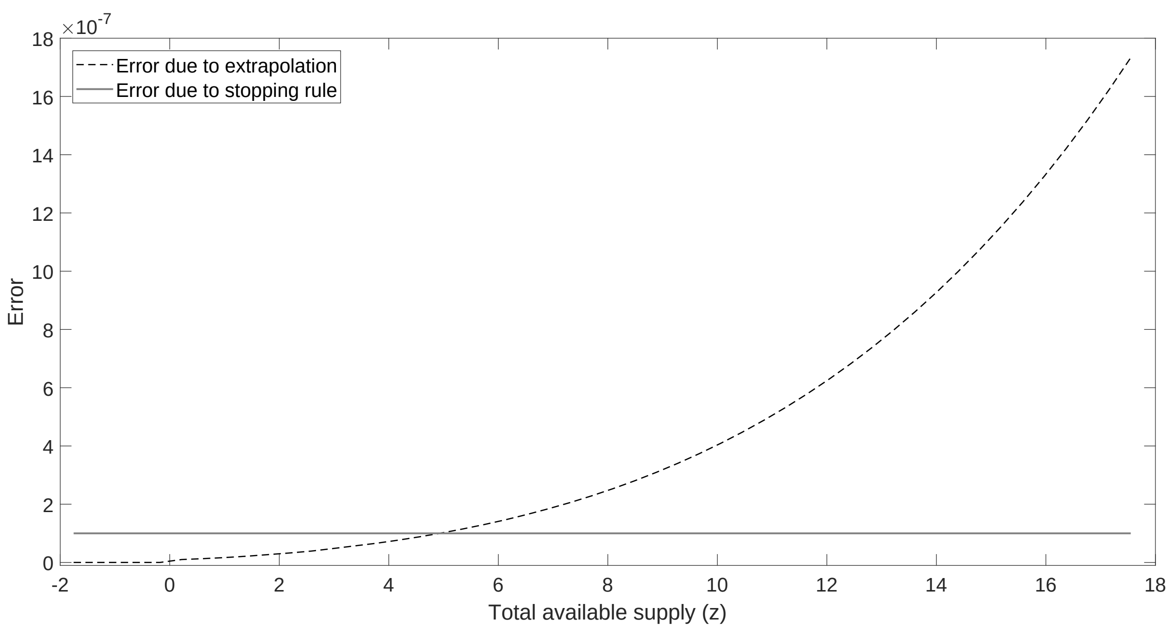

Figure 2.4: Error due to stopping rule and error due to extrapolation.

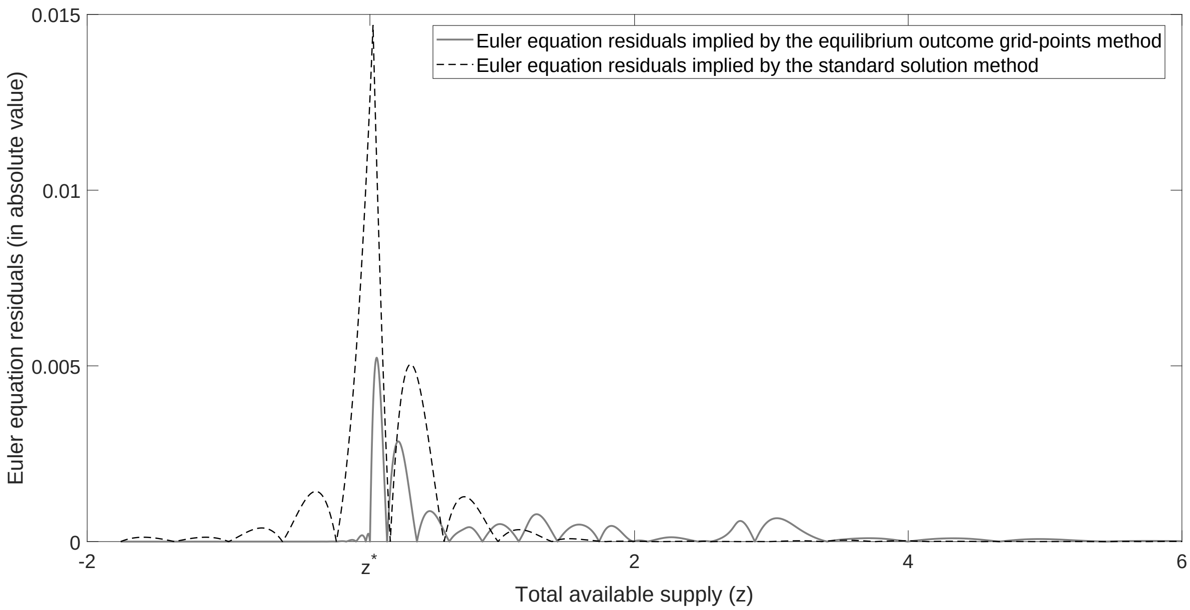

Figure 2.5: Implied Euler equation residuals.

2.5 Conclusions

This chapter provides a new numerical strategy for solving nonlinear rational expectation models with inequality constraints, and i.i.d. shocks with bounded support in cases where the state variable is bounded. It addresses three problems of the standard solution method: robustness to scaling transformation of the SREE function, its error of approximation due to extrapolation in the ergodic set, and interpolation around the kink. If the assumption of bounded support for the shocks is relaxed, the new numerical method still solves two of the problems of the standard solution method: robustness to scaling transformation of the SREE function, and interpolation around the kink. Addressing these problems is relevant for numerical and empirical applications. The approach presented in this chapter can be implemented for several microeconomic and macroeconomic models such as the income-saving models described by Schechtman and Escudero (1977) and Deaton (1991).

Chapter 3

Comments on: "On the behavior of commodity prices when speculative storage is bounded"

3.1 Introduction

Oglend and Kleppe (2017; OK2017 hereafter) in a paper entitled: "On the behavior of commodity prices when speculative storage is bounded" investigate the implications of bounded speculative storage on commodity prices. They assume that speculative storage is bounded from below at zero and above at an exogenous maximum capacity. Under this assumption OK2017 addresses, in Theorem 1, the convergence of a value function iteration operator. Such operator approximates a stationary value function that satisfies the Bellman equation solving the consumption-storage allocation of a representative consumer in a infinite horizon optimization problem.

Furthermore, OK2017 in Theorem 1 establishes that the derivative of this stationary value function coincides with the stationary rational expectations equilibrium (SREE) price function that solves the Euler conditions implied by an optimal speculation problem. This result has also been addressed in the literature for cases where the storage has no upper bound; see for example Benveniste and Scheinkman (1979), Coleman (1989, 1990, 1991), Deaton (1991), Deaton and Laroque (1992, 1995), Bobenrieth et al. (2012), and Rendahl (2015). This literature also provides iterative solution methods based either on the Bellman equation or on the Euler equations that allow approximation of the stationary value function or SREE respectively.

One method that allows approximation of the SREE directly is the time iteration operator described in the next section. Deaton and Laroque (1992) and Rendahl (2015) prove that this time iteration operator coincides with the derivative of the value function iteration operator. Also, assuming that the modulus of the operator is less than one, they prove that such operator satisfies Blackwell's sufficient conditions for a contraction mapping and therefore convergence of this operator to the SREE follows from the application of Banach's fixed point theorem, also known as the contraction mapping theorem.[1]

As an alternative to the time iteration operator, fixed point iteration operator (also described in the next section) is often implemented due to its generally faster convergence. However, to my knowledge, no one has proved that this fixed point iteration operator necessarily converges.

OK2017 in Theorem 2 claims that a fixed point iteration operator converges to SREE. This claim has two problems. First, they assume that the fix point iteration operator corresponds to the derivative of the value function operator, and under this assumption they claim convergence. Second, even if the derivative of the value function operator is replaced by the time iteration operation, contraction mapping arguments and Banach's fixed point theorem do not apply directly to their formulation, because they consider a more general version of the model where the modulus can take values greater than one.

In this chapter, I present a counter-example with modulus less than one, that shows Theorem 2 in OK2017 is false. I chose a set of parameters that satisfy all the assumptions of OK2017. However, for the chosen parameters, the fixed point iteration operator proposed by OK2017 oscillates between two different functions. This oscillation take place over a given domain included in the ergodic set, implying that neither discounting nor monotonicity, the sufficient Blackwell's sufficient conditions for a contraction mapping, are satisfied.

3.2 The theoretical model

Consider a simple competitive commodity market in which storers are risk neutral, and face a constant discount rate $r > 0$. Supply shocks, ω_t, are i.i.d., with compact support $[\underline{\omega}, \overline{\omega}] \in \mathbb{R}$. The state variable is the total available supply at time t defined as $z_t \equiv \omega_t + (1-d)x_{t-1}$, where x_{t-1} is storage at time $t-1$, and $d \in [0,1)$ is the physical deterioration rate of stocks. Price is formed as $p_t = F(c_t)$, where consumption at time t is given by $c_t \equiv z_t - x_t$. The inverse consumer demand: $F : \mathbb{R} \to \mathbb{R}$, is strictly decreasing, and a maximum capacity for storage $C > 0$.

A stationary rational expectations equilibrium (SREE) in this model is a price function f which describes the current price p_t as a function of the state z_t, and which satisfies, for all z_t,

$$p_t = f(z_t) = \min\left\{F(z_t - C), \max\left\{F(z_t), \left(\frac{1-d}{1+r}\right) E_t f\big(\omega_{t+1} + (1-d)\big[z_t - F^{-1}\big(f(z_t)\big)\big]\big)\right\}\right\}. \tag{3.1}$$

Since the ω_t's are i.i.d., f is the solution to the following functional equation:

$$f(z) = \min\left\{F(z - C), \max\left\{F(z), \left(\frac{1-d}{1+r}\right) E f\big(\omega + (1-d)\big[z - F^{-1}\big(f(z)\big)\big]\big)\right\}\right\}. \tag{3.2}$$

[1] See Stokey and Lucas (1989) Theorem 3.3 in p.54 for a description of Blackwell's sufficient conditions for a contraction (discounting and monotonicity), and Theorem 3.2 in p.50 for a description of the contraction mapping theorem.

Alternatively, consider the following relationship $U'(z) \equiv F(z)$, where U is the utility function of a representative consumer, and the Bellman equation:

$$V^*(z) = \max_{0 \le x \le C} U(z - x) + \frac{1}{1+r} EV\left(\omega + (1-d)x\right),$$

as the solution to the optimization problem:

$$\max_{\{x_t\}_{t=0}^{\infty}} \left\{ \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} E_0 U(z_t - x_t) \right\},$$

subject to:

$$\begin{array}{rcl} 0 \le & x_t & \le C, \\ z_{t+1} & = & \omega_{t+1} + (1-d)x_t. \end{array}$$

Existence and uniqueness of the SREE, $f(z)$, and the stationary value function $V^*(z)$, as well as some of their properties are established by OK2017, Theorem 1, p. 55. OK2017 in Theorem 1 also defines the relationship between $V^*(z)$ and $f(z)$ as follows:

$$\frac{dV^*(z)}{dz} \equiv f(z).$$

This result has been established by Benveniste and Scheinkman (1979), and extended in more general contexts for example by Coleman (1989, 1990, 1991), Deaton (1991), Bobenrieth et al. (2012), and Rendahl (2015).

3.2.1 OK2017, Theorem 2

Considering the assumptions of the theoretical model presented above, OK2017 in p. 56, Theorem 2, state the following:

OK 2017, Theorem 2: *Providing that $f_0(z)$ is a continuous, bounded and non-increasing function, the function iteration:*

$$f_{n+1}(z) = \min\left\{F(z-C), \max\left\{F(z), \left(\frac{1-d}{1+r}\right) Ef_n\big(\omega + (1-d)\big[z - F^{-1}\big(f_n(z)\big)\big]\big)\right\}\right\}, \tag{3.3}$$

converges to the SREE $f(z)$.

In the proof of this theorem OK2017 appear to assume that in the recursive operator described by equation (3.3), $f_{n+1}(z) \equiv \frac{dV_{n+1}(z)}{dz}$,[2] where $V_{n+1}(z)$ is the recursive operator given by:

[2] See OK2017 pp. 65–66.

$$V_{n+1}(z) = \max_{0 \leq s \leq C} \left\{ U(z-s) + \frac{1}{1+r} EV_n(\omega + (1-d)s) \right\}. \tag{3.4}$$

However, if $f_{n+1}(z) \equiv \frac{dV_{n+1}(z)}{dz}$, then:

$$f_{n+1}(z) = \min \left\{ F(z-C), \max \left\{ F(z), \left(\frac{1-d}{1+r} \right) Ef_n\big(\omega + (1-d)\big[z - F^{-1}\big(f_{n+1}(z)\big)\big]\big) \right\} \right\}. \tag{3.5}$$

Notice that in the right hand side of equation (3.3) the function f_n appears twice, while in the right hand side of equation (3.5) the function f_n appears only once, being replaced the second time by f_{n+1}. Equation (3.5) is the operator that correctly represents the derivative of the function V_{n+1}. As I mentioned in the introduction of this chapter, this relationship between V_{n+1} and equation (3.5) has been extensively discussed in the literature, see for example Coleman (1989, 1990, 1991), Deaton (1991), Deaton and Laroque (1992, 1995), and Rendahl (2015). Adopting the definitions of Rendahl (2015), I denote the operator described by the equation (3.3) as *fixed point iteration operator* and the operator defined by equation (3.5) as *time iteration operator*.

The fixed point iteration operator is often implemented in the literature instead of the time iteration operator to approximate the SREE described by the equation (3.2). The reason is that the numerical procedure implied by its recursive equation is faster because it avoids the root finding operations required by the implementation of the recursive equation of the time iteration operator. Assuming no restriction on the maximum capacity, and that $0 < \left(\frac{1-d}{1+r}\right) < 1$, Deaton and Laroque (1992) shows that the time iteration operator of equation (3.5) satisfies the Blackwell's sufficient conditions for a contraction mapping and therefore its convergence follows from the application of the Banach's fixed point theorem. However, to my knowledge, no one in the literature, except OK2017, has claimed the convergence of the fixed point iteration operator. Deaton and Laroque (1992) explain why they replace the time iteration operator by the fixed point iteration operator: *"In practice, this is an inconveniently slow algorithm, since it requires, at each iteration, a set of subsidiary iterations to solve for the next function, which is itself going to be modified at the next step. We have found that removing the subsidiary iterations does not prevent convergence."* (Deaton and Laroque, 1992, p. 9). In a companion paper, they elaborate: *"Note that if the innermost $f_n(z)$ on the right hand side of equation (28) were replaced by $f_{n+1}(z)$ the iteration would be a contraction and convergence would be guaranteed. However, such a procedure would require an iterative calculation for each new n, which would greatly increase computation time. And although the iteration defined by equation (28) is not generally a contraction, the procedure always seems to converge in practice."*[3] (Deaton and Laroque, 1995, p. S21). Rendahl

[3]Equation (28) refers to their fixed point operator, similar to the one described by the equation (3.3).

(2015) mentions that *"...there are no guarantees that the sequence of successive guesses obtained under fixed point iteration will eventually converge to the solution, and oscillating or exploding sequences are frequent"*, (Rendahl, 2015, p. 1119–1120).

The next section of this chapter provides a counter-example, consistent with assumptions of OK2017, for which the fixed point iteration operator defined by equation (3.3) does not converge to the SREE as OK2017 Theorem 2 claims. In fact, my counter-example shows that this operator oscillates between two different functions. This oscillation take place over a given domain included in the ergodic set.

3.3 A counter-example

Consider $\beta \equiv \frac{1-d}{1+r}$, for $d = 0$, and $r > 0$, s.t. $0 < \beta < 1$, an inverse consumption demand $F(c) = c^{-\rho}$, $\rho > 0$, $C > 0$, and a two point distribution for $\omega \sim [\underline{\omega}, \overline{\omega}]$, with probabilities α and $(1-\alpha)$ respectively, $0 < \alpha < 1$, $\underline{\omega} < \overline{\omega}$. This two point distribution is in the class of distributions considered by OK2017.

Define:

$$\begin{aligned} p^* &\equiv \beta E F(\omega) = \beta\big[\alpha F(\underline{\omega}) + (1-\alpha)F(\overline{\omega})\big], \\ z^* &\equiv F^{-1}(p^*) = F^{-1}\Big(\beta\big[\alpha F(\underline{\omega}) + (1-\alpha)F(\overline{\omega})\big]\Big), \end{aligned}$$

If the maximum capacity equals zero, p^* is the discounted expected price. If, in addition the current price equals p^*, then z^* is the current total available supply and consumption.

Also, define:

$$\begin{aligned} p^{**} &\equiv \beta\big[\alpha F(\underline{\omega}) + (1-\alpha)F(\overline{\omega} - C)\big], \\ z^{**} &\equiv F^{-1}(p^{**}) = F^{-1}\Big(\beta\big[\alpha F(\underline{\omega}) + (1-\alpha)F(\overline{\omega} - C)\big]\Big). \end{aligned}$$

If the maximum capacity is strictly positive and binding at the maximum shock realization, then p^{**} is the discounted expected price conditional on zero current stocks. More generally, p^{**} is an upper bound for the unconditional discounted expected price. If, in addition the current price equals p^{**}, then z^{**} is the current consumption.

Now, using these new definitions, I impose five assumptions that implicitly involve the modulus β, the maximum capacity C, the parameters of the shock distribution ($\underline{\omega}$, $\overline{\omega}$, α), and the inverse consumption demand function (ρ).

Assumptions: The set of parameters: $\underline{\omega}$, $\overline{\omega}$, α, ρ, β, and C are chosen such that:

A.1: $\underline{\omega} < z^* < \overline{\omega}$, which for this model is equivalent to assume that $F(\overline{\omega}) < p^* < F(\underline{\omega})$.[4]

This assumption ensures that the discounted expected price if the maximum capacity equals zero (p^*), lies in the ergodic set of prices and that z^* lies in the ergodic set of harvest shocks.[5]

A.2: $\beta EF(\omega + C) \geq F(\overline{\omega} - C)$.

This assumption implies that the discounted price if storage is zero and if the distribution of the shocks is shifted to the right by C ($\beta EF(\omega + C)$), cannot be smaller than the price implied by current consumption equal to $\overline{\omega} - C$. Also, this assumption implicitly defines an upper bound for the maximum capacity: $C \leq \overline{\omega} - F^{-1}(\beta EF(\omega + C))$.

By Lemma 1 (in Appendix D) assumptions A.1 and A.2 imply that $\underline{\omega} < z^{**} < z^* < \overline{\omega}$, and that $F(\underline{\omega}) > p^{**} > p^* > F(\overline{\omega})$. Hence, these two assumptions ensure that p^* and p^{**} belong to the ergodic set of prices, and that z^* and z^{**} belong to the ergodic set of total available supply as is shown in Figure 3.1.

A.3: $C \geq z^* - z^{**}$.

This assumption explicitly defines a lower bound for the maximum capacity. In order to make my counter-example works the maximum capacity C cannot be smaller than the difference between z^* and z^{**}. Assumptions A.1 and A.2 imply that such difference is a positive number.

A.4: $\dfrac{\underline{\omega} + z^*}{2} \leq z^{**}$.

This assumption establishes the relative location of z^{**} with respect to $\underline{\omega}$ and z^*. It implies that z^{**} cannot be smaller than the average of $\underline{\omega}$ and z^*.

A.5: $F(z) \geq \beta[\alpha F(\underline{\omega} + z - z^{**}) + (1 - \alpha)F(\overline{\omega} + z - z^{**} - C)]$, $\forall z$ s.t. $z^{**} \leq z \leq z^*$.

As shown in Appendix D, the right hand side of this assumption coincides with the analytical expression of f_n evaluated in the domain $[z^{**}, z^*]$ conditional in positive stocks.

[4] Notice that the first part of this assumption $F(\overline{\omega}) < \beta EF(\omega) \equiv p^*$ is equivalent to the assumption in Theorem 2, Deaton and Laroque (1992). Also, the second part of this assumption $p^* \equiv \beta EF(\omega) < F(\underline{\omega})$, it is true by construction if $\beta < 1$. However, I make this second part as an explicit assumption since Oglend and Kleppe (2017) allow the case with $\beta \geq 1$.

[5] The ergodic set of the harvest shocks is a subset of the ergodic set of the total available supply.

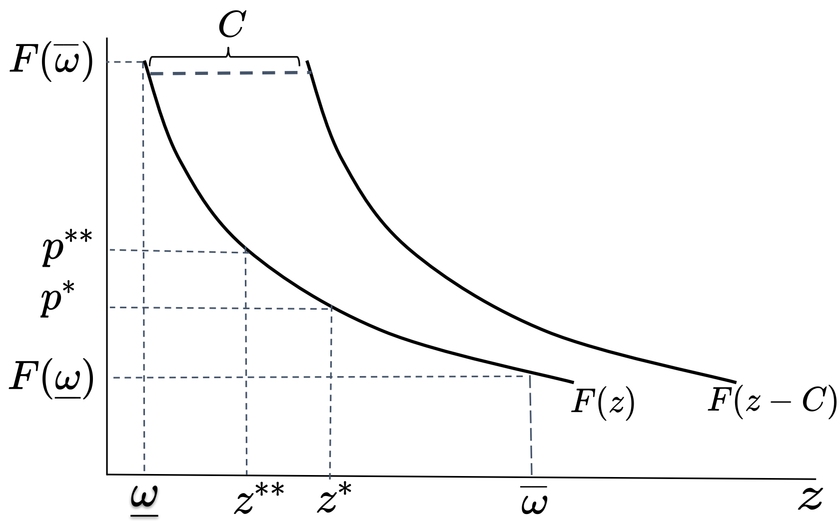

Figure 3.1: Location of z^*, z^{**}, p^*, and p^{**} implied by assumption A.1 and A.2.

Therefore, this assumption implies that $f_n(z)$ is equal to $F(z)$, $\forall z \in [z^{**}, z^*]$ and consequently that the storage implied by the iterate n is equal to zero $\forall z \in [z^{**}, z^*]$.

The following theorem establishes that the fixed point operator proposed by OK2017 in Theorem 2 does not converge.

Consider the OK2017 operator, fixing $d = 0$, as represented in equation (3.3):

$$f_{n+1}(z) = \min\left[F(z-C), \max\left[F(z), \beta E f_n\big(\omega + z - F^{-1}\{f_n(z)\}\big)\right]\right], \quad \forall n \in \mathbb{N}, \tag{3.6}$$

and define the first guess for the function $f_n(z)$ as: $f_0(z)$, a continuous, bounded and non-increasing function, $\forall z \geq \underline{\omega}$.

Theorem. *Under assumptions A.1-A.5, assume that for some $n \in \mathbb{N}$, the function f_n satisfies the following hypotheses:*

(i) $f_n(z) = F(z)$, $\forall z$ *s.t.* $\underline{\omega} \leq z \leq z^*$,

(ii) $f_n(z) = F(z - C)$, $\forall z \geq \overline{\omega}$,

then f_{n+1} satisfies the following properties:

(a) $f_{n+1}(z) = F(z)$, $\forall z$ *s.t.* $\underline{\omega} \leq z \leq z^{**}$,

(b) $f_{n+1}(z) = p^{**}$, $\forall z$ *s.t.* $z^{**} \leq z \leq z^*$,

(c) $f_{n+1}(z) = F(z - C)$, $\forall z \geq \overline{\omega}$,

and f_{n+2} *satisfies the same hypotheses as* f_n.

Proof: See Appendix D.

This theorem establishes that the functions differ over the domain z^{**} to z^*, at iterations n and $n+1$, but at iteration $n+2$ the function $f_{n+2}(z)$ equals function $f_n(z)$ over this domain.

Figure 3.2 shows the iterate f_n satisfying hypotheses (i) and (ii) and Figure 3.3 shows the iterate f_{n+1} satisfying properties (a), (b), and (c). The proof proceeds in two stages. In the first stage, I show that if iterate n satisfies hypotheses (i) and (ii) then iterate $n+1$ satisfies the properties (a), (b), and (c). In the second, I show that if iterate $n+1$ satisfies properties (a), (b), and (c), then iterate $n+2$ satisfies the same hypotheses as does iterate n. In consequence, the elements of this sequence alternate between two different functions. Therefore this sequence of functions does not converge.

Corollary 1 in Appendix D shows that for this counter-example the fixed point iteration operator of OK2017 does not satisfy Blackwell's sufficient conditions for a contraction mapping.

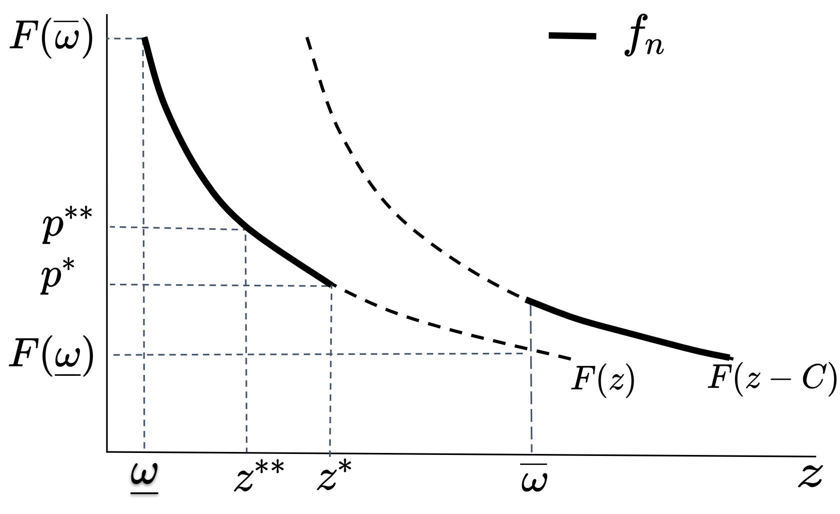

Figure 3.2: Iterate f_n satisfying hypotheses (i) and (ii).

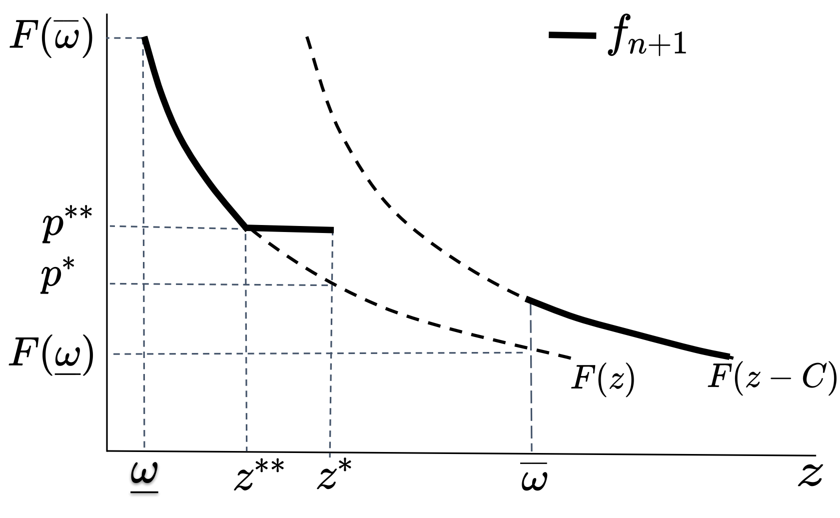

Figure 3.3: Iterate f_{n+1} satisfying properties (a), (b), and (c).

3.4 Numerical Example

Let $f_0 \equiv F(z)$, $\forall z \geq \underline{\omega}$. and $C = \overline{\omega} - z^*$, Then:

$$\begin{aligned} f_1(z) &= \min\left\{F(z-C), \max\left[F(z), \beta E f_0\big(\omega + z - F^{-1}\{f_0(z)\}\big)\right]\right\}, \\ f_1(z) &= \min\left\{F(z-C), \max\left[F(z), \beta E F\big(\omega + z - F^{-1}\{F(z)\}\big)\right]\right\}, \\ f_1(z) &= \min\left\{F(z-C), \max\left[F(z), \beta E F\big(\omega\big)\right]\right\}. \end{aligned} \tag{3.7}$$

Then:

$$f_1(z) = \begin{cases} F(z), & \underline{\omega} \leq z \leq z^*, \\ p^*, & z^* \leq z \leq z^* + C, \\ F(z-C) & z \geq z^* + C. \end{cases} \tag{3.8}$$

Notice that $f_1(z)$ satisfies the hypotheses (i), (ii) of the Theorem. Now I pick a numerical setup that satisfies the assumptions A.1-A.5. I consider the numerical example in OK2017, fixing $d = 0$, assuming a two point distribution for the shocks.

$F(c) = c^{-\rho}$, with $\rho = 4$, $r = 0.05$, $C = 60$, $\underline{\omega} = 70$, $\overline{\omega} = 220$, $\alpha = 0.5$. Then the implied values are: $z^* = 84.0515$, and $z^{**} = 83.5115$. and assumptions A.1-A.5 are satisfied.

3.5 Conclusions

This chapter provides a counter-example that establishes that Theorem 2 of OK2017 is false; their fixed point iteration operator does not always converge to the SREE as claimed.

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Appendix A

The estimated model is normalized at the mean and standard deviation of per capita maize production, assuming net supply shocks $N(0, 1)$ and linear inverse consumption demand $F(c) = a + bc$, where c is interpreted as per capita consumption. For the calculation of consumption demand elasticities, I re-scale the distribution of maize production, setting its mean and standard deviation at μ and σ, respectively, and use the identification Proposition of Deaton and Laroque (1996, Proposition 1, p. 906) to correspondingly re-scale the consumption demand parameters. The re-scaled inverse consumption demand is $F(c) = a - b\frac{\mu}{\sigma} + \frac{b}{\sigma}c$. Therefore, the price elasticity of consumption demand, evaluated at mean real detrended price, is given by $\frac{1}{1-(a-b\frac{\mu}{\sigma})/\overline{p}}$, where $\overline{p}$ denotes the mean of detrended real prices.

Maize production data from USDA/PSD and world population data from the US Census Bureau are available for the period 1961-2012.[6] Figure A.1 shows per capita production which is detrended assuming linear trends, with three subsample periods: 1961-1970, 1971-2000, and 2001-2012, implying three distinct intercept and slope parameters. The values for the mean μ and standard deviation σ used in my calculation of consumption demand elasticities are obtained as the weighted averages of the intercepts and standard deviations of detrended per capita production, respectively, of each of these subsample periods. Figure A.2 shows detrended per capita production for the period 1961-2012.

[6] `http://apps.fas.usda.gov/psdonline/psdQuery.aspx` and `http://www.census.gov/population/international/data/idb/informationGateway.php`, respectively.

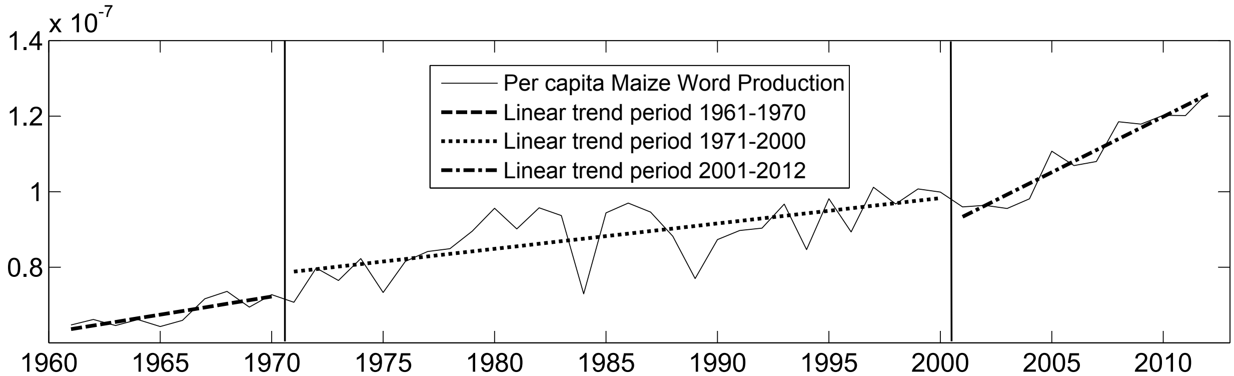

Figure A.1: Per capita Maize World Production 1961-2012. Units in the vertical axis are 1000 MT per capita.

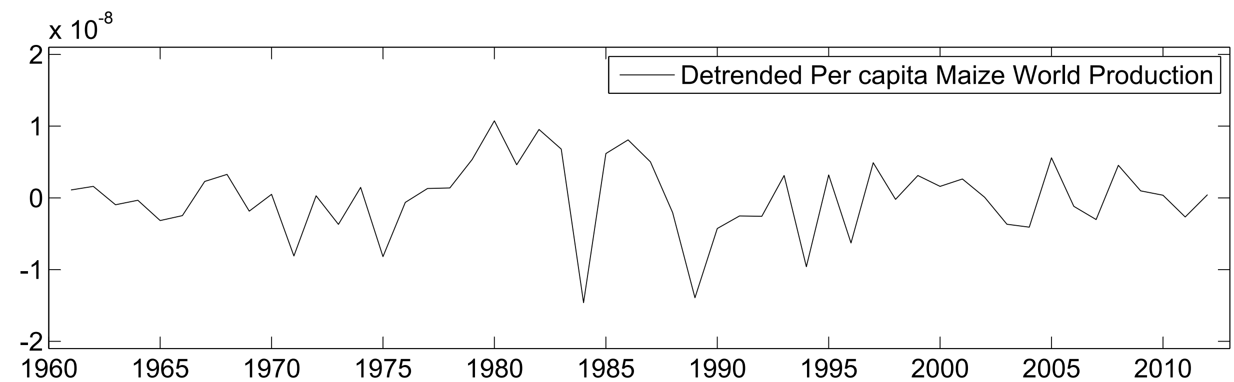

Figure A.2: Detrended Per capita Maize World Production 1961-2012.

Appendix B

Consider a model exactly equal to the one presented in section 2, except that the inverse demand function is $\widetilde{F} \equiv \frac{F}{\lambda}$, with $\lambda > 0$. Also assume that $\widetilde{f}_{<n>}(z) \equiv \frac{f_{<n>}(z)}{\lambda}$.

Proposition 3: *The functions* $\widetilde{f}_{<n+1>}$ *and* $f_{<n+1>}$ *satisfy:* $\widetilde{f}_{<n+1>}(z) = \frac{f_{<n+1>}(z)}{\lambda}$.

Proof of Proposition 3: Since $\left(\frac{F}{\lambda}\right)^{-1}(p) = F^{-1}(\lambda p)$, for $g(z) \equiv \frac{f_{<n+1>}(z)}{\lambda}$, we have:

$$\begin{aligned} &\max\left\{\left(\frac{1-d}{1+r}\right) E\widetilde{f}_{<n>}\big(\omega + (1-d)\big[z - \left(\frac{F}{\lambda}\right)^{-1}\big(g(z)\big)\big]\big), \frac{F(z)}{\lambda}\right\}, \\ &= \max\left\{\frac{1}{\lambda}\left(\frac{1-d}{1+r}\right) Ef_{<n>}\big(\omega + (1-d)\big[z - F^{-1}\big(\lambda g(z)\big)\big]\big), \frac{F(z)}{\lambda}\right\}, \\ &= \frac{1}{\lambda}\max\left\{\left(\frac{1-d}{1+r}\right) Ef_{<n>}\big(\omega + (1-d)\big[z - F^{-1}\big(f_{<n+1>}(z)\big)\big]\big), F(z)\right\}, \\ &= \frac{1}{\lambda} f_{<n+1>}(z) = g(z). \end{aligned}$$

Appendix C

Consider a storage model with a continuous and strictly decreasing inverse demand function $F(c)$, a sequence $\{\omega_t\}_{t\in\mathbb{N}}$ of i.i.d shocks with support $[\underline{\omega}, \overline{\omega}[$, and an interest rate $r > 0$ and a physical deterioration rate of stocks $d \in [0, 1)$. Also, consider the space:

$$\widetilde{G} \equiv \{g : [\underline{\omega}, +\infty[\to \mathbb{R},\ g \geq 0,\ g \text{ continuous}, g \text{ non-increasing}, g(\underline{\omega}) = F(\underline{\omega})\}.$$

For any given $g \in \widetilde{G}$, consider the operator T defined as $g \mapsto Tg$, where:

$$Tg(z) \ \equiv\ \max\left\{\left(\frac{1-d}{1+r}\right) Eg\left[\omega + (1-d)(z - F^{-1}(Tg(z)))\right], F(z)\right\}$$

Proposition 4: If $g \in \widetilde{G}$ is strictly decreasing, so is Tg.

Proof of Proposition 4: Following the section (iii) Theorem 1's proof of Deaton and Laroque (1992), assume that g is strictly decreasing on $[\underline{\omega}, +\infty[$, and by contradiction assume Tg is not strictly decreasing. Since Tg is non-increasing, let $[z', z''[$ be the first interval on which Tg is constant,. i.e., such that Tg is strictly decreasing on $[\underline{\omega}, z'[$, and let m the value of Tg on this interval. Since F is strictly decreasing then for all $z \in [z', z''[$:

$$Tg(z) = m = \beta Eg\left[\omega + (1-d)(z - F^{-1}(m))\right].$$

The right-hand side of the above expression is constant, and since g is strictly decreasing, we must have $g\left[\underline{\omega} + (1-d)(z - F^{-1}(m))\right]$ constant on $[z', z''[$, which contradicts the assumption that g is strictly decreasing on $[\underline{\omega}, +\infty[$.

Appendix D

Lemma 1: $\underline{\omega} < z^{**} < z^{*} < \overline{\omega}$.

Proof of Lemma 1:

By assumption A.1, $\underline{\omega} < z^{*} < \overline{\omega}$, then:

$$\begin{aligned}
&\underline{\omega} < z^{*}, \\
&\Rightarrow F(\underline{\omega}) > F(z^{*}), \\
&\Rightarrow F(\underline{\omega}) > p^{*}, \\
&\Rightarrow F(\underline{\omega}) > \alpha F(\underline{\omega}) + (1-\alpha)p^{*}, \\
&\Rightarrow F(\underline{\omega}) > \alpha F(\underline{\omega}) + (1-\alpha)(\beta EF(\omega)), && \text{(using the definition of } p^{*}\text{)} \\
&\Rightarrow F(\underline{\omega}) > \alpha F(\underline{\omega}) + (1-\alpha)(\beta EF(\omega + C)), && \text{(because } C > 0\text{, and } F \text{ is strictly decreasing)} \\
&\Rightarrow F(\underline{\omega}) > \alpha F(\underline{\omega}) + (1-\alpha)F(\overline{\omega} - C), && \text{(by assumption A.2)} \\
&\Rightarrow F(\underline{\omega}) > \beta\big[\alpha F(\underline{\omega}) + (1-\alpha)F(\overline{\omega} - C)\big], \\
&\Rightarrow \underline{\omega} < F^{-1}\Big(\beta\big[\alpha F(\underline{\omega}) + (1-\alpha)F(\overline{\omega} - C)\big]\Big), \\
&\Rightarrow \underline{\omega} < F^{-1}(p^{**}), \\
&\Rightarrow \underline{\omega} < z^{**}.
\end{aligned}$$

Also: $p^{**} > p^{*}$, (by the definition of p^{*} and p^{**})
$\Rightarrow F(p^{**}) < F(p^{*})$,
$\Rightarrow z^{**} < z^{*}$.

Therefore: $\underline{\omega} < z^{**} < z^{*} < \overline{\omega}$, and $F(\underline{\omega}) > p^{**} > p^{*} > F(\overline{\omega})$.

Lemma 2: Define the storage function implied by iteration n as $z - F^{-1}\{f_n(z)\}$. Then the storage function implied by iteration n is bounded by $0 \leq z - F^{-1}\{f_n(z)\} \leq C$, $\forall n \in \mathbb{N}$, and $\forall z \geq \underline{\omega}$.

Proof of Lemma 2:

$F(z) \leq f_n(z) \leq F(z-C)$,
$\Rightarrow z \geq F^{-1}\{f_n(z)\} \geq z - C$, (Applying F^{-1})
$\Rightarrow -z \leq -F^{-1}\{f_n(z)\} \leq -z + C$, (Multiplying by -1)
$\Rightarrow 0 \leq z - F^{-1}\{f_n(z)\} \leq C$. (Adding z)

Lemma 3: The iterate n evaluated at the maximum shock satisfies the following condition: $f_n(\overline{\omega}) = F(\overline{\omega} - C)$, $\forall n \in \mathbb{N}$.

Proof of Lemma 3:

$$f_{n+1}(\overline{\omega}) = \min\left[F(\overline{\omega}-C), \max\left[F(\overline{\omega}), \beta E f_n\big(\omega + \overline{\omega} - F^{-1}\{f_n(\overline{\omega})\}\big)\right]\right], \quad \forall n \in \mathbb{N}.$$

However:

$\beta E f_n\big(\omega + \overline{\omega} - F^{-1}\{f_n(\overline{\omega})\}\big)$,
$\geq \beta E F\big(\omega + \overline{\omega} - F^{-1}\{f_n(\overline{\omega})\}\big)$, (because $f_n(z) \geq F(z)$, $\forall z > \underline{\omega}$)
$\geq \beta E F\big(\omega + C\big)$, (because $\overline{\omega} - F^{-1}\{f_n(\overline{\omega})\} \leq C$, by Lemma 2)
$\geq F(\overline{\omega} - C)$. (by assumption A.2)

Therefore, $f_{n+1}(\overline{\omega}) = F(\overline{\omega} - C)$, $\forall n \in \mathbb{N}$.

Proof of the Theorem:

First, I prove that f_{n+1} satisfies properties (a)-(c).

Properties (a) and (b):

$f_{n+1}(z) = F(z)$, $\forall z$ s.t. $\underline{\omega} \leq z \leq z^{**}$, and $f_{n+1}(z) = p^{**}$, $\forall z$ s.t. $z^{**} \leq z \leq z^{*}$, respectively.

By hypothesis (i), $\forall z$ s.t. $\underline{\omega} \leq z \leq z^{*}$, $f_n(z) = F(z)$, then:

$$\begin{aligned}
f_{n+1}(z) &= \min\left[F(z-C), \max\left[F(z), \beta E f_n\big(\omega + z - F^{-1}\{f_n(z)\}\big)\right]\right], \\
f_{n+1}(z) &= \min\left[F(z-C), \max\left[F(z), \beta E f_n\big(\omega + z - F^{-1}\{F(z)\}\big)\right]\right], \\
f_{n+1}(z) &= \min\left[F(z-C), \max\left[F(z), \beta E f_n(\omega)\right]\right], \\
f_{n+1}(z) &= \min\left[F(z-C), \max\left[F(z), \beta\big(\alpha f_n(\underline{\omega}) + (1-\alpha) f_n(\overline{\omega})\big)\right]\right], \\
f_{n+1}(z) &= \min\left[F(z-C), \max\left[F(z), \beta\big(\alpha F(\underline{\omega}) + (1-\alpha) F(\overline{\omega}-C)\big)\right],\right] \quad \text{(by hypotheses (i-ii))}, \\
f_{n+1}(z) &= \min\left[F(z-C), \max\left[F(z), p^{**}\right]\right].
\end{aligned}$$

Therefore, using assumption A.3, I conclude that: $f_{n+1}(z) = F(z)$, $\forall z$ s.t. $\underline{\omega} \leq z \leq z^{**}$, and $f_{n+1}(z) = p^{**}$, $\forall z$ s.t. $z^{**} \leq z \leq z^{*}$, thus function f_{n+1} satisfies properties (a) and (b).

Property (c):

$f_{n+1}(z) = F(z-C)$, $\forall z \geq \overline{\omega}$.

By hypothesis (ii), $\forall z \geq \overline{\omega}$, $f_n(z) = F(z-C)$, then:

$$
\begin{aligned}
f_{n+1}(z) &= \min\left[F(z-C), \max\left[F(z), \beta E f_n\big(\omega + z - F^{-1}\{f_n(z)\}\big)\right]\right], \\
f_{n+1}(z) &= \min\left[F(z-C), \max\left[F(z), \beta E f_n\big(\omega + z - F^{-1}\{F(z-C)\}\big)\right]\right], \\
f_{n+1}(z) &= \min\left[F(z-C), \max\left[F(z), \beta E f_n(\omega + C)\right]\right].
\end{aligned}
$$

Notice that: $\beta E f_n(\omega + C) \geq \beta E F(\omega + C) \geq F(\overline{\omega} - C)$, by assumption A.2 and because $f_n(z) \geq F(z)$. Therefore, $\beta E f_n(\omega + C) \geq F(z-C) > F(z)$, $\forall z \geq \overline{\omega}$, and $f_{n+1}(z) = F(z-C)$, thus function f_{n+1} satisfies property (c).

Second, I prove that f_{n+2} satisfies hypotheses (i) and (ii).

Hypothesis (i):

$f_{n+2}(z) = F(z)$, $\forall z$ s.t. $\underline{\omega} \leq z \leq z^*$.

Case 1: by property (a), $\forall z$ s.t. $\underline{\omega} \leq z \leq z^{**}$, $f_{n+1}(z) = F(z)$, then:

$$
\begin{aligned}
f_{n+2}(z) &= \min\left[F(z-C), \max\left[F(z), \beta E f_{n+1}\big(\omega + z - F^{-1}\{f_{n+1}(z)\}\big)\right]\right], \\
f_{n+2}(z) &= \min\left[F(z-C), \max\left[F(z), \beta E f_{n+1}\big(\omega + z - F^{-1}\{F(z)\}\big)\right]\right], \\
f_{n+2}(z) &= \min\left[F(z-C), \max\left[F(z), \beta E f_{n+1}(\omega)\right]\right], \\
f_{n+2}(z) &= \min\left[F(z-C), \max\left[F(z), \beta\big(\alpha f_{n+1}(\underline{\omega}) + (1-\alpha) f_{n+1}(\overline{\omega})\big)\right]\right], \\
f_{n+2}(z) &= \min\left[F(z-C), \max\left[F(z), \beta\big(\alpha F(\underline{\omega}) + (1-\alpha) F(\overline{\omega} - C)\big)\right]\right], \\
&\qquad \text{(by properties (a), (c), and Lemma 3)} \\
f_{n+2}(z) &= \min\left[F(z-C), \max\left[F(z), p^{**}\right]\right].
\end{aligned}
$$

Then, $\forall z$ s.t. $\underline{\omega} \leq z \leq z^{**}$, $p^{**} \leq F(z) \leq F(z-C)$, and therefore: $f_{n+2}(z) = F(z)$.

Case 2: by property (b) $\forall z$ s.t. $z^{**} \leq z \leq z^*$, $f_{n+1}(z) = p^{**}$, then:

$$
\begin{aligned}
f_{n+2}(z) &= \min\left[F(z-C), \max\left[F(z), \beta E f_{n+1}\big(\omega + z - F^{-1}\{f_{n+1}(z)\}\big)\right]\right], \\
f_{n+2}(z) &= \min\left[F(z-C), \max\left[F(z), \beta E f_{n+1}\big(\omega + z - F^{-1}\{p^{**}\}\big)\right]\right], \\
f_{n+2}(z) &= \min\left[F(z-C), \max\left[F(z), \beta E f_{n+1}\big(\omega + z - z^{**}\big)\right]\right], \\
f_{n+2}(z) &= F(z).
\end{aligned}
$$

The last step comes from the fact that: $\forall z$ s.t. $z^{**} \leq z \leq z^*$, $\beta E f_{n+1}\big(\omega + z - z^{**}\big) \leq F(z) < F(z-C)$. Since:

$$\begin{aligned}
\beta E f_{n+1}\big(\omega + z - z^{**}\big) &= \beta\big[\alpha f_{n+1}\big(\underline{\omega} + z - z^{**}\big) + (1-\alpha) f_{n+1}\big(\overline{\omega} + z - z^{**}\big)\big], \\
\beta E f_{n+1}\big(\omega + z - z^{**}\big) &= \beta\big[\alpha F(\underline{\omega} + z - z^{**}) + (1-\alpha) F(\overline{\omega} + z - z^{**} - C)\big], \\
& \quad \text{(by assumption A.4 and properties (a) and (c))} \\
\beta E f_{n+1}\big(\omega + z - z^{**}\big) &\leq F(z) \text{(by assumption A.5).}
\end{aligned}$$

Then, $f_{n+2}(z) = F(z)$, $\forall z$ s.t. $\underline{\omega} \leq z \leq z^{*}$, thus f_{n+2} satisfies hypothesis (i).

Hypothesis (ii):

$f_{n+2}(z) = F(z - C)$, $\forall z \geq \overline{\omega}$.

By property (c) $\forall z \geq \overline{\omega}$, $f_{n+1}(z) = F(z - C)$, then:

$$\begin{aligned}
f_{n+2}(z) &= \min\left[F(z-C), \max\left[F(z), \beta E f_{n+1}\big(\omega + z - F^{-1}\{f_{n+1}(z)\}\big)\right]\right], \\
f_{n+2}(z) &= \min\left[F(z-C), \max\left[F(z), \beta E f_{n+1}\big(\omega + z - F^{-1}\{F(z-C)\}\big)\right]\right], \\
f_{n+2}(z) &= \min\left[F(z-C), \max\left[F(z), \beta E f_{n+1}(\omega + C)\right]\right].
\end{aligned}$$

Notice that: $\beta E f_{n+1}(\omega + C) \geq \beta E F(\omega + C) \geq F(\overline{\omega} - C)$, because $f_{n+1}(z) \geq F(z)$ and by assumption A.2. Therefore, $\beta E f_{n+1}(\omega + C) \geq F(z - C) > F(z)$, $\forall z \geq \overline{\omega}$, and $f_{n+2}(z) = F(z - C)$, thus f_{n+2} satisfies hypothesis (ii).

Corollary 1: Under assumptions A.1-A.5 the fixed point operator proposed by OK2017 does not satisfy Blackwell's sufficient conditions for a contraction mapping: monotonicity and discounting.

Proof of Corollary 1:

Monotonicity:
By the theorem presented in section 3.3, the sequence of functions obtained by the application of the fixed point iteration operator of OK2017 satisfies that $f_n(z) < f_{n+1}(z)$, $\forall z \in [z^{**}, z^{*}]$ and $f_{n+1}(z) > f_{n+2}(z) = f_n(z)$, $\forall z \in [z^{**}, z^{*}]$ contradicting the monotonicity property.

Discounting:
By the theorem presented in section 3.3, the sequence of functions obtained by the application of the fixed point iteration operator of OK2017 satisfies that $f_{n+1}(z) = p^{**}$ $\forall z \in [z^{**}, z^{*}]$. Take a positive constant $a = F(\underline{\omega}) - p^{**}$ and define the operator T such that $f_{n+1}(z) \equiv T f_n(z)$, $\forall n \in \mathbb{N}$, and for all $z \geq \underline{\omega}$. Then:
$T(f_{n+1}(z) + a) = F(\underline{\omega}) > f_{n+1}(z) + \beta a = p^{**} + \beta a$, $\forall z \in [z^{**}, z^{*}]$, in contradiction to the discounting property.