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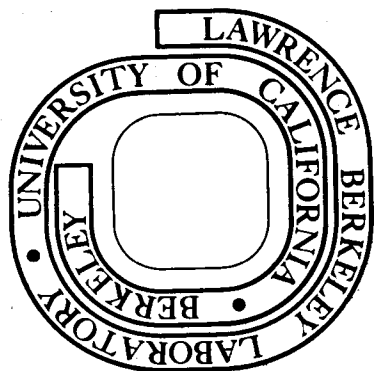1975-08-01

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ON UNDERSTANDING SPIN-FLIP SYNCHROTRON RADIATION

AND THE TRANSVERSE POLARIZATION OF ELECTRONS IN

STORAGE RINGS[*]

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August 18, 1975

ABSTRACT

A mainly didactic discussion is given of the mechanism for
the gradual build up of transverse polarization of electrons and posi-
trons in storage rings. The history and basic results are reviewed
briefly. Then a naive explanation of the polarization in terms of
spontaneous emission via a nonrelativistic magnetic dipole transition in
a moving inertial frame is presented and criticized. Although plausible
and surprisingly good (for electrons and positrons), the elementary dis-
cussion fails, chiefly because the spin-magnetic-moment system cannot
be treated in isolation from the orbital motion. A correct semiclassical
description of radiation by a spin system is then given, in direct analogy
with semiclassical radiation theory for charged particles ignoring spin.
The classical equation of motion for a spin in relativistic motion,
derived originally by Thomas, is used to obtain an effective Hamiltonian
of interaction of a spin with electromagnetic fields. Emission and
absorption of radiation is then described by replacing the classical
electromagnetic fields with the appropriately normalized photon fields.
It is proved in an appendix that the relevant quantum-mechanical matrix
element reduces to this semiclassical form in the limit applicable to
synchrotron radiation (classical orbit and neglect of recoil). The
resulting formulas are applicable to charged particles of arbitrary
g factor and serve as a basis for generalization of the Russian results
for the characteristic time of polarization and its asymptotic value.
These results are of physical interest only for the known case of
g = 2 but serve useful pedagogic purposes, refuting some of the
expectations of the naive explanation. The various differential spectra
in angle and in frequency for numbers of photons and for radiated
power for g = 2 are treated in detail and compared with the
corresponding spectra for ordinary synchrotron radiation.

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I. INTRODUCTION

The emission of synchrotron radiation by a relativistic charged particle subject to transverse acceleration is a much studied and much used phenomenon. Its history as a theoretical possiblity extends back at least to before 1900 with the relativistic generalization of the Larmor power formula by Liénard and others. For a charge in uniform, circular motion, the detailed harmonic content and angular distributions for each harmonic were calculated in 1911 for an Adams Prize Essay by Schott (1912), but they remained an exercise in mathematical physics until the 1940's when the first electron synchrotrons were constructed and synchrotron light was observed. The names of Pomeranchuk, Schiff, and Schwinger are among those who gave modern theoretical discussions of the phenomena in published and unpublished work, with the paper of Schwinger (1949) containing the various theoretical results in their most tractable form. The essentials now occur in numerous advanced texts.[1]

More recent and somewhat less well known is the realization of a gradual polarization of electrons and positrons as they experience a sustained transverse acceleration while orbiting in a storage ring. The mechanism is the emission of spin-flip synchrotron radiation, as first pointed out by Ternov, Loskutov, and Korovina (1961). For initially unpolarized electrons or positrons of charge e, mass m, energy $E = \gamma mc^2$ in uniform motion in a circle of radius ρ, there is a gradual build up of transverse polarization according to

$$P(t) = P_o\left(1 - e^{-t/\tau_o}\right) \tag{1a}$$

where the maximum polarization is

$$P_o = 8/5\sqrt{3} = 0.9238 \tag{1b}$$

and the characteristic time τ_o is

$$\tau_o = \left[\frac{5\sqrt{3}}{8} \frac{e^2 \hbar \gamma^5}{m^2 c^2 \rho^3} \right]^{-1} \tag{1c}$$

(Sokolov and Ternov, 1963). The polarization is perpendicular to both velocity and acceleration, that is, along the direction of the magnetic field responsible for the bending. Positrons are polarized parallel to the magnetic field, electrons antiparallel.

The original work of Sokolov, Ternov, and collaborators was done with exact solutions for a relativistic Dirac electron in a uniform magnetic field. Subsequently, Baier and Katkov generalized the results to motion in inhomogeneous fields. For the spin-flip radiation by relativistic electrons or positrons they obtained (Baier and Katkov, 1967a; Baier, 1971a,b) the transition probability per unit time,

$$W = \frac{5\sqrt{3}}{16} \frac{e^2 \hbar}{m^2 c^5} \gamma^5 |\dot{\underset{\sim}{\beta}}|^3 \left[1 - \frac{2}{9}(\underset{\sim}{\zeta} \cdot \hat{\underset{\sim}{\beta}})^2 + \frac{8\sqrt{3}}{15} \underset{\sim}{\zeta} \cdot (\hat{\underset{\sim}{\beta}} \times \hat{\dot{\underset{\sim}{\beta}}}) \right] \tag{2}$$

where the unit axial vector $\underset{\sim}{\zeta}$ specifies the initial spin direction in the electron's rest frame and $\hat{\underset{\sim}{\beta}}$ and $\hat{\dot{\underset{\sim}{\beta}}}$ are unit vectors in the directions of the local velocity and acceleration, respectively. For a circular orbit with $|\dot{\underset{\sim}{\beta}}| = c/\rho$, Eq. (2) leads to results (1) with the senses of polarization for electrons and positrons already stated.

The amount of spin-flip radiation is extremely small compared to the ordinary (nonflip) synchrotron radiation. The ratio of the powers radiated is (Ternov, Loskutov, and Korovina, 1961; Sokolov and Ternov, 1963)

$$\frac{P_{\text{spin-flip}}}{P_{\text{ordinary}}} = 3\left(\frac{\hbar\gamma^2}{mc\rho}\right)^2 \left(1 \pm \frac{35\sqrt{3}}{64} \right) \tag{3}$$

where the choice of sign depends on the initial spin state of the particle. Only when γ approaches the critical value,

$$\gamma_c = \left(\frac{mc\rho}{\hbar} \right)^{\frac{1}{2}} , \tag{4}$$

will the amount of spin-flip radiation be comparable to the ordinary synchrotron radiation. At present a typical bending radius for an electron storage ring is $\rho \simeq 13$ meters. Hence $\gamma_c \simeq 6 \times 10^6$, while $\gamma < 10^4$, showing that the ratio (3) is of the order of 10^{-11}. The smallness of this ratio is reflected in the relative largeness of the build-up time τ_o.

In practice one must distinguish the ring's effective bending radius ρ from the average orbit radius R, defined is the circumference of the orbit divided by 2π. Let the s be the length along the actual orbit in the storage ring and $\rho(s)$ be the radius of curvature of the orbit at each point. Then by consideration of the accumulation of probabilities it is easy to show that the effective radius of curvature ρ to be inserted in (1c) is

$$\rho^{-3} = \oint \left[\rho(s)\right]^{-3} ds \bigg/ \oint ds . \tag{5}$$

This formula is valid even if $\rho(s)$ changes sign locally around the orbit as would occur with the so-called wiggler magnets, suggested as a means of controlling the characteristic time τ_o (Paterson, Rees, and Wiedemann, 1975). For a storage ring consisting of a set of identical bending magnets of bending radius ρ and straight sections

combining to an orbit of circumference $2\pi R$, the right-hand side of (5) is equal to $(\rho/R)\rho^{-3}$. In practical units the time constant τ_o is then

$$\tau_o(\text{sec}) = 98.66 \frac{[\rho(\text{m})]^3}{[E(\text{GeV})]^5} \cdot \frac{R}{\rho} \quad . \qquad (6)$$

For SPEAR, the storage ring at the Stanford Linear Accelerator Center, $\rho = 12.7$ m, $R = 37.3$ m. At 2 GeV per beam the build-up time is roughly 5 hours, while at 4 GeV per beam it is about 10 minutes. The strong dependence on energy means that the polarization can be utilized as an effective physics tool only in the upper energy range of existing storage rings (SPEAR and DORIS, at Hamburg).

Indications of a build up of the polarization in a single circulating beam were first reported in 1968 by the Orsay group (Belbeoch et al., 1968), with unambiguous evidence from both Novosibirsk and Orsay in 1971.[2] The first observations on polarization with two beams, under conditions similar to actual running for physics, were made at Orsay and presented by LeDuff et al. (1973). More recently observations have been made at SPEAR on the polarization of a single stored beam with $E = 2.4$ GeV (Camerini et al., 1975). The first observation of polarization with stored colliding beams in the reactions $e^+e^- \to \mu^+\mu^-$ and $e^+e^- \to$ hadrons at 3.7 GeV per beam and its use in clarifying the physics of $e^+e^- \to$ hadrons has been reported by Schwitters et al. (1975). Contemporaneously, polarization measurements in the colliding beam reaction $e^+e^- \to \mu^+\mu^-$ at 0.5-0.7 GeV per beam have been communicated from Novosibirsk by Kurdadze et al. (1975).

For all practical purposes the works of Sokolov and Ternov and cf Baier and Katkov, especially the review by Baier (1971b) with its

discussion of both theoretical and practical problems, are more than adequate to describe the radiative polarization of beams in storage rings. Nevertheless, it seems that there is the need for an "anschaulich", didactic discussion of the subject. After all, Schwinger (1954) demonstrated clearly that ordinary synchrotron radiation is an entirely classical phenomenon. He showed that the orbit is classical provided $(\hbar c/E\rho) \ll 1$, where E is the total energy of the particle and ρ is the orbit radius of curvature, and that the first order quantum-mechanical corrections were obtained by replacement of $\omega \to \omega(1 + \hbar\omega/E)$ in the differential transition probability. It follows that for relativistic particles with $1 \ll \gamma \ll \gamma_c$ the orbit can be treated classically and recoil effects can be neglected. This regime of approximation is the basis of the treatment of the spin-flip synchrotron radiation and similar problems by Baier and Katkov (1967a,b, 1968).[3] The works of Schwinger and of Baier and Katkov are important in seeking as classical an understanding as possible of the phenomenon. We focus on the spin itself and seek in its dynamics a simple physical basis for the spin-flip radiation. The words "spin-flip" warn, of course, that the treatment cannot be completely classical--the electron spin must be treated quantum-mechanically--but otherwise it is reasonable to expect that one can obtain an understanding of the phenomenon in simple intuitive terms. It turns out that there are subtleties that prevent the realization of this expectation in its naivest form, but a satisfying elementary explanation can be obtained nevertheless.

The plan of the paper is as follows. Firstly, the most naive orientation is presented. It does surprisingly and deceptively well. Then its shortcomings are described. Next, the familiar semiclassical

treatment of emission of radiation found in texts on quantum mechanics
is outlined and generalized via the classical relativistic equation
of motion of spin to include spin-flip radiation. The effective
interaction Hamiltonian so obtained serves as the basis of a
semiclassical treatment of the radiative polarization for a particle of
charge e and arbitrary g-factor. The proof that the effective inter-
action Hamiltonian leads in the soft-photon limit to the same matrix
element as the Dirac current with γ^μ and $\sigma^{\mu\nu}$ couplings is reserved
for an appendix. The virtue of a treatment with arbitrary g-factor,
seemingly only an academic curiosity, is in its ability to confound
some of the "common sense" notions of the naive orientation. The final
section treats the angular and frequency spectra of the spin-flip
radiation for electrons and positrons (g = 2). These are of
pedagogical, if not practical, value.

II. NAIVE TREATMENT AND ITS SHORTCOMINGS

A. Elementary description

The physicist's appetite for an elementary description of
radiative polarization is whetted by the following facts:

(1) The effect involves spin-flip.

(2) The rate is very slow, as befits a magnetic dipole
transition between states with a small energy difference.

(3) The electrons and positrons are polarized with their
magnetic moments parallel to the magnetic field, corresponding to
the state of lowest energy of an isolated spin system.

(4) Formulas (1c) or (2) smack of magnetic dipole, with
$|\underset{\sim}{\beta}|^3$ providing the factor of ω^3 and $|\underset{\sim}{\mu}|^2/\hbar$ visible in the
product of fundamental constants.

Obviously, he says, go to the rest frame of the orbiting electron
and consider a simple M1 transition from the upper energy level to
the lower. We follow his prescription.

Though we know that for relativistic particles all that
affects the character of the radiation is a segment of trajectory
of length d $\sim \rho/\gamma$, corresponding to an angular deflection
$\Delta\theta \sim 1/\gamma$, for simplicity we consider a particle of charge e and
mass m moving at constant speed v = cβ in a circular orbit of
radius ρ in a uniform static magnetic field B. The orbital

frequency is $\omega_0 = v/\rho = \omega_B/\gamma$, where $\omega_B = eB/mc$ is the nonrelativistic cyclotron frequency. We now consider the fields in an instantaneously comoving inertial frame K' moving with speed $v = c\beta$ tangent to the circle. The magnetic field B appears in this frame as a magnetic field $B' = \gamma B$ in the same direction as $\underset{\sim}{B}$ and an electric field $E' = \gamma\beta B$ in the direction $\underset{\sim}{v} \times \underset{\sim}{B}$, as shown in Fig. 1. Suppose that the spin degree of freedom can be treated nonrelativistically in this frame. With magnetic moment,

$$\underset{\sim}{\mu} = \frac{g}{2} \cdot \frac{e\hbar}{2mc} \underset{\sim}{\sigma} , \qquad (7)$$

the spin system has two energy levels in K' with frequency difference,

$$\omega'_{12} = \left|\frac{g}{2}\right| \cdot \frac{eB'}{mc} = \left|\frac{g}{2}\right|\gamma^2\omega_0 . \qquad (8)$$

The transition probability per unit time for a spontaneous magnetic dipole transition from the upper state to the lower is

$$w' = \frac{4}{3\hbar}\left(\frac{\omega'_{12}}{c}\right)^3 \left|\langle 2|\underset{\sim}{\mu}|1\rangle\right|^2 . \qquad (9)$$

With (7) and (8) this becomes

$$w' = \frac{2}{3}\left|\frac{g}{2}\right|^5 \frac{e^2\hbar}{m^2c^5} \gamma^6 \omega_0^3 . \qquad (10)$$

Time dilatation gives a laboratory transition rate reduced by one power of γ. With $\omega_0 = c/\rho$ for a relativistic particle, (10) then leads to a characteristic time,

$$\tau_{naive} = \left[\frac{2}{3}\left|\frac{g}{2}\right|^5 \frac{e^2\hbar\gamma^5}{m^2c^2\rho^3}\right]^{-1} \qquad (11)$$

to be compared with (1c).

For $|g| = 2$, Eq.(11) agrees with (1c) to within a factor of order unity. Furthermore, spontaneous emission from the "upper" to "lower" energy level leads trivially to 100% polarization with the correct senses for electrons and positrons. Comparison with (2), with its ratio of approximately 25 for the "downwards" transition rate compared to the "upwards" one and its ultimate polarization of 92.4%, indicates that all the essentials are given qualitatively, and even semiquantitatively, by the naive argument. Not bad! The physicist then waves his hands expressively and remarks that of course the spin is not exactly at rest all the time in the moving frame and such neglected refinements can explain away the remaining small discrepancies. The phenomenon is "understood".

B. Criticisms of the simple explanation

There are a number of shortcomings to this naive description. The first is that the polarization is not 100%. The "energetically forbidden" upwards transition occurs, albeit at a much slower rate (for $g = 2$) than the "energetically allowed" transition. The second is that the spectrum of emitted frequencies in the moving frame is not a narrow line at $\omega' = \omega'_{12}$ given by (8), but a broad synchrotron spectrum extending to frequencies of the order $\omega'_{max} \sim \gamma^2\omega_0$, independent of the value of g. The third deficiency is that the inverse characteristic time is not proportional to $|g|^5$ as given by (11), but shows a more complicated dependence, varying as

$|g|^2$ for large $|g|$. The fourth and most dramatic shortcoming is that the degree and sense of polarization depends sensitively on the value of g and is such that for $g < 1/2$ the "upper" energy level is populated preferentially over the "lower" one!

Reasons for the failure of the naive argument are not hard to find. First of all, it is not permissible to consider the spin degree of freedom in isolation from the orbital motion, even in the instantaneously comoving frame. For such considerations to have approximate validity it is required that the energy level spacings associated with other degrees of freedom be large compared to the transition energy of the degree of freedom under study. Thus the hyperfine transition in atomic hydrogen can be treated without regard to the electronic orbital motion, except as an "external" source of magnetic field. In the present circumstances, however, these conditions do not hold. For relativistic circular orbits in the laboratory, Bohr's quantization rule for angular momentum gives the orbital quantum number as

$$n = \gamma mc\rho/\hbar = \gamma\gamma_c^2 \tag{12}$$

where γ_c is the critical value, (4). The spacing between adjacent orbital energy levels is

$$\Delta E = \hbar\omega_o \tag{13}$$

where $\omega_o = c/\rho$ is the orbital frequency. For highly relativistic particles this spacing is very small compared to $\hbar\omega'_{12}$ given by (8), or, more properly for considerations in the laboratory,

$\gamma\hbar\omega'_{12} \sim \gamma^3\hbar\omega_o$. With the spacing between orbital levels very small compared to the transition energy, that transition will inevitably involve some changes in orbital quantum number. In other words, there will occur exchanges of energy between spin and orbital degrees of freedom. Thre is then little significance in the concept of "upper" and "lower" energy states for the spin system alone.

Another way to reach the same conclusion is to consider the conservation of momentum during the emission of a typical "spin-flip" photon. For ordinary synchrotron radiation the photons emerge within angles of the order of $\Delta\theta \sim \gamma^{-1}$ of the path of the particle and possess a broad spectrum of energies up to $\gamma^3\hbar\omega_o$ and somewhat beyond. The same will be demonstrated below for the spin-flip synchrotron radiation. With emission essentially parallel to the particle's direction and a typical momentum of the order of $\gamma^3\hbar\omega_o/c$, the photon will cause the particle's momentum to decrease by an amount,

$$\Delta p = \gamma^3\hbar\omega_o/c = \gamma^3\hbar/\rho \quad . \tag{14}$$

This corresponds to a fractional change in orbital quantum number,

$$\frac{\Delta n}{n} \simeq \frac{\Delta p}{p} \simeq \frac{\gamma^3\hbar}{\gamma mc\rho} = (\gamma/\gamma_c)^2 \quad , \tag{15}$$

and, using (12), to a value of Δn itself of the order of

$$\Delta n \simeq \gamma^3 \quad . \tag{16}$$

This demonstrates that the changes in orbital quantum number from recoil are enormous. With 2.5 GeV electrons, $\gamma \simeq 5 \times 10^3$ and $\Delta n \simeq 10^{11}$. At the quantum level the orbital motion is evidently disturbed by the emission act! The disturbance is neverthess totally

negligible to the orbit and its classical description provided $\gamma \ll \gamma_c$. For the typical conditions of $\rho \simeq 13$ meters and $\gamma \simeq 5 \times 10^3$, Eq.(12) yields $n \simeq 2 \times 10^{17}$ and (15), $\Delta n/n \simeq 5 \times 10^{-7}$. The astronomical value of n shows how classical the orbit is; the minute value of $\Delta n/n$ shows how small the perturbation of the orbit. Note from (12) and (14) that $\Delta n/n$ is just the fractional change in the energy of the particle as it emits the photon. These considerations provide justification for a classical treatment of the problem (given classical trajectory and soft-photon limit).

Yet another shortcoming of the naive argument is the assumption that the particle's motion is nonrelativistic and can be ignored in the moving inertial frame.

At any instant the state of motion of the particle can be specified by its velocity vector $\underset{\sim}{v}$ and the components of acceleration $\underset{\sim}{\dot{v}}$ parallel and perpendicular to it. Equivalently, the instantaneous radius of curvature ρ is related to the transverse acceleration by

$$\rho = v^2/\dot{v}_\perp \tag{17}$$

while the rate of change of <u>speed</u> is equal to the parallel component of acceleration. Only a length of arc of the order of ρ/γ or a time interval $\Delta t \sim \rho/\gamma v$ is relevant in considering the radiation. In practical circumstances this time interval is so short that the radius of curvature and the speed can be treated as constants during it. The arbitrary trajectory can thus be approximated locally as a circular path or radius ρ along which the particle moves at constant speed $v = \beta c$ or angular velocity $\omega_0 = \beta c/\rho$. A suitable choice of coordinates in the laboratory is shown in Fig. 2. The zero

of time is chosen when the particle is at the origin. For a horizontal storage ring the guiding magnetic field is in the vertical (z) direction, in or out of the page, the velocity at $t = 0$ is in the x direction, and the acceleration at that instant in the y direction. The instantaneously comoving inertial frame is defined by a boost in the positive x direction with speed βc. Denoting coordinates in the moving frame with primes, we have the orbit described parametrically in the two frames by

$$\left.\begin{aligned} x &= \rho \sin \omega_0 t \\ y &= \rho(1 - \cos \omega_0 t) \\ z &= 0 \end{aligned}\right\} \text{Laboratory} \tag{18}$$

$$\left.\begin{aligned} x' &= \gamma\rho(\sin \omega_0 t - \omega_0 t) \\ y' &= \rho(1 - \cos \omega_0 t) \\ z' &= 0 \end{aligned}\right\} \text{Moving frame .} \tag{19}$$

The time coordinate in the moving frame is

$$\omega_0 t' = \gamma(\omega_0 t - \beta^2 \sin \omega_0 t) . \tag{20}$$

For laboratory times such that $\gamma\omega_0|t| = O(1)$, the orbit equations (19) and (20) can be approximated as

$$\left.\begin{aligned} x' &\simeq -(\rho/6\gamma^2)(\gamma\omega_0 t)^3 \\ y' &\simeq (\rho/2\gamma^2)(\gamma\omega_0 t)^2 \\ \omega_0 t' &\simeq (\omega_0 t/\gamma)(1 + \gamma^2\omega_0^2 t^2/6) \end{aligned}\right\} \tag{21}$$

The equation of the orbit is thus

$$y' = (\rho/2\gamma^2)(6\gamma^2|x'|/\rho)^{2/3} \quad . \tag{22}$$

This path is shown on the right-hand side of Fig. 2. Note that the unit of length is ρ/γ^2, so the scale is greatly enlarged compared to the laboratory figure. Values of the parameter $\gamma\omega_0 t$ are indicated along the path to show the correspondence with points on the circular arc in the laboratory. In terms of this parameter the components of the velocity and acceleration of the particle in the moving frame are

$$\left.\begin{aligned}
\beta'_x &= -\gamma^2\omega_0^2 t^2/2\gamma' \\
\beta'_y &= \gamma\omega_0 t/\gamma' \\
\dot{\beta}'_x &= -\gamma^2\omega_0(\gamma\omega_0 t)/\gamma'^3 \\
\dot{\beta}'_y &= \gamma^2\omega_0(1 - \gamma^2\omega_0^2 t^2/2)/\gamma'^3
\end{aligned}\right\} \tag{23}$$

where

$$\gamma' = 1 + \gamma^2\omega_0^2 t^2/2 \tag{24}$$

is the ratio of energy to rest energy for the particle in the moving frame.

Since the relevant range of $\gamma\omega_0 t$ is of order unity, (23) and (24) tell us that the particle, while instantaneously at rest in the moving frame at $t = 0$, soon attains speeds close to that of light. It is admittedly not ultrarelativistic in the contributing time interval, but is changing its state of motion rapidly and is

certainly not even approximately at rest for purpose of calculating the radiation.

Two comments in passing:

(1) The path in the moving frame can be thought of as being produced by the combined action of a magnetic field in the z' direction and an electric field in the y' direction. The scale of curvature of the path is ρ/γ^2, as shown in Fig. 2. This means that, although the speed is not constant in this frame, the characteristic orbital angular frequency is $\omega'_0 \sim \gamma^2\omega_0$, of the same order of magnitude as (8), the frequency associated with intrinsic spin.

(2) It is amusing to verify the Lorentz invariance of total radiated power by calculating in the moving frame with Liénard's generalization of the Larmor power formula,

$$P' = \frac{2e^2\gamma'^6}{3c}\left[(\dot{\beta}')^2 - (\beta' \times \dot{\beta}')^2\right] \quad . \tag{25}$$

Substitution from (23) leads to the familiar result,

$$P' = 2e^2\omega_0^2\gamma^4/3c = 2e^2 c\gamma^4\beta^4/3\rho^2 \quad , \tag{26}$$

independent of time, even though the components of velocity and acceleration are time-dependent in the moving frame.

It is hoped that by now the reader is persuaded that the naive consideration of the electron's spin as an isolated, nonrelativistic system in the moving frame is unjustified. Because of ease of exchange of energy between mechanical and spin degrees of freedom no significance can be attached to the labels "upper" and "lower" energy levels for the magnetic moment interaction. Since the motion in the

instantaneously comoving inertial frame becomes somewhat relativistic in the time interval of interest, there is no compelling reason for considering the phenomenon in that frame. The laboratory serves as well and is more familiar. We now proceed to a discussion of a semi-classical derivation of the correct results.

It may be objected that the business of the instantaneously comoving inertial frame is a straw man, that there is a frame where the spin is always at rest, namely the exactly comoving Lorentz frame obtained by a boost with the instantaneous velocity $\underset{\sim}{v}(t)$. The difficulty with such an approach is that discussion of frequency spectra and transition probabilities inevitably requires consideration of nonvanishing time intervals. A time-dependent Lorentz transformation to a noninertial frame seems to present insurmountable problems, and is not "anschaulich", to say the least. The relativistic effects of acceleration, i.e., the Thomas precession, are included automatically in the derivation that follows.

III. SEMICLASSICAL DESCRIPTION

A. <u>Semiclassical radiation theory for charge</u>

The time honored elementary treatment of spontaneous emission proceeds as follows. First consider a nonrelativistic charged particle of mass m and charge e interacting with an <u>external</u> classical electromagnetic field described by scalar and vector potentials $(\Phi, \underset{\sim}{A})$ and also with another given interaction potential U. Its motion is described quantum mechanically by the Schrodinger equation with a Hamiltonian,

$$H = \frac{1}{2m} (\underset{\sim}{p} - e\underset{\sim}{A}/c)^2 + e\Phi + U \quad . \tag{27}$$

Commonly (e.g., in atomic physics) the potential U is absent and the scalar potential is the Coulomb potential of the fixed nuclei. If the vector potential is treated as a perturbation, the Hamiltonian is written as a zeroth order term,

$$H_o = \frac{(\underset{\sim}{p})^2}{2m} + e\Phi + U$$

plus a small interaction term,

$$H_{int} = -e\underset{\sim}{A} \cdot \underset{\sim}{\beta} \tag{28}$$

where the velocity operator is $\underset{\sim}{\beta} = (-i\hbar/mc)\underset{\sim}{\nabla}$ and the potentials are in the radiation gauge with $\underset{\sim}{\nabla} \cdot \underset{\sim}{A} = 0$. The term in A^2 has been neglected. Effects of weak external fields are examined by use of perturbation theory with the states of the unperturbed Hamiltonian H_o as the basis. Phenomena like the Zeeman effect involve static external fields, but one can also treat time-varying applied fields and discuss transitions between different energy levels of the unperturbed system.

It is then an easy step to consider $\underset{\sim}{A}$ in (28) as the vector potential of a plane electromagnetic wave incident on the unperturbed system,

$$\underset{\sim}{A}(\underset{\sim}{r}, t) = \underset{\sim}{\varepsilon} A_o e^{i\underset{\sim}{k} \cdot \underset{\sim}{r} - i\omega t} + c.c. \quad . \tag{29}$$

The constant A_o is initially arbitrary, but is soon chosen to have the value,

$$A_o = (2\pi\hbar c/k)^{\frac{1}{2}} \quad , \tag{30}$$

corresponding to one photon of energy $\hbar\omega$ per unit volume in the incident beam, computed by equating the classical time-averaged Poynting vector to $\hbar c^2 \underset{\sim}{k}$. The substitution of the vector potential (29) into the interaction Hamiltonian (28), followed by a treatment of time-dependent perturbation theory using the method of variation of parameters of Dirac, and leading to a discussion of the photo-electric effect or other transitions involving the absorption of photons, can be traced in almost any book on quantum mechanics.

The derivation involves at some step a resonant enhancement (conservation of energy!) arising from the time integral of the product of two exponentials,

$$e^{i(E_f - E_i)t/\hbar} \cdot e^{-i\omega t} \quad .$$

The first factor comes from the time dependences of the initial and final unperturbed states and the second from the first term in (29). Since $E_f > E_i$ by assumption, the second (complex conjugate) term in (29) gives no contribution to the time integral. However, it takes no prodding to convince the student to consider the opposite situation where $E_i > E_f$. He or she is thus led smoothly to spontaneous emission where the second (complex conjugate) piece in (29) is operative. It is plausible in considering a transition with the emission of a single photon of wave number $\underset{\sim}{k}$ that the same normalization constant (30) enters the vector potential here as for absorption.

For our purposes the "golden-rule" result for the transition probability is not as appropriate as an expression for the differential probability at time t for the emission of a photon of polarization $\underset{\sim}{\varepsilon}$ and wave number $\underset{\sim}{k}$ in an elemental volume d^3k:

$$dp(t) = \left| \frac{1}{i\hbar} \int_{-\infty}^{t} \left\langle \Psi_f(t') \left| H_{int}(t') \right| \Psi_i(t') \right\rangle dt' \right|^2 \frac{d^3k}{(2\pi)^3} \quad . \tag{31}$$

It is customary to extract the time dependence of the initial and final states and so obtain the exponential factor discussed above, but because of our transition to a classical orbit following Baier and Katkov we treat the states and operators in the Heisenberg picture. In the limit as $t \to +\infty$, (31) is the probability of photon emission into d^3k. The energy radiated can be obtained by multiplying by $\hbar\omega$. We are thus led to a result with a classical counterpart, the differential intensity of energy radiated with polarization $\underset{\sim}{\varepsilon}$ per unit solid angle and per unit frequency interval,

$$\frac{d^2I}{d\Omega d\omega} = \hbar\omega \frac{dp(\infty)}{d\Omega d\omega} \quad .$$

With the second term in (29) operative in (28) the interaction becomes

$$(H_{int})_{emission} = -e\sqrt{\frac{2\pi\hbar c}{k}} \underset{\sim}{\varepsilon}^* \cdot \underset{\sim}{\beta} e^{i\omega t - i\underset{\sim}{k}\cdot\underset{\sim}{r}} \quad . \tag{32}$$

This gives

$$\frac{d^2I}{d\Omega d\omega} = \frac{e^2\omega^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} \left\langle \Psi_f(t) \left| \underset{\sim}{\varepsilon}^* \cdot \underset{\sim}{\beta}(t) e^{i\omega t - i\underset{\sim}{k}\cdot\underset{\sim}{r}(t)} \right| \Psi_i(t) \right\rangle dt \right|^2 \quad . \tag{33}$$

Here the velocity $\underset{\sim}{\beta}(t)$ and the coordinate $\underset{\sim}{r}(t)$ are Heisenberg operators. Equation (33) can be compared with its classical analog.[4] The transition to the classical limit is evidently achieved by the

replacement,

$$\left\langle \Psi_f(t) \middle| \varepsilon^* \cdot \beta(t) \, e^{i\omega t - i\mathbf{k}\cdot\mathbf{r}(t)} \middle| \Psi_i(t) \right\rangle$$

$$\longrightarrow \varepsilon^* \cdot \beta(t) \, e^{i\omega t - i\mathbf{k}\cdot\mathbf{r}(t)} \tag{34}$$

where now $\beta(t)$ and $\mathbf{r}(t)$ are given classical quantities. This is just the result of Schwinger (1954) and Baier and Katkov (1967b) in the limit that the orbit is classical (the wave functions localized tightly around the orbit) and the energy of the emitted photon is very small compared to the energy of the particle (the noncommutativity of the various Heisenberg operators can then be neglected).

The result (33) with the replacement (34) can form a starting point for the derivation of the classic results of Schwinger (1949) and others for ordinary synchrotron radiation.[1] The alert reader may have noticed that we began with the nonrelativistic Schrodinger equation and are now discussing extreme relativistic motion! The reason this is permitted is that to the neglect of spin the interaction Hamiltonian (28) is correct relativistically with a suitable velocity operator. In the classical limit, the velocity operator is replaced by the classical velocity. The result is therefore generally applicable for arbitrary speeds provided the trajectory is classical and $\gamma \ll \gamma_c$.

B. Semiclassical radiation theory for spin

  1. Nonrelativistic spin system

A semiclassical treatment of emission and absorption of radiation by a spin system in motion parallels the discussion of the last section. For orientation we first consider a spin $\hbar\mathbf{s}$ with

associated magnetic moment $\mu_0 = ge\hbar\mathbf{s}/2mc$ at rest in interaction with an external magnetic field $\mathbf{B}$. The Hamiltonian of interaction is

$$H_{int} = -\mu_0 \cdot \mathbf{B} = -\left(\frac{ge\hbar}{2mc}\right)\mathbf{s}\cdot\mathbf{B} . \tag{35}$$

The corresponding Heisenberg equation of motion is the familiar result,

$$\frac{d\mathbf{s}}{dt} = \frac{i}{\hbar}\left[H_{int}, \mathbf{s}\right] = \left(\frac{g}{2}\right)\frac{e}{mc}\mathbf{s}\times\mathbf{B} . \tag{36}$$

The interaction Hamiltonian (35) can be used to discuss the effects of static or time-varying magnetic fields on the energy levels and transitions of the spin system in isolation or perhaps with coupling to other (orbital) degrees of freedom. Spontaneous emission can be treated by the ansatz of the previous section--the emitted photon is described by the second term of the vector potential (29) with strength A_0 given by (30). The electric and magnetic fields of the emitted photon are thus

$$\mathbf{E}(\mathbf{r},t) = -i\sqrt{2\pi\hbar\omega}\, \varepsilon^* \, e^{i\omega t - i\mathbf{k}\cdot\mathbf{r}}$$

$$\tag{37}$$

$$\mathbf{B}(\mathbf{r},t) = -i\sqrt{2\pi\hbar\omega}\, (\mathbf{n}\times\varepsilon^*) \, e^{i\omega t - i\mathbf{k}\cdot\mathbf{r}}$$

where $\mathbf{n}$ is a unit vector in the direction of $\mathbf{k}$. With this magnetic field inserted into (35), standard lowest order perturbation theory leads, in the long wavelength limit, to the magnetic dipole transition rate (9).

2. Relativistic spin system

In order to describe radiation by a spin system in relativistic motion we must obtain suitable generalizations of (35) and (36). The relativistic equation of motion for spin is by now relatively well known. It was first derived by Thomas (1927) in his detailed paper on what is called the Thomas precession, was discussed in a particle physics context by Bargmann, Michel, and Telegdi (1959), and is now standard textbook fare.[5] The Thomas-BMT equation of motion for the spin $\underset{\sim}{s}$ of a particle of charge e, mass m, and rest-frame magnetic moment $\underset{\sim}{\mu}_0 = ge\hbar\underset{\sim}{s}/2mc$, in motion with velocity $\underset{\sim}{v} = \underset{\sim}{\beta}c$ in external electromagnetic fields $\underset{\sim}{E}, \underset{\sim}{B}$ can be written in Thomas's original form,

$$\frac{d\underset{\sim}{s}}{dt} = \frac{e}{mc}\underset{\sim}{s} \times \left[\left(a + \frac{1}{\gamma}\right)\underset{\sim}{B} - \frac{a\gamma}{\gamma+1}\underset{\sim}{\beta}(\underset{\sim}{\beta}\cdot\underset{\sim}{B}) - \left(a + \frac{1}{\gamma+1}\right)\underset{\sim}{\beta} \times \underset{\sim}{E}\right] \tag{38}$$

where a is called the magnetic moment anomaly and is defined by

$$a = \frac{g-2}{2} \quad . \tag{39}$$

The spin vector $\underset{\sim}{s}$ describes the spin in its rest system (just as does the Pauli $\underset{\sim}{\sigma}/2$ and the Pauli spinors in the 2-component reduction of the 4-component Dirac spinor), but the time rate of change in (38) is with respect to laboratory time.

Equation (38) is the relativistic generalization of (35). Strictly speaking, it holds only for spatially uniform fields because it lacks $\underset{\sim}{\nabla}(\underset{\sim}{\mu}\cdot\underset{\sim}{B})$ terms, but is an adequate description for sufficiently slowly varying fields or weak fields, whatever their space and time variation. The Thomas-BMT equation can be thought of as

following from an effective Hamiltonian in the same way as (36) follows as a Heisenberg equation of motion from (35). Evidently this effective Hamiltonian is

$$H_{int}^{(eff)} = -\frac{e\hbar}{mc}\underset{\sim}{s}\cdot\left[\left(a + \frac{1}{\gamma}\right)\underset{\sim}{B} - \frac{a\gamma}{\gamma+1}\underset{\sim}{\beta}(\underset{\sim}{\beta}\cdot\underset{\sim}{B}) - \left(a + \frac{1}{\gamma+1}\right)\underset{\sim}{\beta} \times \underset{\sim}{E}\right] . \tag{40}$$

Although (40) is explicit and the most useful form for calcualtion, the terms in the square bracket can be rearranged into a more intuitive, if implicit, form. First we define the magnetic field $\underset{\sim}{B}'$ in the rest frame of the spin,

$$\underset{\sim}{B}' = \gamma(\underset{\sim}{B} - \underset{\sim}{\beta} \times \underset{\sim}{E}) - \frac{\gamma^2}{\gamma+1}\underset{\sim}{\beta}(\underset{\sim}{\beta}\cdot\underset{\sim}{B}) \quad . \tag{41}$$

Then we introduce the Thomas precession angular velocity vector $\underset{\sim}{\omega}_T$:

$$\underset{\sim}{\omega}_T = \frac{\gamma^2}{\gamma+1}(\dot{\underset{\sim}{\beta}} \times \underset{\sim}{\beta}) = \frac{e}{mc}\frac{\gamma}{\gamma+1}\left[\beta^2\underset{\sim}{B} - \underset{\sim}{\beta}(\underset{\sim}{\beta}\cdot\underset{\sim}{B}) - \underset{\sim}{\beta} \times \underset{\sim}{E}\right] . \tag{42}$$

In terms of $\underset{\sim}{B}'$ and $\underset{\sim}{\omega}_T$ the effective Hamiltonian (40) can be written

$$H_{int}^{(eff)} = -\frac{1}{\gamma}\underset{\sim}{\mu}_0\cdot\underset{\sim}{B}' + \hbar\underset{\sim}{\omega}_T\cdot\underset{\sim}{s} \quad . \tag{43}$$

The two terms in (43) have immediate physical interpretations. The first is the expected rest-frame coupling between magnetic moment and magnetic field in that frame, diminished by a factor γ^{-1} to account for the time dilatation seen in the laboratory (remember that (38) is a laboratory equation of motion, even though $\underset{\sim}{s}$ is the rest-frame spin vector). The second term is the contribution to the energy from

the relativistic Thomas precession of axes in the accelerated rest frame.

3. Radiation formula

The semiclassical description of radiation by the spin $\mathbf{s}$ proceeds with the replacement of the classical external fields $\mathbf{E}, \mathbf{B}$ in (40) by the fields (37) of the emitted photon. The effective Hamiltonian for emission then becomes

$$\left[H_{int}^{(eff)}\right]_{emission} = i\sqrt{2\pi\hbar\omega}\,\frac{e\hbar}{mc}\,\mathbf{s}\cdot\mathbf{V}\,e^{i\omega t - i\mathbf{k}\cdot\mathbf{r}} \qquad (44)$$

where

$$\mathbf{V} = -\left(a + \frac{1}{\gamma}\right)\mathbf{n}\times\boldsymbol{\varepsilon}^* - \frac{a\gamma}{\gamma+1}\boldsymbol{\beta}\,\boldsymbol{\beta}\cdot(\mathbf{n}\times\boldsymbol{\varepsilon}^*) - \left(a + \frac{\gamma}{\gamma+1}\right)\boldsymbol{\beta}\times\boldsymbol{\varepsilon}^* . \qquad (45)$$

The matrix element of (44) between particle states (of spin and spatial coordinates) can be used straightforwardly to discuss transitions between states of different spin orientation. For the present purposes we consider the classical limit of the <u>orbital</u> motion, as in going from (31) to (33) and (34). Since we are concerned primarily with electrons and positrons we specialize to spin $\frac{1}{2}$ and write $\mathbf{s} = \mathbf{g}/2$. Comparison of the Hamiltonian (32) for the emission of radiation by a charge e with (44) shows that the formula at the end of the last section can be transcribed with the substitution,

$$\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\beta}(t) \rightarrow -i\,\frac{\hbar k}{2mc}\,\mathbf{g}\cdot\mathbf{V}(t) . \qquad (46)$$

Now the only quantum-mechanical aspect is the Pauli spin vector. The spin analog of (33) and (34) is

$$\frac{d^2 I_{spin}}{d\Omega d\omega} = \frac{e^2\hbar^2\omega^4}{16\pi^2 m^2 c^5}\left|\int_{-\infty}^{\infty} dt\,\langle f|\mathbf{g}|i\rangle\cdot\mathbf{V}(t)\,e^{i\omega t - i\mathbf{k}\cdot\mathbf{r}(t)}\right|^2 \qquad (47)$$

with $\mathbf{V}(t)$ given by (45). The radiation is emitted in the course of a transition from initial spin state i to final spin state f, both states specified in the rest frame of the particle. While in principle nonflip, as well as spin-flip, transitions contribute to the radiation, the nonflip transitions are dominated overwhelmingly by the ordinary charge radiation (see Eq. (3)). Thus only spin-flip transitions need concern us.

The semiclassical (47) and the corresponding result for the transition probability, derived from the Thomas-BMT equation, are shown in the appendix to follow from the matrix element of the Dirac-Pauli current (with $\sigma^{\mu\nu}$ coupling for the anomalous magnetic moment) in the soft-photon limit. Equations (47) and (45) thus provide a straightforward generalization to arbitrary g-factor of the formulas of Baier and Katkov. They are of negligible practical value for $g \neq 2$, but are put to pedagogic use below.

In the appendix, (47) is also compared with a strictly classical expression for radiation emitted by a moving magnetic moment. The two results are not the same. The reasons for the difference are discussed.

IV. SPIN-FLIP SYNCHROTRON RADIATION FOR ARBITRARY g FACTOR

A. *Differential energy, photon number, and transition rates*

We now apply (47) to a calculation of the radiation emitted by a relativistic spin $\frac{1}{2}$ particle of charge e and arbitrary g factor in a spin-flip transition while moving at velocity $\underset{\sim}{\beta}c$ in an instantaneously circular arc of radius ρ. Defining the time integral in (47) to be

$$\underset{\sim}{u}(\omega,\underset{\sim}{n},\underset{\sim}{\epsilon}) = \int dt \, \underset{\sim}{V}(t) \, e^{i\omega t - i\underset{\sim}{k}\cdot\underset{\sim}{r}(t)} \quad , \tag{48}$$

we have the intensity of energy radiated per unit solid angle and per unit frequency interval with polarization $\underset{\sim}{\epsilon}$ in a single passage along the arc,

$$\frac{d^2I}{d\Omega d\omega} = \frac{e^2 \hbar^2 \omega^4}{16\pi^2 m^2 c^5} \left| \langle f | \underset{\sim}{\sigma}\cdot\underset{\sim}{u} | i \rangle \right|^2 \quad . \tag{49}$$

The <u>number</u> of photons emitted per unit solid angle, etc., is obtained by dividing by $\hbar\omega$:

$$\frac{d^2N}{d\Omega d\omega} = \frac{e^2 \hbar \omega^3}{16\pi^2 m^2 c^5} \left| \langle f | \underset{\sim}{\sigma}\cdot\underset{\sim}{u} | i \rangle \right|^2 \quad . \tag{50}$$

The differential <u>transition rate</u> follows from (50) with multiplication by $\omega_o/2\pi$, where $\omega_o = \beta c/\rho$:

$$\frac{d^2w}{d\Omega d\omega} = \frac{e^2 \hbar \omega^3 \omega_o}{32\pi^3 m^2 c^5} \left| \langle f | \underset{\sim}{\sigma}\cdot\underset{\sim}{u} | i \rangle \right|^2 \quad . \tag{51}$$

This last result rigorously depends on the assumption of continuous motion at constant speed in a circular orbit, but in practice holds provided the speed and radius of curvature are sensibly constant over a reasonable segment of path. The modifications for storage-ring orbits with bending sections and straight sections are almost self-evident. For the total rate they have been incorporated in (5) and (6).

We choose the initial spin direction to be along a unit axial vector $\underset{\sim}{\zeta}$ in the rest frame and consider a transition in which the spin direction changes from $+\underset{\sim}{\zeta}$ to $-\underset{\sim}{\zeta}$. The square of the matrix element in (49)-(51), summed over photon polarizations, can thus be written

$$S = \sum_{pol} \left| \langle f | \underset{\sim}{\sigma}\cdot\underset{\sim}{u} | i \rangle \right|^2 \tag{52}$$

$$= \frac{1}{16} \sum_{pol} \sum_{spins} \left| \langle m' | (1 - \underset{\sim}{\sigma}\cdot\underset{\sim}{\zeta})\underset{\sim}{\sigma}\cdot\underset{\sim}{u}(1 + \underset{\sim}{\sigma}\cdot\underset{\sim}{\zeta}) | m \rangle \right|^2 \quad .$$

The sum over spins yields

$$S = \sum_{pol} \left[|\underset{\sim}{u}|^2 - |\underset{\sim}{\zeta}\cdot\underset{\sim}{u}|^2 + 2\underset{\sim}{\zeta}\cdot(\text{Im } \underset{\sim}{u} \times \text{Re } \underset{\sim}{u}) \right] \quad . \tag{53}$$

What remains now is a calculation of $\underset{\sim}{u}$ with $\underset{\sim}{V}(t)$ given by (45) and $\underset{\sim}{\beta}(t)$ and $\underset{\sim}{r}(t)$ found from the circular orbit equations (18), approximated suitably for $\omega_o|t| = O(\gamma^{-1})$. The approximations are essentially the same as for ordinary synchrotron radiation[1] and the integrals encountered the same. The relative complexity of $\underset{\sim}{V}(t)$, expecially for $a \neq 0$, makes the calculation algebraically cumbersome

and not very illuminating. We merely quote results. The choice of coordinate axes and angles is shown in Fig. 3. It is the same set of axes as in Fig. 2, with the x axis being the velocity direction at $t = 0$. Since the radiation pattern is strongly peaked in this direction, we choose the polar angle θ with respect to it, although the angle up from the orbital plane is actually more appropriate.

B. Doubly differential spectrum in frequency and angle

For the sake of compactness in relatively unwieldy formulas we introduce some notation. We define

$$\left.\begin{array}{l} t = \gamma\theta \sin \phi \\[2mm] \eta = \dfrac{\omega}{3\gamma^3\omega_o}(1 + \gamma^2\theta^2 \sin^2 \phi)^{3/2} \\[2mm] \nu = \dfrac{2\omega}{3\gamma^3\omega_o} \end{array}\right\} \qquad . \qquad (54)$$

In terms of these variables and τ_o defined by (1c) the differential transition probability (51), summed over photon polarizations, is

$$\frac{d^2w}{d\Omega d\omega} = \frac{3\sqrt{3}\,\nu^3(1 + t^2)}{40\pi^3\tau_o\gamma^2\omega_o}\left\{\left[(1 + t^2)(1 + a^2t^2)\right.\right.$$

$$+ \zeta_1^2\Big((1 + a)^2 + (a^2 - 1)t^2 + a^2t^4\Big) + \zeta_3^2\, a(2 + a)(1 + 2t^2)$$

$$+ 2\zeta_1\zeta_3 t(1 - at^2)\Big] K_{1/3}^2(\eta)$$

Equation (55) continued

$$+ (1+t^2)\Big[(1+a)^2 + a^2t^2 - \zeta_1^2\Big((1+a)^2 - a^2t^2\Big) - 2\zeta_1\zeta_3 a(1+a)t\Big] K_{2/3}^2(\eta)$$

$$+ 2\sqrt{1 + t^2}\,\Big[\zeta_3(1 + a)(1 + a + 2at^2) - \zeta_1 a^2t(1 + 2t^2)\Big]$$

$$\left. \times K_{1/3}(\eta)\, K_{2/3}(\eta) \right\} \qquad . \qquad (55)$$

This somewhat formidable expression gives the differential rate of emission in angle and frequency. In spite of the polynomials in $\underline{a}$ and $\underline{t}$, the familiar modified Bessel functions show that the spectra are typical of synchrotron radiation, collimated in angle (with $\theta \sin \phi \sim \gamma^{-1}$ in this case) and with a broad spectrum of frequencies extending up to $\omega_{max} \sim \gamma^3\omega_o$. The exponential behavior of the Bessel functions for large η assures that, although the spectrum depends in detail upon the g factor, the range of frequencies is $0 < \omega \lesssim \gamma^3\omega_o$, independent of g. A Lorentz transformation to the instantaneously comoving rest frame shows that the frequencies there are on the range, $0 < \omega' \lesssim \gamma^2\omega_o$, independent of g. This is contrary to the naive expectation that ω'_{12} given by (8) is a typical frequency in the moving frame.

For the physically interesting case of $a = 0$ we consider the frequency-angle spectrum in detail in the next section. But for the present we proceed directly to the integrated rate and the polarization.

C. Angular distribution

The simplest path to the integrated rate is by integration of (55) first over frequencies and then over angles. By means of the definite integral formula,

$$\int_0^\infty x^\lambda K_\mu(x) K_\nu(x)\, dx =$$

$$\frac{2^{\lambda-2}}{\Gamma(\lambda+1)} \Gamma\left(\frac{\lambda+1+\mu+\nu}{2}\right) \Gamma\left(\frac{\lambda+1+\mu-\nu}{2}\right) \Gamma\left(\frac{\lambda+1-\mu+\nu}{2}\right)$$

$$\times\ \Gamma\left(\frac{\lambda+1-\mu-\nu}{2}\right)\ ,$$

the integration over frequencies yields the angular distribution,

$$\frac{dw}{d\Omega} = \frac{16\gamma}{45\pi^2\tau_0}(1+t^2)^{-5}\left\{(1+t^2)(1+a^2t^2)\right.$$

$$+\ \zeta_1^2\Big((1+a)^2 + (a^2-1)t^2 + a^2t^4\Big) + \zeta_3^2\, a(2+a)(1+2t^2)$$

$$+\ 2\zeta_1\zeta_3 t(1-at^2)$$

$$+\ \frac{5}{4}(1+t^2)\Big[(1+a)^2 + a^2t^2 - \zeta_1^2\Big((1+a)^2 - a^2t^2\Big)$$

$$-\ 2\zeta_1\zeta_3 a(1+a)t\Big]$$

$$+\ \frac{105\pi}{256}\sqrt{1+t^2}\Big[\zeta_3(1+a)(1+a+2at^2) - \zeta_1 a^2 t(1+2t^2)\Big]\Bigg\}$$

$$(56)$$

Again, in spite of the polynomials in $\underline{t}$, with terms up to t^4, the relativistic forward cone of radiation is evident by the presence of the factor $(1+t^2)^{-5}$.

D. Total rate, characteristic time, and polarization

The integration over angles is accomplished most simply by noting from Fig. 3 that for small θ, $\theta\sin\phi \simeq \sin\theta\sin\phi = \cos\theta'$, where θ' is a polar angle measured from the z axis. Introducing a corresponding azimuthal angle ϕ' and noting that the distribution (56) is negligible except near $\theta' = \pi/2$, we can write the solid angle element as

$$d\Omega = d\phi'\, d(\cos\theta') \simeq \frac{1}{\gamma}\, d\phi'\, dt\ .$$

The range of t is effectively $(-\infty,\infty)$ for $\gamma \gg 1$. The integration over ϕ' contributes a factor of 2π and terms odd in t do not contribute. The remaining integrals are elementary. The final result for the transition rate is

$$w = \frac{1}{2\tau_0}\left\{1 + \frac{10a}{9} + \frac{34a^2}{45} + (\hat{\beta}\cdot\zeta)^2\left[-\frac{2}{9} - \frac{a}{3} + \frac{a^2}{30}\right]\right.$$

$$+\ \left[(\hat{\beta}\times\hat{\beta})\cdot\zeta\right]^2 a\left(1+\frac{a}{2}\right) + \frac{8}{5\sqrt{3}}(1+a)\left(1+\frac{4a}{3}\right)(\hat{\beta}\times\hat{\beta})\cdot\zeta\Bigg\}\ .$$

$$(57)$$

We have written the rate in a manner independent of the choice of coordinates by introducing the orthogonal unit vectors $\hat{\beta}$, $\hat{\dot\beta}$, $\hat{\beta}\times\hat{\dot\beta}$ along what are the x, y, z axes in Fig. 3. Equation (57) is the generalization to arbitrary g factor of the result (2) of Baier and Katkov (1967a).

The polarization of an initially unpolarized beam grows in time according to Eq. (1), but with a mean life τ that is obtained by (57) by choosing $\underset{\sim}{\zeta}$ along $\hat{\underset{\sim}{\beta}} \times \hat{\underset{\sim}{\dot{\beta}}}$ and summing the rates in the two directions.[6] This yields a characteristic time,

$$\tau = \tau_o \left[1 + \frac{19a}{9} + \frac{113}{90} a^2 \right]^{-1} . \qquad (58)$$

The asymptotic polarization (in the direction opposite to $\hat{\underset{\sim}{\beta}} \times \hat{\underset{\sim}{\dot{\beta}}}$) is

$$P = \frac{8}{5\sqrt{3}} \left[\frac{(1 + a)\left(1 + \frac{4a}{3}\right)}{1 + \frac{19a}{9} + \frac{113a^2}{90}} \right] . \qquad (59)$$

The growth time τ in units of τ_o is shown as a function of a or g in Fig. 4. It decreases for large $|g|$ as g^{-2}, but has a maximum value of $\tau/\tau_o = 8.88$ when $a = -0.8497$ or $g = 0.3186$. The polarization as a function of a or g is shown in Fig. 5. Its behavior is much more interesting than that of τ/τ_o. For large $|g|$ the limiting value is $P = 0.981$. It has a maximum value of 0.99196 at $a = 2.22$ ($g = 6.44$), and shows a dramatic dip to negative values on the range $-1.00 < a < -0.75$ ($0 < g < 0.5$). For $a < -1$ ($g < 0$), the polarization grows gradually up to its asymptotic value at large negative a.

To set the sign of the polarization in the framework of the "upper" and "lower" energy levels of an isolated magnetic moment, we must realize that the "lower" level has $\underset{\sim}{s}$ antiparallel to $\hat{\underset{\sim}{\beta}} \times \hat{\underset{\sim}{\dot{\beta}}}$ for $g > 0$, but parallel for $g < 0$. Figure 5 thus shows that for g factors on the range $-\infty < g < 0.5$ the "upper" energy state is

populated preferentially over the "lower", contrary to naive ideas of spontaneous emission by the isolated spin system.

V. ANGULAR AND FREQUENCY DISTRIBUTIONS FOR $g = 2$

The only physically relevant g factor is $g = 2$, appropriate for electrons and positrons. The total transition rate for spin-flip synchrotron radiation has been discussed in the Introduction. Here we examine the angular and frequency distributions of the radiation. These are of academic interest only because, as we observed in connection with Eq. (3), the energy radiated in the spin-flip transitions is negligible compared with the ordinary synchrotron radiation provided $\gamma \ll \gamma_c$.

A. Angular Distributions of Photons and of Radiated Power

The starting point is Eq. (55), specialized to $a = 0$, for the doubly differential transition rate in angle and frequency:

$$\frac{d^2w}{d\Omega d\omega} = \frac{3\sqrt{3}}{40\pi^3} \frac{\nu^3(1 + t^2)}{\tau_o \gamma^2 \omega_o} \left\{ \left[1 + t^2 + \zeta_1^2(1 - t^2) + 2\zeta_1\zeta_3 t \right] K_{1/3}^2(\eta) \right.$$
$$\left. + (1 - \zeta_1^2)(1 + t^2)K_{2/3}^2(\eta) + 2\zeta_3\sqrt{1 + t^2} K_{1/3}(\eta) K_{2/3}(\eta) \right\} \qquad (60)$$

The notation is defined by Eq. (54), with reference to Fig. 3. The angle variable t is, for small θ, γ times the latitude with respect to the z axis, that is, the angle between the direction of emission and the instantaneous plane of the orbit. It is the traditional synchrotron radiation angle, called ψ by Schwinger (1949) and θ by Jackson (1975).

The angular distribution of photons (number of photons per unit time per unit solid angle[7]) is given by (56) with $\underline{a} = 0$:

$$\frac{dw}{d\Omega} = \frac{16\gamma}{45\pi^2\tau_o}(1+t^2)^{-5}\left\{2\zeta_1^2 + \frac{9}{4}(1-\zeta_1^2)(1+t^2) + 2\zeta_1\zeta_3 t + \frac{105\pi}{256}\zeta_3\sqrt{1+t^2}\right\} \quad . \tag{61}$$

The angular distribution of _radiated power_ (energy per unit time per unit solid angle) is obtained by multiplying (60) by $\hbar\omega$ and then integrating over frequencies, as in going from (55) to (56). The result is[7]

$$\frac{dP}{d\Omega} = \frac{77\sqrt{3}}{256\pi}\frac{\gamma^4\hbar\omega_o}{\tau_o}(1+t^2)^{-\frac{13}{2}}\left\{2\zeta_1^2 + \frac{24}{11}(1-\zeta_1^2)(1+t^2) + 2\zeta_1\zeta_3 t + \frac{2^{13}\sqrt{3}}{3^3 \cdot 77\pi}\zeta_3\sqrt{1+t^2}\right\} \quad . \tag{62}$$

These angular distributions can be compared with the angular distribution of radiated power for the ordinary (nonflip) synchrotron radiation,

$$\frac{dP_{ordinary}}{d\Omega} = \frac{\gamma^5}{32\pi}\left(\frac{e^2 c}{\rho^2}\right)\frac{(7+12t^2)}{(1+t^2)^{7/2}} \quad . \tag{63}$$

We see that in the relativistic domain all the angular distributions are confined to angles of the order of γ^{-1} away from the instantaneous orbital plane, with $\underline{t} = \gamma\psi$ as the natural variable. The spin-flip angular distributions are somewhat narrower than the nonflip, the power decreasing as $|t|^{-11}$ compared to $|t|^{-5}$ at large $|t|$.

This is a reflection of the harder photon spectrum of the spin-flip, magnetic radiation with an overall additional factor of ω^2 in its frequency spectrum relative to that emitted by a charge. Similarly, the difference in $\underline{t}$ dependence between the number distribution (61) and the energy distribution (62) is explained by the fact that the softer photons have a broader distribution in angle than the harder ones.

B. Total Transition Rate and Total Spin-Flip Power Radiated

The total transition rate of Baier and Katkov is obtained by specialization of (57) to $a = 0$ or integration of (61) over angles with $d\Omega = \gamma^{-1} dt\, d\phi'$. The result is Eq. (2), which in the present notation is

$$w = \frac{1}{2\tau_o}\left[1 - \frac{2}{9}\zeta_1^2 + \frac{8}{5\sqrt{3}}\zeta_3\right] \quad : \tag{64}$$

The total spin-flip power, from (62), is

$$P_{spin\text{-}flip} = \frac{16}{5\sqrt{3}}\frac{\gamma^3\hbar\omega_o}{\tau_o}\left[1 - \frac{1}{6}\zeta_1^2 + \frac{35\sqrt{3}}{64}\zeta_3\right] \quad . \tag{65}$$

The ordinary radiated power is

$$P_{ordinary} = \frac{2}{3}\left(\frac{e^2}{\hbar\rho}\right)\gamma^4\hbar\omega_o \quad . \tag{66}$$

This leads to a ratio of spin-flip to ordinary power of

$$\frac{P_{spin\text{-}flip}}{P_{ordinary}} = 3\left(\frac{\hbar\gamma^2}{mc\rho}\right)^2\left[1 - \frac{1}{6}\zeta_1^2 + \frac{35\sqrt{3}}{64}\zeta_3\right] \quad , \tag{67}$$

in agreement with (3) for $\zeta_1 = 0$, $\zeta_3 = \pm 1$.

C. Frequency Distributions

The frequency spectrum of the radiation is found from (60) by integration over angles. This is not quite as easy a task as integration over frequency at fixed angle. The angular integral of (60) can be written

$$\frac{dw}{d\omega} = \frac{3\sqrt{3}}{10\pi^2}\frac{\nu^3}{\gamma^3\omega_o\tau_o}\int_0^\infty dt\left\{\left[2\zeta_1^2(1 + t^2) + (1 - \zeta_1^2)(1 + t^2)^2\right]K_{1/3}^2(\eta)\right.$$

$$\left. + (1 - \zeta_1^2)(1 + t^2)^2 K_{2/3}^2(\eta) + 2\zeta_3(1 + t^2)^{3/2} K_{1/3}(\eta) K_{2/3}(\eta)\right\}$$

$$(68)$$

where $\eta = (\nu/2)(1 + t^2)^{3/2}$ and $\nu = 2\omega/3\gamma^3\omega_o$. The modified Bessel functions of order $1/3$ and $2/3$ are related to the Airy function $Ai(\xi)$ and its derivative. The appropriate integrals have been evaluated in another connection by Aspnes (1966). Expressing his Airy function forms in terms of Bessel functions, we obtain the relevant integrals:

$$\int_0^\infty (1 + t^2) K_{1/3}^2\left(\frac{\nu}{2}(1 + t^2)^{3/2}\right)dt = \frac{\pi}{\sqrt{3}\nu}\int_\nu^\infty K_{1/3}(s)\,ds$$

$$\int_0^\infty (1 + t^2)^2 K_{1/3}^2\left(\frac{\nu}{2}(1 + t^2)^{3/2}\right)dt = \frac{\pi}{2\sqrt{3}\nu}\left[K_{2/3}(\nu)\right.$$

$$\left. + \int_\nu^\infty K_{1/3}(s)\,ds\right]$$

Equation (69) continued next page

Equation (69) continued

$$\int_0^\infty (1 + t^2)^2 K_{2/3}^2\left(\frac{\nu}{2}(1 + t^2)^{3/2}\right)dt =$$

$$\frac{\pi}{2\sqrt{3}\nu}\left[3K_{2/3}(\nu) - \int_\nu^\infty K_{1/3}(s)\,ds\right]$$

$$\int_0^\infty (1 + t^2)^{3/2} K_{1/3}\left(\frac{\nu}{2}(1 + t^2)^{3/2}\right) K_{2/3}\left(\frac{\nu}{2}(1 + t^2)^{3/2}\right)dt =$$

$$\frac{\pi}{\sqrt{3}\nu} K_{1/3}(\nu) .$$

$$(69)$$

With these integrals and the conversion to the dimensionless frequency variable ν, the number of photons per unit time per unit interval in ν takes the form,

$$\frac{dw}{d\nu} = \frac{9}{10\pi}\frac{\nu^2}{\tau_o}\left[(1 - \zeta_1^2) K_{2/3}(\nu) + \zeta_1^2\int_\nu^\infty K_{1/3}(s)ds + \zeta_3 K_{1/3}(\nu)\right] .$$

$$(70)$$

The corresponding expression for the spin-flip power radiated per unit interval in ν is

$$\frac{dP_{spin-flip}}{d\nu} = \frac{27}{20\pi}\left(\frac{\gamma^3\hbar\omega_o}{\tau_o}\right)\nu^3\left[(1 - \zeta_1)^2 K_{2/3}(\nu)\right.$$

$$\left. + \zeta_1^2\int_\nu^\infty K_{1/3}(s)ds + \zeta_3 K_{1/3}(\nu)\right] .$$

$$(71)$$

This can be compared with the frequency spectrum of the ordinary synchrotron radiation,

$$\frac{dP_{ordinary}}{d\nu} = P_{ordinary} \left[\frac{9\sqrt{3}}{8\pi} \nu \int_{\nu}^{\infty} K_{5/3}(s)ds \right] \qquad (72)$$

with the total power given by (66).

The normalized frequency distributions of the number of photons emitted per unit time in spin-flip transitions are shown in Fig. 6 for the "down" transition ($\zeta_3 = +1$, $\zeta_1 = \zeta_2 = 0$) and the "up" transition ($\zeta_3 = -1$, $\zeta_1 = \zeta_2 = 0$). The spectrum for the predominant "down" transition peaks around $\nu \simeq 1.5$ and extends to well beyond $\nu = 4$. The weaker "up" transition consists of somewhat softer photons, with a maximum at $\nu \simeq 0.7$. The areas are respectively 0.962 and 0.038, the "down" transition being 25.25 times as probable as the "up".

A graphical comparison of the separately normalized power spectra for the spin-flip and the nonflip synchrotron radiations is given in Fig. 7. For the ordinary radiation the quantity plotted is the coefficient of $P_{ordinary}$ in (72). For the spin-flip radiation it is $27\sqrt{3}\,\nu^3/128\pi$ times the square-bracket in (71) with $\zeta_1 = \zeta_2 = 0$, $\zeta_3 = \pm 1$. All the power spectra fall exponentially for large ν, but for $\nu \lesssim 1$ their behaviors are very different. The ordinary synchrotron radiation spectrum is proportional to $\nu^{1/3}$ for small ν, while the spin-flip spectra vary as $\nu^{7/3}$. The spin-flip radiation involves harder photons, as already mentioned in discussion of the angular distributions. The presence of an extra factor of ω^2 in the frequency distribution of radiation arising from

a magnetic moment in motion as compared to that for a charge in motion is a general feature, classically and quantum-mechanically.[8]

VI. SUMMARY

The primary purpose of this paper is didactic: to present as intuitive an interpretation as possible of the gradual transverse polarization of electron and positron beams as they orbit in storage rings. A naive description of the process, utilizing a moving inertial frame, is shown to be deficient in several respects, even though it appears superficially to give roughly correct answers, at least for electrons and positrons. The basic reason for its failure (and hence the absence of a truly simple description) is that the spin system cannot be treated in isolation because it is imbedded in a virtual continuum of states associated with the mechanical motion of the particle.

A semiclassical description of the radiative process is given by analogy with the well-known semiclassical treatment of radiation by a charged particle. The classical relativistic equation of motion for a spin in arbitrary motion in electromagnetic fields (the Thomas-BMT equation) yields an effective Hamiltonian for the coupling of a spin to electromagnetic fields. In analogy with the substitution,

$$e\boldsymbol{\beta}\cdot\mathbf{A}_{external} \rightarrow e\boldsymbol{\beta}\cdot\mathbf{A}_{photon} \quad ,$$

in the conventional transition to emission processes in the interaction Hamiltonian for a charged particle, we replace the external $\mathbf{E}$ and $\mathbf{B}$ fields in the Thomas-BMT effective Hamiltonian with the corresponding fields for a photon. Perturbation theory then yields an essentially classical expression for the transition probability

with quantum mechanics entering only via the matrix element of the Pauli spin operator. It is proved in the Appendix that this result is equal to the quantum-mechanical expression in the limit of soft photons and the neglect of recoil on the path of the particle. Both of these qualifications are appropriate for $1 \ll \gamma \ll \gamma_c \equiv (mc\rho/\hbar)^{\frac{1}{2}}$.

Some new results are derived concerning the spin-flip synchrotron radiation, the characteristic time of growth of the transverse polarization, and the ultimate polarization for a charged particle of spin $\frac{1}{2}$ with arbitrary g factor. Since electrons and positrons are the only particles likely to show detectable polarizations by this mechanism, these results are of no practical interest. They serve a pedagogic purpose, however, since they permit the upsetting of one of the key concepts of the naive description, namely, that the polarization arises from spontaneous emission as the spin moves from its "upper" to its "lower" state in the magnetic field. It is found that for $g < 0.5$ the opposite is true.

The angular and frequency distributions of numbers of photons and of radiated power are presented for the physically interesting circumstance of $g = 2$. They are compared with the corresponding spectra for the ordinary synchrotron radiation. This again is of limited practical value because of the minuteness of the spin-flip radiation, but may serve a pedagogic end.

Finally we remark that our concern has been with the basic phenomenon and mechanism of transverse polarization by spin-flip synchrotron radiation. Important practical aspects of the secular motion of spins in e^+e^- storage rings and of various mechanisms of detection of the transverse polarization can be found in the papers by Baier (1971a,b) and Schwitters (1974) and the references therein.

APPENDIX

In this appendix we establish explicitly that the appropriate quantum-mechanical matrix element of the Dirac-Pauli electromagnetic current for a particle of spin $\frac{1}{2}$ and arbitrary magnetic moment reduces in the soft-photon limit to the semiclassical results (32) and (44)-(45) for the nonflip and spin-flip transitions, respectively. We also examine the connection of this result to a purely classical expression for the radiation emitted by a moving classical magnetic moment.

A. Soft-photon limit of the matrix element of the Dirac-Pauli current

The relevant operator for the transition of a spin $\frac{1}{2}$ particle of charge $\underline{e}$ and mass $\underline{m}$ from one state to another with the emission of a photon of momentum $\underline{k}$ and polarization $\underline{\varepsilon}$ is[9]

$$\left[H_{int} \right]_{emission} = e\sqrt{\frac{2\pi}{\omega}}\, \varepsilon_\mu^*\, j^\mu\, e^{i\omega t - i\underline{k}\cdot\underline{r}} \qquad (A1)$$

where j^μ is the Dirac-Pauli electromagnetic current operator for unit charge. The appropriate matrix element is that of (33), with (32) replaced by (A1). In the large-quantum-number (classical orbit) and soft-photon limit the quantities in (A1) become c-numbers, except for a Dirac spinor product in the current. Comparison of (A1) with (32) and (44)-(45) shows that the correspondences between semiclassical and quantum-mechanical results for the nonflip and spin-flip transitions are

$$\underline{\varepsilon}^*\cdot\underline{\beta} \;\leftrightarrow\; -\left[\varepsilon_\mu^*\, j^\mu\right]_{nonflip,\,radiation\ gauge}$$

$$\langle \chi_f | \underline{\sigma}\cdot\underline{V} | \chi_i \rangle \;\leftrightarrow\; \frac{2m}{\hbar\omega}\left[\varepsilon_\mu^*\, j^\mu\right]_{spin-flip,\,radiation\ gauge}. \qquad (A2)$$

The matrix element of the current j^μ, including a Pauli term as well as the normal Dirac part, is

$$\epsilon_\mu j^\mu = \bar{u}(p')\left[\not{\epsilon} + i\frac{a}{2m}\not{\epsilon}\not{k}\right]u(p) \ .\qquad (A3)$$

In writing (A3) the particle is assumed to be a point particle of charge $\underline{e}$, mass $\underline{m}$ and anomalous moment $\underline{a}$, defined by (39). The Dirac notation is that of Bjorken and Drell (1964). The spinors are plane wave spinors normalized to one particle per unit volume. The 4-momenta are $p^\mu = \left(p_0 = \gamma(t)m, \ \underline{p} = \gamma(t)m\underline{\beta}(t)\right)$ where $\underline{\beta} = d\underline{r}/dt$ is the instantaneous classical velocity, and $p'^\mu = p^\mu - k^\mu$. The limit $k^\mu \to 0$ is to be taken, keeping only lowest order nonvanishing terms.

The explicit verification of the correspondences (A2) follows straightforwardly upon reduction of (A3) to two-component form, using

$$\not{B} = \begin{pmatrix} B_0 & -\underline{\sigma}\cdot\underline{B} \\ \underline{\sigma}\cdot\underline{B} & -B_0 \end{pmatrix}\qquad (A4)$$

for any 4-vector $B^\mu = (B_0, \underline{B})$, and

$$u(p) = \left(\frac{E+m}{2E}\right)^{\frac{1}{2}}\begin{pmatrix} \chi_i \\ \dfrac{\underline{\sigma}\cdot\underline{p}}{E+m}\chi_i \end{pmatrix}\qquad (A5)$$

and an analogous expression for $u(p')$. Without approximation, the result of this two-component reduction is

$$-\epsilon_\mu^* j^\mu = \langle \chi_f | A + i\underline{\sigma}\cdot\underline{B} | \chi_i \rangle \qquad (A6)$$

where

$$A = \sqrt{\frac{(E'+m)(E+m)}{4EE'}}\ \underline{\epsilon}^*\cdot\left[\left(1 - \frac{a\omega}{2m}\right)\frac{\underline{p}}{E+m} + \left(1 + \frac{a\omega}{2m}\right)\frac{\underline{p}'}{E'+m}\right.$$
$$\left. + \frac{a\,\underline{k}\times(\underline{p}\times\underline{p}')}{2m(E+m)(E'+m)}\right]\qquad (A7)$$

and

$$\underline{B} = \sqrt{\frac{(E'+m)(E+m)}{4EE'}}\left\{\left(1 - \frac{a\omega}{2m}\right)\frac{\underline{\epsilon}^*\times\underline{p}}{E+m} - \left(1 + \frac{a\omega}{2m}\right)\frac{\underline{\epsilon}^*\times\underline{p}'}{E'+m}\right.$$
$$- \frac{a}{2m}\underline{k}\times\underline{\epsilon}^*$$
$$\left. + \frac{a}{2m(E+m)(E'+m)}\left[\underline{p}\cdot(\underline{k}\times\underline{\epsilon}^*)\underline{p}' + \underline{p}'\cdot(\underline{k}\times\underline{\epsilon}^*)\underline{p} - \underline{p}\cdot\underline{p}'(\underline{k}\times\underline{\epsilon}^*)\right]\right\}$$
$$(A8)$$

Using $E' = E - \omega$ and $\underline{p}' = \underline{p} - \underline{k}$, we can now eliminate E' and $\underline{p}'$ and keep only the lowest order nonvanishing contributions to A and $\underline{B}$ as $\omega = |\underline{k}|$ becomes negligible compared to E and $|\underline{p}|$. For A we find

$$\lim_{\omega\to 0} A = \underline{\epsilon}^*\cdot\frac{\underline{p}}{E} = \underline{\epsilon}^*\cdot\underline{\beta}\qquad (A9)$$

as expected. The corrections are of order ω/E. The anomaly $\underline{a}$ does not enter until $O(\omega^2/mE)$.[10] The soft-photon limit of $\underline{B}$ is proportional to $|\underline{k}|$. We thus consider the analog of the the semiclassical $\underline{V}$ of (45) and (A2), namely

$$\lim_{\omega \to 0}\left(-\frac{2m}{\omega}\underset{m}{B}\right) = \left(a + \frac{m}{E}\right)\frac{\underset{m}{k} \times \underset{m}{\varepsilon}^*}{\omega} - \frac{a\,\underset{m}{p}\,\underset{m}{p}\cdot(\underset{m}{k} \times \underset{m}{\varepsilon}^*)}{E(E + m)\omega}$$

$$- \left(a + \frac{m}{E + m}\right)\frac{\underset{m}{p} \times \underset{m}{\varepsilon}^*}{E} \quad . \tag{A10}$$

With $E = \gamma m$, $\underset{m}{p} = \gamma m \underset{m}{\beta}$, and $\underset{m}{k} = \omega \underset{m}{n}$, the right-hand side of (A10) becomes identical with (45). This establishes that the semiclassical results derived from the Thomas-BMT equation of motion are correct quantum-mechanically for $\gamma \ll \gamma_c$.

B. **Comparison with a purely classical expression**

A localized magnetic moment $\underset{m}{\mu}$ in motion gives rise to radiation whose spectrum of radiated energy with polarization $\underset{m}{\varepsilon}$, frequency ω, and wave vector $\underset{m}{k} = \omega \underset{m}{n}$ is (Jackson, 1975, Eq.(14.74))

$$\frac{d^2 I_{classical}}{d\omega d\Omega} = \frac{\omega^4}{4\pi^2 c^3}\left| \int \left[(\underset{m}{n} \times \underset{m}{\varepsilon}^*)\cdot\underset{m}{\mu}(t) + \underset{m}{\varepsilon}^*\cdot(\underset{m}{\beta}(t) \times \underset{m}{\mu}(t))\right] \right.$$

$$\left. \times e^{i\omega t - i\underset{m}{k}\cdot\underset{m}{r}(t)} dt \right|^2 \quad . \tag{A11}$$

The first term in the amplitude is evidently proportional to $\underset{m}{\mu}\cdot\underset{m}{B}_{rad}$ while the second is proportional to $\underset{m}{d}\cdot\underset{m}{E}_{rad}$, where $\underset{m}{d} = \underset{m}{\beta} \times \underset{m}{\mu}$ is the electric dipole moment associated with the moving magnetic moment.

As it stands, (A11) bears only a slight resemblance to (47) with $\underset{m}{V}(t)$ given by (45) or (A10). This is because the magnetic moment $\underset{m}{\mu}(t)$ is the moment observed in the laboratory. In (47) the spin matrix element is taken in the rest frame of the particle. To make a meaningful comparison it is therefore necessary to express $\underset{m}{\mu}$

in terms of $\underset{m}{\mu}_0$, the rest frame magnetic moment. Since magnetization (magnetic moment density) and the negative of electric polarization (electric dipole moment density) transform under Lorentz transformations in the same way as $\underset{m}{B}$ and $\underset{m}{E}$, we find that

$$\underset{m}{\mu}\,\delta^{(3)}(\underset{m}{r} - \underset{m}{R}) = \gamma\left[\underset{m}{\mu}_0 - \frac{\gamma}{\gamma + 1}\underset{m}{\beta}(\underset{m}{\beta}\cdot\underset{m}{\mu}_0)\right]\delta^{(3)}(\underset{m}{r}' - \underset{m}{R}') \quad .$$

But the Dirac delta functions (inverse volumes) transform as[11]

$$\delta^{(3)}(\underset{m}{r}' - \underset{m}{R}') = \gamma^{-1}\delta^{(3)}(\underset{m}{r} - \underset{m}{R}) \quad .$$

Hence the moving moment $\underset{m}{\mu}$ is given in terms of the rest-frame moment $\underset{m}{\mu}_0$ as

$$\underset{m}{\mu} = \underset{m}{\mu}_0 - \frac{\gamma}{\gamma + 1}\underset{m}{\beta}(\underset{m}{\beta}\cdot\underset{m}{\mu}_0) \quad . \tag{A12}$$

The square bracketed quantity in the amplitude in (A11) then can be written

$$(\underset{m}{n} \times \underset{m}{\varepsilon}^*)\cdot\underset{m}{\mu} + \underset{m}{\varepsilon}^*\cdot(\underset{m}{\beta} \times \underset{m}{\mu})$$

$$= \underset{m}{\mu}_0\cdot\left[\underset{m}{n} \times \underset{m}{\varepsilon}^* - \frac{\gamma}{\gamma + 1}\underset{m}{\beta}\,\underset{m}{\beta}\cdot(\underset{m}{n} \times \underset{m}{\varepsilon}^*) - \underset{m}{\beta} \times \underset{m}{\varepsilon}^*\right] \quad . \tag{A13}$$

The corresponding expression from (44) and (45) is

$$\frac{e\hbar}{2mc}\underset{m}{\sigma}\cdot\underset{m}{V}$$

$$= \frac{e\hbar}{2mc}\underset{m}{\sigma}\cdot\left[\left(a + \frac{1}{\gamma}\right)\underset{m}{n} \times \underset{m}{\varepsilon}^* - \frac{a\gamma}{\gamma + 1}\underset{m}{\beta}\,\underset{m}{\beta}\cdot(\underset{m}{n} \times \underset{m}{\varepsilon}^*) - \left(a + \frac{1}{\gamma + 1}\right)\underset{m}{\beta} \times \underset{m}{\varepsilon}^*\right] \quad . \tag{A14}$$

Comparison of (A13) and (A14) shows that the classical expression has the same structure as the terms proportional to $\underline{a}$ in (A14). This is quite understandable when we realize that the term involving $\underline{a}$ in the

current (A3) is the Pauli term, $(\bar{\psi}\sigma_{\mu\nu}F^{\mu\nu}\psi)$. The second rank tensor $(\bar{\psi}\sigma_{\mu\nu}\psi)$ is the quantum-mechanical analog of the classical magnetization tensor $M_{\mu\nu}$ with the same Lorentz group properties.

It is instructive to write

$$\underset{\sim}{\mu}_0 = (1 + a) \frac{e\hbar}{2mc} \underset{\sim}{\sigma} \tag{A15}$$

in (A13) and then consider the difference between $\underset{\sim}{V}_{\text{classical}}$, namely $(1 + a)$ times the square bracket in (A13), and $\underset{\sim}{V}_{q-m}$ in (A14):

$$\Delta\underset{\sim}{V} = \underset{\sim}{V}_{\text{classical}} - \underset{\sim}{V}_{q-m}$$

$$= \left[\left(1 - \frac{1}{\gamma}\right)\underset{\sim}{n} \times \underset{\sim}{\varepsilon}^* - \frac{\gamma}{\gamma + 1} \underset{\sim}{\beta}\,\underset{\sim}{\beta}\cdot(\underset{\sim}{n} \times \underset{\sim}{\varepsilon}^*) - \frac{\gamma}{\gamma + 1} \underset{\sim}{\beta} \times \underset{\sim}{\varepsilon}^* \right] . \tag{A16}$$

With the identifications $\underset{\sim}{n} \times \underset{\sim}{\varepsilon}^* \to \underset{\sim}{B}$ and $\underset{\sim}{\varepsilon}^* \to \underset{\sim}{E}$, (A16) is seen to be proportional to the Thomas precession frequency (42). The difference between the classical (A13) and the quantum-mechanical (A14) is precisely the matrix element of the Thomas precession energy, $\hbar\underset{\sim}{\omega}_T\cdot\underset{\sim}{s}$, in the effective Hamiltonian (43).

We can now see clearly the difference between the purely classical treatment and the semiclassical or fully quantum-mechanical treatment. It hinges on the spin being a dynamical variable. In the classical approach, the magnetic moment, and by implication the spin since it is proportional via (A15), is a prescribed function of time. It is coupled to the radiation fields with an interaction proportional to $M_{\lambda\sigma}F^{\lambda\sigma}$. In terms of rest frame quantities this interaction becomes the first term in (43). In contrast, consideration

of spin as a dynamical variable leads to the complete effective Hamiltonian (43), with a direct coupling of the spin $\underset{\sim}{s}$ to the electromagnetic fields (provided the particle is charged) in addition to the coupling via the magnetic moment. This direct coupling, the Thomas precessional energy, gives a contribution to the radiation different in detail from the classical result (A13). Only in the limits $|a| \gg 1$ or $\beta \ll 1$ do the classical and quantum-mechanical expressions approach each other.

FOOTNOTES

1. See, for example, Jackson (1975), Sect. 14.6, or Landau and Lifshitz (1971), Sect. 74.

2. The results from VEPP-2 at Novosibirsk are summarized in Sect. 6 of Baier (1971b) which is a slightly updated version of Baier (1971a), with the addition of these experimental observations. The results of the Orsay storage ring group are contained in the report by Potaux (1971) to the accelerator conference in Geneva.

3. A summary of the work of Baier and Katkov on the classical regime and lowest order quantum corrections for ordinary and spin-flip synchrotron radiation can be found in Sect. 59 of Berestetskii, Lifshitz, and Pitaevskii (1971), written in collaboration with Baier.

4. Jackson (1975), Eq. (14.67).

5. See, for example, Barut (1964), Sect. II.4; Hagedorn (1963), Chapter 9; Jackson (1975), Sect. 11.11; Sard (1970), Sect. 5.4.

6. See the solutions for the temporal behaviors of the components of the polarization vector given by Baier (1971a,b), Sect. 3, esp. Eq. (3.23) ff.

7. Strictly, the number of photons per unit time is not an instantaneous rate but actually the number of photons per passage of the particle times the repetition rate $\omega_o/2\pi$. Similarly, the radiated power is energy per passage times $\omega_o/2\pi$.

8. Compare Eqs. (33) and (A11) for the classical expressions and see Low (1954) and Gell-Mann and Goldberger (1954) for the original discussions of the quantum-mechanical soft-photon theorem for radiation by a particle possessing a charge and a magnetic moment.

9. For economy of notation we lapse into units in which $\hbar = c = 1$, $e^2 = 1/137$, and use $(\omega, \underset{\sim}{k})$ as the photon's 4-vector, with $(E, \underset{\sim}{p})$ and $(E', \underset{\sim}{p}')$ as the charged particle's 4-momenta before and after emission.

10. It is somewhat curious to note that neglect of terms of order ω/E, with $\omega \sim \gamma^3\omega_o$, require $\gamma^2 \ll \gamma_c^2$, while neglect of terms of order ω^2/mE involving $\underline{a}$ require $\gamma^5 \ll \gamma_c^4$.

11. This is just the FitzGerald-Lorentz contraction.

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FIGURE CAPTIONS

Fig. 1. Orbit of a positively charged particle with a uniform magnetic field B into the page is a circular path of radius ρ traversed at constant speed v. In the frame moving with velocity $\underset{\sim}{v}$ to the right the orbit is retrograde, caused by a magnetic field $B' = \gamma B$ and a crossed electric field $E' = \gamma \beta B$ with directions as shown on the right.

Fig. 2. Segment of particle orbit as seen in the laboratory and in the instantaneously comoving inertial frame. In the laboratory the path is the arc of a circle of radius ρ, traversed at constant angular speed ω_o. In the moving frame it has a cusp at the origin. The tick marks and numbers along the path give the values of the laboratory time parameter, $\gamma \omega_o t$. Note that the length scale in the moving frame is ρ/γ^2.

Fig. 3. Coordinate system used in the calculations. The orbit lies in the x-y plane with x and y axes defined by the directions of $\underset{\sim}{\beta}$ and $\underset{\sim}{\dot\beta}$ at t = 0. The unit vector $\underset{\sim}{n}$ specifies the direction of the photon wave vector $\underset{\sim}{k}$.

Fig. 4. Characteristic time τ for growth of transverse polarization in units of the electron-positron time τ_o, Eq. (1c), as a function of anomaly $\underline{a}$ (bottom scale) or g factor (top scale).

Fig. 5. Asymptotic transverse polarization P as a function of the anomaly $\underline{a}$ or g factor. Positive values of P correspond to a preponderance of spins in the direction of $\hat\beta \times \hat{\dot\beta}$ (the direction of the guiding magnetic field for e > 0). For g < 0.5, the particles' magnetic moments end up preferentially

opposite to the magnetic field, contrary to naive

expectations.

Fig. 6. Normalized frequency spectra $\tau_o dw/d\nu$ for the number of

photons emitted per unit interval in the dimensionless

frequency variable $\nu = 2\omega/3\gamma^3\omega_o$. The dominant "down"

transition corresponds to a spin-flip from $\zeta_3 = +1$ to

$\zeta_3 = -1$ (spin finally in the direction opposite to $\hat{\beta} \times \hat{\dot{\beta}}$).

The small "up" transition is in the reverse direction.

Fig. 7. Log-log plot of separately normalized ordinary (nonflip) and

spin-flip power frequency spectra as functions of the dimen-

sionless variable $\nu = 2\omega/3\gamma^3\omega_o$. The actual spin-flip

power is much smaller than the ordinary power provided

$\gamma \ll \gamma_c$ (see Eq. (3) or (67)). At low frequencies

($\nu \ll 1$), the nonflip distribution varies as $\nu^{1/3}$, while the

spin-flip distributions vary as $\nu^{7/3}$. At high frequencies

($\nu \gg 1$) all spectra vanish exponentially (times different

powers.
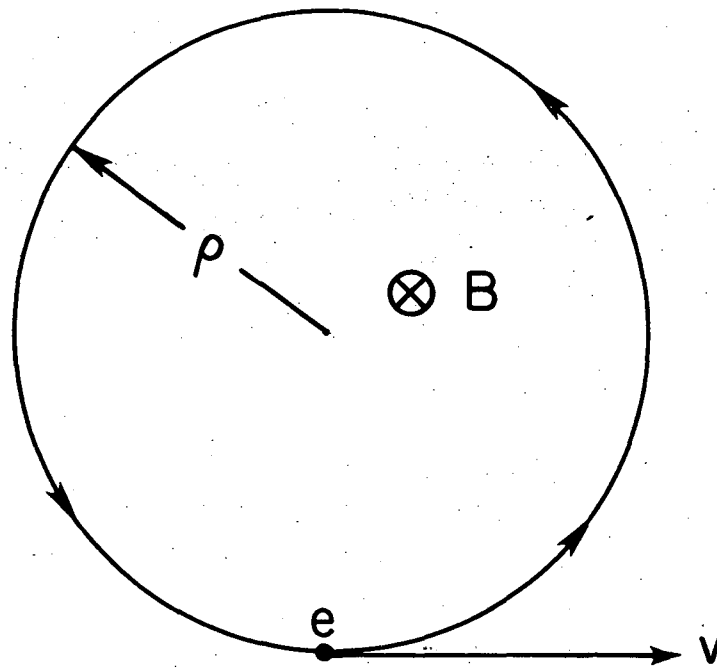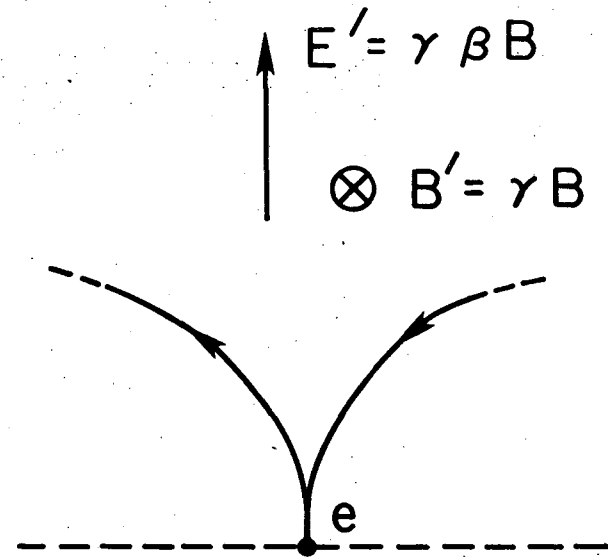
Laboratory frame   Instantaneously co-moving
inertial frame

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Fig. 1.

$$\omega_0 = \frac{\beta c}{\rho}$$

$\omega_0 t$ ρ

Laboratory frame

Instantaneously co-moving inertial frame

$+2.0$ $2\rho/\gamma^2$ -2.0

$+1.5$ ρ/γ^2 -1.5

$+1.0$ -1.0

$+0.5$ -0.5

$-\rho/\gamma^2$ ρ/γ^2
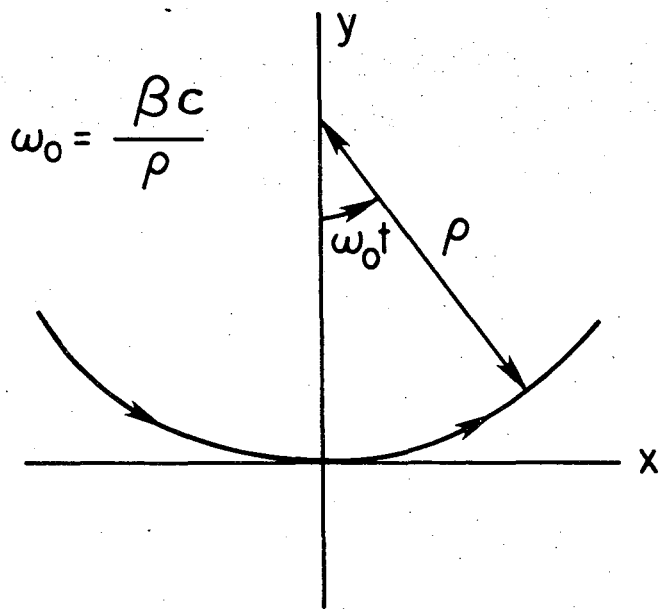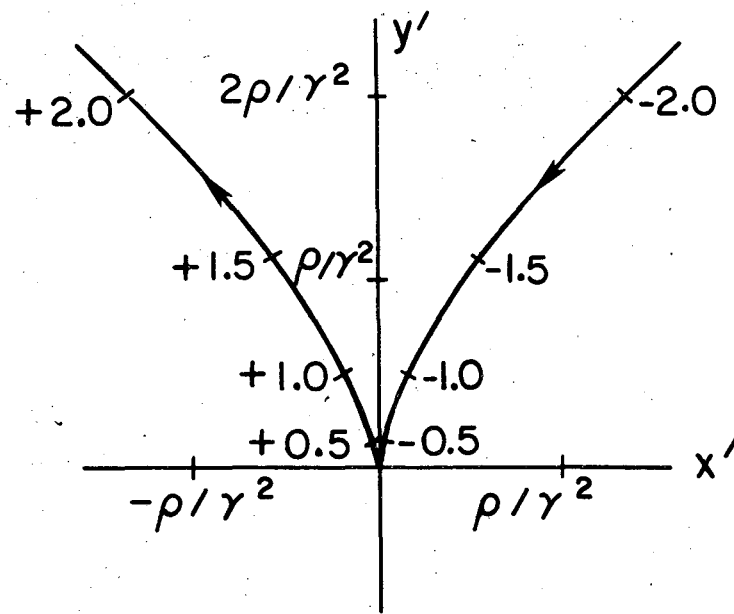
Fig. 2.
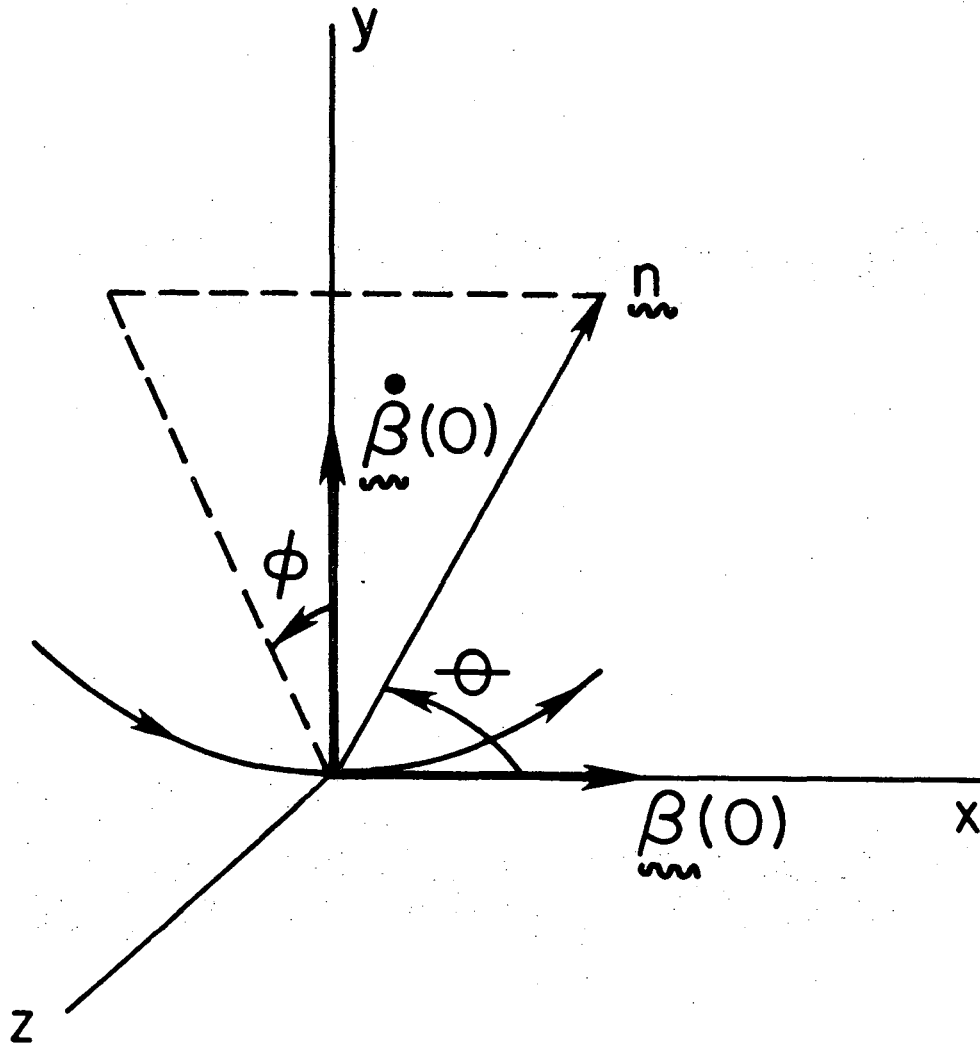
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Fig. 3.

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Fig. 4.

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Fig. 5.

Fig. 6.

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Fig. 7.

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