#28-98

Resolution of a Financial Puzzle

November 1998

Michael J. Brennan
Anderson Graduate School of Management
University of California, Los Angeles

and

Yihong Xia
Anderson Graduate School of Management
University of California, Los Angeles

Finance Working Paper
sponsored by

CAPITAL MANAGEMENT SCIENCES

Resolution of a Financial Puzzle

M.J. Brennan*and Y. Xia†

The apparent inconsistency between the Tobin Separation Theorem and the advice of popular investment advisors pointed out by Canner *et al* (1997) is shown to be explicable in terms of the hedging demands of optimising long-term investors in an environment in which the investment opportunity set is subject to stochastic shocks.

*Irwin and Goldyne Hearsh Professor of Banking and Finance, University of California, Los Angeles, and Professor of Finance, London Business School.

†Doctoral candidate, University of California, Los Angeles.

In a recent article, Canner, Mankiw and Weil (1997) have pointed out that the portfolio recommendations of popular financial advisors are apparently inconsistent with the Tobin (1958) separation theorem that, if the distribution of returns belongs to the elliptical class[1], then the proportions in which different risky assets are held in the optimal portfolio of risky assets is independent of the investor's risk aversion. The authors report that the financial advisors they study recommend that the ratio of bonds to stocks increase as the investor's risk aversion increases. They consider several possible explanations for this phenomenon, including the absence of a real riskless asset, non mean-variance preferences, differences between the historical and the subjective distribution of returns, constraints on short sales, and considerations raised by the fact that investors have multi-period horizons, but conclude that these explanations are unsatisfactory, leaving an apparent puzzle. In this paper we show that the variation in the ratio of bonds to stocks recommended by the financial advisors is quite consistent with a model of portfolio optimization in a dynamic context. The reason for the violation of the separation principle is that bonds are not just any risky asset but have the particular property that their returns covary negatively with expectations about future interest rates. This covariation, which plays no role in the one period context of the Tobin separation theorem, is important for an investor with a multi-period horizon, as the classic paper of Merton (1973) recognizes[2]. The model of time varying interest rates and expected stock returns is presented in Section I and some representative calculations are offered in Section II.

[1]Tobin originally stated the theorem for mean variance preferences. The necessary condition for investors to have mean-variance preferences for arbitrary utility functions is that the distribution of returns belong to the elliptical class of which the normal distribution is a member.

[2]Canner *et al* (1997) recognize the possible importance of the dynamic considerations induced by a multi-period horizon, but attempt to model the dynamic portfolio decision in a *static* fashion by considering only a buy and hold policy, adjusting the mean vector and covariance matrix for the length of the horizon: they conclude that "it appears impossible to reconcile the advice of financial advisors with the textbook CAPM by changing the time horizon".

1 The Model

Uncertainty about future interest rates is represented by the Hull and White (1996) two-factor extension of the Vasicek (1977) model of the term structure of interest rates[3], while the equity risk premium in excess of the instantaneously riskless interest rate is assumed to be a constant. Thus, the instantaneously riskless interest rate, $r(t)$, is assumed to follow a bivariate Markov process:

$$\begin{aligned} dr &= [\theta(t) + \lambda_r + u - kr]dt + \sigma_r dz_r \\ du &= [\lambda_u - bu]dt + \sigma_u dz_u \end{aligned} \tag{1}$$

where dz_r and dz_u are increments to standard Brownian motions, and λ_r and λ_u are the risk premia per unit exposure to innovations in r and u. In this model of interest rates u affects the stochastic 'target' towards which the riskless rate r is adjusting; this second element in the state vector allows for independent variation in the short and long end of the yield curve which is apparent in the changing slope of the yield curve[4]. Given the stochastic process (1), the price at time t of a bond paying \$1 at time T is given by:

$$P(t,T) = A(t,T)e^{-B(t,T)r - C(t,T)u} \tag{2}$$

where:

$$\begin{aligned} B(t,T) &= \frac{1}{a}[1 - e^{-a(T-t)}] \\ C(t,T) &= \frac{1}{a(a-b)}e^{-a(T-t)} - \frac{1}{b(a-b)}e^{-b(T-t)} + \frac{1}{ab} \end{aligned}$$

and $A(t,T)$ is a complicated function of $\theta(\tau), \tau = t, T$, that can be chosen so that the resulting bond yields fit any given term structure[5].

Let P_j denote the price at time t of a bond maturing at time T_j, and let π_S denote the risk premium on the equity security which is assumed to be

[3]The two factor version of the model is more realistic than the original single factor model in that it allows for independent variation in long and short term yields.

[4]Litterman and Scheinkman (1991) show that two factors capture around 98% of the variance of bond returns.

[5]The continuously compounded yield to maturity at time t on a bond maturing at time T is $\frac{1}{T-t}\left(-\ln A(t,T) + B(t,T)r + c(t,T)u\right)$.

constant. Then, using (2), the joint stochastic process for the stock price, S, and the bond prices, P_j $(j = 1, 2)$, is given by:

$$\frac{dS}{S} = (r + \pi_S)dt + \sigma_S dz_s \tag{3}$$

$$\frac{dP_j}{P_j} = (r + \pi_j)dt - B(t, T_j)\sigma_r dz_r - C(t, T_j)\sigma_u dz_u \tag{4}$$

where:

$$\pi_j \equiv -\lambda_r B(t, T_j) - \lambda_u C(t, T_j) \tag{5}$$

We consider the problem of an investor who is concerned with maximizing the expected value of an iso-elastic utility function defined over period T wealth, W_T:

$$Max\ E[\frac{W_T^{1-\gamma}}{1-\gamma}|\Im_t] \tag{6}$$

where $\Im_t \equiv \{r_\tau, u_\tau, S_\tau, \tau \leq t; W_t\}$ denotes the information available at time t. The investor is assumed to be able to invest in the equity security and in two (or more) bonds of different maturities. Let Σ denote the $(3x3)$ covariance matrix of the returns on the three risky assets, $\underline{\pi}$ the $(3x1)$ vector of risk premia, and $\underline{\sigma}_{Rr}$ and $\underline{\sigma}_{Ru}$ the $(3x1)$ vectors of covariances of the asset returns with the innovations in r and u[6].

Then the Bellman equation for optimality may be written as:

$$\begin{aligned} 0 \ = \ & \max_{\underline{x}}[-J_t + WJ_W(r + \underline{x}'\underline{\pi}) + J_r(\theta(t) + u - kr) - J_u bu + \frac{1}{2}J_{rr}\sigma_r^2 \\ & + \frac{1}{2}J_{uu}\sigma_u^2 + \frac{1}{2}J_{WW}W^2\underline{x}'\Sigma\underline{x} + J_{Wr}W\ \underline{x}'\underline{\sigma}_{Rr} + J_{Wu}W\ \underline{x}'\underline{\sigma}_{Ru} \\ & + J_{ru}\sigma_{ru}] \end{aligned} \tag{7}$$

where $J(W, r, u, t, T)$ is the indirect utility of wealth function which satisfies the terminal boundary condition:

$$J(W, r, u, T, T) = \frac{W^{1-\gamma}}{1-\gamma} \tag{8}$$

[6]Expressions for Σ, $\underline{\sigma}_{Rr}$, and $\underline{\sigma}_{Ru}$ are given in the Appendix.

The first order condition in (7) implies that the vector of optimal risky asset proportions, $\underline{x}^*$, is given by:

$$\underline{x}^* = \frac{-J_W}{WJ_{WW}}\Sigma^{-1}\underline{\pi} - \frac{J_{Wr}}{WJ_{WW}}\Sigma^{-1}\underline{\sigma}_{Rr} - \frac{J_{Wu}}{WJ_{WW}}\Sigma^{-1}\underline{\sigma}_{Ru} \tag{9}$$

The following proposition which characterizes the investor's optimal portfolio strategy may be verified by substitution in equations (7) and (8)[7]:

Proposition 1 *(i) The solution to the optimal control problem (7) subject to the terminal boundary condition (8) is:*

$$J(W, r, u, t, T) = \frac{F(t,T)W^{*1-\gamma}}{1-\gamma} \tag{10}$$

where $F(t,T)$ is the solution to an ordinary differential equation with boundary $F(T,T) = 1$, and $W^* \equiv W/P(r,u,t,T)$ is the forward value of the investor's wealth measured in time T dollars.

(ii) The vector of optimal risky asset allocations given by:

$$\underline{x}^* = \frac{1}{\gamma}\Sigma^{-1}\underline{\pi} + (1-\frac{1}{\gamma})\Sigma^{-1}\begin{pmatrix}\sigma_{SP}\\ \sigma_{P_1P}\\ \sigma_{P_2P}\end{pmatrix} \tag{11}$$

where P denotes the price of a bond which matures at the horizon, T, and σ_{SP} and σ_{P_jP} denote the covariances between the returns on the T period bond and the return on the stock and the T_j period bond respectively.

(iii) Since the return on the bond that matures at time T is an exact linear combination of the returns on the other two bonds, the portfolio allocation vector, $\underline{x}^*$, may also be written as:

$$\underline{x}^* = \frac{1}{\gamma}\Sigma^{-1}\underline{\pi} + (1-\frac{1}{\gamma})\begin{pmatrix}0\\ \beta_1\\ \beta_2\end{pmatrix} \tag{12}$$

where β_1, and β_1 are the coefficients from the multiple regression of the return on the $T-$ period bond on the other two bonds[8]*.*

[7] The solution to the portfolio problem may be obtained constructively using the martingale pricing technique suggested by Cox and Huang (1989).

[8] $\beta_1 = (C_2B_0 - B_2C_0)/(B_1C_2 - C_1B_2), \beta_2 = (-C_1B_0 + B_1C_0)/(B_1C_2 - C_1B_2)$ where $B_0 \equiv B(t,T)$, $B_1 \equiv B(t,T_1)$ etc.

Equations (11) and (12) clearly show, as in Merton (1973), that the vector of optimal portfolio proportions is a linear combination of the mean variance efficient portfolio, $\Sigma^{-1}\underline{\pi}$, and of a hedge portfolio where the relative weights depend on the risk aversion parameter, γ. In general, the portfolio weights in the hedge portfolio are proportional to the coefficients from a regression of the return on the T period bond on the returns of the two bonds held by the investor.

If one of the discount bonds held by the investor matures on the horizon date,T, then (assuming without loss of generality that it is bond 1) $T_1 = T$, and the hedge portfolio takes on a particularly simple form, since the regression of the return on the T period bond on the returns of the two bonds held by the investor becomes degenerate and all of the weight is placed on the T period bond; the hedge portfolio is then simply the T period bond, and the portfolio holdings are given by[9]:

$$\underline{x}^* = \frac{1}{\gamma}\Sigma^{-1}\underline{\pi} + (1 - \frac{1}{\gamma})\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{13}$$

As the investor's risk aversion is increased, the weight on the T period bond increases until, in the limit, as $\gamma \to \infty$,the investor allocates the whole of her wealth to this bond and, as a result, bears no risk.

While equations (12) and (13) are inconsistent with the Tobin Separation Theorem, they do place other restrictions on the characteristics of optimal portfolios. It is straightforward to verify the following:

Corollary: *(i) If the stock holding is positive for any level of risk aversion then it is decreasing in the risk aversion coefficient,* γ.

(ii) If the stock holding is positive and $T = T_1$ *or* T_2, *then* $\alpha(\gamma) \equiv (x_2 + x_3)/x_1$, *the ratio of bonds to stock held in the optimal portfolio, is increasing in the risk aversion coefficient,* γ.

(iii) The proportions of the portfolio allocated to the riskless asset ("cash"), and to each of the bonds, are linear functions of the proportion allocated to stock.

[9]In this case the investor's opportunity set can be summarized by the price of the $T-$*period* bond. Merton (1971) also analyses a model in which changes in the investment opportunity set are perfectly correlated with an asset price.

2 Some Illustrative Calculations

Table 1 repeats the evidence on the advisors' recommendations reported by Canner *et al.*(1997). Note first that, consistent with the Corollary (i), the recommended allocations to stock are positive and decreasing in the level of risk aversion. Corollary (ii) predicts that if the maturity of one of the bonds coincides with the investor's horizon, then the ratio of bonds to stock held in the optimal portfolio will be increasing in the investor's risk aversion. This is precisely the pattern observed for all four advisors, and the pattern that Canner *et al.* regarded as a puzzle. Finally, since the Corollary predicts that the bond allocation will be a linear function of the stock allocation, the fourth column reports the bond allocation for the moderate investor that would be obtained by linear interpolation from the bond and stock allocations for the other two investors. In three cases the interpolated allocation which is consistent with the theory falls below the actual recommendation. However, *The New York Times*' recommendation does correspond to the linearity restriction imposed by the theory.

As an illustration of the potential role of dynamic hedging on portfolio allocation rules, optimal portfolios were constructed for investors with decision horizons of 1, 10, and 20 years when the set of available assets included a riskless asset, bonds of maturities of 5, 10, and 20 years, and an equity portfolio. The variance-covariance matrix of asset returns was estimated from the time series of monthly returns on the value weighted market index and on constant maturity Treasury bonds of these maturities for the period January 1942 to December 1997 provided by CRSP[10]. The vector of mean returns was constructed by assuming that the pure expectations theory of the term structure holds so that $\lambda_u = \lambda_r = 0$, and that the equity market risk premium was 4%.[11]

Table 2 reports the cash, stock and aggregate bond allocations for investors with different horizons and risk aversions, and different sets of bonds in which to invest. In all cases it is assumed that the investor can invest in a

[10]The covariance matrix of the theory contains covariances with pure discount bonds of different maturities. We have approximated this with covariances of constant maturity coupon bonds.

[11]While this is significantly below the historical average risk premium, Brown *et al.*(1995) observe that survival bias would make the historical data consistent with an equity risk premium of *4%* if the probability of a stock market surviving over the long run is *80%*.

bond with a maturity equal to his horizon, and therefore the risky asset allocations are computed using equation (13). The allocations are independent of the investment horizon, and the stock allocations are virtually identical for the 10 year horizon investor whether the bond portfolio consists of 1 and 10 year bonds or 10 and 20 year bonds[12]; this is what we should expect if all bond returns are subject to the same two stochastic shocks as in equation (4), since then the returns on any two bonds are spanned by the returns on any other two. However the amount optimally allocated to bonds does depend on which maturity bonds are being considered. For example, an investor with a risk aversion parameter of 4 and an investment horizon of 10 years will optimally allocate 37% of his wealth to bonds if he considers only the 1 and 10 year bonds, while the optimal allocation rises to 52% if he considers the 10 and 20 year bonds. The bond allocation is increasing in the investor's risk aversion[13].

Overall, the portfolio allocations shown in Table 2 are not too different from the advisors' recommendations shown in Table 1; in particular, the ratio of bonds to stocks is increasing in the investor's risk aversion. Thus, the puzzle that Canner *et al.* identified can be resolved by consideration of the portfolio hedging demands of an investor with a more than one period horizon when the investment opportunity set is stochastic; this was first discussed by Merton (1973). However, the usefulness of the advisors's recommendations could certainly be enhanced by more specific advice about the maturity of the bonds to be included in the portfolio.

3 Appendix

Σ, the $(3x3)$ covariance matrix of the returns on the three risky assets is given by:

$$\Sigma = \begin{pmatrix} \sigma_S^2 & \sigma_{P_1S} & \sigma_{P_2S} \\ \sigma_{P_1S} & \sigma_{P_1}^2 & \sigma_{P_1P_2} \\ \sigma_{P_2S} & \sigma_{P_1P_2} & \sigma_{P_2}^2 \end{pmatrix}$$

where $\sigma_{PS} = -\sigma_{Sr}B(t,T_j) - \sigma_{Su}C(t,T_j)$, $\sigma_{P_j,P_k} = \sigma_r^2 B(t,T_j)B(t,T_k) + \sigma_{ru}[B(t,T_j)C(t,T_k)+B(t,T_k)C(t,T_j)]+\sigma_u^2 C(t,T_j)C(t,T_k)$, $\sigma_{P_j}^2 = \sigma_r^2 B(t,T_j)^2+$

[12]The stock allocations show some small variation depending on the set of bond investments considered.

[13]Not shown in the table is that the optimal allocation between the two bonds changes with the investor's horizon as shown in equation (13).

$2\sigma_{ru}[B(t,T_j)^2 + C(t,T_j)^2] + \sigma_u^2 C(t,T_j)^2$, and σ_{ru} denotes the instantaneous correlation between the innovations in r and u *etc.*

4 References

Brown, S. J; Goetzmann, W. N; Ross, S. A.,1995, Survival, *Journal of Finance,* 1995, 853-873.

Canner.N., N.G. Mankiw, and D.N. Weil, 1997, An Asset Allocation Puzzle, *American Economic Review*, 87, 181-191.

Cox, J.C., and C-f. Huang, 1989, Optimum Consumption and Portfolio Policies When Asset Prices Follow a Diffusion Process, *Journal of Economic Theory*, 49, 33-83.

Hull, J., and A. White,1996, *Hull-White on Derivatives*, RISK Publications, London.

Litterman, R., and J. Scheinkman, 1991, Common Factors affecting Bond Returns, *Journal of Fixed Income*, 1, 54-61.

Merton, R.C., 1973, An Intertemporal Capital Asset Pricing Model, *Econometrica*, 41, 867-887.

Tobin, J., 1958, Liquidity Preference as Behavior towards Risk, *Review of Economic Studies*, 25, 68-85.

Vasicek, O., 1977, An Equilibrium Characterization of the Term Structure, *Journal of Financial Economics*, 5, 177-188.

Advisor and Investor Type	Cash	Stock	Bonds	Linearly interpolated Bond Allocation	Ratio of Bonds to Stock
Fidelity					
Conservative	50%	20%	30%		150%
Moderate	20	40	40	30	100
Aggressive	5	65	30		46
Merrill Lynch					
Conservative	20	45	35		78
Moderate	5	55	40	30	73
Aggressive	5	75	20		27
Jane Bryant Quinn					
Conservative	50	20	30		150
Moderate	10	50	40	18.75	80
Aggressive	0	100	0		0
The New York Times					
Conservative	20	40	40		100
Moderate	10	60	30	30	50
Aggressive	0	80	20		25

Asset Allocations Recommended by Financial Advisors and
Evidence on the Linearity Hypothesis
Source: Canner *et al.* (1997)

Table 1

		1 and 10 year bonds		10 and 20 year bonds	
	Horizon	1 year	10 years	10 years	20 years
γ					
2	Cash	20%	20%	-9%	-9%
2	Bonds	-26	-26	3	3
2	Stock	105	105	106	106
2	Stock/ Bonds	*n/a*	*n/a*	353	353
3	Cash	14	14	-6	-6
3	Bonds	16	16	35	35
3	Stock	70	70	71	71
3	Stock/ Bonds	437	437	203	203
4	Cash	10	10	-5	-5
4	Bonds	37	37	52	52
4	Stock	53	53	53	53
4	Stock/ Bonds	143	143	102	102
6	Cash	7	7	-3	-3
6	Bonds	58	58	68	68
6	Stock	35	35	35	35
6	Stock/ Bonds	60	60	51	51

Portfolio Allocations for Investors with Different Horizons and Risk Aversion

The investor is assumed to be allowed to invest in a cash security, an equity security and two bonds. The first two columns assume that the investor invests in 1 and 10 year bonds; the second two columns that she invests in 10 and 20 year bonds. The equity risk premium (π_S) is assumed to be 4%, and the pure expectations hypothesis is assumed to hold so that $\lambda_u = \lambda_r = 0$. The variance-covariance matrices of asset returns are derived from monthly returns on the value-weighted equity market index and constant maturity bonds for the period January 1942 to December 1997 from CRSP

Table 2